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Master Thesis

Towards a general theory of homotopic Hopf-Galois extensions

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Introduction

When on the 30th may 1832, the night before the duel that was about to put an end to his life, Evariste Galois (1811-1832) summarized his notes, he was certainly not imagining how many books and articles of the twentieth and the twenty-first centuries would associate his name to such a variety of topics that are so far from his original work. Since antiquity, up to the nineteenth century, no one has been in position to solve the problem of polynomial equations of degree (strictly) greater than 4. It is only in the nineteenth century that mathematicians reached the answer: It is impossible to solve by radicals a general polynomial equation of degree greater or equal to 5, while few formulas where found for particular equations. What is interesting about this quest to solve such equations is not the set of formulas itself, but all mathematical theory needed to solve the problem. Many of the fundamental notions of modern algebra that Galois initiated, such as groups, rings, fields or complex numbers, had to be developed. It is this theoretical machinery, rather than the few formulas to solve specific cubic or quadratic equations, that has continued to grow up to this day. The seed that Galois planted then has now become a forest.

This paper is an attempt to give rise to yet another tree, using the branches of mathematics that are homotopy theory and category theory. The problem is as follow: Let $f : A \hookrightarrow B$ be an extension of commutative rings and let G be a finite subgroup of

$$\{g : S \to S \mid g \text{ is an } R\text{-algebra automorphism}\},$$

where B has the A-algebra structure induced by f. This gives rise to two maps

- $i : A \hookrightarrow B^G$ defined to be the inclusion of A into the ring B^G of fixed elements of B under all elements of G, and

- $h : B \otimes_A B \to \prod_G B$ defined to be the commutative ring homomorphism specified by $h(x \otimes y) := (x \cdot g(y))_{g \in G}$, where $\prod_G B$ is the set of all sequences $(x_g)_{g \in G}$ in B,

which, according to S.U. Chase, D.K. Harrison and A. Rosenberg (cf. [4]), are such that $f : A \hookrightarrow B$ becomes a G-Galois extension of commutative rings if and only if i and h are bijective. We want to extend this fact to the case where $f : A \to B$ becomes a morphism of commutative monoids in a closed symmetric monoidal model category $(\mathcal{C}, \otimes, 1)$, and where the group G becomes a commutative Hopf monoid in $\mathcal{C}$. The first difficulty is to know what the induced maps i and h become in this context, and how the monoid G must act on A and B. Using the model category structure of $\mathcal{C}$, we may then generalize the notion of Galois extension to the case where i and h are weak equivalences, in order to obtain an appropriate definition of homotopic Hopf-Galois extensions. The strategy is then to establish different properties related to homotopic Hopf-Galois extensions which generalize some of the results found in the articles [4] and [19] of S.U. Chase and J. Rognes respectively, and this, while moving toward the goal of establishing an homotopic Hopf-Galois correspondence theorem which encompasses the corresponding theorem for the special case of commutative rings.

Before being in position to solve this problem, we first have to establish the needed theory on closed symmetric monoidal model categories and on Galois extensions of commutative rings. We start in chapter 1 by studying model categories in general. We shall provide the definition and some examples of a model category, before moving on to the study of the induced homotopy category and its left and right derived functors. We shall finally introduce Quillen pairs and Quillen equivalences, and terminate with an important theorem that explains how these two notions induce adjoint pairs and categorical equivalences respectively.

In chapter 2, we shall restrict our attention to monoidal model categories and cofibrantly generated model categories, for which we shall need to generalize the notions of monoids, modules and algebras to specific objects of a monoidal category. We end the chapter with a theorem that provides a condition under which the structure of a cofibrantly generated monoidal model category is preserved when we restrict our attention to its underlying categories of modules and algebras.

Chapter 3 establishes Quillen equivalences between some specific monoidal categories. This requires in particular a study on simplicial and cosimplicial objects which we shall need for the rest of the paper.

In chapter 4, we introduce the Galois theory of commutative rings based on the classical Galois theory of field. This will involve a reinterpretation of what is usually meant by a Galois extension in terms of morphisms i and h, followed by their generalization to commutative rings. We end the chapter by providing a Galois correspondence theorem for this context.

It is only at this stage that we may begin the study, in chapter 5, of the above question on the generalization of Galois extensions to homotopic Hopf-Galois extensions.

For full comprehension of the text, the reader is expected to have a solid knowledge of general topology, algebraic topology, as well as graduate level algebra which encompasses rings and modules theory, basic homological algebra, and the Galois theory of finite extension of fields. A previous knowledge of basic category theory is highly recommended but not essential in the sense that most of the necessary notions are summarized in the first section; the beginner with the appropriate mathematical maturity should be able to follow the text with the help of a book to complement the theory.

I would like to thank Kathryn Hess Bellwald for her guidance, support and availability; and without whom this project would not have reached its present form.

Chapter 1

Model categories

This chapter is an introduction to the theory of model categories, which was first developed by D.G. Quillen. Model categories are categories endowed with a supplementary structure given by three classes of morphisms and a set of five axioms. This provides enough structure to develop a generalized homotopy theory on the level of categories. It of course encompasses traditional homotopy theory on the category of topological spaces and can be applied to any of the more algebraic categories that satisfy the supplementary axioms.

After a concise reminder of the basic theory of category theory needed for this paper in section 1.1, we will provide, in section 1.2, the definition of a model category followed by some standard examples such as topological spaces or chain complexes of modules. Proving that a particular category has a model structure is usually long and very technical. For this reason, we won't be detailing these examples. We will instead proceed, in section 1.3, to the study of the homotopy relations in model categories. These are the relations of *right homotopy* and *left homotopy*. We will see under which conditions these two relations form equivalence relations and coincide to give rise to a generalized general notion of homotopy. From this homotopic relation, it then becomes possible to derive another category, the *homotopy category*, whose objects are the same as in the original model category, but whose morphisms become the equivalent classes of the original morphisms under the homotopic equivalence relation; this is done in 1.4. We will end this chapter, in section 1.5, by introducing derived functors. These functors are the direct generalization of the left and right derived functors used in traditional homological algebra. They will provide us with a final important result (cf. theorem 1.5.12) which establishes categorical equivalences between the homotopy categories of two model categories via the notion of Quillen equivalences.

1.1 Reminder on category theory

In this section we review the necessary notions and constructions we will need for the rest of the paper. For reference and notational purposes, we briefly state the fundamental definitions of category theory.

Definition 1.1.1. A *(large) category* $\mathcal{C}$ consists of a pair $(Ob\mathcal{C}, Mor\mathcal{C})$, where $\mathcal{C} = Ob\mathcal{C}$ is the class of *objects* of $\mathcal{C}$ and $Mor\mathcal{C}$ the class of *morphisms*, *arrows* or *maps* of $\mathcal{C}$, such that:

- For every morphism $f \in Mor\mathcal{C}$, there exists a *domain* $X \in \mathcal{C}$ and a *codomain* $Y \in \mathcal{C}$ associated to it. We note $f : X \to Y$ and denote by $\mathcal{C}(X, Y) = Mor_{\mathcal{C}}(X, Y)$ the class of all morphisms having X as domain and Y as codomain.

- For every objects $X, Y, Z \in \mathcal{C}$, there exists a *composition* map

$$\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z) \ : \ (f,g) \mapsto g \circ f$$

  which is associative and such that for every object $X \in \mathcal{C}$ there is an identity morphism $id_X : X \to X$ satisfying $f \circ id_X = f$ for every $f \in \mathcal{C}(X,Y)$ and $id_X \circ g = g$ for all $g \in \mathcal{C}(Y,X)$.

We call *small categories* those categories which exist within the context of set theory, ie. in which the class of objects, as well as the class of morphisms between any two given objects, form sets. Furthermore, we say that a category $\mathcal{C}$ is *finite* if $Ob\mathcal{C}$ is a finite set and $\mathcal{C}(X,Y)$ is a finite set for any objects X, Y in $\mathcal{C}$.

Notations 1.1.2. The most frequently encountered categories are denoted

- $\mathcal{S}et$ for the category whose objects are sets and whose morphisms are set maps,
- $\mathcal{T}op$ for the category whose objects are topological spaces and whose morphisms are continuous functions,
- $\mathcal{T}op_*$ for the category whose objects are pointed topological spaces and whose morphisms are pointed continuous functions,
- $\mathcal{G}r$ for the category whose objects are groups and whose morphisms are group homomorphisms,
- $\mathcal{A}b$ for the category whose objects are abelian groups and whose morphisms are abelian group homomorphisms,
- $\mathcal{R}ng$ for the category whose objects are rings and whose morphisms are ring homomorphisms,
- $\mathcal{F}ld$ for the category whose objects are fields and whose morphisms are field homomorphisms,
- ${}_R\mathcal{M}od$ (resp. $\mathcal{M}od_R$) for the category whose objects are left (resp. right) R-modules and whose morphisms are left (resp. right) R-modules homomorphisms,
- $\mathcal{C}h$ (resp. $\mathcal{C}h^+$) for the category of chain complexes (resp. $\mathbb{N}$-graded chain complexes) of abelian groups whose morphisms are the obvious collections of abelian group homomorphisms in each dimension that make the appropriate squares commute,
- $\mathcal{C}h_k$ (resp. $\mathcal{C}h_k^+$) for the category of chain complexes (resp. $\mathbb{N}$-graded chain complexes) of k-modules for a commutative ring k.

Furthermore, for a given category $\mathcal{C}$ and an object A in $\mathcal{C}$, we note

- $\mathcal{C}^{op}$ the *opposite category* whose objects are the same as $\mathcal{C}$ and whose morphisms are the reversed arrows of $\mathcal{C}$,
- $A \downarrow \mathcal{C}$ the *under category* whose objects are all morphisms $A \to X$ in $\mathcal{C}$ with $X \in \mathcal{C}$ and whose morphisms from a given object $f : A \to X$ to a given object $g : A \to Y$ are the morphisms $h : X \to Y$ in $\mathcal{C}$ which satisfy $hf = g$,
- $\mathcal{C} \downarrow A$ the *over category* whose objects are all morphisms $X \to A$ in $\mathcal{C}$ with $X \in \mathcal{C}$ and whose morphisms from a given object $f : X \to A$ to a given object $g : Y \to A$ are the morphisms $h : X \to Y$ in $\mathcal{C}$ which satisfy $gh = f$.

- $\mathcal{C}^{\rightarrow}$ the *arrow category* whose objects are the morphisms of $\mathcal{C}$ and whose morphisms between any two objects f, g in $\mathcal{C}^{\rightarrow}$ are the commutative squares of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

in $\mathcal{C}$.

Definition 1.1.3. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a pair of mappings

$$F = F_{ob} : Ob\mathcal{C} \to Ob\mathcal{D} \qquad \text{and} \qquad F = F_{mor} : Mor\mathcal{C} \to Mor\mathcal{D},$$

such that

- if $g \circ f$ is defined in $\mathcal{C}$, then $F(g) \circ F(f)$ is defined in $\mathcal{D}$ and $F(g \circ f) = F(g) \circ F(f)$,
- for each object X in $\mathcal{C}$ we have $F(id_X) = id_{F(X)}$.

Definitions 1.1.4. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A *natural transformation* from F to G is a map $\tau : Ob\mathcal{C} \to Mor\mathcal{D}$ such that for every $X \in Ob\mathcal{C}$ and every morphism $(f : X \to Y) \in Mor\mathcal{C}$ the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\tau_X"'] & F(Y) \arrow[d, "\tau_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y), \end{tikzcd}$$

with $\tau_X := \tau(X)$ and $\tau_Y := \tau(Y)$, commutes.

If in addition τ_X is an *isomorphism*, that is if there exists a morphism $g : G(X) \to F(X)$ in $\mathcal{C}$ such that $g \circ \tau_X = id_{G(X)}$ and $\tau_X \circ g = id_{F(X)}$, we say that τ is a *natural isomorphism.*

Moreover, we say that F is an *equivalence of categories* if there exists a functor $F' : \mathcal{D} \to \mathcal{C}$ such that the composites FF' and $F'F$ are related to the appropriate identity functors by natural isomorphisms.

Notation 1.1.5. For two categories $\mathcal{C}$ and $\mathcal{D}$, we denote by $\mathcal{C}^{\mathcal{D}}$ the category whose objects are the functors from $\mathcal{D}$ to $\mathcal{C}$ and whose morphisms are all the natural transformations between these functors.

Definitions 1.1.6. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *epimorphism* if for every morphisms $g, g' : Y \to Z$ in $\mathcal{C}$ with $gf = g'f$ we have $g = g'$. On the other hand, f is a *monomorphism* if for every morphisms $g, g' : Z \to X$ in $\mathcal{C}$ with $fg = fg'$ we have $g = g'$.

Furthermore, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *full* (respectively *faithful*), if for each pair (X, Y) of objects of $\mathcal{C}$ the map

$$\mathcal{C}(X, Y) \;\to\; \mathcal{D}(F(X), F(Y))$$

induced by F is an epimorphism (respectively a monomorphism). We then say that a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if the inclusion functor $i : \mathcal{C}' \to \mathcal{C}$ is full (the functor i is always faithful).

Definition 1.1.7. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors between categories $\mathcal{C}$ and $\mathcal{D}$. The functors F and G are said to be *adjoints* or to form an *adjoint pair* if there is a natural isomorphism φ from

$$\mathcal{D}(F(-), -) : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}et \qquad \text{to} \qquad \mathcal{C}(-, G(-)) : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}et;$$

in other words if for any object $X \in \mathcal{C}$ and for any object $Y \in \mathcal{D}$ there is an isomorphism $\mathcal{D}(F(X), Y) \cong \mathcal{C}(X, G(Y))$ such that the diagram

$$\begin{tikzcd}
(f : F(X) \to Y) \arrow[r, "\varphi_{X,Y}", "\cong"'] \arrow[d] & (f^\sharp : X \to G(Y)) \arrow[d] \\
(g^\flat : F(X') \to Y') \arrow[r, "\varphi_{X',Y'}", "\cong"'] & (g : X' \to G(Y')).
\end{tikzcd}$$

commutes for any given morphisms $f \in \mathcal{D}(F(X), Y)$ and $g \in \mathcal{C}(X', G(Y'))$. We denote this

$$F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$$

and say that F is the *left adjoint* of G and G the *right adjoint* of F. Since any two left adjoints of G (respectively any two right adjoints of F) are canonically naturally isomorphic, we then speak of *the* left adjoint or *the* right adjoint of a functor (provided they exist). In addition, the adjoint pair $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ induces two natural transformations

$$\eta : Id_{\mathcal{C}} \to GF \qquad \text{and} \qquad \varepsilon : FG \to Id_{\mathcal{D}}$$

which are respectively called *unit* and *counit* of the adjunction. The adjunction (F, G, φ) is uniquely determined by $(F, G, \eta, \varepsilon)$ via the components

$$\eta_X = \varphi(id_{F(X)}) \qquad \text{and} \qquad \varepsilon_Y = \varphi^{-1}(id_{G(Y)}).$$

It can equally well be described by the two identities

$$\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, equal] & FGF \arrow[d, "\varepsilon F"] \\
& F
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
GFG \arrow[d, "G\varepsilon"'] & G \arrow[l, "\eta G"'] \arrow[dl, equal] \\
G &
\end{tikzcd}$$

(cf. [17] section IV.1 for more details).

Definition 1.1.8. Let $\mathcal{C}$ be a category and $\mathcal{D}$ a small category. We consider the *diagonal functor*

$$\Delta \;\; \mathcal{C} \to \mathcal{C}^{\mathcal{D}} \;\; : \;\; X \mapsto \Delta(X), \;\; f \mapsto \Delta(f),$$

where $\Delta(X) \in \mathcal{C}^{\mathcal{D}}$ is the functor defined by

$$\Delta(X)(D) = X \quad \text{for every } D \in Ob\mathcal{D} \qquad \text{and} \qquad \Delta(X)(g) = id_X \quad \text{for every } g \in Mor\mathcal{D},$$

and where $\Delta(f)$, with $f \in \mathcal{C}(X, X')$, is a natural transformation from $\Delta(X)$ to $\Delta(X')$.

Definition 1.1.9. Let $\mathcal{C}$ be a category, $\mathcal{D}$ a small category, and $F \in \mathcal{C}^{\mathcal{D}}$. A *limit* for F is an object $limF = lim_{\mathcal{D}}$ of $\mathcal{C}$ with a natural transformation $\tau : \Delta(limF) \to F$ such that for any object $X \in \mathcal{C}$ and any natural transformation $\sigma : \Delta(X) \to F$ there exists a unique morphism $f \in \mathcal{C}(X, limF)$ whose natural transformation $\Delta(f)$ makes the following diagram commute.

$$\begin{tikzcd}
& \Delta(X) \arrow[dl, dashed, "\exists! \Delta(f)"'] \arrow[d, "\sigma"] \\
\Delta(limF) \arrow[r, "\tau"'] & F
\end{tikzcd}$$

If $limF$ exists for every functor $F \in \mathcal{C}^{\mathcal{D}}$, we say that $\mathcal{C}$ *has all small limits* or that $\mathcal{C}$ is *complete*; in this case $lim(-) : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ becomes a functor. If in addition $\mathcal{D}$ is finite, we say that $\mathcal{C}$ *has all finite limits.*

Examples 1.1.10. **(1)** If $\mathcal{D} = \{X_i\}_{i \in I}$ is a set of objects without any other arrows than the required identity morphisms, then the limit $limF = \prod_{i \in I} X_i$ is the *product* of the X_i's.

(2) If $\mathcal{D} = \{A \to B \leftarrow C\}$, then $limF$ is the *pullback* of the diagram

$$\begin{tikzcd} limF \arrow[r, dashed, "i'"] \arrow[d, dashed, "j'"] & F(C) \arrow[d, "j"] \\ F(A) \arrow[r, "i"'] & F(B). \end{tikzcd}$$

We denote it $limF = F(A) \times_{F(B)} F(C)$ and say that i' (resp. j') is the *base change* of i (resp. j) along j (resp. i).

(3) If $\mathcal{D} = \emptyset$, then $limF$ (if it exists) is a *terminal object* of $\mathcal{C}$.

(4) If $\mathcal{D} = \{\ldots \to n \to \ldots \to 2 \to 1 \to 0\}$, so that $F : \mathcal{D} \to \mathcal{C}$ is a *tower* in $\mathcal{C}$, then $limF$ is the *inverse limit* of the tower F.

Definition 1.1.11. Let $\mathcal{C}$ be a category, $\mathcal{D}$ a small category, and $F \in \mathcal{C}^{\mathcal{D}}$. A *colimit* for F is an object $colimF = colim_{\mathcal{D}}$ of $\mathcal{C}$ with a natural transformation $\tau : F \to \Delta(limF)$ such that for any object $X \in \mathcal{C}$ and any natural transformation $\sigma : F \to \Delta(X)$ there exists a unique morphism $f \in \mathcal{C}(colimF, X)$ whose natural transformation $\Delta(f)$ makes the following diagram commute.

$$\begin{tikzcd} F \arrow[r, "\sigma"] \arrow[d, "\tau"'] & \Delta(X) \\ \Delta(colimF) \arrow[ur, dashed, "\exists! \Delta(f)"'] & \end{tikzcd}$$

If $colimF$ exists for every functor $F \in \mathcal{C}^{\mathcal{D}}$, we say that $\mathcal{C}$ *has all small colimits* or that $\mathcal{C}$ is *cocomplete*; in this case $colim(-) : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ becomes a functor. If in addition $\mathcal{D}$ is finite, we say that $\mathcal{C}$ *has all finite limits*. A category that is both complete and cocomplete is sometimes called *bicomplete*.

Examples 1.1.12. **(1)** If $\mathcal{D} = \{X_i\}_{i \in I}$ is a set of objects without any other arrows than the required identity morphisms, then the colimit $colimF = \coprod_{i \in I} X_i$ is the *coproduct* of the X_i's.

(2) If $\mathcal{D} = \{A \leftarrow B \to C\}$, then $colimF$ is the *pushout* of the diagram

$$\begin{tikzcd} F(B) \arrow[r, "i"] \arrow[d, "j"] & F(C) \arrow[d, dashed, "j'"] \\ F(A) \arrow[r, dashed, "i'"'] & colimF. \end{tikzcd}$$

We denote it $colimF = F(A) \vee_{F(B)} F(C)$ and say that i' (resp. j') is the *cobase change* of i (resp. j) along j (resp. i).

(3) If $\mathcal{D} = \emptyset$, then $colimF$ (if it exists) is a *initial object* of $\mathcal{C}$.

(4) If $\mathcal{D} = \{0 \to 1 \to 2 \to \ldots \to n \to \ldots\}$, so that $F : \mathcal{D} \to \mathcal{C}$ is a *telescope* in $\mathcal{C}$, then $colimF$ is the *direct limit* of the telescope F.

The following results of general category theory will be needed for later. However their proofs are out of the scope of this paper and will not be given here (cf. [17] and [1] for proofs).

Proposition 1.1.13. *Let $\mathcal{C}$ be a category, $\mathcal{D}$ a small category. If $\mathcal{C}$ is bicomplete, then $\mathcal{C}^{\mathcal{D}}$ too and we have adjoint pairs*

$$\Delta : \mathcal{C} \Longleftrightarrow \mathcal{C}^{\mathcal{D}} : lim \qquad \text{and} \qquad colim : \mathcal{C}^{\mathcal{D}} \Longleftrightarrow \mathcal{C} : \Delta.$$

Proposition 1.1.14. *Let $F : \mathcal{C} \Longleftrightarrow \mathcal{C}' : G$ be a pair of adjoint functors. Then*

(1) *F preserves colimits; that is for any functor $\varphi \in \mathcal{C}^{\mathcal{D}}$, the existence of $colim\varphi$ implies the existence of $colim(F \circ \varphi)$, and $F(colim\varphi) \cong colim(F \circ \varphi)$.*

(2) *G preserves limits; that is for any functor $\varphi \in \mathcal{C}'^{\mathcal{D}}$, the existence of $lim\varphi$ implies the existence of $lim(G \circ \varphi)$, and $G(lim\varphi) \cong lim(G \circ \varphi)$.*

1.2 Model categories: definition and examples

We now enter the core of the subject. In this section we define what a model category is and give few common examples. Showing that a category is actually a model category is not trivial at all; it is usually long and very technical. For these reasons, and for the sake of continuity in the theory development, the proofs will not be given here.

Before establishing the notion of model category, we first need to define what lifts, in a more general context than those used for the definitions of Hurewicz fibrations and cofibrations, are. We also need the notion of a retract in a category, from which we shall establish a useful result called "the retract argument".

Definitions 1.2.1. Let $\mathcal{C}$ be a category and $\mathcal{L}$ a set of morphisms in $\mathcal{C}$. A morphism $f : A \to B$ in $\mathcal{C}$ satisfies the *left lifting property* with respect to $\mathcal{L}$, a fact we denote $f \in LLP(\mathcal{L})$, if for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"] \arrow[ur, dashed, "\exists \hat{k}"] & D \end{tikzcd}$$

in $\mathcal{C}$ with $g \in \mathcal{L}$, there exists a morphism $\hat{k} : B \to C$ such that $g\hat{k} = k$ and $\hat{k}f = h$.

Dually, f satisfies the *right lifting property* with respect to $\mathcal{L}$, a fact we denote $f \in RLP(\mathcal{L})$, if for every commutative diagram

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ D \arrow[r, "k"] \arrow[ur, dashed, "\exists \hat{k}"] & B \end{tikzcd}$$

in $\mathcal{C}$ with $g \in \mathcal{L}$, there exists a morphism $\hat{k} : D \to A$ such that $f\hat{k} = k$ and $\hat{k}g = h$.

More generally, we say that a commutative square diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"] & D \end{tikzcd}$$

has a lift, or *satisfies the lifting property*, if there is a morphism $\hat{k} : B \to C$, called the *lift*, such that $h = \hat{k}f$ and $k = g\hat{k}$.

Definition 1.2.2. A morphism $f : A \to B$ in a category $\mathcal{C}$ is a *retract* of a morphism $g : C \to D \in Mor\mathcal{C}$ if there is in $\mathcal{C}$ a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "j"'] & D \arrow[r, "s"'] & B \end{tikzcd}$$

such that $ri = id_A$ and $sj = id_B$.

Proposition 1.2.3 (the retract argument)**.** *Let $\mathcal{C}$ be a category and let $f = pi$ be a factorization in $\mathcal{C}$.*

- *If i has the left lifting property with respect to f, then f is a retract of p.*
- *If p has the right lifting property with respect to f, then f is a retract of i.*

Proof. Suppose that f has the left lifting property with respect to p with

$$f :\ A \xrightarrow{i} B \xrightarrow{p} C.$$

Then, the commutative square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "p"] \\ C \arrow[r, equal] & C \end{tikzcd}$$

has a lift $r : C \to B$ which fits into the commutative diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ C \arrow[r, "r"'] & B \arrow[r, "p"'] & C, \end{tikzcd}$$

so that f is a retract of i. The proof of the second assertion is dual. □

We may now define what suitable structure we need on a category in order to develop homotopy theory.

Definition 1.2.4. A *model category* is a category $\mathcal{C}$ with three classes of morphisms

$$WE = WE_{\mathcal{C}}, \quad Fib = Fib_{\mathcal{C}} \quad \text{and} \quad Cof = Cof_{\mathcal{C}} \quad \text{in} \quad Mor\mathcal{C},$$

each of them being closed under composition and containing all identity morphisms, such that the following axioms are satisfied:

(M_1) $\mathcal{C}$ has all finite limits and colimits.

(M_2) If $f, g \in Mor\mathcal{C}$ such that $g \circ f$ is defined in $\mathcal{C}$ and such that two of the morphisms $f, g, g \circ f$ are in WE, then the third one also is.

(M_3) If f is a retract of g and g is in WE (resp. Fib and Cof), then f also belongs to WE (resp. Fib and Cof).

(M_4) Any commutative square diagram of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

in $\mathcal{C}$ has a lift if either $i \in Cof$ and $p \in Fib \cap WE$, or $i \in Cof \cap WE$ and $p \in Fib$.

(M_5) Each morphism $f \in Mor\mathcal{C}$ can be factored in two ways:

- $f = pi$ with $i \in Cof$ and $p \in Fib \cap WE$,
- $f = p'i'$ with $i' \in Cof \cap WE$ and $p' \in Fib$.

All morphisms in WE are called *weak equivalences* and are denoted with an arrow $\xrightarrow{\sim}$, all morphisms in Fib are called *fibrations* and are denoted with an arrow $\twoheadrightarrow$, and all morphisms in Cof are called *cofibrations* and are denoted with an arrow $\hookrightarrow$. Finally, the fibrations (resp. cofibrations) that are also weak equivalences are called *acyclic fibrations* (resp. *acyclic cofibrations*).

Remarks 1.2.5. (1) Since WE, Fib and Cof are closed under composition and contain all identity morphisms, we may also view them as subcategories of $\mathcal{C}$.

(2) Axiom (M_1) implies the existence of an initial object $\emptyset$ and a terminal object $*$ in $\mathcal{C}$. We say that an object A in $\mathcal{C}$ is *fibrant* if $A \to *$ is a fibration, and dually that A is *cofibrant* is $\emptyset \to A$ is a cofibration.

(3) The factorizations of morphisms provided by (M_5) are not always functorial; being functorial meaning that there is a functor

$$(p, i) \ : \ \mathcal{C}^{\to} \to \mathcal{C}^{\to} \times \mathcal{C}^{\to} \ : \ f \mapsto (p, i)(f) = (p(f), i(f)),$$

where $\mathcal{C}^{\to}$ denotes the category of morphisms in $\mathcal{C}$, such that $f = p(f) \circ i(f)$ for any $f \in Ob\mathcal{C}^{\to}$.

(4) The set of axioms $(M_1) - (M_5)$ is not minimal since Fib, WE determine Cof and Cof, WE determine Fib.

Let's now provide few important examples:

Example 1.2.6. The category $\mathcal{T}op$ can be provided with a model category structure by defining a morphism $f : X \to Y$ to be

- a *weak equivalence* if it is a *weak homotopy equivalence*, ie. $\pi_n(f) : \pi_n(X) \xrightarrow{\cong} \pi_n(Y)$ is an isomorphism for every $n \in \mathbb{N}$,
- a *fibration* if it is a *Serre fibration*, ie. any diagram of the form

$$\begin{tikzcd} S^n \arrow[r] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ S^n \times I \arrow[r] & Y, \end{tikzcd}$$

  with $n \in \mathbb{N}$, has the lifting property,
- a *cofibration* if $f : X \to Y$ is a retract of a morphism $g : X \to Y'$, where Y' is obtained from X by attaching cells.

The above model category is the one that comes up the most frequently in everyday algebraic topology. However, in a topological situation where we require that weak equivalences correspond to homotopy equivalences, we rather use the following model category structure.

Example 1.2.7. The category $\mathcal{T}op$ can be provided with another model category structure by defining a morphism $f : X \to Y$ to be

- a *weak equivalence* if it is a homotopy equivalence,
- a *fibration* if it is a *Hurewicz fibration*, ie. any diagram of the form

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r] & Y, \end{tikzcd}$$

  with $n \in \mathbb{N}$ and $Z \in Ob\mathcal{T}op$, has the lifting property,
- a *cofibration* if $f : X \to Y$ is a *closed Hurewicz cofibration*, ie. $f : X \to Y$ is an injection that satisfies the homotopic extension property with $f(X)$ closed in Y.

As one might expect, it turns out that other strictly algebraic categories are model categories as in the following example.

Example 1.2.8. Let R be a unitary associative ring and consider ${}_R\mathcal{M}od$ the category of left R-modules. Let ${}_R\mathcal{C}h^+$ be the category of non negatively graded chain complexes of left R-modules, in which an object M is a collection $\{M_n\}_{n\in\mathbb{N}}$ of R-modules together with boundary mappings $\partial_n : M_n \to M_{n-1}$ satisfying $\partial_{n-1} \circ \partial_n = 0$, and in which a morphism $f : M \to N$ is a collection of morphisms $f_n : M_n \to N_n$ in ${}_R\mathcal{M}od$ such that $f_{n-1}\partial_n^M = \partial_n^N f_n$ for each $n \geq 1$. The category ${}_R\mathcal{C}h^+$ can be provided with a model category structure by defining a morphism $f : M \to N$ to be

- a *weak equivalence* if it induces an equivalence in homology,
- a *fibration* if for each $n \geq 1$ the map $f_n : M_n \to N_n$ is an epimorphism of ${}_R\mathcal{M}od$,
- a *cofibration* if for each $n \geq 0$ the map $f_n : M_n \to N_n$ is a monomorphism of ${}_R\mathcal{M}od$ which has a projective R-module as its cokernel.

Based on any given model category, it is also possible to construct many other model categories. The most basic examples of such model structures are build, from a given model category $\mathcal{C}$, on the opposite category $\mathcal{C}^{op}$, the under category $A \downarrow \mathcal{C}$ and the upper category $\mathcal{C} \downarrow A$.

Example 1.2.9. Given a model category $\mathcal{C}$, the opposed category $\mathcal{C}^{op}$ (cf. 1.1.2) can be provided with a model category structure by defining a morphism $f^{op} : Y \to X$ to be

- a *weak equivalence* if the corresponding morphism $f : X \to Y$ is a weak equivalence in $\mathcal{C}$,
- a *fibration* if the corresponding morphism $f : X \to Y$ is a cofibration in $\mathcal{C}$,
- a *cofibration* if the corresponding morphism $f : X \to Y$ is a fibration in $\mathcal{C}$.

This shows in particular that the five axioms $(M_1) - (M_5)$ are self-dual in the sense that for any given statement S about model categories and its dual S^*, obtained by reversing all arrows and permuting the words "fibration" and "cofibration", we have that S is true for all model categories if and only if S^* is.

Example 1.2.10. Given a model category $\mathcal{C}$, the under category $A \downarrow \mathcal{C}$ (cf. 1.1.2) can be provided with a model category structure by defining a morphism $h : (A \to X) \to (A \to Y)$ to be

- a *weak equivalence* if the corresponding morphism $h : X \to Y$ is a weak equivalence in $\mathcal{C}$,
- a *fibration* if the corresponding morphism $h : X \to Y$ is a fibration in $\mathcal{C}$,
- a *cofibration* if the corresponding morphism $h : X \to Y$ is a cofibration in $\mathcal{C}$.

We can provide a model category structure on $\mathcal{C}\downarrow A$ in a similar way.

We end this section by establishing the following basic properties.

Proposition 1.2.11. *Let $\mathcal{C}$ be a model category. Then*

(1) $Fib = RLP(Cof \cap WE)$ *and* $Fib \cap WE = RLP(Cof)$,

(2) $Cof = LLP(Fib \cap WE)$ *and* $Cof \cap WE = LLP(Fib)$,

(3) Fib *and* $Fib \cap WE$ *are stable under base change,*

(4) Cof *and* $Cof \cap WE$ *are stable under cobase change.*

Proof. (1-2) Axiom (M_4) implies that all (acyclic) fibrations and all (acyclic) cofibrations in $\mathcal{C}$ already have the desired lifting property, so that all four inclusions "$\subseteq$" are satisfied. Since the argument for the four reversed inclusions are similar, we only prove

$$Cof \ \supseteq \ LLP(Fib \cap WE).$$

Suppose that $f : A \to B$ has the left lifting property with respect to all acyclic fibrations. By (M_5) we can factor f as

$$f :\ A \overset{}{\underset{i}{\hookrightarrow}} C \overset{\sim}{\underset{p}{\twoheadrightarrow}} B.$$

By assumption, the square diagram

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f"'] & C \arrow[d, "p"', "\sim"] \\ B \arrow[r, "id_B"] & B \end{tikzcd}$$

has a lift $g : B \to C$. This implies, by the commutative diagram

$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "f"'] & A \arrow[r, "id"] \arrow[d, "i"'] & A \arrow[d, "f"'] \\ B \arrow[r, "g"] & C \arrow[r, "p"] & B, \end{tikzcd}$$

that f is a retract of i, so that f is a cofibration by (M_3).

(3-4) Since the four assertions use the same argument, we only have to prove one, say Cof is stable under cobase change. Let $i : A \hookrightarrow B$ be a cofibration. We choose a morphism $f : A \to A'$ in $\mathcal{C}$ and construct a pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & A' \arrow[d, "j"'] \\ B \arrow[r, "g"] & B'. \end{tikzcd}$$

We need to prove that j is a cofibration. In order to do that, since $Cof = LLP(Fib \cap WE)$ it is enough to show that $j \in LLP(Fib \cap WE)$. Let $p : C \overset{\sim}{\twoheadrightarrow} D$ be an acyclic fibration and consider the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "a"] \arrow[d, "j"'] & C \arrow[d, two heads, "p"', "\sim"] \\ B' \arrow[r, "b"] & D, \end{tikzcd} \tag{$*$}$$

where $a, b \in Mor\mathcal{C}$, which we can enlarge to the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & A' \arrow[r, "a"] \arrow[d, "j"'] & C \arrow[d, two heads, "p"', "\sim"] \\ B \arrow[r, "g"] & B' \arrow[r, "b"] & D. \end{tikzcd} \tag{$**$}$$

Since i is a cofibration, the diagram $(**)$ has a lift $h : B \to C$. Finally, the universal property of pushouts implies that the morphisms $h : B \to C$ and $a : A' \to C$ induce the desired lift in diagram $(*)$. $\square$

Remark 1.2.12. The first two properties imply that if, for a given model category, we choose Cof and WE, then Fib is pinned down by 1.2.11.(1) Similarly, if we choose Fib and WE, then Cof is pinned down by 1.2.11.(2) This shows that the five model category axioms $(M_1) - (M_5)$ are overdetermined.

1.3 Homotopy relation in model categories

Now that a more general context has been established for homotopy theory, we need to define what the actual homotopy relation is for a model category. In the special case of topological spaces, the homotopy relation can be defined in two ways. If we consider two homotopic continuous functions $f, g : A \to X$ between topological spaces, a homotopy is a continuous mapping $H : A \times I \to X$ which fits into the commutative diagram

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[dr, "f"'] & A \times I \arrow[d, "H"] & A \arrow[l, hook', "i_1"'] \arrow[dl, "g"] \\ & X & \end{tikzcd},$$

where I denotes the interval $[0, 1] \subseteq \mathbb{R}$, $A \times I$ the cylinder on A, and $i_t(a) = (a, t)$. Another way to do this is to use the path space $X^I = \{\lambda : I \to X \mid \lambda \text{ is continuous}\}$, endowed with the compact-open topology, and to define a homotopy $K : A \to X^I$ between f and g to be a continuous map that fits into the commutative diagram

$$\begin{tikzcd} X & X^I \arrow[l, two heads, "p_0"'] \arrow[r, two heads, "p_1"] & X \\ & A \arrow[ul, "f"] \arrow[u, "K"] \arrow[ur, "g"'] & \end{tikzcd},$$

where $p_t(\lambda) = \lambda(t)$. It is clear that both way are equivalent in this context. We want to define homotopy relations in model categories in an appropriate way, ie. so as to generalize the above two definitions. However, the two approaches will not always coincide so as the necessity to study in which cases they will.

Throughout this section, we fix a model category $\mathcal{C} = (\mathcal{C}, WE, Fib, Cof)$ and two objects A and X in $\mathcal{C}$. We start with the first approach, by establishing the notion of cylinder object in our model category $\mathcal{C}$.

Definition 1.3.1. Consider the pushout of $\emptyset \to A$ with itself.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d, "i'_0"] \arrow[ddr, bend left, "id_A"] & \\ A \arrow[r, "i'_1"'] \arrow[drr, bend right, "id_A"'] & A \vee A \arrow[dr, "\nabla"] & \\ & & A, \end{tikzcd}$$

where $\nabla : A \vee A \to A$ is the folding map. A *cylinder object* on A is an object $A \wedge I$ of $\mathcal{C}$ together with a factorization

$$\begin{tikzcd} A \vee A \arrow[r, "i"] \arrow[rr, bend right, "\nabla"'] & A \wedge I \arrow[r, "p", "\sim"'] & A \end{tikzcd}$$

of ∇, where p is a weak equivalence. We define

$$i_0 := i \circ i'_0 \qquad \text{and} \qquad i_1 := i \circ i'_1.$$

A cylinder object $A \wedge I$ is *good* if $i \in Cof$, and is *very good* if in addition $p \in Fib \cap WE$.

Remark 1.3.2. By (M_5) there exists at least one very good cylinder object on A.

Property 1.3.3. *If A is cofibrant and $A \wedge I$ is a good cylinder object on A, then $i_0, i_1 : A \to A \wedge I$ are acyclic cofibrations.*

Proof. Let's check this for i_0. Since the identity morphism id_A factors as $A \xrightarrow{i_0} A \wedge I \underset{\sim}{\xrightarrow{p}} A$, it follows from (M_2) that i_0 is a weak equivalence. Since $A \vee A$ is defined by the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook, "\text{cof}"'] & A \arrow[d, "i'_0"] \\ A \arrow[r, "i'_1"'] & A \vee A, \end{tikzcd}$$

it follows from 1.2.11.(4) that i'_0 is a cofibration. By assumption, $i : A \vee A \hookrightarrow A \wedge I$ is a cofibration, so that the composition $i_0 = i \circ i'_0$ is also a cofibration. The argument is the same for i_1. □

From this, we may define the first type of homotopy relation in $\mathcal{C}$.

Definition 1.3.4. Let $f, g : A \to X$ be two morphisms in $\mathcal{C}$. A *left homotopy* from f to g is a morphism $H : A \wedge I \to X$ which makes the diagram

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[dr, "f"'] & A \wedge I \arrow[d, "H"] & A \arrow[l, hook', "i_1"'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

commute, where $A \vee A \xrightarrow{i} A \wedge I \underset{\sim}{\xrightarrow{p}} A$ is a cylinder object on A. We say that H is *good* (resp. *very good*) if $A \wedge I$ is. We denote the existence of a left homotopy from f to g by $f \sim_l g$, we say that f and g are *left homotopic* and we write $\pi^l(A, X)$ for the set of left homotopy classes in $\mathcal{C}(A, X)$.

Example 1.3.5. If $\mathcal{C}$ is the model category of 1.2.6, then a possible choice of cylinder object for a topological space A is the usual cylinder $A \times I$. In that case, the notion of left homotopy related to this particular cylinder object coincide with the usual notion of homotopy.

Remark 1.3.6. If $f \sim_l g$ via H, then (M_2) implies the equivalence $f \in WE \Leftrightarrow g \in WE$. Indeed, we saw above that i_0 and i_1 are weak equivalences. Then if $f = Hi_0 \in WE$, we have $H \in WE$, so that $g = Hi_1 \in WE$.

We shall now establish the main results on the relation of left homotopy.

Proposition 1.3.7. *If $f \sim_l g : A \to X$, then there is a good left homotopy from f to g. If in addition X is fibrant, then there is a very good left homotopy from f to g.*

Proof. We can apply (M_5) to the morphism $A \vee A \to A \wedge I$, where $A \wedge I$ is the cylinder object of some homotopy, in order to obtain a factorization

$$A \vee A \hookrightarrow A \wedge I' \xrightarrow{\sim} A \wedge I \xrightarrow{\sim} A,$$

so that $A \wedge I'$ is a cylinder of a good homotopy. Now choose a good homotopy $H : A \wedge I \to X$ from f to g. Applying (M_5) and (M_2) to the morphism $A \wedge I \xrightarrow{\sim} A$ we obtain a factorization

$$A \wedge I \overset{\sim}{\hookrightarrow} A \wedge I' \overset{\sim}{\twoheadrightarrow} A.$$

From this, since X is fibrant, we get a diagram

$$\begin{tikzcd} A \wedge I \arrow[r, "H"] \arrow[d, hook, "\sim"] & X \arrow[d, two heads] \\ A \wedge I' \arrow[r] & * \end{tikzcd}$$

which by (M_4) induces the desired very good left homotopy $H' : A \wedge I' \to X$. $\square$

Proposition 1.3.8.

(1) *If A is cofibrant, then $\sim_l$ is an equivalence relation on $\mathcal{C}(A, X)$ for any $X \in \mathcal{C}$, ie. $\pi^l(A, X)$ is the set of equivalence classes of $\mathcal{C}(A, X)$ induced by $\sim_l$.*

(2) *If A is cofibrant and $p : X \overset{\sim}{\twoheadrightarrow} Y$ is an acyclic fibration, then p induces a bijection*

$$p_* : \ \pi^l(A, X) \to \pi^l(A, Y) \ : \ [f] \mapsto [pf].$$

(3) *If X is fibrant, then*

$$h \sim_l k : A \to B \qquad \textit{and} \qquad f \sim_l g : B \to X \qquad \textit{imply} \qquad fh \sim_l gk : A \to X.$$

This induces a map

$$\pi^l(A,B) \times \pi^l(B,X) \to \pi^l(A,X) \;:\; ([h],[f]) \mapsto [fh].$$

Proof. (1) Since we can take A itself as a cylinder object for A, it is clear that f is a left homotopy from f to itself; this proves reflexivity. Consider now the morphism $s : A \vee A \to A \vee A$ that switches components. The obvious identity $g \vee f = (f \vee g)s$ implies that if $f \sim_l g$, then $g \sim_l f$; this proves symmetry. Assume now that we have $f \sim_l g \sim_l h$. We can choose a good left homotopy $H : A \wedge I \to X$ from f to g with $Hi_0 = f, Hi_1 = g$, and a good left homotopy $H' : A \wedge I' \to X$ from g to h with $H'i'_0 = g, H'i'_1 = h$. Let $A \wedge I''$ be the pushout

$$\begin{tikzcd} A \arrow[r, hook, "i'_0", "\sim"'] \arrow[d, hook, "i_1"', "\sim"] & A \wedge I' \arrow[d] \\ A \wedge I \arrow[r] & A \wedge I'', \end{tikzcd}$$

where, since A is cofibrant, i_1 and i'_0 are acyclic cofibration (cf. 1.3.3). By 1.2.11.(4) and the universal property of pushouts, it follows that $A \wedge I''$ is a cylinder object for A. We can then apply the universal property of pushouts again to $H : A \wedge I \to X$ and $H' : A \wedge I' \to X$ in order to obtain the desired left homotopy $H'' : A \wedge I'' \to X$; this proves transitivity.

(2) The map p_* is well defined since if $f, g : A \to X$ are two morphisms and H is a left homotopy from f to g, then pH is a left homotopy from pf to pg. Let's show that p_* is surjective. Let $[f] \in \pi^l(A,Y)$. Since A is cofibrant, we have a diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "p"', "\sim"] \\ A \arrow[r, "f"] & Y \end{tikzcd}$$

which, by (M_4), has a lift $g : A \to X$. We clearly have $p_*[g] = [pg] = [f]$, so that the surjectivity of p_* is proven. Let's now show that p_* is injective. Let $f, g : A \to X$ and suppose that $pf \sim_l pg : A \to Y$. By 1.3.7, we can choose a good left homotopy $H : A \wedge I \to Y$ from pf to pg. By (M_4), the diagram

$$\begin{tikzcd} A \vee A \arrow[r, "f \vee g"] \arrow[d, hook] & X \arrow[d, two heads, "p"', "\sim"] \\ A \wedge I \arrow[r, "H"] & Y \end{tikzcd}$$

has a lift H', which is none other than the desired left homotopy from f to g.

(3) It is enough to show that if $h \sim_l k : A \to B$ and $f \sim_l g : B \to X$, then fh and gk represent the same element of $\pi^l(A,X)$. For this, we only have to check that $fh \sim_l gh : A \to X$ and that $gh \sim_l gk : A \to X$. The second left homotopy is obtained by composing the left homotopy between h and k with g. It remains to prove the first one. Since X is fibrant, by 1.3.7 there exists very good left homotopy $H : B \wedge I \to X$ between f and g. We choose a good cylinder object for A:

$$A \vee A \overset{j}{\hookrightarrow} A \wedge I \overset{\sim}{\to} A.$$

By (M_4), the diagram

$$\begin{tikzcd} A \vee A \arrow[r, "h \vee h"] \arrow[d, hook, "j"'] & B \vee B \arrow[r, "i"] & B \wedge I \arrow[d, two heads, "\sim"'] \\ A \wedge I \arrow[r, "\sim"] & A \arrow[r, "h"] & B \end{tikzcd}$$

has a lift $k : A \wedge I \to B \wedge I$. The desired left homotopy is none other than Hk. □

We now get to the second approach. The following definitions and results are essentially dual to what we saw above.

Definition 1.3.9. Consider the pullback of $X \to *$ with itself:

$$\begin{tikzcd} X \arrow[dr, "\Delta"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\ & X \times X \arrow[r, "q_1'"'] \arrow[d, "q_0'"] & X \arrow[d] \\ & X \arrow[r] & *, \end{tikzcd}$$

where $\Delta : X \to X \times X$ is the diagonal map. A *path object* on X is an object X^I of $\mathcal{C}$ together with a factorization

$$\begin{tikzcd} X \arrow[r, "j", "\sim"'] \arrow[rr, bend right, "\Delta"'] & X^I \arrow[r, "q"] & X \times X \end{tikzcd}$$

of Δ, where j is a weak equivalence. We define

$$q_0 := q_0' \circ q \qquad \text{and} \qquad q_1 := q_1' \circ q.$$

A path object X^I is *good* if $q \in Fib$, and is *very good* if in addition $j \in Cof \cap WE$.

Remark 1.3.10. By (M_5) there exists at least one very good path object on X.

Property 1.3.11. *If X is fibrant and X^I is a good cylinder object on X, then $q_0, q_1 : X^I \to X$ are acyclic fibrations.*

Proof. The argument is strictly dual to 1.3.3. □

Definition 1.3.12. Let $f, g : A \to X$ be two morphisms in $\mathcal{C}$. A *right homotopy* from f to g is a morphism $H : A \to X^I$ which makes the diagram

$$\begin{tikzcd} X & X^I \arrow[l, two heads, "q_0"'] \arrow[r, two heads, "q_1"] & X \\ & A \arrow[ul, "f"] \arrow[u, "H"] \arrow[ur, "g"'] & \end{tikzcd}$$

commute, where $X \underset{\sim}{\xrightarrow{j}} X^I \xrightarrow{q} X \times X$ is a path object on X. We say that H is *good* (resp. *very good*) if X^I is. We denote the existence of a right homotopy from f to g by $f \sim_r g$, we say that f and g are *right homotopic*, and we write $\pi^r(A, X)$ for the set of right homotopy classes in $\mathcal{C}(A, X)$.

Example 1.3.13. If $\mathcal{C}$ is the model category of 1.2.6, then a possible choice of path object of a topological space X is the topological space X^I of all paths in X endowed with the compact-open topology. In that case, the notion of right homotopy related to this particular path object coincide with the usual notion of homotopy.

Proposition 1.3.14. *If $f \sim_r g : A \to X$, then there is a good right homotopy from f to g. If in addition A is cofibrant, then there is a very good right homotopy from f to g.*

Proof. Strictly dual to 1.3.7. □

Proposition 1.3.15.

(1) *If X is fibrant, then $\sim_r$ is an equivalence relation on $\mathcal{C}(A, X)$ for any $A \in \mathcal{C}$, ie. $\pi^r(A, X)$ is the set of equivalence classes of $\mathcal{C}(A, X)$ induced by $\sim_r$.*

(2) *If X is fibrant and $i : A \overset{\sim}{\hookrightarrow} B$ is an acyclic cofibration, then i induces a bijection*

$$i^* : \ \pi^r(B, X) \to \pi^r(A, X) \ : \ [f] \mapsto [fi].$$

(3) *If A is cofibrant, then*

$$f \sim_r g : A \to X \qquad \text{and} \qquad h \sim_r k : X \to Y \qquad \text{imply} \qquad hf \sim_r kg : A \to Y.$$

This induces a map

$$\pi^r(A, X) \times \pi^r(X, Y) \to \pi^r(A, Y) \ : \ ([f], [h]) \mapsto [hf].$$

Proof. Strictly dual to 1.3.8. □

We are now able to establish in which case the left and right homotopic relations coincide.

Theorem 1.3.16. *Let $f, g : A \longrightarrow X$ be two morphisms in $\mathcal{C}$.*

(1) *If A is cofibrant and $f \sim_l g$, then $f \sim_r g$.*

(2) *If X is fibrant and $f \sim_r g$, then $f \sim_l g$.*

(3) *If both A is cofibrant and X is fibrant, there is an equivalence relation $\sim$ on $\mathcal{C}$ such that*

$$f \sim_l g \ \Leftrightarrow \ f \sim g \ \Leftrightarrow \ f \sim_r g$$

We then say that f and g are homotopic and we denote by $\pi(A, X)$ the set of homotopy classes in $\mathcal{C}$.

Proof. (1) By 1.3.7, there is a good cylinder object

$$A \vee A \xhookrightarrow{i_0 \vee i_1} A \wedge I \xrightarrow[\sim]{p} A$$

for A, where by 1.3.3 i_0 and i_1 are acyclic cofibrations, as well as a good homotopy $H : A \wedge I \to X$ from f to g. Consider a good path object

$$X \xrightarrow[\sim]{j} X^I \xrightarrow{(q_0, q_1)} X \times X$$

for X (cf. 1.3.10). By (M_4), the diagram

$$\begin{tikzcd} A \arrow[r, "jf"] \arrow[d, hook, "i_0"', "\sim"] & X^I \arrow[d, two heads, "(q_0,q_1)"] \\ A \wedge I \arrow[r, "(fp,H)"] & X \times X \end{tikzcd}$$

has a lift $K : A \wedge I \to X^I$, which composed with i_1 is the desired right homotopy $Ki_1 : A \to X^I$ from f to g.

(2) The proof is dual to (1).

(3) This is simply a consequence of (1) and (2). □

Corollary 1.3.17. *If A is cofibrant, X is fibrant and cofibrant, and Y is fibrant, then*

$$f \sim g : A \to X \qquad \text{and} \qquad h \sim k : X \to Y \qquad \text{imply} \qquad hf \sim kg : A \to Y.$$

Proof. This is an easy consequence of theorem 1.3.16 and propositions 1.3.8 and 1.3.15. □

Theorem 1.3.16 allows to establish a more general notion of homotopy equivalence, which in fact coincides with the notion of weak equivalence as the following theorem states.

Definition 1.3.18. If A and X are both fibrant and cofibrant, then a morphism $f : A \to X$ is a *homotopy equivalence* if there exists a morphism $g : X \to A$ such that $gf \sim id_A$ and $fg \sim id_X$. In that case we say that g is an *homotopy inverse* of f.

Theorem 1.3.19. *Let $f : A \to X$ be a morphism in $\mathcal{C}$ with A and X both fibrant and cofibrant. Then f is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. (⇒) Suppose that $f : A \to X$ is a weak equivalence. Applying (M_5) to f we obtain a composite

$$\begin{tikzcd} A \arrow[r, hook, "q", "\sim"'] & C \arrow[r, two heads, "p", "\sim"'] & X, \end{tikzcd}$$

in which by (M_2), the morphism p is also a weak equivalence. Since q is an acyclic cofibration and A is fibrant, the diagram

$$\begin{tikzcd} A \arrow[r, "id_A"] \arrow[d, hook, "q"', "\sim"] & A \arrow[d, two heads] \\ C \arrow[r] & * \end{tikzcd}$$

has by (M_4) a lift $r : C \to A$ such that $rq = id_A$. By 1.3.15.(2), q induces a bijection

$$q^* \; : \; \pi^r(C, C) \to \pi^r(A, C) \; : \; [g] \mapsto [gq].$$

We then have $q^*([qr]) = [qrq] = [q]$, so that $qr \sim_r id_C$ which shows that r and q are homotopy equivalences inverse to each other. We can use the dual argument to show that p is also a homotopy equivalence, so that $f = pq$ is itself a homotopy equivalence.

(⇐) Suppose now that $f : A \to X$ is a homotopy equivalence, in other words that f has a homotopic inverse $g : X \to A$. Applying (M_5) to f we obtain a composite

$$\begin{tikzcd} A \arrow[r, hook, "q", "\sim"'] & C \arrow[r, two heads, "p"] & X, \end{tikzcd}$$

in which C is both fibrant and cofibrant. By (M_2), in order to prove that f is a weak equivalence, it is enough to show that p is a weak equivalence. Let $H : X \wedge I \to X$ be a homotopy from fg to id_X. By (M_4) the diagram

$$\begin{tikzcd} X \arrow[r, "qg"] \arrow[d, hook, "i_0"', "\sim"] & C \arrow[d, two heads, "p"] \\ X \wedge I \arrow[r, "H"] & X \end{tikzcd}$$

has a lift $H' : X \wedge I \to C$. Let $s := H'i_1$, so that $ps = id_X$. We know that q is a weak equivalence, so that by the above argument q has a homotopy inverse r. Composing the established equality $pq = f$ on the right with r gives $p \sim fr$ (cf. 1.3.8.(3)). Since in addition we have $s \sim qg$ via the homotopy H', it follows from 1.3.8.(3) and 1.3.15.(3) that

$$sp \sim qgp \sim qgfr \sim qr \sim id_C,$$

so that by 1.3.6, sp is a weak equivalence. The commutative diagram

$$\begin{tikzcd} C \arrow[r, "id_C"] \arrow[d, "p"] & C \arrow[r, "id_C"] \arrow[d, "sp"] & C \arrow[d, "p"] \\ X \arrow[r, "s"] & C \arrow[r, "p"] & X \end{tikzcd}$$

shows that p is a retract of sp (cf. 1.2.2), so that by (M_3) the morphism p is a weak equivalence; and so is f. $\square$

1.4 The homotopy category of a model category

We fix a model category $\mathcal{C} = (\mathcal{C}, WE, Fib, Cof)$. The next step is to construct an induced model category $Ho(\mathcal{C})$ whose objects are the same as $\mathcal{C}$, but whose morphisms are homotopy equivalences of morphisms in $\mathcal{C}$ in the case where left and right homotopy relations coincide, that is when the domain and codomain of a morphism are both fibrant and cofibrant.

The following categories will be used as tools for defining $Ho(\mathcal{C})$ and constructing a canonical functor $\gamma : \mathcal{C} \to Ho(\mathcal{C})$.

Notations 1.4.1. We define:

- $\mathcal{C}_c$ to be the full subcategory of $\mathcal{C}$ generated by the cofibrant objects of $\mathcal{C}$.
- $\mathcal{C}_f$ to be the full subcategory of $\mathcal{C}$ generated by the fibrant objects of $\mathcal{C}$.
- $\mathcal{C}_{cf}$ to be the full subcategory of $\mathcal{C}$ generated by the objects of $\mathcal{C}$ that are both fibrant and cofibrant.
- $\pi\mathcal{C}_c$ to be the category of cofibrant objects of $\mathcal{C}$ whose morphisms are the right homotopy classes of $\mathcal{C}$.
- $\pi\mathcal{C}_f$ to be the category of fibrant objects of $\mathcal{C}$ whose morphisms are the left homotopy classes of $\mathcal{C}$.
- $\pi\mathcal{C}_{cf}$ to be the category of both fibrant and cofibrant objects of $\mathcal{C}$ whose morphisms are the homotopy classes of $\mathcal{C}$.

The next step is to define fibrant and cofibrant replacements of an object $\mathcal{C}$. These notions give rise to two functors Q and R which, by restriction to $\pi\mathcal{C}_f$ and $\pi\mathcal{C}_c$, allow to define the functor $\gamma : \mathcal{C} \to Ho(\mathcal{C})$.

Construction 1.4.2. For each object X in $\mathcal{C}$, we apply (M_5) to the morphism $\emptyset \to X$ in order to obtain a factorization

$$\emptyset \hookrightarrow QX \underset{\sim}{\overset{p_X}{\twoheadrightarrow}} X,$$

on which we impose that if X is itself cofibrant then $QX = X$; such a factorization is a *cofibrant model*, or *cofibrant replacement*, of X. Dually, we apply (M_5) to the morphism $X \to *$ in order to obtain a factorization

$$X \underset{\sim}{\overset{i_X}{\hookrightarrow}} RX \twoheadrightarrow *,$$

on which we impose that if X is itself fibrant then $RX = X$; such a factorization is a *fibrant model*, or *fibrant replacement*, of X.

Proposition 1.4.3. *For any morphisms $f : X \to Y$ in $\mathcal{C}$, there is a morphism $\tilde{f} : QX \to QY$ that makes the following diagram commutative:*

$$\begin{tikzcd} QX \arrow[r, "\tilde{f}"] \arrow[d, "p_X"', "\sim"] & QY \arrow[d, "p_Y"', "\sim"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The induced morphism $\tilde{f}$ depends, up to left homotopy or up to right homotopy, only on f, and is a weak equivalence if and only if f is. In addition if Y is fibrant, then $\tilde{f}$ depends, up to left homotopy or up to right homotopy, only on the left homotopy class of f.

Proof. The induced morphism $\tilde{f}$ is obtained by applying (M_4) to the diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, hook] & QY \arrow[d, "\sim"', "p_Y"] \\ QX \arrow[r, "fp_X"] & Y, \end{tikzcd}$$

in other words $\tilde{f}$ is a lift in the above diagram. This, with 1.3.8.(2), imply that $\tilde{f}$ is uniquely determined up to left homotopy by f. In addition, since QX is cofibrant, 1.3.16.(1) implies that two morphisms which are left homotopic are also right homotopic. It follows that $\tilde{f}$ is uniquely determined up to right homotopy by f. The assertion about weak equivalences is an easy consequence of (M_2). Finally, if Y is fibrant, then so is QY and we only have to apply 1.3.8.(3) to obtain the final assertion. ☐

Remark 1.4.4. From the uniqueness statements of 1.4.3 it follows that if $f = id_X$, then $\tilde{f} \sim_r id_{QX}$. In addition, this also implies that if we consider two morphisms $f : X \to Y$ and $g : Y \to Z$ we have $\widetilde{gf} \sim_r \tilde{g}\tilde{f}$. We therefore have a functor

$$Q \ : \ \mathcal{C} \to \pi\mathcal{C}_c \ : \ X \mapsto QX \ , \ (f : X \to Y) \mapsto ([\tilde{f}]_r : QX \to QY).$$

Proposition 1.4.5. *For any morphism $f : X \to Y$ in $\mathcal{C}$, there is a morphism $\bar{f} : RX \to RY$ that makes the following diagram commutative:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_X"', "\sim"] & Y \arrow[d, hook, "i_Y"', "\sim"] \\ RX \arrow[r, "\bar{f}"] & RY \end{tikzcd}$$

The induced morphism $\bar{f}$ depends, up to left homotopy or up to right homotopy, only on f, and is a weak equivalence if and only if f is. In addition if X is cofibrant, then $\bar{f}$ depends, up to left homotopy or up to right homotopy, only on the right homotopy class of f.

Proof. Strictly dual to 1.4.3. □

Remark 1.4.6. From the uniqueness statements of 1.4.5 it follows that if $f = id_X$, then $\bar{f} \sim_l id_{RX}$. In addition, this also implies that if we consider two morphisms $f : X \to Y$ and $g : Y \to Z$ we have $\overline{gf} \sim_l \bar{g}\bar{f}$. We therefore have a functor

$$R \ : \ \mathcal{C} \to \pi\mathcal{C}_f \ : \ X \mapsto RX \ , \ (f : X \to Y) \mapsto ([\bar{f}]_l : RX \to RY).$$

Proposition 1.4.7. *The restriction to $\mathcal{C}_f$ of the functor $Q : \mathcal{C} \to \pi\mathcal{C}_c$ induces a functor*

$$Q' \ : \ \pi\mathcal{C}_f \to \pi\mathcal{C}_{cf} \ : \ X \mapsto QX \ , \ ([f]_g : X \to Y) \mapsto ([\tilde{f}] : QX \to QY).$$

Dually, the restriction to $\mathcal{C}_c$ of the functor $R : \mathcal{C} \to \pi\mathcal{C}_f$ induces a functor

$$R' \ : \ \pi\mathcal{C}_c \to \pi\mathcal{C}_{cf} \ : \ X \mapsto RX \ , \ ([f]_r : X \to Y) \mapsto ([\bar{f}] : RX \to RY).$$

Proof. Since the two statements are dual to one another, we only have to prove one, say the second one. If we suppose that $f, g : X \to Y$ are two morphisms in $\mathcal{C}$ between two cofibrant objects and that they represent the same morphisms in $\pi\mathcal{C}_c$, we must show that $Rf = Rg$ in $\pi\mathcal{C}_{cf}$. It is sufficient to prove this for the special case where f and g are directly related by a right homotopy, ie. $f \sim_r g$. This, however, is a direct consequence of 1.4.5. □

Definition 1.4.8. The *homotopy category* $Ho(\mathcal{C}) = Ho\mathcal{C}$ of a model category $\mathcal{C}$ is the category defined by

$$\begin{cases} ObHo(\mathcal{C}) = Ob\mathcal{C}, \\ Ho(\mathcal{C})(X,Y) = \pi\mathcal{C}_{cf}(R'QX, R'QY) = \pi(RQX, RQY). \end{cases}$$

There is then a functor

$$\gamma = \gamma_{\mathcal{C}} \ : \ \mathcal{C} \to Ho(\mathcal{C}) \ : \ X \mapsto X \ , \ (f : X \to Y) \mapsto [RQ(f) : RQX \to RQY].$$

Remark 1.4.9. In the case where X and Y are both fibrant and cofibrant, by construction the map $\gamma : \mathcal{C}(X,Y) \to Ho(\mathcal{C})(X,Y)$ is surjective. This induces a bijection

$$\pi(X,Y) \cong Ho(\mathcal{C})(X,Y).$$

Example 1.4.10. The homology category $Ho(\mathcal{T}op)$ of the model category $\mathcal{T}op$ as given in 1.2.7 is equivalent to the usual homotopy category of topological spaces which has homotopy equivalence classes of continuous functions as morphisms.

The next proposition help us understand, via the functor γ, what the morphisms in $Ho(\mathcal{C})$ are with respect to $Mor\mathcal{C}$. This will provide a useful corollary.

Proposition 1.4.11. *For any morphism f in a model category $\mathcal{C}$, $\gamma(f)$ is an isomorphism in $Ho(\mathcal{C})$ if and only if f is a weak equivalence in $\mathcal{C}$. Furthermore, the class of all morphisms of $Ho(\mathcal{C})$ is generated by the class*

$$\{\gamma(f) \in MorHo(\mathcal{C}) \mid f \in Mor\mathcal{C}\} \ \cup \ \{\gamma(f)^{-1} \in MorHo(\mathcal{C}) \mid f \in WE_{\mathcal{C}} \subseteq Mor\mathcal{C}\}.$$

Proof. Let $f : X \to Y$ be a weak equivalence in $\mathcal{C}$. Then by construction $RQ(f)$ can be represented by a morphism $f' : RQX \to RQY$ which is also a weak equivalence in $\mathcal{C}$. By 1.3.19, f' has an inverse up to left or right homotopy, and represents an isomorphism in $\pi\mathcal{C}_{cf}$ which is none other than $\gamma(f)$. Inversely, if $\gamma(f)$ is an isomorphism in $Ho(\mathcal{C})$, then f' has an inverse up to homotopy and is therefore a weak equivalence by 1.3.19. It follows by construction that f is a weak equivalence.

For any object X of $\mathcal{C}$ the morphism $i_{QX}p_X^{-1} : X \to RQX$ (cf. 1.4.2) is a weak equivalence in $\mathcal{C}$. It follows from the above statement that the morphism

$$\gamma(i_{QX})\gamma(p_X)^{-1} = \gamma(i_{QX}p_X^{-1}) \qquad \text{in} \quad Ho(\mathcal{C})$$

is an isomorphism from X to RQX in $Ho(\mathcal{C})$. Furthermore, for any two objects X and Y in $\mathcal{C}$ the functor γ induces an epimorphism (cf. 1.4.9)

$$\mathcal{C}(RQX, RQY) \to Ho(\mathcal{C})(RQX, RQY).$$

Consequently, any morphism $f : X \to Y$ in $Ho(\mathcal{C})$ can be represented as

$$f = \gamma(p_Y)\gamma(i_{QY})^{-1}\gamma(f')\gamma(i_{QX})\gamma(p_X)^{-1}$$

for some morphism $f' : RQX \to RQY$ in $\mathcal{C}$. □

Corollary 1.4.12. *Let $\mathcal{D}$ be a category and $\mathcal{C}$ a model category. If $F, G : Ho(\mathcal{C}) \to \mathcal{D}$ are two functors and $\tau : F\gamma \to G\gamma$ is a natural transformation, then τ, with the obvious appropriate identifications, also gives a natural transformation from F to G.*

Proof. We need to check that for any given morphism $h : X \to Y$ in $Ho(\mathcal{C})$, the appropriate diagram $D(h)$

$$\begin{tikzcd}
F(X) \arrow[r, "\tau_X"] \arrow[d, "F(h)"'] & G(X) \arrow[d, "G(h)"] \\
F(Y) \arrow[r, "\tau_Y"] & G(Y)
\end{tikzcd}$$

commutes. By assumption $D(h)$ commutes if $h = \gamma(f)$ for some morphism f in $\mathcal{C}$ or if $h = \gamma(g)^{-1}$ for some weak equivalence in $\mathcal{C}$. If we write $h = h_1h_2$, it is clear that $D(h)$ commutes if both $D(h_1)$ and $D(h_2)$ commute. This fact proves the desired result, since by 1.4.11 any morphism in $Ho(\mathcal{C})$ is a composite of morphisms of the form $\gamma(f)$ and $\gamma(g)^{-1}$. □

An even stronger result than proposition 1.4.11 can be established (cf. 1.4.15) via the notion of localization. This provides the functor γ with a universal property with respect to $WE_{\mathcal{C}}$.

Definition 1.4.13. Let $\mathcal{C}$, $\mathcal{D}$ be categories and $W \subseteq Mor\mathcal{C}$ a class of morphisms in $\mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{D}$ is a *localization* of $\mathcal{C}$ with respect to W if

(1) $F(f)$ is an isomorphism for any $f \in W$, and

(2) for any functor $G : \mathcal{C} \to \mathcal{D}'$ which sends elements of W on isomorphisms of $\mathcal{D}'$, there exists a unique functor $G' : \mathcal{D} \to \mathcal{D}'$ such that $G'F = G$.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[dr, "\forall G"'] & \mathcal{D} \arrow[d, dashed, "\exists! G'"] \\
& \mathcal{D}'
\end{tikzcd}$$

The second condition implies that $\mathcal{D} \cong \mathcal{D}'$ and that all localizations of $\mathcal{C}$ with respect to W, if they exist, are canonically isomorphic.

Lemma 1.4.14. *Let $\mathcal{D}$ be a category, $\mathcal{C}$ a model category, and $F : \mathcal{C} \to \mathcal{D}$ a functor sending weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. If $f, g : A \to X$ are are left or right homotopic in $\mathcal{C}$, then $F(f) = F(g)$ in $\mathcal{D}$.*

Proof. Since the other case is dual, we only need to assume that f and g are left homotopic. We choose (cf. 1.3.7) a good left homotopy $H : A \wedge I \to X$ from f to g, so that $A \wedge I$ is a good cylinder object for A.

$$A \vee A \xhookrightarrow{i_0 \vee i_1} A \wedge I \xrightarrow[\sim]{w} A$$

By construction, $wi_0 = wi_1 = id_A$ so that $F(w)F(i_0) = F(w)F(i_1)$. By assumption, since w is a weak equivalence, $F(w)$ is an isomorphism. It follows that $F(i_0) = F(i_1)$, so that

$$F(f) = F(H)F(i_0) = F(H)F(i_1) = F(g).$$

□

Theorem 1.4.15. *If $\mathcal{C}$ is a model category and WE the class of weak equivalences of $\mathcal{C}$, then the functor $\gamma : \mathcal{C} \to Ho(\mathcal{C})$ is a localization of $\mathcal{C}$ with respect to WE.*

Proof. We need to verify the two conditions of 1.4.13. The first one has already been proven in 1.4.11. For the second one, let's consider a functor $G : \mathcal{C} \to \mathcal{D}$ which sends weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. We must construct a unique functor $G' : Ho(\mathcal{C}) \to \mathcal{D}$ that verifies $G'\gamma = G$. Since the objects of $Ho(\mathcal{C})$ are the same as the objects of $\mathcal{C}$, the effect of G' on objects is clear. Let's now pick a morphism $f : X \to Y$ in $Ho(\mathcal{C})$ which is well defined up to homotopy (cf. 1.3.16) and represented by a morphism $f' : RQX \to RQY$ in $\mathcal{C}$. By 1.4.14 $G(f')$ only depends on the homotopy class of f'. Letting $G'(f)$ be

$$G'(f) := G(p_Y)G(i_{QY})^{-1}G(f')G(i_{QX})G(p_X)^{-1},$$

clearly defines a functor from $Ho(\mathcal{C})$ to $\mathcal{D}$. If f is the image by γ of a morphism $h : X \to Y$ in $\mathcal{C}$, then, after an appropriate altering of f' up to right homotopy (cf. 1.4.3 and 1.4.5), we obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "h"'] & QX \arrow[l, "p_X"'] \arrow[r, "i_{QX}"] \arrow[d, "\tilde{h}"'] & RQX \arrow[d, "f'"'] \\
Y & QY \arrow[l, "p_Y"'] \arrow[r, "i_{QY}"] & RQY,
\end{tikzcd}$$

on which we may apply G to see that $G'(f) = G(h)$, so that $G'\gamma = G$. The uniqueness of G' is simply a consequence of 1.4.11. □

This result allows to interpret $Ho(\mathcal{C})$ in a more conceptual way which, surprisingly enough, only depends on WE. This shows the importance of the class of weak equivalences as it carries all the homotopic information of a model category.

1.5 Derived functors

In this section, for any given functor $F : \mathcal{C} \to \mathcal{D}$ from a model category $\mathcal{C}$ to a category $\mathcal{D}$, we study its left and right derived functors $LF, RF : Ho(\mathcal{C}) \to \mathcal{D}$, and its total left and right derived functors $\mathbb{L}F, \mathbb{R}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$. This will lead to the notions of Quillen pairs and Quillen equivalences, and will provide a criteria for two homotopy categories to be equivalent.

Definition 1.5.1. Let $\mathcal{D}$ be a category, $\mathcal{C}$ a model category, and $F : \mathcal{C} \to \mathcal{D}$ a functor. A *left derived functor* for F is a pair (LF, τ) which consists of a functor $LF : Ho(\mathcal{C}) \to \mathcal{D}$ and natural transformation $\tau : LF \circ \gamma \to F$ such that for any pair

$$(G : Ho(\mathcal{C}) \to \mathcal{D},\ \sigma : G \circ \gamma \to F),$$

there exists a unique natural transformation $\sigma' : G \to LF$ such that $\tau\sigma'\gamma = \sigma$.

$$\begin{tikzcd} & LF\gamma \arrow[d, "\tau"] \\ G\gamma \arrow[ur, dashed, "\exists!\sigma'\gamma"] \arrow[r, "\forall\sigma"'] & F \end{tikzcd}$$

Dually, a *right derived functor* for F is a pair (RF, τ) which consists of a functor $RF : Ho(\mathcal{C}) \to \mathcal{D}$ and natural transformation $\tau : F \to RF \circ \gamma$ such that for any pair

$$(G : Ho(\mathcal{C}) \to \mathcal{D},\ \sigma : F \to G \circ \gamma),$$

there exists a unique natural transformation $\sigma' : RF \to G$ such that $\sigma'\gamma\tau = \sigma$.

$$\begin{tikzcd} F \arrow[r, "\forall\sigma"] \arrow[d, "\tau"'] & G\gamma \\ RF\gamma \arrow[ur, dashed, "\exists!\sigma'\gamma"'] & \end{tikzcd}$$

Remarks 1.5.2. (1) By the universal property of the above definition, two left (resp. right) derived functors for F are canonically naturally isomorphic; we will then talk about *the* left (resp. right) derived functor of F.

(2) If $F : \mathcal{C} \to \mathcal{D}$ is a functor from a model category $\mathcal{C}$ to a category $\mathcal{D}$ that sends weak equivalences to isomorphisms, then by 1.4.15 there exists a unique functor $F' : Ho(\mathcal{C}) \to \mathcal{D}$ such that $F = F'\gamma$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[dr, "F"'] & Ho(\mathcal{C}) \arrow[d, dashed, "\exists!F'"] \\ & \mathcal{D} \end{tikzcd}$$

In this case it is clear that (F', τ), with the identity natural transformation $\tau : F'\gamma \to F$, is a left and right derived functor of F, ie. $RF = F' = LF$.

We now give a result (cf. 1.5.4 below) that generalizes this last remark; for this we need the following lemma.

Lemma 1.5.3. *Let $\mathcal{C}$ be a model category, $\mathcal{D}$ a category, and let $F : \mathcal{C}_c \to \mathcal{D}$ (cf. 1.4.1) be a functor such that $F(f)$ is an isomorphism whenever f is an acyclic cofibration between objects of $\mathcal{C}_c$. If $f, g : A \to B$ are morphisms in $\mathcal{C}_c$ such that $f \sim_r g$ in $\mathcal{C}$, then $F(f) = F(g)$.*

Proof. By 1.3.14, we may choose a right homotopy $H : A \to B^I$ from f to g such that B^I is a very good path object for B; in other words we have

$$\begin{tikzcd} B \arrow[r, hook, "j", "\sim"'] & B^I \arrow[r, two heads, "q"] & B \times B. \end{tikzcd}$$

Since the above morphism $j : B \to B^I$ is an acyclic cofibration and since B is by assumption cofibrant, the path object B^I is itself cofibrant. By assumption, it follows that $F(j)$ is defined and is an isomorphism. By construction we have

$$F(q_0)F(j) = F(q_1)F(j) = F(id_B),$$

so that $F(q_0) = F(q_1)$. Finally, the relations $f = q_0H$ and $g = q_1H$ give

$$F(f) = F(q_0)F(H) = F(q_1)F(H) = F(g),$$

which is none other than the desired equality. □

Proposition 1.5.4. *Let $\mathcal{C}$ be a model category, $\mathcal{D}$ a category, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that $F(f)$ is an isomorphism whenever f is a weak equivalence between cofibrant objects in $\mathcal{C}$. Then the left derived functor (LF, τ) of F exists, and for each cofibrant object X of $\mathcal{C}$ the morphism*

$$\tau_X : LF(X) \to F(X)$$

is an isomorphism in $\mathcal{D}$.

Dually, if F is such that $F(f)$ is an isomorphism whenever f is a weak equivalence between fibrant objects in $\mathcal{C}$. Then the right derived functor (RF, τ') of F exists, and for each fibrant object X of $\mathcal{C}$ the morphism

$$\tau'_X : F(X) \to RF(X)$$

is an isomorphism in $\mathcal{D}$.

Proof. Since both assertions are dual to each other, we only have to prove the first one. By 1.5.3 the functor F identifies right homotopic morphisms between cofibrant objects of $\mathcal{C}$; this fact induces a functor $F' : \pi\mathcal{C}_c \to \mathcal{D}$ (cf. 1.4.1). By assumption, if g is a morphism in $\pi\mathcal{C}_c$ which is represented by a weak equivalence in $\mathcal{C}$, then $F'(g)$ is an isomorphism in $\mathcal{D}$. We saw earlier that there is a functor $Q : \mathcal{C} \to \pi\mathcal{C}_c$ that sends any weak equivalence f in $\mathcal{C}$ to a right homotopy class $g = Q(f)$ which is represented by a weak equivalence in $\mathcal{C}$ (cf. 1.4.3 and 1.4.4). From this, it follows that the composite functor $F'Q$ sends weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. By the universal property of $Ho(\mathcal{C})$ (cf. 1.5.2.(2)), $F'Q$ induces a functor

$$LF : Ho(\mathcal{C}) \to \mathcal{D},$$

and we have a natural transformation

$$\tau \ : \ (LF)\gamma \to F \ : \ (X \in \mathcal{C}) \mapsto (F(p_X) : LF(X) = F(QX) \to F(X)).$$

If X is cofibrant then $QX = X$ and the morphism τ_X becomes the identity id_X; in particular τ_X is an isomorphism.

What remains to prove is that the pair (LF, τ) is universal in the sense of 1.5.1. So we consider a functor $G : Ho(\mathcal{C}) \to \mathcal{D}$, a natural transformation $\sigma : G\gamma \to F$, an hypothetical natural transformation $\sigma' : G \to LF$, as well as for each object X of $\mathcal{C}$ the following commutative diagram:

$$\begin{tikzcd}
G(QX) \arrow[r, "\sigma'_{QX}"] \arrow[d, "G\gamma(p_X)"'] & LF(QX) \arrow[r, "\tau_{QX}=id"] \arrow[d, "LF\gamma(p_X)=id"] & F(QX) \arrow[d, "F(p_X)"] \\
G(X) \arrow[r, "\sigma'_X"] & LF(X) \arrow[r, "\tau_X=F(p_X)"] & F(QX)
\end{tikzcd}$$

If σ' is to satisfy 1.5.1 we must have

$$\tau_{QX} \circ \sigma'_{QX} = \sigma_{QX} \qquad \text{so that} \qquad \sigma'_X = \sigma_{QX} G(\gamma p_X)^{-1},$$

which proves that there is at most one natural transformation σ' satisfying the required universal property of 1.5.1; in other words, if such a natural transformation exists it is unique. Now if we define σ'_X to be

$$\sigma'_X = \sigma_{QX} G(\gamma p_X)^{-1} \qquad \text{for every object } X \in \mathcal{C},$$

we obtain a natural transformation $\sigma' : G \to LF$, so that its existence is proven. $\square$

Conclusion 1.5.5. The left derived functor $LF : Ho(\mathcal{C}) \to \mathcal{D}$ of the above proposition is defined

- on objects by $LF(X) = F(QX)$, where $QX \overset{\sim}{\twoheadrightarrow} X$ is a fixed cofibrant model,
- on morphisms by $LF(f) = F(Qf)$, where $Qf = \tilde{f}$ is as in 1.4.3.

Dually, the right derived functor $RF : Ho(\mathcal{C}) \to \mathcal{D}$ of the above proposition is defined

- on objects by $RF(X) = F(RX)$, where $X \overset{\sim}{\hookrightarrow} RX$ is a fixed fibrant model,
- on morphisms by $RF(f) = F(Rf)$, where $Rf = \bar{f}$ is as in 1.4.5.

We shall now consider the left and right derived functors of a particular kind of functor.

Definition 1.5.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between model categories. The *total left derived functor* of F, denoted $\mathbb{L}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$, is the left derived functor of the composite

$$\gamma_{\mathcal{D}} F : \mathcal{C} \to Ho(\mathcal{D}),$$

where $\gamma_{\mathcal{D}} : \mathcal{D} \to Ho(\mathcal{D})$ is the canonical functor for $\mathcal{D}$ (cf. 1.4.8).

Dually, the *total right derived functor* of F, denoted $\mathbb{R}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$, is the right derived functor of the same composite $\gamma_{\mathcal{D}} F$.

Remark 1.5.7. Again, these functors are unique up to canonical natural isomorphism.

Example 1.5.8. Let R be a unitary associative ring, and ${}_R\mathcal{C}h^+$ the model category of chain complexes given in 1.2.8. Consider a right R-module M, a left R-module N and its corresponding chain complex $K(N,0) \in {}_R\mathcal{C}h^+$ whose only nontrivial module is N in degree 0. The functor $M \otimes - : {}_R\mathcal{M}od \to \mathcal{A}b = {}_{\mathbb{Z}}\mathcal{M}od$ induces a functor $F : {}_R\mathcal{C}h^+ \to {}_{\mathbb{Z}}\mathcal{C}h^+$ for which a total left derived functor $\mathbb{L}F$ exists. We then have natural isomorphisms

$$H_n(\mathbb{L}F(K(N,0))) \cong Tor_n^R(M,N) \qquad \text{for each} \quad n \in \mathbb{N},$$

where $Tor_n^R(M,-)$ is the usual n'th left derived functor of $M \otimes -$ from homological algebra.

After strengthening the notion of adjoint pairs to Quillen pairs and Quillen equivalences, we shall study what relations these pairs induce on the level of homotopy categories. These relations are in fact given by the total left and right derived functors of the given pair (cf. 1.5.12).

Definition 1.5.9. Let $\mathcal{C} = (\mathcal{C}, WE_{\mathcal{C}}, Fib_{\mathcal{C}}, Cof_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, WE_{\mathcal{D}}, Fib_{\mathcal{D}}, Cof_{\mathcal{D}})$ be model categories. A pair of adjoint functors

$$F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$$

is a *Quillen pair* if one of the following three equivalent conditions is satisfied:

i) F preserves cofibrations and G preserves fibrations.

i') G preserves fibrations and acyclic fibrations.

i'') F preserves cofibrations and acyclic cofibrations.

If in addition we have

> *ii)* $(f : F(A) \to B) \in WE_{\mathcal{D}}$ if and only if its adjoint $f^\sharp : A \to G(B) \in WE_{\mathcal{C}}$ for all $A \in Ob\mathcal{C}$ and $B \in Ob\mathcal{D}$,

then we say that $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ is a *Quillen equivalence.*

Let's first prove the above equivalences:

Proof. $(i \Leftrightarrow i'')$ Suppose that F preserves cofibrations and acyclic cofibrations. Let $f : A \to B$ be an acyclic cofibration in $\mathcal{C}$ and $g : C \to D$ be a fibration in $\mathcal{D}$, and consider the adjoint commutative diagrams

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, hook, "f"', "\sim"] & G(C) \arrow[d, "G(g)"] \\ B \arrow[r, "v"] & G(D) \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, "u^\flat"] \arrow[d, "F(f)"'] & C \arrow[d, "g"] \\ F(B) \arrow[r, "v^\flat"] & D. \end{tikzcd}$$

The fact that F preserves acyclic cofibrations implies the existence of a lift $w : F(B) \to C$ in the right-hand diagram whose adjoint $w^\sharp : B \to G(C)$ is a lift in the left-hand diagram. This means that

$$G(g) \in RLP(WE_{\mathcal{C}} \cap Cof_{\mathcal{C}}) = Fib_{\mathcal{C}} \qquad \text{(cf. 1.2.11)},$$

so that G preserves fibrations. To prove the converse, we only have to follow the steps of the above argument in reverse order and remember that $WE \cap Cof = LLP(Fib)$ (cf. 1.2.11).

$(i \Leftrightarrow i')$ It is simply dual to $(i \Leftrightarrow i'')$. □

Example 1.5.10. The adjoint pair

$$W : \mathcal{C} \times \mathcal{C} \Longleftrightarrow \mathcal{C} : \Delta,$$

where $W(A \times B) = A \vee B$ and $\Delta(A) = (A, A)$, is a Quillen pair (the three distinct classes WE, Fib and Cof of the model category $\mathcal{C} \times \mathcal{C}$ are are simply the products $WE_{\mathcal{C}} \times WE_{\mathcal{C}}, Fib_{\mathcal{C}} \times Fib_{\mathcal{C}}$ and $Cof_{\mathcal{C}} \times Cof_{\mathcal{C}}$ respectively).

Before getting to theorem 1.5.12 below, we need the following lemma.

Lemma 1.5.11. *If $F : \mathcal{C} \to \mathcal{D}$ is a functor between model categories that sends all acyclic cofibrations between cofibrant objects to weak equivalences, then F preserves all weak equivalences between cofibrant objects.*

Proof. Consider a weak equivalence $f : A \to B$ in $\mathcal{C}$ between cofibrant objects. Applying (M_5) to the morphism $f \vee id_B : A \vee B \to B$, we get

$$\begin{tikzcd} A \vee B \arrow[r, hook, "i"] & C \arrow[r, two heads, "p", "\sim"'] & B. \end{tikzcd}$$

Since A and B are cofibrant the morphisms

$$i_0 = i \circ inc_0 : A \to C \qquad \text{and} \qquad i_1 = i \circ inc_1 : B \to C$$

are cofibrations in $\mathcal{C}$ (cf. 1.3.3). Since pi_0 and pi_1 are weak equivalences and p is a weak equivalence, it follows from (M_2) that i_0 and i_1 are also weak equivalences. By assumption $F(i_0)$, $F(i_1)$ and $F(pi_1) = F(id_B) = id_{F(B)}$ are then weak equivalences in $\mathcal{D}$, so that $F(p)$ is a weak equivalence as well. It follows that

$$F(pi_0) = F(pi \circ inc_0) = F(f)$$

is a weak equivalence in $\mathcal{D}$. □

Theorem 1.5.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.*

(1) *A Quillen pair $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ induces an adjoint pair*

$$\mathbb{L}F : Ho(\mathcal{C}) \Longleftrightarrow Ho(\mathcal{D}) : \mathbb{R}G.$$

(2) *A Quillen equivalence $F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G$ induces a category equivalence*

$$\mathbb{L}F : Ho(\mathcal{C}) \cong Ho(\mathcal{D}) : \mathbb{R}G.$$

Proof. (1) Since F preserves acyclic cofibrations (cf. 1.5.9), lemma 1.5.11 tells us that F preserves all weak equivalences between cofibrant objects. In addition, the functor $\gamma_{\mathcal{D}} : \mathcal{D} \to Ho(\mathcal{D})$ is a localization with respect to $WE_{\mathcal{D}}$ (cf. 1.4.15), so that

$$F' := \gamma_{\mathcal{D}} \circ F : \mathcal{C} \to Ho(\mathcal{D})$$

sends weak equivalences in $\mathcal{C}$ to isomorphisms in $Ho(\mathcal{D})$. This, by 1.5.4, guaranties the existence of $LF' = \mathbb{L}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$. The dual argument demonstrate the existence of $\mathbb{R}G : Ho(\mathcal{D}) \to Ho(\mathcal{C})$, so that the total derived functors $\mathbb{L}F$ and $\mathbb{R}G$ exist.

Furthermore, since F is a left adjoint, it preserves colimits (cf. 1.1.14) and therefore initial objects. On the other hand, since G is a right adjoint, it preserves limits and therefore terminal objects. It then follows that F sends cofibrant objects in $\mathcal{C}$ to cofibrant objects in $\mathcal{D}$ and that G sends fibrant objects in $\mathcal{D}$ to fibrant objects in $\mathcal{C}$.

Now, let's choose a cofibrant object A in $\mathcal{C}$ and a fibrant object X in $\mathcal{D}$, and show that the adjunction isomorphism $\mathcal{C}(A, G(X)) \cong \mathcal{D}(F(A), X)$ preserves the homotopy equivalence relation (cf. 1.3.16) and gives a bijection

$$\pi(A, G(X)) \cong \pi(F(A), X). \qquad (*)$$

If $f, g : A \to G(X)$ represent the same class in $\pi(A, G(X))$, then $f \sim_l g$ via a left homotopy $H : A \wedge I \to G(X)$ in which the cylinder object $A \wedge I$ is good (cf. 1.3.7) and therefore cofibrant by 1.3.3. It then follow, using the fact that F preserves cofibrations and acyclic cofibrations, that $F(A \wedge I)$ is a cylinder object for $F(A)$, so that

$$H^\flat : F(A \wedge I) \to X$$

is a left homotopy between $f^\flat$ and $g^\flat$. Since X is fibrant we obtain $f^\flat \sim g^\flat$, and a dual argument with right homotopies shows that $f^\flat \sim g^\flat$ implies $f \sim g$ and proves $(*)$.

Let the functor $Q : \mathcal{C} \to \pi\mathcal{C}_c$ be as in 1.4.4 and the functor $S : \mathcal{D} \to \pi\mathcal{D}_f$ be as in 1.4.6. It follows from the construction of $\mathbb{L}F$ given in the proof of 1.5.4 and its dual for $\mathbb{R}G$ that the isomorphism $(*)$ gives a bijection

$$\begin{tikzcd}
Ho(\mathcal{C})(A, \mathbb{R}G(X)) \arrow[r, "(\gamma p_A)^*", "\cong"'] & Ho(\mathcal{C})(QA, G(SX)) \arrow[d, leftrightarrow, "\cong"] & \\
& Ho(\mathcal{D})(F(QA), SX) \arrow[r, "(\gamma i_X)_*^{-1}", "\cong"'] & Ho(\mathcal{D})(\mathbb{L}F(A), X),
\end{tikzcd} \qquad (**)$$

for any object A of $\mathcal{C}$ and any object X of $\mathcal{D}$. This bijection gives a natural isomorphism of functors from $\mathcal{C}^{op} \times \mathcal{D}$ to $\mathcal{S}et$, and by 1.4.12 a natural isomorphism of functors from $Ho(\mathcal{C})^{op} \times Ho(\mathcal{D})$ to $\mathcal{S}et$ which is none other than the desired adjunction between $\mathbb{L}F$ and $\mathbb{R}G$.

(2) Let A be a cofibrant object of $\mathcal{C}$. Since the morphism $i_{F(A)} : F(A) \to SF(A)$ is a weak equivalence in $\mathcal{D}$ (cf. 1.4.2), its adjoint $i^\sharp_{F(A)} : A \to GSF(A)$ in $\mathcal{C}$ is also a weak equivalence by assumption. Let

$$\varepsilon_A := id^\sharp_{\mathbb{L}F(A)} : A \longrightarrow \mathbb{R}G(\mathbb{L}F(A)) \qquad \in \; Ho(\mathcal{C})$$

be the adjoint of the identity morphism $id_{\mathbb{L}F(A)}$ in $Ho(\mathcal{D})$. It follows from $(**)$ that ε_A is an isomorphism. Now since every object of $Ho(\mathcal{C})$ is isomorphic to a cofibrant object of $\mathcal{C}$, it follows that ε_A is an isomorphism for any object A of $Ho(\mathcal{C})$. This fact implies that the functor $\mathbb{R}G\mathbb{L}F$ is naturally isomorphic to the identity functor of $Ho(\mathcal{C})$. Finally, a dual argument shows that $\mathbb{L}F\mathbb{R}G$ is naturally isomorphic to the identity functor of $Ho(\mathcal{D})$ so that the desired result is proven. □

We shall apply the above theorem to the construction of the homotopy pushout and the homotopy pullback functors.

Example 1.5.13. Let $\mathcal{C}$ be a model category and $\mathcal{D}$ the small category $\{a \leftarrow b \rightarrow c\}$. In the category $\mathcal{C}^{\mathcal{D}}$ of functors $\mathcal{D} \rightarrow \mathcal{C}$, an object X is pushout data

$$X(a) \longleftarrow X(b) \longrightarrow X(c)$$

in $\mathcal{C}$, and a morphism $f : X \rightarrow Y$ is a commutative diagram

$$\begin{tikzcd} X(a) \arrow[d, "f_a"] & X(b) \arrow[l] \arrow[r] \arrow[d, "f_b"] & X(c) \arrow[d, "f_c"] \\ Y(a) & Y(b) \arrow[l] \arrow[r] & Y(c). \end{tikzcd} \qquad (*)$$

Given a morphism $f : X \rightarrow Y$ in $\mathcal{C}^{\mathcal{D}}$, let $\partial_b(f) := X(b)$ and define objects $\partial_a(f)$ and $\partial_c(f)$ of $\mathcal{C}$ by the respective pushout diagrams

$$\begin{tikzcd} X(b) \arrow[r] \arrow[d, "f_b"'] & X(a) \arrow[d] \\ Y(b) \arrow[r] & \partial_a(f) \end{tikzcd} \qquad \begin{tikzcd} X(b) \arrow[r] \arrow[d, "f_b"'] & X(c) \arrow[d] \\ Y(b) \arrow[r] & \partial_c(f). \end{tikzcd} \qquad (**)$$

The commutative diagram $(*)$ induces morphisms

$$i_a(f) : \partial_a(f) \rightarrow Y(a), \qquad i_b(f) : \partial_b(f) \rightarrow Y(b) \qquad \text{and} \qquad i_c(f) : \partial_c(f) \rightarrow Y(c).$$

We then have the following result:

We can define a morphism $f : X \rightarrow Y$ in $\mathcal{C}^{\mathcal{D}}$ to be

- *a weak equivalence if the morphisms f_a, f_b and f_c are weak equivalences in $\mathcal{C}$,*
- *a fibration if the morphisms f_a, f_b and f_c are fibrations in $\mathcal{C}$,*
- *a cofibration if the morphisms $i_a(f)$, $i_b(f)$ and $i_c(f)$ are cofibrations in $\mathcal{C}$,*

in order to provide $\mathcal{C}^{\mathcal{D}}$ with the structure of a model category. In this case the adjoint functors

$$colim : \mathcal{C}^{\mathcal{D}} \Longleftrightarrow \mathcal{C} : \Delta \qquad \textit{(cf. 1.1.13)}$$

form a Quillen pair, so that the total derived functors $\mathbb{L}colim$ and $\mathbb{R}\Delta$ exist and form an adjoint pair

$$\mathbb{L}colim : Ho(\mathcal{C}^{\mathcal{D}}) \Longleftrightarrow Ho(\mathcal{C}) : \mathbb{R}\Delta. \qquad \textit{(cf. 1.5.12)}$$

Proof. Axiom (M_1) is a consequence of 1.1.13. Axioms (M_2) and (M_3) follow from the corresponding axioms for $\mathcal{C}$. Let's prove the cofibration-acyclic fibration part of (M_4). Consider the commutative diagram

$$\begin{tikzcd}
(A(a) \leftarrow A(b) \rightarrow A(c)) \arrow[r] \arrow[d, hook, "f"'] & (X(a) \leftarrow X(b) \rightarrow X(c)) \arrow[d, two heads, "\sim"', "p"] \\
(B(a) \leftarrow B(b) \rightarrow B(c)) \arrow[r] & (Y(a) \leftarrow Y(b) \rightarrow Y(c))
\end{tikzcd}$$

in $\mathcal{C}^{\mathcal{D}}$, where f is a cofibration and p an acyclic cofibration. This diagram consists of three commutative squares

$$\begin{tikzcd}
A(a) \arrow[r] \arrow[d, "f_a"'] & X(a) \arrow[d, "p_a"] & A(b) \arrow[r] \arrow[d, "f_b"'] & X(b) \arrow[d, "p_b"] & A(c) \arrow[r] \arrow[d, "f_c"'] & X(c) \arrow[d, "p_c"] \\
B(a) \arrow[r] & Y(a) & B(b) \arrow[r] & Y(b) & B(c) \arrow[r] & Y(c).
\end{tikzcd} \qquad (***)$$

The facts that f is a fibration and p an acyclic cofibration respectively imply that f_b is a fibration and $i_b(p) = p_b$ is an acyclic cofibration in $\mathcal{C}$, so that by axiom (M_4) in $\mathcal{C}$ the middle square diagram $(***)$ has a lift. This in turn induces morphisms

$$u : \partial_a(f) \to X(a) \qquad \text{and} \qquad v : \partial_c(f) \to X(c).$$

We now have two commutative diagrams

$$\begin{tikzcd}
\partial_a(f) \arrow[r, "u"] \arrow[d, "i_a(f)"'] & X(a) \arrow[d, "p_a"] & \partial_c(f) \arrow[r, "v"] \arrow[d, "i_c(f)"'] & X(c) \arrow[d, "p_c"] \\
B(a) \arrow[r] & Y(a) & B(c) \arrow[r] & Y(c)
\end{tikzcd}$$

which, by (M_4) in $\mathcal{C}$, have lifts that induce the two remaining lifts in $(***)$. The proof of the acyclic cofibration-fibration part of (M_4) is similar.

Let's now prove the acyclic cofibration-fibration part of (M_5). Consider a morphism $f : A \to B$ in $\mathcal{C}^{\mathcal{D}}$. We may use axiom (M_5) in $\mathcal{C}$ to factor the morphism $f_b : A(b) \to B(b)$ as

$$\begin{tikzcd} A(b) \arrow[r, hook, "\sim"] & Y \arrow[r, two heads] & B(b). \end{tikzcd}$$

Define X and Z to be respectively the pushouts of

$$A(a) \leftarrow A(b) \rightarrow Y \qquad \text{and} \qquad Y \leftarrow A(b) \rightarrow A(c),$$

so that we have a commutative diagram

$$\begin{tikzcd}
A(a) \arrow[d, hook, "\sim"'] & A(b) \arrow[l] \arrow[r] \arrow[d, hook, "\sim"'] & A(c) \arrow[d, hook, "\sim"'] \\
X \arrow[d, "k"'] & Y \arrow[l] \arrow[r] \arrow[d, two heads] & Z \arrow[d, "l"] \\
B(a) & B(b) \arrow[l] \arrow[r] & B(c)
\end{tikzcd}$$

in which the lower outside vertical arrows k and l are constructed using the universal property of pushouts. We can now use (M_5) in $\mathcal{C}$ in order to factor k and l as respectively

$$X \overset{\sim}{\hookrightarrow} X' \twoheadrightarrow B(a) \qquad \text{and} \qquad Z \overset{\sim}{\hookrightarrow} Z' \twoheadrightarrow B(c),$$

and thus obtain an object $X' \leftarrow Y \rightarrow Z'$ of $\mathcal{C}^{\mathcal{D}}$ which is an intermediate object for the desired factorization of f. A similar argument establishes the second part of (M_5), so that $\mathcal{C}^{\mathcal{D}}$ has the desired model category structure.

What remains to show is a direct application of 1.1.13 and 1.5.12 with the fact that the functor Δ preserves both fibrations and acyclic fibrations. $\square$

The functor $\mathbb{L}colim$ thus constructed is the *homotopy pushout functor*. It follows from 1.5.4 that $\mathbb{L}colim(X)$ is isomorphic to $colim(X)$ if X is a cofibrant object of $\mathcal{C}^{\mathcal{D}}$.

Dually, it is possible to construct the functor $\mathbb{R}lim$ as in the following example.

Example 1.5.14. Let $\mathcal{C}$ be a model category and $\mathcal{D}$ the small category $\{a \rightarrow b \leftarrow c\}$. Given a morphism $f : X \rightarrow Y$ in $\mathcal{C}^{\mathcal{D}}$, ie.

$$\begin{tikzcd} X(a) \arrow[r] \arrow[d, "f_a"] & X(b) \arrow[d, "f_b"] & X(c) \arrow[l] \arrow[d, "f_c"] \\ Y(a) \arrow[r] & Y(b) & Y(c), \arrow[l] \end{tikzcd} \qquad (*)$$

let $\delta_b(f) := X(b)$ and define objects $\delta_a(f)$ and $\delta_c(f)$ of $\mathcal{C}$ by the respective pullback diagrams

$$\begin{tikzcd} \delta_a(f) \arrow[r] \arrow[d] & X(b) \arrow[d, "f_b"] \\ Y(a) \arrow[r] & Y(b) \end{tikzcd} \qquad \begin{tikzcd} \delta_c(f) \arrow[r] \arrow[d] & X(b) \arrow[d, "f_b"] \\ Y(c) \arrow[r] & Y(b). \end{tikzcd}$$

The commutative diagram $(*)$ induces morphisms

$$p_a(f) : X(a) \rightarrow \delta_a(f), \qquad p_b(f) : X(b) \rightarrow \delta_b(f) \qquad \text{and} \qquad p_c(f) : X(c) \rightarrow \delta_c(f).$$

We then have the following result:

We can define a morphism $f : X \rightarrow Y$ in $\mathcal{C}^{\mathcal{D}}$ to be

- *a weak equivalence if the morphisms f_a, f_b and f_c are weak equivalences in $\mathcal{C}$,*
- *a fibration if the morphisms $p_a(f)$, $p_b(f)$ and $p_c(f)$ are fibrations in $\mathcal{C}$,*
- *a cofibration if the morphisms f_a, f_b and f_c are cofibrations in $\mathcal{C}$,*

in order to provide $\mathcal{C}^{\mathcal{D}}$ with the structure of a model category. In this case the adjoint functors

$$\Delta : \mathcal{C} \Longleftrightarrow \mathcal{C}^{\mathcal{D}} : lim \qquad \textit{(cf. 1.1.13)}$$

form a Quillen pair, so that the total derived functors $\mathbb{R}lim$ and $\mathbb{L}\Delta$ exist and form an adjoint pair

$$\mathbb{L}\Delta : Ho(\mathcal{C}) \Longleftrightarrow Ho(\mathcal{C}^{\mathcal{D}}) : \mathbb{R}lim. \qquad \textit{(cf. 1.5.12)}$$

Proof. Strictly dual to the proof given in 1.5.13. $\square$

The functor $\mathbb{R}lim$ thus constructed is the *homotopy pullback functor*. It follows from 1.5.4 that $\mathbb{R}lim$ is isomorphic to $lim(X)$ if X is a cofibrant object of $\mathcal{C}^{\mathcal{D}}$.

Chapter 2

Monoidal model categories

Now that the desired context for homotopy theory has been established, the next step is to add some algebraic structure to it: the structure of a closed symmetric monoidal category. This monoidal structure, appropriately combined with the homotopy theory of model categories, will give rise to the theoretical context in which we shall develop the general notion of homotopic Hopf-Galois extensions.

In order to do this, we are first going to study, in section 2.1, the closed symmetric monoidal structure of a monoidal category. This will allow to generalize the classical notions of modules and algebras over a ring without the need to stay within the restricting context of sets, using instead commutative diagrams to define the appropriate laws of compositions. In section 2.2, we shall treat an important special case of model categories, namely cofibrantly generated model categories. In section 2.3, we will combine model and closed symmetric monoidal structures to give rise to monoidal model categories. We will end the chapter by showing, in section 2.4, that the subcategories of modules and algebras of a given monoidal model category are actually cofibrantly generated, and by giving enough conditions under which the homotopy categories of these subcategories are equivalent.

2.1 Monoidal categories

We start by defining what a monoidal category is; it is a category endowed with a law of composition, the tensor product, which acts both on the objects and the morphisms of the category. For this reason, it can be defined as a functor which takes two objects (or morphisms) into another one modulo some appropriate conditions of compatibility.

Definition 2.1.1. Let $\mathcal{C}$ be a category, $1 = 1_{\mathcal{C}}$ an object of $\mathcal{C}$ and $\otimes = \otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ a bifunctor for which we write

$$\begin{aligned} \otimes_{Ob}(A, B) &= A \otimes B \in Ob\mathcal{C}, && \text{for every objects } A, B \in \mathcal{C}, \\ \otimes_{Mor}(f, g) &= f \otimes g \in Mor\mathcal{C}, && \text{for every morphisms } f, g \in Mor\mathcal{C}. \end{aligned}$$

The triple $(\mathcal{C}, \otimes, 1)$ is a *monoidal category* if the following conditions hold:

- The product $\otimes$ is associative up to isomorphism; ie. there is a family of isomorphisms

$$\{\alpha_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C \mid A, B, C \in \mathcal{C}\}$$

which make the diagram

$$\begin{tikzcd} A\otimes(B\otimes C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "f\otimes(g\otimes h)"'] & (A\otimes B)\otimes C \arrow[d, "(f\otimes g)\otimes h"'] \\ A'\otimes(B'\otimes C') \arrow[r, "\alpha_{A',B',C'}"] & (A'\otimes B')\otimes C' \end{tikzcd}$$

commute for each morphisms $f \in \mathcal{C}(A, A')$, $g \in \mathcal{C}(B, B')$ and $h \in \mathcal{C}(C, C')$.

- There is in $\mathcal{C}$ a family of natural isomorphisms

$$\{\rho_A : A \otimes 1 \longrightarrow A \text{ and } \lambda_A : 1 \otimes A \longrightarrow A \mid A \in \mathcal{C}\}$$

which make the diagram

$$\begin{tikzcd} A\otimes 1 \arrow[r, "\rho_A"] \arrow[d, "f\otimes id_1"'] & A \arrow[d, "f"] & 1\otimes A \arrow[l, "\lambda_A"'] \arrow[d, "id_1\otimes f"] \\ A'\otimes 1 \arrow[r, "\rho_{A'}"] & A' & 1\otimes A' \arrow[l, "\lambda_{A'}"'] \end{tikzcd}$$

commute for each morphism $f \in \mathcal{C}(A, A')$.

- We have coherence of isomorphisms α, ρ and λ in the sense that $\lambda_1 = \rho_1$ and that the diagrams

$$\begin{tikzcd} A\otimes(B\otimes(C\otimes D)) \arrow[r, "\alpha_{A,B,C\otimes D}"] \arrow[d, "id_A\otimes\alpha_{B,C,D}"'] & (A\otimes B)\otimes(C\otimes D) \arrow[r, "\alpha_{A\otimes B,C,D}"] & ((A\otimes B)\otimes C)\otimes D \arrow[d, "\alpha_{A,B,C}\otimes id_D"] \\ A\otimes((B\otimes C)\otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & (A\otimes(B\otimes C))\otimes D \end{tikzcd}$$

and

$$\begin{tikzcd} A\otimes(1\otimes B) \arrow[r, "\alpha_{A,1,B}"] \arrow[d, "id_A\otimes\lambda_B"'] & (A\otimes 1)\otimes B \arrow[d, "\rho_A\otimes id_B"] \\ A\otimes B \arrow[r, equal] & A\otimes B \end{tikzcd}$$

commute for any choice of objects A, B, C, D in $\mathcal{C}$.

The operation $\otimes$ is then a *tensor product* on $\mathcal{C}$ and the object 1 a *unit* for $(\mathcal{C}, \otimes)$.

In addition, we say that the monoidal category $(\mathcal{C}, \otimes, 1)$ is *symmetric* if for every objects $A, B \in \mathcal{C}$ we have isomorphisms

$$\gamma_{A,B} : A \otimes B \cong B \otimes A$$

which make the following diagrams commute:

$$\gamma_{B,A} \circ \gamma_{A,B} = id \qquad\qquad \lambda_A \circ \gamma_{A,1} : A \otimes 1 \cong A$$

$$\begin{tikzcd} A\otimes(B\otimes C) \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "id_A\otimes\gamma_{B,C}"'] & (A\otimes B)\otimes C \arrow[r, "\gamma_{A\otimes B,C}"] & C\otimes(A\otimes B) \arrow[d, "\alpha_{C,A,B}"] \\ A\otimes(C\otimes B) \arrow[r, "\alpha_{A,C,B}"] & (A\otimes C)\otimes B \arrow[r, "\gamma_{A,C}\otimes id_B"] & (C\otimes A)\otimes B. \end{tikzcd}$$

Finally, we say that a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is *closed* if every functor of the form

$$- \otimes B : \mathcal{C} \longrightarrow \mathcal{C} \qquad \text{with} \qquad B \in Ob\mathcal{C}$$

is the left adjoint of an endofunctor

$$(-)^B : \mathcal{C} \longrightarrow \mathcal{C}.$$

This latter functor induces an internal Hom functor

$$[-,-] : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$$

defined by $[B, C] := C^B$ on objects of $\mathcal{C}$, and by

$$[-,-](f^{op} : B' \to B,\ g : C \to C') \ = \ g^f : C^B \to C'^{B'} : h \mapsto g \circ h \circ f,$$

on morphisms of $\mathcal{C}$.

Remark 2.1.2. In a monoidal category $(\mathcal{C}, \otimes, 1)$, the functoriality of $\otimes$ implies that

- $id_{A \otimes B} = id_A \otimes id_B$ for every objects A, B in $\mathcal{C}$,
- $(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$ whenever these compositions are defined in $\mathcal{C}$.

The structure of a monoidal category can be found in numerous situations encountered in classical mathematics. Instead of stating a great number of them, we shall restrict ourselves to the fundamental cases of sets and modules, and briefly mention how braids also fits into this model.

Examples 2.1.3. (1) The category $\mathcal{S}et$ has a closed symmetric monoidal structure where $\otimes : \mathcal{S}et \times \mathcal{S}et \to \mathcal{S}et$ is given on objects and on morphisms by

$$\otimes(A, B) = A \otimes B := A \times B \qquad \text{and} \qquad \otimes(f, g) = f \otimes g := f \times g$$

respectively, where $f \times g : A \times B \to A' \times B'$ is defined by $(f \times g)(a, b) = (f(a), g(b))$, and the unit object is a singleton $1 = \{*\}$. Furthermore, for any given object B in $\mathcal{S}et$ the right adjoint functor $(-)^B : \mathcal{S}et \to \mathcal{S}et$ is simply given by

$$(-)^B(C) \ = \ C^B \ = \ \mathcal{S}et(B, C).$$

(2) For a given unitary ring R, the category ${}_R\mathcal{M}od_R$ of R-bimodules is endowed with a closed symmetric monoidal structure given by $({}_R\mathcal{M}od_R, \otimes_R, A)$, where $\otimes_R$ denotes the usual tensor product on R-modules. Again, the right adjoint functor $(-)^N : {}_R\mathcal{M}od_R \to {}_R\mathcal{M}od_R$ is given for an R-bimodule N by

$$(-)^N(M) \ = \ M^N \ = \ {}_R\mathcal{M}od_R(N, M),$$

which of course correspond to the closed symmetric monoidal structure given above on $\mathcal{S}et$. In particular, for $R = \mathbb{Z}$ we have a closed symmetric monoidal category $(\mathcal{A}b, \otimes_{\mathbb{Z}}, \mathbb{Z})$.

(3) Another example of monoidal category is the n-strand braid group. An *n-strand braid* consists of a permutation $\tau \in S_n$ and a sequence $(\alpha_1, \ldots, \alpha_n)$ of paths

$$\alpha_i : \ I \longrightarrow \mathbb{R} \times I \times I, \qquad \text{with} \quad I = [0, 1] \subseteq \mathbb{R},$$

the *strands*, such that

- $\alpha_i(0) = (i, 0, 1)$ (begining of a strand i).

- $\alpha_i(1) = (\tau(i), 0, 0)$ (end of a strand i).
- If $\alpha_i = (\alpha_i^1, \alpha_i^2, \alpha_i^3)$, then $s < t$ imply $\alpha_i^3(s) > \alpha_i^3(t)$
  (any strand of the braid is decreasing in its third variable).
- If $i \neq j$, then $\alpha_i(I) \cap \alpha_j(I) = \emptyset$ (the strands do not intersect).

Two n-strand braids $\alpha = (\tau; \alpha_1, \dots, \alpha_n)$ and $\beta = (\sigma; \beta_1, \dots, \beta_n)$ are said to be *isotopic* if there exists a continuous function

$$H : \ (\mathbb{R} \times I \times I) \times I \ \longrightarrow \ \mathbb{R} \times I \times I$$

such that

- $H(-, 0) = id_{\mathbb{R}\times I \times I}$,
- $H(-, t)$ is an homeomorphism for every $t \in I$,
- $(\tau; H(-, t) \circ \alpha_1, \dots, H(-, t) \circ \alpha_n)$ is a braid for every $t \in I$,
- $H(-, 1) \circ \alpha_i = \beta_i$ for every $i \leq n$;

in other words if α can be continuously transformed into β in such a way that each stage of the transformation is always a braid. We denote by $[\alpha]$ the isotopy class of a braid α, and we define

$$\mathcal{B}_n \ := \ \{ \ [\alpha] \mid \alpha \text{ is an } n\text{-strand braid } \},$$

as well as

$$\mathcal{B} \ := \ \bigcup_{n\in\mathbb{N}} \mathcal{B}_n.$$

For two n-strand braids $\alpha = (\tau; \alpha_1, \dots, \alpha_n)$ and $\beta = (\sigma; \beta_1, \dots, \beta_n)$, we define the braid

$$\alpha \star \beta \ := \ (\sigma \circ \tau; \ \alpha_1 * \beta_{\tau(1)}, \ \dots, \ \alpha_n * \beta_{\tau(n)}),$$

by concatenation of each strand of α with the corresponding strand of β. Since

$$[\alpha] = [\alpha'] \quad \text{and} \quad [\beta] = [\beta'] \quad \Rightarrow \quad [\alpha \star \beta] = [\alpha' \star \beta'],$$

we may define for each n a law of composition $\cdot$ on $\mathcal{B}_n$ by

$$[\alpha] \cdot [\beta] \ := \ [\alpha \star \beta],$$

with which $(\mathcal{B}_n, \cdot)$ becomes a group whose unit element is the isotopy class of the braid

$$\varepsilon_n \ = \ (id_{S_n}; \ \{1\} \times \{0\} \times I, \ \dots, \ \{n\} \times \{0\} \times I).$$

We may then view $\mathcal{B}$ as a category by defining

$$Ob\mathcal{B} = \mathbb{N} \qquad \text{and} \qquad \mathcal{B}(n, m) = \begin{cases} \mathcal{B}_n & \text{if} \quad n = m, \\ \emptyset & \text{if} \quad n \neq m, \end{cases}$$

where the composition of two morphisms is given by the product $\cdot$ of the corresponding braid classes. From this, we may consider the tensor product $\otimes$ on $\mathcal{B}$ defined by

$$\otimes_{Ob} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \qquad \text{with} \quad \otimes_{Ob} (m, n) = m \otimes n := m + n.$$

$$\text{and} \quad \otimes_{Mor} : \mathcal{B}_m \times \mathcal{B}_n \longrightarrow \mathcal{B}_{m+n} \qquad \text{with} \quad \otimes_{Mor} (f, g) = f \otimes g,$$

where $f \otimes g$ is the braid class obtained by juxtaposing a braid in g on the right of a braid in f. The unit of $\otimes$ is $0 \in \mathbb{N} = \mathcal{B}$, and the category $(\mathcal{B}, \otimes, 0)$ is clearly monoidal but not symmetric.

The most fundamental generalized algebraic object to be defined within the context of a monoidal category is certainly the notion of monoid.

Definition 2.1.4. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. A *monoid* in $\mathcal{C}$ is an object $M \in Ob\mathcal{C}$ together with a *multiplication morphism* $\mu : M \otimes M \to M$ and a *unit morphism* $\eta : 1 \to M$ which make the two following diagrams commute:

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[r, "\mu \otimes id_M"] \arrow[d, "\alpha_{M,M,M}"'] & M \otimes M \arrow[r, "\mu"] & M \arrow[d, equal] \\
M \otimes (M \otimes M) \arrow[r, "id_M \otimes \mu"] & M \otimes M \arrow[r, "\mu"] & M
\end{tikzcd} \qquad \text{(associative law)}$$

$$\begin{tikzcd}
1 \otimes M \arrow[r, "\eta \otimes id_M"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "\mu"] & M \otimes 1 \arrow[l, "id_M \otimes \eta"'] \arrow[dl, "\rho_M"] \\
& M &
\end{tikzcd} \qquad \text{(unit law)}$$

If in addition we have $\mu \circ sym = \mu$, where $sym : M \otimes M \to M \otimes M$ is the map that permutes components, we say that M is *commutative*.

We can turn the class $\mathcal{M}on_{\mathcal{C}}$ of all monoids in $\mathcal{C}$ into a category by defining a morphism of monoids $f : (M, \mu, \eta) \to (M', \mu', \eta')$ to be a morphism $f : M \to M'$ in $\mathcal{C}$ such that both diagrams

$$\begin{tikzcd}
M \otimes M \arrow[r, "f \otimes f"] \arrow[d, "\mu"'] & M' \otimes M' \arrow[d, "\mu'"'] \\
M \arrow[r, "f"'] & M'
\end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd}
1 \arrow[r, equal] \arrow[d, "\eta"] & 1 \arrow[d, "\eta'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}$$

commute. An object of $\mathcal{M}on_{\mathcal{C}}$ is sometimes called $\mathcal{C}$*-monoid*.

Example 2.1.5. A monoid, in the traditional sense of the term, is obviously a monoid in the monoidal category $(\mathcal{S}et, \times, \{*\})$. With the more general definition given above, the monoid structure of a monoid $M = (M, \mu, \eta)$ is given by the multiplication

$$\mu(x, y) = xy \qquad \text{for every} \quad x, y \in M$$

with unit $\eta(\{*\}) \in M$. The associative and unit laws then become

$$(xy)z = x(yz) \qquad \text{and} \qquad \eta(\{*\})x = x = x\eta(\{*\})$$

respectively, for all $x, y, z \in M$. In particular, this is true for groups, rings and fields.

The example above shows how a (generalized) monoid extends the notion of a traditional ring. Following this line of thought, we may also extend the notion of a module over a given ring.

Definition 2.1.6. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category and (R, μ, η) a $\mathcal{C}$-monoid. A *left R-module* in $\mathcal{C}$ is an object $N \in Ob\mathcal{C}$ together with a morphism $l : R \otimes N \to N$ which makes the diagram

$$\begin{tikzcd}
(R \otimes R) \otimes N \arrow[r, "\mu \otimes id_N"] \arrow[d, "\alpha_{R,R,N}"'] & R \otimes N \arrow[r, "l"] & N \arrow[d, equal] \\
R \otimes (R \otimes N) \arrow[r, "id_R \otimes l"] & R \otimes N \arrow[r, "l"] & N \\
& 1 \otimes N \arrow[u, "\eta \otimes id_N"] \arrow[ur, "\lambda_N"'] &
\end{tikzcd}$$

commute; if that is the case we say that l is a *left R-action* on N. The class ${}_R\mathcal{M}od$ of left R-modules in $\mathcal{C}$ forms a category by defining a morphism $f : (N, l) \to (N', l')$ of left R-modules to be a morphism $f : N \to N'$ in $\mathcal{C}$ which fits into the commutative diagram

$$\begin{tikzcd}
R \otimes N \arrow[r, "id_R \otimes f"] \arrow[d, "l"'] & R \otimes N' \arrow[d, "l'"] \\
N \arrow[r, "f"'] & N';
\end{tikzcd}$$

in which case we say that the left R-actions l and l' are *R-equivariant.*

Dually, a *right R-module* in $\mathcal{C}$ is an object $M \in Ob\mathcal{C}$ together with a morphism $r : M \otimes R \to M$ which makes the diagram

$$\begin{tikzcd}
(M \otimes R) \otimes R \arrow[r, "r \otimes id_R"] \arrow[d, "\alpha_{M,R,R}"'] & M \otimes R \arrow[r, "r"] & M \arrow[d, equal] \\
M \otimes (R \otimes R) \arrow[r, "id_M \otimes \mu"] & M \otimes R \arrow[r, "r"] & M \\
& M \otimes 1 \arrow[u, "id_M \otimes \eta"] \arrow[ur, "\rho_M"'] &
\end{tikzcd}$$

commute; if that is the case we say that r is a *right R-action* on N. Again, the class $\mathcal{M}od_R$ of right R-modules in $\mathcal{C}$ forms a category by defining a morphism $f : (M, r) \to (M', r')$ of right R-modules to be a morphism $f : M \to M'$ in $\mathcal{C}$ which fits into the commutative diagram

$$\begin{tikzcd}
M \otimes R \arrow[r, "f \otimes id_R"] \arrow[d, "r"'] & M' \otimes R \arrow[d, "r'"] \\
M \arrow[r, "f"'] & M';
\end{tikzcd}$$

in which case we say that the right R-actions r and r' are *R-equivariant.*

For another monoid S in $\mathcal{C}$ we denote by ${}_R\mathcal{M}od_S$ the category of (R, S)-bimodules consisting of all objects of $\mathcal{C}$ that are both left R-modules and right S-modules with the obvious appropriate morphisms. In case the monoid R is commutative, we have an isomorphism of categories

$${}_R\mathcal{M}od \;\cong\; \mathcal{M}od_R \;\cong\; {}_R\mathcal{M}od_R;$$

we then speak of *the category of R-modules* (in $\mathcal{C}$) that we denote by any of the three forms above.

Example 2.1.7. As the above definition generalizes the traditional notion of R-module for a (commutative) ring R, such an R-module is simply an R-module in the monoidal category $(\mathcal{A}b, \times, 0)$ where R is seen as a (commutative) monoid in $\mathcal{A}b$.

From this, a natural generalization of the tensor product and the hom functor over a given ring, as encountered in homological algebra, follows.

Construction 2.1.8. Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category and (R, μ, η) a monoid in $\mathcal{C}$. Similarly to the definition of the traditional tensor product, we may define another tensor product, denoted $\otimes_R$, of a right R-module and a left R-module

$$- \otimes_R - : \mathcal{M}od_R \times_R \mathcal{M}od \to \mathcal{C}$$

using a universal property. This is done via the notion of coequalizer.

Given a pair $f, g : A \to B$ of morphisms in $\mathcal{C}$ having the same domain and codomain, we call *coequalizer* of (f, g) a morphism $u : B \to E$ such that

- uf = ug,
- for any morphism $h : B \to C$ such that $hf = hg$, there exists in $\mathcal{C}$ a unique morphism $h' : E \to C$ such that $h = h'u$.

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \arrow[r, "u"] \arrow[dr, "\forall h"'] & E \arrow[d, dashed, "\exists ! h'"] \\ & & C \end{tikzcd}$$

In this case we also say that E is the *coequalizer* of f and g. We say that $\mathcal{C}$ *has coequalizers* if a coequalizer exits for each pair $f, g : A \to B$ in $\mathcal{C}$.

Dually, given a pair $f, g : A \to B$ of morphisms in $\mathcal{C}$ having the same domain and codomain, we call *equalizer* of (f, g) a morphism $u : E \to A$ such that

- fu = gu,
- for any morphism $h : C \to A$ such that $fh = gh$, there exists in $\mathcal{C}$ a unique morphism $h' : C \to E$ such that $uh' = h$.

$$\begin{tikzcd} C \arrow[d, dashed, "\exists ! h'"'] \arrow[dr, "\forall h"] & & \\ E \arrow[r, "u"'] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$$

In this case we also say that E is the *equalizer* of f and g. We say that $\mathcal{C}$ *has equalizers* if an equalizer exits for each pair $f, g : A \to B$ in $\mathcal{C}$.

Now if $\mathcal{C}$ has coequalizers, we define $M \otimes_R N$ to be the coequalizer in $\mathcal{C}$ of the two maps $M \otimes R \otimes N \to M \otimes N$ induced by the right action of R on M and the left action of R on N. This new tensor product $\otimes_R$ has R as unit. In case the monoid R is commutative, the categories ${}_R\mathcal{M}od$ and $\mathcal{M}od_R$ are equivalent, so that *tensoring over* R, ie. using $\otimes_R$, turns $\mathcal{M}od_R$ into a symmetric monoidal category with unit R.

If in addition $\mathcal{C}$ has equalizers, there is also a Hom object of R-modules, denoted $[M,N]_R$, which is defined to be the equalizer of the two morphisms $[M,N] \to [R \otimes M, N]$, the first of them being induced by the action of R on M and the second as the composition

$$[M,N] \xrightarrow{R\otimes -} [R \otimes M, R \otimes N] \xrightarrow{\alpha} [R \otimes M, N],$$

with α induced by the action of R on N. From now on, we shall always assume that the main monoidal category in which we are working has equalizers and coequalizers.

We can also provide an R-module with some additional structure to obtain an R-algebra.

Definition 2.1.9. If $(\mathcal{C}, \otimes, 1)$ is a monoidal category and R a commutative monoid in $\mathcal{C}$, an *R-algebra* $A \in Ob\mathcal{C}$ is defined to be a monoid in the category $\mathcal{M}od_R$ of R-modules. This is equivalent to giving the object A a monoid structure (A, μ_A, η_A) together with an R-action given by a morphism $f : R \to A$ which makes the following diagram commute:

$$\begin{tikzcd}
R \otimes A \arrow[r, "f\otimes id_A"] \arrow[d, "sym"'] & A \otimes A \arrow[r, "\mu_A"] & A \arrow[d, equal] \\
A \otimes R \arrow[r, "id_A \otimes f"] & A \otimes A \arrow[r, "\mu_A"] & A
\end{tikzcd}$$

From 2.1.4 and 2.1.6, we obviously have a category $\mathcal{M}on_{\mathcal{M}od_R}$ of R-algebras that we more concisely denote by $\mathcal{A}lg_R$.

Example 2.1.10. This new notion obviously generalizes the more traditional notion of a k-algebra for a commutative ring k: The ring k and a k-algebra A are clearly monoids in the category $(\mathcal{S}et, \times, \{*\})$ (cf. 2.1.5), and the action of k on A is given by the scalar multiplication of the ring k on the k-algebra A.

Another important notion of category theory we are going to be using is the notion of monad. It consists of an endofunctor which pocesses the structure of a monoid in the category of endofunctors of a given category. It leads to the notion of T-algebra which we are going to use within the context of cofibrantly generated model categories.

Definition 2.1.11. Let $\mathcal{C}$ be a category. An endofunctor $T : \mathcal{C} \to \mathcal{C}$ has composites

$$T^2 = T \circ T : \mathcal{C} \to \mathcal{C} \qquad \text{and} \qquad T^3 = T \circ T^2 : \mathcal{C} \to \mathcal{C}.$$

For a natural transformation $\mu : T^2 \to T$ with components $\mu_x : T^2x \to Tx$ for every $x \in \mathcal{C}$, we denote by $T\mu : T^3 \to T^2$ the natural transformation with components $(T\mu)_x := T(\mu_x)$, and by $\mu T : T^3 \to T^2$ the natural transformation with components $(\mu T)_x := \mu_{Tx}$.

A *monad* or a *triple* $T = (T, \mu, \eta)$ in $\mathcal{C}$ consists in an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with two natural transformations

$$\mu : T^2 \to T \qquad \text{and} \qquad \eta : Id_{\mathcal{C}} \to T$$

which make the two diagrams below commute:

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
IdT \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & TId \arrow[l, "T\eta"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}$$

We refer to the first diagram as the *associative law* and to the second as the *unit law.* we call μ the *multiplication* and η the *unit* of the monad T. This is equivalent to saying that the monad T is a monoid in the category of endofunctors of $\mathcal{C}$ with the product being the composition of endofunctors and the unit the identity functor $Id_{\mathcal{C}}$.

Remark 2.1.12. Every adjunction

$$(F, G, \eta, \varepsilon) \ = \ (F : \mathcal{C} \Longleftrightarrow \mathcal{D} : G),$$

where η is the unit and ε the counit, gives rise to a monad $(GF, G\varepsilon F, \eta)$ in $\mathcal{C}$ (cf. [17] section VI.1).

Definition 2.1.13. Let $T = (T, \mu, \eta)$ be a monad in a category $\mathcal{C}$. A T-*algebra* is a pair (x, h) consisting of an object x in $\mathcal{C}$, the *underlying object* of the algebra, and a morphism $h : Tx \to x$ in $\mathcal{C}$, the *structure map* of the algebra, which make both diagrams below commute:

$$\begin{tikzcd} T^2x \arrow[r, "Th"] \arrow[d, "\mu x"'] & Tx \arrow[d, "h"] \\ Tx \arrow[r, "h"] & x \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\eta x"] \arrow[dr, "id_x"'] & Tx \arrow[d, "h"] \\ & x \end{tikzcd}$$

We refer to the first diagram as the *associative law* and to the second as the *unit law.* We obtain the category $\mathcal{C}^T$ of all T-algebras in $\mathcal{C}$ by defining a morphism of T-algebras $f : (x, h) \to (x', h')$ to be a morphism $f : x \to x'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} Tx \arrow[r, "h"] \arrow[d, "Tf"'] & x \arrow[d, "f"] \\ Tx' \arrow[r, "h'"] & x' \end{tikzcd}$$

commutes.

The following result provides an adjoint pair of canonical functors between a category $\mathcal{C}$ and its corresponding category $\mathcal{C}^T$ for any monad T in $\mathcal{C}$.

Proposition 2.1.14. *Let $\mathcal{C}$ be a category and (T, μ, η) a monad in $\mathcal{C}$. We have an adjoint pair*

$$(F, U, \delta, \varepsilon) \ = \ (F : \mathcal{C} \Longleftrightarrow \mathcal{C}^T : U),$$

where U is the forgetful functor and F the free functor defined respectively by the commutative squares

$$\begin{tikzcd} (x,h) \arrow[r, mapsto, "U"] \arrow[d, "f"'] & x \arrow[d, "f"] \\ (x',h') \arrow[r, mapsto, "U"] & x' \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} x \arrow[r, mapsto, "F"] \arrow[d, "f"'] & (Tx, \mu_x) \arrow[d, "F(f)"] \\ x' \arrow[r, mapsto, "F"] & (Tx', \mu_{x'}), \end{tikzcd}$$

with $\delta = \eta$ and $\varepsilon(x, h) = h$ for each T-algebra (x, h). In addition, the monad defined in $\mathcal{C}$ by the adjunction (cf. 2.1.12) is the given monad (T, μ, η).

Proof. It is clear that U is a functor and that the pair $(Tx, \mu_x : T(Tx) \to Tx)$ is a T-algebra, by the associative and unit laws of the monad T, so that F is indeed a functor. We then have

$$UFx = U(Tx, \mu_x) = Tx, \qquad \text{so that} \quad UF = T.$$

The unit η of the given monad T is therefore the natural transformation $\eta = \delta : Id_{\mathcal{C}} \to UF = T$. On the other hand we have

$$FU(x,h) = F(x) = (Tx, \mu_x),$$

while the associative law of a T-algebra (x,h) states that the structure map $h : Tx \to x$ is in fact a morphism $(Tx, \mu_x) \to (x,h)$ in $\mathcal{C}^T$. The resulting transformation

$$\varepsilon_{(x,h)} = h : (Tx, \mu_x) = FU(x,h) \to (x,h)$$

is natural by definition of a morphism of T-algebras. From the unit law for the monad T and the unit law for a T-algebra, we respectively obtain both commutative diagrams

$$\begin{tikzcd} Tx \arrow[r, "T\eta_x"] \arrow[dr, equal] & TTx \arrow[d, "\mu_x"] \\ & Tx \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, equal] & Tx \arrow[d, "h"] \\ & x, \end{tikzcd}$$

so that by 1.1.7, δ and ε define an adjunction. This adjunction determines a monad in $\mathcal{C}$: its endofunctor UF is the original T, its unit δ the original unit η, and since its multiplication $\mu' = U\varepsilon F$ is such that

$$\mu'_x = U\varepsilon(Tx, \mu_x) = U\mu_x = \mu_x,$$

it follows that $\mu' = \mu$ is the original multiplication of T. □

2.2 Cofibrantly generated model categories

We are now going to treat a particular kind of model category, namely cofibrantly generated categories. As its name suggests, a cofibrantly generated model category is a category whose model structure is completely determined by a set of cofibrations and a set of acyclic cofibrations. From Quillen's small object argument, that we shall establish in details, a cofibrantly generated model category has the convenience that the factorizations of axiom (M_5) become functorial. Another advantage of such a model category is that a property which holds for its sets of generating cofibrations and generating acyclic cofibrations usually holds for every morphisms in that category. Most of model categories found in the literature, such as topological spaces and simplicial sets (cf. section 3.1), are cofibrantly generated. The difficult part of the definition of a cofibrantly generated model category is to formulate the notion of relative smallness. For this we need to consider the following set-theoretic concepts.

Definition 2.2.1. An *ordinal* γ is an ordered isomorphism class of well ordered sets; it can be identified with the well ordered set of all preceding ordinals. We use the same symbol to denote its associated category

$$\gamma = \{0 = \emptyset \to 1 \to 2 \to \ldots \to \alpha \to \ldots\} \qquad \text{with} \quad \alpha < \gamma,$$

where $\emptyset$ denotes its initial ordinal. A *limit ordinal* is an ordinal which is not a successor of another ordinal. Furthermore, an ordinal κ is a *cardinal* if its cardinality, ie. the smallest ordinal, denoted $|\kappa|$, which is in bijection with κ, is strictly larger than that of any preceding ordinal. For a cardinal κ and a limit ordinal β, we say that β is *κ-filtered* if

$$A \subseteq \beta \qquad \text{and} \qquad |A| \leq \kappa \qquad \Longrightarrow \qquad SupA < \beta.$$

As an example, for any $\kappa < \infty$ the limit ordinal $\omega = \aleph_0 = |\mathbb{N}|$ is always κ-filtered. It is worth noting that if λ is a limit ordinal, then $colim_{\alpha<\lambda} = \lambda$.

From this, we may define relative smallness, which involves the notion of transfinite composition of a γ-sequence for an ordinal γ.

Definition 2.2.2. Let $\mathcal{C}$ be a cocomplete category and γ an ordinal. A *γ-sequence* in $\mathcal{C}$ is a functor $X : \gamma \to \mathcal{C}$, ie. a diagram

$$X(\emptyset) \to X(1) \to X(2) \to \ldots \to X(\alpha) \to \ldots \qquad \text{in } \mathcal{C} \text{ with } \quad \alpha < \gamma,$$

which preserves colimits; in other words it is a functor X such that for every limit ordinal $\beta < \gamma$ the induced morphism $colim_{\alpha<\beta} X(\alpha) \to X(\beta) = X(colim_{\alpha<\beta})$ is an isomorphism. The composed morphism

$$X(\emptyset) \to colim_{\alpha<\gamma} X(\alpha)$$

is called the *transfinite composition* of the maps of the γ-sequence.

Moreover, a morphism class $I \subseteq Mor\mathcal{C}$ is said to be *closed under transfinite composition* if for every ordinal γ and every γ-sequence $X : \gamma \to \mathcal{C}$ with maps $X(\alpha) \to X(\alpha+1)$ in I for all $\alpha < \gamma$, the induced morphism $X(\emptyset) \to colim_\gamma X$ is also in I. In this case, and for a cardinal κ, an object A in $\mathcal{C}$ is said to be *κ-small relative to I* if for every κ-filtered ordinal γ and every γ-sequence $X : \gamma \to I$, the induced set map

$$colim_{\alpha<\gamma} \mathcal{C}(A, X(\alpha)) \longrightarrow \mathcal{C}(A, colim_\gamma X)$$

is a bijection, ie. an isomorphism in $\mathcal{S}et$ (we can effectively speak of sets here since $\mathcal{C}$ is cocomplete (cf. 1.1.11)). Finally, an object B in $\mathcal{C}$ is said to be *small relative to I* if there exists a cardinal κ such that B is κ-small relative to I. In the special case where $I = Mor\mathcal{C}$, we simply say that B is *small* in $\mathcal{C}$.

Example 2.2.3. Any set is small relative to the category $\mathcal{S}et$; in particular, a set $A \in Ob\mathcal{S}et$ is $|A|$-small relative to $Mor\mathcal{S}et$.

To see this, let γ be an $|A|$-filtered ordinal, $X : \gamma \to \mathcal{S}et$ a γ-sequence in $\mathcal{S}et$, and denote $j_\alpha : X_\alpha \to X_{\alpha+1}$ the morphisms composing that sequence. We need to check that the set map

$$\varphi_A : \ colim_{\alpha<\gamma} \mathcal{C}(A, X_\alpha) \longrightarrow \mathcal{C}(A, colim_{\alpha<\gamma} X_\alpha)$$

is bijective. Let $f : A \to colim_{\alpha<\gamma} X_\alpha$. We know that

$$colim_{\alpha<\gamma} X_\alpha \ = \ (\coprod_\alpha X_\alpha)/\sim,$$

where the equivalence relation $\sim$ is given by

$$x \sim j_\alpha(x) \qquad \text{for every } \alpha < \gamma \text{ and every } x \in X_\alpha.$$

Then, for every a in A there is an ordinal $\beta(a) < \gamma$ such that $f(a) \in colim_{\alpha<\gamma} X_\alpha$ is represented by an element in $X_{\beta(a)}$, since

$$\begin{cases} |\{\beta(a)\}_{a\in A}| \ \leq \ |A| \\ \gamma \text{ is } |A|\text{-filtered} \end{cases} \qquad \text{imply that} \qquad \sup_{a\in A} \beta(a) < \gamma.$$

Define $\beta := \sup_{a\in A} \beta(a) < \gamma$. Then for every a in A, $f(a)$ is represented by an element in X_β, so that there exists a map $f' : A \to X_\beta$ which fits into the commutative diagram

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[dr, "f"'] & X_\beta \arrow[d] \\ & colim_{\alpha<\gamma} X_\alpha. \end{tikzcd}$$

It follows that φ_A is surjective.

It remains to check the injectivity. Let g and g' be two elements of $colim_{\alpha<\gamma}\mathcal{C}(A, X_\alpha)$ with

$$\varphi_A(g) \;=\; \varphi_A(g') \;\in\; \mathcal{C}(A, colim_{\alpha<\gamma}X_\alpha).$$

By the same argument as above, there exists a $\beta < \gamma$ such that every $\varphi_A(g)(a) = \varphi_A(g')(a)$ is represented by the same element of X_β. This leads to the existence of a map $g'' : A \to X_\beta$ which fits into the commutative diagram

$$\begin{tikzcd} A \arrow[r, "g''"] \arrow[dr, "\varphi_A(g)"'] & X_\beta \arrow[d] & A \arrow[l, "g''"'] \arrow[dl, "\varphi_A(g')"] \\ & colim_{\alpha<\gamma}X_\alpha & \end{tikzcd},$$

so that $g = g''$ in $colim_{\alpha<\gamma}\mathcal{C}(A, X_\alpha)$.

The next step is to define, related to any set I of morphisms of a cocomplete category $\mathcal{C}$, three classes of morphisms $I{-}inj$, $I{-}cof$ and $I{-}cell$ in $\mathcal{C}$. From this, we shall establish the transfinite version of Quillen's small object argument.

Definition 2.2.4. Let $\mathcal{C}$ be a cocomplete category and $I \subseteq Mor\mathcal{C}$ a class of morphisms in $\mathcal{C}$. We define:

- $I{-}inj := RLP(I)$ to be the class of *I-injectives*,
- $I{-}cof := LLP(I{-}inj)$ to be the class of *I-cofibrations*,
- $I - cell$ to be the class of the (possibly transfinite) compositions of cobase changes of morphisms in I; in other words it is the class of morphisms $f : A \to B$ in $\mathcal{C}$ for which there exist an ordinal γ and a γ-sequence $X : \gamma \to \mathcal{C}$ such that $X(\emptyset) = A$, each $X(\alpha) \to X(\alpha{+}1)$ is a cobase change of a morphism in I, and the composition $X(\emptyset) \to colim_{\alpha<\gamma}X(\alpha)$ is isomorphic to f. The morphisms in $I{-}cell$ are called *regular I-cofibrations*.

Remarks 2.2.5. (1) We have

$$I \;\subseteq\; I{-}cell \;\subseteq\; I{-}cof.$$

Indeed, it is clear that $I \subseteq I{-}cof$ and $I \subseteq I{-}cell$; and to verify the inclusion $I{-}cell \subseteq I{-}cof$, let $X : \gamma \to \mathcal{C}$ be a γ-sequence composed of morphisms $f_\alpha : X_\alpha \to X_{\alpha+1}$ such that for every $\alpha < \gamma$ we have a pushout square

$$\begin{tikzcd} W_\alpha \arrow[r] \arrow[d, "g_\alpha"'] & X_\alpha \arrow[d, "f_\alpha"] \\ Z_\alpha \arrow[r] & X_{\alpha+1}, \end{tikzcd}$$

with $g_\alpha \in I$. Then $g_\alpha \in I{-}cof = LLP(I{-}inj)$ for every $\alpha < \gamma$. Applying the same argument used to prove 1.2.11.(4), we obtain that f_α is in $I{-}cof$ for every $\alpha < \gamma$. This means that every commutative square

$$\begin{tikzcd} X_\alpha \arrow[r, "i_\alpha"] \arrow[d, "f_\alpha"'] & W \arrow[d, "g"] \\ X_{\alpha+1} \arrow[r, "j_\alpha"] & Z, \end{tikzcd}$$

with $g \in I{-}inj$, has a lift $i_{\alpha+1} : X_{\alpha+1} \to W$. This lift fits into the commutative square

$$\begin{tikzcd} X_{\alpha+1} \arrow[r, "i_{\alpha+1}"] \arrow[d, "f_{\alpha+1}"'] & W \arrow[d, "g"] \\ X_{\alpha+2} \arrow[r, "j_{\alpha+1}"] & Z, \end{tikzcd}$$

with $f_{\alpha+1} \in I{-}cof = LLP(I{-}inj)$. Since $g \in I{-}inj$, the above diagram has a lift $i_{\alpha+2} : X_{\alpha+2} \to W$ which fits into the commutative diagram

$$\begin{tikzcd} X_\alpha \arrow[r, "i_\alpha"] \arrow[d, "f_\alpha"'] & W \arrow[dd, "g"] \\ X_{\alpha+1} \arrow[d, "f_{\alpha+1}"'] & \\ X_{\alpha+2} \arrow[uur, "i_{\alpha+2}"] \arrow[r, "j_{\alpha+1}"'] & Z, \end{tikzcd}$$

so that $f_{\alpha+1} f_\alpha$ is in $I{-}cof$ as well. By induction, the transfinite composition $X_0 \to colim_{\alpha<\gamma} X_\alpha$ is in $I{-}cof$ and the desired inclusion is verified. In particular, $I{-}cof$ is closed under transfinite composition.

(2) Since by 1.2.11 we have

$$Cof \;=\; LLP(Fib \cap WE) \qquad \text{and} \qquad Cof \cap WE \;=\; LLP(Fib),$$

the argument used in (1) imply that these two morphism classes are closed under transfinite composition as well.

Proposition 2.2.6 (the small object argument)**.** *Let $\mathcal{C}$ be a cocomplete category and I a set of morphisms in $\mathcal{C}$ whose domains are small relative to $I{-}cell$.*

(1) *Any morphism f in $\mathcal{C}$ has a functorial factorization $f = qj$ with $q \in I{-}inj$ and $j \in I{-}cell$; in other words there is a functor*

$$(q, j) \;:\; \mathcal{C}^{\to} \to \mathcal{C}^{\to} \times \mathcal{C}^{\to} \;:\; f \mapsto (q, j)(f) = (q(f), j(f)), \qquad \textit{(cf. 1.1.2)}$$

such that $f = q(f) \circ j(f)$ for any $f \in Ob\mathcal{C}^{\to}$.

(2) *Every I-cofibration is a retract of a regular I-cofibration.*

Proof. (1) Let $f : A \to B$ be a morphism in $\mathcal{C}$. Choose for each morphism i in I a cardinal κ_i such that the domain of i is κ_i-small with respect to $I{-}cell$. Let κ be the smallest cardinal which is strictly greater than every $\{\kappa_i\}_{i\in I}$, and let γ be a κ-filtered ordinal. We are going to define a γ-sequence $X : \gamma \to \mathcal{C}$ such that $X_\alpha \to X_{\alpha+1}$ is in $I{-}cell$ for every $\alpha < \gamma$, so that the transfinite composition $X_0 \to colim_\gamma X$ is in $I{-}cell$ as well by definition.

Let $X_0 = A$ and S_0 be the class of all pairs of morphisms $(g : W \to X_0 = A,\ h : Z \to B)$ in $\mathcal{C}$ which make the square

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "i"'] & A = X_0 \arrow[d, "f"] \\ Z \arrow[r, "h"] & B \end{tikzcd}$$

commute for a morphism $i : W \to Z$ in I. We then define $X_1 \in Ob\mathcal{C}$ to be the pushout, and q_1 the induced morphism, of

$$\begin{tikzcd}
\coprod_{S_0} W \arrow[r, "\coprod_{S_0} g"] \arrow[d, "\coprod_{S_0} i"'] & A = X_0 \arrow[d, "f_0"] \arrow[ddr, bend left, "f=q_0"] & \\
\coprod_{S_0} Z \arrow[r] \arrow[drr, bend right, "\coprod_{S_0} h"'] & X_1 \arrow[dr, dashed, "q_1"] & \\
 & & B,
\end{tikzcd}$$

with $f = q_0 \in I{-}cell$ and $\coprod_{S_0} i \in I{-}cell$. Now suppose given

$$\begin{tikzcd}
 & B & & \\
X_0 \arrow[ur, "q_0"] & X_1 \arrow[u, "q_1"'] & \cdots & X_\alpha . \arrow[ull, "q_\alpha"']
\end{tikzcd}$$

We define S_α to be the class of all pairs $(g : W \to X_\alpha,\ h : Z \to B)$ of morphisms in $\mathcal{C}$ which make the square

$$\begin{tikzcd}
W \arrow[r, "g"] \arrow[d, "i"'] & X_\alpha \arrow[d, "q_\alpha"] \\
Z \arrow[r, "h"] & B
\end{tikzcd}$$

commute for a morphism $i : W \to Z$ in I, and we define $X_{\alpha+1}$ to be the pushout, with induced morphism $q_{\alpha+1}$, of

$$\begin{tikzcd}
\coprod_{S_\alpha} W \arrow[r, "\coprod_{S_\alpha} g"] \arrow[d, "\coprod_{S_\alpha} i"'] & X_\alpha \arrow[d, "f_\alpha"] \arrow[ddr, bend left, "q_\alpha"] & \\
\coprod_{S_\alpha} Z \arrow[r] \arrow[drr, bend right, "\coprod_{S_\alpha} h"'] & X_{\alpha+1} \arrow[dr, dashed, "q_{\alpha+1}"] & \\
 & & B,
\end{tikzcd}$$

with $f_\alpha \in I{-}cell$ and $\coprod_{S_\alpha} i \in I{-}cell$. For a limit ordinal β, we let $X_\beta := colim_{\alpha<\beta} X_\alpha$ and $q_\beta : X_\beta \to B$ be the morphisms induced by all q_α's such that $\alpha < \beta$. We define $j(f)$ to be the transfinite composition $A = X_0 \to colim_{\alpha<\gamma} X_\alpha$ and $q(f) : colim_{\alpha<\gamma} X_\alpha \to B$ to be the morphism induced by the q_α's. We then have $f = q(f) \circ j(f)$ since the triangle

$$\begin{tikzcd}
A = X_0 \arrow[rr, "j(f)"] \arrow[dr, "f=q_0"'] & & colim_{\alpha<\gamma} X_\alpha \arrow[dl, "q(f)"] \\
 & B &
\end{tikzcd}$$

commutes by construction. In addition, since $X_\alpha \to X_{\alpha+1}$ is in $I{-}cell$ for every $\alpha < \gamma$, we know that $j(f)$ also is. It remains to show that $q(f)$ is I-injective.

In order to do this, let's consider the commutative diagram

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "i"'] & colim_{\alpha<\gamma} X_\alpha \arrow[d] \\
Z \arrow[r] & B
\end{tikzcd}$$

with $i \in I$. Since γ is κ-filtered and W is κ_i-small with respect to $I-cell$, we have a bijection

$$colim_{\alpha<\gamma}\mathcal{C}(W, X_\alpha) \xrightarrow{\cong} \mathcal{C}(W, colim_{\alpha<\gamma} X_\alpha),$$

so that there is a $\beta < \gamma$ with a factorization

$$\begin{tikzcd}
W \arrow[rr, "g'"] \arrow[dr, "g"'] & & X_\beta \arrow[dl] \\
& colim_{\alpha<\gamma} X_\alpha &
\end{tikzcd} .$$

Finally, we obtain by construction the commutative diagram

$$\begin{tikzcd}
W \arrow[rr, "g'"] \arrow[dd, "i"'] & & X_\beta \arrow[dl] \arrow[d] \arrow[dd, bend left, "q_\beta"] \\
& X_{\beta+1} \arrow[r] \arrow[dr] & colim_{\alpha<\gamma} X_\alpha \arrow[d] \\
Z \arrow[ur] \arrow[rr, "h"'] & & B
\end{tikzcd} ,$$

which induces the commutative diagram

$$\begin{tikzcd}
W \arrow[rr, "g"] \arrow[dd, "i"'] & & colim_{\alpha<\gamma} X_\alpha \arrow[dd, "q(f)"] \\
& X_{\beta+1} \arrow[ur] & \\
Z \arrow[ur] \arrow[rr, "h"'] & & B
\end{tikzcd} ,$$

so that $q(f) \in I-inj$ as desired.

(2) Let f be a morphism in $I-cof$. From what we just proved, we have a factorization

$$\begin{tikzcd}
\bullet \arrow[r, "j(f)"] \arrow[dr, "f"'] & \bullet \arrow[d, "q(f)"] \\
& \bullet
\end{tikzcd}$$

with $j(f) \in I{-}cell$ and $q(f) \in I{-}inj$. Since $I{-}cof = LLP(I{-}inj)$ by definition, f has the left lifting property with respect to $q(f)$, so that we have a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r, "j(f)"] \arrow[d, "f"'] & \bullet \arrow[d, "q(f)"] \\ \bullet \arrow[r, "id"] & \bullet \end{tikzcd}$$

which has a lift s with $q(f)s = id$ and $sf = j(f)$. It follows that we have a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r, "id"] \arrow[d, "f"'] & \bullet \arrow[r, "id"] \arrow[d, "j(f)"'] & \bullet \arrow[d, "f"'] \\ \bullet \arrow[r, "s"'] \arrow[rr, bend right, "id"'] & \bullet \arrow[r, "q(f)"'] & \bullet, \end{tikzcd}$$

and consequently f is a retract of $j(f) \in I{-}cell$ as desired. $\square$

The idea of a cofibrantly generated model category is to have two sets I and J of cofibrations and acyclic cofibrations which contain all the necessary information to characterize the model structure of the category via their related injectives, cofibrations and regular cofibrations. Note that in order to be able to apply the small object argument to such a model category, we require I and J to be sets, and not just classes of maps as in definition 2.2.4.

Definition 2.2.7. A model category $(\mathcal{C}, WE, Fib, Cof)$ is said to be *cofibrantly generated* if it is complete and cocomplete (ie. bicomplete), and if there exists a set of cofibrations I and a set of acyclic cofibrations J such that

- $Fib = J{-}inj$,
- $Fib \cap WE = I{-}inj$,
- the domain of each morphism in I (resp. J) is small relative to $I{-}cell$ (resp. $J{-}cell$).

The morphisms in I are the *generating cofibrations* and the ones in J *the generating acyclic cofibrations*.

Remarks 2.2.8. **(1)** By 1.2.11, it is clear that $Cof = I{-}cof$ and $Cof \cap WE = J{-}cof$.
(2) According to the small object argument, any morphism f of a cofibrantly generated category may be functorially factored as $f = pi$ with

$$p \in Fib, \quad i \in Cof \cap WE \qquad \text{and also} \qquad p \in Fib \cap WE, \quad i \in Cof.$$

Definition 2.2.9. Let $\mathcal{C}$ be a cofibrantly generated model category and T a monad in $\mathcal{C}$. We want to provide $\mathcal{C}^T$ with a model category structure. In order to do this we define a morphism in $\mathcal{C}^T$ to be

- a *weak equivalence* if the underlying morphism in $\mathcal{C}$ is a weak equivalence,
- a *fibration* if the underlying morphism in $\mathcal{C}$ is a fibration,
- a *cofibration* if it has the left lifting property with respect to the class of acyclic fibrations in $\mathcal{C}^T$.

We shall now establish under which conditions the free T-algebra functor $\mathcal{C} \to \mathcal{C}^T$ for a given monad T in a cofibrantly generated model category $\mathcal{C}$ preserves the model structure of $\mathcal{C}$ on $\mathcal{C}^T$, ie. when the model structure on $\mathcal{C}^T$ induced from the model structure of $\mathcal{C}$ via the free T-algebra functor coincide with the model structure given in 2.2.9.

Proposition 2.2.10. *Let $\mathcal{C}$ be a cofibrantly generated model category and T a monad in $\mathcal{C}$ such that its underlying functor commutes with direct limits. Let I be a set of generating cofibrations and J a set of generating acyclic cofibrations for $\mathcal{C}$. Denote by I^T and J^T the respective images of I and J under the free T-algebra functor $\mathcal{C} \to \mathcal{C}^T$, and assume that the domains of all morphisms in I^T and J^T are small relative to I^T-cell and J^T-cell respectively. If one of the two following conditions is satisfied:*

(1) *every morphism in J^T-cell is a weak equivalence,*

(2) *every object in $\mathcal{C}$ is fibrant and every T-algebra X in $\mathcal{C}$ has a good path object X^I in $\mathcal{C}^T$,*

then $\mathcal{C}^T$ is a cofibrantly generated model category with I^T as a set of generating cofibrations and J^T as a set of generating acyclic cofibrations for $\mathcal{C}^T$.

Proof. The fact that all limits and colimits existing in $\mathcal{C}$ also exist in $\mathcal{C}^T$ is a consequence of commutativity of the underlying functor T with direct limits, a fact about limits and colimits we shall take for granted for the sake of continuity, so that (M_1) is satisfied for $\mathcal{C}^T$. Axioms (M_2) and (M_3) are clearly satisfied by definition 2.2.9. The half of (M_4) which requires that cofibrations have the left lifting property with respect to acyclic fibrations is given by definition of a cofibration in $\mathcal{C}^T$ (cf. 2.2.9). Let's now show (M_5).

By 2.2.9, the adjunction

$$U : \mathcal{C}^T \Longleftrightarrow \mathcal{C} : F,$$

where U is the forgetful functor and F the free T-algebra functor, preserves fibrations and acyclic fibrations, so that all morphisms in I^T or J^T is a cofibration in $\mathcal{C}^T$. This implies that

$$(I^T-cof \ \cup \ J^T-cof) \ \subseteq \ Cof_{\mathcal{C}^T}. \tag{$*$}$$

Since I is a set of generating cofibrations in $\mathcal{C}$ and the adjunction preserves fibrations and acyclic fibrations, a morphism is in I^T-inj precisely when it is in $Fib_{\mathcal{C}} \cap WE_{\mathcal{C}}$, and consequently in $Fib_{\mathcal{C}^T} \cap WE_{\mathcal{C}^T}$ by definition, so that

$$I^T-inj \ = \ Fib_{\mathcal{C}^T} \cap WE_{\mathcal{C}^T}.$$

By assumption, we may now apply the small object argument to obtain a (functorial) factorization

$$\begin{aligned} f = pi \qquad & \text{with} \quad p \ \in \ I^T-inj \ = \ (Fib_{\mathcal{C}^T} \cap WE_{\mathcal{C}^T}) \\ & \text{and} \quad i \ \in \ I^T-cell \ \subseteq \ I^T-cof \ \subseteq \ Cof_{\mathcal{C}^T}, \end{aligned}$$

which establishes the cofibration-acyclic fibration part of axiom (M_5).

The acyclic cofibration-fibration part of (M_5) needs hypothesis (1) or (2). Applying the small object argument to J^T gives a functorial factorization of any morphism f in $\mathcal{C}^T$ as

$$f = pi \qquad \text{with} \quad i \in J^T-cell \quad \text{and} \quad p \in J^T-inj. \tag{$**$}$$

By preservation of fibrations and acyclic fibrations, and since J is a set of generating acyclic cofibrations in $\mathcal{C}$, from the same argument as above we obtain

$$J^T-inj \ = \ Fib_{\mathcal{C}^T}.$$

If we assume hypothesis (1); by $(*)$ we have

$$J^T\!-\!cell \;\subseteq\; (WE_{\mathcal{C}^T} \cap I^T\!-\!cof) \;\subseteq\; (WE_{\mathcal{C}^T} \cap Cof_{\mathcal{C}^T}),$$

so that $i \in WE_{\mathcal{C}^T} \cap Cof_{\mathcal{C}^T}$ and $p \in Fib_{\mathcal{C}^T}$ in factorization $(**)$. If we now assume hypothesis (2) and let the morphism $i : X \to Y$ of $(**)$ be in $J^T\!-\!cof$, what remains to show is that i is a weak equivalence in $\mathcal{C}^T$, or equivalently that i is a weak equivalence in $\mathcal{C}$. Since X is fibrant and $Fib_{\mathcal{C}^T} = J^T\!-\!inj$, we obtain a retraction r of i by lifting in the commutative square

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "i"'] & X \arrow[d, two heads] \\ Y \arrow[r] \arrow[ur, dashed, "r"'] & *. \end{tikzcd}$$

By hypothesis (2), Y possesses a good path object Y^I, and since $i \in LLP(I^T\!-\!inj = Fib_{\mathcal{C}^T})$, the commutative square

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "i"'] & Y \arrow[r] & Y^I \arrow[d] \\ Y \arrow[rr, "{(id_Y, ir)}"'] \arrow[urr, dashed, "H"] & & Y \times Y \end{tikzcd}$$

has a lift H, so that id_Y is right homotopic to $i \circ r$ via H. Therefore, $id_Y = i \circ r$ in the homotopy category $Ho(\mathcal{C})$. Finally, since a morphism in $\mathcal{C}$ is a weak equivalence if and only if it is sent by γ on an isomorphisms in $Ho(\mathcal{C})$ (cf. 1.4.11), this shows that i is a weak equivalence in $\mathcal{C}$, or equivalently in $\mathcal{C}^T$, so that axiom (M_5) is proven.

It remains to prove the other half of (M_4), ie. that any acyclic cofibration $f : A \overset{\sim}{\hookrightarrow} B$ has the left lifting property with respect to $Fib_{\mathcal{C}^T}$, or equivalently that

$$Fib_{\mathcal{C}^T} \cap WE_{\mathcal{C}^T} \;\subseteq\; J^T\!-\!cof \qquad \text{since} \qquad Fib_{\mathcal{C}^T} = J^T\!-\!inj.$$

The small object argument provides a factorization

$$A \xrightarrow[\sim]{i} W \xrightarrow{p} B, \qquad \text{of} \qquad f \qquad \text{with} \quad i \;\in\; J^T\!-\!cell \;\subseteq\; WE_{\mathcal{C}^T} \cap (I^T\!-\!cof)$$
$$\text{and} \quad p \;\in\; J^T\!-\!inj \;=\; Fib_{\mathcal{C}^T}.$$

In addition, p is a weak equivalence since f is. It follows that the commutative square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, hook, "f"', "\sim"] & W \arrow[d, two heads, "p"', "\sim"] \\ B \arrow[r, "id_B"'] & B \end{tikzcd}$$

has a lift, so that $f \in J^T\!-\!cof$ as desired. $\square$

2.3 Monoidal model categories

We are now going to combine the structure of a model category with the algebraic structure of a closed symmetric monoidal category. These two structures have to be compatible in a way expressed by the pushout product axiom and the unit axiom. The pushout product axiom guarantees that for cofibrant objects the tensor product is an invariant of the weak equivalences, so that it becomes a product in the homotopy category. The unit axiom is especially important when dealing with the homotopy category of a monoidal model category. It ensures that the monoidal structure on the model category induces a monoidal structure on the homotopy category.

In certain situations, a third axiom, namely the monoid axiom, may also be required. This third axiom is a crucial ingredient for lifting the model category structure to monoids, modules and algebras.

Definition 2.3.1. A model category $\mathcal{C}$ is a *monoidal model category* if it is a closed symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ which satisfies the following axioms:

- *(pushout product axiom):* If $f : A \to B$ and $g : X \to Y$ are cofibrations, then the induced morphism
$$A \otimes Y \bigvee_{A\otimes X} B \otimes X \longrightarrow B \otimes Y$$
is also a cofibration. If in addition f or g is a weak equivalence, then so is the induced morphism.

- *(unit axiom):* If $q : 1^c \xrightarrow{\sim} 1$ is a cofibrant replacement of the unit object, ie. 1^c is cofibrant and $q \in WE$, then for any cofibrant object A the morphism
$$q \otimes id_A : 1^c \otimes A \longrightarrow 1 \otimes A \cong A$$
is a weak equivalence.

Definition 2.3.2. If $(\mathcal{C}, \otimes, 1)$ is a monoidal model category and I a class of morphisms in $\mathcal{C}$, we denote by $I \otimes \mathcal{C}$ the class
$$I \otimes \mathcal{C} := \{A \otimes Z \to B \otimes Z \in Mor\mathcal{C} \mid (A \to B) \in I,\ Z \in Ob\mathcal{C}\}.$$
Then $\mathcal{C}$ satisfies the *monoid axiom* if
$$[(WE \cap Cof) \otimes \mathcal{C}]-cell \ \subseteq\ WE.$$

Remark 2.3.3. In the special case where every object in $\mathcal{C}$ is cofibrant, the monoid axiom is a direct consequence of the pushout product axiom: The fact that $\otimes$ preserves colimits in each of its variables implies that the initial object $\emptyset \in Ob\mathcal{C}$ acts like a zero for the tensor product in the sense that
$$A \otimes \emptyset \ \cong\ \emptyset \ \cong\ \emptyset \otimes B.$$
With this, the pushout product axiom says that for any acyclic cofibration $A \to B$ and for any object Z in $\mathcal{C}$ (which is cofibrant), the induced morphism $A \otimes Z \to B \otimes Z$ is also an acyclic cofibration, so that
$$[(WE \cap Cof) \otimes \mathcal{C}] \ \subseteq\ WE \cap Cof \ \subseteq\ WE.$$
Since the class $WE \cap Cof$ is closed under cobase change and transfinite composition (cf. 1.2.11.(4) and 2.2.5.(2)), we obtain
$$[(WE \cap Cof) \otimes \mathcal{C}]-cell \ \subseteq\ (WE \cap Cof)-cell \ =\ WE \cap Cof \ \subseteq\ WE,$$
so that the monoid axiom is verified.

In case a monoidal model category is cofibrantly generated, we saw that the fibrations can be detected by checking the right lifting property against a set of maps (the set of generating acyclic cofibrations) as opposed to the whole class of cofibrations. A similar idea holds for the pushout product and the monoid axioms as well.

Proposition 2.3.4. *Let $(\mathcal{C}, \otimes, 1)$ be a cofibrantly generated model category endowed with a closed symmetric monoidal structure.*

(1) *If the pushout product axiom holds for a set of generating cofibrations and a set of generating acyclic cofibrations, then it holds in general.*

(2) *If $(J \otimes \mathcal{C})-cell \subseteq WE$ for a set of generating acyclic cofibrations J, then the monoid axiom holds in general.*

Proof. (1) Let's consider a morphism $i : A \to B$ in $\mathcal{C}$, and denote by $G(i)$ the class of all morphisms $j : K \to L$ in $\mathcal{C}$ for which the pushout product

$$p : \ (A \otimes L) \vee_{A \otimes K} (B \otimes K) \longrightarrow B \otimes L$$

is a cofibration. From the adjunction between the tensor product functor $Z \otimes -$ and the Hom functor $[Z, -]$ for any object Z in $\mathcal{C}$, p has the left lifting property with respect to a morphism $f : X \to Y$ if and only if j has the left lifting property with respect to the morphism

$$q : \ [B, X] \longrightarrow [B, Y] \times_{[A,Y]} [A, X].$$

Therefore, a morphism is in $G(i)$ if and only if it has the left lifting property with respect to q for every morphism $f : X \to Y$ in $WE \cap Fib$. By 2.2.5.(1) and an argument as in 1.2.11.(4), this implies that $G(i)$ is closed under cobase change, transfinite composition and retracts. If $i : A \to B$ is a generating cofibration, $G(i)$ contains by assumption all generating cofibrations, so that

$$Cof \subseteq G(i)$$

by the closure properties and the small object argument. If again we use the same argument after reversing the roles of i and an arbitrary cofibration $j : K \to L$, we obtain

$$Cof \subseteq G(j),$$

so that the pushout product axiom is proven for two arbitrary cofibrations. The same argument can be used to show that the pushout product is an acyclic cofibration when one of its constituents is.

(2) By the small object argument, every morphism in $WE \cap Cof$ is a morphisms in $J-cell$. Thus every morphism in

$$[(WE \cap Cof) \otimes \mathcal{C}] - cell$$

is a retract of a morphism in $(J \otimes \mathcal{C})-cell$, ie. of a weak equivalence, so that the monoid axiom is satisfied by axiom (M_3). □

2.4 Algebras and modules in monoidal model categories

In this section, we establish the main two results of the chapter. The first one shows under which conditions the subcategories of modules and algebras of a cofibrantly generated monoidal category are cofibrantly generated, while the second result establishes some categorical equivalences between their homotopy categories. In order to do this, we first need to state how the three classes of morphisms that are fibrations, cofibrations and weak equivalences must be defined for the model structures to be preserved. This is done in the following way.

Definition 2.4.1. Let $\mathcal{C}$ be a monoidal model category. In the category $\mathcal{M}on_{\mathcal{C}}$, the categories ${}_R\mathcal{M}od$ and $\mathcal{M}od_R$ for a fixed monoid R in $\mathcal{C}$, and the category $\mathcal{A}lg_R$ for a fixed commutative monoid in $\mathcal{C}$, a morphism is defined to be:

- a *weak equivalence* if it is a weak equivalence in the underlying category $\mathcal{C}$,
- a *fibration* if it is a fibration in the underlying category $\mathcal{C}$,
- a *cofibration* if it has the left lifting property with respect to the set of all acyclic fibrations. In particular, a morphism which happen to be a cofibration in the underlying category $\mathcal{C}$ is a cofibration.

With this structure, the categories mentioned in 2.4.1 are model categories according to the following result.

Theorem 2.4.2. *Let $(\mathcal{C}, \otimes, 1)$ be a cofibrantly generated monoidal model category. Suppose that every object in $\mathcal{C}$ is small relative to the whole category and that the monoid axiom is satisfied.*

(1) *For a fixed monoid R in $\mathcal{C}$, the categories ${}_R\mathcal{M}od$ and $\mathcal{M}od_R$ are cofibrantly generated model categories.*

(2) *For a fixed commutative monoid R in $\mathcal{C}$, the category $({}_R\mathcal{M}od, \otimes_R, R) \cong (\mathcal{M}od_R, \otimes_R, R)$ is a cofibrantly generated monoidal model category satisfying the monoid axiom.*

(3) *For a fixed commutative monoid R in $\mathcal{C}$, the category $\mathcal{A}lg_R$ is a cofibrantly generated model category.*

Proof. (1) By 2.4.1, the unit axiom is trivially verified for ${}_R\mathcal{M}od$ and $\mathcal{M}od_R$. Consider the endofunctor $T_R : \mathcal{C} \to \mathcal{C}$ defined on objects by $T_R(M) = M \otimes R$ and on morphisms by $T_R(f) = f \otimes id_R$. This functor has a monad structure

$$T_R = (T_R, \mu_R, \eta_R)$$

whose multiplication and unit morphisms μ_R and η_R are induced by the multiplication μ and the unit η of the given monoid R. After unrolling the definitions, it becomes clear that the category $\mathcal{M}od_R$ of right R-modules is precisely the category $\mathcal{C}^{T_R}$ of T_R-algebras. Defining I to be a set of generating cofibrations, J a set of generating acyclic cofibrations for $\mathcal{C}$, and I^T, J^T their respective images under the free algebra functor T_R, we obtain from the assumed monoid axiom for $\mathcal{C}$ that

$$J^T\!-cof \ \subseteq\ WE_{\mathcal{C}} \ =\ WE_{\mathcal{M}od_R}, \qquad \text{so that} \qquad J^T\!-cell \ \subseteq\ WE_{\mathcal{C}} \ =\ WE_{\mathcal{M}od_R}.$$

This allows to apply 2.2.10 and obtain the desired result for the category $\mathcal{M}od_R$ of right R-modules. The same argument using the functor ${}_RT : \mathcal{C} \to \mathcal{C}$ defined by

$${}_RT(M) = R \otimes M \qquad \text{and} \qquad {}_RT(f) = id_R \otimes f$$

establishes the corresponding result for left R-modules.

(2) From (1) we know that $\mathcal{M}od_R$ is a cofibrantly generated model category, and from 2.1.8 that $(\mathcal{M}od_R, \otimes_R, R)$ has a closed symmetric monoidal structure. The unit axiom is trivially satisfied by definition 2.4.1. By 2.3.4, it remains to check the pushout product axiom and the monoid axiom for a set of generating cofibrations I_R and a set of generating acyclic cofibrations J_R for $\mathcal{M}od_R$. Let $I_R = I^T$ and $J_R = J^T$ as in (1). Then every generating cofibration in I_R is of the form

$$id_R \otimes f :\ R \otimes A \longrightarrow R \otimes B$$

for a cofibration $f : A \to B$ in $\mathcal{C}$ (in fact for f in the set of generating cofibrations I of $\mathcal{C}$ by 2.2.10). For the pushout product of two such maps, let's say

$$id_R \otimes f : \; R \otimes A \longrightarrow R \otimes B \qquad \text{and} \qquad id_R \otimes g : \; R \otimes X \longrightarrow R \otimes Y,$$

we obtain

$$(R \otimes A) \otimes_R (R \otimes Y) \bigvee_{(R\otimes A)\otimes_R(R\otimes X)} (R \otimes B) \otimes_R (R \otimes X) \longrightarrow (R \otimes B) \otimes_R (R \otimes Y),$$

which is in fact a generating cofibration

$$R \otimes (A \otimes Y) \bigvee_{R\otimes(A\otimes X)} R \otimes (B \otimes X) \longrightarrow R \otimes (B \otimes Y) \qquad \text{in } I_R,$$

induced by the cofibration

$$A \otimes Y \bigvee_{A\otimes X} B \otimes X \longrightarrow B \otimes Y \qquad \text{in } \mathcal{C}$$

(the latter being the pushout product of the cofibrations f and g in $\mathcal{C}$). This, together with the same argument applied to the set of generating acyclic cofibrations J_R, verifies the pushout product axiom for I_R, J_R, and consequently for $\mathcal{M}od_R$.

Let's now check the monoid axiom. As J is the arbitrary set of generating acyclic cofibrations in $\mathcal{C}$ chosen in (1) as above, J_R consists of morphisms

$$f \otimes id_R : \; A \otimes R \longrightarrow B \otimes R$$

for morphisms $f : A \to B$ in J, so that we have the inclusion

$$J_R \otimes_R \mathcal{M}od_R \; \subseteq \; J \otimes \mathcal{C}.$$

From the fact that the forgetful functor $U : \mathcal{M}od_R \to \mathcal{C}$ preserves colimits, we obtain

$$(J_R \otimes_R \mathcal{M}od_R)-cell \; \subseteq \; (J \otimes \mathcal{C})-cell \; \subseteq \; WE_{\mathcal{C}} \; = \; WE_{\mathcal{M}od_R},$$

so that the monoid axiom is verified for $\mathcal{M}od_R$.

(3) From (2) we know that $(\mathcal{M}od_R, \otimes_R, R)$ is a cofibrantly generated monoidal model category satisfying the monoid axiom. If 1 is cofibrant in $\mathcal{C}$, the morphism $f : \emptyset \to 1$ is a cofibration in $\mathcal{C}$ so that the induced morphism

$$id_R \otimes f : \; R \otimes \emptyset = \emptyset \longrightarrow R \otimes 1 = R$$

is a cofibration in $\mathcal{M}od_R$, and consequently R, the unit for $\otimes_R$, is cofibrant in $\mathcal{M}od_R$. From this observation, and for the sake of simplifying the notations, we may identify the category $\mathcal{A}lg_R$ of R-algebras (ie. of monoids in $\mathcal{M}od_R$) with the category $\mathcal{M}on_{\mathcal{C}}$ of monoids in $\mathcal{C}$.

Let's consider the free monoid functor $T_{\mathcal{C}} : \mathcal{C} \to \mathcal{M}on_{\mathcal{C}}$ defined by

$$T_{\mathcal{C}}(X) := \coprod_{n\in\mathbb{N}} X^{\otimes n} = 1 \coprod X \coprod (X \otimes X) \coprod (X \otimes X \otimes X) \coprod \ldots,$$

with multiplication given by juxtaposition, ie. by

$$T_{\mathcal{C}}(X) \otimes T_{\mathcal{C}}(X) \;\cong\; \textstyle\coprod_{m\geq 0} X^{\otimes m} \otimes \coprod_{n\geq 0} X^{\otimes n} \;\cong\; \coprod_{m,n\geq 0} X^{\otimes m} \otimes X^{\otimes n} \xrightarrow{mult} \coprod_{k\geq 0} X^{\otimes n},$$

with

$$mult[(x_1 \otimes \ldots \otimes x_m) \otimes (y_1 \otimes \ldots \otimes y_n)] \;:=\; x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n.$$

Since $T_{\mathcal{C}}$ is left adjoint to the forgetful functor $U_{\mathcal{C}} : \mathcal{M}on_{\mathcal{C}} \to \mathcal{C}$ (cf. [17] Chap.VII Thm.2), $T_{\mathcal{C}}$ is a monad in $\mathcal{C}^{T_{\mathcal{C}}}$ with $\mathcal{C}^{T_{\mathcal{C}}} = \mathcal{M}on_{\mathcal{C}}$.

We now want to check the conditions of 2.2.10 in order to prove that the category $\mathcal{C}^{T_{\mathcal{C}}} = \mathcal{M}on_{\mathcal{C}}$ is a cofibrantly generated model category. Because the monoidal product $\otimes$ is closed symmetric, $\otimes$ commutes with colimits, so that the underlying functor of the monad $T_{\mathcal{C}}$ commutes with colimits as required for 2.2.10. Since every object in $\mathcal{C}$ is small relative to $Mor\mathcal{C}$, it follows that every object in $\mathcal{M}on_{\mathcal{C}}$ is small relative to $Mor\mathcal{M}on_{\mathcal{C}}$. What remains to show is condition (1) of 2.2.10 that every regular $T_{\mathcal{C}}(J)$-cofibration, for a set of generating acyclic cofibrations of $\mathcal{C}$, is a weak equivalence. This however is a direct consequence of lemma 6.2 in [20]. □

Definition 2.4.3. Let $f : R \to S$ be a morphism of monoids in a monoidal category $(\mathcal{C}, \otimes, 1)$. The *restriction of scalars* of f is the functor

$$Res_f : \mathcal{M}od_S \to \mathcal{M}od_R \; : \; M \mapsto Res_f(M), \; g \mapsto Res_f(g)$$

which sends an S-module M to itself viewed as an R-module via the multiplication

$$m \cdot r := m \cdot f(r),$$

and consequently a morphism g of S-modules to its obvious restriction to R-modules.

Moreover, the *extension of scalars* of f is the functor

$$Ext_f : \mathcal{M}od_R \to \mathcal{M}od_S \; : \; M \mapsto Ext_f(M) = M \otimes_R S, \; g \mapsto Ext_f(g)$$

which sends an R-module M to the S-module $M \otimes_R S$ with multiplication

$$(m \otimes s) \cdot s' := m \otimes ss',$$

and consequently a morphism g of R-modules to its obvious extension to S-modules.

It turns out that these two functors form an adjoint pair

$$Ext_f \; : \; \mathcal{M}od_R \Longleftrightarrow \mathcal{M}od_S \; : \; Res_f.$$

This holds for right modules as well (with the appropriate side modifications).

Theorem 2.4.4. *Let $(\mathcal{C}, \otimes, 1)$ be a monoidal model category and R a monoid in $\mathcal{C}$.*

(1) *Suppose that for any cofibrant left R-module N the functor $- \otimes_R N : \mathcal{M}od_R \to \mathcal{C}$ preserves weak equivalences. Then, for a weak equivalence $R \xrightarrow{\sim} S$ in $\mathcal{M}on_{\mathcal{C}}$, the total derived functors of restriction and extension of scalars induce equivalences of categories*

$$Ho(\mathcal{M}od_R) \cong Ho(\mathcal{M}od_S).$$

(2) *Dually, suppose that for any cofibrant right R-module M the functor $M \otimes_R - : {}_R\mathcal{M}od \to \mathcal{C}$ preserves weak equivalences. Then, for a weak equivalence $R \xrightarrow{\sim} S$ in $\mathcal{M}on_{\mathcal{C}}$, the total derived functors of restriction and extension of scalars induce equivalences of categories*

$$Ho({}_R\mathcal{M}od) \cong Ho({}_S\mathcal{M}od).$$

(3) *Suppose that R is commutative, that 1 is cofibrant in $\mathcal{C}$ and that for any cofibrant left R-module N the functor $- \otimes_R N : \mathcal{M}od_R \to \mathcal{C}$ preserves weak equivalences. Then, for a weak equivalence of commutative monoids $R \xrightarrow{\sim} S$, the total derived functors of restriction and extension of scalars induce an equivalence of categories*

$$Ho(\mathcal{A}lg_R) \; \cong \; Ho(\mathcal{A}lg_S).$$

Proof. (1) For the weak equivalence $f : R \xrightarrow{\sim} S$, we have an adjoint pair

$$Ext_f \ : \ \mathcal{M}od_R \Longleftrightarrow \mathcal{M}od_S \ : \ Res_f. \tag{$*$}$$

Since all weak equivalences and fibrations are defined in the underlying category $\mathcal{C}$, it follows that Res_f preserves fibrations and acyclic fibrations. By assumption, for the cofibrant left R-module N, the morphism

$$N \cong R \otimes_R N \xrightarrow{f \otimes_R id_N} S \otimes_R N$$

is a weak equivalence. Thus, for a fibrant left S-module Y which fits into

$$\emptyset \hookrightarrow N \cong R \otimes_R N \xrightarrow{\sim} S \otimes_R N \longrightarrow Y \twoheadrightarrow *,$$

a R-module morphism $N \to Y$ is by (M_2) a weak equivalence if and only if its adjoint S-module morphism

$$S \otimes_R N \longrightarrow Y$$

is a weak equivalence. The pair $(*)$ is therefore a Quillen equivalence and an application of theorem 1.5.12 yields the desired result.

(2) The argument is the same as above.

(3) Using the fact that cofibrant R-algebras are also cofibrant as R-modules by 2.4.2, the same argument shows the desired result. $\square$

Chapter 3

Equivalences of monoidal model categories

In chapter 2, we saw under which conditions it is possible to extend the model structure of a cofibrantly generated monoidal model category to the underlying categories of monoids, modules and algebras. The goal here is to construct functors between these categories, and to use them in order to establish Quillen equivalences between the categories of $\mathbb{N}$-graded rings, modules and algebras with their corresponding simplicial categories. This motivates the introduction of simplicial and cosimplicial categories, as well as the study of the correspondence there is between simplicial abelian groups and chain complexes of abelian groups.

In section 3.1, we start by providing a brief introduction to simplicial categories, of which simplicial set and simplicial abelian groups are special cases. This will lead, in section 3.2, to the study of the relation there is between simplicial abelian groups and $\mathbb{N}$-graded chain complexes via the normalizing functor $N : s\mathcal{A}b \to \mathcal{C}h^+$. We shall then introduce monoidal functors and monoidal natural transformations, before studying the particular examples that are the shuffle and the Alexander-Whitney maps. In section 3.3, we proceed to the construction of various left adjoint functors to the right adjoint of a weak Quillen equivalence. We will use these functors to establish the desired Quillen equivalences.

3.1 Simplicial categories

We give here the definition of simplicial and cosimplicial categories and state few fundamental examples we will be using later. For any given category $\mathcal{C}$, we may form its simplicial category $s\mathcal{C}$ and its cosimplicial category $cos\mathcal{C}$ of all the contravariant, respectively covariant, functors from a fixed small category, namely the delta category Δ, to $\mathcal{C}$. We first need define the delta category.

Definition 3.1.1. Let $[n]$ denote the ordered sequence $(0, 1, \ldots, n)$ for every $n \in \mathbb{N} = \{0, 1, \ldots\}$. The Δ *category* is the category whose objects are every $[n]$ with $n \in \mathbb{N}$, and whose morphisms are given by

$$\Delta([m], [n]) := \{f : [m] \to [n] \mid i < j \Rightarrow f(i) \leq f(j)\} \qquad \text{for every} \quad m, n \in \mathbb{N}.$$

Among them we define the maps

$$\delta_i : [n-1] \to [n] \qquad \text{by} \qquad \delta_i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geq i, \end{cases} \qquad \text{for} \qquad 0 \leq i \leq n,$$

$$\sigma_i : [n+1] \to [n] \qquad \text{by} \qquad \sigma_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i, \end{cases} \qquad \text{for} \qquad 0 \leq i \leq n.$$

Remarks 3.1.2. (1) The above δ_i's and σ_i's satisfy the following conditions:

$$\begin{aligned} \delta_j \delta_i &= \delta_i \delta_{j-1} && \text{for} \quad i < j, \\ \sigma_j \sigma_i &= \sigma_i \sigma_{j+1} && \text{for} \quad i \leq j, \\ \sigma_j \delta_i &= \delta_i \sigma_{j-1} && \text{for} \quad i < j, \\ \sigma_j \delta_i &= \delta_{i-1} \sigma_j && \text{for} \quad i > j+1, \\ \sigma_i \delta_i = id &= \sigma_i \delta_{i+1} && \text{for all} \quad i \geq 0. \end{aligned}$$

(2) We can see that for $f \in \Delta([m],[n])$ with $f \neq id_{[m]}$, for every elements $0 \leq i_s < \ldots < i_1 \leq n$ in $[n]$ which are not in $Im(f) = f([m])$, and for every elements $0 \leq j_1 < \ldots < j_t < m$ of $[m]$ which satisfy $f(j_{t_k}) = f(j_{t_k}+1)$, we have a unique factorization

$$f = \delta_{i_1} \ldots \delta_{i_s} \sigma_{j_1} \ldots \sigma_{j_t} \qquad \text{with} \qquad n - t + s = m.$$

In particular, the set

$$\{\delta_i \mid 0 \leq i \leq n\} \cup \{\sigma_j \mid 0 \leq j \leq m\}$$

generates $\Delta([m],[n])$.

These generating maps are essentials to consider, as they determine the structure of each object in simplicial and cosimplicial categories.

Definition 3.1.3. Let $\mathcal{C}$ be a category, and consider the category $\mathcal{C}^{\Delta^{op}}$ (cf. 1.1.2 and 1.1.5) of all functors $\Delta^{op} \to \mathcal{C}$ whose morphisms are all the natural transformations between them. For a given functor $F : \Delta^{op} \to \mathcal{C}$, we define the *faces* $d_i := F(\delta_i)$ and the *degeneracies* $s_j := F(\sigma_j)$, which by 3.1.2 satisfy the following conditions:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{for} \quad i < j, \\ s_i s_j &= s_{j+1} s_i && \text{for} \quad i \leq j, \\ d_i s_j &= s_{j-1} d_i && \text{for} \quad i < j, \\ d_i s_j &= s_j d_{i-1} && \text{for} \quad i > j+1, \\ d_i s_i = id &= d_{i+1} s_i && \text{for all} \quad i \geq 0. \end{aligned}$$

We call $s\mathcal{C} := \mathcal{C}^{\Delta^{op}}$ the *simplicial category* of $\mathcal{C}$; its objects are of the form

$$F = F_\bullet : \qquad \begin{tikzcd} F_0 \arrow[r, bend right, "s_0"'] & F_1 \arrow[l, bend right, "d_0, d_1"'] \arrow[r, bend right, "s_0, s_1"'] & F_2 \arrow[l, bend right, "d_0, d_1, d_2"'] \arrow[r, bend right, "s_0, s_1, s_2"'] & F_3 \arrow[l, bend right, "d_0, d_1, d_2, d_3"'] \; \ldots\ldots\ldots \end{tikzcd} \quad ,$$

in $\mathcal{C}$ with $F_n = F([n])$, they are called *simplicial objects* of $\mathcal{C}$ in $s\mathcal{C}$; and its morphisms $f : F \to F'$ are collections $\{f_k\}_{k\in\mathbb{N}}$ of morphisms $f_k : F_k \to F'_k$ in $\mathcal{C}$ commuting with the faces and degeneracies, ie

$$f_k d_i = d_i f_{k+1} \qquad \text{and} \qquad f_k s_i = s_i f_{k-1},$$

they are called *simplicial morphisms* of $\mathcal{C}$ in $s\mathcal{C}$. Furthermore,

- the elements of F_k for $k \geq 0$ are the *k-simplices* or the *simplices of F of dimension k*,
- the simplices x of F which can be written as $x = s_i y$ for a simplex y and a degeneracy s_i are said to be *degenerated*; all others simplices are *non-degenerated*.

Dually, we may consider the category $\mathcal{C}^{\Delta}$ of all functors $\Delta \to \mathcal{C}$ whose morphisms are all the natural transformations between them. For a given functor $G : \Delta \to \mathcal{C}$, we define the *cofaces* $d^i := G(\delta_i)$ and the *codegeneracies* $s^j := G(\sigma_j)$, which by 3.1.2 satisfy the following conditions:

$$\begin{aligned} d^j d^i &= d^i d^{j-1} && \text{for} \quad i < j, \\ s^j s^i &= s^i s^{j+1} && \text{for} \quad i \leq j, \\ s^j d^i &= d^i s^{j-1} && \text{for} \quad i < j, \\ s^j d^i &= d^{i-1} s^j && \text{for} \quad i > j+1, \\ s^i d^i = id &= s^i d^{i+1} && \text{for all} \quad i \geq 0. \end{aligned}$$

We call $cos\mathcal{C} := \mathcal{C}^{\Delta}$ the *cosimplicial category* of $\mathcal{C}$; its objects are of the form

$$G = G^{\bullet} : \qquad \begin{tikzcd} G^0 \arrow[r, bend right, "{d^0,d^1}"'] & G^1 \arrow[l, bend right, "s^0"'] \arrow[r, bend right, "{d^0,d^1,d^2}"'] & G^2 \arrow[l, bend right, "{s^0,s^1}"'] \arrow[r, bend right, "{d^0,d^1,d^2,d^3}"'] & G^3 \arrow[l, bend right, "{s^0,s^1,s^2}"'] \; \cdots\cdots \end{tikzcd} ,$$

in $\mathcal{C}$ with $G^n = G([n])$, they are called *cosimplicial objects* of $\mathcal{C}$ in $cos\mathcal{C}$; and its morphisms $g : G \to G'$ are simply collections $\{g^k\}_{k\in\mathbb{N}}$ of morphisms $g^k : G^k \to G'^k$ in $\mathcal{C}$ commuting with the cofaces and codegeneracies, they are called *cosimplicial morphisms* of $\mathcal{C}$ in $cos\mathcal{C}$. Finally,

- the elements of G^k for $k \geq 0$ are the *k-cosimplices* or the *cosimplices of G of codimension k*,
- the cosimplices x of G which can be written as $x = d^i y$ for a cosimplex y and a coface d^i are said to be *codegenerated*; all others cosimplices are *non-codegenerated*.

Remarks 3.1.4. (1) From remark 3.1.2.(2) above, it is clear that for any morphism f in Δ^{op} we have

$$F(f) = s_{j_t} \dots s_{j_1} d_{i_s} \dots d_{i_1};$$

and dually for any f in Δ

$$G(f) = d^{i_1} \dots d^{i_s} s^{j_1} \dots s^{j_t}.$$

(2) Any simplex x in F can be written uniquely as

$$x = s_{j_r} \dots s_{j_1} y,$$

with a non-degenerated simplex y and degeneracies indexed by $0 \leq j_1 < \dots < j_r$, and dually for a cosimplex x in G.

This directly leads to the definitions of simplicial and cosimplicial sets.

Example 3.1.5. We may of course apply the definitions of simplicial and cosimplicial categories to the most usual categories such as

- $\mathcal{S}et$ whose simplicial category $s\mathcal{S}et$ is the category of *simplicial sets* and whose cosimplicial category $cos\mathcal{S}et$ is the category of *cosimplicial sets*,
- $\mathcal{A}b$ whose simplicial category $s\mathcal{A}b$ is the category of *simplicial abelian groups* and whose cosimplicial category $cos\mathcal{A}b$ is the category of *cosimplicial abelian groups*.

A fundamental example of a simplicial set is $\Delta[n]$. It is often used in the construction of other simplicial or cosimplicial sets, such as the nerve of a category defined below.

Example 3.1.6. Let $n \in \mathbb{N}$. We define the simplicial set $\Delta[n] \in Obs\mathcal{S}et$ to be the functor $\Delta[n] : \Delta^{op} \to \mathcal{S}et$ defined on objects of Δ^{op} by

$$\Delta[n]([m]) = \Delta([m],[n]) = (\Delta[n])_m,$$

and on morphisms $f : [m] \to [m']$ in Δ^{op} by

$$\Delta[n](f) \ : \ \Delta([m'],[n]) \to \Delta([m],[n]) \ : \ g \mapsto g \circ f.$$

The faces and degeneracies are explicitly given by

$$\begin{aligned} & d_i = \Delta[n](\delta_i) : \ \Delta[n]_m \to \Delta[n]_{m-1} \ : \ f \mapsto f \circ \delta_i, \\ \text{and} \quad & s_i = \Delta[n](\sigma_i) : \ \Delta[n]_m \to \Delta[n]_{m+1} \ : \ f \mapsto f \circ \sigma_i. \end{aligned}$$

Moreover, we can identify each morphism $f \in \Delta([m],[n]) = (\Delta[n]_m)$ with its image $Im(f) = f([m])$ in order to see an m-simplexe of $\Delta[n]$ as a sequence

$$f = (a_0, \ldots, a_m) \qquad \text{with} \quad a_i = f(i) \quad \text{and} \quad 0 \leq a_1 \leq \ldots \leq a_m \leq n,$$

so that the faces and degeneracies act on simplices by

$$\begin{aligned} & d_i(a_0, \ldots, a_m) = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m), \\ \text{and} \quad & s_i(a_0, \ldots, a_m) = (a_0, \ldots, a_i, a_i, a_{i+1}, \ldots, a_m). \end{aligned}$$

Consequently,

$$\Delta[n]_m = \{(a_0, \ldots, a_m) \mid 0 \leq a_1 \leq \ldots \leq a_m \leq n \text{ with } a_i \in \mathbb{N}\},$$

where the non-degenerated simplices are given by the sequences without repetitions. Since every element of $\Delta[n]$ can be obtained after applying faces and degeneracies to $[n] = \{0, \ldots, n\} \in \Delta[n]_n$, we refer to $\Delta[n]$ as the *free simplicial set generated by* $[n]$.

Example 3.1.7. We can see $[n] \in \Delta[n]_n$ as a category whose objects and morphisms are given by

$$Ob[n] = \{0, 1, \ldots, n\} \qquad \text{and} \qquad [n](i,j) = \{t_{i,j} : i \to j \mid i \leq j \text{ and } t_{i,i} = id_i\}$$

respectively. For any small category $\mathcal{C}$, we define the *nerve* of $\mathcal{C}$ to be the simplicial set $\mathcal{N}(\mathcal{C}) : \Delta^{op} \to \mathcal{S}et$ defined on objects and morphisms by

$$\begin{aligned} \mathcal{N}(\mathcal{C})([n]) \ &= \ \mathcal{C}at([n], \mathcal{C}), \\ \mathcal{N}(\mathcal{C})(f^{op}) \ &: \ \mathcal{C}at([n'], \mathcal{C}) \to \mathcal{C}at([n], \mathcal{C}) \ : \ g \mapsto g \circ f, \end{aligned}$$

where $\mathcal{C}at$ denotes the category of small categories and functors between them. Consequently,

- $\mathcal{N}(\mathcal{C})_0 \;=\; \mathcal{C}at([0],\mathcal{C}) \;\cong\; Ob\mathcal{C}$,
- $\mathcal{N}(\mathcal{C})_1 \;=\; \mathcal{C}at([1],\mathcal{C}) \;\cong\; Mor\mathcal{C}$,
- $\mathcal{N}(\mathcal{C})_2 \;=\; \mathcal{C}at([2],\mathcal{C}) \;\cong\; \{\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \mid f, g \in Mor\mathcal{C}\}$, etc.

In particular, if $\mathcal{C}$ is cocomplete, we may see each $\mathcal{N}(\mathcal{C})_n$ as an n-sequence in $\mathcal{C}$.

We end this section with a fact that will be used to define the Γ functor in the next section (cf. 3.2.2.(2)).

Remark 3.1.8. Any functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor

$$\begin{aligned} sF :\; s\mathcal{C} \to s\mathcal{D} \;:\; A_\bullet = \{A_n\}_{n\in\mathbb{N}} &\;\mapsto\; sF(A_\bullet) = \{F(A_n)\}_{n\in\mathbb{N}} \\ (d_i : A_n \to A_{n-1}) &\;\mapsto\; (F(d_i) : F(A_n) \to F(A_{n-1})) \\ (s_i : A_n \to A_{n+1}) &\;\mapsto\; (F(s_i) : F(A_n) \to F(A_{n+1})), \end{aligned}$$

which acts the obvious way on $Mors\mathcal{C}$.

Examples 3.1.9. (1) For the forgetful functor $U : \mathcal{G}r \to \mathcal{S}et$, sU is simply the functor that forgets the group structure in each level, and similarly for $U : \mathcal{A}b \to \mathcal{S}et$ or other common algebraic structures.

(2) We define the functor

$$<->:\; \mathcal{S}et \to \mathcal{A}b \;:\; X \mapsto <X> = \bigoplus_{x\in X} \mathbb{Z}x$$

to be the functor which to each set X associates the free abelian group it generates. In that case

$$s <-> \;:\; s\mathcal{S}et \to s\mathcal{A}b \;:\; K_\bullet \mapsto s <K_\bullet>,$$

where $s <K_\bullet>_n = \bigoplus_{x\in K_n} \mathbb{Z}x \; = <K_n>$.

3.2 Chain complexes and simplicial abelian groups

We are now going study the correspondence between chain complexes and simplicial abelian groups. The goal here is not to enter too much into the details of the results given, but rather to understand the constructions (any good book on simplicial categories, such as [7], will provide the reader with proofs). There is in fact a categorical equivalence between the category $s\mathcal{A}b$ of simplicial abelian groups and the category $\mathcal{C}h^+$ of $\mathbb{N}$-graded chain complexes of abelian groups. It is given by the functors $N : s\mathcal{A}b \to \mathcal{C}h^+$ and $\Gamma : \mathcal{C}h^+ \to s\mathcal{A}b$ as defined below.

Definition 3.2.1. Let $A \in Obs\mathcal{A}b$ be a simplicial abelian group with faces d_i and degeneracies s_i. The *(ordinary) chain complex* CA of A is the non-negative chain complex (A_*, d_*) defined in each degree by $(CA)_n = A_n$ with differentials being the alternative sums of the faces, ie.

$$d \;=\; d_n \;:=\; \sum_{i=0}^{n} (-1)^i d_i \;:\; (CA)_n \to (CA)_{n-1}.$$

This chain complex has a canonic subchain complex DA, called the *complex of degenerate simplices*, defined in each degree $(DA)_n$ to be the (abelian) subgroup of $A_n = (CA)_n$ generated by all degenerate n-simplices in A_n. From this, we define the *normalized chain complex* NA to be the quotient chain complex of CA with DA, given in each degree as the abelian group

$$(NA)_n = (CA)_n/(DA)_n$$

with the obvious quotiented differencial maps.

The category of differential non-negatively graded (or $\mathbb{N}$-graded) chain complexes of abelian groups will be denoted $\mathcal{C}h^+$.

Remarks 3.2.2. **(1)** It is clear by definition that DA is acyclic, so that the canonical projection $CA \to NA$ is a quasi-isomorphism, ie. an isomorphism in homology.

(2) From the above definition, we have a *normalization functor* $N : s\mathcal{A}b \to \mathcal{C}h^+$, between the category of simplicial abelian groups and the category of non-negatively graded chain complexes of abelian groups. It is an equivalence of categories whose inverse $\Gamma : \mathcal{C}h^+ \to s\mathcal{A}b$ can be defined on a complex C in $\mathcal{C}h^+$ by

$$(\Gamma C)_n = \mathcal{C}h^+(N\Delta^n, C) \qquad \text{with} \quad N\Delta^n := N(s < \Delta[n] >).$$

The corresponding isomorphisms $\eta_A : A \to \Gamma NA$ and $\varepsilon_C : N\Gamma C \to C$ are defined by

$$\begin{aligned} A_n \ni a \ &\mapsto \ (N\bar{a} : N\Delta^n \mapsto NA) \ \in \ (\Gamma NA)_n \\ \text{and} \qquad \Gamma(\varepsilon_C) &= \eta^{-1}_{\Gamma C} : \ \Gamma N \Gamma C \to \Gamma C \end{aligned}$$

respectively, where $\bar{a} : \Delta[n] \to A$ is the unique morphism of simplicial sets which sends the generating n-simplex of $\Delta[n]$ (cf. 3.1.6) to $a \in A_n$.

In addition, from the usual tensor product $\otimes_{\mathbb{Z}}$ for abelian groups, we may define symmetric tensor products on the categories $\mathcal{C}h^+$ and $s\mathcal{A}b$.

Definition 3.2.3. The tensor product of two positive chain complexes of abelian groups C and D is defined from the usual tensor product of abelian groups by

$$(C \otimes D)_n \ := \bigoplus_{p+q=n} C_q \otimes D_p,$$

with differentials given on homogeneous elements by the formula

$$d(x \otimes y) \ = \ dx \otimes y \ + \ (-1)^{|x|} x \otimes dy,$$

where $|x|$ denotes the dimension of x. On the other hand, the tensor product of simplicial abelian groups is defined dimensionwise, ie.

$$(A \otimes B)_n := A_n \otimes B_n \qquad \text{and} \qquad d(x \otimes y) := dx \otimes dy.$$

Remarks 3.2.4. **(1)** Both tensor products are symmetric monoidal. Their symmetry is clear, the respective unit objects are

- the free abelian group of rank one (ie. $\mathbb{Z}$) viewed as a complex concentrated in degree zero $\mathbb{Z}[0] : \quad 0 \to \mathbb{Z} \to 0$,
- the free abelian group of rank one (ie. $\mathbb{Z}$) viewed as a constant simplicial abelian group, ie. the simplicial abelian group which is isomorphic to $\mathbb{Z}$ in each level and whose faces and degeneracies are the identity morphism of $\mathbb{Z}$ in $\mathcal{A}b$,

the associativity and unit morphisms for $\mathcal{C}h^+$ and $s\mathcal{A}b$ are the obvious ones, the commutativity isomorphism for $s\mathcal{A}b$ is the obvious one, and the commutativity isomorphism for $\mathcal{C}h^+$ is given by the formula

$$\tau_{C,D} : \ C \otimes D \ \to D \otimes C \ : \ x \otimes y \mapsto (-1)^{|x||y|} y \otimes x.$$

(2) The unit objects of the categories $s\mathcal{A}b$ and $\mathcal{C}h^+$ are preserved under the normalization functor N and its inverse Γ. However, the equivalence of categories given above by N and Γ does not send one tensor product to the other.

Before introducing the shuffle and the Alexander-Whitney maps, it is worth knowing what a monoidal functor is. We then provide few remarks that will be useful for understanding the constructions of the next section.

Definition 3.2.5. A *(lax) monoidal functor* $F = (F, \varphi, \mu) : (\mathcal{C}, \otimes, 1) \to (\mathcal{D}, \wedge, 1')$ between monoidal categories $(\mathcal{C}, \otimes, 1)$ and $(\mathcal{D}, \wedge, 1')$ is an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ equipped with a unit morphism $\mu : 1' \to F(1)$ and natural morphisms

$$\varphi_{A,B} : \; FA \wedge FB \to F(A \otimes B) \qquad \text{in } \mathcal{D} \text{ for every objects } A, B \text{ in } \mathcal{C},$$

which make the following three diagrams commute for every objects A, B, C in $\mathcal{C}$:

$$\begin{tikzcd}
(FA \wedge FB) \wedge FC \arrow[r, "\alpha'"] \arrow[d, "\varphi \wedge id"'] & FA \wedge (FB \wedge FC) \arrow[d, "id \wedge \varphi"] \\
F(A \otimes B) \wedge FC \arrow[d, "\varphi"'] & FA \wedge F(B \otimes C) \arrow[d, "\varphi"] \\
F((A \otimes B) \otimes C) \arrow[r, "F(\alpha)"] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
1' \wedge FA \arrow[r, "\lambda"] \arrow[d, "\mu \wedge id"'] & FA \arrow[d, "F\lambda"] \\
F1 \wedge FA \arrow[r, "\varphi"] & F(1 \otimes A)
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \wedge 1' \arrow[r, "\rho"] \arrow[d, "id \wedge \mu"'] & FA \arrow[d, "F\rho"] \\
FA \wedge F1 \arrow[r, "\varphi"] & F(A \otimes 1).
\end{tikzcd}$$

A monoidal functor (F, φ, μ) is *strong monoidal* (resp. *strict monoidal*) if the morphisms μ and $\varphi_{A,B}$ are isomorphisms (resp. identities). It is clear that the composite of monoidal functors is monoidal.

Moreover, for a monoidal functor $(R, \varphi, \mu) : (\mathcal{C}, \otimes, 1) \to (\mathcal{D}, \wedge, 1')$ between monoidal categories, and assuming R has a left adjoint $L : \mathcal{D} \to \mathcal{C}$, we consider the adjoint $\tilde{\mu} : L(1') \to 1$ of μ and the canonic morphism $\tilde{\varphi}_{A,B} : L(A \wedge B) \to LA \otimes LB$ adjoint to the composite

$$A \wedge B \xrightarrow{\eta_A \wedge \eta_B} RLA \wedge RLB \xrightarrow{\varphi_{LA,LB}} R(LA \otimes LB),$$

which can equivalently be defined as the composition

$$\tilde{\varphi}_{A,B} : \quad L(A \wedge B) \xrightarrow{L(\eta_A \wedge \eta_B)} L(RLA \wedge RLB) \xrightarrow{L(\varphi_{LA,LB})} LR(LA \otimes LB) \xrightarrow{\varepsilon_{LA \otimes LB}} LA \otimes LB,$$

where η and ε respectively denote the unit and counit of the adjunction. In this case, R is said to be a *(lax) comonoidal functor*.

Furthermore, a *(lax) monoidal natural transformation*

$$\tau : \; (F, \varphi, \mu) \to (G, \psi, \nu) \; : \; (\mathcal{C}, \otimes, 1) \to (\mathcal{D}, \wedge, 1')$$

between two monoidal functors is a natural transformation $\tau : F \to G$ between the underlying ordinary functors F and G such that the diagrams

$$\begin{tikzcd}
FA \wedge FB \arrow[r, "\varphi"] \arrow[d, "\tau_A \wedge \tau_B"'] & F(A \otimes B) \arrow[d, "\tau_{A \wedge B}"'] \\
GA \wedge GB \arrow[r, "\psi"] & G(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
1' \arrow[r, "\mu"] \arrow[d, equal] & F(1) \arrow[d, "\tau_1"] \\
1' \arrow[r, "\nu"] & G(1),
\end{tikzcd}$$

for every objects A, B in $\mathcal{C}$, commute in $\mathcal{D}$. The composite of monoidal natural transformations is natural as well.

Finally, in the dual case of natural transformations between comonoidal functors, we talk about *(lax) comonoidal natural transformations.*

Example 3.2.6. If $(\mathcal{C}, \otimes, 1)$ is a symmetrical monoidal category, the endofunctors $X \otimes -$ and $- \otimes X$ for any monoid (X, μ_X, η_X) in $\mathcal{C}$ are monoidal. To see why, it suffices to define the maps $\varphi_{A,B}$ as the composites

$$(X \otimes A) \otimes (X \otimes B) \xrightarrow{\cong} (X \otimes X) \otimes (A \otimes B) \xrightarrow{\mu_X \otimes id} X \otimes (A \otimes B),$$

and similarly for $- \otimes X$. The required diagrams effectively commute from the associativity of μ_X. In particular, if the monoid X is commutative, the endofunctors

$$\begin{array}{rrcl} & M \otimes_X - & : & (\mathcal{M}od_X, \otimes_X, X) \longrightarrow (\mathcal{M}od_X, \otimes_X, X) \\ \text{and} & - \otimes_X M & : & (\mathcal{M}od_X, \otimes_X, X) \longrightarrow (\mathcal{M}od_X, \otimes_X, X) \end{array}$$

are monoidal for any X-module M.

Definitions 3.2.7. Let A and B be simplicial abelian groups. We call *shuffle maps* the morphisms in $\mathcal{C}h^+$ defined by

$$\nabla = \nabla_{A,B} : \ CA \otimes CB \to C(A \otimes B) \ : \ a \otimes b \mapsto \nabla(a, b) = \sum_{\mu,\nu} sgn(\mu, \nu) \cdot s_\nu a \otimes s_\mu b$$

on simplices $a \in A_p$ and $b \in B_q$ whose image $\nabla(a \otimes b)$ is in $C_{p+q}(A \otimes B) = A_{p+q} \otimes B_{p+q}$, where the sum is taken over all (p, q)-*shuffles*, that is, permutations of the set

$$\{0, \dots, p + q - 1\}$$

which leave the first p elements and the last q elements in their original order, ie. a (p, q)-shuffle is of the form

$$(\mu, \nu) = (\mu_1, \dots, \mu_p, \nu_1, \dots, \nu_q) \qquad \begin{array}{ll} \text{with} & \mu_1 < \dots < \mu_q \\ \text{and} & \nu_1 < \dots < \nu_q, \end{array}$$

and the associated degeneracies are given by

$$s_\mu b = s_{\mu_p} \dots s_{\mu_1} b \qquad \text{and} \qquad s_\nu a = s_{\nu_q} \dots s_{\nu_1} a.$$

On the other hand, we define the *Alexander-Whitney maps* to be the morphisms in $\mathcal{C}h^+$ defined by

$$AW = AW_{A,B} : \ C(A \otimes B) \to CA \otimes CB \ : \ a \otimes b \mapsto \bigoplus_{p+q=n} \tilde{d}^p(a) \otimes d_0^q(b)$$

on n-simplices $a \in A_n$ and $b \in B_n$, where the *front face* $\tilde{d}^p : A_{p+q} \to A_p$ is induced by the injective monotone morphism

$$\tilde{\delta}^p : \ [p] \to [p + q] \ : \ i \mapsto i \qquad \text{in} \quad \Delta^{op},$$

and the *back face* $d_0^q : B_{p+q} \to B_q$ is induced by the injective monotone morphism

$$\delta_0^p : \ [q] \to [p + q] \ : \ i \mapsto p + i \qquad \text{in} \quad \Delta^{op}.$$

If we consider all $\nabla_{A,B}$ and $AW_{A,B}$ for every objects A and B in $s\mathcal{A}b$, we obtain natural transformations

$$\begin{aligned} \nabla :\ s\mathcal{A}b \times s\mathcal{A}b \to Mor\mathcal{C}h^{+}\ &:\ C(-)\otimes C(-) \to C(-\otimes -), \\ \text{and} \qquad AW :\ s\mathcal{A}b \times s\mathcal{A}b \to Mor\mathcal{C}h^{+}\ &:\ C(-\otimes -) \to C(-)\otimes C(-), \end{aligned}$$

that we call *shuffle map* and *Alexander-Whitney map* as well.

Finally, for the inverse functor $\Gamma :\ \mathcal{C}h^{+} \to s\mathcal{A}b$ of N (cf. 3.2.2), and for every objects C, D in $\mathcal{C}h^{+}$, we will denote the morphism

$$\varphi = \varphi_{C,D} :\ \Gamma C \otimes \Gamma D \to \Gamma(C \otimes D) \qquad \text{in} \quad s\mathcal{A}b$$

to be the composite

$$\Gamma C \otimes \Gamma D \xrightarrow[\eta(\Gamma C \otimes \Gamma D)]{} \Gamma N(\Gamma C \otimes \Gamma D) \xrightarrow[\Gamma(AW_{\Gamma C,\Gamma D})]{} \Gamma[N(\Gamma C) \otimes N(\Gamma D)] \xrightarrow[\Gamma(\varepsilon_C \otimes \varepsilon_D)]{} \Gamma(C \otimes D),$$

where η and ε are respectively the unit and counit of the adjoint pair

$$N :\ s\mathcal{A}b \Longleftrightarrow \mathcal{C}h^{+}\ :\Gamma.$$

Remarks 3.2.8. **(1)** The shuffle map is a monoidal natural transformation whose unit map is the unique chain complex morphism

$$\eta :\ \mathbb{Z}[0] \to C(\mathbb{Z})$$

which is the identity in dimension 0. Consequently, the Alexander-Whitney map is its corresponding comonoidal transformation whose unit is η^{-1}.

(2) Both ∇ and AW preserve the subcomplexes of degenerate simplices, so that they induce maps

$$\begin{aligned} \nabla^{N} = \nabla^{N}_{A,B} &:\ NA \otimes NB \to N(A \otimes B) \\ \text{and} \qquad AW^{N} = AW^{N}_{A,B} &:\ N(A \otimes B) \to NA \otimes NB, \end{aligned}$$

on objects A, B in $s\mathcal{A}b$, for which we shall also use the same names if the context is clear. These restricted natural transformations are again monoidal and comonoidal respectively, and the corresponding restricted unit morphisms are isomorphisms.

(3) For each simplicial abelian groups A and B, the composite morphism

$$AW_{A,B} \circ \nabla_{A,B} :\ CA \otimes CB \to CA \otimes CB$$

differs from $id_{(CA \otimes CB)}$ only by degenerate simplices, so that

$$AW^{N}_{A,B} \circ \nabla^{N}_{A,B} = id_{(NA \otimes NB)}\ :\ NA \otimes NB \to NA \otimes NB.$$

The other composites

$$\nabla_{A,B} \circ AW_{A,B} :\ C(A \otimes B) \to C(A \otimes B) \qquad \text{and} \qquad \nabla^{N}_{A,B} \circ AW^{N}_{A,B} :\ N(A \otimes B) \to N(A \otimes B)$$

are naturally chain homotopic to the identities $id_{C(A \otimes B)}$ and $id_{N(A \otimes B)}$ respectively. In particular, all four morphisms

$$AW_{A,B} \circ \nabla_{A,B}, \qquad AW^{N}_{A,B} \circ \nabla^{N}_{A,B}, \qquad \nabla_{A,B} \circ AW_{A,B} \qquad \text{and} \qquad \nabla^{N}_{A,B} \circ AW^{N}_{A,B}$$

are quasi-isomorphisms of chain complexes.

(4) The shuffle maps ∇ and ∇^N are also symmetric in the sense that for any objects A and B in $s\mathcal{A}b$, the square diagrams

$$\begin{tikzcd} CA \otimes CB \arrow[r, "sym"] \arrow[d, "\nabla_{A,B}"'] & CB \otimes CA \arrow[d, "\nabla_{A,B}"'] \\ C(A \otimes B) \arrow[r, "C(sym)"'] & C(B \otimes A) \end{tikzcd} \qquad \begin{tikzcd} NA \otimes NB \arrow[r, "sym"] \arrow[d, "\nabla^N_{A,B}"'] & NB \otimes NA \arrow[d, "\nabla^N_{A,B}"'] \\ N(A \otimes B) \arrow[r, "N(sym)"'] & N(B \otimes A) \end{tikzcd}$$

commute, where *sym* is the obvious symmetry isomorphism of either simplicial abelian groups or chain complexes. That is not the case however for AW and AW^N.

(5) The functor Γ is not symmetric monoidal since the Alexander-Whitney map AW, and consequently the morphism φ as in 3.2.5, is not symmetric. However, we can turn the comonoidal structure on the normalization functor AW^N given by the Alexander-Whitney map into a monoidal structure on the adjoint functor Γ by defining

$$\varphi_{C,D}:\ \Gamma C \otimes \Gamma D \longrightarrow \Gamma(C \otimes D)$$

to be the composite

$$\Gamma C \otimes \Gamma D \xrightarrow{\eta_{\Gamma C \otimes \Gamma D}} \Gamma N(\Gamma C \otimes \Gamma D) \xrightarrow{\Gamma(AW_{\Gamma C, \Gamma D})} \Gamma[N(\Gamma C) \otimes N(\Gamma D)] \xrightarrow{\Gamma(\varepsilon_C \otimes \varepsilon_D)} \Gamma(C \otimes D).$$

for every chain complexes C and D.

(6) For all simplicial abelian groups $A = \Gamma C$ and $B = \Gamma D$, being via Γ the images of chain complexes C and D, the normalized Alexander-Whitney map $AW^N_{A,B}$ is surjective. It follows, since the unit η and counit ε of the (N, Γ)-adjunction are isomorphisms (N and Γ are inverse of each other), that the morphism $\varphi_{C,D}$ is a surjection as well.

3.3 Equivalences between categories of algebras and modules

In this section, we shall introduce the notions of *weak* and *strong monoidal Quillen equivalences* between two monoidal model categories. A weak monoidal Quillen equivalence provides the basic properties necessary for lifting Quillen equivalences between given categories to their corresponding categories of monoids and modules. The right adjoint functor of such an equivalence being monoidal, we shall construct various induced left adjoints on the corresponding categories of monoids and modules. This will lead to the main result of the chapter (cf. 3.3.8).

Before going into the definition of weak monoidal Quillen equivalences, we first need to define the notion of *Quillen invariance*.

Definition 3.3.1. Let $\mathcal{C}$ be a model category, $\mathcal{D}$ a category, and let

$$L:\ \mathcal{C} \Longleftrightarrow \mathcal{D}\ : R$$

be an adjoint pair. An object of $\mathcal{D}$ is a *cell object* if it can be obtained from the initial object $\emptyset$ of $\mathcal{D}$ by a (possibly transfinite) composition of cobase changes of morphisms of the form $L(f)$ for cofibrations f in $\mathcal{C}$. We say that the functor R *creates a model structure* if the following two conditions are satisfied:

- the category $\mathcal{D}$ supports a model structure in which a morphism $f : X \to Y$ is a weak equivalence, resp. a fibration, if and only if the morphism $R(f)$ is a weak equivalence, resp. a fibration, in $\mathcal{C}$,
- every cofibrant object in $\mathcal{D}$ is a retract of a cell object.

Now suppose that $(\mathcal{C}, \otimes, 1)$ is a monoidal model category such that the forgetful functors

$$\mathcal{M}od_A \longrightarrow \mathcal{C} \qquad \text{and} \qquad {}_A\mathcal{M}od \longrightarrow \mathcal{C}, \qquad \text{for all monoids } A \text{ in } \mathcal{C},$$

create model structures. We say that *right Quillen invariance holds* for $\mathcal{C}$ if for every weak equivalence $f : R \to S$ of $\mathcal{C}$-monoids, the restriction and extension of scalars along f induce a Quillen equivalence

$$Ext_f = - \otimes_R S \ : \ \mathcal{M}od_R \Longleftrightarrow \mathcal{M}od_S \ : \ Res_f,$$

and that *left Quillen invariance holds* for $\mathcal{C}$ if for every weak equivalence $f : R \to S$ of $\mathcal{C}$-monoids, the restriction and extension of scalars along f induce a Quillen equivalence

$$Ext_f = S \otimes_R - \ : \ {}_R\mathcal{M}od \Longleftrightarrow {}_S\mathcal{M}od \ : \ Res_f.$$

Remarks 3.3.2. (1) A typical example of the creation of a model structure occurs when the model category structure on $\mathcal{C}$ is cofibrantly generated.

(2) A sufficient condition for right Quillen invariance in $\mathcal{C}$ is that, for every cofibrant right R-module M, the functor

$$M \otimes_R - : \ {}_R\mathcal{M}od \longrightarrow \mathcal{C}$$

preserves weak equivalences in $\mathcal{C}$; and similarly for left Quillen invariance.

Definition 3.3.3. A *weak monoidal Quillen pair* between monoidal categories $(\mathcal{C}, \otimes, 1)$ and $(\mathcal{D}, \wedge, 1')$ consists of a Quillen pair

$$L : \ \mathcal{D} \Longleftrightarrow \mathcal{C} \ : R$$

with a (lax) monoidal structure

$$\varphi_{X,Y} : RX \wedge RY \to R(X \otimes Y), \qquad \nu : 1' \to R(1)$$

on the right adjoint R, such that the following conditions are satisfied:

- The comonoidal morphisms

  $$\tilde{\varphi}_{A,B} : L(A \wedge B) \to LA \otimes LB, \qquad \text{for every cofibrant objects } A \text{ and } B \text{ in } \mathcal{D},$$

  are weak equivalences in $\mathcal{C}$.
- For any cofibrant replacement $q : 1'_c \xrightarrow{\sim} 1'$, the composite morphism

  $$L(1'_c) \xrightarrow{L(q)} L(1') \xrightarrow{\tilde{\nu}} 1$$

  is a weak equivalence in $\mathcal{C}$.

A *strong monoidal Quillen pair* is a weak monoidal Quillen pair for which the comonoidal morphisms $\tilde{\varphi}$ and $\tilde{\nu}$ are isomorphisms. A weak (resp. strong) monoidal Quillen pair is a *weak* (resp. *strong*) *monoidal Quillen equivalence* if the underlying Quillen pair is a Quillen equivalence.

We may now construct various left adjoints for the right adjoint functor of a weak monoidal Quillen pair.

Construction 3.3.4. Let $(\mathcal{C}, \otimes, 1)$ and $(\mathcal{D}, \wedge, 1')$ be monoidal categories, and let $R : \mathcal{C} \to \mathcal{D}$ be the right adjoint of a weak monoidal Quillen pair. Then R induces various functors on the underlying categories of monoids and modules. More precisely, for a monoid $A = (A, \mu, \eta)$ in $\mathcal{C}$ the monoid structure on RA is given by the composite morphisms

$$RA \wedge RA \xrightarrow{\varphi_{A,A}} R(A \otimes A) \xrightarrow{R(\mu)} RA \qquad \text{and} \qquad 1' \xrightarrow{\nu} R(1) \xrightarrow{R(\eta)} RA.$$

For a right A-module M in $\mathcal{C}$ with action morphism $\alpha : M \otimes A \to M$, the object RM of $\mathcal{D}$ becomes a right RA-module via the composite morphism

$$RM \wedge RA \xrightarrow{\varphi_{M,A}} R(M \otimes A) \xrightarrow{R(\alpha)} RM,$$

and similarly for a left A-module N. In this context, R has a left adjoint $L : \mathcal{D} \to \mathcal{C}$ which inherits a comonoidal structure

$$\tilde{\varphi} : \ L(A \wedge B) \to LA \otimes LB \qquad \text{and} \qquad \tilde{\nu} : \ L(1') \to 1,$$

the pair being of course strong monoidal if $\tilde{\varphi}$ and $\tilde{\nu}$ are isomorphisms. In this latter case, the left adjoint L becomes a strong monoidal functor via the inverses

$$\tilde{\varphi}^{-1} : \ LA \otimes LB \to L(A \wedge B) \qquad \text{and} \qquad \tilde{\nu}^{-1} : \ 1 \to L(1').$$

Via these morphisms, L then lifts to a functor on monoids and modules (and consequently algebras) which are of course left adjoint to the monoid and module valued (and particularly algebra valued) versions of R. This treats the case of strong monoidal Quillen pairs, but we want to treat the more general case of weak monoidal Quillen pairs, in which the functors induced by R on monoids, modules and algebras still have left adjoints. These latter functors, however, are not usually given by the original left adjoint functor L defined here. We shall construct more functors below.

(1) We assume here that the forgetful functor $U_{\mathcal{C}} : \mathcal{M}on_{\mathcal{C}} \to \mathcal{C}$ from $\mathcal{C}$-monoids to $\mathcal{C}$ creates a model structure on $\mathcal{M}on_{\mathcal{C}}$ (cf. 3.3.1). In particular, the category $Alg_{\mathcal{C}}$ of $\mathcal{C}$-algebras is cocomplete and the forgetful functor $U_{\mathcal{C}}$ has a left adjoint free monoid functor $T_{\mathcal{C}} : \mathcal{C} \to \mathcal{M}on_{\mathcal{C}}$ (cf. [17] Chap.VII Thm.2) defined by

$$T_{\mathcal{C}}(X) \ := \ \coprod_{n \in \mathbb{N}} X^{\otimes n} \ = \ 1 \coprod X \coprod (X \otimes X) \coprod (X \otimes X \otimes X) \coprod \ldots,$$

with multiplication given by juxtaposition, ie. by

$$TX \otimes TX \ \cong \ \textstyle\coprod_{m \geq 0} X^{\otimes m} \otimes \coprod_{n \geq 0} X^{\otimes n} \ \cong \ \coprod_{m,n \geq 0} X^{\otimes m} \otimes X^{\otimes n} \xrightarrow{mult} \coprod_{k \geq 0} X^{\otimes n},$$

with

$$mult[(x_1 \otimes \ldots \otimes x_m) \otimes (y_1 \otimes \ldots \otimes y_n)] \ := \ x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n,$$

and similarly for $T_{\mathcal{D}}$. From this, the monoid valued lift $R : \mathcal{M}on_{\mathcal{C}} \to \mathcal{M}on_{\mathcal{D}}$ has a left adjoint

$$L^{mon} : \ \mathcal{M}on_{\mathcal{D}} \to \mathcal{M}on_{\mathcal{C}},$$

whose value for a $\mathcal{D}$-monoid B can be defined as the coequalizer (cf. 2.1.8) of the $\mathcal{C}$-monoid morphisms

$$T_{\mathcal{C}}(LT_{\mathcal{D}}B) = T_{\mathcal{C}}(LT_{\mathcal{D}}U_{\mathcal{D}}B) \rightrightarrows T_{\mathcal{C}}(LB) \dashrightarrow L^{mon}B,$$

the first of them being obtained by precomposing the adjunction unit

$$T_{\mathcal{D}}B = T_{\mathcal{D}}U_{\mathcal{D}}B \longrightarrow B$$

with the composite functor $T_{\mathcal{C}}L$, and the second being the unique morphism in $\mathcal{M}on_{\mathcal{C}}$ which restricts to the $\mathcal{C}$-morphism

$$L(T_{\mathcal{D}}B) \;\cong\; \coprod_{n\geq 0} L(B^{\wedge n}) \xrightarrow{\quad \amalg \tilde{\varphi} \quad} \coprod_{n\geq 0} (LB)^{\otimes n} \;\cong\; T_{\mathcal{C}}(LB).$$

Since the functor R preserves the underlying objects, the monoid left adjoint functor L^{mon} and the original left adjoint functor L are related by a natural isomorphism

$$L^{mon} \circ T_{\mathcal{D}} \;\cong\; T_{\mathcal{C}} \circ L \;:\; \mathcal{D} \longrightarrow \mathcal{M}on_{\mathcal{C}}.$$

(2) From the above construction for monoids, the right and left module valued functors

$$R :\; \mathcal{M}od_A \to \mathcal{M}od_{RA} \qquad \text{and} \qquad R :\; {}_A\mathcal{M}od \to {}_{RA}\mathcal{M}od,$$

for a given $\mathcal{C}$-monoid A, have left adjoint functors

$$L^A :\; \mathcal{M}od_{RA} \to \mathcal{M}od_A \qquad \text{and} \qquad {}^A L :\; {}_{RA}\mathcal{M}od \to {}_A\mathcal{M}od \qquad \text{respectively},$$

as soon as free R-modules and coequalizers of R-modules exist. Since the functor R preserves the underlying objects, the left adjoint functors L^A and ${}^A L$ are related to the original left adjoint functor L by natural isomorphisms

$$\begin{aligned} L^A \circ (- \wedge RA) \;&\cong\; (- \otimes A) \circ L \;:\; \mathcal{D} \longrightarrow \mathcal{M}od_A, \\ {}^A L \circ (RA \wedge -) \;&\cong\; (A \otimes -) \circ L \;:\; \mathcal{D} \longrightarrow {}_A\mathcal{M}od, \end{aligned}$$

where $X \otimes A$ is the free right A-module generated by an object X in $\mathcal{C}$, $A \otimes X$ the free left A-module generated by X, and similarly for $Y \wedge RA$ and $RA \wedge Y$ with $Y \in \mathcal{D}$.

(3) For a given monoid B in $\mathcal{D}$, the functor $R : \mathcal{C} \to \mathcal{D}$ induces a functor

$$R :\; \mathcal{M}od_{(L^{mon}B)} \to \mathcal{M}od_B$$

which is the composite functor

$$R : \quad \mathcal{M}od_{(L^{mon}B)} \xrightarrow{\;R\;} \mathcal{M}od_{R(L^{mon}B)} \xrightarrow{\;\eta^*\;} \mathcal{M}od_B,$$

where η^* is the restriction of scalars along the adjunction unit $\eta : B \to R(L^{mon}B)$. We denote by

$$L_B : \quad \mathcal{M}od_B \longrightarrow \mathcal{M}od_{(L^{mon}B)}$$

the left adjoint to the functor $R : \mathcal{M}od_{(L^{mon}B)} \to \mathcal{M}od_B$, which is the composite

$$\mathcal{M}od_B \xrightarrow{\;-\otimes_B R(L^{mon}B)\;} \mathcal{M}od_{R(L^{mon}B)} \xrightarrow{\;L^{(L^{mon}B)}\;} \mathcal{M}od_{(L^{mon}B)};$$

it is related to the original left adjoint functor L by a natural isomorphism

$$L_B \circ (- \wedge B) \;\cong\; (- \otimes L^{mon}B) \circ L \;:\; \mathcal{D} \longrightarrow \mathcal{M}od_{L^{mon}B}.$$

The same construction for left modules gives a left adjoint functor

$${}_B L : \quad {}_B\mathcal{M}od \longrightarrow {}_{(L^{mon}B)}\mathcal{M}od$$

to the restricted functor $R :\; {}_{(L^{mon}B)}\mathcal{M}od \to\; {}_B\mathcal{M}od$ which satisfy

$${}_B L \circ (B \wedge -) \;\cong\; (L^{mon}B \otimes -) \circ L \;:\; \mathcal{D} \longrightarrow_{L^{mon}B} \mathcal{M}od.$$

Remark 3.3.5. In case the monoidal adjoint pair $L : \mathcal{D} \Longleftrightarrow \mathcal{C} : R$ is strong monoidal, the functors L^{mon}, L_B and ${}_BL$ are given by L which is monoidal via φ^{-1}, and the functors

$$L^A : \mathcal{M}od_{RA} \to \mathcal{M}od_A \qquad \text{and} \qquad {}^AL : {}_{RA}\mathcal{M}od \to {}_A\mathcal{M}od$$

are given by the formulas

$$L^A(M) = L(M) \otimes_{LRA} A \qquad \text{and} \qquad {}^AL(N) = A \otimes_{LRA} L(N),$$

where A is a LRA-module via the adjunction counit $\varepsilon : LRA \to A$. In general however, the functor L does not pass to monoids and modules.

The following result states under which conditions the left adjoints L_B, ${}_BL$, L^A, ${}^AL$ and L^{mon} of the functor R induce Quillen equivalences. The proof will not be given here; it is essentially covered in section 5 of [21].

Lemma 3.3.6. *Let $R : (\mathcal{C}, \otimes, 1) \to (\mathcal{D}, \wedge, 1')$ be the right adjoint of a weak monoidal Quillen equivalence, and suppose that the unit objects 1 and $1'$ are cofibrant.*

(1) *If B is a cofibrant monoid in $\mathcal{D}$ such that the forgetful functors create model structures for B-modules and $L^{mon}B$-modules, then*

$$L_B : \mathcal{M}od_B \Longleftrightarrow \mathcal{M}od_{L^{mon}B} : R \qquad \textit{and} \qquad {}_BL : {}_B\mathcal{M}od \Longleftrightarrow {}_{L^{mon}B}\mathcal{M}od : R$$

are Quillen equivalences.

(2) *If right (respectively left) Quillen invariance holds for $\mathcal{C}$ and $\mathcal{D}$, then for any fibrant monoid A in $\mathcal{C}$ such that the forgetful functors for A-modules and RA-modules create model structures, the adjoint pair*

$$L^A : \mathcal{M}od_{RA} \Longleftrightarrow \mathcal{M}od_A : R \qquad (\textit{respectively} \quad {}^AL : {}_{RA}\mathcal{M}od \Longleftrightarrow {}_A\mathcal{M}od : R)$$

is a Quillen equivalence. If in addition R preserves weak equivalences between monoids and if the forgetful functors for modules over any monoid create model structures, then the Quillen equivalence above holds for any monoid A in $\mathcal{C}$.

(3) *If the forgetful functors for monoids in $\mathcal{C}$ and $\mathcal{D}$ create model structures, then the adjoint pair*

$$L^{mon} : \mathcal{M}on_{\mathcal{D}} \Longleftrightarrow \mathcal{M}on_{\mathcal{C}} : R$$

is a Quillen equivalence.

Assuming 3.3.6, we may now proceed to the main theorem of the chapter. For clarity reasons, let us fix some notation.

Notation 3.3.7. For a category $\mathcal{C}$, we denote by $\mathcal{DGC}$, respectively $\mathcal{GC}$, the category whose objects are $\mathbb{N}$-differential-graded (resp. $\mathbb{N}$-graded) objects of $\mathcal{C}$ and whose morphisms are the appropriate collections of morphisms in $\mathcal{C}$ making the obvious required squares commute. For example:

- $\mathcal{DGA}b = \mathcal{C}h^+$ denotes the category of $\mathbb{N}$-graded chain complexes of abelian groups,
- $\mathcal{DGM}od_k = \mathcal{C}h_k^+$ denotes the category of $\mathbb{N}$-graded chain complexes of k-modules for a commutative ring k,
- $\mathcal{DGA}lg_k =: \mathcal{DGA}_k$ denotes the category of $\mathbb{N}$-differential-graded k-algebras for a commutative ring k,

- $\mathcal{DGA}lg_R =: \mathcal{DGA}_R$ denotes the category of $\mathbb{N}$-differential-graded R-algebras for a monoid R in $\mathcal{C}$,
- $\mathcal{GM}od_R$ denotes the category of $\mathbb{N}$-graded R-modules for a monoid R in $\mathcal{C}$.

Theorem 3.3.8. *Let $N : s\mathcal{A}b \to \mathcal{C}h^+$ and $\Gamma : \mathcal{C}h^+ \to s\mathcal{A}b$ be the functors defined in 3.2.1 and 3.2.2.*

(1) *For an $\mathbb{N}$-differential-graded algebra R, there is a Quillen equivalence*

$$\mathcal{GM}od_R \;\cong_Q\; s\mathcal{M}od_{\Gamma R}$$

between the categories of $\mathbb{N}$-graded R-modules and simplicial modules over the simplicial algebra ΓR.

(2) *For a simplicial ring A, there is a Quillen equivalence*

$$\mathcal{DGM}od_{NA} \;\cong_Q\; \mathcal{DGM}od_A$$

between the categories of $\mathbb{N}$-differential-graded NA-modules and simplicial A-modules.

(3) *For a commutative ring k, there is a Quillen equivalence*

$$\mathcal{DGA}_k \;\cong_Q\; s\mathcal{A}lg_k$$

between the categories of $\mathbb{N}$-differential-graded k-algebras and simplicial k-algebras. In particular, for $k = \mathbb{Z}$, there is a Quillen equivalence

$$\mathcal{DGR}ng \;\cong_Q\; s\mathcal{R}ng$$

between the categories of $\mathbb{N}$-differential-graded rings and simplicial rings.

(4) *For a simplicial commutative ring A, there is a Quillen equivalence*

$$\mathcal{DGA}lg_{NA} \;\cong_Q\; s\mathcal{A}lg_A$$

between the categories of $\mathbb{N}$-differential-graded NA-algebras and simplicial A-algebras.

Remark 3.3.9. Part (3) is a special case of part (4) for a constant commutative simplicial ring A.

Proof. (3) Let k be a commutative ring. Consider the normalization functor N and its inverse Γ as an adjoint pair

$$\Gamma : \; \mathcal{C}h_k^+ \Longleftrightarrow s\mathcal{M}od_k \;\; : N$$

between the category of $\mathbb{N}$-graded chain complexes of k-modules and the category of simplicial k-modules, where N is the right adjoint of Γ. We may also view N as a monoidal functor (N, φ, ν) with all $\varphi_{A,B}$ being the shuffle maps

$$\nabla_{A,B} : \; NA \otimes NB \to N(A \otimes B)$$

for every simplicial k-modules A and B. By remark 3.2.8.(3), $\nabla_{A,B}$ is a chain homotopy equivalence whose inverse is the Alexander-Whitney map

$$AW_{A,B} : \; N(A \otimes B) \to NA \otimes B$$

for any simplicial k-modules A and B. Since the functor Γ sends quasi-isomorphisms in $\mathcal{C}h_k^+$ to weak equivalences in $s\mathcal{M}od_r$, and since the unit and counit of the adjunction between N and Γ are isomorphisms, the corresponding comonoidal maps

$$\tilde{\varphi}_{C,D} = \tilde{\nabla}_{C,D} : \Gamma(C \otimes D) \to \Gamma C \otimes \Gamma D, \qquad \text{for every } C, D \text{ in } \mathcal{C}h_k^+,$$

are weak equivalences. This means that N is the right adjoint of a weak monoidal Quillen equivalence between the categories $s\mathcal{M}od_k$ and $\mathcal{C}h_k^+$. In addition, the unit objects of $s\mathcal{M}od_k$ and $\mathcal{C}h_k^+$ (cf. 3.2.4.(1)) are both cofibrant, so that by 3.3.6.(3) we have a Quillen equivalence

$$\mathcal{DGA}_k \;=\; \mathcal{M}on_{(\mathcal{C}h_k^+)} \;\cong_Q\; \mathcal{M}on_{(s\mathcal{M}od_k)} \;=\; s\mathcal{A}lg_k$$

as desired. The Quillen equivalence

$$\mathcal{DGR}ng \;\cong_Q\; s\mathcal{R}ng$$

is now obvious if $k = \mathbb{Z}$.

(2) Let $A \in Ob s\mathcal{R}ng$ be a simplicial ring. Quillen invariance holds for the category $s\mathcal{R}ng$ of simplicial rings, and the normalization functor

$$N : \; s\mathcal{M}od_k \longrightarrow \mathcal{C}h_k^+$$

preserves all weak equivalences, so that Quillen invariance also holds for $\mathcal{C}h_k^+$ by remark 3.3.2.(2). Consequently, we can apply 3.3.8.(2) for $k = \mathbb{Z}$ in order to prove that the functor

$$N : \; s\mathcal{M}od_k = s\mathcal{M}od_{\mathbb{Z}} = s\mathcal{A}b \;\longrightarrow\; \mathcal{C}h_k^+ = \mathcal{C}h_{\mathbb{Z}}^+ = \mathcal{C}h^+$$

is the right adjoint of a Quillen equivalence

$$\mathcal{DGM}od_{NA} \;\cong_Q\; \mathcal{DGM}od_A$$

between the categories of $\mathbb{N}$-differential-graded NA-modules and the category of simplicial A-modules.

(1) Let k be a commutative ring. This time we view the normalizing functor $N : s\mathcal{M}od_k \to \mathcal{C}h_k^+$ as the left adjoint of the pair

$$N : \; s\mathcal{M}od_k \Longleftrightarrow \mathcal{C}h_k^+ \; : \; \Gamma.$$

The monoidal structure of Γ (cf. 3.2.8.(5)) is made in such a way that the comonoidal transformation $\tilde{\varphi}$, given by

$$\tilde{\varphi}_{A,B} : \; L(A \otimes B) \longrightarrow LA \otimes LB$$

for the left adjoint N, is none other than the Alexander-Whitney map

$$AW_{A,B} : \; N(A \otimes B) \longrightarrow NA \otimes NB \qquad \text{for every objects } A, B \text{ in } s\mathcal{M}od_k.$$

Since this last morphism is a chain homotopy equivalence for every A, B in $s\mathcal{M}od_k$ whose homotopy inverse is the shuffle map ∇, the functor Γ becomes the right adjoint of a weak monoidal Quillen equivalence. As we already saw above, the unit objects of $s\mathcal{M}od_k$ and $\mathcal{C}h_k^+$ are both cofibrant, so that we can apply 3.3.6.(2) to obtain, for a monoid R in $\mathcal{C}h_k^+$ (ie. a $\mathbb{N}$-differential-graded k-algebra R), a Quillen equivalence

$$s\mathcal{M}od_{\Gamma R} \;\cong\; \mathcal{DGM}od_R$$

between the category of simplicial ΓR-modules and the category of $\mathbb{N}$-differential-graded R-modules.

(4) Let A be a simplicial commutative ring. By remark 3.2.8.(4), the shuffle map ∇, as well as its extension to normalized chain complexes

$$\nabla : \; NK \otimes NL \longrightarrow N(K \otimes L) \qquad \text{for every objects } K, L \text{ in } Obs\mathcal{A}b,$$

are (lax) symmetric monoidal. This is not the case however of the Alexander-Whitney map; that is, AW is not symmetric. Consequently:

- The normalized chain NA forms a differential-graded algebra which is commutative in the sense that
$$xy \;=\; (-1)^{|x||y|}yx,$$
for homogeneous elements x and y in NA.

- The functor $N : s\mathcal{A}b \to \mathcal{C}h^+$ inherits a monoidal structure when viewed as a functor from simplical A-modules with tensor product over A to $\mathbb{N}$-differential-graded NA-modules with tensor product over NA. More precisely there is a unique natural chain complex morphism
$$\nabla^A : NM \otimes_{NA} NM' \longrightarrow N(M \otimes_A M') \qquad \text{for } M, M' \in Ob\mathcal{M}od_A$$
such that the square
$$\begin{tikzcd} NM \otimes NM' \arrow[r, "\nabla"] \arrow[d] & N(M \otimes M') \arrow[d] \\ NM \otimes_{NA} NM' \arrow[r, "\nabla^A"] & N(M \otimes_A M'), \end{tikzcd}$$
where the vertical arrows are the canonical quotient maps, commutes. In addition, since A is commutative, ∇^A becomes a symmetric monoidal functor from A-modules to NA-modules.

Now let $L^A : \mathcal{DGM}od_{NA} \to s\mathcal{M}od_A$ denote the left adjoint of N when viewed as a functor from A-modules to NA-modules. The comonoidal morphism for L^A has the form

$$L^A(W \otimes_{NA} W') \longrightarrow L^A(W) \otimes_A L^A(W') \qquad \text{for } W, W' \in Ob\mathcal{DGM}od_{NA}.$$

Proposition 3.16 in [21] imply that the adjoint pair

$$L^A : \; \mathcal{DGM}od_{NA} \Longleftrightarrow \mathcal{M}od_A : \; N$$

is a weak monoidal Quillen pair which is in fact, by 3.3.6.(2), a Quillen equivalence. Finally, we can apply 3.3.6.(3) to obtain a Quillen equivalence

$$\mathcal{DGA}_{NA} \;=\; \mathcal{M}on_{(\mathcal{M}od_{NA})} \;\cong_Q\; \mathcal{M}on_{(\mathcal{M}od_A)} \;=\; s\mathcal{A}lg_A$$

as desired. $\square$

Chapter 4

Galois theory of commutative rings

The main goal of this chapter is to make the connection, in the generalizing process, between the classical Galois theory of finite extensions of fields and the theory of homotopic Hopf-Galois extensions of chapter 5. This will be done be reinterpreting the classical elements of the Galois correspondence for fields as isomorphisms, before generalizing these isomorphisms to the case of commutative rings. In the next chapter, these isomorphisms will in turn generalize to weak equivalences in a monoidal model category.

We shall start with a short reminder on classical Galois theory, in which we define the classical Galois correspondence and state the fundamental Galois theorem for the case of finite extensions of fields. In section 4.2, we reinterpret these classical elements into algebra isomorphisms which easily generalize to the case of commutative rings. We end the section by establishing different characterizations of the notion of a Galois extension of commutative ring (cf. theorem 4.2.9); these are due to the work of Chase, Harrison and Rosenberg (cf. [4]). In section 4.3 we provide a topological example, that of normal covering spaces. In section 4.4 we return to the theory and show how the Galois extensions of a commutative ring R are preserved and reflected under the functor $T \otimes_R -$ for a faithfully flat R-algebra T. In the last section, we finally proceed to the generalization of the Galois theorem for finite Galois extensions of fields to the case of connected commutative rings.

4.1 Reminder on classical Galois theory

This is just a review of the basic elements of Galois theory for fields. We provide here the definitions and results needed. For more details see [16] or [1].

Reminder 4.1.1. We say that a field L is an *extension* of a field K if K is a subfield of L. In particular, L is a K-vector space whose dimension, the *degree of* L *over* K, is denoted $[L : K]$. We say that the extension $K \subseteq L$ is *finite* if $[L : K]$ is. For $l_1, \ldots, l_n \in L$, we denote

- $K[l_1, \ldots, l_n]$ to be the smallest subring of L containing K and $\{l_1, \ldots, l_n\}$, which equivalently is the ring of all polynomial expressions in $l_1, \ldots, l_n$ with coefficients in K,
- $K(l_1, \ldots, l_n)$ to be the smallest subfield of L containing K and $\{l_1, \ldots, l_n\}$, which equivalently is the field of all rational polynomial expressions in $l_1, \ldots, l_n$ with coefficients in K.

Furthermore, for any field extension $K \subseteq L$ and any $l \in L$, we consider the surjective ring homomorphism $ev : K[X] \to K[l]$ which to each polynomial $f \in K[X]$ associates the element

$f(l)$ in the ring $K[l] \subseteq K(l) \subseteq L$. Since $Ker(ev)$ has to be a prime ideal of $K[X]$ ($Im(ev) = K[l] \cong K[X]/Ker(ev)$ being entire), it can only be of one of the following forms:

- $Ker(ev) = \{0\}$. In this case ev is injective, $K[l] \cong K[X]$, $[K(l) : K] = \infty$, and there is no non-zero polynomial f in $K[X]$ such that $f(l) = 0$. We then say that l is *transcendent* over K.

- $Ker(ev)$ is a (prime) ideal of $K[X]$ generated by an irreductible polynomial in $K[X]$. This means that there exists non-zero polynomials f in $K[X]$ such that $f(l) = 0$. Among them, the polynomial of smallest degree whose leading coefficient is the unit of K is the *minimal polynomial* of l over K. We denote it $min(l, K)$. It follows that $Ker(ev)$ is the principal ideal $< min(l, K) >$ generated by $min(l, K)$, and consequently that $min(l, K)$ is irreductible in $K[X]$. We also have

$$K[l] \;\cong\; \frac{K[X]}{Ker(ev)} \;=\; \frac{K[X]}{< min(l,K) >} \;\cong\; K(l),$$

$$deg(l) \;=\; [K(l) : K] \;=\; deg(min(l, K)),$$

and we say that l is *algebraic* over K.

A field extension $K \subseteq L$ is said to be *algebraic* if every element l of L is algebraic over K. It is worth noting that any finite extension L of K is algebraic, since if L contained a tanscendent element l over K, we would have

$$[L : K] \;\geq\; [K(l) : K] \;=\; \infty.$$

Moreover, an algebraic field extension $K \subseteq L$ is

- *normal* if the roots of $min(l, K)$ are all simple for every l in L,

- *separable* if the minimal polynomials $min(l, K) \in K[X]$ of every l in L factors entirely as polynomials of degree 1 in $L[X]$.

Finally, for a given field extension $K \subseteq L$, a field homomorphism $f : L \to \bar{L}$ into an algebraic closure $\bar{L}$ of L is a *K-homomorphism* if it fixes every element of K, ie. $f(k) = k$ for every k in K. In the special case where $K \subseteq L$ is an algebraic extension, a K-homomorphism is necessary an automorphism of L. We denote the group of K-automorphism of L, with multiplication given by composition, by $Aut_K(L)$; and more generally the group of K-homomorphisms by $End_K(L)$.

Here are two fundamental properties on normal and separable extensions.

Properties 4.1.2.

(1) *If $K \subseteq M \subseteq L$ are field extensions such that L is a normal extension of K, then L is a normal extension of M.*

(2) *If $K \subseteq M \subseteq L$ are field extensions such that L is a separable extension of K, then L is a separable extension of M.*

Proof. (1) By assumption, every l in L has a minimal polynomial $min(l, K)$ in $K[X]$, and since $K[X] \subseteq M[X]$, the extension $M \subseteq L$ is algebraic. In addition, the minimal polynomial $min(l, M)$ of any l in L divides $min(l, K)$ in $M[X]$, and since $min(l, K)$ factors as polynomials of degree 1 in $L[X]$, it follows that $min(l, M)$ does too.

(2) The extension $M \subseteq L$ is algebraic by (1). In addition, the minimal polynomial $min(l, M)$ of any l in L is a factor of $min(l, K)$ in $M[X]$, so that all of its roots in L are distinct. □

The above properties allow to define a Galois extension not only in terms of normality and separability, but also in terms of its Galois correspondence (cf. [1] for more details).

Definition 4.1.3. Let $K \subseteq L$ be a field extension. We say that $K \subseteq L$ is a *Galois extension* if it is normal and separable; or equivalently if we have a *Galois correspondence*

$$\{M \in Ob\mathcal{F}ld \mid K \subseteq M \subseteq L\} \underset{L^{(-)}}{\overset{Gal[L:-]}{\rightleftarrows}} \{H \in Ob\mathcal{G}r \mid H \leq Aut_K(L)\}$$

between the set of intermediate field extensions of $K \subseteq L$ and the set of subgroups of $Aut_K(L)$, where

$$Gal[L : M] := Aut_K(M) \qquad \text{and} \qquad L^H := \{l \in L \mid h(l) = l,\ \forall h \in H\},$$

such that $K = L^G$ for $G := Gal[L : K]$. We call

$$G = Gal[L : K] = Aut_K(L)$$

the *Galois group* of the extension, we say that L is a *G-Galois (field) extension* of K, and we call $K \subseteq L$ a *G-Galois extension (of fields)*.

Remarks 4.1.4. **(1)** 4.1.2 obviously imply that $M \subseteq L$ is a Galois field extension as well, so that $Gal[L : M]$ is well defined as the Galois group of $M \subseteq L$.

(2) The Galois correspondence satisfy

$$L^{Gal[L:M]} = M \qquad \text{and} \qquad Gal[L : L^H] \geq H$$

for an intermediate field extension $K \subseteq M \subseteq L$ and a subgroup H of $Aut_K(L)$. In the special case where the extension $K \subseteq L$ is finite, the second relation becomes an equality.

We now state the fundamental result of classical Galois theory. We shall omit its proof since its generalization to connected commutative rings (cf. 4.5.6) will be established at the end of the chapter.

Theorem 4.1.5 (Galois theorem for fields)**.** *Let $K \subseteq L$ be a finite G-Galois extension of fields.*

(1) *The Galois correspondence is an isomorphism in $\mathcal{S}et$.*

(2) *For every intermediate field extension $K \subseteq M \subseteq L$, we have*

$$[L : M] \ = \ \#Gal[L : M] \ = \ \#Aut_K(L),$$

where the prefix # denotes the cardinality of the group it precedes.

(3) *For every intermediate field extension $K \subseteq M \subseteq L$, we have*

$$f \cdot Gal[L : M] \cdot f^{-1} \ = \ Gal[L : f(M)] \qquad \textit{for any } f \textit{ in } Gal[L : M].$$

(4) *For every intermediate field extension $K \subseteq M \subseteq L$ such that the extension $K \subseteq M$ is normal (and therefore Galois), $Gal[L : M]$ is a normal subgroup of $Gal[L : K]$ and*

$$Gal[M : K] \ \cong \ Gal[L : K]/Gal[L : M].$$

4.2 Galois extensions of commutative rings

We now want to generalize the notion of Galois extension for fields to commutative rings. The simple replacement of all fields by commutative rings in definition 4.1.3 would give rise to a notion that would end up being too weak. Instead of doing this, we shall provide different equivalent definitions which turn out to generalize well. These equivalences, among others we won't be using in this paper, were mainly established by Chase, Harrison and Rosenberg in [4] who used them to develop the Galois theory of commutating rings.

We first start by reinterpreting the classical notion of G-Galois extension, as seen in the previous section, in a way that will easily be expanded to the case of commutative rings. Consider a field extension $K \subseteq L$ and a finite group G such that every element of G is a K-automorphism of L. This gives rise to a G-action

$$G \times L \to L \ : \ (a, l) \mapsto al = a \cdot l := a(l) \qquad \text{with} \qquad a \cdot k = k \qquad \text{for every } k \in K,$$

or equivalently a group homomorphism

$$\varphi : \ G \to Aut_K(L) \ : \ a \mapsto (\varphi(a) : l \mapsto a(l) = a \cdot l) \qquad \text{with} \qquad g|_K = id_K,$$

where of course

$$e_G \cdot l = l \qquad \text{and} \qquad a \cdot (b \cdot l) = (ab) \cdot l \qquad \text{for every } a, b \in G \text{ and } l \in L$$

by definition of a G-action.

Definition 4.2.1. The L-algebra $L{<}G{>}$ is defined to be the L-vector space

$$L{<}G{>} := \bigoplus_{a \in G} La \ = \ \{\sum_{a \in G} x_a \cdot a \mid x_a \in L, \ \forall a \in G\},$$

with addition and scalar multiplication given componentwise, and multiplication given by

$$(\sum_{a \in G} x_a \cdot a)(\sum_{b \in G} y_b \cdot b) \ := \ \sum_{a,b \in G} (x_a \cdot a(y_b)) \cdot ab.$$

With the usual L-algebra structure of $End_K(L)$ given by

- $(f + g)(l) := f(l) + g(l)$ for every f, g in $End_K(L)$ and every l in L,
- $(f \cdot g)(l) := f(l) \cdot g(l)$ for every f, g in $End_K(L)$ and every l in L,
- $l \cdot f(l') := f(ll')$ for every f in $End_K(L)$ and every l, l' in L,

we then have an L-algebra homomorphism

$$j : \ L{<}G{>} \to End_K(L) \ : \ x_a \cdot a \mapsto (l \mapsto x_a \cdot a(l)),$$

which provides L with a structure of $L{<}G{>}$-module.

We shall now establish a necessary and sufficient condition under which the map j is an isomorphism. The proof uses the Dedekind lemma below.

Lemma 4.2.2 (Dedekind). *Let $K \subseteq L$ be a field extension. For any K-algebra A, the set*

$$Alg_K(A, L) = \{f : A \to L \mid f \textit{ is a } K\textit{-algebra homomorphism}\}$$

is a linearly independent subset of the L-vector space

$$Hom_K(A, L) = \{f : A \to L \mid f \textit{ is a } K\textit{-module homomorphism}\},$$

whose structure is given by

- $(f+g)(a) = f(a) + g(a)$ *for every* $a \in A$,
- $(f \cdot g)(a) = f(a) \cdot g(a)$ *for every* $a \in A$,
- $(kf)(a) = f(ka)$ *for every* $a \in A$ *and every* $k \in K$.

Proof. Suppose that $Alg_K(A, L)$ is linearly dependent in $Hom_K(A, L)$, and consider a linear combination

$$\sum_{i=1}^{n} l_i f_i \equiv 0 \qquad \text{with} \quad l_i \in L \quad \text{and} \quad f_i \in Alg_K(A, L)$$

such that

$$i \neq j \;\Rightarrow\; f_i \neq f_j \qquad \text{and} \qquad l_i \neq 0 \qquad \text{for every} \quad i = 1, \ldots, n,$$

in other words such that n is minimal for this dependence relation. Since all f_i's are in $Alg_K(A, L)$, we have

$$f_i(ab) = f_i(a) \cdot f_i(b) \qquad \text{for all } i\text{'s and every } a, b \text{ in } A.$$

It follows that

$$\begin{aligned} \sum_{i=1}^{n-1} (l_i f_i(a) - l_i f_n(a)) \cdot f_i(b) &= \sum_{i=1}^{n} (l_i f_i(a) - l_i f_n(a)) \cdot f_i(b) \\ &= \sum_{i=1}^{n} l_i f_i(ab) - f_n(a) \sum_{i=1}^{n} l_i f_i(b) \\ &= 0 \qquad \text{for every} \quad a, b \in A, \end{aligned}$$

so that by minimality of n we have

$$l_i f_i(a) = l_i f_n(a) \qquad \text{for every} \quad a \in A \quad \text{and} \quad i = 1, \ldots, n.$$

Consequently, $f_i(a) = f_n(a)$ for every $a \in A$ and $i = 1, \ldots, n$ which contradicts our assumption and proves the desired result. □

Proposition 4.2.3. *The map* $j : L < G > \rightarrow End_K(L)$ *is an isomorphism if and only if* G *is embedded into* $Aut_K(L)$ *(via* φ*) and* $K \subseteq L$ *is a* G*-Galois extension.*

Proof. ($\Leftarrow$) Assume that $G \subseteq Aut_K(L)$ and that the extension $K \subseteq L$ is G-Galois; meaning in particular that the group G is finite. Let

$$x = \sum_{a \in G} x_a \cdot a \qquad \text{and} \qquad y = \sum_{a \in G} y_a \cdot a$$

be two elements of $L{<}G{>}$ such that $j(x) = j(y)$. Since j is L-linear, we have

$$\sum_{a \in G} x_a \cdot j(a) \;=\; j(\sum_{a \in G} x_a \cdot a) \;=\; j(\sum_{a \in G} y_a \cdot a) \;=\; \sum_{a \in G} y_a \cdot j(a). \qquad (*)$$

Furthermore, every $j(a)$ for a in G is a K-algebra endomorphism of L. This implies, by 4.2.2, that the set $\{j(a) \mid a \in G\}$ is linearly independent in $End_K(L)$, so that $x_a = y_a$ for every $a \in G$ and consequently $x = y$ according to $(*)$; this proves the injectivity of j.

For the bijectivity, we know that $K = L^G$ for a finite group G. Let $n := \#G$ be the cardinality of G. We then have isomorphisms

$$\begin{aligned} L{<}G{>} &\cong Hom_{L<G>}(L{<}G{>}, L{<}G{>}) \cong Hom_{L<G>}(L^{\otimes n}, L^{\otimes n}) \\ &\cong Hom_{L<G>}(L, L)^{\otimes n^2} \cong [Hom_L(L, L)^G]^{\otimes n^2} \\ &\cong K^{\otimes n^2}, \end{aligned}$$

so that

$$n^2 \;=\; dim_K L{<}G{>} = \; dim_K L \cdot dim_L L{<}G{>} = \; dim_K L \cdot n,$$

and consequently $dim_K L = n$. It follows that

$$dim_K End_K(L) \;=\; n^2 \;=\; dim_K L{<}G{>},$$

so that j is an isomorphism.

($\Rightarrow$) Assume that $j : L < G > \to End_K(L)$ is an isomorphism. If $\varphi : G \to Aut_K(L)$ was not injective, there would exist two distinct elements $a \neq b$ in G such that $a(l) = b(l)$ for every $l \in L$, ie. such that

$$j(a) = j(1_L \cdot a) = j(1_L \cdot b) = j(b);$$

and consequently j would not be injective. From this contradiction, it follows that G embeds in $Aut_K(L)$ via φ.

Now if the extension $K \subseteq L$ wasn't G-Galois, there would exist an $x \in L \backslash K$ such that $g(x) = x$ for every $g \in G$, so that for every $f \in j(L{<}G{>}) \cong End_K(L)$ we would have

$$\begin{aligned} x \cdot f(l) \;&=\; x \cdot (x_a \cdot a(l))_{a \in G} \;=\; (x_a \cdot x \cdot a(l))_{a \in G} \\ &=\; (x_a \cdot a(x) \cdot a(l))_{a \in G} \;=\; (x_a \cdot a(xl))_{a \in G} \\ &=\; f(xl) \qquad \text{for every} \quad l \in L. \end{aligned}$$

This would mean that x commute with every element f of the group $End_K(L)$, in other words that x would be in the center of $End_K(L)$. The latter being just K this would mean that x belongs to K, which contradicts our assumption. The extension $K \subseteq L$ is therefore G-Galois. $\square$

The next step of the construction is to define a map h which turns out to be dual to j. As a consequence, the map h becomes an isomorphism under the same necessary and sufficient condition of 4.2.3.

Definition 4.2.4. The L-algebra $\prod_G L$ is defined to be the set of all maps $f : G \to L$, ie. the set of all sequences $(x_a)_{a \in G}$ in L, endowed with

- addition defined by $(f + g)(a) := f(a) + g(a)$ for every $f, g \in \prod_G L$ and $a \in G$, ie. by

  $$(x_a)_{a \in G} + (y_a)_{a \in G} = (x_a + y_a)_{a \in G}$$

  on sequences,

- multiplication defined by $(f \cdot g)(a) := f(a) \cdot g(a)$ for every $f, g \in \prod_G L$ and $a \in G$, ie. by

  $$(x_a)_{a \in G} \cdot (y_a)_{a \in G} = (x_a y_a)_{a \in G}$$

  on sequences, and

- scalar multiplication defined by $(l \cdot f)(a) := l \cdot f(a)$ for every $f \in \prod_G L$ and $l \in L$, ie. by

  $$l \cdot (x_a)_{a \in G} = (l x_a)_{a \in G}$$

  on sequences.

With the L-algebra structure on $L \otimes_K L$ given by $l(x \otimes y) := lx \otimes y$, where $\otimes_K$ denotes the usual tensor product over the field K, we then have an L-algebra homomorphism

$$h : \; L \otimes_K L \to \prod_G L \;:\; x \otimes y \mapsto (a \mapsto x \cdot a(y))_{a \in G} = (x \cdot a(y))_{a \in G}.$$

Remark 4.2.5. Since G is finite, we have

$$\begin{tikzcd}
L < G > \arrow[d, leftrightarrow, "\cong"'] & & Hom_L(L, Hom_K(L,L)) \cong End_K(L) \arrow[d, leftrightarrow, "\cong"] \\
Hom_L(\prod_G L, L) \arrow[rr, "j=Hom_L(h,L)"] & & Hom_L(L \otimes_K L, L),
\end{tikzcd}$$

so that h is simply the dual of j.

Proposition 4.2.6. *The map $h : L \otimes_K L \to \prod_G L$ is an isomorphism if and only if G is embedded into $End_K(L)$ (via φ) and $K \subseteq L$ is a G-Galois extension.*

Proof. From the duality of 4.2.5, h is an isomorphism if and only if j is. The desired result therefore follows from 4.2.3. □

This new interpretation of G-Galois extensions, via the maps j and h, can easily be generalized to a good notion of G-Galois extension $R \subseteq S$ of commutative rings. We shall also define the trace $Tr : S \to R$ associated to such an extension.

Definition 4.2.7. We say that an inclusion of commutative rings $R \subseteq S$ is an *extension* if R is a subring of S. In this case, the inclusion $inc : R \hookrightarrow S$ is an injective homomorphism of commutating rings which gives S the structure of an R-algebra whose scalar multiplication is given by the ring multiplication. In addition, we define

$$End_R(S) := \{f : S \to S \mid f \text{ is an } R\text{-algebra endomorphism}\},$$
$$Aut_R(S) := \{f : S \to S \mid f \text{ is an } R\text{-algebra automorphism}\},$$

and for a finite subgroup G of $Aut_R(S)$ we have maps

- $i : R \hookrightarrow S^G$ defined to be the inclusion of R into the ring S^G of fixed elements of S under all elements of G, where the elements of R remain fixed under G since

$$f(r) = f(r \cdot 1) = r \cdot f(1) = r \cdot 1 = r$$

  for every f in G and every r in R.

- $j : S{<}G{>} \to End_R(S)$ defined to be the ring homomorphism specified by

$$j(x_a \cdot a)(s) := x_a \cdot a(s), \qquad \text{as in 4.2.1,}$$

  where $S{<}G{>}$ has the S-algebra structure given in 4.2.1,

- $h : S \otimes_R S \to \prod_G S$ defined to be the commutative ring homomorphism specified by

$$h(x \otimes y) := (x \cdot a(y))_{a \in G}, \qquad \text{as in 4.2.4,}$$

  where $\prod_G S$ has the S-algebra structure given in 4.2.4 and $\otimes_R$ denotes the usual tensor product over R.

We then say that $R \subseteq S$ is a *G-Galois extension (of commutative rings)* if i and h are bijective.

Moreover, for two G-Galois extensions S and S' of a commutative ring R, we define a *morphism of G-Galois extensions* to be an R-algebra homomorphism

$$\varphi : S \to S'$$

which is G-equivariant (cf. 2.1.6); in other words such that

$$\varphi(g \cdot s) = g \cdot \varphi(s) \qquad \text{for every} \quad s \in S,\ g \in G.$$

This defines the category $\mathcal{G}al(R, G)$ of G-Galois extensions of a commutative ring R.

Finally, for any G-Galois extension of commutative rings $R \subseteq S$ we may define the *trace*

$$Tr:\ S \to R \qquad \text{by} \qquad Tr(y) := \sum_{g \in G} g(y),$$

which is well defined from the following facts:

- The map $Tr : S \to S$ is clearly well defined.
- Since $R = S^G$ via i, every a in G is R-linear, and consequently Tr is R-linear. Then, for any r in $S^G = R$ we have

  $$Tr(r) \ = \ r \cdot Tr(1_R) \ = \ r \cdot (\underbrace{1_R + \ldots + 1_R}_{|G| \text{ times}}) \ \in \ R,$$

  so that $Tr(S^G) \subseteq R$.
- For any element s in S and any R-algebra automorphism $a \in G$ with $a(s) \neq s$, there is a unique inverse element a^{-1} in G such that

  $$a(s) + a^{-1}(s) \ = \ a(s) - a(s) \ = \ 0_S \ = \ 0_R \ \in \ R,$$

  so that $Tr(S \backslash S^G) \subseteq R$.

Examples 4.2.8. **(1)** From what we have done so far (cf. 4.2.3 and 4.2.6), a G-Galois extension of fields is obviously a special case of a G-Galois extension of commutative rings.

(2) For any commutative ring R, we have a *trivial G-Galois extension* $S = \prod_G R$, on which the action of $G \leq Aut_R(\prod_G R)$ is given by

$$g((x_a)_{a \in G}) = (g(x_a))_{a \in G} \qquad \text{for every} \quad g \in G,\ x_a \in R,$$

so that i and h are bijective since

$$S \otimes_R S \ = \ \prod_G R \otimes_R \prod_G R \ \cong \ \prod_G \prod_G (R \otimes_R R) \ \cong \ \prod_G \prod_G R \ = \ \prod_G S.$$

More generally, any G-Galois extension S of R which is isomorphic to $\prod_G R$ is said to be *trivial.*

The following result is very useful. It provides few equivalences of the notion of G-Galois extensions for commutative rings. Other characterizations can be found in [4].

Theorem 4.2.9. *Let $R \subseteq S$ be an extension of commutative rings, and G a finite subgroup of $Aut_R(S)$ with $e := e_G = id_S$. The following conditions are equivalent:*

(1) *The extension $R \subseteq S$ is G-Galois, ie. the maps $i : R \hookrightarrow S^G$ and $h : S \otimes_R S \to \prod_G S$ are bijective.*

(2) *The maps $i : R \hookrightarrow S^G$ and $h : S \otimes_R S \to \prod_G S$ are surjective.*

(3) *The maps $i : R \hookrightarrow S^G$ and $j : S < G > \to End_R(S)$ are bijective and S is a finitely generated projective R-module.*

(4) *The map* $i : R \hookrightarrow S^G$ *is bijective and there exist* $x_1, \ldots, x_n, y_1, \ldots, y_n$ *in* S *such that*

$$\sum_{i=1}^{n} x_i a(y_i) \;=\; \delta_{a,e} \;=\; \begin{cases} 1, & \text{if} \quad a = e, \\ 0, & \text{if} \quad a \neq e. \end{cases}$$

Proof. $(2 \Leftrightarrow 4)$ The group G naturally acts on $S \otimes_R S$ and $\prod_G S$ by

$$g(x \otimes y) = x \otimes g(y) \qquad \text{and} \qquad g(x_a)_{a \in G} = (x_{ag})_{a \in G}$$

respectively, where G acts on $\prod_G S$ by index shift. It follows that

$$\begin{aligned} h(g(x \otimes y)) &= h(x \otimes g(y)) \;=\; (x \cdot ag(y))_{a \in G} \;=\; g \cdot (x \cdot a(y))_{a \in G} \\ &= g \cdot h(x \otimes y), \end{aligned}$$

ie. h is compatible with the G-action given above. Now suppose that

$$(1_S, 0, \ldots, 0) \in Im(h) \subseteq \prod_G S,$$

where $1 = 1_S$ is at position $e = e_G$. There exists then an element $\sum_i x_i \otimes y_i$ in $S \otimes_R S$ such that

$$\begin{aligned} h(\sum_i x_i \otimes y_i) &= \sum_i h(x_i \otimes y_i) \;=\; \sum_i (x_i \cdot a(y_i))_{a \in G} \\ &= (1, 0, \ldots, 0); \end{aligned}$$

and consequently

$$\sum_i (x_i \cdot a(y_i)) \;=\; \begin{cases} 1 & \text{if } a = e, \\ 0 & \text{if } a \neq e. \end{cases}$$

From the compatibility of the G-action we have

$$h(g(\sum_i x_i \otimes y_i)) \;=\; g(h(\sum_i x_i \otimes y_i)) \;=\; (0, \ldots, 0, \underbrace{1}_{g\text{th place}}, 0, \ldots, 0).$$

This implies that for any element $(z_a)_{a \in G}$ in $\prod_G S$ we have

$$\begin{aligned} h(\sum_{a \in G} z_a \cdot g(\sum_i x_i \otimes y_i)) &= \sum_{a \in G} z_a \cdot h(g(\sum_i x_i \otimes y_i)) \\ &= \sum_{a \in G} z_a \cdot (0, \ldots, 0, \underbrace{1}_{g\text{th place}}, 0, \ldots, 0) \\ &= (z_a)_{a \in G}, \end{aligned}$$

so that h is surjective. Conversely, if h is surjective then $(1, 0, \ldots, 0)$ is clearly an element of $Im(h)$. In addition, the equivalence between the surjectivity of i and the bijectivity of i is clear.

$(1 \Rightarrow 2)$ This is trivial.

$(2 \Rightarrow 3)$ We can assume (4). Let's first show that S is a finitely generated projective R-module. Consider the trace

$$Tr : \; S \to R \qquad \text{defined by} \qquad Tr(x) := \sum_{g \in G} g(x) \quad \text{as in 4.2.7,}$$

and define the maps

$$\varphi_i : \; S \longrightarrow R \qquad \text{by} \qquad \varphi_i := Tr(zy_i) \qquad \text{for every} \quad i = 1, \ldots, n.$$

We then have

$$\begin{aligned}\sum_{i=1}^{n} x_i \varphi_i(z) &= \sum_{i=1}^{n} x_i \cdot Tr(zy_i) = \sum_{i=1}^{n}\sum_{a\in G} x_i \cdot a(z)\cdot a(y_i) \\ &= \sum_{i=1}^{n} a(z) \sum_{a\in G} x_i \cdot a(y_i) = \sum_{i=1}^{n} a(z)\cdot \delta_{a,e} \\ &= e(z) = z, \qquad \text{for every} \quad z \in S,\end{aligned}$$

so that the pair $(x_i, \varphi_i)_{i=1}^n$ forms a dual basis for S. By the dual basis lemma (cf. for example [15] Lemma 2.9), the R-module S is finitely generated projective.

Now by localization, we may assume that S is a finitely generated free R-module with basis $x'_1, \ldots, x'_n$ (cf. [16] section X.4 theorem 4.4); and since we can write the preimage of $(1, 0, \ldots, 0)$ under h by

$$\sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n} x'_i \otimes y'_i \qquad \text{for appropriate } y'_i \text{ 's in } S,$$

we may omit the primes and consider that the x_i 's form a basis for S. From the above calculation we get

$$x_j = \sum_{i=1}^{n} x_i \cdot \varphi_i(x_j),$$

so that

$$Tr(x_j y_i) = \varphi_i(x_j) = \delta_{i,j} \tag{$*$}$$

since $(\varphi_i)_{i=1}^n$ is the dual basis of $(x_j)_{j=1}^n$. Consider the matrices

$$A := (a(x_i))_{a,i} \qquad \text{and} \qquad B := (b(y_j))_{j,b}.$$

Then

$$AB = \left(\sum_{i=1}^{n} a(x_i) b(y_i)\right)_{a,b} = (\delta_{a,b})_{a,b} = I$$

by (4), and

$$\begin{aligned} BA &= \left(\sum_{a\in G} a(x_i) a(y_j)\right)_{i,j} = \left(\sum_{a\in G} a(x_i y_j)\right)_{i,j} \\ &= (Tr(x_i y_j))_{i,j} = (\delta_{i,j})_{i,j} = I \end{aligned}$$

by $(*)$. This means that A is an invertible matrix, and consequently that the map $j : S\!<\!G\!>\to End_R(S)$ is an isomorphism.

$(3 \Rightarrow 1)$ Since S is a finitely generated projective R-module, we may again assume that S is free over R with basis $x_1, \ldots, x_n$ as above. Again, the map $j : S\!<\!G\!>\to End_R(S)$ is bijective if and only if the matrix

$$A = (a(x_i))_{a,i}$$

is invertible. This, by duality, is equivalent to the bijectivity of the map $h : S \otimes_R S \to \prod_G S$. $\square$

4.3 The example of normal covering maps

We detail here a topological example of the newly established notion of Galois extension.

Definition 4.3.1. Let $p : Y \to X$ be an epimorphism in $\mathcal{T}op$. An open set U of X is said to be *evenly covered* by p if its inverse image $p^{-1}(U)$ can be written as a disjoint union

$$p^{-1}(U) \;=\; \coprod_{\alpha \in I} V_\alpha$$

of open sets V_α in Y, such that the restrictions

$$\{p|_{V_\alpha} : V_\alpha \to U\}_{\alpha \in I}$$

are homeomorphisms, ie. isomorphisms in $\mathcal{T}op$. The collection $\{V_\alpha\}_{\alpha\in I}$ is then a partition of $p^{-1}(U)$ into *slices.* If every point x of X has a neighborhood U_x which is evenly covered by p, then p is a *covering map* and Y is the *covering space* of X.

Moreover, two covering maps $p : Y \to X$ and $p' : Y \to X$ between the same spaces Y and X are said to be *equivalent* if there exists an homeomorphism $h : Y \to Y$ such that the diagram

$$\begin{tikzcd} & Y \arrow[d, "p'"] \\ Y \arrow[ur, "h"] \arrow[r, "p"'] \arrow[ur, phantom, "\cong"', shift right=2] & X \end{tikzcd}$$

commutes; this clearly induces an equivalence relation on the set of all covering maps from Y to X. We denote by $G(p)$ the group of all homeomorphisms $h : Y \to Y$ involved in the equivalent class of the covering map $p : Y \to X$, whose multiplication is given by composition; $G(p)$ is the group of *deck transformations* of p.

Finally, a covering map (or space) $p : Y \to X$ is said to be *normal*, or sometimes *regular*, if for every element x in X and every pair of elements y and y' in $p^{-1}(x) \subseteq Y$ there is a deck transformation h in $G(p)$ such that $h(y) = y'$.

Remarks 4.3.2. (1) If $p : Y \to X$ is a covering map, then for each x in X the subspace $p^{-1}(x)$ of Y is discrete since each slice V_α of $p^{-1}(U_x)$ is open in Y and intersects the set $p^{-1}(x)$ in a single point.

(2) By the unique lifting property of a covering map $p : Y \to X$, ie. the fact that a commutative square

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r, "f"'] \arrow[ur, dashed, "\exists! f'"] & X \end{tikzcd}$$

of morphisms in $\mathcal{T}op$ with Z path connected, has a unique lift $f' : Z \to Y$ (cf. [11] proposition 1.34), a deck transformation $h : Y \to Y$ is completely determined by where it sends a single point assuming that Y is path connected. In particular, only the identity deck transformation, ie. the unit element in $G(p)$, can fix a point of Y.

Example 4.3.3. The map

$$p :\ \mathbb{R} \to \mathbb{S}^1 \ :\ x \mapsto e^{2\pi i x}$$

is a normal covering map. As even covering of $\mathbb{S}^1$, we can take

$$\mathcal{U} = \{\ \mathbb{S}^1\backslash\{1\},\ \mathbb{S}^1\backslash\{-1\}\ \}.$$

The inverse images of its elements are

$$p^{-1}(\mathbb{S}^1\backslash\{1\}) = \bigcup_{n\in\mathbb{Z}}(n, n+1) \qquad \text{and} \qquad p^{-1}(\mathbb{S}^1\backslash\{-1\}) = \bigcup_{n\in\mathbb{Z}}(n-1/2, n+1/2),$$

so that the restrictions

$$p|_{(n,n+1)} : (n, n+1) \to \mathbb{S}^1\backslash\{1\} \qquad \text{and} \qquad p|_{(n-1/2,n+1/2)} : (n-1/2, n+1/2) \to \mathbb{S}^1\backslash\{-1\}$$

are effectively homeomorphisms. For this covering map, which projects a vertical helix onto a circle, the deck transformations are the vertical translations taking the helix onto itself; more formally these are the homeomorphisms

$$h : \mathbb{R} \to \mathbb{R} : x \mapsto x + n \qquad \text{with} \quad n \in \mathbb{Z},$$

so that $G(p) \cong \mathbb{Z}$ in this case. The normality of p is then clear.

The term "normal" for covering spaces is motivated by the following result (cf. [11] proposition 1.39) that we won't be needing here.

Proposition 4.3.4. *Let $p : (Y, y_0) \to (X, x_0)$ be a covering map from path-connected covering pointed space (Y, y_0) to a path-connected and locally path-connected pointed space (X, x_0). Let H be the subgroup*

$$H \;:=\; \pi_1 p(\pi_1(Y, y_0)) \;\leq\; \pi_1(X, x_0),$$

where $\pi_1 : \mathcal{T}op_* \to \mathcal{G}r$ denotes the usual fundamental group functor from the category of pointed spaces to the category of groups.

(1) *The covering map p is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$.*

(2) *The group $G(p)$ is isomorphic to the quotient $N(H)/H$, where $N(H)$ denotes the normalizer of H in $\pi_1(X, x_0)$.*

In particular, if p is normal then $G(p)$ is isomorphic to $\pi_1(X, x_0)$.

Definition 4.3.5. Given an action of a group G on a space Y, we can form the quotient space Y/G in which each point y of Y is identified with every element of its orbit

$$Gy = \{g \cdot y \mid g \in G\}.$$

The topological space Y/G is the *orbit space* of the action, to which we associate its *orbit projection*

$$\pi : Y \to Y/G \qquad \text{defined by} \quad \pi(y) = Gy.$$

Proposition 4.3.6. *Let $p : Y \to X$ be a covering map from a path-connected covering space.*

(1) *The action of the deck transformation group $G(p)$ on Y is such that every $y \in Y$ has a neighborhood U which satisfy*

$$g_1(U) \cap g_2(U) \neq \emptyset \quad \Rightarrow \quad g_1 = g_2 \qquad \textit{for every } g_1, g_2 \in G(p);$$

in other words all the images $g(U)$ for $g \in G(p)$ are disjoint in Y.

(2) *The associated orbit projection $\pi : Y \to Y/G(p)$ is a normal covering map whose group of deck transformations is $G(p)$.*

Proof. (1) Let y be an element of Y and U be a neighborhood of y in Y which project homeomorphically via p to an open set V in X. If $g_1(U) \cap g_2(U) \neq \emptyset$ for some deck transformations $g_1, g_2 \in G(p)$, then there exist elements x_1, x_2 in U such that $g_1(x_1) = g_2(x_2)$. Since x_1 and x_2 must lie in the same set $p^{-1}(x)$ for an x in X, and that $p^{-1}(x)$ intersects U in only one point, we must have $x_1 = x_2$ in U. It follows that $g_1^{-1}g_2$ fixes the point $x_1 = x_2$, so that $g_1^{-1}g_2 = id$ by 4.3.2.(2), and consequently $g_1 = g_2$.

(2) Let $U \subseteq Y$ be as in (1). Then $\pi : Y \to Y/G(p)$ simply identifies all the disjoint homeomorphic sets $\{g(U)\}_{g\in G(p)}$ to a single open set $\pi(U)$ in $Y/G(p)$. By definition of the quotient topology on $Y/G(p)$, all restrictions

$$\pi|_{g(U)} :\ g(U) \longrightarrow p(U) \qquad \text{for} \quad g \in G(p)$$

are homeomorphisms, so that π is a covering space. Furthermore, each element of $G(p)$ acts as a deck transformation for π, so that $G(p) \subseteq G(\pi)$. In addition, the covering map π is normal since for every elements g_1, g_2 in $G(p)$ the map $g_2g_1^{-1}$ takes $g_1(U)$ to $g_2(U)$.

It remains to show the inclusion $G(\pi) \subseteq G(p)$. If f is an element of $G(\pi)$, then for any y in Y the points y and $f(y)$ are in the same orbit. There is then a deck transformation $g \in G(p)$ such that $g(y) = f(y)$. Since Y is path-connected, it follows by 4.3.2.(2) that $f = g \in G(p)$, and consequently $G(\pi) \subseteq G(p)$. □

Proposition 4.3.7. *Let $p : Y \to X$ be a normal covering map from a path-connected covering space whose group of deck transformations $G(p)$ is finite, and consider the canonical morphism*

$$\xi :\ Y \times G(p) \longrightarrow Y \times_X Y \ :\ (y, g) \mapsto (y, g(y))$$

in $\mathcal{T}op$ induced by the pullback diagram

$$\begin{tikzcd}
Y \times G(p) \arrow[drr, bend left, "\xi_2"] \arrow[dr, dashed, "\xi"] \arrow[ddr, bend right, "\xi_1"'] & & \\
& Y \times_X Y \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\
& Y \arrow[r, "p"'] & X,
\end{tikzcd}$$

where

- *$G(p)$ is given the discrete topology,*
- *$\xi_1 : (y, g) \mapsto y$ the canonical projection, and*
- *$\xi_2 : (y, g) \mapsto y \cdot g$ the right action of $G(p)$ on Y.*

Then ξ is an homeomorphism, ie. an isomorphism in $\mathcal{T}op$.

Proof. From the fact that the covering map p is normal, it follows that ξ is surjective. In addition,

$$\begin{aligned}
\xi(y, g) = \xi(y', g') \quad &\Leftrightarrow \quad (y, g(y)) = (y', g'(y')) \\
&\Leftrightarrow \quad y = y' \text{ and } g(y) = g'(y) \\
&\Leftrightarrow \quad y = y' \text{ and } g = g',
\end{aligned}$$

where the last equivalence is a consequence of 4.3.2.(2) and the path-connectedness of Y. The morphism ξ is therefore injective. □

Construction 4.3.8. Let $p : Y \to Z$ be a covering map whose covering space Y is a path-connected compact Hausdorff space and whose group of deck transformations $G = G(p)$ is finite. From 4.3.6 we know that its associated orbit projection $\pi : Y \to X := Y/G$ is a normal covering map whose group of deck transformations is G, and from 4.3.7 that the induced morphism

$$\xi : Y \times G \longrightarrow Y \times_X Y$$

in $\mathcal{T}op$, with G endowed with the discrete topology, is an isomorphism.

Dually, let $R := C(X)$ and $S := C(Y)$ be the commutative rings of real continuous functions on X and Y respectively. From a standard result of algebra, we have isomorphisms

$$X \;\cong\; \{\text{maximal ideals of } R\} \qquad \text{and} \qquad Y \;\cong\; \{\text{maximal ideals of } S\}.$$

The group G acts on S from the left by

$$G \times S \to S \;:\; (g, s) \mapsto g * s,$$

where

$$g * s : \; Y \to \mathbb{R} \qquad \text{is defined by} \qquad (g * s)(y) := s(g(y)),$$

and the canonical map $C(\pi) : R \to T$ dual to π identifies R with the invariant ring S^G via the isomorphism

$$S^G \;=\; C(Y)^G \;\cong\; C(Y/G) \;=\; C(X) \;=\; R.$$

The map ξ is therefore dual to the canonical homomorphism

$$h : \; S \otimes_R S \to \prod_G S \;:\; s \otimes s \mapsto (g \mapsto s \cdot g(t)) = (s \cdot (g * t))_{g \in G}$$

via

$$\begin{tikzcd}
 & C(Y) \arrow[d] \arrow[dr] & & \\
C(X) \arrow[ur, "C(\pi)"] \arrow[dr, "C(\pi)"'] & C(Y) \otimes_{C(X)} C(Y) \arrow[r, dashed, "\exists!"] & C(Y \times_X Y) \arrow[r, "C(\xi)"] & C(G \times Y) \\
 & C(Y) \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

since

$$C(Y) \otimes_{C(X)} C(Y) = S \otimes_R S \qquad \text{and} \qquad C(G \times Y) = C(\prod_G Y) \cong \prod_G C(Y) = \prod_G S.$$

Finally, since ξ is an isomorphism, the morphism h is an isomorphism via the antiequivalence between the category of compact Hausdorff spaces and the category of commutative rings.

4.4 Faithful flatness

In this section, we shall study under which condition the functor $T \otimes_R -$, for an R-algebra T, preserves and reflects a G-Galois extension $R \subseteq S$. This condition is the faithful flatness of T.

Definition 4.4.1. Let R be a commutative ring. We say that an R-module M is *flat* over R if the functor

$$M \otimes_R - : \ \mathcal{M}od_R \to \mathcal{M}od_R$$

is *exact*, ie. it preserves exact sequences. On the other hand, we say that M is *faithful* over R if the above functor reflects exact sequences, ie. the inverse image of an exact sequence is exact; or equivalently if $M/\mathfrak{m}M \neq 0$ for every maximal ideal $\mathfrak{m}$ of R. If these two conditions are satisfied for M, we say that M is *faithfully flat* over R.

Lemma 4.4.2. *Let R be a commutative ring, M a faithfully flat R-module, and $\varphi : A \to B$ an homomorphism of R-modules. Then φ is an isomorphism if and only if the induced morphism*

$$M \otimes_R \varphi : \ M \otimes_R A \to M \otimes_R B$$

is an isomorphism. This statement remains true if we replace both occurrences of "isomorphism" by either "monomorphism" or "epimorphism".

Proof. An homomorphism of R-modules $\varphi : A \to B$ is an isomorphism if and only if we have an exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0. \tag{$*$}$$

This sequence is sent by the functor $M \otimes_R -$ to the sequence

$$0 \longrightarrow M \otimes_R A \xrightarrow{M\otimes_R\varphi} M \otimes_R B \longrightarrow 0, \tag{$**$}$$

which is in turn exact by the flatness of M, so that $M \otimes_R \varphi$ is an isomorphism. To show the converse, it suffices to use the same argument in reverse order, using this time the faithfulness of M.

In order to show the last statement, we can proceed the same way after replacing the sequences $(*)$ and $(**)$ by respectively the sequences

$$0 \longrightarrow A \xrightarrow{\varphi} B \qquad \text{and} \qquad 0 \longrightarrow M \otimes_R A \xrightarrow{M\otimes_R\varphi} M \otimes_R B$$

for monomorphisms, and the sequences

$$A \xrightarrow{\varphi} B \longrightarrow 0 \qquad \text{and} \qquad M \otimes_R A \xrightarrow{M\otimes_R\varphi} M \otimes_R B \longrightarrow 0$$

for epimorphisms. □

Proposition 4.4.3. *Let R be a commutative ring, T an R-algebra which is faithfully flat as an R-module, and let S be a commutative ring extension of R endowed with a G-action for a finite group G which acts on S by R-automorphisms. If $T \otimes_R S$ is a G-Galois extension of $T \cong T \otimes_R R$, then S is a G-Galois extension of R.*

Proof. Since T is flat, the canonical injection of commutative rings

$$0 \hookrightarrow R \xhookrightarrow{inc} S$$

induces by 4.4.2 an injection of R-algebras

$$0 \hookrightarrow T \otimes_R R \xhookrightarrow{T\otimes inc} T \otimes_R S,$$

so that the R-algebra $T = T \otimes_R R$ can be considered as a subalgebra of $T \otimes_R S$. Consider the map $h : S \otimes_R S \to \prod_G S$ associated with the commutative ring extension $R \subseteq S$, and the isomorphism

$$h_T : \ (T \otimes_R S) \otimes_T (T \otimes_R S) \longrightarrow \textstyle\prod_G (T \otimes_R S)$$

associated with the G-Galois extension $T \subseteq T \otimes_R S$. Notice that $h_T \cong T \otimes_R h$ since

$$T \otimes_R (S \otimes_R S) \ \cong \ (T \otimes_R S) \otimes_R S \ \cong \ (T \otimes_R S) \otimes_T (T \otimes_R S)$$

and

$$T \otimes_R \prod_G S \ \cong \ \prod_G (T \otimes_R S).$$

This, with 4.4.2, implies that h is an isomorphism. In addition, we have

$$T \otimes_R R \ \cong \ T \ \cong \ (T \otimes_R S)^G \ \cong \ T^G \otimes_R S^G \ = \ T \otimes_R S^G$$

since $T \subseteq T \otimes_R S$ is a G-Galois extension. It follows that we have an isomorphism

$$T \otimes_R R \xrightarrow{\cong} T \otimes_R S^G$$

which, by 4.4.2, induces an isomorphism $R \cong S^G$. $\square$

Lemma 4.4.4. *Any G-Galois extension of commutative rings $R \subseteq S$ is faithfully flat over R.*

Proof. By 4.2.9.(3), S is a projective R-module so that the functor $S \otimes_R -$ is exact and consequently S is flat over R (cf. for example [18] section 2.4).

Now let $\mathfrak{m}$ be a maximal ideal of R, and define

$$R_{\mathfrak{m}} := (R \backslash \mathfrak{m})^{-1} R \qquad \text{and} \qquad S_{\mathfrak{m}} := (S \backslash \mathfrak{m})^{-1} S$$

to be localizations of R and S respectively (cf. for example [16] section II.4). We then have $R_{\mathfrak{m}} \subseteq S_{\mathfrak{m}}$, and since $R_{\mathfrak{m}} \neq \{0\}$ it follows that $S_{\mathfrak{m}} \neq \{0\}$. This, by Nakayama Lemma (cf. for example [16] section X.4), implies that $S/\mathfrak{m}S \neq \{0\}$; or equivalently that S is faithful over R. $\square$

Lemma 4.4.5. *Let $R \subseteq S$ be a G-Galois extension of commutative rings, and consider the trace*

$$Tr : \ S \to R \qquad \textit{defined by} \qquad Tr(y) := \sum_{g \in G} g(y) \quad \textit{as in 4.2.7.}$$

Then:

(1) *The trace $Tr : S \to R$ is surjective.*

(2) *The R-submodule R of S is a direct summand of S.*

Proof. (1) Consider the diagonal ring

$$\Delta(S) := \{(x_a)_{a \in G} \in \prod_G S \mid x_a = x_b, \ \forall a, b \in G\}.$$

We have a commutative diagram

$$\begin{tikzcd} S \otimes_R S \arrow[r, "h", "\cong"'] \arrow[d, "S \otimes_R Tr"'] & \prod_G S \arrow[d, "Tr_S"] \\ S \otimes_R R \arrow[r, "k", "\cong"'] & \Delta(S), \end{tikzcd}$$

where h is an isomorphism by assumption, where the map $k : s \otimes r \mapsto (sr, \ldots, sr)$ is an isomorphism via the obvious isomorphisms

$$S \otimes_R R \;\cong\; S \;\cong\; \Delta(S),$$

and where

$$Tr_S : \; (x_a)_{a \in G} \longmapsto \textstyle\sum_{g \in G} g(x_a)_{a \in G} = \sum_{g \in G} (g(x_a))_{a \in G}$$

is the trace associated to the trivial G-Galois extension $S \subseteq \prod_G S$ (cf. 4.2.8.(2)) which, by the way G acts on $\prod_G S$, gives

$$Tr_S(x, 0, \ldots, 0) \;=\; (x, \ldots, x) \qquad \text{for every} \quad x \in S.$$

In particular, Tr_S is surjective, so that $S \otimes_R Tr$ is surjective as well via the isomorphisms h and k. Now from 4.4.4 we know that S is faithfully flat over R, which means that Tr is surjective.

(2) We have a short exact sequence

$$0 \longrightarrow R \xrightarrow{inc} S \longrightarrow M \longrightarrow 0, \tag{$*$}$$

and by (1) we can choose an element c in S such that $Tr(c) = 1_R$. Define the homomorphism

$$r : \; S \longrightarrow R \qquad \text{by} \qquad r(x) = Tr(cx).$$

Since

$$\begin{aligned}
(r \circ inc)(x) \;&=\; r(x) \;=\; Tr(cx) \;=\; \sum_{a \in G} a(cx) \;=\; \sum_{a \in G} a(c)a(x) \\
&=\; \sum_{a \in G} a(c)x \;=\; (\sum_{a \in G} a(c)) \cdot x \;=\; Tr(c) \cdot x \;=\; x \\
&=\; id(x),
\end{aligned}$$

it follow that r is a retraction in $(*)$, so that $S \cong R \oplus M$ as desired. □

Proposition 4.4.6. *Let $R \subseteq S$ be a G-Galois extension of commutative rings, and T an R-algebra which is faithfully flat as an R-module. Then $T \subseteq T \otimes_R S$ is again a G-Galois extension.*

Proof. We already saw in the proof of 4.4.3 that the map

$$h_T : \; (T \otimes_R S) \otimes_S (T \otimes_R S) \longrightarrow \textstyle\prod_G T \otimes_R S$$

is an isomorphism since $h_T \cong T \otimes_R h$ and $h : S \otimes_R S \to \prod_G S$ is an isomorphism by assumption. By 4.4.5.(2), there exists an R-module M such that $R \oplus M \cong S$, so that

$$T \otimes_R S \;\cong\; T \otimes_R (R \oplus M) \;\cong\; (T \otimes_R R) \oplus (T \otimes_R M) \;\cong\; T \oplus (T \otimes_R M).$$

This means that T is a direct summand of $T \otimes_R S$ as well. In particular T embeds into $T \otimes_R S$.

Now consider the trace $Tr : S \to R$ and choose an element c in S such that $Tr(c) = 1_R$ as in 4.4.5. Let x be an element of $(T \otimes_R S)^G$. Then

$$\begin{aligned}
x \;&=\; (T \otimes_R Tr)(1_R \otimes c) \cdot x \;=\; \sum_{a \in G} (1_R \otimes a(c)) \cdot x \\
&=\; \sum_{a \in G} (T \otimes_R a)[(1_R \otimes c) \cdot x] \;=\; (T \otimes_R Tr)[(1_R \otimes c)x] \\
&\in\; Im(T \otimes_R Tr) \;=\; T \otimes_R R \;=\; T,
\end{aligned}$$

so that the map $i_T : T \to (T \otimes_R S)^G$ is surjective. The desired result now follows from 4.2.9. □

Conclusion 4.4.7. We can summarize 4.4.3 and 4.4.6 in the following way:

Let $R \subseteq S$ be an extension of commutative rings, G a finite group which acts on S by R-automorphisms, and let T be an R-algebra which is faithfully flat as an R-module. Then S is a G-Galois extension of R if and only if $T \otimes_R S$ is a G-Galois extension of $T \cong T \otimes_R R$.

4.5 Galois correspondence for commutative rings

In this section, we establish the main result of the Galois theory of commutative rings for the case of *connected* commutative rings (cf. definition 4.5.3) as developed in [4]. Because this result should be a generalization of 4.1.5, it would be tempting to find a bijection

$$\{\text{subgroups of } G\} \longleftrightarrow \{R\text{-subalgebras of } S\}$$

for any G-Galois extension of commutative rings $R \subseteq S$, where S has the canonical R-algebra structure given by the inclusion $R \hookrightarrow S$. This however is not possible. To see why, it suffices to consider the example where

$$R = \mathbb{Z} \qquad \text{and} \qquad S = \prod_G R = R \times R = \mathbb{Z} \times \mathbb{Z}$$

is the trivial G-Galois extension with a group G of cardinality two: While the commutative ring S has an infinite number of R-algebras, the group G only has two subgroups (namely G and $\{e_G\}$). In order to obtain a satisfying result, we need a more general notion of separability (than the one given in 4.1.1).

Definition 4.5.1. Let R be a commutative ring. An R-algebra S is *separable over R* if S is projective as an $(S \otimes_R S)$-module whose structure is given by

$$(s \otimes t) \cdot x := sxt \qquad \text{for} \quad s, t, x \in S \quad \text{with} \quad (s \otimes t) \in (S \otimes_R S).$$

Example 4.5.2. If $K \subseteq L$ is a finite field extension, then L is a separable K-algebra if and only if the extension $K \subseteq L$ is separable in the sense of 4.1.1.

Definition 4.5.3. An element e of a ring is said to be *idempotent* if $e^2 = e$. Furthermore, we say that a ring is *connected* when its only idempotent elements are its units 0 and 1.

In order for theorem 4.5.6 to fully generalize 4.1.5, it is essential to check that every field is connected. The following proposition guaranties that this is indeed the case.

Proposition 4.5.4. *If a ring is not connected, then it is not entire. In particular, every field is connected.*

Proof. Let e be an idempotent element of a ring R which is not zero or one. Since $e^2 = e$, we have $e(1-e) = 0$. From the fact that $1 \neq e \neq 0$ it follows that $e \neq 0 \neq 1-e$ so that the elements e and $1-e$ are zero divisors of R, ie. R is not entire. The second assertion follows from the fact that a field is entire by definition. $\square$

Using yet another characterization of a Galois extension provided by [4] (cf. [4] thm 1.3) which we state below (cf. 4.5.5), we may extend theorem 4.1.5 to connected commutative rings.

Lemma 4.5.5. *Let $R \subseteq S$ be an extension of commutative rings, and let G be a finite subgroup of $Aut_R(S)$ such that $S^G = R$. Then $R \subseteq S$ is a G-Galois extension if and only if S is separable over R and, for any non-zero idempotent element $e \in S$ and any $a, b \in G$ with $a \neq b$, there exists an element s in S such that*

$$e \cdot a(s) \;\neq\; e \cdot b(s).$$

Theorem 4.5.6 (Galois theorem for connected commutative rings)**.** *Let $R \subseteq S$ be a G-Galois extension of commutative rings with S connected.*

(A) *Let $H \leq G$ be a subgroup of G, and let $U := S^H$ be the subalgebra of H-invariant elements of S. Then*

- **(1)** *U is separable over R,*
- **(2)** *S is an H-Galois extension of U,*
- **(3)** *$H = \{g \in G \mid g(u) = u, \ \forall u \in U\}$,*
- **(4)** *if H is a normal subgroup of G, then U is a G/H-Galois extension of R.*

(B) *Conversely, if R is connected and $U \subseteq S$ is a separable R-subalgebra of S, then there exists a subgroup H of G such that*

$$U = S^H \qquad \text{and} \qquad H = \{g \in G \mid g(u) = u, \ \forall u \in U\}.$$

Proof. (A.2) Since S is a G-Galois extension of R, 4.2.9.(4) is satisfied and we can choose elements $x_1, \dots, x_n, y_1, \dots, y_n$ in S such that

$$\sum_{i=1}^{n} x_i a(y_i) \;=\; \delta_{a,e} \qquad \text{with } a \in G \text{ and } e = e_G.$$

Obviously, this equality remains true for every a in H. This, with the fact that $U = S^G$, implies that the corresponding condition 4.2.9.(4) for the extension $U \subseteq S$ is satisfied and theorem 4.2.9 makes it then an H-Galois extension.

(A.1) By (A.2) and 4.2.9.(3), the U-module S is projective, so that $S \otimes_R S$ is projective over $U \otimes_R U$. Applying (A.2) and 4.5.5, it follows that S is projective over $S \otimes_R S$ and consequently over $U \otimes_R U$. By 4.4.5.(2), U is a direct summand of S as a U-module, and consequently as a $U \otimes_R U$-module. This means that U is projective over $U \otimes_R U$.

(A.3) Let

$$H' \;:=\; \{g \in G \mid g(u) = u, \ \forall u \in U\}.$$

Since $U = S^H$ by definition, the inclusion $H \subseteq H'$ is clear. Furthermore, by definition of H' we have

$$s^{H'} \;=\; U \;=\; S^H.$$

Applying (A.2) to (U, H) and (U, H'), we obtain an H-Galois extension S of U and an H'-Galois extension S of U, so that

$$\prod_H S \;\cong\; S \otimes_U S \;\cong\; \prod_{H'} S$$

via the corresponding maps

$$h_H : \; S \otimes_U S \longrightarrow \prod_H S \qquad \text{and} \qquad h_{H'} : \; S \otimes_U S \longrightarrow \prod_{H'} S.$$

This force H and H' to have the same cardinality, ie. $H = H'$.

(A.4) Suppose that H is a normal subgroup of G. The quotient group G/H acts on $U = S^H$ by
$$aH \cdot u \;=\; a(u) \qquad \text{with } a \in G \text{ and } u \in U,$$
since every element of U remains fixed under H, and therefore
$$U^{G/H} \;=\; U^G \;=\; (S^H)^G \;=\; S^G \;=\; R. \tag{$*$}$$
By 4.2.9, there are elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ in S such that
$$\sum_{i=1}^{n} x_i a(y_i) \;=\; \delta_{a,e} \qquad \text{for every } a, e \in G \text{ with } e = e_G.$$
Let $c \in S$ be a preimage of $1 = 1_R$ via the trace $Tr : S \to R$ (cf. 4.4.5), so that
$$\sum_{a \in H} a(c) \;=\; 1,$$
and define
$$x_i' \;:=\; \sum_{a \in H} a(x_i c) \qquad \text{and} \qquad y_i' \;:=\; \sum_{a \in H} a(y_i) \qquad \text{for} \quad i \leq n.$$
Then all x_i' 's and y_i' 's are in $S^H = U$ since
$$\begin{aligned} h(x_i') \;&=\; \sum_{a \in H} ha(x_i c) \;=\; \sum_{a \in H} a(x_i c) \;=\; x_i', \\ \text{and} \qquad h(y_i') \;&=\; \sum_{a \in H} ha(y_i) \;=\; \sum_{a \in H} a(y_i) \;=\; y_i'. \end{aligned}$$
A direct calculation then shows that for any $g \in G$
$$\sum_{i=1}^{n} x_i' g(y_i') \;=\; \begin{cases} 1 & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$
This implies that the R-algebra U with acting group G/H would satisfy 4.2.9.(4) if the map
$$i_U : \; R \hookrightarrow U^{G/H}$$
was bijective. This however is a direct consequence of $(*)$, so that U is a G/H-Galois extension of R by 4.2.9.

(B) The argument uses the theory of separability, which we don't want to develop here, in an essential way. A proof of this result can be found in [4] as theorem 2.2.a. $\square$

Conclusion 4.5.7. For any G-Galois extension of commutative rings $R \subseteq S$ with S connected, we have a bijection
$$\{\text{subgroups of } G\} \longleftrightarrow \{\text{separable } R\text{-subalgebras of } S\},$$
with correspondences
$$H \longmapsto S^H \qquad\qquad G^U \longleftarrow\!\shortmid\; U,$$
where
$$\begin{aligned} S^H &= \{s \in S \mid h(s) = s, \ \forall h \in H\}, \\ G^U &= \{g \in G \mid g(u) = u, \ \forall u \in U\}. \end{aligned}$$

This correspondence preserves the action of G in the sense that for a separable R-subalgebra U of S and an element g of G we have

$$G^{g(U)} \;\cong\; gUg^{-1}.$$

Consequently, a subgroup H of G is normal in G if and only if

$$g(S^H) \;\cong\; G^{g(S^H)} \;\cong\; gS^Hg^{-1} \;\cong\; gHg^{-1} \;=\; H \;=\; S^H \qquad \text{for every} \quad g \in G,$$

and this if and only if

$$g(S^H) \;=\; S^H \qquad \text{for every} \quad g \in G,$$

in which case S^H is a G-Galois extension of R. By 4.5.4, this generalizes the Galois theorem for fields as stated in 4.1.5.

Chapter 5

Homotopic Hopf-Galois extensions

We now extend the notion of Galois extensions of commutative rings, developed within the context of a particular monoidal model category, to monoidal model categories in general. More precisely, we shall view a G-Galois extension of commutative rings $f : R \to S$ as a morphism of commutative monoids, in the monoidal category $(\mathcal{A}b, \times, \{*\})$, where the group G becomes a particular kind of commutative monoid H (namely, a *commutative Hopf monoid*). The G-action on S then becomes *the H-coaction of S over R*, and the G-Galois extension f an *homotopic H-Hopf-Galois extension* which we shall more concisely call an *H-Hopf-Galois extension*. As soon as these notions are established, we shall attempt to generalize the theory of chapter 4 to this new context.

We start, in section 5.1, by establishing the notion of H-Hopf-Galois extension $f : A \to B$. This will be done by means of the coequalizer $B \otimes_A B$ and the totalization of a fibrant replacement of the generalized cobar complex $C^\bullet(H; B)$, giving rise to two morphisms

$$h : \ B \otimes_A B \longrightarrow B \otimes H \qquad \text{and} \qquad i : \ A \longrightarrow C(H; B)$$

which correspond to the morphisms h and i defined in 4.2.7. We shall then provide the example of the trivial H-Hopf-Galois extensions $A \to A \otimes H$, as well as a characterization of an H-Hopf-Galois extension $A \to B$ in terms of its Amitsur complex $C(B/A)$ and a canonical morphism $\eta : \ A \to C(B/A)$.

In section 5.2, we establish a new notion of faithfulness that corresponds to what was defined in 4.4 for commutative rings. After introducing dualizablility and faithfulness, and studying some of their basic properties, we shall characterize H-Hopf-Galois extensions in terms of these two notions.

Section 5.3 provides an answer to the question: When is an H-Hopf-Galois extension $g : A \to C$ preserved and/or reflected under a functor $B \otimes_A -$ for an A-module B? The answer is indeed accessible and might be formulated in terms of dualizablility and faithfulness.

We terminate the chapter, in section 5.4, by suggesting what might be the next step towards the development of this general theory of Hopf-Galois extensions. More precisely, the goal would be to demonstrate a Hopf-Galois correspondence theorem that would encompass 4.1.5 and 4.5.6.

5.1 The notion of Hopf-Galois extension

In this section, we mainly establish the notion of Hopf-Galois extension, provide an example (the trivial Hopf-Galois extension), and demonstrate another characterization of that notion similarly to what has been done in chapter 4.2 (the example was 4.2.8.(2) and the characterization 4.2.9).

Throughout the section, we work within a cofibrantly generated (closed symmetric) monoidal model category $\mathcal{C} = (\mathcal{C}, \otimes, 1)$ as defined in chapter 2. We shall consider a commutative monoid $A = (A, \mu_A, \eta_A)$ in $\mathcal{C}$, and a commutative A-algebra $B = (B, \mu_B, \eta_B, \alpha_{AB})$ in the category $c\mathcal{A}lg_A$ of commutative A-algebras, where

- $\mu_B : B \otimes B \to B$ is the multiplication of B as defined in 2.1.4,
- $\eta_B : 1 \to B$ is the unit of B as defined in 2.1.4,
- $\alpha_{AB} : A \otimes B \to B$ is the action of A on B as defined in 2.1.6.

In the category of A-modules (and in particular of A-algebras), we have a tensor product over A, denoted $\otimes_A$, which is defined as the coequilazer induced from the left and right actions of A on each factor (cf. 2.1.8). In particular, we have a multiplication

$$\mu_B^A : \ B \otimes_A B \longrightarrow B$$

of B over A, which is induced by $\mu_B : B \otimes B \to B$ from the universal property of the coequalizer $B \otimes_A B$ in the following way.

$$\begin{tikzcd}
B \otimes A \otimes B \arrow[r, shift left] \arrow[r, shift right] & B \otimes B \arrow[r, "coeq"] \arrow[dr, "\mu_B"'] & B \otimes_A B \arrow[d, dashed, "\exists ! \mu_B^A"] \\
& & B
\end{tikzcd}$$

We shall also assume the existence of a unit $\eta_B^A : A \to B$ which makes the diagram

$$\begin{tikzcd}
A \otimes_A B \arrow[r, "\eta_B^A \otimes_A id_B"] \arrow[dr, leftrightarrow, "\cong"'] & B \otimes_A B \arrow[d, "\mu_B^A"] & B \otimes_A A \arrow[l, "id_B \otimes_A \eta_B^A"'] \arrow[dl, leftrightarrow, "\cong"] \\
& B &
\end{tikzcd}$$

commute in the category of A-modules.

Besides A and B, we also need to consider a third ingredient, that of a commutative Hopf monoid as defined below. We shall denote it H.

Definition 5.1.1. A *(commutative) Hopf monoid* $H = (H, \mu_H, \eta_H, \delta_H, \varepsilon_H)$ in $\mathcal{C}$ is a (commutative) monoid $H = (H, \mu_H, \eta_H)$ in $\mathcal{C}$ equipped with a *counit* $\varepsilon_H : H \to 1$ and a *coproduct* $\delta_H : H \to H \otimes H$, in the category of (commutative) $\mathcal{C}$-monoids, which make the following diagrams commute

$$\begin{tikzcd}
H \arrow[r, "\delta_H"] \arrow[d, equal] & H \otimes H \arrow[r, "\delta_H \otimes id_H"] & (H \otimes H) \otimes H \arrow[d, "\cong"] \\
H \arrow[r, "\delta_H"] & H \otimes H \arrow[r, "id_H \otimes \delta_H"] & H \otimes (H \otimes H)
\end{tikzcd}$$

$$\begin{tikzcd}
& H \arrow[dl, "\lambda_H"', "\cong"] \arrow[d, "\delta_H"] \arrow[dr, "\rho_H", "\cong"'] & \\
1 \otimes H & H \otimes H \arrow[l, "\varepsilon_H \otimes id_H"] \arrow[r, "id_H \otimes \varepsilon_H"'] & H \otimes 1
\end{tikzcd}$$

the first diagram being the *coassociativity* and the second the *counity* of H.

Remark 5.1.2. We are not assuming that the coproduct δ_H is cocommutative; in the sense that we don't necessarily have

$$sym \circ \delta_H \;\cong\; \delta_H,$$

where sym denotes the morphism that permutes both factors of $H \otimes H$.

Now that the three ingredients A, B and H have been defined, we need them to be related in some way. The first two are already related by the action α_{AB}. Similarly, we shall relate H to A and B by a coaction β_{HB} of H on B viewed as a module over A.

Definition 5.1.3. We say that H *coacts on B over A* if there is a morphism

$$\beta_{HB} :\; B \longrightarrow B \otimes H$$

in the category $c\mathcal{A}lg_A$ of commutative A-algebras, where $B \otimes H$ has the canonical A-algebra structure induced by the A-algebra structure of B, such that the two following conditions are satisfied:

- The following diagram commutes.

$$\begin{tikzcd}
B \arrow[r, "\beta_{HB}"] \arrow[d, equal] & B \otimes H \arrow[rr, "id_B \otimes \delta_H"] & & B \otimes (H \otimes H) \arrow[d, "\cong"] \\
B \arrow[r, "\beta_{HB}"] & B \otimes H \arrow[rr, "\beta_{HB} \otimes id_H"] \arrow[d, "id_B \otimes \varepsilon_H"] & & (B \otimes H) \otimes H \\
& B \otimes 1 \arrow[ul, "\cong"] & &
\end{tikzcd}$$

  The commutativity of the large rectangle is the *coassociativity* and the commutativity of the lower triangle the *counity.*

- The morphism

$$h :\; B \otimes_A B \longrightarrow B \otimes H$$

  defined to be the composition

$$\begin{tikzcd}
B \otimes_A B \arrow[r, "id_B \otimes_A \beta_{HB}"] & B \otimes_A (B \otimes H) \arrow[d, "\cong"] & \\
& (B \otimes_A B) \otimes H \arrow[r, "\mu_B^A \otimes id_H"] & B \otimes H
\end{tikzcd}$$

  is a cofibration in $\mathcal{C}$.

The next step is to establish the notion of the totalization of a cosimplicial object in $cos\mathcal{C}$; it involves, once again, the notion of equalizer as defined in 2.1.8.

Definition 5.1.4. Let $X : \Delta \to \mathcal{C}$ be a cosimplicial object. The *totalization* or *total object* $Tot(X)$ of X is the object of $\mathcal{C}$ defined to be the equalizer of the maps

$$\begin{tikzcd}
Tot(X) \arrow[r] & \prod_{[n]\in Ob\Delta} [\Delta[n], X^n] \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & \prod_{f:[n]\to[k]\in Mor\Delta} [\Delta[n], X^k],
\end{tikzcd}$$

where

- $[\Delta[n], -]$ is the adjoint of the functor

$$- \otimes \Delta[n] : \ cos\mathcal{C} \longrightarrow cos\mathcal{C},$$

where the tensor product is defined relative to the functorial reedy cosimplicial frame on $\mathcal{C}^{\Delta^{op}}$ as defined in [12].16.7.8,

- $\varphi := \prod_f \varphi_f$, with φ_f defined for every $f : [n] \to [k] \in Mor\Delta$ as

$$\varphi_f := [id_{\Delta[n]}, X(f)] : \ [\Delta[n], X^n] \longrightarrow [\Delta[n], X^k],$$

- $\psi := \prod_f \psi_f$, with ψ_f defined for every $f : [n] \to [k] \in Mor\Delta$ as

$$\psi_f := [f_*, id_{X^k}] : \ [\Delta[k], X^k] \longrightarrow [\Delta[n], X^k],$$

with

$$f_* : \ \Delta[n] \longrightarrow \Delta[k] \ : \ g \mapsto f \circ g.$$

Remark 5.1.5. According to [12] thm 18.6.6.(2) and 18.6.7.(2), we have a functor

$$Tot(-) : \ cos\mathcal{C} \longrightarrow \mathcal{C},$$

called the *totalization functor*, which preserves weak equivalences between fibrant objects, where fibrations, cofibrations and weak equivalences of cosimplicial objects respectively refer to reedy fibrations, reedy cofibrations and reedy weak equivalences as defined in [12].15.3.3.(3). In particular, a cosimplicial map is a weak equivalence of $cos\mathcal{C}$ if and only if it is a weak equivalence of $\mathcal{C}$ in each of its codegrees.

The totalization functor may be applied to a cosimplical object called the *Hopf cobar complex of B over A*. This gives rise to an object $C(H;B)$ and a map $i : \ A \longrightarrow C(H;B)$ in $\mathcal{C}$ which we shall use to establish the notion of Hopf-Galois extension.

Definition 5.1.6. Suppose that H coacts on B over A. The *Hopf cobar complex* $C^\bullet(H;B)$ is the cosimplicial commutative A-algebra

$$C^\bullet(H;B) : \ \Delta \longrightarrow c\mathcal{A}lg_A$$

with

- $C^\bullet(H;B)^n \ = \ C^n(H;B) \ := \ B \otimes \underbrace{H \otimes \ldots \otimes H}_{n \text{ times}}$ in each codegree n,

- the coface maps $d^i : \ C^n(H;B) \to C^{n+1}(H;B)$ defined in each codegree n by

$$d^i \ := \begin{cases} \beta_{HB} \otimes id_H^{\otimes n}, & \text{for } i = 0, \\ id_B \otimes id_H^{\otimes i-1} \otimes \delta_H \otimes id_H^{\otimes n-i}, & \text{for } 0 < i < n, \\ id_B \otimes id_H^{\otimes n} \otimes \eta_H, & \text{for } i = n, \end{cases}$$

- the codegeneracy maps $s^i : \ C^n(H;B) \to C^{n-1}(H;B)$ defined in each codegree n by

$$s^i \ := \begin{cases} \alpha_{HB} \otimes id_H^{\otimes n-1}, & \text{for } i = 0, \\ id_B \otimes id_H^{\otimes i-1} \otimes \varepsilon_H \otimes id_H^{\otimes n-i}, & \text{for } 0 < i < n, \\ id_B \otimes id_H^{\otimes n-1} \otimes \varepsilon_H, & \text{for } i = n. \end{cases}$$

Furthermore, we suppose the existence of a functorial fibrant replacement $RC^\bullet(H;B)$ (cf. 1.4.2) of $C^\bullet(H;B)$ in the category of cosimplicial commutative A-algebras, and we define

$$C(H;B) \ := \ Tot(RC^\bullet(H;B))$$

to be its totalization. The algebra unit $\eta_B^A : A \to B$ induces a *coaugmentation*

$$coaug: \ A \longrightarrow C^\bullet(H;B),$$

which is the morphism of cosimplicial commutative A-algebras, with A seen as the constant cosimplical commutative A-algebra, defined in each codegree n as

$$coaug^n \ := \ \eta_B^A \otimes \eta_H^{\otimes n}.$$

Upon totalization of the induced map $A \to RC^\bullet(H;B)$, this coaugmentation induces a morphism of commutative A-algebras

$$i: \ A \longrightarrow C(H;B).$$

Remark 5.1.7. Let $\mathcal{H}$ be the category of right H-comodules in the category of commutative A-algebras, and let $c\mathcal{A}lg_A$ denote the category of commutative A-algebras. By 5.1.5 and the functoriality of R, we have a functor

$$C(H;-): \ \mathcal{H} \longrightarrow c\mathcal{A}lg_A$$

which preserves weak equivalences.

We are now in position to generalize Galois extensions of commutative rings, as studied in chapter 4, to the context of the monoidal category $\mathcal{C}$.

Definition 5.1.8. A morphism $f: A \to B$ of commutative monoids in $\mathcal{C}$, where the commutative monoid B is given the natural A-algebra structure induced by f, is an *homotopic H-Hopf-Galois extension*, or more concisely an *H-Hopf-Galois extension*, if H is a commutative Hopf monoid which coacts on B over A such that the induced maps

$$h: \ B \otimes_A B \longrightarrow B \otimes H \qquad \text{and} \qquad i: \ A \longrightarrow C(H;B)$$

are weak equivalences in $\mathcal{C}$.

Remark 5.1.9. From the assumption made on h in 5.1.3, if f is an H-Hopf-Galois extension, then h is in fact an acyclic cofibration.

Example 5.1.10. The map

$$f = id_A \otimes \eta_H: \ A \longrightarrow A \otimes H$$

is a H-Hopf-Galois extension, provided the induced morphisms

$$i: A \to C(H; A \otimes H) \qquad \text{and} \qquad (\mu_H \otimes id_H) \circ (id_H \otimes \delta_H)$$

are weak equivalences. Indeed, the object $A \otimes H$ is a commutative monoid in $\mathcal{C}$, whose multiplication and unit morphisms are respectively given by

$$\begin{aligned}
&\mu_{(A\otimes H)} \ : \ A \otimes H \otimes A \otimes H \xrightarrow{\ \cong\ } A \otimes A \otimes H \otimes H \xrightarrow{\mu_A \otimes \mu_H} A \otimes H,\\
\text{so that} \quad &\mu^A_{(A\otimes H)} \ : \ (A \otimes H) \otimes_A (A \otimes H) \xrightarrow{\ \cong\ } A \otimes H \otimes H \xrightarrow{id_A \otimes \mu_H} A \otimes H,\\
\text{and by} \quad &\eta_{(A\otimes H)} \ : \ 1 \xrightarrow{\eta_A} A \xrightarrow{id_A \otimes \eta_H} A \otimes H,
\end{aligned}$$

because

- the diagram

$$\begin{tikzcd}
(A\otimes H\otimes A\otimes H)\otimes A\otimes H \arrow[r, "\mu_{(A\otimes H)}\otimes id_{(A\otimes H)}"] \arrow[d, "\cong"'] & A\otimes H\otimes A\otimes H \arrow[r, "\mu_{(A\otimes H)}"] & A\otimes H \arrow[d, equal] \\
A\otimes H\otimes (A\otimes H\otimes A\otimes H) \arrow[r, "id_{(A\otimes H)}\otimes\mu_{(A\otimes H)}"] & A\otimes H\otimes A\otimes H \arrow[r, "\mu_{(A\otimes H)}"] & A\otimes H
\end{tikzcd}$$

commutes, since it can be decomposed as

$$\begin{tikzcd}
(A\otimes H)^{\otimes 3} \arrow[r, "id_A\otimes sym\otimes id_H\otimes id_A\otimes id_H"] \arrow[d, "id_A\otimes id_H\otimes id_A\otimes sym\otimes id_H"] & A^{\otimes 2}\otimes H^{\otimes 2}\otimes A\otimes H \arrow[r, "\mu_A\otimes\mu_H\otimes id_A\otimes id_H"] \arrow[d, "(534)\in S_6"] & (A\otimes H)^{\otimes 2} \arrow[d, "id_A\otimes sym\otimes id_H"'] \\
A\otimes H\otimes A^{\otimes 2}\otimes H^{\otimes 2} \arrow[r, "(243)\in S_6"] \arrow[d, "id_A\otimes id_H\otimes\mu_A\otimes\mu_H"] & A^{\otimes 3}\otimes H^{\otimes 3} \arrow[r, "\mu_A\otimes id_A\otimes\mu_H\otimes id_H"] \arrow[d, "id_A\otimes\mu_A\otimes id_H\otimes\mu_H"] & A^{\otimes 2}\otimes H^{\otimes 2} \arrow[d, "\mu_A\otimes\mu_H"'] \\
(A\otimes H)^{\otimes 2} \arrow[r, "id_A\otimes sym\otimes id_H"] & A^{\otimes 2}\otimes H^{\otimes 2} \arrow[r, "\mu_A\otimes\mu_H"] & A\otimes H,
\end{tikzcd}$$

where the upper left square commutes by permutation of the symmetric group S_6, the upper right and the lower left squares commute by the naturality of the symmetry *sym* which permutes two factors, and where the lower right square is made up of the two diagrams

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[r, "\mu_A\otimes id_A"] \arrow[d, "\cong"'] & A\otimes A \arrow[r, "\mu_A"] & A \arrow[d, equal] \\
A\otimes (A\otimes A) \arrow[r, "id_A\otimes\mu_A"] & A\otimes A \arrow[r, "\mu_A"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
(H\otimes H)\otimes H \arrow[r, "\mu_H\otimes id_H"] \arrow[d, "\cong"'] & H\otimes H \arrow[r, "\mu_H"] & H \arrow[d, equal] \\
H\otimes (H\otimes H) \arrow[r, "id_H\otimes\mu_H"] & H\otimes H \arrow[r, "\mu_H"] & H
\end{tikzcd}$$

which commute by assumption, and

- the diagram

$$\begin{tikzcd}
1\otimes A\otimes H \arrow[r, "\eta_A\otimes id_{(A\otimes H)}"] & A\otimes A\otimes H \arrow[r, "id_A\otimes\eta_H\otimes id_{(A\otimes H)}"] & A\otimes H\otimes A\otimes H \arrow[d, "\cong"] \\
 & & A\otimes A\otimes H\otimes H \arrow[d, "\mu_A\otimes\mu_H"] \\
 & & A\otimes H \arrow[uull, "\cong"', "\lambda_{A\otimes H}"]
\end{tikzcd}$$

commutes since it is made up of the commutative diagrams

$$\begin{tikzcd}
1\otimes A \arrow[r, "\eta_A\otimes id_A"] & A\otimes A \arrow[d, "\mu_A"] \\
 & A \arrow[ul, "\cong"', "\lambda_A"]
\end{tikzcd}
\qquad
\begin{tikzcd}
1\otimes H \arrow[r, "\eta_H\otimes id_H"] & H\otimes H \arrow[d, "\mu_H"] \\
 & H, \arrow[ul, "\cong"', "\lambda_H"]
\end{tikzcd}$$

and similarly for

$$\begin{tikzcd}
A\otimes H\otimes 1 \arrow[r, "id_{A\otimes H}\otimes\eta A\otimes H"] & A\otimes H\otimes A\otimes H \arrow[d, "\mu_{A\otimes H}"] \\
& A\otimes H \arrow[ul, "\cong", "\rho_A"']
\end{tikzcd}$$

by the symmetry of $\mathcal{C}$.

In addition, $A\otimes H$ has an A-module (hence A-algebra) structure given by the action

$$\alpha = \mu_A \otimes id_H \;:\; A\otimes A\otimes H \longrightarrow A\otimes H,$$

and H coacts on $A\otimes H$ over A via the coaction

$$\beta = id_A \otimes \delta_H \;:\; A\otimes H \longrightarrow A\otimes H\otimes H,$$

because

- the diagram

$$\begin{tikzcd}
A\otimes H \arrow[r, "id_A\otimes\delta_H"] \arrow[d, equal] & A\otimes H\otimes H \arrow[r, "id_{(A\otimes H)}\otimes\delta_H"] & A\otimes H\otimes(H\otimes H) \arrow[d, "\cong"] \\
A\otimes H \arrow[r, "id_A\otimes\delta_H"] & A\otimes H\otimes H \arrow[r, "id_A\otimes\delta_H\otimes id_H"] & (A\otimes H\otimes H)\otimes H
\end{tikzcd}$$

  can be reduced to the diagram

$$\begin{tikzcd}
H \arrow[r, "\delta_H"] \arrow[d, equal] & H\otimes H \arrow[r, "id_H\otimes\delta_H"] & H\otimes(H\otimes H) \arrow[d, "\cong"] \\
H \arrow[r, "\delta_H"] & H\otimes H \arrow[r, "\delta_H\otimes id_H"] & (H\otimes H)\otimes H,
\end{tikzcd}$$

  which commutes by assumption, and

- the diagram

$$\begin{tikzcd}
A\otimes H\otimes 1 \arrow[dr, "\cong", "\rho_{(A\otimes H)}"'] & A\otimes H\otimes H \arrow[l, "id_{(A\otimes H)}\otimes\varepsilon_H"'] \\
& A\otimes H \arrow[u, "id_A\otimes\delta_H"']
\end{tikzcd}$$

  can be reduced to the diagram

$$\begin{tikzcd}
H\otimes 1 \arrow[dr, "\cong", "\rho_H"'] & H\otimes H \arrow[l, "id_H\otimes\varepsilon_H"'] \\
& H, \arrow[u, "\delta_H"']
\end{tikzcd}$$

  which commutes by assumption.

The coaction $\beta = id_A \otimes \delta_H$, which is a morphism of commutative A-algebras by construction, induces the morphism

$$h: \ (A \otimes H) \otimes_A (A \otimes H) \longrightarrow A \otimes H \otimes H,$$

which factors as

$$\begin{tikzcd}
(A \otimes H) \otimes_A (A \otimes H) \arrow[r, "id_A \otimes id_H \otimes_A id_A \otimes \delta_H"] & (A \otimes H) \otimes_A (A \otimes H \otimes H) \arrow[d, "\cong"] & \\
& (A \otimes H \otimes_A A \otimes H) \otimes H \arrow[d, "\cong"] & \\
& (A \otimes H \otimes H) \otimes H \arrow[r, "(id_A \otimes \mu_H) \otimes id_H"] & A \otimes H \otimes H.
\end{tikzcd}$$

By assumption, this implies that h is a weak equivalence, and consequently that $f : A \to A \otimes H$ is a H-Hopf-Galois extension.

Conjecture 5.1.11. In example 5.1.10, we should be able to remove the hypothesis that $(\mu_H \otimes id_H) \circ (id_H \otimes \delta_H)$ and $i : A \to C(H; A \otimes H)$ are weak equivalences, as this is probably the case. This would then provide a generalization of the trivial extensions defined in 4.2.8 and in paragraph 5.1 of [19]. *We shall then assume that the map*

$$f = id_A \otimes \eta_H : \ A \longrightarrow A \otimes H$$

is a H-Hopf-Galois extension for every commutative $\mathcal{C}$-monoid A.

We now establish another cosimplicial object, the *Amitsur complex*, from which we may characterize H-Hopf-Galois extensions. Such a characterization is given as theorem 5.1.16 at the end of the section. Its proof requires two supplementary results (cf. 5.1.14 and 5.1.15) we shall prove below.

Definition 5.1.12. Assume that H coacts on B over A. The *Amitsur complex* $C^\bullet(B/A)$ is the cosimplicial commutative A-algebra

$$C^\bullet(B/A) : \ \Delta \longrightarrow c\mathcal{A}lg_A$$

with

- $C^\bullet(B/A)^n \ = \ C^n(B/A) \ := \ \underbrace{B \otimes_A \ldots \otimes_A B}_{n+1 \text{ times}}$ in each codegree n,
- the coface maps $d^i : \ C^n(B/A) \to C^{n+1}(B/A)$ defined in each codegree n by
$$d^i \ := \begin{cases} \eta_B^A \otimes_A id_B^{\otimes_A n+1}, & \text{for } i = 0, \\ id_B^{\otimes_A i} \otimes_A \eta_B^A \otimes_A id_B^{\otimes_A n-i+1}, & \text{for } 0 < i < n, \\ id_B^{\otimes_A n+1} \otimes_A \eta_B^A, & \text{for } i = n, \end{cases}$$
where $\eta_B^A : \ A \longrightarrow B = C^0(B/A)$ is the unit map defined above,
- the codegeneracy maps $s^i : \ C^n(B/A) \to C^{n-1}(B/A)$ defined in each codegree n by
$$s^i \ := \begin{cases} \mu_B^A \otimes_A id_B^{\otimes_A n-1}, & \text{for } i = 0, \\ id_B^{\otimes_A i} \otimes_A \mu_B^A \otimes_A id_B^{\otimes_A n-i-1}, & \text{for } 0 < i < n, \\ id_B^{\otimes_A n-1} \otimes_A \mu_B^A, & \text{for } i = n. \end{cases}$$
where $\mu_B^A : \ B \otimes_A B \longrightarrow B = C^0(B/A)$ is the multiplication map defined above.

Furthermore, we assume the existence of a functorial fibrant replacement $RC^\bullet(B/A)$ of $C^\bullet(B/A)$ (cf. 1.4.2) in the category of commutative A-algebras, and we define the *completion of A along B* to be the totalization

$$C(B/A) \;:=\; Tot(RC^\bullet(B/A)).$$

The algebra unit $\eta_B^A : A \to B$, upon totalization of the induced map $A \to RC^\bullet(B/A)$, induces a canonical *completion map* of commutative A-algebras

$$\eta : \; A \longrightarrow C(B/A).$$

If η is a weak equivalence in $\mathcal{C}$, we say that A *is complete along* B.

Now that the maps i and η, connecting A with $C(H;B)$ and $C(B/A)$ respectively, have been defined, we may also define a map $h' : \; C(B/A) \longrightarrow C(H;B)$ which completes the triangle

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\eta"'] & C(H;B) \arrow[d, "h'"] \\ & C(B/A). \end{tikzcd}$$

It actually happens that this triangle commutes as proposition 5.1.14 below will show.

Definition 5.1.13. Let $C^\bullet(B/A)$ be the Amitsur complex and $C^\bullet(H;B)$ the Hopf cobar complex, with A, B and H as defined above. We define

$$h^\bullet : \; C^\bullet(B/A) \longrightarrow C^\bullet(H;B) \quad \in \quad Mor(sc\mathcal{A}lg_A)$$

to be the canonical morphism of cosimplicial commutative A-algebras given in each codegree n by the map

$$h^n : \; \underbrace{B \otimes_A \ldots \otimes_A B}_{n+1 \text{ times}} \;\longrightarrow\; B \otimes \underbrace{H \otimes \ldots \otimes H}_{n \text{ times}},$$

which is the composition of all composites

$$\begin{tikzcd} B^{\otimes_A(i+1)} \otimes H^{\otimes j} \arrow[r, "\cong"] & B^{\otimes_A(i-1)} \otimes_A (B \otimes_A B) \otimes H^{\otimes j} \arrow[d, "(id_B^{\otimes_A(i-1)} \otimes_A h \otimes id_H^{\otimes j})"] & \\ & B^{\otimes_A(i-1)} \otimes_A (B \otimes H) \otimes H^{\otimes j} \arrow[r, "\cong"] & B^{\otimes_A i} \otimes H^{\otimes(j+1)} \end{tikzcd}$$

for $j = 0, \ldots, n-1$, with $i + j = n$ and $h : B \otimes_A B \to B \otimes H$ as in 5.1.3. Upon totalization of the induced map $RC^\bullet(B/A) \to RC^\bullet(H;B)$, we obtain a morphism of commutative A-algebras

$$h' : \; C(B/A) \longrightarrow C(H;B).$$

Proposition 5.1.14. *Let A, B, H be as above, with H coacting on B over A, and consider the induced morphisms*

- $i : \; A \longrightarrow C(H;B)$ *as defined in 5.1.6,*
- $\eta : \; A \longrightarrow C(B/A)$ *as defined in 5.1.12, and*
- $h' : \; C(B/A) \longrightarrow C(H;B)$ *as defined in 5.1.13.*

Then $i \;=\; h' \circ \eta$.

Proof. By the functoriality of $TotR(-)$, we only have to check the commutativity of the triangle

$$\begin{tikzcd} A \arrow[r, "\bar{\eta}"] \arrow[dr, "\bar{i}"'] & C^\bullet(B/A) \arrow[d, "\bar{h}"] \\ & C^\bullet(H;B), \end{tikzcd} \tag{$*$}$$

where $\bar{\eta}$, $\bar{i}$ and $h^\bullet$ are the morphisms which induce η, i and h' respectively by application of the composite functor $Tot(R(-))$. Since

- $\bar{i}$ is defined in each codegree n as $\bar{i}^n = \eta_B^A \otimes \eta_H^{\otimes n}$,
- $\bar{\eta}$ is defined in each codegree n as $\bar{\eta}^n = (\eta_B^A)^{\otimes_A (n+1)}$,
- $\bar{h}$ is defined in each codegree n as $\bar{h}^n = h^n$,

we only have to check the commutativity of $(*)$ on the first factor, ie. that $\bar{h}^0 \circ \bar{\eta}^0 = \bar{i}^0$, or equivalently that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\eta_B^A \otimes_A \eta_B^A"] \arrow[dr, "\eta_B^A"'] & B \otimes_A B \arrow[d, "\mu_B^A"] \\ & B. \end{tikzcd} \tag{$**$}$$

By definition of the unit $\eta_B^A : A \to B$, we have a commutative diagram

$$\begin{tikzcd} B \otimes_A A \arrow[r, "id_B \otimes \eta_B^A"] \arrow[dr, leftrightarrow, "\cong"'] & B \otimes_A B \arrow[d, "\mu_B^A"] \\ & B, \end{tikzcd}$$

which we may combine with the obvious commutative diagram

$$\begin{tikzcd} A \cong A \otimes_A A \arrow[r, "\eta_B^A \otimes_A id_A"] \arrow[dr, "\eta_B^A"'] & B \otimes_A A \arrow[d, leftrightarrow, "\cong"] \\ & B \end{tikzcd}$$

in order to obtain the commutativity of $(**)$ as desired. □

The desired characterization of Hopf-Galois extensions may finally be established.

Lemma 5.1.15. *Let A, B, H be as above with B cofibrant in $c\mathcal{A}lg_A$ and H coacting on B over A. If the induced map*

$$h : \ B \otimes_A B \longrightarrow B \otimes H$$

is a weak equivalence, then $h' : C(B/A) \to C(H;B)$ is a weak equivalence as well, and the map $\eta_B^A : A \to B$ is a H-Hopf-Galois extension if and only if A is complete along B.

Proof. Since by assumption h is an acyclic cofibration and B is cofibrant in the category $c\mathcal{A}lg_A$ of commutative A-algebras, the cosimplicial map

$$h^\bullet : \ C^\bullet(B/A) \longrightarrow C^\bullet(H;B), \qquad \text{as defined in 5.1.13,}$$

is a weak equivalence in $cos\mathcal{C}$. Consequently, it is in each codegree a weak equivalence in $c\mathcal{A}lg_A$, and the induced map

$$h' : \ C(B/A) \longrightarrow C(H;B), \qquad \text{as defined in 5.1.13,}$$

is by 5.1.5 a weak equivalence in $c\mathcal{A}lg_A$. Finally, since $i = h' \circ \eta$ by 5.1.14, axiom (M_2) implies that i is a weak equivalence if and only if η is one as was to be shown. □

Theorem 5.1.16. *Let $f : A \to B$ be a morphism of commutative monoids in $\mathcal{C}$ with B cofibrant in $c\mathcal{A}lg_A$, where B is given the natural A-algebra structure induced by f, and let H be a commutative Hopf monoid in $\mathcal{C}$ which coacts on B over A. Then the following conditions are equivalent:*

(1) *The map $f : A \to B$ is an H-Hopf-Galois extension.*

(2) *The induced maps $h : B \otimes_A B \to B \otimes H$ and $\eta : A \to C(B/A)$ are weak equivalences.*

Proof. $(1 \Rightarrow 2)$ By definition, the induced maps $h : B \otimes_A B \to B \otimes H$ and $i : A \to C(H;B)$ are weak equivalences. It follows by 5.1.15 that the induced map $h' : C(B/A) \to C(H;B)$ is a weak equivalence as well. In addition, we know from 5.1.14 that $i = h' \circ \eta$. Therefore, the fact that η is a weak equivalence is a consequence of the model category axiom (M_2) for $\mathcal{C}$.

$(2 \Rightarrow 1)$ By 5.1.15, the induced map $h' : C(B/A) \to C(H;B)$ is a weak equivalence. Therefore, the fact that i is a weak equivalence follows from 5.1.14 and the model category axiom (M_2) for $\mathcal{C}$. □

5.2 Dualizability and faithfulness

Following a similar idea to what has been done in section 4.4, the goal of this section is to define a generalized notion of faithful flatness, and to study how we may use this new notion to characterize Hopf-Galois extensions. In order to do that, we shall also define and study another important ingredient: the notion of dualizablility.

We work again in the context of a cofibrantly generated monoidal model category $(\mathcal{C}, \otimes, 1)$. For every object X of $\mathcal{C}$, we have by assumption a functor

$$[X, -] \ : \ \mathcal{C} \longrightarrow \mathcal{C} \ : \ Y \mapsto [X, Y] \qquad \text{cf. 2.1.8,}$$

which is the right adjoint of $- \otimes X$. We then have natural isomorphisms

$$\mathcal{C}(X \otimes Y, Z) \ \cong \ \mathcal{C}(X, [Y, Z]) \qquad \text{for every objects} \quad X, Y, Z \quad \text{in } \mathcal{C},$$

and in addition we impose the natural isomorphisms

$$[X \otimes Y, Z] \ \cong \ [X, [Y, Z]] \qquad \text{for every objects} \quad X, Y, Z \quad \text{in } \mathcal{C}.$$

Now for any such objects, we may consider the morphism

$$id_{[X,Y]} : \ [X, Y] \longrightarrow [X, Y],$$

whose adjoint is the *evaluation morphism*

$$ev_{XY}: \; X \otimes [X,Y] \;\cong\; [X,Y] \otimes X \;\longrightarrow\; Y.$$

From the evaluation morphism, we can form the composite morphism

$$([X,Y] \otimes Z) \otimes X \xrightarrow{\cong} (X \otimes [X,Y]) \otimes Z \xrightarrow{ev_{XY} \otimes id_Z} Y \otimes Z,$$

whose adjoint is the canonical natural map

$$\nu = \nu_{XYZ}: \; [X,Y] \otimes Z \longrightarrow [X, Y \otimes Z].$$

Within the monoidal model category $(\mathcal{M}od_A, \otimes_A, A)$ for a commutative $\mathcal{C}$-monoid A (cf. 2.4.2), the same approach leads to a morphism

$$\nu^A = \nu^A_{XYZ}: \; [X,Y]_A \otimes_A Z \longrightarrow [X, Y \otimes_A Z]_A \qquad \text{cf. 2.1.8},$$

obtained from the evaluation morphism

$$ev^A_{XY}: \; X \otimes_A [X,Y]_A \;\longrightarrow\; Y.$$

The appropriate notion of dualizability may now be defined in terms of the maps ν and ν^A.

Definition 5.2.1. Let $DX := [X, 1] \in Ob\mathcal{C}$ be the *functional dual* of an object X in $\mathcal{C}$. We say that X is *dualizable* if the canonical map

$$\nu_{X1X}: \; DX \otimes X \longrightarrow [X,X]$$

is a weak equivalence in $\mathcal{C}$. In those cases where ν_{X1X} is an isomorphism or the identity morphism, we say respectively that X is *strongly* or *strictly* dualizable.

More generally, for a module $M \in Ob\mathcal{C}$ over a commutative monoid $A \in Ob\mathcal{C}$, we define $D_A M := [M, A]_A$ to be the *functional dual of M over A*, ie. the equalizer of the two morphisms

$$[M,A] \overset{\varphi}{\underset{\psi}{\rightrightarrows}} [A \otimes M, A],$$

where φ is induced by the action of A on M, and where ψ is the composition

$$[M,A] \xrightarrow{A \otimes -} [A \otimes M, A \otimes A] \xrightarrow{[A \otimes M, \alpha_{AA}]} [A \otimes M, A]$$

as defined in 2.1.8. We say that M *is dualizable over* A, or that M *is A-dualizable*, if the canonical map

$$\nu^A_{MAM}: \; D_A M \otimes_A M \longrightarrow [M,M]_A,$$

is a weak equivalence within the monoidal model category $(\mathcal{M}od_A, \otimes_A, A)$. In those cases where ν^A_{MAM} is an isomorphism or the identity morphism, we say respectively that X is *strongly* or *strictly* A-dualizable or dualizable over A.

Remark 5.2.2. **(1)** In the definition of a dualizable A-module, the commutativity of the monoid A is necessary.

(2) Being dualizable is equivalent to being 1-dualizable as the 1-action is simply given by the unit morphism of the underlying monoid.

We shall need the following result on dualizability as given in [19] under lemma 3.3.2.

Lemma 5.2.3. *Let X, Y, Z be objects of $\mathcal{C}$, A a commutative monoid of $\mathcal{C}$, and let M, N, P be three A-modules.*

(1) *If X or Z is dualizable, then the canonical map*

$$\nu_{XYZ} : \ [X, Y] \otimes Z \longrightarrow [X, Y \otimes Z]$$

is a weak equivalence in $\mathcal{C}$.

(2) *If X is dualizable, then DX too and the canonical map*

$$\rho_X : \ X \longrightarrow DDX = [[X, 1], 1],$$

defined as the right adjoint of the evaluation

$$ev_X = ev_{X1} : \ X \otimes [X, 1] \xrightarrow{\cong} [X, 1] \otimes X \longrightarrow 1,$$

is a weak equivalence in $\mathcal{C}$.

(3) *If M or P is A-dualizable, then the canonical map*

$$\nu^A_{MNP} : \ [M, N]_A \otimes_A P \longrightarrow [M, N \otimes_A P]_A$$

is a weak equivalence in $\mathcal{C}$.

(4) *If M is A-dualizable, then $D_A M$ too and the canonical map*

$$\rho^A_M : \ M \longrightarrow D_A D_A M = [[M, 1]_A, 1]_A,$$

defined as the right adjoint of the evaluation

$$ev^A_M = ev^A_{MA} : \ M \otimes_A [M, A]_A \xrightarrow{\cong} [M, A]_A \otimes_A M \longrightarrow A,$$

is a weak equivalence in $\mathcal{C}$.

The following result provides a very useful isomorphism between Hom objects over different commutative monoids.

Lemma 5.2.4. *Let $f : A \to B$ be a morphism of commutative monoids in $\mathcal{C}$. For any A-module M and any B-module N, there is an isomorphism*

$$[M, N]_A \ \cong \ [B \otimes_A M, N]_B,$$

where N is also viewed as an A-module via the restriction of scalars $Res_f : \mathcal{M}od_B \to \mathcal{M}od_A$ (cf. 2.4.3), and where $B \otimes_A M$ has a B-module structure whose action is given on the first factor by the multiplication of B.

Proof. By the universal property of equalizers, it suffices to find two composable morphisms

$$\varphi : \ [M, N]_A \longrightarrow [B \otimes_A M, N]_B \qquad \text{and} \qquad \psi : \ [B \otimes_A M, N]_B \longrightarrow [M, N]_A,$$

in order to obtain

$$\psi \circ \varphi \ = \ id_{[M,N]_A} \qquad \text{and} \qquad \varphi \circ \psi \ = \ id_{[B \otimes_A M, N]_B}.$$

The morphism φ may be given as the composite

$$\begin{tikzcd}
{[M,N]_A} \arrow[r, "\cong"] & A\otimes_A [M,N]_A \arrow[r, "f\otimes_A [M,N]_A"] & B\otimes_A [M,N]_A \arrow[d, "\sigma_{MN}"] & \\
& & {[B\otimes_A M, B\otimes_A N]_B} \arrow[r, "{[B\otimes_A M,\lambda_N]}"] & {[B\otimes_A M, N]_B},
\end{tikzcd}$$

where σ_{MN} corresponds by adjunction to the composite

$$B\otimes_A [M,N]_A \otimes_A (B\otimes_A M) \xrightarrow{\varphi^A_{[M,N]_A,M}} B\otimes_A ([M,N]_A\otimes_A M) \xrightarrow{B\otimes_A ev} B\otimes_A N,$$

with $\varphi^A_{[M,N]_A,M}$ given by the lax monoidal structure of the functor $B\otimes_A -$ as given in 3.2.6. Moreover, the morphism ψ may be given by the composite

$$[B\otimes_A M,N]_B \xrightarrow{k} [B\otimes_A M,N]_A \xrightarrow{[\lambda_M,N]_A} [M,N]_A,$$

where k is given, via the universal property of equalizers, by the restriction of h in the commutative diagram

$$\begin{tikzcd}
{[B\otimes_A M,N]_B} \arrow[r] \arrow[d, dashed, "k"'] \arrow[dr, "Res(h)"] & {[B\otimes_A M,N]} \arrow[r, shift left] \arrow[r, shift right] & {[B\otimes B\otimes M,N]} \\
{[B\otimes_A M,N]_A} \arrow[r, "h"] & {[B\otimes_A M,N]} \arrow[r, shift left] \arrow[r, shift right] & {[A\otimes B\otimes M,N]}.
\end{tikzcd}$$

□

In the isomorphism $[M,N]_A \;\cong\; [B\otimes_A M,N]_B$, there is in fact a connection between the A-dualizability of M and the B-dualizability of $B\otimes_A M$ as the following result shows.

Proposition 5.2.5. *Let $f : A\to B$ be a morphism of commutative monoids in $\mathcal{C}$, and let M be an A-module dualizable over A. Then the B-module $B\otimes_A M$ is dualizable over B.*

Proof. By 5.2.4, $B\otimes_A M$ is a B-module whose action is defined on the first factor by the multiplication of B, and we have an isomorphism

$$[M,N]_A \;\cong\; [B\otimes_A M,N]_B.$$

In addition, the canonical map

$$\nu^B_{(B\otimes_A M,B,B\otimes_A M)} \;:\; D_B(B\otimes_A M)\otimes_B (B\otimes_A M) \;\longrightarrow\; [B\otimes_A M, B\otimes_A M]_B$$

factors as the composite

$$\begin{tikzcd}
{[B\otimes_A M,B]\otimes_B (B\otimes_A M)} \arrow[r, "\cong"] & {[M,B]_A\otimes_A M} \arrow[d, "\nu", "\sim"'] & \\
& {[M,B\otimes_A M]_A} \arrow[r, "\cong"] & {[B\otimes_A M, B\otimes_A M]_B},
\end{tikzcd}$$

where the map ν is a weak equivalence by 5.2.3.(3) since M is a dualizable A-module by assumption. □

We now move on to faithfulness. Another notion we shall need as well is a generalized version of injective modules as we define below.

Definition 5.2.6. Let A be a commutative monoid in $\mathcal{C}$. An A-module M is said to be *faithful*, or *faithful over* A, if for every morphism of A-modules $f : X \to Y$, f is a weak equivalence if and only if $f \otimes_A id_M$ (or equivalently $id_M \otimes_A f$) is a weak equivalence.

Furthermore, a map of commutative monoids $g : A \to B$ is *faithful* if B, endowed with the canonical A-module structure given by

$$A \otimes B \xrightarrow{g \otimes id_B} B \otimes B \xrightarrow{\mu_B} B,$$

is faithful over A.

Finally, an A-module N is said to be *injective*, or *injective over* A, if for every weak equivalence of A-modules $f : X \to Y$, the induced map $[f, N]_A : [Y, N]_A \to [X, N]_A$ is a weak equivalence. If in addition we also have the reverse implication, ie. that for every weak equivalence $[f, N]_A$ the map f is a weak equivalence, we shall then say that N is *cofaithful*, or *cofaithful over* A.

Remark 5.2.7. By 2.3.3, any cofibrant A-module M is so that if f is a acyclic cofibration, then $f \otimes id_M$ and $id_M \otimes f$ are weak equivalences.

Properties 5.2.8. *Let $f : A \to B$ be a morphism of commutative monoids.*

(1) *If M is a faithful A-module, then $B \otimes_A M$ is a faithful B-module.*

(2) *If f is faithful and M is an A-module such that the B-module $B \otimes_A M$ is faithful over B, then M is faithful over A.*

Proof. (1) Consider a weak equivalence $g : C \to D$ in $\mathcal{C}$. Then the composite

$$\begin{tikzcd} g \otimes_B id_{(B\otimes_A M)} : \quad C \otimes_B (B \otimes_A M) \arrow[r, "\cong"] & C \otimes_A M \arrow[d, "g\otimes_A id_A", "\sim"'] & \\ & D \otimes_A M \arrow[r, "\cong"] & D \otimes_B (B \otimes_A M) \end{tikzcd}$$

is a weak equivalence since M is faithful over A. Conversely, a weak equivalence

$$h \otimes_B id_{(B\otimes_A M)} : \ E \otimes_B (B \otimes_A M) \longrightarrow F \otimes_B (B \otimes_A M)$$

is isomorphic to a weak equivalence

$$h \otimes_B id_M : \ E \otimes_A M \longrightarrow F \otimes_A M,$$

which in turns implies that $h : E \to F$ is a weak equivalence by faithfulness of M.

(2) Consider a weak equivalence $g : C \to D$ in $\mathcal{C}$. Since $B \otimes_A M$ is faithful over B we have a weak equivalence

$$\begin{tikzcd} g \otimes_A id_M : \quad C \otimes_A M \arrow[r, "\cong"] & C \otimes_B B \otimes_A M \arrow[d, "g\otimes_B id_{(B\otimes_A M)}", "\sim"'] & \\ & D \otimes_B B \otimes_A M \arrow[r, "\cong"] & D \otimes_A M. \end{tikzcd}$$

Conversely, a weak equivalence

$$h \otimes_A id_M : \ E \otimes_A M \longrightarrow F \otimes_A M$$

induces, by faithfulness of f, a weak equivalence

$$\begin{tikzcd} E \otimes_A B \otimes_A M \arrow[r, "\cong"] & E \otimes_A M \otimes_A B \arrow[d, "h\otimes_A id_M \otimes_A id_B", "\sim"'] & \\ & F \otimes_A M \otimes_A B \arrow[r, "\cong"] & F \otimes_A B \otimes_A M, \end{tikzcd}$$

which in turns induces a weak equivalence $h : E \to F$ by faithfulness of $B \otimes_A M$. $\square$

Proposition 5.2.9. *Let $f : A \to B$ be a faithful morphism of commutative monoids in $\mathcal{C}$ with B dualizable over A, and let M be a faithful A-module. If $B \otimes_A M$ is a dualizable B-module, then M is a dualizable A-module.*

Proof. We need to check that the map

$$\nu^A_{MAM} : \ [M, A]_A \otimes_A M \longrightarrow [M, M]_A,$$

is a weak equivalence in $\mathcal{C}$. Consider the commutative diagram

$$\begin{tikzcd} B \otimes_A [M, A]_A \otimes_A M \arrow[r, "id_B \otimes_A \nu^A_{MAM}"] \arrow[d, "\cong"'] & B \otimes_A [M, M]_A \arrow[d, "\cong"] \\ {[M, A]_A \otimes_A B \otimes_A M} \arrow[d, "\nu^A_{MAB} \otimes_A id_M"'] & {[M, M]_A \otimes_A B} \arrow[d, "\nu^A_{MMB}"] \\ {[M, A \otimes_A B]_A \otimes_A M} \arrow[d, "\cong"'] & {[M, M \otimes_A B]_A} \arrow[d, "\cong"] \\ {[M, B]_A \otimes_A M} \arrow[d, "\cong"'] & {[M, B \otimes_A M]_A} \arrow[d, "\cong"] \\ {[B \otimes_A M, B]_B \otimes_B (B \otimes_A M)} \arrow[r, "\nu^B_{(B\otimes_A M)B(B\otimes_A M)}"] & {[B \otimes_A M, B \otimes_A M]_B,} \end{tikzcd}$$

where the lowest vertical isomorphisms are given by 5.2.4. Since $B \otimes_A M$ is assumed to be dualizable over B, the lowest horizontal map is a weak equivalence by definition. In addition, B is dualizable over A, so that 5.2.3.(3) and the faithfulness of M imply that the canonical maps

$$\nu^A_{MAB} \otimes_A id_M \qquad \text{and} \qquad \nu^A_{MMB}$$

are weak equivalences. Consequently, the upper horizontal map is a weak equivalence by (M_2), and the faithfulness of B over A implies that the map ν^A_{MAM} is a weak equivalence. $\square$

In 4.2.7, besides i and h, we also defined a ring homomorphism $j : S{<}G{>} \to End_R(S)$ which provided another way to characterize Galois extensions of commutative rings (cf. 4.2.9). The same thing can be done here.

Definition 5.2.10. For a morphism of commutative monoids $f : A \to B$, and a commutative Hopf monoid H which coacts on B over A, we define the map

$$j : \ DH \otimes B \longrightarrow [B, B]_A$$

to be the composite

$$\begin{tikzcd} DH \otimes B \arrow[r, "\nu_{H1B}"] \arrow[rrdd, "j"'] & {[H,B]} \arrow[r, "\cong"] & {[B \otimes H, B]_B} \arrow[d, "{[h,B]_B}"] \\ & & {[B \otimes_A B, B]_B} \arrow[d, "\cong"] \\ & & {[B,B]_A,} \end{tikzcd}$$

where $h : B \otimes_A B \to B \otimes H$ is the canonical morphism induced from f (cf. 5.1.3), and where both isomorphisms are given by 5.2.4.

Proposition 5.2.11. *Let $f : A \to B$ be a morphism of commutative monoids, and H a commutative Hopf monoid which coacts on B over A. Suppose that B is dualizable over itself.*

(1) *If B is injective over itself and if h is a weak equivalence, then j is a weak equivalence as well.*

(2) *If B is cofaithful over itself, then h is a weak equivalence if and only if j is a weak equivalence.*

Proof. This is an easy consequence of 5.2.6, 5.2.3.(2) and the model category axiom (M_2) as stated in 1.2.4. □

Proposition 5.2.12. *Let $f : A \to B$ be a morphism of commutative monoids, and H a commutative Hopf monoid which coacts on B over A. Suppose that the induced morphism*

$$h : \; B \otimes_A B \longrightarrow B \otimes H, \qquad \textit{as defined in 5.1.3},$$

is a weak equivalence (and consequently an acyclic cofibration).

(1) *For a cofibrant B-module M, there is a canonical weak equivalence*

$$h_M : \; M \otimes_A B \longrightarrow M \otimes H.$$

(2) *If H is dualizable, for a cofibrant injective B-module M, there is a canonical weak equivalence*

$$j_M : \; DH \otimes M \longrightarrow [B, M]_A.$$

Proof. (1) The map h_M is defined to be the composite

$$M \otimes_A B \xrightarrow{\cong} M \otimes_B B \otimes_A B \xrightarrow{id_M \otimes_B h} M \otimes_B B \otimes H \xrightarrow{\cong} M \otimes H,$$

which is a weak equivalence by (M_2) and 5.2.7 since h is an acyclic cofibration by assumption.

(2) The map j_M is defined to be the composite

$$\begin{tikzcd} DH \otimes M \arrow[r, "\nu_{H1M}", "\sim"'] & {[H,M]} \arrow[r, "\cong"] & {[B \otimes H, M]_B} \arrow[d, "{[h,M]_B}", "\sim"'] \\ & & {[B \otimes_A B, M]_B} \arrow[r, "\cong"] & {[B,M]_A,} \end{tikzcd}$$

where ν_{H1M} is a weak equivalence by 5.2.3.(1) and the dualizability of H, and where $[h, M]_B$ is a weak equivalence by injectivity of M. The fact that j_M is a weak equivalence then follows from axiom (M_2). □

Remark 5.2.13. According to 5.2.3.(1), we may also transfer the dualizability of H on M in 5.2.12.(2).

For a morphism of commutative monoids $f : A \to B$ with a commutative Hopf monoid H coacting on B over A, there is in $\mathcal{C}$ a canonical map ν' from $M \otimes_A C(H; B)$ to $C(H; M \otimes_A B)$. We shall provide two conditions under which this map becomes a weak equivalence.

Proposition 5.2.14. *Let $f : A \to B$ be an H-Hopf-Galois extension. Then, for a B-module M which is faithful over A, the canonical map*

$$\nu' : \ M \otimes_A C(H; B) \longrightarrow C(H; M \otimes_A B)$$

is a weak equivalence.

Proof. By the universal property of totalizations, the weak equivalence $i_M : M \to C(H; M \otimes H)$, induced by the trivial H-Hopf-Galois extension $M \to M \otimes H$ of M, factors as the composite

$$\begin{tikzcd} M \arrow[r, "\cong"] & M \otimes_A A \arrow[r, "id_M \otimes_A i", "\sim"'] & M \otimes_A C(H; B) \arrow[d, "\nu'"] & \\ & & C(H; M \otimes_A B) \arrow[r, "C(H; h_M)", "\sim"'] & C(H; M \otimes H), \end{tikzcd}$$

where

- the map $id_M \otimes_A i$ is a weak equivalence since i is a weak equivalence and M is faithful, and
- the map $C(H; h_M)$ is by 5.1.7 a weak equivalence since h_M is a weak equivalence.

The fact that ν' is a weak equivalence then follows from axiom (M_2). □

The second condition requires the following assertion.

Conjecture 5.2.15. Let $f : A \to B$ be a morphism of commutative monoids, and H a commutative Hopf monoid which coacts on B over A. For any A-dualizable A-module M, we have a weak equivalence

$$TotR([M, C^\bullet(H; B)]_A) \xrightarrow{\sim} [M, TotR(C^\bullet(H; B))]_A.$$

Proposition 5.2.16. *Let $f : A \to B$ be a morphism of commutative monoids, and H a commutative Hopf monoid which coacts on B over A. For any A-dualizable A-module M, the canonical map*

$$\nu' : \ M \otimes_A C(H; B) \longrightarrow C(H; M \otimes_A B)$$

is a weak equivalence.

Proof. For each $n \in \mathbb{N}$, consider the weak equivalence

$$\begin{tikzcd} C^n(H; [D_A M, B]_A) \arrow[r, equal] & {[[M, A]_A, B]_A \otimes H^{\otimes n}} \arrow[d, "\sim"] & \\ & {[[M, A]_A, B \otimes H^{\otimes n}]_A} \arrow[r, equal] & {[D_A M, C^n(H; B)]_A.} \end{tikzcd}$$

After application of the composite functor $TotR(-)$, we obtain by 5.1.7 and 5.2.15 a weak equivalence

$$C(H;[D_AM,B]_A) \xrightarrow{\sim} [D_AM, C(H;B)]_A,$$

which fits into the commutative diagram

$$\begin{tikzcd}
M \otimes_A C(H;B) \arrow[r, "\rho_M^A \otimes_A id_{C(H;B)}", "\sim"'] \arrow[d, "\nu'"'] & D_AD_AM \otimes_A C(H;B) \arrow[r, "\nu^A_{(D_AM)AC(H;B)}", "\sim"'] & {[D_AM, C(H;B)]_A} \\
C(H; M \otimes_A B) \arrow[r, "C(H;\rho_M^A \otimes_A id_B)", "\sim"'] & C(H; D_AD_AM \otimes_A B) \arrow[r, "C(H;\nu^A_{(D_AM)AB})", "\sim"'] & C(H;[D_AM,B]_A), \arrow[u, "\sim"']
\end{tikzcd}$$

where the four horizontal maps are weak equivalences by 5.1.7, 5.2.7 and the A-dualizability of M. It follows from axiom (M_2) that ν' is a weak equivalence. □

We may finally proceed to the characterization of a Hopf-Galois extension in terms of dualizability and faithfulness. We shall assume the following result based on proposition 6.2.1 in [19], which we conjecture is true also in this context.

Conjecture 5.2.17. *Let $f : A \to B$ be an H-Hopf-Galois extension. Then B is a dualizable A-module.* This result is the generalization to Hopf-Galois extensions of the fact seen in 4.2.9 that for each Galois extension of commutative rings $R \subseteq S$, the ring S is a finitely generated R-module.

Theorem 5.2.18. *Let $f : A \to B$ be a morphism of commutative monoids, and H a cofibrant commutative Hopf monoid which coacts on B over A. Then f is a faithful H-Hopf-Galois extension if and only if the induced map $h : B \otimes_A B \to B \otimes H$ is a acyclic cofibration and the A-module B is faithful and dualizable over A.*

Proof. ($\Rightarrow$) This is a consequence of 5.2.17 and the definition of an H-Hopf-Galois extension.

($\Leftarrow$) Suppose that h is a weak equivalence and that B is dualizable and faithful over A. We must check that the canonical map $i : A \to C(H;B)$ is a weak equivalence which, by faithfulness of B, is equivalent to the fact that

$$id_B \otimes_A i \ : \ B \cong B \otimes_A A \longrightarrow B \otimes_A C(H;B)$$

is a weak equivalence. As in the proof of 5.2.14, we have a commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\cong"] \arrow[drrr, bend right=10, "\sim", "i_B"'] & B \otimes_A A \arrow[r, "id_B \otimes_A i"] & B \otimes_A C(H;B) \arrow[r, "\nu'", "\sim"'] & C(H; B \otimes_A B) \arrow[d, "C(H;h)", "\sim"'] \\
& & & C(H; B \otimes H),
\end{tikzcd}$$

where

- the map $C(H;h)$ is a weak equivalence by 5.1.7,
- the map ν' is a weak equivalence by 5.2.16, and
- the map i_B, induced after application of the composite functor $TotR(-)$ by the trivial H-Hopf-Galois extension $B \to B \otimes H$, is a weak equivalence by definition.

It follows by (M_2) that $i_B \otimes_A i$ is a weak equivalence, and consequently that i is a weak equivalence by faithfulness of B. □

5.3 Cobase changes of Hopf-Galois extensions

The goal of this section is to establish when an H-Hopf-Galois extension $g : A \to C$ is preserved and reflected under a functor $B \otimes_A -$ for an A-module B. Conveniently enough, the condition under which this happens to be the case may be expressed in terms of dualizability and faithfulness.

Let us fix again a cofibrantly generated monoidal model category $(\mathcal{C}, \otimes, 1)$. For two morphisms of commutative monoids $f : A \to B$ and $g : A \to C$, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "f'"] \\ B \arrow[r, "g'"'] & B \otimes_A C, \end{tikzcd}$$

where $f' = f \otimes_A id_C$, $g' = id_B \otimes_A g$, and where B and C have the A-module structures induced by f and g respectively. In this situation, we shall say that f' *is the cobase change of* f *along* g, and that g' *is the cobase change of* g *along* f. We shall first establish under which conditions H-Hopf-Galois extensions are preserved under cobase changes.

Proposition 5.3.1. *Let $f : A \to B$ be a morphism of commutative monoids with B cofibrant. If $g : A \to C$ is a faithful H-Hopf-Galois extension, then the cobase change $g' : B \to B \otimes_A C$ of g along f is a faithful H-Hopf-Galois extension as well.*

Proof. By assumption, the commutative Hopf monoid H coacts on C over A, so that the coaction $\beta_{HC} : C \to C \otimes H$ fits into the commutative diagram

$$\begin{tikzcd} C \arrow[r, "\beta_{HC}"] \arrow[d, equal] & C \otimes H \arrow[r, "id_C \otimes \delta_H"] & C \otimes (H \otimes H) \arrow[d, "\cong"] \\ C \arrow[r, "\beta_{HC}"] \arrow[dr, "\cong"'] & C \otimes H \arrow[r, "\beta_{HC} \otimes id_H"] \arrow[d, "id_C \otimes \varepsilon_H"] & (C \otimes H) \otimes H \\ & C \otimes 1 & . \end{tikzcd}$$

This coaction uniquely extends to g' via the functor $B \otimes_A -$, which sends g to g' and the commutative diagram above to the commutative diagram

$$\begin{tikzcd} (B \otimes_A C) \arrow[r, "\beta_{H(B \otimes_A C)}"] \arrow[d, equal] & (B \otimes_A C) \otimes H \arrow[r, "id_{(B \otimes_A C)} \otimes \delta_H"] & (B \otimes_A C) \otimes (H \otimes H) \arrow[d, "\cong"] \\ (B \otimes_A C) \arrow[r, "\beta_{H(B \otimes_A C)}"] \arrow[dr, "\cong"'] & (B \otimes_A C) \otimes H \arrow[r, "\beta_{H(B \otimes_A C)} \otimes id_H"] \arrow[d, "id_{(B \otimes_A C)} \otimes \varepsilon_H"] & ((B \otimes_A C) \otimes H) \otimes H \\ & (B \otimes_A C) \otimes 1 & , \end{tikzcd}$$

where $\beta_{H(B \otimes_A C)} = id_B \otimes_A \beta_{HC}$, so that H coacts on $B \otimes_A C$ over B as well.

We shall prove that g' is a faithful H-Hopf-Galois extension by invoking 5.2.18. We know that C is a dualizable A-module by 5.2.17, and is faithful by hypothesis. By 5.2.5 and 5.2.8.(1), it follows that $B \otimes_A C$ is a dualizable and faithful B-module. It remains to verify that the canonical map

$$h: \quad (B \otimes_A C) \otimes_B (B \otimes_A C) \longrightarrow (B \otimes_A C) \otimes H$$

is a weak equivalence. This map fits into the commutative diagram

$$\begin{tikzcd}
B \otimes_A C \otimes_A C \arrow[r, "id_B \otimes_A h'", "\sim"'] \arrow[d, "\cong"'] & B \otimes_A (C \otimes H) \arrow[d, "\cong"] \\
(B \otimes_A C) \otimes_B (B \otimes_A C) \arrow[r, "h"] & (B \otimes_A C) \otimes H,
\end{tikzcd}$$

where h' is an acyclic cofibration by assumption, so that $id_B \otimes_A h'$ is a weak equivalence. It follows from (M_2) that h is a weak equivalence. The required hypotheses of 5.2.18 for $g' : B \to B \otimes_A C$ to be an H-Hopf-Galois extension are therefore verified. $\square$

Proposition 5.3.2. *Let $f : A \to B$ be a morphism of commutative monoids with B faithful and dualizable over A, and let $g : A \to C$ be an H-Hopf-Galois extension. Then the cobase change $g' : B \to B \otimes_A C$ of g along f is an H-Hopf-Galois extension as well.*

Proof. The commutative Hopf monoid H coacts on $B \otimes_A C$ over B and makes the canonical map

$$h: \quad (B \otimes_A C) \otimes_B (B \otimes_A C) \longrightarrow (B \otimes_A C) \otimes H$$

into a weak equivalence as in the proof of 5.3.1. It remains to verify that the canonical map

$$i: \; B \longrightarrow C(H; B \otimes_A C)$$

is a weak equivalence. This map fits into the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\cong"] \arrow[drr, bend right, "i"'] & B \otimes_A A \arrow[r, "id_B \otimes_A i'", "\sim"'] & B \otimes_A C(H; C) \arrow[d, "\nu'", "\sim"'] \\
& & C(H; B \otimes_A C),
\end{tikzcd}$$

where

- the canonical map ν' is a weak equivalence by 5.2.16 since B is dualizable over A,
- the canonical map i', and consequently $id_B \otimes_A i'$ by faithfulness of B over A, is a weak equivalence since $g : A \to C$ is an H-Hopf-Galois extension.

It then follows, from axiom (M_2), that i is a weak equivalence as desired. $\square$

Conversely, we may study under which conditions cobase changes reflect (faithful) H-Hopf-Galois extensions.

Proposition 5.3.3. *Let $A \to B$ and $g : A \to C$ be morphisms of commutative monoids with B faithful and dualizable over A, let H be a dualizable commutative Hopf monoid that coacts on C over A, and consider the cobase change $g' : B \to B \otimes_A C$ of g along f.*

(1) *If g' is an H-Hopf-Galois extension, then g is an H-Hopf-Galois extension as well.*

(2) *If g' is a faithful H-Hopf-Galois extension, then g is a faithful H-Hopf-Galois extension as well.*

Proof. (1) Suppose that $g' : B \to B \otimes_A C$ is an H-Hopf-Galois extension. We must check that the two canonical maps

$$i : \ A \longrightarrow C(H;C) \qquad \text{and} \qquad h : \ C \otimes_A C \longrightarrow C \otimes H$$

are weak equivalences.

By assumption, we have a weak equivalence $i' : B \to C(H; B \otimes_A C)$, for the H-Hopf-Galois extension g', which fits into the commutative diagram

$$\begin{tikzcd} B \arrow[r, "\cong"] \arrow[drr, bend right, "\sim", "i'"'] & B \otimes_A A \arrow[r, "id_B \otimes_A i"] & B \otimes_A C(H;C) \arrow[d, "\sim"', "\nu'"] \\ & & C(H; B \otimes_A C), \end{tikzcd}$$

where the map ν' is a weak equivalence by 5.2.16 since B is dualizable over A. It follows from axiom (M_2) that the map $id_B \otimes_A i$ is a weak equivalence, and consequently that i is a weak equivalence by faithfulness of B over A.

Furthermore, the canonical map $h : C \otimes_A C \to C \otimes H$ fits, as in the proof of 5.3.1, into the commutative square

$$\begin{tikzcd} B \otimes_A C \otimes_A C \arrow[r, "id_B \otimes_A h"] \arrow[d, "\cong"'] & B \otimes_A (C \otimes H) \arrow[d, "\cong"] \\ (B \otimes_A C) \otimes_B (B \otimes_A C) \arrow[r, "h'", "\sim"'] & (B \otimes_A C) \otimes H, \end{tikzcd}$$

where the canonically induced map

$$h' : \ (B \otimes_A C) \otimes_B (B \otimes_A C) \longrightarrow (B \otimes_A C) \otimes H$$

is a weak equivalence by assumption. From axiom (M_2), it follows that $id_B \otimes_A h$ is a weak equivalence, and consequently that h is a weak equivalence by faithfulness of B over A.

(2) From (1), it remains to check that if g' is faithful then g is faithful. This however is a direct consequence of 5.2.8.(2). $\square$

Notice that reflecting (faithful) H-Hopf-Galois extensions under cobase changes requires stronger conditions than preserving them under cobase changes. As a consequence, we may improve 5.3.3 in the following way.

Theorem 5.3.4. *If $A \to B$ and $g : A \to C$ are morphisms of commutative monoids such that B is faithful and dualizable over A, and if H is a dualizable commutative Hopf monoid that coacts on C over A, then*

(1) *the map g is an H-Hopf-Galois extension if and only if its cobase change along f is an H-Hopf-Galois extension,*

(2) *the map g is a faithful H-Hopf-Galois extension if and only if its cobase change along f is a faithful H-Hopf-Galois extension.*

Proof. This is simply a combination of 5.3.1, 5.3.2 and 5.3.3. $\square$

The reason for the term *cobase change* to be used in this context comes from the following probable assertion.

Conjecture 5.3.5. For two morphisms of commutative monoids $f : A \to B$ and $g : A \to C$, we may form their pushout

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "f'"] \arrow[ddr, bend left, "l"] & \\ B \arrow[r, "g'"'] \arrow[drr, bend right, "k"'] & B \otimes_A C \arrow[dr, dashed, "p"] & \\ & & D, \end{tikzcd}$$

which corresponds to the monoid $B \otimes_A C$ since for any two morphisms of commutative monoids

$$k:\ B \longrightarrow D \qquad \text{and} \qquad l:\ C \longrightarrow D \qquad \text{with} \qquad lg = kf,$$

the unique morphism $p : B \otimes_A C \to D$ is given by the coequalizer

$$\begin{tikzcd} B \otimes A \otimes C \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & B \otimes C \arrow[r] \arrow[d, "k \otimes l"] & B \otimes_A C \arrow[d, dashed, "\exists ! p"] \\ & D \otimes D \arrow[r, "\mu_D"] & D, \end{tikzcd}$$

where μ_D is a morphism of commutative A-algebras by the commutativity of D, and where φ and ψ are respectively defined as the composites

$$B \otimes A \otimes C \xrightarrow{id_B \otimes f \otimes id_C} B \otimes B \otimes C \xrightarrow{\mu_B \otimes id_C} B \otimes C$$

and

$$B \otimes A \otimes C \xrightarrow{id_B \otimes g \otimes id_C} B \otimes C \otimes C \xrightarrow{id_B \otimes \mu_C} B \otimes C.$$

If that is the case, the above results provide conditions under which H-Hopf-Galois extensions are preserved under pushouts.

5.4 Towards a Hopf-Galois correspondence theorem

This section is a proposed sketch of a future Hopf-Galois correspondence theorem. We shall first generalize the cobar complexes $C(H; B)$, where B is a right H-comodule, to cobar complexes of the form $C(B; H; B')$ with B' being a left H-comodule. This allows to formulate a trivial example of Hopf-Galois extensions for a right H-comodule. We shall then provide a sufficiently strong notion, that of allowability, in order to formulate what might be the desired correspondence theorem. This might certainly be achieved within the context of a cofibrantly generated monoidal model category $(\mathcal{C}, \otimes, 1)$, which we fix for the rest of the section. As above, we shall consider a commutative monoid A in $\mathcal{C}$, a commutative Hopf monoid H in $\mathcal{C}$, and two commutative A-algebras B and B' such that H coacts on them over A in such a way that B becomes a right H-comodule and B' a left H-comodule.

Definition 5.4.1. The *Hopf cobar complex* $C^\bullet(B; H; B')$ is the cosimplicial commutative A-algebra

$$C^\bullet(B; H; B') :\ \Delta \longrightarrow c\mathcal{A}lg_A$$

with

- $C^\bullet(B;H;B')^n \;=\; C^n(B;H;B') \;:=\; B\otimes \underbrace{H\otimes\ldots\otimes H}_{n \text{ times}} \otimes B'$ in each codegree n,
- the coface maps $d^i:\ C^n(B;H;B')\to C^{n+1}(B;H;B')$ defined in each codegree n by

$$d^i \;:=\; \begin{cases} \beta_{HB}\otimes id_H^{\otimes n}\otimes id_{B'}, & \text{for } i=0,\\ id_B\otimes id_H^{\otimes i-1}\otimes \delta_H\otimes id_H^{\otimes n-i}\otimes id_{B'}, & \text{for } 0<i<n,\\ id_B\otimes id_H^{\otimes n}\otimes \beta_{HB'}, & \text{for } i=n,\end{cases}$$

- the codegeneracy maps $s^i:\ C^n(B;H;B')\to C^{n-1}(B;H;B')$ defined in each codegree n by

$$s^i \;:=\; \begin{cases} \alpha_{HB}\otimes id_H^{\otimes n-1}\otimes id_{B'}, & \text{for } i=0,\\ id_B\otimes id_H^{\otimes i-1}\otimes \varepsilon_H\otimes id_H^{\otimes n-i}\otimes id_{B'}, & \text{for } 0<i<n,\\ id_B\otimes id_H^{\otimes n-1}\otimes \alpha_{HB'}, & \text{for } i=n.\end{cases}$$

Furthermore, we suppose the existence of a functorial fibrant replacement $RC^\bullet(B;H;B')$ of $C^\bullet(B;H;B')$ in the category of cosimplicial commutative A-algebras, and we define

$$C(B;H;B') \;:=\; Tot(RC^\bullet(B;H;B'))$$

to be its totalization.

Remarks 5.4.2. **(1)** It is clear that $C(B;H;1)\cong C(H;B)$.

(2) Since H is a commutative Hopf monoid, it has a canonical structure of (left and right) H-comodule.

From this, we should be able to establish the following properties and example.

Properties 5.4.3. *Under a possible supplementary condition,*

(1) *there are weak equivalences $C(H;H;1)\sim * \sim C(1;H;H)$,*

(2) *there are weak equivalences $C(B;H;H)\sim B$ and $C(H;H;B')\sim B'$,*

(3) *there are projections $\pi: C(B;H;B')\to B$ and $\pi': C(B;H;B')\to B'$.*

Example 5.4.4. Let M be a right H-comonoid of commutative algebras. Then the morphism

$$\tau_M = C(M;H;\eta_H):\ C(M;H;1)\longrightarrow C(M;H;H)$$

is an H-Hopf-Galois extension, where

- the morphism $\beta: C(M;H;H)\to C(M;H;H)\otimes H$ comes from $id_M\otimes id_{H^{\otimes n}}\otimes\delta_H$,
- the morphism $i: C(M;H;1)\to C(C(M;H;H);H;1)$ is a weak equivalence by 5.4.3.(3),
- the morphism $h: C(M;H;H)\otimes_{C(M;H;1)} C(M;H;H)\to C(M;H;H)\otimes H$, defined as the composite

$$\begin{tikzcd} C(M;H;H)\otimes_{C(M;H;1)} C(M;H;H) \arrow[d, "id\otimes\beta"] & \\ C(M;H;H)\otimes_{C(M;H;1)} C(M;H;H)\otimes H \arrow[r, "\mu\otimes id"] & C(M;H;H)\otimes H, \end{tikzcd}$$

is a weak equivalence as it fits into the commutative diagram

$$\begin{tikzcd}
C(M;H;H)\otimes_{C(M;H;1)} C(M;H;H) \arrow[r, "\sim"] \arrow[d, "id\otimes\beta"] & M\otimes H \arrow[dd, equal] \\
C(M;H;H)\otimes_{C(M;H;1)} C(M;H;H)\otimes H \arrow[d, "\mu\otimes id"] & \\
C(M;H;H)\otimes H \arrow[r, "\sim"] & M\otimes H.
\end{tikzcd}$$

The above results, together with the following condition of allowability, should be sufficient for the desired correspondence theorem (as stated below) to be verified.

Definition 5.4.5. Let $\bar{H}$ be a commutative Hopf monoid in $\mathcal{C}$, and let $q: H\to\bar{H}$ be a morphism of commutative Hopf monoids. We say that q, or equivalently $\bar{H}$, is allowable if the induced map

$$\pi_H:\ C(H;\bar{H};1)\longrightarrow H,$$

where H has the $\bar{H}$-comodule structure induced by q, has a section $H\to C(H;\bar{H};1)$.

Theorem 5.4.6 (Hopf-Galois correspondence)**.** *Let $f: A\to B$ be an H-Hopf-Galois extension, and let $\bar{H}$ be a commutative Hopf monoid in $\mathcal{C}$. If $q: H\to\bar{H}$ is allowable, then the $\bar{H}$-Hopf-Galois extension*

$$\tau_B:\ C(B;\bar{H};1)\longrightarrow C(B;\bar{H};\bar{H}),\qquad \textit{as given in 5.4.4},$$

fits into a commutative diagram

$$\begin{tikzcd}
C(B;\bar{H};1) \arrow[r, "\pi_B", "\sim"'] \arrow[dr, bend right, "\tau_B"'] & B \arrow[d, "\iota_B", "\sim"'] \\
& C(B;\bar{H};\bar{H}),
\end{tikzcd}$$

where the morphism of $\bar{H}$-comodules ι_B is a weak equivalence. In addition, if $C(1;\bar{H};H)$ is endowed with a natural structure of Hopf monoid, then the induced map

$$i_{\bar{H}}:\ A\longrightarrow C(\bar{H};B)\cong C(B;\bar{H};1)$$

is a $C(1;\bar{H};H)$-Hopf-Galois extension.

$$\begin{tikzcd}
A \arrow[r, "i_{\bar{H}}"] \arrow[d, "i_H"', "\sim"] & C(B;\bar{H};1) \arrow[r, "\pi_B"] \arrow[d, equal] & B \arrow[d, "\iota_B"', "\sim"] \\
C(B;H;1) \arrow[r, "C(B;q;1)"] & C(B;\bar{H};1) \arrow[r, "C(B;\bar{H};\eta_{\bar{H}})"] & C(B;\bar{H};\bar{H})
\end{tikzcd}$$

This result, however, may only be conjectured for now and constitutes a topic for future research.

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