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Calculus of Variations

The second order pullback equation

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Abstract Let f, g be two closed k-forms over $\mathbb{R}^n$. The pullback equation studies the existence of a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi^*(g) = f.$$

We prove two types of results. The first one sharpens some of the existing regularity results. The second one discusses the possibility of choosing the map φ as the gradient of a function $\Phi : \mathbb{R}^n \to \mathbb{R}$. We show that this is a very rare event unless the two forms are constant.

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1 Introduction

Let f, g be two closed k-forms over $\mathbb{R}^n$. The *pullback equation*, systematically investigated in [4], studies the existence of a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\varphi^*(g) = f. \tag{1}$$

A natural question is to know if this diffeomorphism can be chosen as the gradient of a function $\Phi : \mathbb{R}^n \to \mathbb{R}$. Indeed when $k = n$ and $g \equiv 1$, the pullback equation becomes, by abuse of notations, the *prescribed Jacobian equation*

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$$\det \nabla \varphi = f.$$

In this context the requirement $\varphi = \nabla \Phi$ transforms the equation into the *Monge-Ampère equation*

$$\det \nabla^2 \Phi = f.$$

We should point out that one of the main differences between the first and the second order problems is that we cannot proceed by composition in the second order case. This comes from the fact that the gradient structure is not preserved by composition, not even in the constant case.

Our article is organized as follows. After a brief recalling of the notations for exterior and differential forms, we start our discussion (cf. Sect. 3) with the first order case where we do not require that the map is a gradient. We obtain two results, one for k-forms of rank k (cf. Theorem 3) and the other one for 2-forms (cf. Theorem 6). We discuss in this introduction only the second one which corresponds to the classical Darboux theorem for forms of non-maximal rank. Our theorem improves, on all the existing results, the regularity of φ. Indeed it is classically proved that if $f, g \in C^r$, then $\varphi \in C^{r-1}$; however in [3,4] it is established, using elliptic regularity, that if $f, g \in C^{r,\alpha}$, then $\varphi \in C^{r,\alpha}$ provided $0 < \alpha < 1$. We show here (cf. Theorem 6) that we can get, by elementary means, the result of [3] (see also [4]) even when $\alpha = 0$ or $\alpha = 1$.

We next turn to the second order case where we impose that $\varphi = \nabla \Phi$. We will obtain two types of results. The first ones concern the analytical problem (cf. Sect. 4) and the second ones the algebraic problem (cf. Sect. 5) where the forms f and g are constant forms.

In fact, apart from the cases $k = 0$ (cf. Proposition 8), $k = 1$ (cf. Corollary 10) which are elementary and from the case $k = n$ mentioned above, one cannot expect to find solutions of (1) of the form $\varphi = \nabla \Phi$. We give two simple examples showing this fact in the symplectic case (cf. Proposition 12) and when $k = n - 1$ (cf. Proposition 11), which are, besides the cases $k = 0, 1, n$, the only cases where (1) can be systematically solved, see [4].

In this context the contrast with the algebraic case (cf. Sect. 5) is striking. Before describing our results let us first explain our terminology. By algebraic we mean that the forms f and g are constant forms and the map φ is a linear map $\varphi(x) = Ax$ with A invertible. Requiring that $\varphi = \nabla \Phi$ means that we want A to be symmetric. We will show that, contrary to the non-constant case, this can be achieved when $k = 2$ (cf. Theorem 19), when $k = n - 1$ (cf. Corollary 15 and Proposition 16) or more generally for k-forms (cf. Theorem 17) having rank k.

We rephrase the above result when $k = 2$ in terms of matrices. The two forms f and g can be seen as $n \times n$ skew symmetric matrices F and G. Theorem 20 states that if F and G are also invertible (this necessarily implies that n is even), then there exists A invertible so that

$$A^t G A = F \quad \text{and} \quad A^t = A. \tag{2}$$

The above result (without requiring the symmetry of A) is standard, cf. for example Corollary 2.5.14 in [6] or Corollary 2.3.1 in [9]. The decomposition (2) has an interesting equivalent formulation (cf. Theorem 20). It states that any invertible matrix X (here n is even) can be written as

$$X = S B$$

where S is symmetric and B is *symplectic* which means that

$$B^t J B = J$$

where J is the standard symplectic matrix namely

$$J = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}$$

It is interesting to compare the above decomposition with the standard polar decomposition which states that any matrix X can be written as

$$X = S\,O$$

where S is symmetric and O is orthogonal i.e. O preserves the identity matrix I (namely $O^t O = O^t I O = I$).

2 Notations

We gather here the notations we will use throughout this article. For more details, see [4].

2.1 Exterior forms

Let $0 \leq k \leq n$ be an integer. An exterior k-form will be denoted by

$$f = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

The set of exterior k-forms over $\mathbb{R}^n$ is denoted by $\Lambda^k(\mathbb{R}^n)$. If $k = 0$, we set

$$\Lambda^0\left(\mathbb{R}^n\right) = \mathbb{R}.$$

(i) The *exterior product* of $f \in \Lambda^k(\mathbb{R}^n)$ with $g \in \Lambda^l(\mathbb{R}^n)$, denoted by $f \wedge g$, is defined as usual and it belongs to $\Lambda^{k+l}(\mathbb{R}^n)$. The *scalar product* between two k-forms f and g is denoted by

$$\langle g; f \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} f_{i_1 \cdots i_k}.$$

The *Hodge star operator* associates to $f \in \Lambda^k(\mathbb{R}^n)$ a form $(*f) \in \Lambda^{n-k}(\mathbb{R}^n)$. The *interior product* of $f \in \Lambda^k(\mathbb{R}^n)$ with $g \in \Lambda^l(\mathbb{R}^n)$ is defined by

$$g \lrcorner f = (-1)^{n(k-l)} * (g \wedge (*f)).$$

(ii) Let $A \in \mathbb{R}^{n \times m}$ be a matrix (with n rows and m columns) and $f \in \Lambda^k(\mathbb{R}^n)$ be given by

$$f = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

The *pullback of f by A*, denoted $A^*(f)$, is defined by

$$A^*(f) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} A^{i_1} \wedge \cdots \wedge A^{i_k} \in \Lambda^k(\mathbb{R}^m)$$

where A^j is the jth row of A and is identified with

$$A^j = \sum_{k=1}^{m} A_k^j e^k \in \Lambda^1(\mathbb{R}^m).$$

If $k = 0$, we then let

$$A^*(f) = f.$$

(iii) We next recall the notion of *rank* (also called rank of order 1 in [4]) of $f \in \Lambda^k(\mathbb{R}^n)$. We first associate to the linear map

$$g \in \Lambda^1\left(\mathbb{R}^n\right) \to g \lrcorner f \in \Lambda^{k-1}\left(\mathbb{R}^n\right)$$

a matrix $\overline{f} \in \mathbb{R}^{\binom{n}{k-1}\times n}$ such that, by abuse of notations,

$$g \lrcorner f = \overline{f}\, g \quad \text{for every } g \in \Lambda^1\left(\mathbb{R}^n\right).$$

Explicitly, using the lexicographical order for the columns (index below) and the rows (index above) of the matrix $\overline{f}$, we have

$$(\overline{f})_i^{j_1\cdots j_{k-1}} = f_{i\, j_1\cdots j_{k-1}}$$

for $1 \leq i \leq n$ and $1 \leq j_1 < \cdots < j_{k-1} \leq n$. The rank of the k-form f is then the rank of the $\binom{n}{k-1} \times n$ matrix $\overline{f}$. We then write

$$\operatorname{rank}[f] = \operatorname{rank}\left(\overline{f}\right).$$

We have the following elementary result (cf. Proposition 2.37 of [4]): let $f \in \Lambda^2(\mathbb{R}^n)$ then rank$[f]$ is even and rank $[f] = 2m$ if and only if

$$f^m \neq 0 \quad \text{and} \quad f^{m+1} = 0$$

where $f^m = \underbrace{f \wedge \cdots \wedge f}_{m-times}$.

2.2 Differential forms

Let $0 \leq k \leq n$ and $\Omega \subset \mathbb{R}^n$ be an open set. A differential k-form $f : \Omega \to \Lambda^k$ will be written as

$$f = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where $f_{i_1\cdots i_k} : \Omega \to \mathbb{R}$, for every $1 \leq i_1 < \cdots < i_k \leq n$. We also, by abuse of notations, identify, when necessary, dx^i with e^i. When $f_{i_1\cdots i_k} \in C^r(\Omega)$, for every $1 \leq i_1 < \cdots < i_k \leq n$, we will write $f \in C^r\left(\Omega; \Lambda^k\right)$. The differential forms obey pointwise the laws of the exterior algebra. For instance the exterior product is defined pointwise as

$$(f \wedge g)(x) = f(x) \wedge g(x).$$

(i) The *exterior derivative* of $f \in C^1(\Omega;\Lambda^k)$ denoted df belongs to $C^0(\Omega;\Lambda^{k+1})$ and is defined by

$$df = \sum_{1\le i_1<\cdots<i_k\le n} \sum_{m=1}^{n} \frac{\partial f_{i_1\cdots i_k}}{\partial x_m}\, dx^m \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

If $k=n$, then $df=0$. The k-form f is said to be closed if $df=0$ in Ω.

(ii) Let $U \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ be open and $\varphi = (\varphi^1,\ldots,\varphi^n) \in C^1(U;V)$. Let $f \in C^0(V;\Lambda^k(\mathbb{R}^n))$. Then the *pullback* of f by φ, denoted $\varphi^*(f)$, belongs to $C^0\left(U;\Lambda^k(\mathbb{R}^m)\right)$ and is defined by

$$\varphi^*(f) = \sum_{1\le i_1<\cdots<i_k\le n} (f_{i_1\cdots i_k}\circ\varphi)\, d\varphi^{i_1}\wedge\cdots\wedge d\varphi^{i_k}$$

where $d\varphi^s$ is the exterior derivative of the 0-form φ^s, i.e.

$$d\varphi^s = \sum_{l=1}^{m} \frac{\partial\varphi^s}{\partial x_l}\, dx^l.$$

This is a generalization of the definition of the pullback for exterior forms (constant forms). Indeed if $\varphi(x) = Ax$, where $A \in \mathbb{R}^{n\times m}$ is a matrix, and f is constant, then

$$\varphi^*(f) = A^*(f).$$

3 The first order pullback equation

3.1 Frobenius theorem

In the main theorems of the present section we will need the classical Frobenius theorem however with a kind of Cauchy data. First of all we recall some definitions and notations that will be used in Frobenius theorem.

Definition 1 Let $U \subset \mathbb{R}^n$ be an open set.

(i) For $a,b \in C^1(U;\mathbb{R}^n)$, $[a,b] \in C^0(U;\mathbb{R}^n)$ stands for the *Lie bracket* of a and b and is defined by $[a,b] = ([a,b]_1,\ldots,[a,b]_n)$, where

$$[a,b]_i = \sum_{m=1}^{n}\left(a_m\frac{\partial b_i}{\partial x_m} - b_m\frac{\partial a_i}{\partial x_m}\right).$$

(ii) Let $a^{m+1},\ldots,a^n \in C^1(U;\mathbb{R}^n)$. We say that the family $\{a^{m+1},\ldots,a^n\}$ is *involutive* in U if, for every $m+1\le i,j\le n$, there exist $c_{ij}^p \in C^0(U)$, $m+1\le p\le n$, verifying

$$[a^i,a^j] = \sum_{p=m+1}^{n} c_{ij}^p a^p \quad \text{in } U.$$

(iii) For $a \in C^1(U;\mathbb{R}^n)$ we define φ_t^a as the unique solution of

$$\begin{cases} \frac{d}{dt}\varphi_t^a = a\circ\varphi_t^a \\ \varphi_0^a = \mathrm{id}. \end{cases}$$

In the sequel we will write, for $1 \leq m < n$,

$$x = \left(x', x_{m+1}, \dots, x_n\right) = (x_1, \dots, x_m, x_{m+1}, \dots, x_n) \in \mathbb{R}^m \times \mathbb{R}^{n-m}.$$

Theorem 2 *Let $1 \leq m < n, r \geq 1$ be integers and $x_0 \in \mathbb{R}^n$. Let $a^{m+1} \cdots, a^n$ be a C^r involutive family in a neighborhood of x_0 and $h \in C^r(\mathbb{R}^m; \mathbb{R}^n)$ with $h(x_0') = x_0$ be such that*

$$\left\{\frac{\partial h}{\partial x_1}(x_0'), \dots, \frac{\partial h}{\partial x_m}(x_0'), a^{m+1}(x_0), \dots, a^n(x_0)\right\} \quad \text{is linearly independent.}$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^r(U; \varphi(U))$ such that $\varphi(x_0) = x_0$

$$\varphi(x_1, \dots, x_m, (x_0)_{m+1}, \dots, (x_0)_n) = h(x_1, \dots, x_m) \quad \text{for every } x \in U$$

and, for every $m + 1 \leq i \leq n$,

$$\frac{\partial \varphi}{\partial x_i} \in \operatorname{span}\{(a^{m+1} \circ \varphi), \dots, (a^n \circ \varphi)\} \quad \text{in } U.$$

Proof With no loss of generality we can assume that $x_0 = 0$. We claim that

$$\varphi(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = \varphi_{x_{m+1}}^{a^{m+1}} \circ \cdots \circ \varphi_{x_n}^{a^n}(h(x_1, \dots, x_m))$$

has all the desired properties. Indeed $\varphi \in C^r$ near 0, $\varphi(0) = 0$ and

$$\nabla\varphi(0) = \left(\frac{\partial h}{\partial x_1}(0), \dots, \frac{\partial h}{\partial x_m}(0), a^{m+1}(0), \dots, a^n(0)\right).$$

Hence φ is a C^r diffeomorphism near 0. Finally, using the involutivity of the family $a^{m+1}, \dots, a^n$, we have (cf. for example [10 p. 41]) that, for every $m + 1 \leq i \leq n$,

$$\frac{\partial \varphi}{\partial x_i} \in \operatorname{span}\{(a^{m+1} \circ \varphi), \dots, (a^n \circ \varphi)\} \quad \text{near } 0$$

which concludes the proof. □

3.2 The case of k-forms of rank k

The following result improves Theorem 15.1 in [4] (when $k < n$).

Theorem 3 *Let $2 \leq k \leq n, r \geq 1$ be integers and $x_0 \in \mathbb{R}^n$. Let f and g be two C^r k-forms verifying, in a neighborhood of x_0,*

$$df = dg = 0 \quad \text{and} \quad \operatorname{rank}[f] = \operatorname{rank}[g] = k.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^r(U; \varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(g) = f \quad \text{in } U.$$

In the proof of the above theorem we will also need the following lemma (cf. Lemma 4.7 in [4]).

Lemma 4 *Let $V \subset \mathbb{R}^n$ be an open set, $g \in C^0\left(V; \Lambda^k\right)$ and $a \in C^0(V; \mathbb{R}^n)$ be such that*

$$a \lrcorner g = 0 \quad \text{in } V.$$

Let $1 \leq j \leq n$ be fixed, $U \subset \mathbb{R}^n$ be an open set, $\varphi \in \operatorname{Diff}^1(U; \varphi(U))$ be such that $\varphi(U) \subset V$ and

$$\frac{\partial \varphi}{\partial x_j} = a \circ \varphi \quad \text{in } U.$$

Then, in U and for every $1 \leq i_1 < \cdots < i_k \leq n$

$$(\varphi^* (g))_{i_1 \cdots i_k} = 0 \quad \text{if } j \in \{i_1, \ldots, i_k\} .$$

Proof (Theorem 3) With no loss of generality we can assume $x_0 = 0$. We can also assume that

$$f = dx^1 \wedge \cdots \wedge dx^k$$

Indeed if φ_1 solves

$$\varphi_1^*(g) = dx^1 \wedge \cdots \wedge dx^k$$

and φ_2 solves

$$\varphi_2^*(f) = dx^1 \wedge \cdots \wedge dx^k$$

then $\varphi = \varphi_1 \circ (\varphi_2)^{-1}$ verifies $\varphi^*(g) = f$. Up to permuting the coordinates we can also suppose that $g_{1\cdots k}(0) \neq 0$. Since $\operatorname{rank}[g] = k$ (see (iii) of Sect. 2.1), it is easy to find a neighborhood V of 0 and $a^i \in C^r(V; \mathbb{R}^n)$, $k+1 \leq i \leq n$, such that, for every $x \in V$,

$$\{a^{k+1}(x), \ldots, a^n(x)\} \text{ is linearly independent}$$

and

$$\operatorname{span}\{a^{k+1}(x), \ldots, a^n(x)\} = \ker \overline{g}(x) .$$

Then exactly as the proof of Theorem 4.5 of [4], we have that the family $\{a^{k+1}, \ldots, a^n\}$ is involutive in V. Let us use the abbreviation $0_l = (0, \ldots, 0) \in \mathbb{R}^l$. Let $h : \mathbb{R}^k \to \mathbb{R}^k$ ($h = \left(h^1, \ldots, h^k\right)$) be such that $h(0) = 0$ and

$$g_{1\cdots k}(h(x_1, \ldots, x_k), 0_{n-k}) \det \nabla h(x_1, \ldots, x_k) = 1$$

for every $x_1, \ldots, x_k$ small enough, or equivalently

$$h^*(i^*(g)) = dx^1 \wedge \cdots \wedge dx^k \tag{3}$$

where $i : \mathbb{R}^k \to \mathbb{R}^n$ given by $i(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0_{n-k})$ is the inclusion map. For example h defined by $h^i(x) = x_i$, $1 \leq i \leq k-1$ and

$$\int_0^{h^k(x_1,\ldots,x_k)} g_{1\cdots k}(x_1, \ldots, x_{k-1}, t, 0_{n-k}) dt = x_k$$

has all the desired properties. We claim that, for $x = (x_1, \ldots, x_n)$ small enough,

$$\varphi(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = \varphi_{x_{k+1}}^{a^{k+1}} \circ \cdots \circ \varphi_{x_n}^{a^n} (h(x_1, \ldots, x_k), 0_{n-k})$$

has all the required properties, where we recall that φ_t^a stands for the unique solution of

$$\begin{cases} \frac{d}{dt} \varphi_t^a = a \circ \varphi_t^a \\ \varphi_0^a = \operatorname{id} . \end{cases}$$

First of all, note that $\varphi(0) = 0$ and φ is C^r near 0. Then observing that

$$\nabla\varphi(0) = \begin{pmatrix} h^1_{x_1}(0) & \cdots & h^1_{x_k}(0) & (a^{k+1}_1)(0) & \cdots & (a^n_1)(0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h^k_{x_1}(0) & \cdots & h^k_{x_k}(0) & (a^{k+1}_k)(0) & \cdots & (a^n_k)(0) \\ 0 & \cdots & 0 & (a^{k+1}_{k+1})(0) & \cdots & (a^n_{k+1})(0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & (a^{k+1}_n)(0) & \cdots & (a^n_n)(0) \end{pmatrix}$$

we deduce that

$$\begin{aligned} \det\nabla\varphi(0) &= det\nabla h(0) \cdot (a^{k+1} \wedge \cdots \wedge a^n)_{(k+1)\cdots n}(0) \\ &= 1/g_{1\cdots k}(0) \cdot (a^{k+1} \wedge \cdots \wedge a^n)_{(k+1)\cdots n}(0) \neq 0 \end{aligned}$$

where we have used Corollary 25 for the last inequality. Hence φ is a C^r diffeomorphism near 0. Recalling that the family $a^{k+1}, \ldots, a^n$ is involutive near 0, we have, by Theorem 2 (more precisely its proof), that, for every $k+1 \leq i \leq n$,

$$\frac{\partial\varphi}{\partial x_i} \in \operatorname{span}\{(a^{m+1} \circ \varphi), \ldots, (a^n \circ \varphi)\} \quad \text{near } 0.$$

Therefore near 0, we have, for every $k+1 \leq i \leq n$,

$$\frac{\partial\varphi}{\partial x_i} \in \ker(\overline{g}) \circ \varphi.$$

Using Lemma 4, we deduce that, near 0

$$(\varphi^*(g))_{i_1\cdots i_k} = 0 \quad \text{for every } (i_1, \cdots, i_k) \neq (1 \cdots k).$$

In other words, near 0,

$$\varphi^*(g) = \lambda\, dx^1 \wedge \cdots \wedge dx^k,$$

where λ is a C^{r-1} function. On one hand, noticing that $\varphi(x_1, \ldots, x_k, 0_{n-k}) = (h(x_1, \ldots, x_k), 0_{n-k})$, or equivalently $\varphi \circ i = i \circ h$, we obtain from (3) that

$$\begin{aligned} \lambda(x_1, \ldots, x_k, 0_{n-k})\, dx^1 \wedge \cdots \wedge dx^k &= i^*(\varphi^*(g)) = (\varphi \circ i)^*(g) = (i \circ h)^*(g) \\ &= h^*(i^*(g)) = dx^1 \wedge \cdots \wedge dx^k. \end{aligned}$$

We thus obtain that

$$\lambda \equiv 1 \quad \text{on } x_{k+1} = \cdots = x_n = 0.$$

On the other hand, since $dg = 0$ and hence $d(\varphi^*(g)) = d(\lambda\, dx^1 \wedge \cdots \wedge dx^k) = 0$, we immediately deduce that

$$\lambda = \lambda(x_1, \ldots, x_k).$$

Combining these last two observations we directly deduce that $\lambda \equiv 1$, which concludes the proof. □

Remark 5 In the case $k = n-1$, the previous proof is nothing else than an application of the well-known method of characteristics and the classical Cartan lemma. Indeed let f be a closed $(n-1)$-form which is C^r in a neighborhood of 0 and such that $f_{1\cdots(n-1)}(0) \neq 0$. Let $h \in \operatorname{Diff}^r(\mathbb{R}^{n-1};\mathbb{R}^{n-1})$ be such that $h(0)=0$ and

$$\det \nabla h(x_1,\ldots,x_{n-1}) = f_{1\cdots(n-1)}(x_1,\ldots,x_{n-1},0), \quad \text{near } 0.$$

For example one can take $h^i(x_1,\ldots,x_{n-1}) = x_i$ for $1 \leq i \leq n-2$ and

$$h^{n-1}(x_1,\ldots,x_{n-1}) = \int_0^{x_{n-1}} f(x_1,\ldots,x_{n-2},t,0)\,dt.$$

Since $f_{1\cdots(n-1)}(0) \neq 0$, we can find, using the method of characteristics, a C^r function φ^i, $1 \leq i \leq n-1$, such that, in a neighborhood of 0,

$$d\varphi^i \wedge f = 0 \quad \text{and} \quad \varphi^i(x_1,\ldots,x_{n-1},0) = h^i(x_1,\ldots,x_{n-1}).$$

Define $\varphi = (\varphi^1,\ldots,\varphi^{n-1},x_n)$. Then φ is easily seen to be a C^r diffeomorphism near 0 and verifies $\varphi(0)=0$. We claim that, near 0,

$$\varphi^*(dx^1 \wedge \cdots \wedge dx^{n-1}) = f.$$

Indeed, since $d\varphi^i \wedge f = 0$ for every $1 \leq i \leq n-1$, we deduce using Cartan lemma (see e.g. Theorem 2.42 in [4]) the existence of a C^{r-1} function λ such that

$$\varphi^*(dx^1 \wedge \cdots \wedge dx^{n-1}) = d\varphi^1 \wedge \cdots \wedge d\varphi^{n-1} = \lambda f. \tag{4}$$

It remains to show that $\lambda \equiv 1$ to have the claim. Since

$$\varphi(x_1,\ldots,x_{n-1},0) = (h(x_1,\ldots,x_{n-1}),0),$$

we immediately deduce that $\lambda \equiv 1$ on $x_n = 0$. In particular $\lambda(0) = 1 \neq 0$. Using (4) we hence have that

$$\frac{1}{\lambda(\varphi^{-1})} dx^1 \wedge \cdots \wedge dx^{n-1} = (\varphi^{-1})^*(f).$$

Since $df = 0$ we directly deduce that $1/\lambda(\varphi^{-1})$ (and hence $\lambda(\varphi^{-1})$) does not depend of x_n. Combining this with the fact that $\lambda(\varphi^{-1}) \equiv 1$ on $x_n = 0$ (since $\varphi^{-1}(\{x_n=0\}) \subset \{x_n=0\}$ and $\lambda = 1$ on $x_n = 0$) we deduce that $\lambda(\varphi^{-1}) \equiv 1$ and therefore $\lambda \equiv 1$, which proves the claim.

3.3 The case $k=2$

We now turn to the case $k=2$.

Theorem 6 *Let $1 \leq 2m \leq n$, $r \geq 1$ be integers and $x_0 \in \mathbb{R}^n$. Let f and g be two C^r 2-forms verifying, in a neighborhood of x_0,*

$$df = dg = 0 \quad \textit{and} \quad \operatorname{rank}[f] = \operatorname{rank}[g] = 2m.$$

Then there exist a neighborhood U of x_0 and $\varphi \in \operatorname{Diff}^r(U;\varphi(U))$ such that $\varphi(x_0) = x_0$ and

$$\varphi^*(g) = f \quad \textit{in } U.$$

Remark 7 When $2m = n$, the result is weaker than the one in [2] (see also Theorem 14.1 in [4]). It is however better, when $2m < n$, than Theorem 14.3 in [4].

Proof As in the previous proof we can assume that $x_0 = 0$ and

$$f = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}.$$

Up to permuting the coordinates we can also suppose that

$$(g^m)_{1\cdots(2m)}(0) \neq 0.$$

Since $\operatorname{rank}[g] = 2m$ (see (iii) of Sect. 2.1) it is easy to find a neighborhood V of 0 and $a^i \in C^r(V;\mathbb{R}^n)$, $2m+1 \leq i \leq n$, such that for every $x \in V$

$$\{a^{2m+1}(x), \ldots, a^n(x)\} \text{ is linearly independent}$$
$$\operatorname{span}\{a^{2m+1}(x), \ldots, a^n(x)\} = \ker \overline{g}(x).$$

Then, exactly as the proof of Theorem 4.5 of [4], we have that the family $\{a^{2m+1}, \ldots, a^n\}$ is involutive in V. Define for ϵ small enough,

$$\widetilde{g} \in C^r((-\epsilon,\epsilon)^{2m}; \Lambda^2(\mathbb{R}^{2m}))$$

as (recall the abbreviation $0_l = (0, \ldots, 0) \in \mathbb{R}^l$)

$$\widetilde{g}(x_1, \ldots, x_{2m}) = \sum_{1\leq i<j\leq 2m} g_{ij}(x_1, \ldots, x_{2m}, 0_{n-2m})dx^i \wedge dx^j,$$

or in other words, $\widetilde{g} = i^*(g)$, where $i : \mathbb{R}^{2m} \to \mathbb{R}^n$ is the inclusion map given by $i(x_1, \ldots, x_{2m}) = (x_1, \ldots, x_{2m}, 0_{n-2m})$. Note that $(\widetilde{g})^m(0) \neq 0$ and therefore $\widetilde{g}$ has rank $2m$ near 0 (cf. (iii) of Sect. 2.1). Using Theorem 14.1 of [4] there exists a C^r local diffeomorphism $h : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ ($h = (h^1, \ldots, h^{2m})$) such that $h(0) = 0$ and

$$h^*(i^*(g)) = h^*(\widetilde{g}) = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} \quad \text{near } 0. \tag{5}$$

(We even have that $h \in C^{r,\alpha}$ for any $0 < \alpha < 1$, since $\widetilde{g} \in C^r \subset C^{r-1,\alpha}$, but we do not need this.) We claim that, for $x = (x_1, \ldots, x_n)$ small enough,

$$\varphi(x_1, \ldots, x_{2m}, x_{2m+1}, \ldots, x_n) = \varphi^{a^{2m+1}}_{x_{2m+1}} \circ \cdots \circ \varphi^{a^n}_{x_n}(h(x_1, \ldots, x_{2m}), 0_{n-2m})$$

has all the required properties, where we recall that φ^a_t stands for the unique solution of

$$\begin{cases} \frac{d}{dt}\varphi^a_t = a \circ \varphi^a_t \\ \varphi^a_0 = \mathrm{id}. \end{cases}$$

First of all, note that $\varphi(0) = 0$ and φ is C^r near 0. Then observing that

$$\nabla\varphi(0) = \begin{pmatrix} h^1_{x_1}(0) & \cdots & h^1_{x_{2m}}(0) & (a^{2m+1}_1)(0) & \cdots & (a^n_1)(0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ h^{2m}_{x_1}(0) & \cdots & h^{2m}_{x_{2m}}(0) & (a^{2m+1}_{2m})(0) & \cdots & (a^n_{2m})(0) \\ 0 & \cdots & 0 & (a^{2m+1}_{2m+1})(0) & \cdots & (a^n_{2m+1})(0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & (a^{2m+1}_n)(0) & \cdots & (a^n_n)(0) \end{pmatrix}$$

we deduce that

$$\det \nabla\varphi(0) = \det \nabla h(0) \cdot (a^{2m+1} \wedge \cdots \wedge a^n)_{(2m+1)\cdots n}(0) \neq 0$$

where we have used Corollary 26 for the last inequality. Hence φ is a C^r diffeomorphism near 0. Recalling that the family $a^{2m+1}, \ldots, a^n$ is involutive near 0, we have, by Theorem 2 (more precisely its proof), that, for every $m+1 \leq i \leq n$,

$$\frac{\partial\varphi}{\partial x_i} \in \operatorname{span}\{(a^{m+1} \circ \varphi), \ldots, (a^n \circ \varphi)\} \quad \text{near } 0.$$

Therefore near 0, we have, for every $2m+1 \leq i \leq n$,

$$\frac{\partial\varphi}{\partial x_i} \in \ker(\overline{g}) \circ \varphi.$$

Using Lemma 4, we deduce that, near 0,

$$(\varphi^*(g))_{ij} = 0 \quad \text{for every } 2m+1 \leq i < j \leq n.$$

In other words, near 0,

$$\varphi^*(g) = \sum_{1\leq i<j\leq 2m} \lambda_{ij} dx^i \wedge dx^j \tag{6}$$

where λ_{ij} are C^{r-1} functions. On one hand, noticing that $\varphi \circ i = i \circ h$, i.e.

$$\varphi(x_1, \ldots, x_{2m}, 0_{n-2m}) = (h(x_1, \ldots, x_{2m}), 0_{n-2m}),$$

we have, using (5) and (6), that

$$\sum_{1\leq i<j\leq 2m} \lambda_{ij}(x_1, \ldots, x_{2m}, 0_{n-2m})\, dx^i \wedge dx^j = i^*(\varphi^*(g))(x_1, \ldots, x_{2m})$$
$$= h^*(i^*(g))(x_1, \ldots, x_{2m}) = h^*(\widetilde{g})(x_1, \ldots, x_{2m}) = \sum_{i=1}^m dx^{2i-1} \wedge dx^{2i}.$$

We therefore obtain, if $x_{2m+1} = \cdots = x_n = 0$, that

$$\lambda_{ij} = \begin{cases} 1 & \text{if } (i,j) = (2l-1, 2l) \text{ for some } 1 \leq l \leq m \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

On the other hand, since $dg = 0$ and hence

$$d(\varphi^*(g)) = d\left(\sum_{1\leq i<j\leq 2m} \lambda_{ij} dx^i \wedge dx^j\right) = 0$$

we immediately deduce that

$$\lambda_{ij} = \lambda_{ij}(x_1, \ldots, x_{2m}).$$

Combining these last two observations we directly deduce that λ_{ij} verifies (7) for every x. This ends the proof. $\square$

4 The second order pullback equation

4.1 The cases $k = 0$ and $k = 1$

Proposition 8 *Let $r \geq 2$, $n \geq 1$, be integers, $x_0 \in \mathbb{R}^n$ and $f, g \in C^r(\mathbb{R}^n)$ be such that*

$$g(x_0) = f(x_0) \quad \textit{and} \quad \nabla g(x_0), \nabla f(x_0) \neq 0.$$

Then there exist a neighborhood U of x_0 and $\Phi \in C^r(U)$ such that

$$g(\nabla\Phi) = f \quad \textit{in } U, \quad \nabla\Phi \in \mathrm{Diff}^{r-1}(U; \nabla\Phi(U)) \quad \textit{and} \quad \nabla\Phi(x_0) = x_0.$$

Remark 9 (i) We should point out that the result is weaker, from the point of view of regularity, than the one for first order (see Theorem 13.1 in [4]). Indeed it can be proved that there exist a neighborhood U of x_0 and $\varphi \in \mathrm{Diff}^r(U; \varphi(U))$ such that

$$g(\varphi) = f \quad \text{in } U \quad \text{and} \quad \varphi(x_0) = x_0.$$

The proposition cannot be improved as the elementary example $g(x) = x_1$ shows. Indeed in this case we have

$$g(\nabla\Phi) = \frac{\partial\Phi}{\partial x_1} = f$$

and therefore no gain of regularity in the variables $x_2, \ldots, x_n$ can be expected in general.

(ii) A similar remark applies to the next corollary (see Corollary 13.3 in [4]).

Proof With no loss of generality we can assume that $x_0 = 0$. We split the proof into two cases.

Case 1 There exist $1 \leq i, j \leq n$ such that $i \neq j$ and $f_{x_i}(0), g_{x_j}(0) \neq 0$. Without loss of generality (the proof being exactly the same for the other cases) we can assume that $i = n-1$ and $j = n$, that is, $f_{x_{n-1}}(0), g_{x_n}(0) \neq 0$. Let $h \in C^\infty(\mathbb{R}^{n-1})$ be defined by

$$h(x_1, \ldots, x_{n-1}) = \sum_{i=1}^{n-2} x_i^2.$$

Since $g_{x_n}(0) \neq 0$, $\nabla h(0) = 0$ and $g(0) = f(0)$, by classical results about first order equations (cf. for example [7]), there exist a neighborhood U of 0 and a (unique) $\Phi \in C^r(U)$ verifying

$$\begin{cases} g(\nabla\Phi) = f & \text{in } U \\ \Phi(x_1, \ldots, x_{n-1}, 0) = h(x_1, \ldots, x_{n-1}) & \text{for } (x_1, \ldots, x_{n-1}, 0) \in U \\ \nabla\Phi(0) = 0. \end{cases}$$

If we show that $\det \nabla^2\Phi(0) \neq 0$, the proof will be finished, taking U smaller if necessary. Differentiating $g(\nabla\Phi) = f$ in 0 we find, since $\Phi = h$ on $x_n = 0$,

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \Phi_{x_1 x_n}(0) \\ 0 & 1 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \Phi_{x_{n-1} x_n}(0) \\ \Phi_{x_1 x_n}(0) & \cdots & \cdots & \cdots & \Phi_{x_{n-1} x_n}(0) & \Phi_{x_n x_n}(0) \end{pmatrix} \begin{pmatrix} g_{x_1}(0) \\ \vdots \\ \vdots \\ \vdots \\ g_{x_{n-1}}(0) \\ g_{x_n}(0) \end{pmatrix} = \begin{pmatrix} f_{x_1}(0) \\ \vdots \\ \vdots \\ \vdots \\ f_{x_{n-1}}(0) \\ f_{x_n}(0) \end{pmatrix}.$$

On one hand, we deduce from the previous equation that

$$\Phi_{x_{n-1}x_n}(0)\, g_{x_n}(0) = f_{x_{n-1}}(0)$$

which implies, since $g_{x_n}(0)\,,\, f_{x_{n-1}}(0) \neq 0$, that

$$\Phi_{x_{n-1}x_n}(0) \neq 0.$$

On the other hand, an easy calculation gives that the determinant of the previous $n \times n$ matrix (which is precisely $\nabla^2 \Phi(0)$) is

$$(-1)^n (\Phi_{x_{n-1}x_n})^2.$$

We therefore find that $\det \nabla^2 \Phi(0) \neq 0$.

Case 2 There exists $1 \le i \le n$ such that

$$f_{x_i}(0)\,,\, g_{x_i}(0) \neq 0 \quad \text{and} \quad f_{x_j}(0)\,,\, g_{x_j}(0) = 0 \quad \text{for } j \neq i.$$

Without loss of generality (the proof being exactly the same for the other cases) we can assume that $i = n$, that is,

$$f_{x_n}(0)\,,\, g_{x_n}(0) \neq 0 \quad \text{and} \quad f_{x_j}(0) = g_{x_j}(0) = 0 \quad \text{for every } 1 \le j \le n-1.$$

Let $h \in C^\infty(\mathbb{R}^{n-1})$ be defined by

$$h(x_1, \ldots, x_{n-1}) = \sum_{i=1}^{n-1} x_i^2.$$

Since $g_{x_n}(0) \neq 0$, $\nabla h(0) = 0$ and $g(0) = f(0)$, by classical results about first order equations (cf. for example [7]), there exist a neighborhood U of 0 and a (unique) $\Phi \in C^r(U)$ verifying

$$\begin{cases} g(\nabla\Phi) = f & \text{in } U \\ \Phi(x_1, \ldots, x_{n-1}, 0) = h(x_1, \ldots, x_{n-1}) & \text{for } (x_1, \ldots, x_{n-1}, 0) \in U \\ \nabla\Phi(0) = 0. \end{cases}$$

If we show that $\det \nabla^2 \Phi(0) \neq 0$, the proof will be finished, taking U smaller if necessary. Differentiating $g(\nabla\Phi) = f$ in 0 we find, since $\Phi = h$ on $x_n = 0$,

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \Phi_{x_1 x_n}(0) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & \Phi_{x_{n-1}x_n}(0) \\ \Phi_{x_1 x_n}(0) & \cdots & \cdots & \Phi_{x_{n-1}x_n}(0) & \Phi_{x_n x_n}(0) \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ g_{x_n}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ f_{x_n}(0) \end{pmatrix}$$

We deduce from the previous equation that $\Phi_{x_i x_n}(0)\, g_{x_n}(0) = 0$, $1 \le i \le n-1$, which implies, since $g_{x_n}(0) \neq 0$, that

$$\Phi_{x_i x_n}(0) = 0, \quad 1 \le i \le n-1.$$

Similarly we also get that $\Phi_{x_n x_n}(0) \neq 0$. Then, noticing that the determinant of the previous $n \times n$ matrix (which is precisely $\nabla^2 \Phi(0)$) is

$$\Phi_{x_n x_n}(0)$$

we have the claim. $\square$

Corollary 10 *Let $n, r \geq 1$ be integers, $x_0 \in \mathbb{R}^n$ and f, g be C^r 1-forms such that*

$$df = dg = 0 \text{ near } x_0 \quad \text{and} \quad f(x_0), g(x_0) \neq 0.$$

Then there exist a neighborhood U of x_0 and $\Phi \in C^{r+1}(U)$ such that

$$(\nabla\Phi)^*(g) = f \text{ in } U, \quad \nabla\Phi \in \operatorname{Diff}^r(U; \nabla\Phi(U)) \quad \text{and} \quad \nabla\Phi(x_0) = x_0.$$

Proof By Poincaré lemma (cf. for example Corollary 8.6 of [4]) there exist a neighborhood V of x_0 and $G, F \in C^{r+1}(V)$ such that

$$dF = f \quad \text{and} \quad dG = g \quad \text{in } V.$$

Adding if necessary a constant, we can also assume that $F(x_0) = G(x_0)$. We are then in a position to apply Proposition 8 to get $U \subset V$, a neighborhood of x_0, and $\Phi \in C^{r+1}(U)$ such that

$$G(\nabla\Phi) = F \quad \text{in } U \quad \nabla\Phi \in \operatorname{Diff}^r(U; \nabla\Phi(U)) \quad \text{and} \quad \nabla\Phi(x_0) = x_0.$$

Applying the exterior derivative to both sides of $G(\nabla\Phi) = F$ (which is equivalent to $(\nabla\Phi)^*(G) = F$) we get that

$$(\nabla\Phi)^*(dG) = dF,$$

which is precisely our claim. □

4.2 Counterexamples

We start with a counterexample for $(n-1)$-forms.

Proposition 11 *Let $f \in C^\infty(\mathbb{R}^3; \Lambda^2)$ be given by*

$$f = (1 + x_3)\, dx^1 \wedge dx^2 + (2x_1x_3 + x_2)\, dx^1 \wedge dx^3.$$

Then there exists no $\Phi \in C^3(\mathbb{R}^3)$ such that, near 0,

$$(\nabla\Phi)^*(dx^1 \wedge dx^2) = f$$

although there exists a local C^∞ diffeomorphism φ such that

$$\varphi^*(dx^1 \wedge dx^2) = f.$$

Proof Since $df = 0$ and $f(0) \neq 0$, there exists (cf. Theorem 15.3 in [4]) a local C^∞ diffeomorphism φ such that

$$\varphi^*(dx^1 \wedge dx^2) = f.$$

It remains to show that there exists no $\Phi \in C^2$ such that

$$d(\Phi_{x_1}) \wedge d(\Phi_{x_2}) = f.$$

For the sake of contradiction suppose that such a Φ exists. We therefore must have

$$d(\Phi_{x_1}) \wedge f = d(\Phi_{x_2}) \wedge f = 0$$

which is equivalent to the two following equations

$$-(2x_1x_3 + x_2)\Phi_{x_1x_2} + (1 + x_3)\Phi_{x_1x_3} = 0 \tag{8}$$
$$-(2x_1x_3 + x_2)\Phi_{x_2x_2} + (1 + x_3)\Phi_{x_2x_3} = 0. \tag{9}$$

Computing $\frac{\partial}{\partial x_2}$ (8) $-$ $\frac{\partial}{\partial x_1}$ (9) we directly obtain that

$$\Phi_{x_1 x_2} - 2x_3 \Phi_{x_2 x_2} = 0 \tag{10}$$

which is equivalent to

$$\frac{\partial}{\partial x_2}(\Phi_{x_1} - 2x_3 \Phi_{x_2}) = 0$$

or to

$$\Phi_{x_1} - 2x_3 \Phi_{x_2} = h(x_1, x_3) \tag{11}$$

for some function h. Combining (8) and (10) we obtain that

$$-2x_3(2x_1x_3 + x_2)\Phi_{x_2x_2} + (1 + x_3)\Phi_{x_1x_3} = 0$$

and multiplying (9) by $2x_3$ we get that

$$-2x_3\,(2x_1x_3 + x_2)\,\Phi_{x_2x_2} + 2x_3\,(1 + x_3)\,\Phi_{x_2x_3} = 0.$$

Hence the last two equations imply directly that, near 0,

$$\Phi_{x_1x_3} = 2x_3 \Phi_{x_2x_3}\,.$$

Differentiating (11) with respect to x_3 and using the previous equation we obtain

$$h_{x_3} = \Phi_{x_1x_3} - 2x_3\Phi_{x_2x_3} - 2\Phi_{x_2} = -2\Phi_{x_2}.$$

Since h does not depend on x_2, we immediately get from the previous equation that

$$\Phi_{x_2x_2} = 0.$$

Combining this with (9) and (10) we find that

$$\Phi_{x_1x_2} = \Phi_{x_2x_3} = 0 \quad \text{near } 0$$

and hence $d(\Phi_{x_2}) = 0$ and finally

$$0 = d(\Phi_{x_1}) \wedge d(\Phi_{x_2}) = f$$

which is the desired contradiction. □

We now turn to a counterexample for symplectic forms.

Proposition 12 *Let $f \in C^\infty(\mathbb{R}^4; \Lambda^2)$ be defined by*

$$f = (1 + x_3)\, dx^1 \wedge dx^2 + x_2\, dx^1 \wedge dx^3 + 2\, dx^3 \wedge dx^4.$$

Then there exists no $\Phi \in C^3(\mathbb{R}^4)$ such that near 0

$$(\nabla\Phi)^* \left(dx^1 \wedge dx^2 + dx^3 \wedge dx^4\right) = f$$

although there exists a local C^∞ diffeomorphism φ such that

$$\varphi^* \left(dx^1 \wedge dx^2 + dx^3 \wedge dx^4\right) = f.$$

Proof Since $df = 0$ and rank $[f(0)] = 4$ (since $f^2(0) \neq 0$) there exists (cf. Theorem 14.1 in [4]) a local C^∞ diffeomorphism φ such that

$$\varphi^*\left(dx^1 \wedge dx^2 + dx^3 \wedge dx^4\right) = f.$$

We now show that we cannot choose $\varphi = \nabla\Phi$. For the sake of contradiction suppose that such a Φ exists. Then it has to satisfy the following six equations

$$\begin{cases} \Phi_{x_1x_1}\Phi_{x_2x_2} - \Phi_{x_1x_2}\Phi_{x_1x_2} + \Phi_{x_1x_3}\Phi_{x_2x_4} - \Phi_{x_2x_3}\Phi_{x_1x_4} = 1 + x_3 \\ \Phi_{x_1x_1}\Phi_{x_2x_3} - \Phi_{x_1x_3}\Phi_{x_1x_2} + \Phi_{x_1x_3}\Phi_{x_3x_4} - \Phi_{x_3x_3}\Phi_{x_1x_4} = x_2 \\ \Phi_{x_1x_1}\Phi_{x_2x_4} - \Phi_{x_1x_4}\Phi_{x_1x_2} + \Phi_{x_1x_3}\Phi_{x_4x_4} - \Phi_{x_3x_4}\Phi_{x_1x_4} = 0 \\ \Phi_{x_1x_2}\Phi_{x_2x_3} - \Phi_{x_2x_2}\Phi_{x_1x_3} + \Phi_{x_2x_3}\Phi_{x_3x_4} - \Phi_{x_3x_3}\Phi_{x_2x_4} = 0 \\ \Phi_{x_1x_2}\Phi_{x_2x_4} - \Phi_{x_2x_2}\Phi_{x_1x_4} + \Phi_{x_2x_3}\Phi_{x_4x_4} - \Phi_{x_3x_4}\Phi_{x_2x_4} = 0 \\ \Phi_{x_1x_3}\Phi_{x_2x_4} - \Phi_{x_2x_3}\Phi_{x_1x_4} + \Phi_{x_3x_3}\Phi_{x_4x_4} - \Phi_{x_3x_4}\Phi_{x_3x_4} = 2. \end{cases} \tag{12}$$

In particular, writing the second, third, fourth and fifth equations of (12) in matrix form, we get

$$\begin{pmatrix} \Phi_{x_3x_4} - \Phi_{x_1x_2} & -\Phi_{x_3x_3} & \Phi_{x_1x_1} & 0 \\ \Phi_{x_4x_4} & -\Phi_{x_3x_4} - \Phi_{x_1x_2} & 0 & \Phi_{x_1x_1} \\ -\Phi_{x_2x_2} & 0 & \Phi_{x_3x_4} + \Phi_{x_1x_2} & -\Phi_{x_3x_3} \\ 0 & -\Phi_{x_2x_2} & \Phi_{x_4x_4} & -\Phi_{x_3x_4} + \Phi_{x_1x_2} \end{pmatrix} \cdot \begin{pmatrix} \Phi_{x_1x_3} \\ \Phi_{x_1x_4} \\ \Phi_{x_2x_3} \\ \Phi_{x_2x_4} \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{13}$$

An easy calculation gives that the determinant of the matrix on the left-hand side of (13) is equal to

$$\left(\Phi_{x_1x_1}\Phi_{x_2x_2} - \Phi_{x_1x_2}^2 - \Phi_{x_3x_3}\Phi_{x_4x_4} + \Phi_{x_3x_4}^2\right)^2.$$

Subtracting the first equation of (12) from the last equation of (12), it follows that

$$\left(\Phi_{x_1x_1}\Phi_{x_2x_2} - \Phi_{x_1x_2}^2 - \Phi_{x_3x_3}\Phi_{x_4x_4} + \Phi_{x_3x_4}^2\right)^2 = (-1 + x_3)^2.$$

Hence, for $x_3 \neq 1$, (13) is easily seen to be equivalent to

$$\begin{cases} (-1 + x_3)\,\Phi_{x_1x_3} = x_2(\Phi_{x_1x_2} + \Phi_{x_3x_4}) \\ (-1 + x_3)\,\Phi_{x_1x_4} = x_2\Phi_{x_4x_4} \\ (-1 + x_3)\,\Phi_{x_2x_3} = x_2\Phi_{x_2x_2} \\ \Phi_{x_2x_4} = 0. \end{cases} \tag{14}$$

Differentiating the second equation of (14) with respect to x_2 and using that $\Phi_{x_2x_4} = 0$ we obtain that

$$\Phi_{x_4x_4} = 0.$$

Inserting this last equation in the second equation of (14) we get that

$$\Phi_{x_1x_4} = 0.$$

Hence, since $\Phi_{x_1x_4} = \Phi_{x_2x_4} = \Phi_{x_4x_4} = 0$, the last equation of (12) becomes

$$-(\Phi_{x_3x_4})^2 = 2$$

which is the desired contradiction. □

5 The second order case for exterior forms

5.1 The case of k-forms of rank k and the symplectic case

We start with the case $k = 1$ (see also Corollary 10).

Proposition 13 *Let $n \geq 3$ be an integer and $f, g \in \Lambda^1(\mathbb{R}^n)$ be such that $f, g \neq 0$. Then there exists $A \in \mathbb{R}^{n\times n}$ such that* $\det A > 0,\ A^t = A$ *and*

$$A^*(g) = f.$$

Remark 14 When $n = 2$ the previous proposition is still verified except for the conclusion $\det A > 0$. Indeed, for $g = e^1$ and $f = e^2$, any symmetric A verifying $A^*(g) = f$ necessarily satisfies $\det A = -1 < 0$.

Proof Step 1 We first show that we can assume that $g = e^1$. Indeed suppose that for any $h \in \Lambda^1(\mathbb{R}^n)$, $h \neq 0$ there exists a symmetric matrix A such that

$$\det A > 0 \quad \text{and} \quad A^*(e^1) = h.$$

Let $g, f \in \Lambda^1(\mathbb{R}^n)$. Take (cf. for example Proposition 2.24 (i) of [4]) an invertible matrix X such that $X^*(g) = e^1$. By hypothesis there exists a symmetric matrix A such that

$$\det A > 0 \quad \text{and} \quad A^*(e^1) = (X^{-t})^*(f).$$

Replacing e^1 by $X^*(g)$ in the last equation we deduce that

$$(XAX^t)^*(g) = f.$$

The matrix XAX^t has then all the desired properties.

Step 2 We show the proposition when $g = e^1$. We split the discussion into two cases.

Case 1 $f_2 = \cdots = f_n = 0$. Then, noting that $f_1 \neq 0$ since $f \neq 0$, the diagonal matrix A defined by

$$A_i^i = \begin{cases} f_1 & \text{if } i = 1, 2 \\ 1 & \text{if } i \geq 3 \end{cases}$$

has all the desired properties.

Case 2 There exists $l \geq 2$, with $f_l \neq 0$. Take $k \in \{2, \ldots n\} \setminus \{l\}$ (here we use that $n \geq 3$). It is then easily seen that the matrix A defined by

$$A_j^i = \begin{cases} f_j & \text{if } i = 1 \\ f_i & \text{if } j = 1 \\ 0 & \text{if } 2 \leq i, j \text{ and } i \neq j \\ 0 & \text{if } i = j = l \\ -1 & \text{if } i = j = k \\ 1 & \text{if } i = j \text{ and } i \notin \{1, k, l\} \end{cases}$$

is symmetric, satisfies $A^*(e^1) = f$ and $\det A = (f_l)^2$, which concludes the proof. $\square$

Corollary 15 *Let $n \geq 3$ be an integer and $f, g \in \Lambda^{n-1}(\mathbb{R}^n)$ be such that $f, g \neq 0$. Then there exists $A \in \mathbb{R}^{n\times n}$ such that* $\det A > 0,\ A^t = A$ *and*

$$A^*(g) = f.$$

Proof By the previous proposition there exists $A \in \mathbb{R}^{n\times n}$ such that $\det A > 0$, $A^t = A$ and

$$A^*(*g) = *f.$$

Using Proposition 2.19 of [4], the previous equation becomes

$$\det A\left[*((A^{-t})^*(g))\right] = *f$$

where we recall that $*$ is the usual Hodge star operator. Therefore letting

$$B = (\det A)^{\frac{1}{n-1}}\, A^{-t}$$

we have that B is symmetric, $\det B > 0$ and

$$B^*(g) = f$$

which ends the proof. □

We give another way of proving Corollary 15. This proof uses the method of characteristics for first order linear partial differential equations, since the method to obtain Φ uses the idea of Remark 5. In order to simplify the notations in the next proposition we write, for $f \in \Lambda^{n-1}(\mathbb{R}^n)$

$$f_{\widehat{i}} = f_{1\cdots(i-1)(i+1)\cdots n} \quad 1 \le i \le n.$$

Proposition 16 *Let $n \ge 2$ be an integer and $f \in \Lambda^{n-1}(\mathbb{R}^n)$ be such that $f_{\widehat{n}} \neq 0$. Then $\Phi : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\Phi(x) = G\left(x_1 + (-1)^n \frac{f_{\widehat{1}}}{f_{\widehat{n}}} x_n, \ldots, x_i + (-1)^{n-i+1}\frac{f_{\widehat{i}}}{f_{\widehat{n}}} x_n, \ldots, x_{n-1} + \frac{f_{\widehat{n-1}}}{f_{\widehat{n}}} x_n\right),$$

verifies

$$(\nabla\Phi)^*(dx^1 \wedge \cdots \wedge dx^{n-1}) = f \quad \text{in } \mathbb{R}^n$$

for any $G \in C^2(\mathbb{R}^{n-1})$ satisfying

$$\det \nabla^2 G = f_{\widehat{n}} \quad \text{in } \mathbb{R}^{n-1}.$$

Proof First notice that

$$\nabla^2\Phi = \begin{pmatrix} \Phi_{x_1x_1} & \cdots & \Phi_{x_1x_{n-1}} & \Phi_{x_1x_n} \\ \vdots & \ddots & \vdots & \vdots \\ \Phi_{x_1x_{n-1}} & \cdots & \Phi_{x_{n-1}x_{n-1}} & \Phi_{x_{n-1}x_n} \\ \Phi_{x_1x_n} & \cdots & \Phi_{x_{n-1}x_n} & \Phi_{x_nx_n} \end{pmatrix}$$

where

$$\Phi_{x_ix_j} = \begin{cases} G_{x_ix_j} & \text{if } 1 \le i \le j \le n-1 \\ \sum_{k=1}^{n-1}(-1)^{n+1-k}\frac{f_{\widehat{k}}}{f_{\widehat{n}}} G_{x_ix_k} & \text{if } 1 \le i \le n-1 \text{ and } j = n \\ \sum_{k.l=1}^{n-1}(-1)^{k+l}\frac{f_{\widehat{k}} f_{\widehat{l}}}{(f_{\widehat{n}})^2} G_{x_kx_l} & \text{if } i = j = n. \end{cases}$$

We write the above identity as

$$\nabla^2\Phi = \begin{pmatrix} G^1 & \cdots & G^{n-1} & \sum_{j=1}^{n-1}(-1)^{n+1-j}\frac{f_{\widehat{j}}}{f_{\widehat{n}}} G^j \\ \Phi_{x_1x_n} & \cdots & \Phi_{x_{n-1}x_n} & \Phi_{x_nx_n} \end{pmatrix}$$

where G^i stands for

$$\begin{pmatrix} G_{x_i x_1} \\ \vdots \\ G_{x_i x_{n-1}} \end{pmatrix}.$$

Hence, for $1 \leq i \leq n-1$,

$$\begin{aligned}
&\left((\nabla\Phi)^*\left(dx^1 \wedge \cdots \wedge dx^{n-1}\right)\right)_{1\cdots(i-1)(i+1)\cdots n} \\
&\quad = \det\left(G^1, \ldots, G^{i-1}, G^{i+1}, \ldots, G^{n-1}, \sum_{j=1}^{n-1} (-1)^{n+1-j} \frac{f_{\widehat{j}}}{f_{\widehat{n}}} G^j\right) \\
&\quad = \det\left(G^1, \ldots, G^{i-1},\ G^{i+1}, \ldots, G^{n-1},\ (-1)^{n+1-i} \frac{f_{\widehat{i}}}{f_{\widehat{n}}} G^i\right) \\
&\quad = \frac{f_{\widehat{i}}}{f_{\widehat{n}}} \det\left(G^1, \ldots, G^{n-1}\right) = f_{\widehat{i}}.
\end{aligned}$$

Since we also have

$$\left((\nabla\Phi)^*\left(dx^1 \wedge \cdots \wedge dx^{n-1}\right)\right)_{1\cdots(n-1)} = \det(G^1, \ldots, G^{n-1}) = f_{\widehat{n}},$$

the proposition is proved. □

We now discuss the more general case of k-forms of rank k.

Theorem 17 *Let $1 \leq k \leq n-1$ be two integers and $f, g \in \Lambda^k(\mathbb{R}^n)$ be such that* $\operatorname{rank}[f] = \operatorname{rank}[g] = k$. *Then there exists $A \in \mathrm{GL}(n)$ such that*

$$A^*(g) = f \quad \text{and} \quad A^t = A.$$

Proof Step 1 Let us first assume that $g = e^1 \wedge \cdots \wedge e^k$. Since $\operatorname{rank}[f] = k$ it follows by classical results (combining Propositions 2.24 (i) and 2.43 (ii) in [4]) that there exists $B \in \mathrm{GL}(n)$ such that

$$B^*(e^1 \wedge \cdots \wedge e^k) = B^1 \wedge \cdots \wedge B^k = f.$$

Let $H \in \mathbb{R}^{k\times k}$ be the submatrix of B obtained by extracting the first k rows and columns, i.e.

$$H = \left(B^i_j\right)^{1\leq i\leq k}_{1\leq j\leq k}.$$

Using Lemma 27 there exists $S \in \mathrm{GL}(k)$ such that

$$SH = (SH)^t \tag{15}$$

and $\det S = 1$, which means that

$$S^*\left(e^1 \wedge \cdots \wedge e^k\right) = (\det S)\ e^1 \wedge \cdots \wedge e^k = e^1 \wedge \cdots \wedge e^k. \tag{16}$$

Let $I_m \in \mathbb{R}^{m\times m}$ denote the identity matrix and $O_{l,m} \in \mathbb{R}^{l\times m}$ (with l rows and m columns) the zero matrix. We then define $Q \in \mathbb{R}^{n\times n}$ by

$$Q = \begin{pmatrix} S & O_{k,n-k} \\ O_{n-k,k} & I_{n-k} \end{pmatrix}$$

and A by $A = QB$. Then A has the form

$$A = QB = \begin{pmatrix} SH & A_{(k+1,n)}^{(1,k)} \\ A_{(1,k)}^{(k+1,n)} & A_{(k+1,n)}^{(k+1,n)} \end{pmatrix}$$

where

$$A_{(l,m)}^{(i,j)} \in \mathbb{R}^{(j-i+1)\times(m-l+1)}$$

denotes the block obtained by extracting the rows i to j and the columns l to m of A. Using (16) we obtain that

$$\begin{aligned} A^* \left(e^1 \wedge \cdots \wedge e^k\right) &= B^* \left(Q^* \left(e^1 \wedge \cdots \wedge e^k\right)\right) = B^* \left(S^* \left(e^1 \wedge \cdots \wedge e^k\right)\right) \\ &= B^* \left(e^1 \wedge \cdots \wedge e^k\right) = B^1 \wedge \cdots \wedge B^k = f. \end{aligned}$$

Note that this equation is independent of the last $n-k$ rows of A. Since A is invertible, we have that the first k rows of A are linearly independent. Hence, using (15) and Lemma 28, we can redefine the last $n-k$ rows of A to obtain that $A^t = A$, $A \in \mathrm{GL}(n)$ and

$$A^* \left(e^1 \wedge \cdots \wedge e^k\right) = f.$$

Step 2 Let now g be an arbitrary exterior k-form of rank k. By Step 1 we have for every $B \in \mathrm{GL}(n)$ that there exists $A \in \mathrm{GL}(n)$ such that

$$A^* \left(e^1 \wedge \cdots \wedge e^k\right) = B^*(f) \quad \text{and} \quad A^t = A.$$

As in Step 1 we can find $B_1 \in \mathrm{GL}(n)$ such that

$$B_1^*(e^1 \wedge \cdots \wedge e^k) = g.$$

We then apply Lemma 18 (with $B_2 = I$) to find $A \in \mathrm{GL}(n)$ such that

$$A^*(g) = f \quad \text{and} \quad A^t = A.$$

The theorem is therefore established. □

In the above theorem we used the following elementary lemma. As already mentioned the lemma cannot be obtained by straight composition of symmetric matrices, since the product of such matrices is, in general, not symmetric.

Lemma 18 *Let* $0 \le k \le n$ *be integers and* $g, h \in \Lambda^k(\mathbb{R}^n)$ *be such that for every* $B \in \mathrm{GL}(n)$ *there exists* $A \in \mathbb{R}^{n\times n}$ *such that*

$$A^*(g) = B^*(h) \quad \textit{and} \quad A^t = A.$$

Then for every $B_1, B_2 \in \mathrm{GL}(n)$ *there exists* $A \in \mathbb{R}^{n\times n}$ *such that* $A^t = A$ *and*

$$A^*(B_1^*(g)) = B_2^*(h).$$

Proof By hypothesis there exists a symmetric matrix C such that

$$C^*(g) = (B_2 B_1^t)^*(h)$$

or equivalently

$$(B_1[B_1^{-1} C B_1^{-t}] B_1^t)^*(g) = (B_1^t)^*(B_2^*(h))$$

or equivalently

$$(B_1^{-1} C B_1^{-t})^*(B_1^*(g)) = B_2^*(h).$$

Hence $A = B_1^{-1} C B_1^{-t}$ has all the desired properties. □

We now turn our attention to the symplectic case where we have the following result.

Theorem 19 *Let n be even and $g, f \in \Lambda^2(\mathbb{R}^n)$ be such that* $\operatorname{rank}[g] = \operatorname{rank}[f] = n$. *Then there exists $A \in \mathrm{GL}(n)$ such that*

$$A^*(g) = f \quad \textit{and} \quad A^t = A.$$

Proof Let $G \in \mathbb{R}^{n \times n}$ (and similarly for F) be defined by

$$G = \left(g_{ij}\right)_{1 \le i, j \le n}$$

with the usual convention that $g_{ij} = -g_{ji}$ if $i \ge j$. With these notations the theorem reads as: for any $G, F \in \mathrm{GL}(n)$ such that $G^t = -G$, $F^t = -F$ there exists $A \in \mathrm{GL}(n)$ such that

$$A^t G A = F \quad \text{and} \quad A^t = A.$$

But this is exactly what will be established in Theorem 20 and the remark following it. □

5.2 Equivalent formulation in terms of matrices

We now prove a theorem on matrices. But let us first recall that a matrix B is called symplectic if

$$B^t J B = J$$

where J is the standard symplectic matrix namely

$$J = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

Theorem 20 *Let n be even. Then the following two statements hold and they are equivalent.*

(i) *For every $F \in \mathrm{GL}(n)$ such that $F^t = -F$ there exists $A \in \mathrm{GL}(n)$ such that*

$$A^t J A = F \quad \textit{and} \quad A^t = A. \tag{17}$$

(ii) *For every $X \in \mathrm{GL}(n)$ there exist $S \in \mathrm{GL}(n)$ with $S^t = S$ and a symplectic matrix $B \in \mathrm{GL}(n)$ such that*

$$X = SB.$$

Remark 21 Statement (i) is in fact more general (and we will prove it in this more general framework). We will indeed prove that for every $G, F \in \mathrm{GL}(n)$ such that $G^t = -G$, $F^t = -F$ there exists $A \in \mathrm{GL}(n)$ such that

$$A^t G A = F \quad \text{and} \quad A^t = A.$$

Similarly, Statement (ii) is more general and indeed will be proved in the following form. The symplectic matrix B is then replaced by a matrix B such that

$$B^t G B = G$$

where $G \in \mathrm{GL}(n)$ with $G^t = -G$.

Proof (i) The following proof has been given to us by D. Kressner and B.C. Vandereycken [8]. According to Sect. 6 in [11], there exists an invertible matrix X such that

$$\widetilde{G} = X^t G X \quad \text{and} \quad \widetilde{F^{-1}} = X^t F^{-1} X$$

are both block diagonal

$$\widetilde{G} = \begin{pmatrix} \widetilde{G}_1 & 0 & \cdots & 0 \\ 0 & \widetilde{G}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \widetilde{G}_s \end{pmatrix} \quad \widetilde{F^{-1}} = \begin{pmatrix} \left(\widetilde{F^{-1}}\right)_1 & 0 & \cdots & 0 \\ 0 & \left(\widetilde{F^{-1}}\right)_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \left(\widetilde{F^{-1}}\right)_s \end{pmatrix}$$

with, for $1 \le i \le s$, $\widetilde{G}_i, \left(\widetilde{F^{-1}}\right)_i \in \mathbb{R}^{2m_i \times 2m_i}$ of the type

$$\widetilde{G}_i = \begin{pmatrix} 0 & S_i \\ -S_i & 0 \end{pmatrix} \quad \left(\widetilde{F^{-1}}\right)_i = \begin{pmatrix} 0 & R_i \\ -R_i & 0 \end{pmatrix}$$

where $S_i, R_i \in \mathbb{R}^{m_i \times m_i}$ are both symmetric and invertible. Now proceeding blockwise, one easily obtains the result.

(ii) Let $X \in \mathrm{GL}(n)$. Since $X^{-t} G X^{-1}$ is skew-symmetric and invertible, there exists by (i) a matrix $A \in \mathrm{GL}(n)$ such that $A^t = A$ and

$$A^t G A = X^{-t} G X^{-1}$$

or equivalently

$$(AX)^t G (AX) = G.$$

Thus $B = AX$ has the desired property, $S = A^{-1}$ is symmetric and

$$X = A^{-1} A X = S B$$

which is the required decomposition.

(iii) Let us now show that (i) and (ii) are equivalent statements. We already proved that (i) $\Rightarrow$ (ii) so let us establish the reverse implication. Let G and F be two invertible skew-symmetric matrices. By classical result (cf. for example Proposition 2.24 of [4]) there exists $X \in \mathrm{GL}(n)$ such that

$$X^t F X = G.$$

Writing $X = SB$ with S symmetric and B such that $B^t G B = G$, we find that the previous equation is equivalent to

$$F = S^{-t} B^{-t} G B^{-1} S^{-1}$$

and therefore, writing $A = S^{-1}$ which is symmetric, we get

$$A^t G A = F$$

which is the desired result. □

We have a very similar result for k-forms of rank k.

Theorem 22 *Let $1 \leq k \leq n$ be integers. Then the following statement holds and is equivalent to that of Theorem 17. For every $g \in \Lambda^k(\mathbb{R}^n)$ such that* $\operatorname{rank}[g] = k$ *and every $X \in \mathrm{GL}(n)$ the following decomposition holds*

$$X = SB$$

where S is symmetric and B verifies

$$B^*(g) = g.$$

Proof Obviously it is enough to prove that the above statement is equivalent to the one of Theorem 17.

Step 1 $(\Rightarrow)$. Let $X \in \mathrm{GL}(n)$. Since $(X^{-1})^*(g)$ is a k-form with rank $[g] = k$, there exists by hypothesis $A \in \mathrm{GL}(n)$ such that $A^t = A$ and

$$A^*(g) = (X^{-1})^*(g)$$

or equivalently

$$(AX)^*(g) = g.$$

Thus

$$X = A^{-1}(AX)$$

is the desired decomposition.

Step 2 $(\Leftarrow)$. Let g and f be two k-forms of rank k. By classical result (combining Propositions 2.24 (i) and 2.43 (ii) in [4]) there exists $X \in \mathrm{GL}(n)$ such that

$$X^*(f) = g.$$

Writing $X = SB$ with S symmetric and B satisfying

$$B^*(g) = g$$

we have

$$g = (SB)^*(f) = B^*(S^*(f))$$

and therefore, noticing that $(B^{-1})^*(g) = g$,

$$(S^{-1})^*(g) = f$$

which is the desired claim, since S is symmetric. □

Acknowledgments We would like to thank D. Kressner and B.C. Vandereycken for providing the proof of (i) in Theorem 20.

6 Appendix

6.1 Appendix 1

We start with a well-known elementary result.

Lemma 23 *Let $a^1, \ldots, a^k, b^1, \ldots, b^k \in \Lambda^1(\mathbb{R}^n)$ be two families of linearly independent k exterior 1-forms. Then*

$$\operatorname{span}\{a^1, \ldots, a^k\} = \operatorname{span}\{b^1, \ldots, b^k\} \tag{18}$$

if and only if there exists $c \neq 0$ *such that*

$$a^1 \wedge \cdots \wedge a^k = c\, b^1 \wedge \cdots \wedge b^k. \tag{19}$$

Proof Suppose first that (18) holds true. Then there exists an invertible matrix $C \in GL(k)$ with entries c_{ij} such that, for $1 \leq i \leq k$,

$$a^i = \sum_{j=1}^{k} c_{ij} b^j. \tag{20}$$

Thus we obtain (19) with $c = \det C$. On the other hand if (19) holds true, then it follows that

$$a^i \wedge b^1 \wedge \cdots \wedge b^k = 0 \quad \text{for } 1 \leq i \leq k.$$

This easily implies that a^i must be of the form (20). □

We now give some algebraic results that have been used in the proof of Theorems 3 and 6.

Lemma 24 *Let* $1 \leq k < n$ *be two integers and* $a^1, \ldots, a^n \in \Lambda^1(\mathbb{R}^n)$ *be linearly independent and such that*

$$\langle a^i; a^j \rangle = 0 \qquad \text{for every } 1 \leq i \leq k < j \leq n.$$

Then

$$(a^1 \wedge \cdots \wedge a^k)_{1\cdots k} \neq 0 \quad \Leftrightarrow \quad (a^{k+1} \wedge \cdots \wedge a^n)_{(k+1)\cdots n} \neq 0.$$

Proof With no loss of generality (cf. Lemma 23) we can assume that $a^1, \ldots, a^n$ satisfy

$$\langle a^i; a^j \rangle = \delta_{ij} \quad \text{for every} 1 \leq i, j \leq n.$$

In other words (identifying 1-forms with vectors) letting $A \in \mathbb{R}^{n\times n}$ be the matrix whose ith row is a^i, we have $A \in O(n)$. In particular we have

$$A^*(e^i) = a^i \quad \text{for every } 1 \leq i \leq n.$$

Using Proposition 2.19 of [4] we then have

$$\begin{aligned} *(a^1 \wedge \cdots \wedge a^k) = *(A^*(e^1 \wedge \cdots \wedge e^k)) &= \tfrac{1}{\det A}\, A^*(e^{k+1} \wedge \cdots \wedge e^n) \\ &= \pm\, a^{k+1} \wedge \cdots \wedge a^n. \end{aligned}$$

We therefore find

$$\begin{aligned} (a^1 \wedge \cdots \wedge a^k)_{1\cdots k} &= (e^1 \wedge \cdots \wedge e^k) \lrcorner (a^1 \wedge \cdots \wedge a^k) \\ &= *((e^1 \wedge \cdots \wedge e^k) \wedge (*(a^1 \wedge \cdots \wedge a^k)) \\ &= \pm * ((e^1 \wedge \cdots \wedge e^k) \wedge a^{k+1} \wedge \cdots \wedge a^n) \\ &= \pm (a^{k+1} \wedge \cdots \wedge a^n)_{(k+1)\cdots n} \end{aligned}$$

which proves the lemma. □

Corollary 25 *Let* $1 \leq k < n$ *be integers and* $f \in \Lambda^k(\mathbb{R}^n)$ *be such that* $\operatorname{rank}[f] = k$ *and* $f_{1\cdots k} \neq 0$. *Let also* $a^{k+1}, \ldots, a^n \in \Lambda^1(\mathbb{R}^n)$ *be such that*

$$\operatorname{span}\left\{a^{k+1}, \ldots, a^n\right\} = \ker \overline{f}.$$

Then

$$(a^{k+1} \wedge \cdots \wedge a^n)_{(k+1)\cdots n} \neq 0.$$

Proof With no loss of generality (cf. Lemma 23) we can assume that $a^{k+1}, \ldots, a^n$ satisfy

$$\langle a^i; a^j \rangle = \delta_{ij} \quad \text{for every } k+1 \leq i, j \leq n.$$

We then choose $a^1, \ldots, a^k \in \Lambda^1(\mathbb{R}^n)$ such that

$$\langle a^i; a^j \rangle = \delta_{ij} \quad \text{for every } 1 \leq i, j \leq n.$$

If we show that

$$f = \lambda\, a^1 \wedge \cdots \wedge a^k$$

for some scalar $\lambda \neq 0$, the corollary will be proved using Lemma 24. Let $B \in \mathrm{O}(n)$ be the matrix whose ith column is equal to a^i and $A = B^{-1} = B^t$ (and therefore A is the matrix whose ith row is equal to a^i). Using Lemma 4 (with $\varphi(x) = Bx$) we deduce that

$$B^*(f) = \lambda\, e^1 \wedge \cdots \wedge e^k,$$

for some scalar $\lambda \neq 0$. Hence

$$f = \lambda\, A^*(e^1 \wedge \cdots \wedge e^k) = \lambda\, a^1 \wedge \cdots \wedge a^k,$$

which proves the claim. □

Corollary 26 *Let* $1 < 2m < n$ *be integers and* $w \in \Lambda^2(\mathbb{R}^n)$ *be such that* $\operatorname{rank}[w] = 2m$ *and* $(w^m)_{1\cdots(2m)} \neq 0$. *Let also* $a^{2m+1}, \ldots, a^n \in \Lambda^1(\mathbb{R}^n)$ *be such that*

$$\operatorname{span}\left\{a^{2m+1}, \ldots, a^n\right\} = \ker \overline{w}.$$

Then

$$\left(a^{2m+1} \wedge \cdots \wedge a^n\right)_{(2m+1)\cdots n} \neq 0.$$

Proof With no loss of generality (cf. Lemma 23) we can assume that $a^{2m+1}, \ldots, a^n$ satisfy

$$\langle a^i; a^j \rangle = \delta_{ij} \quad \text{for every } 2m+1 \leq i, j \leq n.$$

We then choose $a^1, \ldots, a^m \in \Lambda^1(\mathbb{R}^n)$ such that

$$\langle a^i; a^j \rangle = \delta_{ij} \quad \text{for every } 1 \leq i, j \leq n.$$

If we show that

$$w^m = \lambda a^1 \wedge \cdots \wedge a^{2m} \tag{21}$$

for some scalar $\lambda \neq 0$, the corollary will be proved using Lemma 24 and the hypothesis $(w^m)_{1\cdots(2m)} \neq 0$. To show (21), it is enough to prove that

$$w = \sum_{1 \leq i < j \leq 2m} c_{ij}\, a^i \wedge a^j$$

for some $c_{ij} \in \mathbb{R}$. Indeed, if w has the form of the previous equation, then computing w^m we deduce that

$$w^m = \lambda a^1 \wedge \cdots \wedge a^{2m}$$

for some scalar λ. Hence we get that $\lambda \neq 0$ because $w^m \neq 0$ (since $\operatorname{rank}[w] = 2m$). We finally show (21). Since $\{a^1, \ldots, a^n\}$ is a basis of $\mathbb{R}^n$ we have that

$$w = \sum_{1 \le i < j \le n} c_{ij} a^i \wedge a^j$$

for some $c_{ij} \in \mathbb{R}$. It remains to show that $c_{ij} = 0$ for $j > 2m$ to have the claim. Let $s > 2m$. In what follows we make the convention that $c_{ij} = -c_{ji}$. Using Proposition 2.16 of [4] and the fact that $\langle a^i; a^j \rangle = \delta_{ij}$, we easily deduce that

$$0 = a^s \lrcorner\, w = \sum_{1 \le i < j \le n} c_{ij}\, a^s \lrcorner\, (a^i \wedge a^j) = \sum_{1 \le r \le n} \pm c_{sr}\, a^r.$$

This implies that $c_{sr} = 0$ for every $1 \le r \le n$ and every $s > 2m$ and hence proves the claim. $\square$

6.2 Appendix 2

We conclude with some results that have been used in Theorem 17. In the sequel we let $I_m \in \mathbb{R}^{m \times m}$ denote the identity matrix and $O_{l,m} \in \mathbb{R}^{l \times m}$ (with l rows and m columns) the zero matrix.

Lemma 27 *Let $A \in \mathbb{R}^{n \times n}$. Then there exists $S \in \mathbb{R}^{n \times n}$ such that* $\det S = 1$ *and*

$$SA = (SA)^t.$$

Proof There exist $P, Q \in \mathrm{GL}(n)$ and an integer $0 \le r \le n$ such that (cf. for instance [1], Chapter 4, Proposition 2.9)

$$PAQ = \begin{pmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r,n-r} \end{pmatrix}.$$

Let $c, d \in \mathbb{R} \setminus \{0\}$ be given by

$$\det P = c \quad \text{and} \quad \det Q = d$$

and let us define the diagonal matrix $R \in \mathrm{GL}(n)$ by

$$R = \begin{pmatrix} \frac{c}{d} & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Note that $B \in \mathbb{R}^{n \times n}$ defined by

$$B = PAQR \quad \Leftrightarrow \quad A = P^{-1} B (QR)^{-1} \tag{22}$$

satisfies $B^t = B$. We now set

$$S = (QR)^{-t} P.$$

Obviously $\det S = 1$ and we obtain, using (22) and the symmetry of B, that

$$\begin{aligned}(SA)^t &= A^t S^t = (QR)^{-t} B^t P^{-t} P^t (QR)^{-1} = (QR)^{-t} B (QR)^{-1} \\ &= (QR)^{-t} P A = SA\end{aligned}$$

which concludes the proof of the lemma. $\square$

Lemma 28 *Let $0 \le k \le n$ and let $A \in \mathbb{R}^{k\times k}$, $E \in \mathbb{R}^{k\times(n-k)}$ be such that $A^t = A$ and*

$$\operatorname{rank}[(A\ \ E)] = k$$

where the matrix $(A\ \ E) \in \mathbb{R}^{k\times n}$ is obtained by combining A and E as

$$(A\ \ E)^i_j = \begin{cases} A^i_j & \text{if } 1 \le j \le k \\ E^i_{j-k} & \text{if } k+1 \le j \le n. \end{cases}$$

Then there exists $Q \in \mathbb{R}^{(n-k)\times(n-k)}$ such that $Q^t = Q$ and

$$\begin{pmatrix} A & E \\ E^t & Q \end{pmatrix} \in \mathrm{GL}(n).$$

Proof *Step 1* Let $(E^t)^1 = E^t_1$ be the first row of E^t. It is enough to show that there exists $s = (s_1, \dots, s_{n-k}) \in \mathbb{R}^{1\times(n-k)}$ such that

$$\operatorname{rank}[B] = k+1 \tag{23}$$

where $B \in \mathbb{R}^{(k+1)\times n}$ is given by

$$B = \begin{pmatrix} A & E \\ E^t_1 & s \end{pmatrix}. \tag{24}$$

Then we can apply induction on k, supposing that the lemma holds true for $k+1$, and noticing that the case $k = n$ is trivial.

Step 2 The hypothesis $\operatorname{rank}[(A\ \ E)] = k$ is equivalent to the existence of a nonzero minor of order k, or, also equivalently, to the existence of k linearly independent columns of $(A\ \ E)$. Hence there exist $0 \le r \le \min(k, n-k)$ and

$$\begin{cases} L_{k-r} = (l_1, \dots, l_{k-r}) \in \mathbb{N}^{k-r} & \text{with } 1 \le l_1 < \cdots < l_{k-r} \le k \\ J_r = (j_1, \dots, j_r) \in \mathbb{N}^r & \text{with } 1 \le j_1 < \cdots < j_r \le n-k \end{cases} \tag{25}$$

such that

$$\det\left(A_{l_1}\ \dots\ A_{l_{k-r}}\ E_{j_1}\ \dots\ E_{j_r}\right) \neq 0. \tag{26}$$

We now distinguish two cases according to how these linearly independent columns are distributed.

Case 1 Suppose that there exists an $r \le n-k-1$ such that (25) and (26) are satisfied. Note that this is always the case if $k \le n-k-1$, or equivalently $2k < n$.

Case 2 For every $0 \le r \le n-k-1$ and every L_{k-r} and J_r the identity (26) does not hold true. Or in other words, the only possibility for (25) and (26) to be satisfied is with $r = n-k$.

We will deal with Case 1 in Step 3 and with Case 2 in Step 4.

Step 3 (Case 1) In this case there exists $i \in \{1, \dots, n-k\}$ such that $i \notin \{j_1, \dots, j_r\}$. Without loss of generality we may assume that $i = n-k$. We define s in (24) by

$$s = \lambda e_{n-k} = (0, \dots, 0, \lambda).$$

Then, developing the determinant with respect to the last line, we obtain that

$$\begin{aligned} &\det\begin{pmatrix} A_{l_1} & \cdots & A_{l_{k-r}} & E_{j_1} & \cdots & E_{j_r} & E_{n-k} \\ E_1^{l_1} & \cdots & E_1^{l_{k-r}} & 0 & \cdots & 0 & \lambda \end{pmatrix} \\ &\quad = \pm\lambda \det\left[\left(A_{l_1}\ \dots\ A_{l_{k-r}}\ E_{j_1}\ \dots\ E_{j_r}\right)\right] + O(1) \end{aligned}$$

Thus, if we choose λ large enough, we obtain from (26) that B has a nonzero minor of order $k+1$, which proves (23) in the present case.

Step 4 (Case 2) We thus assume that Case 2 holds true.

Step 4.1 Since $r = n-k$ we must have that $(j_1, \ldots, j_{n-k}) = (1, \ldots, n-k)$ and there exists $(l_1, \ldots, l_{2k-n}) \in \mathbb{N}^{2k-n}$ such that

$$\det \left(A_{l_1} \ \ldots \ A_{l_{2k-n}} \ E_1 \ \ldots \ E_{n-k} \right) \neq 0. \tag{27}$$

Thus $\operatorname{rank}[A] \geq 2k-n$. But we must also have $\operatorname{rank}[A] \leq 2k-n$, because we have excluded Case 1, and therefore

$$\operatorname{rank}[A] = 2k-n. \tag{28}$$

Using again (27) we also obtain that

$$\operatorname{rank}[(A \ \ E_1)] = 2k-n+1. \tag{29}$$

We claim that we can choose $s = (0, \ldots, 0)$ in (24).

Step 4.2 Let $M \in \mathrm{GL}(k)$ be defined by

$$M = \left(E_1 \ \ldots \ E_{n-k} \ A_{l_1} \ \ldots \ A_{l_{2k-n}} \right).$$

Then we see that

$$E = M \begin{pmatrix} I_{n-k} \\ O_{2k-n,n-k} \end{pmatrix} \quad \Leftrightarrow \quad M^{-1}E = \begin{pmatrix} I_{n-k} \\ O_{2k-n,n-k} \end{pmatrix}.$$

Showing (23) is equivalent to $\operatorname{rank}\left[B'\right] = k+1$ where

$$B' = \begin{pmatrix} M^{-1} & O_{k,1} \\ O_{1,k} & 1 \end{pmatrix} B = \begin{pmatrix} A_1' & I_{n-k} \\ A_2' & O_{2k-n,n-k} \\ E_1^t & O_{1,n-k} \end{pmatrix}$$

where $A_1' \in \mathbb{R}^{(n-k)\times k}$, $A_2' \in \mathbb{R}^{(2k-n)\times k}$ and $A' \in \mathbb{R}^{k\times k}$, are given by

$$M^{-1}A = A' = \begin{pmatrix} A_1' \\ A_2' \end{pmatrix}.$$

Step 4.3 We claim that

$$\operatorname{rank}\left[\begin{pmatrix} A_2' \\ E_1^t \end{pmatrix}\right] = 2k-n+1. \tag{30}$$

Let us define $N \in \mathrm{GL}(n)$ by

$$N = \begin{pmatrix} I_k & O_{k,n-k} \\ -A_1' & I_{n-k} \end{pmatrix}.$$

We have that

$$\begin{aligned} M^{-1}(A \ \ E)N &= \begin{pmatrix} A_1' & I_{n-k} \\ A_2' & O_{2k-n,n-k} \end{pmatrix} \begin{pmatrix} I_k & O_{k,n-k} \\ -A_1' & I_{n-k} \end{pmatrix} \\ &= \begin{pmatrix} O_{n-k,k} & I_{n-k} \\ A_2' & O_{2k-n,n-k} \end{pmatrix} \end{aligned}$$

This equation and the hypothesis $\operatorname{rank}[(A \ \ E)] = k$ imply that

$$\operatorname{rank}[A_2'] = 2k-n. \tag{31}$$

From (28) and rank $\left[A'\right] = \text{rank}\,[A]$ it also follows that

$$\text{rank}\left[A'\right] = 2k - n. \tag{32}$$

Moreover, using (29) and that $A^t = A$, we also obtain

$$\begin{aligned}\text{rank}\left[\begin{pmatrix} A' \\ E_1^t \end{pmatrix}\right] &= \text{rank}\left[\begin{pmatrix} M^{-1} & O_{k,1} \\ O_{1,k} & 1 \end{pmatrix}\begin{pmatrix} A \\ E_1^t \end{pmatrix}\right] = \text{rank}\left[\begin{pmatrix} A \\ E_1^t \end{pmatrix}\right] \\ &= \text{rank}\left[\begin{pmatrix} A^t & E_1 \end{pmatrix}\right] = \text{rank}\,[(A \;\; E_1)] = 2k - n + 1.\end{aligned}$$

Using this identity, (31) and (32) we get

$$\text{rank}\left[\begin{pmatrix} A_2' \\ E_1^t \end{pmatrix}\right] = \text{rank}\left[\begin{pmatrix} A' \\ E_1^t \end{pmatrix}\right] = 2k - n + 1$$

which was the claim of this step.

Step 4.4 We now show that rank $\left[B'\right] = k + 1$. We obtain that

$$B'N = \begin{pmatrix} A_1' & I_{n-k} \\ A_2' & O_{2k-n,n-k} \\ E_1^t & O_{1,n-k} \end{pmatrix} N = \begin{pmatrix} O_{n-k,k} & I_{n-k} \\ A_2' & O_{2k-n,n-k} \\ E_1^t & O_{1,n-k} \end{pmatrix}.$$

It follows from the special form of $B'N$ and (30) that $\text{rank}[B'N] = (n-k) + (2k-n+1) = k+1$ and therefore

$$\text{rank}\,[B] = \text{rank}\left[B'\right] = \text{rank}[B'N] = k + 1$$

which concludes the proof of the lemma. □

References

1. Artin, M.: Algebra. Prentice Hall, Englewood Cliffs (1991)
2. Bandyopadhyay, S., Dacorogna, B.: On the pullback equation $\varphi^*(g) = f$, Ann. Inst. H. Poincaré Anal. Non Linéaire **26**, 1717–1741 (2009)
3. Bandyopadhyay, S., Dacorogna, B., Kneuss, O.: The pullback equation for degenerate forms. Discrete Contin. Dyn. Syst. Ser. A **27**, 657–691 (2010)
4. Csató, G., Dacorogna, B., Kneuss, O.: The Pullback Equation for Differential Forms. Birkhäuser, Boston (2012)
5. Dacorogna, B., Moser, J.: On a partial differential equation involving the Jacobian determinant. Ann. Inst. H. Poincaré Anal. Non Linéaire **7**, 1–26 (1990)
6. Horn, R.A., Johnson, C.A.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
7. John, F.: Partial Differential Equations. Springer, New York (1982)
8. Kressner, D., Vandereycken, B.C.: Private communication (2012).
9. Serre, D.: Matrices: Theory and Applications. Springer, New York (2002)
10. Taylor, M.E.: Partial Differential Equations, vol. 1. Springer, New York (1996)
11. Thompson, R.C.: Pencils of complex and real symmetric and skew matrices. Linear Algebra Appl. **147**, 323–371 (1991)