Irreducibility in algebraic groups and regular unipotent elements

Donna Testerman and Alexandre Zalesski

1. Introduction

Let H be a reductive linear algebraic group defined over an algebraically closed field F. Throughout this text 'reductive' will mean 'connected reductive'. A unipotent element $u \in H$ is said to be *regular* if the dimension of its centralizer $C_H(u)$ coincides with the rank of H (or, equivalently, u is contained in a unique Borel subgroup of H). Regular unipotent elements of a reductive algebraic group exist in all characteristics (see [**22**]) and form a single conjugacy class. These play an important role in the general theory of algebraic groups. In this paper we study reductive subgroups of H containing a regular unipotent element of H. We find that such subgroups are irreducible in the sense of Serre [**17**], who has generalized the common notions of an irreducible, a completely reducible or a reducible linear group as follows.

Definition 1.1. Let H be a reductive linear algebraic group. A subgroup G of H is called *H-irreducible* if G is contained in no proper parabolic subgroup, *H-completely reducible* (hereafter referred to as H-cr) if, whenever G belongs to a parabolic subgroup P of H, then G belongs to a Levi subgroup of P, and *H-reducible* if G lies in a proper parabolic subgroup of H.

The main result of the paper states that for G, H reductive groups, if G is a closed subgroup of H containing a regular unipotent element of H, then G is H-irreducible. This result is new even for the classical situation where H is the general linear group. Closed subgroups of simple algebraic groups containing a regular unipotent element were studied, and maximal such subgroups classified by Saxl and Seitz in [**16**]; however, they did not treat the irreducibility phenomenon that is the subject of this paper.

The set of regular unipotent elements is a dense open set in the variety of unipotent elements of H. In a sense they are the 'largest' unipotent elements, while the nontrivial elements of root groups are the 'smallest'. In many situations, it is useful to know the subgroups containing elements of this or other special kinds. (See Saxl's survey [**15**] for an overview and bibliography.)

The problem of determining closed subgroups of simple algebraic groups H containing a regular unipotent element has already attracted considerable attention. In [**23**], I. Suprunenko determined closed irreducible semisimple subgroups of $\mathrm{GL}(n, F)$ containing a regular unipotent element. When $\mathrm{char}(F)$ is 0 or a large enough prime, a regular unipotent element lies in a closed subgroup of H isomorphic to (P)SL$(2,F)$. A primary problem solved in [**25, 14**] was to classify all situations when a unipotent element is contained in a closed subgroup of type A_1; the case of regular unipotent elements was crucial, and in some sense the most difficult. Properties of the centralizer C of a regular unipotent element and of its normalizer $N_G(C)$ were investigated in [**13**]. The most extensive study of the overgroups of regular unipotent elements to date was carried out by Saxl and Seitz in [**16**], where they determined maximal (not necessarily connected) positive-dimensional subgroups of H containing a regular unipotent element. As a maximal positive-dimensional subgroup is either the normalizer of a reductive subgroup, or is a parabolic subgroup (by the Borel-Tits theorem [**2**]), and as each parabolic subgroup contains a representative of every unipotent class (and so in particular contains a regular unipotent element), their article is concerned with determining reductive maximal subgroups containing a regular unipotent element.

As an application of our main result, we are able to deduce from [**16**, Theorems A and B] the classification of semisimple subgroups G of simple algebraic groups H such that G contains a regular unipotent element of H. (See Theorem 1.4 below.) Indeed, given such a subgroup $G \subset H$, one embeds G in a maximal positive-dimensional subgroup, which is one of the groups given by the results of [**16**]. If M° is reductive, then G lies in the semisimple group $[M^\circ, M^\circ]$ and we proceed inductively. Otherwise, M is a parabolic subgroup and one is faced with the question of whether G lies in a Levi factor of M in order to again argue inductively. Our main result (Theorem 1.2) solves this problem by showing that G cannot lie in a proper parabolic subgroup of H.

THEOREM 1.2. *Let G be a reductive subgroup of the reductive group H containing a regular unipotent element of H. Then G is not contained in any proper parabolic subgroup of H. In other words, G is H-irreducible.*

The special case $H = \mathrm{SL}(n,F)$ seems worth stating explicitly:

COROLLARY 1.3. *Let $G \subset \mathrm{GL}(n,F)$ be a reductive linear algebraic group. Suppose that G contains an element whose Jordan normal form consists of a single block. Then G is irreducible.*

We first note that the theorem is clearly true if $\mathrm{char}(F) = 0$, since the all FG-modules are then completely reducible and so if G lies in a parabolic subgroup, it necessarily lies in a Levi factor of this group.

Observe that one cannot drop the hypothesis that G is connected. Moreover, a similar statement for finite reductive groups is false: there exists a reducible representation $\rho : \mathrm{PSL}(2,p) \to \mathrm{SL}(p,F)$, where F is of characteristic $p > 0$, such that the image of ρ contains a unipotent element with a single Jordan block matrix.

As mentioned above, we will apply the above results and the main result of [**16**] to obtain the following classification of semisimple subgroups of simple groups H containing a regular unipotent element.

THEOREM 1.4. *Let G be a closed semisimple subgroup of the simple algebraic group H, containing a regular unipotent element of H. Then either the pair H, G is as given in Table* 1 *below or G is isomorphic to* (P)SL$(2,F)$ *and $p = 0$ or $p > h$, where h is the Coxeter number for H. Moreover, for each pair of root systems (Φ_H, Φ_G) as in the table, respectively, for (Φ_H, A_1, p), with $p = 0$ or $p > h$, there*

TABLE 1. Semisimple subgroups $G \subset H$ containing a regular unipotent element

H	G
A_6 A_5	$G_2, p \neq 2$ $G_2, p = 2$
C_3	$G_2, p = 2$
B_3	$G_2, p \neq 2$
D_4	$G_2, p \neq 2$ B_3
E_6	F_4
$A_{n-1}, n > 1$	$C_{n/2}, n$ even $B_{(n-1)/2}, n$ odd, $p \neq 2$
$D_n, n > 4$	B_{n-1}

exists a closed simple subgroup $X \subset H$ of type Φ_G, respectively A_1, containing a regular unipotent element of H.

The conjugacy classes of such subgroups can be deduced from the known structure of maximal connected subgroups of H (see [**19**] and [**11**]).

Our methods for proving the main theorem differ according to whether H is of classical or of exceptional type. In the former case we use results on indecomposable representations of simple algebraic groups applied to our group G and the natural H-module. These include general results such as Lemma 2.2, as well as more special results on splitting certain G-modules of composition length 2 obtained by McNinch [**12**]. For the exceptional groups H, we use in many instances the classification results on maximal subgroups and subgroups of type A_1 obtained by Seitz, Liebeck and Testerman [**11, 25**].

It is well-known that the Jordan normal form of the Kronecker product of two unipotent Jordan blocks is not similar to a Jordan block matrix. It is probably worth mentioning the following generalization of this fact to arbitrary simple algebraic groups: if X and Y are non-abelian commuting reductive subgroups of a simple algebraic group H, then the product XY contains no regular unipotent element of H. This is a special case of Proposition 2.3.

Notation and conventions. We will write $[X, X]$, or X', for the derived subgroup of a group X. The order of an element $x \in X$ is denoted by $|x|$. We use $C_X(M)$ to denote the centralizer in X of a subset $M \subset X$.

Below F is an algebraically closed field of characteristic $p \geq 0$. The term 'a simple algebraic group' designates (unless otherwise stated) a simply-connected simple algebraic group defined over F. To simplify the language we often write $G = A_n, B_n$ etc. instead of the more precise 'G is a simple simply connected linear algebraic group of type A_n, B_n' etc. If G is an algebraic group then G^0 is its connected component. If G is reductive, then $[G, G]$ coincides with the semisimple component of G. All FG-modules under consideration are rational.

As usual, G determines its root system Φ and the weight lattice $\Omega(G)$. We fix a Borel subgroup B of G and a maximal torus $T \subset B$. We denote the corresponding set of simple roots by Δ and the positive roots by Φ^+; dominant weights are denoted by $\Omega^+(G)$. We label Dynkin diagrams as in [**3**] and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$, with associated fundamental dominant weights $\omega_1, \dots, \omega_\ell$. The 1-dimensional connected unipotent group normalized by T, with action given by the character $\alpha \in \Phi$, is denoted by U_α and its elements by $x_\alpha(t)$ $(t \in F)$. For $\mu \in \Omega^+(G)$, we let $V(\mu)$ denote the irreducible FG-module of highest weight μ, and $W(\mu)$ the indecomposable Weyl module of highest weight μ. By a 'classical group', we mean a simple simply connected algebraic group of type A_n $(n \geq 1)$, B_n $(n > 2)$, C_n $(n > 1)$ or D_n $(n > 3)$. Except when $G = B_n$ and $p = 2$, we take the so-called 'natural' module for G to be the irreducible module with highest weight ω_1. In the exceptional case, the natural module for $G = B_n$ is the $(2n+1)$-dimensional reducible FG-module equipped with a nondegenerate quadratic form, whose associated bilinear form has a 1-dimensional radical.

Acknowledgements. We are very grateful to I. Suprunenko and the anonymous referee for carefully reading the manuscript and for providing a detailed list of comments which allowed us to correct several inaccuracies in the first version.

2. Linear representations

Until stated otherwise, we assume that H is a simple algebraic group. We first recall some results from the representation theory of simple algebraic groups and establish a result (Proposition 2.3) which will reduce the problem to the study of simple subgroups. In addition, we will prove Theorem 1.2 in the case where H is the group $\mathrm{SL}(n, F)$.

Lemma 2.1. *Let $G \subset H$ be reductive algebraic groups, and let $u \in G$ be unipotent. If u is regular in H then so are all regular unipotent elements of G.*

Proof. Let x be a regular unipotent element in G. Then u lies in the closure of the G-class of x, and this of course lies in the closure of the H-class of x. If x is not regular in H, this class has dimension strictly less than the dimension of the H-class of u which is a contradiction. Therefore x is a regular element in H as claimed. □

We require some additional notation; let $\Phi(H)$ denote the root system of H and U_α the root subgroups (with respect to a fixed maximal torus T_H of H) for $\alpha \in \Phi(H)$. Fix a base $\Delta(H) \subset \Phi(H)$ and let $\Phi^+(H)$ denote the corresponding set of positive roots, $B_H \supset T_H$ the corresponding Borel subgroup of H, with $R_u(B_H) = U$ and U_1 the subgroup $\Pi_{\beta \in \Phi^+(H) \setminus \Delta(H)} U_\beta$. Finally, we also recall that the centralizer of a regular unipotent element contains no non-central semisimple elements, see [**20**, Ch.III, 1.14(a)].

We will need the following standard result from the literature, see e.g. Humphreys [**7**, 12.4]:

Lemma 2.2. *Let E be an indecomposable module for a simple algebraic group of composition length* 2. *Let μ, λ be the highest weights of E/L, L, resp., where L is the maximal submodule of E. Then either $\lambda > \mu$ or $\mu > \lambda$, and in the latter case E is isomorphic to $W(\mu)/M$, where M is a submodule of $W(\mu)$.*

Actually we get that $W(\mu)$ is reducible if there exists an indecomposable module E as in the above lemma with $\mu > \lambda$.

PROPOSITION 2.3. *Let H be a simple algebraic group and let $X, Y \subset H$ be proper subgroups such that $[X, Y] = 1$ and such that X and Y each contain a semisimple element of $H \setminus Z(H)$. Then XY contains no regular unipotent elements of H.*

We begin with the following:

LEMMA 2.4. *Let H be a simple algebraic group and $u, u' \in U = R_u(B_H)$. Suppose that u is regular and $uu' = u'u$. Then either u' is regular or $u' \in U_1$.*

PROOF. Note first that the lemma is true for groups of rank 2. For this, observe that A_2, $B_2 \cong C_2$, G_2 are the only groups in question. In the first case verifying the lemma is a matter of elementary matrix computations. If G is of type $B_2 = C_2$, the explicit commutator formula for $[u, u_1]$ is available in [**21**, Lemma 33], and for G of type G_2 a similar formula is written down in [**5**, p. 192]. Using these, one easily arrives at the conclusion. (One can also use the commutator formulas for the root subgroups in [**6**, Theorem 1.12.1].)

In general, express u, u' as follows: $u = \Pi_{\alpha \in \Delta(H)} x_\alpha(t_\alpha) \cdot \Pi_{\beta \in \Phi^+(H) \setminus \Delta(H)} x_\beta(t_\beta)$, and $u' = \Pi_{\alpha \in \Delta(H)} x_\alpha(t'_\alpha) \cdot \Pi_{\beta \in \Phi^+(H) \setminus \Delta(H)} x_\beta(t'_\beta)$, where $t_\alpha, t_\beta, t'_\alpha, t'_\beta \in F$.

Suppose the contrary, that is, u' is not regular and $u' \notin U_1$. By [**20**, Ch.III, 1.13] u' is regular if and only if $t'_\alpha \neq 0$ for all $\alpha \in \Delta(H)$. Therefore, there is $\beta \in \Delta(H)$ such that $t'_\beta = 0$. Moreover, as $u' \notin U_1$, there exists a pair of roots $\delta, \gamma \in \Delta(H)$ adjacent to each other in the Dynkin diagram of H, and such that $t'_\gamma = 0$, $t'_\delta \neq 0$. Let $P = \langle B, U_{-\gamma}, U_{-\delta} \rangle$. Then P is a parabolic subgroup of H. Let L be its standard Levi subgroup with $L' := \langle U_{\pm\gamma}, U_{\pm\delta} \rangle$. Let $h : P \to L$ be the natural surjection. Set $\overline{u} = h(u)$, $\overline{u}' = h(u')$. Then $\overline{u}, \overline{u}' \in L'$ and $\overline{u}\overline{u}' = \overline{u}'\overline{u}$. Let $\Phi_L, \Delta_L, \Phi_L^+$ be the root system, simple root system and the set of positive roots for L', with respect to the maximal torus $T_H \cap L$ and Borel subgroup $B_H \cap L$. Then $\Delta_L = \{\gamma, \delta\}$ and

$$\overline{u} = x_\gamma(t_\gamma) x_\delta(t_\delta) \cdot \Pi_{\beta \in (\Phi_L^+ \setminus \Delta_L)} x_\beta(t_\beta)$$

is regular in L' and

$$\overline{u'} = x_\delta(t'_\delta) \cdot \Pi_{\beta \in \Phi_L^+ \setminus \Delta_L} x_\beta(t'_\beta)$$

is not regular. In addition, $\overline{u}\overline{u}' = \overline{u}'\overline{u}$. This is a contradiction, as the result holds in the rank two group L'. □

Proof of Proposition 2.3. Suppose the contrary, and let $u \in XY$ be a regular unipotent element. Then $u = u_1 u_2 = u_2 u_1$ for some unipotent elements $u_1 \in X, u_2 \in Y$. Moreover, neither of u_1, u_2 is regular, as each of X, Y contains a non-central semisimple element. As $uu_i = u_i u$ for $i = 1, 2$, Lemma 2.3 implies that $u_1, u_2 \in U_1$. Then $u = u_1 u_2 \in U_1$, which is false by [**20**, Ch.III, 1.13]. □

The above proof shows slightly more: a regular element u has no factorization $u = u_1 u_2$, where $C_H(u_i)$ contains a semisimple element of $H \setminus Z(H)$.

Theorem 1.2 will follow directly from the following result and Proposition 2.3.

Theorem 2.5. *Let $G \subset H$ be a simple closed subgroup of the simple algebraic group H, and suppose G contains a regular unipotent element of H. Then G is H-irreducible, that is, G is not contained in any parabolic subgroup of H.*

We first establish an elementary lemma.

Lemma 2.6. *Let $P \subset H$ be a proper parabolic subgroup, properly containing a Borel subgroup of H. Let $L \subset P$ be a Levi subgroup and $\pi : P \to L$ the natural surjection. Then for $u \in P$ a regular unipotent element of H, $u \notin L$ and $\pi(u)$ is a regular unipotent element of L.*

Proof. Let $T_H \subset B_H$ be as above. We can assume that $B_H \subset P$. Then $L = \langle T_H, U_{\pm\alpha} : \alpha \in J\rangle$ for some proper non-empty subset J of $\Delta(H)$. Using the notation of the proof of Lemma 2.4, we have $u = \Pi_{\alpha\in\Delta(H)} x_\alpha(t_\alpha) \cdot \Pi_{\beta\in\Phi^+(H)\setminus\Delta(H)} x_\beta(t_\beta)$. By [**20**, Ch.III, 1.13] u regular implies $t_\alpha \neq 0$ for all $\alpha \in \Delta(H)$. This implies the first assertion, as if $u \in L$ then $t_\alpha = 0$ whenever $\alpha \notin J$. The second assertion also follows since $\pi(u) = \Pi_{\alpha\in J} x_\alpha(t_\alpha) \cdot \Pi_{\beta\in\Phi_L^+\setminus J} x_\beta(t_\beta)$, where Φ_L^+ are the set of roots in $\Phi^+(H)$ which are linear combinations of the roots in J. □

Lemma 2.7. *Assume* $\mathrm{char}(F) = 0$ *and let $G \subset H$ be a reductive subgroup of the simple algebraic group H such that $u \in G$ is a regular unipotent element of H. Then G is H-irreducible*

Proof. This follows directly from Lemma 2.6 and the fact that all FG-modules are completely reducible. We apply this to the FG-modules induced by the action of G on the unipotent radical of any parabolic subgroup containing G to see that G lies in a Levi factor, contradicting Lemma 2.6. □

In Lemmas 2.8 and 2.9 below we consider two special cases for $p = 2$. The claim (2) of Lemma 2.8 is stated without proof in [**16**, p. 373]. We provide a proof here for the sake of completeness. Lemma 2.9 would follow from Lemma 2.8 as soon as one shows that every indecomposable representation of $G = G_2$ of degree 7 in characteristic 2 preserves a quadratic form. However, it seems more simple to argue directly.

Lemma 2.8. *Let $G = B_n$ and $p = 2$. Let $\phi : G \to H = A_{2n}$ be an indecomposable representation.*

(1) *The composition factors of ϕ are of dimension $1, 2n$. If the socle of the FG-module corresponding to ϕ is one-dimensional then $\phi(G)$ stabilizes a nondegenerate quadratic form on F^{2n+1} and hence $\phi(G) = \mathrm{SO}(2n+1, F) \subset H = \mathrm{SL}(2n+1, F)$.*

(2) $\mathrm{SO}(2n+1, F)$*, and hence $\phi(G)$, contains no regular unipotent element of H, equivalently, no matrix similar to J_{2n+1}, the unipotent Jordan block of size $2n+1$.*

Proof. (1) It is well-known that the minimal dimension of a non-trivial FG-module is $2n$ with highest weight $\mu = 2^k\omega_1$, and that there is no $2n+1$-dimensional irreducible FG-module. Therefore, the composition length of G on the natural module V for H is 2. Let μ be the highest weight of the non-trivial FG-composition factor of V. By [**9**, II.12.9, II.10.17(2)], the number of non-equivalent non-split extensions of V_0 by $V_{2^k\omega_1}$ is equal to the number of non-equivalent non-split extensions of V_0 by V_{ω_1}. So every non-split extension of V_0 by $V_{2^k\omega_1}$ can be obtained by a Frobenius twist from a non-split extension of V_0 by V_{ω_1}. Therefore, it suffices to deal with $\mu = \omega_1$. Replacing if necessary V by its dual, by Lemma 2.2, we deduce that V is a quotient of $W(\omega_1)$, the Weyl module with highest weight ω_1, whose dimension is $2n+1$. So $V \cong W(\omega_1)$. Furthermore, there is an indecomposable FG-module of dimension $2n+1$ with quadratic form Q defining the orthogonal group $\mathrm{O}(2n+1,F)$, and this module fixes a non-zero vector, as per our earlier discussion concerning the natural B_n-module. The group $\mathrm{SO}(2n+1,F)$ is known to be of type B_n, see [**1**, Sections 23.4 and 23.5]. By the above remarks, this representation is also a twist of the Weyl module $W(\omega_1)$. As $\mathrm{O}(2n+1,F)$ is stable under the Frobenius endomorphism, it follows that $\phi(G)$ coincides with $\mathrm{SO}(2n+1,F)$. This implies (1).

(2) Suppose the contrary, and let $u \in G$ be a unipotent element having one Jordan block in its action on V, so V is uniserial as an $F\langle u\rangle$-module. So the socle of the FG-module V is 1-dimensional and hence $\phi(G)$ preserves a quadratic form on V by (1). Then u stabilizes a totally singular subspace W of dimension $n-1$. Then $\mathrm{Stab}_G(W)$ is a parabolic subgroup P of G, and $u \in P$. Then $X := W^\perp/W$ is a vector space of dimension 3, and the quadratic form Q induces on X a non-degenerate quadratic form defining therefore an orthogonal group $\mathrm{O}(3,F)$. As both W, $W^\perp$ are u-stable, and V is uniserial for u, so is X. Therefore, $\mathrm{SO}(3,F)$ contains a uniserial element u', say, which is the projection of u. Obviously, the order of u' is 4. However, this is false as $\mathrm{SO}(3,F) \cong \mathrm{SL}(2,F)$, so all unipotent elements of $\mathrm{SL}(2,F)$ are of order 2. □

LEMMA 2.9. *Let $p=2$ and $G=G_2$. Suppose that $G \subset H = \mathrm{SL}(7,F)$. Then G contains no regular unipotent element of H.*

PROOF. It is well-known that the minimal dimension of a non-trivial FG-module is 6 with highest weight $\mu = 2^k\omega_1$, and that there is no 7-dimensional irreducible FG-module. Therefore, the composition length of G on the natural module V for H is 2. Let μ be the highest weight of the non-trivial FG-composition factor of V. Arguing as in the proof of the previous lemma, applying again [**9**, II.12.9, II.10.17(2)] and Lemma 2.2, we see that we may assume that V is a quotient of the Weyl module of highest weight $\mu=\omega_1$; but this latter is of dimension 7, so we have that V is isomoprhic to the Weyl module of highest weight ω_1.

We now apply a result of [**24**] which describes the action of the fundamental root group elements of $G=G_2$ on $W(\omega_1)$. Let $E_{ij} \in \mathrm{GL}(7,F)$ denote the matrix with 1 at the position (i,j) and zero elsewhere. As $p=2$, [**24**, the proof of Theorem 3.0, p.43] shows that the matrix of $x_{\alpha_1}(1)x_{\alpha_2}(1)$ (a regular unipotent in G) with respect to a fixed basis of $W(\omega_1)$ is $(1+E_{12}+E_{45}+E_{67}+E_{35})(1+E_{23}+E_{56}) = 1+E_{12}+E_{45}+E_{67})+E_{35}+E_{23}+E_{56}+(E_{13}+E_{46}+E_{36}$. This is not a regular unipotent in $\mathrm{SL}(7,F)$ as the term E_{34} does not occur in this expression. Since regular unipotent elements of G_2 form a single G_2-conjugacy class, the result follows from Lemma 2.1. □

The following lemma is the result [**23**, 1.9], which is crucial in our analysis.

LEMMA 2.10. *Let $\phi : G \to H = \mathrm{SL}(n,F)$, $n>1$, be a non-trivial irreducible representation of the simple algebraic group G, with highest weight $\lambda = a_1\omega_1 + \cdots + a_\ell\omega_\ell$. Suppose that $\phi(G)$ contains a regular unipotent element of H. Then one of the following holds (where $k \geq 0$ is an integer and $k=0$ if $p=0$):*

(i) $G=A_1$, $\lambda = p^k m\omega_1$ and $n=m+1 \leq p$ if $p>0$;

(ii) $G=A_\ell$, $\ell>1$, $\lambda = p^k\omega_1$ or $p^k\omega_\ell$ and $n=\ell+1$;

(iii) $G = C_\ell$, $\ell > 2$, $\lambda = p^k\omega_1$ *and* $n = 2\ell$;

(iv) $G = C_2$, $\lambda = p^k\omega_1$ *and* $n = 4$, *or* $\lambda = p^k\omega_2$ *and* $n = 5$ *for* $p \neq 2$ *and* $n = 4$ *otherwise;*

(v) $G = B_\ell$, $\ell > 2$, $\lambda = p^k\omega_1$ *and* $n = 2\ell + 1$ *for* $p \neq 2$ *and* $n = 2\ell$ *otherwise;*

(vi) $G = G_2$, $p \neq 3$, $\lambda = p^k\omega_1$, *and* $n = 7$ *if* $p \neq 2$ *and* 6 *otherwise;*

(vii) $G = G_2$, $p = 3$ *and* $\lambda = p^k\omega_1$ *or* $p^k\omega_2$ *and* $n = 7$.

Proposition 2.11. *Theorem* 2.5 *is true for* $H = \mathrm{SL}(n, F)$.

Proof. Suppose the contrary, that is, that G is H-reducible, so G acts reducibly on the natural FH-module V. Let $u \in G$ be a unipotent element that is regular in H. This is equivalent to saying that $\dim V^u = 1$, where V^u is the fixed point subspace of u on V. It follows that every FG-submodule of V is indecomposable. Let $0 = V_0 \subset V_1 \subset \cdots \subset V_t = V$ be a composition series for the FG-module V; we have $t > 1$. Then $u|_{V_2}$ is a regular unipotent element in $\mathrm{SL}(V_2)$. Note that as V_2 is indecomposable, and $u|_{V_1}$ and $u|_{V_2/V_1}$ are regular elements, we may assume that the composition factors have highest weights as specified in Lemma 2.10.

We will apply the results of Jantzen [**8**] and McNinch [**12**]. One of them asserts that any FG-module of dimension m is completely reducible if $m \leq p \cdot \ell$, where ℓ is the rank of G (see [**8**, Theorem A] for the case $G = A_1$ and [**12**, Corollary 1.1.1] for the general case). We first note that for $G = A_1$, since u has order p, the dimension of V_2 is at most p. In particular, the above criterion shows that V_2 is completely reducible, contradicting our assumptions. Now we turn to the other representations of Lemma 2.10. We have $\dim V_2 \leq 2(\ell+1)$, respectively 4ℓ, 10, $2(2\ell+1)$, for G as in (ii), respectively (iii), (iv), (v) of Lemma 2.10. Again applying the criterion of [**12**] and recalling that V_2 is indecomposable, we reduce to the following configurations.

(a) $G = A_\ell$, $\ell > 1$ and $p = 2$;

(b) $G = C_\ell$ and $p \leq 3$;

(c) $G = B_\ell$ and $p \leq 3$;

(d) $G = G_2$.

For the cases (a) - (d), we use a stronger result [**12**, Theorem 1], which asserts that an FG-module W of dimension at most $p{\cdot}C$ is completely reducible (where $C = \ell(\ell+1)/2$ for G of type A_ℓ, $\ell(\ell-1)$ for types B_ℓ, C_ℓ and 3 for type G_2), unless the highest weights of the composition factors of W occur in [**12**, Table 5.1.1]. Applying this to the cases (a) - (d), we obtain a contradiction to the indecomposability of V_2 unless either $p = 2$, $G \cong B_\ell$ or C_ℓ and the highest weights of the composition factors are 0 and $2^k\omega_1$, or $G \in \{C_2, G_2\}$ and $p \leq 3$. In the first case, note that there is a surjective homomorphism $B_\ell \to C_\ell$ when $p = 2$, and the highest weight of the irreducible B_ℓ-module induced by the irreducible C_ℓ-module of highest weight ω_1 is ω_1 as well. Thus it suffices to consider only the B_ℓ case. Thus V_2 is an indecomposable B_ℓ-module of composition length 2 with factors of highest weights $2^k\omega_1$ and 0. By Lemma 2.8, the image of B_ℓ in H (and hence the subgroup $C_\ell \subset H$) contains no regular element of H.

Let $G = C_2$. Consider first the case where $p = 2$. Then $|u| = 4$, and hence $\dim V_2 \leq |u| = 4$. But there exists no reducible nontrivial FG-module of dimension 4, so $p > 2$. Now let $p = 3$. Note that the central involution of C_2 is nontrivial in any irreducible representation of dimension 4. It follows that either both composition factors of V_2 are of dimension 4, or V_2 has no composition factor of dimension 4. In the latter case $\dim V_2 \leq 6$, and applying again [12] we get a contradiction. Suppose that $\dim V_2 = 8$. By Lemma 2.10, the highest weights of the composition factors of V_2 are $\mu := 3^k\omega_1$ and $\lambda := 3^m\omega_1$, and we may assume $k \leq m$. By [**12**, Lemma 2.3.3(b)], we can assume that $k = 0$, and $m \geq 1$ by [**12**, Lemma 2.3.1(b)]. Set $\nu = \lambda - \mu = (q-1)\omega_1$, for $q = 3^m$. We now normalize the inner product on $\mathbf{Z}\,\Phi$ so that for $\alpha \in \Phi$ a long root, we have $(\alpha, \alpha) = 1$. Then we will apply [**18**, (6.2)], which shows that $2(\lambda + \omega_1 + \omega_2, \nu) - (\nu, \nu)$ must lie in $(p/2)\,\mathbf{Z} = (3/2)\,\mathbf{Z}$. But a direct calculation shows that $2(\lambda + \omega_1 + \omega_2, \nu) - (\nu, \nu) = \frac{(q-1)(q+5)}{4}$. (The result [**18**, (6.2)] is a consequence of the strong linkage principle.)

Finally, for the case $G = G_2$ and $p \leq 3$, we see that u has order 9 if $p = 3$, and order 8 if $p = 2$ (see Table 2). Then $\dim V \leq 9$, and again by [**12**], V is a completely reducible FG-module unless $p = 2$, and by dimensions we have that V_2 is a twist of the 7-dimensional indecomposable considered in Lemma 2.9. But in this case Lemma 2.9 shows that $u|_{V_2}$ is not regular. This completes the proof. □

3. The case where H is classical

In this section we will establish Theorem 2.5 for the remaining classical groups.

PROPOSITION 3.1. *Theorem* 2.5 *is true if H is classical.*

PROOF. Suppose first that H is of type B_n, respectively C_n. Then a regular unipotent element of H is regular in D, where $D = \mathrm{SL}(2n+1, F)$, resp. $\mathrm{SL}(2n, F)$. Therefore, by Proposition 2.11, G is irreducible on V, and hence cannot be contained in a Levi subgroup of H.

Let H be of type D_n for $n > 3$, and let V be the natural FH-module. By Lemma 2.7 we may assume $\mathrm{char}(F) = p > 0$. Let $u \in G$ be a regular unipotent element of H. By [**16**, Lemma 1.2], the Jordan normal form of u on V consists of two blocks with sizes $2n-1, 1$ if p is odd, and $2n-2, 2$ if $p = 2$.

Let V^u be the fixed point space of u on V. Obviously, $\dim V^u = 2$. We deduce two auxiliary observations from this.

(i) If $X = X_1 \oplus X_2$, where X, X_1, X_2 are u-stable subspaces of V, then the dimension of X_1 or X_2 is at most 1 if $p > 2$, and at most 2 if $p = 2$. (This follows by looking at V^u and V/V^u.)

(ii) If $u^p = 1$ then $p > 5$. Indeed, if $p = 2$ then we have $2n - 2 \leq 2$, which is a contradiction. If $p > 2$, we have $2n - 1 \leq p$, which implies the claim as $n > 3$.

We argue by contradiction and suppose that G is contained in a proper parabolic subgroup of H. Then G stabilizes a non-zero totally singular subspace of V. Let W be a maximal G-stable totally singular subspace of V, $k = \dim W$, and let P be the stabilizer of W in H. Then P is a parabolic subgroup of H and $G \subset P$. Let L be a Levi subgroup of P, so $L = (\mathrm{SL}(k, F) \times D_{n-k}) \cdot T_H$ if $n - k > 1$, and $L = \mathrm{SL}(k, F) \cdot T_H$ if $n - k \leq 1$. Let $\pi : P \to L$ be the natural projection of P onto L. By Lemma 2.6, $\pi(u)$ is a regular unipotent element of L. Then $\pi(u) \in [L, L] = \mathrm{SL}(k, F) \times D_{n-k}$ if $n - k > 1$, otherwise $\pi(u) \in \mathrm{SL}(k, F)$. Denote by τ the further projection of $\pi(G)$ into $\mathrm{SL}(k, F)$. Then if $k > 1$, $\tau(G)$ contains a regular unipotent of $\mathrm{SL}(k, F)$ and so $\tau(G)$ is irreducible in $\mathrm{SL}(k, F)$ by Proposition 2.11; this is trivially true if $k = 1$. Set $U := W^\perp/W$. It is well-known that W and $V/W^\perp$ are dual G-modules.

Suppose first that $U = 0$. As mentioned above, $\tau(G)$ is irreducible on W; in particular, $\tau(G)$ belongs to the list of Lemma 2.10. For groups G of type A_1, C_ℓ, B_ℓ and G_2, all irreducible representations are self-dual [**18**, 1.8], so we have a self-extension here; however, every self-extension splits ([**9**, II.2.12(1)]), which means that V is the direct sum of two G-submodules each of dimension equal to $(\dim V)/2$, which contradicts the observation (i) above. For the remaining configuration of Lemma 2.10, $G = A_\ell$ with $\ell > 1$. Then the highest weights of G on W and $V/W^\perp$ are $p^i\omega_1$, $p^i\omega_\ell$; then this extension splits by Lemma 2.2 and we once again have a contradiction.

We now have that $\dim U > 0$, so $\dim U \geq 2$. We first show that $p > 2$. Suppose $p = 2$. Then the Jordan normal form of u has a block J_{2n-2} of size $2n-2$ on V, and $\tau(u)$ has a block (on W) of size at most $k \leq n-1$. This implies that $|u| > |\tau(u)|$, which is false. Thus, we assume until the end of the proof that $p > 2$.

Suppose that $\dim U = 2$. Then G acts trivially on U, since U is equipped with a non-degenerate G-invariant symmetric bilinear form; hence the restriction of G to $W^\perp$ is an extension of τ by a trivial representation of G. We show that this extension splits. Indeed, if $G \cong A_1$ then $|u| = p$, and hence $\dim \tau \leq p$. By statement (ii) above, $p > 5$ here. If $\dim W < p$, then the splitting follows from [**12**, Corollary 1.1.1]. If $\dim W = p$, then the highest weight of τ is $p^j(p-1)\omega_1$ for some integer $j \geq 0$, and we can use the linkage principle [**7**, 3.6]. The dominant weights linked in A_1 to 0 are of shape $ip - 2$ for some integer $i > 0$. This is not equal to $p^j(p-1)$ for $p > 2$.

Continuing with the case $\dim U = 2$, we must consider the groups of rank greater than 1. Note that $\dim W = k \leq \ell+1$, 2ℓ, 5, $2\ell+2$, 7, 7 in the cases (ii) - (vii), respectively, of Lemma 2.10. Then $\dim W^\perp = k+2 \leq \ell+3$, $2\ell+1$, 7, $2\ell+3$, 9, 9, respectively. By [**12**, Corollary 1.1.1], if $\dim W^\perp \leq \ell p$ then $W^\perp$ is a completely reducible FG-module, which contradicts (i) above. Therefore, we have only to deal with the cases where $\dim W^\perp > \ell p$. This yields the inequalities $\ell+3 > \ell p$, $2\ell+2 > \ell p$, $7 > 2p$, $2\ell+3 > \ell p$, $9 > 2p$, $9 > 2p$, respectively. Recalling that $p > 2$, it follows that the possible configurations are when $p = 3$ in the cases (iv) and (vii).

Consider the case (vii). Then $p = 3$, so $k = 7$, $|u| = 9$, and hence $2n-1 \leq 9$, which violates $k = 7$ (since here we have $\dim V = 2\dim W + 2 = 2k+2$). Finally, suppose that $p = 3$ in case (iv). Then G is of type C_2 or B_2 (they are isomorphic). Then $W^\perp$ is completely reducible by [**12**, Theorem 1], giving a contradiction as above. (Recall that $W^\perp$ is an extension of W by a trivial module and W is as in Lemma 2.10(iv).)

We have now reduced to the case $\dim U > 2$. Let $\sigma : G \to \mathrm{SO}(U)$ denote the representation of G induced by ϕ. Note that $\sigma \neq 1$ as $\sigma(G)$ contains a regular unipotent element of D_{n-k}. If $\dim U = 4$ then $\mathrm{SO}(U)$ is a semisimple group of type A_1A_1, so $G \cong A_1$, and hence $p > 5$ (by (ii) above) and $k \leq p$. As the Jordan form of u on V has a block of size $2n-1$, it follows that $2n-1 \leq p$. As $n = k+2$, we have $2k+3 \leq p$, and hence $\dim W^\perp = k+4 \leq (p+5)/2$. But since $p > 5$, $(p+5)/2 \leq p$, and [**12**, Corollary 1.1.1] implies that $W^\perp$ is a completely reducible FG-module, contradicting (i).

We consider one further special case, that is when $\dim U = 6$, and show that $W^\perp$ is a completely reducible FG-module. We have $D_{n-k} = D_3 \cong A_3$; so $|u| = |\sigma(u)| = p$, or $p = 3$ and $|u| = 9$. If $|u| = p$ then $p > 5$ by (ii) above. In this case,

$2n-1 \leq p$ and $n = k+3$ imply that $\dim W^{\perp} = k+6 \leq \frac{p+7}{2} \leq p$, and so $W^{\perp}$ is a completely reducible FG-module as claimed. Thus we have $p = 3$ and $|u| = 9$. But then $2n-1 \leq 9$ implies that $k \leq 2$. If $k = 2$ then $|\tau(u)| = 3$, and hence $|u| = 3$, a contradiction. So $k = 1$ and $\dim W^{\perp} = 7$. As $D_3 \cong A_3$, by Lemma 2.10, G is of type A_3 or C_2. In the first case, [**12**, Corollary 1.1.1] implies that $W^{\perp}$ is completely reducible; hence we may asume $G = C_2$. Let X be the 4-dimensional natural module for A_3. Then, the highest weight of $X|_G$ is $3^j\omega_1$. It is well-known that U is the wedge square of X and that G acts reducibly with composition factors of dimensions 5 and 1 on $\wedge^2 X$. The highest weight of the non-trivial factor is $3^i\omega_2$. It follows that U is completely reducible (as it is obviously self-dual). By [**9**, II.12.9, II.10.17(2)] and Lemma 2.2, $W^{\perp}$ is completely reducible again as claimed. Using the self-duality of V, in all cases, we have $\dim V^u > 2$, which is a contradiction.

We now consider the remaining cases, where $\dim U \geq 8$. Then $\sigma(G)$ contains a regular unipotent element of $\mathrm{SO}(U) \cong D_{n-k}$. So the Jordan form of $\sigma(u)$ consists of blocks of size 1 and $2n-2k-1$. We show that σ is an irreducible representation of G.

Indeed, suppose the contrary, that $\sigma(G)$ acts reducibly on U. By maximality of W, there is a proper $\sigma(G)$-invariant non-degenerate subspace U' of U (recall that $p > 2$), and hence $\sigma(G)$ stabilizes an orthogonal decomposition $U = U' \oplus U''$. Considering the Jordan form of $\sigma(u)$, we may without loss of generality assume that $\dim U'' = 1$. Let Z be the preimage of U'' under the mapping $W^{\perp} \to W^{\perp}/W$. Then $\dim Z = k+1$. We claim that Z is a completely reducible FG-module. Indeed, if $|u| = p$ then $2n-1 \leq p$ implies $k = \dim W \leq n-4 \leq \frac{p-7}{2}$, and the splitting follows from [**12**, Corollary 1.1.1]. If $|u| > p$, and so $G \neq A_1$, again by *loc.cit*, we can assume that $\dim Z > p\ell$, equivalently, $\dim W > p\ell - 1$. As above, the dimension of W is at most $\ell+1$, 2ℓ, 5, $2\ell+1$, 7, 7 in the cases (ii) - (vii) of Lemma 2.10, respectively. As $p > 2$, this is at most $p\ell - 1$ unless $p = 3$ and $G = G_2$, and then $|u| \leq 9$. As $2n-1 \leq |u|$, it follows that $2n = 10$, and hence $\dim W = 1$. Therefore, Z is a reducible FG-module of dimension 2, and hence trivial, so completely reducible as claimed. Now set $Z = W \oplus Z_1$, where Z_1 is a 1-dimensional, nondegenerate, G-invariant subspace. Then G embeds in $\mathrm{SO}(Z_1^{\perp})$, a simple group of type B_{n-1}. Moreover, the image of G in this B_{n-1} subgroup must contain a unipotent element of the B_{n-1} with a Jordan block of size $2n-1$ on $Z_1^{\perp}$, that is, a regular unipotent element. Since we have already established the result in case $H = B_{n-1}$, we see that the image of G lies in no proper parabolic subgroup of $\mathrm{SO}(Z_1^{\perp})$. But $W \subseteq Z_1^{\perp}$, hence a contradiction.

Thus, σ is irreducible; so either $G = D_{n-k}$ or by [**16**, Theorem B(iv)], $G \cong A_1$, or $G \cong B_3$ and σ is a Frobenius twist of the spinor representation of G. In the first case, when $G = D_{n-k}$, we see that $k = 1$ since $\tau(G)$ must contain a regular unipotent element of $\mathrm{SL}(k, F)$ (see Lemma 2.10). Now since $k = 1$, we see that $V|_G$ has precisely three composition factors, namely a twist of the natural module for G, and 2 trivial modules; but then applying Lemma 2.2 and [**9**, II.12.9, II.10.17(2)], we deduce that V is a completely reducible FG-module contradicting $\dim V^u = 2$. Now for the remaining two cases, observe that σ is tensor indecomposable as $\sigma(u)$ is regular unipotent in D_{n-k} (see [**16**, 1.5]). In addition, tensor indecomposable irreducible representations of $\mathrm{SL}(2, F)$ of even dimension are symplectic, which rules out the case with $G = A_1$. Therefore, G is of type B_3, and $\dim U = 8$. Note that the composition length of $W^{\perp}$ equals 2, and the composition factors are given by τ

TABLE 2. The maximal order of unipotent elements in the exceptional groups

p	2	3	5	7	11	13	17	19	23	29
E_8	2^5	3^4	5^3	7^2	11^2	13^2	17^2	19^2	23^2	29^2
E_7	2^5	3^3	5^2	7^2	11^2	13^2	17^2			
E_6	2^4	3^3	5^2	7^2	11^2					
F_4	2^4	3^3	5^2	7^2	11^2					
G_2	2^3	3^2	5^2							

and σ. By Lemma 2.10, τ is of highest weight 0 or $p^i\omega_1$ for some integer $i \geq 0$, and σ is of highest weight $p^j\omega_3$. Then $W^\perp$ splits as σ is faithful and τ is not faithful for $p > 2$. Therefore, $W^\perp = W \oplus Y$, where $Y \cong U$ is a G-stable subspace of $W^\perp$. As above, this implies that $\dim V^u > 2$, giving our final contradiction. □

4. The case where H is exceptional

In this section we will establish Theorem 2.5 in case H be a simple algebraic group of exceptional type. By Lemma 2.7, we may assume $\mathrm{char}(F) = p > 0$ and we let $o(H)$ be the maximum order of a unipotent element of H. This coincides with the order of a regular unipotent element of H. The value $o(H)$ is explicitly computed in [**25**]; we give these values when $o(H)$ exceeds p, in Table 2.

We will rely heavily on [**10**, Theorem 1], where sufficient conditions for a semisimple subgroup of a simple exceptional algebraic group H to be H-cr are given.

PROPOSITION 4.1. *Let H be a simple algebraic group of exceptional type and G a simple closed subgroup of H. If G contains a regular unipotent element of H, then G does not lie in any proper parabolic subgroup of H. (So Theorem* 2.5 *is true for G of exceptional type.)*

PROOF. Arguing by contradiction, we suppose that $u \in G$, for u a regular unipotent element of H, and $G \subseteq P$, a proper parabolic subgroup of H. As a proper Levi factor of H cannot contain a regular element of H (Lemma 2.6), G does not lie in a Levi factor of P. Hence, we may use [**10**, Theorem 1] to reduce to a small number of possibilities, where Table 3 (taken from [**10**]) gives the maximal value $N(G, H)$ of the prime p for which we must consider the pair (G, H). If there is no value of p in the column corresponding to H, then G necessarily lies in a Levi factor of H for all p.

We now compare the above restrictions on p with the information in Table 2, where we give the orders of the regular unipotent elements in the exceptional groups. For $u \in H$ regular, $u \in G$ implies that the order of u is at most the order of a regular unipotent element in G, as regular unipotent elements are dense in the variety of unipotent elements of G. If G is of type A_n, the regular unipotent elements have order equal to the minimal power p^a with $p^a > n$. The regular unipotent elements in C_n are regular in A_{2n-1} (in the natural representation of C_n). As mentioned in the proof of Proposition 3.1, the regular unipotent elements in D_n acting on the natural $2n$-dimensional representation space have exactly two Jordan blocks of

TABLE 3. $N(G,H)$

	$H=G_2$	F_4	E_6	E_7	E_8
$G=A_1$	3	3	5	7	7
A_2		3	3	5	5
B_2		2	3	3	5
G_2		2	3	7	7
A_3			2	2	2
B_3		2	2	2	2
C_3		2	2	2	2
B_4, C_4, D_4			2	2	2

sizes $(2n-1,1)$ if $p \neq 2$, respectively, of sizes $(2n-2,2)$ if $p=2$. So the regular unipotent elements in D_n have order equal to the minimal p^a with $p^a \geq 2n-1$, respectively $p^a \geq 2n-2$. Finally, we recall that the regular unipotent elements in B_n are again regular in D_{n+1}, under the natural embedding of B_n in D_{n+1}. Combining all of these results and comparing the orders, we see that $o(H) > o(G)$ unless G is of type G_2, $p=5$ and H is of type E_7.

We now consider this possibility in detail. Let P be proper a parabolic subgroup of H minimal with respect to containing G, and let $P=QL$ be a Levi decomposition of P, where $Q=R_u(P)$. If L' has a simple factor of type A_k for some k, then the minimality of P implies that G has a $k+1$-dimensional irreducible representation. If L' has a factor of type D_k for some k, again the minimality of P implies that there exists an irreducible FG-module of dimension m for some $m \leq 2k$, and on which G stablizes a nondegenerate quadratic form. Given that $p=5$, we reduce therefore to the following configurations.

a) L' is of type D_4, G stabilizes a nonsingular 1-space of the natural module for L' and acts irreducibly on a non-degenerate complement to this space,

b) L' is of type A_6 and G acts irreducibly on the natural module for L', or

c) L' is of type E_6.

In the first two cases, we will show that the semidirect product GQ has a unique class of complements to Q, which implies that G is conjugate to a subgroup of L', contradicting Lemma 2.6. The main tool here is [**10**, 1.7]. We refer to the table of highest weights of composition factors of $Q|_{L'}$ given in the proof of [**10**, 3.4], as well as to [**10**, 2.10] for the restriction of these composition factors to the image of G in L'. Then for any such composition factor, say of highest weight μ, we use the known information on the corresponding Weyl module $W(\mu)$ for G, when $p=5$, and we see that $\mathrm{Hom}_{FG}(\mathrm{rad}(W(\lambda), F) = 0$. Then [**10**, 1.7] shows that there is indeed a unique class of complements to Q in GQ.

So it remains to consider the case L' of type E_6. By minimality of P, $\pi(G)$ is L'-irreducible. Then [**16**, Theorem A] implies that $\pi(G) \subset M$, a maximal subgroup of type F_4. The Borel-Tits theorem [**2**] shows that $\pi(G)$ is X-irreducible in every intermediate subgroup $X \subset L'$, so $\pi(G)$ lies in no proper parabolic subgroup of $M = F_4$. But then [**16**, Theorem A] provides a contradiction. □

Proof of Theorem 1.2. As every unipotent element of G is contained in the semisimple subgroup $[G, G]$, we can assume that G is semisimple. By Proposition 2.3, G is simple. So the result follows from Theorem 2.5. □

5. Proof of Theorem 1.4

We can now apply the main theorem of [**16**] and Theorem 1.2 to determine the semisimple closed subgroups of H which contain a regular unipotent element.

Let $G \subset H$ be a semisimple closed subgroup and suppose $u \in G$ for some regular unipotent element of H. Let $M \subset H$ be a maximal closed subgroup with $G \subseteq M$, necessarily of positive dimension. Then Theorem 1.2 implies that M is not a proper parabolic subgroup of H, and so M° is reductive. Moreover, $u \in G = G^\circ \subset M^\circ$ and by Proposition 2.3, we see that $[M^0, M^0]$ is in fact a simple group. Now, if M° is normalized by a maximal torus of H, then its root system corresponds to a subsystem of $\Phi(H)$ and therefore M° does not contain regular unipotent elements of H. In addition, we once again apply what is known about the Jordan block structure of regular unipotent elements in the classical groups. Then by [**16**, Theorem A, Theorem B], we deduce that one of the following holds:

(i) $M^\circ = A_1$ and $p > h$, the Coxeter number for H.

(ii) $M^\circ = F_4 \subset E_6 = H$.

(iii) $M^\circ = B_\ell \subset D_{\ell+1} = H$.

(iv) $M^\circ = C_\ell \subset A_{2\ell-1} = H$.

(v) $M^\circ = B_\ell \subset A_{2\ell} = H$, $p > 2$.

(vi) $M^\circ = B_3 \subset D_4 = H$. (Here there are three conjugacy classes of such subgroups, interchanged by the graph automorphisms of H.)

(vii) $M^\circ = G_2 \subset B_3 = H$, $p > 2$.

(viii) $M^\circ = G_2 \subset C_3 = H$, $p = 2$.

(ix) $M^\circ = A_2 \subset C_4 = H$, $p = 2$.

All of the above examples actually give rise to subgroups containing regular unipotent elements, except the example of (ix). Here G acts irreducibly on V, the natural 8-dimensional module for H, and $V|_G$ has highest weight $2^j(\omega_1 + \omega_2)$, for some j. However, Lemma 2.10 implies that the regular unipotent elements in G are not regular unipotent in $\mathrm{GL}(V)$ and hence are not regular in H.

We must now descend within in each of the above configurations. So we choose a maximal positive-dimensional subgroup of M° arising as an example in the Saxl-Seitz result. Then it is straightforward to see that this gives rise precisely to the list of Theorem 1.4.

References

[1] A. Borel, Linear algebraic groups, 2nd edition, Springer Verlag, 1991.

[2] A. Borel and J. Tits, *Eléments unipotents et sous-groupes paraboliques de groupes réductifs*, Invent. Math. **12** (1971), 95-104.

[3] N. Bourbaki, *Groupes et Algèbres de Lie*, Chaps. IV–VI, Hermann, Paris, 1968.

[4] J. Dieudonné, *La géométrie des groupes classiques*, Springer-Verlag, Berlin, 1971.

[5] H. Enomoto, *The characters of the finite Chevalley group $G_2(q)$, $q = 3^f$*, Japan. J. Math. **2** (1976), 191-247.

[6] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of Finite Simple Groups*, Vol. 3, Part I, Chapter A, Amer. Math. Soc. Providence, R.I. 1998.
[7] J. Humphreys, *Modular representations of finite groups of Lie type*, Cambridge Univ. Press, Cambridge, 2006.
[8] J.C. Jantzen, *Low dimensional representations of reductive groups are semisimple, In: "Algebraic groups and related subjects: a volume in honor of R.W.Richardson"* (eds G. Lehrer *et al*), Cambridge University Press, 1996.
[9] J.C. Jantzen, *Representations of algebraic groups*, Pure and Applied Mathematics Vol. 131, Academic press, Orlando, FL, 1987.
[10] M.W. Liebeck and G.M. Seitz, *Reductive subgroups of exceptional algebraic groups*, Memoirs Amer. Math. Soc., Vol. 580, Amer. Math. Soc., Providence,RI, 1996.
[11] M.W. Liebeck and G.M. Seitz, *The maximal subgroups of positive dimension in exceptional algebraic groups*, Memoirs Amer. Math. Soc., Vol. 802, Amer. Math. Soc., Providence, RI, 2004.
[12] G.J. McNinch, *Dimensional criteria for semisimplicity of representations*, Proc. London Math. Soc. (3) **76** (1998), 95-149.
[13] G.J. McNinch and D.M. Testerman, *Nilpotent centralizers and Springer isomorphisms*, J. of Pure and Applied Algebra, **213** (2009), 1346-1363.
[14] R. Proud, J. Saxl, D. Testerman, *Subgroups of type A_1 containing a fixed unipotent element in an algebraic group*, J. of Algebra, **231**,(2000), 53-66.
[15] J. Saxl, *Overgroups of special elements in simple algebraic groups and finite groups of Lie type. In: "Algebraic groups and their representations"*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 517, Kluwer Acad. Publ., Dordrecht, 1998", 291-298",
[16] J. Saxl and G.M. Seitz, *Subgroups of algebraic groups containing regular unipotent elements*, J. London Math. Soc. (2) **55** (1997), 370-386.
[17] J.-P. Serre, *Complète réductibilité. In: "Séminaire Bourbaki*, 2003/2004", *Astérisque* **299** (2005), Exp. No 932, viii, 195-217.
[18] G.M. Seitz, *The Maximal Subgroups of Classical Algebraic Groups*, Memoirs Amer. Math. Soc., Vol. 365, Amer. Math. Soc., Providence, RI, 1987.
[19] G. M. Seitz, *Maximal Subgroups of Exceptional Algebraic Groups*, Memoirs Amer. Math. Soc., Vol. 441, Amer. Math. Soc., Providence, RI, 1991.
[20] T. Springer and R. Steinberg, *Conjugacy classes, In: "Seminar on algebraic groups and related finite groups*, Lecture notes in Math. Vol.131, Springer-Verlag, Berlin, 1970", E-1 - E-99.
[21] R. Steinberg, *Lectures on Chevalley groups*, Yale Univ., 1967.
[22] R. Steinberg, *Regular elements of semisimple algebraic groups*, Inst. Hautes Études Sci. Publ. Math. **25** (1965), 49-80.
[23] I.D. Suprunenko, *Irreducible representations of simple algebraic groups containing matrices with big Jordan blocks*, Proc. London Math. Soc. **71** (1995), 281-332.
[24] D. Testerman, *Irreducible subgroups of exceptional algebraic groups*, Memoirs Amer. Math. Soc., Vol. 390, Amer. Math. Soc., Providence, RI 1988.
[25] D. Testerman, *A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups*, J. Algebra **177** (1995), 34-76.

EPFL, FSB-MATHGEOM, Station 8, CH-1015 Lausanne, Switzerland
E-mail address: donna.testerman@epfl.ch

Departimento di Matematica e Applicazioni, University of Milano-Bicocca, 53 via R. Cozzi, Milan, 20125, Italy
E-mail address: alexandre.zalesski@gmail.com