Singular Padé-Chebyshev Reconstruction: The Case When Jump Locations are Unknown

Arnel L. Tampos * Jose Ernie C. Lope † Jan S. Hesthaven ‡

Abstract— **We present a singularity-based approach to resolve the Gibbs phenomenon that obstructs the reconstruction of a function with jump discontinuities by a truncated Chebyshev series or a Padé-Chebyshev approximation. In this paper, we consider the more difficult case where the locations of the jump discontinuities are not known. The identification of unknown singularities is carried out using a Padé-Chebyshev approximation. We provide numerical examples to illustrate the method, including an application on postprocessing computational data corrupted by the Gibbs phenomenon.**

Keywords: ***Gibbs phenomenon, function reconstruction, Padé-Chebyshev approximation***

1 Introduction

Approximation of smooth functions by Fourier series or by truncated orthogonal polynomial expansions in general is known to be exponentially convergent and highly accurate ([2], [6]). For functions with singularities, however, convergence of a partial sum of orthogonal series is adversely affected in the area over which the singularities occur, a problem which has come to be known as the Gibbs phenomenon. This phenomenon manifests in an oscillatory behavior at the vicinity of the jumps and thus presents an obstruction in the reconstruction of a discontinuous function.

An exposition on the nature of the Gibbs phenomenon and some remediation schemes to counter its effect can be found in [5], [7], and [8]. A class of techniques aimed at resolving the Gibbs phenomenon comprises Padé-type approximations (e.g. [1], [2], [3], [7], [9], [12]). These methods extend the standard Padé approximation by making use of orthogonal polynomials as basis in lieu of the canonical basis with which the numerator and denominator of a Padé approximant are expanded. A Padé-type approximant enjoys the advantage of utilizing rational functions, which are broader than polynomials and can have singularities, and hence there is a stronger likelihood that it will capture the singularities of the function being approximated ([2], [7]).

Some Padé-based methods work without requiring information about the jump locations. However, locating jump discontinuities can become a relevant issue when the actual function is not explicitly known. In many cases, for instance, involving spectral approximations of nonsmooth solutions to some partial differential equations, the solution comes in the form of computational data that are contaminated by Gibbs phenomenon. As these data are noisy, the standard procedure is to postprocess them to correct the phenomenon. One way this can be done, as demonstrated in [1], [7], and [14], is to use Padé-type approximation. This Padé postprocessing approach, however, may turn out to be less successful unless fed with some information about the possible jump positions which, as noted in [7], can be advantageous for its effective implementation. As computational data may not show explicitly the existence and whereabouts of possible jumps, to somehow locate them can become imperative.

A study by Driscoll and Fornberg ([2], [3]) reveals just how significant the knowledge of the jump locations can be in correcting the Gibbs phenomenon. Realizing that the poles available in a rational approximant do not intrinsically and adequately reproduce the jump behaviors of a discontinuous function f, they devised an approach that incorporates the jump locations into the approximation process. A similar approach that imbibes this concept in the context of Padé-Chebyshev approximation is discussed in [12]-[14].

This paper is anchored on the Singular Padé - Chebyshev (SPC) approximation introduced in [12] and further discussed in [13] and [14], a brief review of which is presented in the next section. Section 3 discusses a Padé-based approach in identifying singularities of the function. Section 4 focuses on some numerical results of the SPC implementation in reconstructing some test functions and postprocessing computational data contaminated by the Gibbs phenomenon.

*Division of Physical Sciences and Mathematics, University of the Philippines Visayas, Miag-ao, Iloilo, Philippines. Email: altampos@yahoo.com

†Institute of Mathematics, University of the Philippines Diliman, Quezon City, Philippines. Email: ernie@math.upd.edu.ph

‡Division of Applied Mathematics, Brown University, Providence, R.I., USA. Email: Jan.Hesthaven@brown.edu

2 A Singularity-based Padé-Chebyshev Approximation

For a function $f : [-1,1] \to \mathbb{R}$ belonging to L^2_ω space with Chebyshev weight function ω, the Chebyshev series expansion $\mathcal{T}(f)$ of f is defined by

$$\mathcal{T}(f) = \sum_{n=0}^{\infty} c_n T_n, \quad c_n = \frac{\langle f, T_n\rangle_{L^2_\omega}}{\|T_n\|^2_{L^2_\omega}}, \tag{1}$$

where the T_ns are the Chebyshev polynomials of the first kind defined as $T_n(x) = \cos(n\theta)$, $\theta = \cos^{-1}(x)$,

$$\langle f, T_n\rangle_{L^2_\omega} = \int_{-1}^{1} f(x)T_n(x)\frac{1}{\sqrt{1-x^2}}dx$$

and

$$\|T_k\|_{L^2_\omega} = \begin{cases} \sqrt{\pi}, & n = 0 \\ \sqrt{\frac{\pi}{2}}, & n \geq 1. \end{cases}$$

Alternatively, we may express (1) as

$$f(x) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n T_n(x), \tag{2}$$

with

$$c_n = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}}dx, \quad n = 0,1,2,\ldots. \tag{3}$$

The coefficients c_n may be approximated using the following Gauss-Chebyshev quadrature rule

$$\int_{-1}^{1} h(x)\omega(x)dx \cong \sum_{k=1}^{m} A_k h(x_k), \tag{4}$$

where $\{x_k\}$ are the zeros of the Chebyshev polynomials $T_m(x) = \cos(m\theta)$, $h(x) = f(x)T_n(x)$, $\omega(x) = \frac{1}{\sqrt{1-x^2}}$, and $A_k = \frac{\pi}{m}$ for all k.

By the definition of T_n and the fact that $\cos(n\theta) = \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right)$, we can introduce the variable $z = e^{i\theta}$ and transform (2) into

$$f(z) = \frac{1}{2}\left(\sum_{n=0}^{\infty}{}' c_n z^n + \sum_{n=0}^{\infty}{}' c_n z^{-n}\right),$$

where the primed sum indicates that the first term in the summation is halved. Let

$$g(z) = \sum_{k=0}^{\infty}{}' c_n z^n. \tag{5}$$

Then (2) becomes

$$f(z) = \frac{1}{2}\left(g(z) + g\left(z^{-1}\right)\right).$$

We refer to $g(z)$ as the *transformed Chebyshev series associated with* $f(z)$, and consequently with $f(x)$. As $f(x) = \Re[g(z)]$, approximating f is tantamount to approximating $g(z)$. Thus, truncating the series $g(z)$ up to degree N determines a *Chebyshev approximant to* f *of order* N and we denote this orthogonal polynomial approximant by $Cheb(N)$. As pointed earlier, approximating a discontinuous function by a truncated orthogonal series such as $Cheb(N)$ suffers from the Gibbs phenomenon, a problem which may be circumvented using some Padé-based techniques.

For a piecewise analytic function f defined on $[-1,1]$ with associated transformed Chebyshev series $g(z)$, its Padé-Chebyshev approximant of order (N,M) can be defined by the rational function

$$\mathcal{R}_{(N,M)}(z) = \frac{P_N(z)}{Q_M(z)} = \frac{\sum_{k=0}^{N} p_k z^k}{\sum_{k=0}^{M} q_k z^k}, \tag{6}$$

where $z = e^{i\cos^{-1}(x)}$ and Q_M is not identically zero, such that

$$Q_M(z)g(z) - P_N(z) = \mathcal{O}\left(z^{N+M+1}\right), \quad z \to 0.$$

We denote the approximant defined in (6) by $PC(N,M)$.

A drawback in a Padé-type approximant such as (6), as observed in the case of the Fourier-Padé approximation ([2], [3]), lies in its inability to sufficiently resolve the Gibbs phenomenon. Driscoll and Fornberg discussed in [2] and [3] a Padé-based correction of this phenomenon that takes into account the singularities of the function. Applying their argument in our context, a jump discontinuity of a function at a point $x = \xi$ can be incorporated into the series $g(z)$ by a logarithm term which takes the form of

$$\log\left(1 - \frac{z}{e^{i\theta_\xi}}\right), \tag{7}$$

where $0 \leq \theta_\xi = \cos^{-1}(\xi) \leq \pi$. This logarithmic singularity is utilized to redefine (6) in the same way as it is handled in [2] and [3].

Let $f(x)$ be a piecewise analytic function defined on $[-1,1]$ with s jump locations at $x = \xi_k \in [-1,1]$, $k = 1,\ldots,s$, and consider its associated transformed Chebyshev series (5). In view of (7), the Padé-Chebyshev approximant (6) may now be modified as

$$\mathcal{R}(z) = \frac{P_N(z) + \sum_{k=1}^{s} R_{V_k}(z)\log\left(1 - \frac{z}{e^{i\theta_k}}\right)}{Q_M(z)}, \tag{8}$$

where $z = e^{i\cos^{-1}(x)}$ and

$$P_N(z) = \sum_{j=0}^{N} p_j z^j, \quad Q_M(z) = \sum_{j=0}^{M} q_j z^j \neq 0,$$

$$R_{V_k}(z) = \sum_{j=0}^{V_k} r_j^{(k)} z^j, \quad k = 1, \dots, s,$$

such that

$$Q_M(z)g(z) - [P_N(z) + U(z)] = \mathcal{O}\left(z^{\eta+1}\right),$$

with

$$U(z) = \sum_{k=1}^{s} R_{V_k}(z) \log\left(1 - \frac{z}{e^{i\theta_k}}\right)$$

and

$$\eta = N + M + s + \sum_{k=1}^{s} V_k.$$

The function in (8) defines the *Singularity-based Padé-Chebyshev (SPC) approximant* to f of order $(N, M, V_1, \dots, V_s)$ and we denote this approximant by $SPC\,(N, M, V_1, \dots, V_s)$.

Proposition 1. *The approximant defined in (8) exists and is unique.*
Proof: (See [12].)

The unknown coefficients of polynomials P_N, Q_M, and R_{V_k}s are then computed through the following linear system of $\eta + 1$ equations in $\eta + 2$ variables:

$$\sum_{j=0}^{M}{}^{*} c_{N-j+t} q_j - \sum_{j=0}^{V_1} a_{N-j+t}^{(1)} r_j^{(1)} - \dots - \sum_{j=0}^{V_s} a_{N-j+t}^{(s)} r_j^{(s)} = 0,$$

$$\sum_{j=0}^{M}{}^{*} c_{l-j} q_j - \sum_{j=0}^{V_1} a_{l-j}^{(1)} r_j^{(1)} - \dots - \sum_{j=0}^{V_s} a_{l-j}^{(s)} r_j^{(s)} = p_l,$$

where $t = 1, \dots, \eta - N$, $l = 0, \dots, N$, and the asterisk-marked summation indicates that the term with c_0 is halved. We note that in this system, $c_n = 0$, for $n < 0$. It should be noted too that the $a_n^{(k)}$ are the coefficients in the Taylor expansion of

$$\log\left(1 - \frac{z}{e^{i\theta_k}}\right) = \sum_{n=1}^{\infty} \left(-\frac{1}{n e^{in\theta_k}}\right) z^n$$

and $a_n^{(k)} = 0$, for $n \leq 0$. Accordingly, $\mathcal{R}(z)$ defined by (8) approximates $g(z)$ which implies that the real part of $\mathcal{R}$ approximates $f(x)$.

Proposition 2. *For the SPC approximant defined in (8), we have*

$$|f(z) - \Re(\mathcal{R}(z))| \leq \frac{1}{|Q_M(z)|} \sum_{l > \eta} |b_l|$$

where $z = e^{i\cos^{-1}(x)}$, $\eta = N + M + s + \sum_{k=1}^{s} V_k$, and

$$b_l = \sum_{j=0}^{M}{}^{*} c_{l-j} q_j - \sum_{j=0}^{V_1} a_{l-j}^{(1)} r_j^{(1)} - \dots - \sum_{j=0}^{V_s} a_{l-j}^{(s)} r_j^{(s)},$$

$a_n^{(k)}$ being the coefficients in the Taylor expansion of $\log\left(1 - \frac{z}{e^{i\theta_k}}\right)$, $c_n = 0$, for $n < 0$, and ,$a_n^{(k)} = 0$ for $n \leq 0$.
Proof: (See [12].)

3 Approximate Jump Locations of a Discontinuous Function

There have been studies on locating jump discontinuities of a discontinuous function ([3], [4], [7]) and some of these explore the connection between jump locations and the differentiated series expansion of the function. Estimating jump locations using Padé approximation is introduced in [3] and its applicability is based on the idea that a Padé approximation of the differentiated series expansion of a discontinuous function f likely leads to an ordinary pole at a jump location. As our approach is founded on Padé-Chebyshev approximation, we further pursue this idea to generate information about the jump locations of discontinuous functions.

For the derivative of f, a Padé-Chebyshev approximant of order (N, M) may be defined as

$$\mathcal{R}_{f'}(z) = \frac{(P_{f'})_N(z)}{(Q_{f'})_M(z)}, \tag{9}$$

where $z = e^{i\cos^{-1}(x)}$ and

$$(P_{f'})_N(z) = \sum_{j=0}^{N} (p_{f'})_j z^j,$$

$$(Q_{f'})_M(z) = \sum_{j=0}^{M} (q_{f'})_j z^j \neq 0,$$

such that

$$(Q_{f'})_M(z) g'(z) - (P_{f'})_N(z) = \mathcal{O}\left(z^{N+M+1}\right).$$

Finding the unknown coefficients of polynomials $(P_{f'})_N$ and $(Q_{f'})_M$ is tantamount to solving the following linear system:

$$\begin{cases} \displaystyle\sum_{j=0}^{M} i(N + \lambda - j + 1) c_{N+\lambda-j+1} (q_{f'})_j = 0, \\ \qquad \lambda = 0, 1, 2, \dots, M - 1, \\ \displaystyle\sum_{j=0}^{M} i(\mu - j) c_{\mu-j} (q_{f'})_j = (p_{f'})_\mu, \\ \qquad \mu = 1, 2 \dots, N, \end{cases}$$

where $i=\sqrt{-1}$ and the expansion coefficients $c_k=0$ for each $k<0$. We remark that $\mathcal{R}_{f'}$ is a Padé-Chebyshev (PC) approximant that approximates $g'(z)$ which is the derivative of the transformed Chebyshev series associated with $f(x)$. Consequently, the real part of $\mathcal{R}_{f'}$ approximates $f'(x)$.

Recalling the definition of the Chebyshev polynomial, we know that $\theta=\cos^{-1}(x)$ with $x\in[-1,1]$ and $\theta\in[0,\pi]$. This defines a mapping from $[-1,1]$ onto $[0,\pi]$. The transformation $z=e^{i\theta}$ consequently maps $[-1,1]$ to the upper half of the unit circle in the complex plane at which $\left|e^{i\theta}\right|=1$. Now consider the Padé-Chebyshev approximant $R_{f'}$ to g'. Let z_0 be a zero of $(Q_{f'})_M$ or a pole of $R_{f'}$. We have $z_0=e^{i\theta_0}$ for some $\theta_0\in[0,\pi]$. By the inverse mapping, $|z_0|=1$ implies that z_0 corresponds to a point x_0 in $[-1,1]$. As z_0 is a singularity, x_0 must be a jump of $f(x)$ in $[-1,1]$. Furthermore, since $z_0=\cos\theta_0+i\sin\theta_0$, the jump must be located at $x_0=\cos\theta_0=\Re(z_0)$. The preceding discussion is summarized in the following proposition.

Proposition 3. *A pole z_0 of $R_{f'}$ for which $|z_0|=1$ corresponds to a jump discontinuity of $f(x)$ in $[-1,1]$ which occurs at $x=\Re(z_0)$.*

This provides a simple criterion by which we may be able to locate a jump discontinuity of a piecewise continuous function using the Padé-Chebyshev approximant of its differentiated series expansion. As stated, we only need to consider those zeros of $(Q_{f'})_M$ for which the modulus is equal to (or approximately) 1 in order to identify the zeroth-order jumps of the function.

The result of this section can be extended naturally to allow the identification of a first-order jump in a continuous function with discontinuous first derivative and also higher-order jumps for functions belonging to class $C^k[-1,1]$ of k times continuously differentiable functions.

4 Numerical Results

The SPC method applied to classical test functions such as the signum function and the absolute value function $f(x)=|x|$ effectively resolves the Gibbs phenomenon present in both the $Cheb(N)$ and $PC(N,M)$ approximants of these functions (see [12] and [13]). In this section, we continue to demonstrate the efficacy of the method by reconstructing the following piecewise smooth functions:

1. $f_1(x)=\begin{cases}\sqrt{1-x^2}, & 0\le x\le 1\\ 0, & -1/2\le x<0\\ -x-1, & -1\le x<-1/2\end{cases}$

2. $f_2(x)=\begin{cases}-2\sin^{-1}x, & -1\le x<-\frac{2}{3}\\ \exp\left(-x^2\right)+x, & -\frac{2}{3}\le x<0.2\\ \cos(3\pi x)+2x, & 0.2\le x\le 1.\end{cases}$

We then show how the method recovers a function from a computational data set that is contaminated by the Gibbs phenomenon. In this regard, we consider reconstructing a function that is given in terms of computational data from the numerical solution to the following viscous Burgers' equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\epsilon\frac{\partial^2 u}{\partial x^2},\ x\in[-1,1],\ \epsilon=0.001 \qquad (10)$$

with boundary conditions

$$u(-1,t)=u(1,t)=0 \qquad (11)$$

and initial condition

$$u(x,0)=-\tanh\left(\frac{x+0.5}{2\epsilon}\right)+1. \qquad (12)$$

The numerical implementation for function reconstruction was done in Scilab, an open-source software.

4.1 Reconstructing f_1

By definition, f_1 has discontinuities at $x=0$ and $x=-\frac{1}{2}$. With reference to Table 1, it is interesting to note that the approximate locations of these jumps may be obtained using (9) and Proposition 3.

Approximating Jump Locations of f_1		
Zeros of $Q_{f_1'}$	Modulus	Jump at
$-0.50105\pm0.86647i$	1.00091	$x\approx-0.5$
$-0.00012\pm1.00007i$	1.00007	$x\approx 0$
$-0.49876\pm1.00989i$	1.12634	*
$-0.31860\pm1.45791i$	1.49231	*
$0.77977\pm2.78340i$	2.89057	*
0.88246	0.88246	*
1.03498	1.03498	*
1.20644	1.20644	*
1.76114	1.76114	*
362.86831	362.86831	*

Table 1: Zeros of $Q_{f_1'}$ of the $PC(15,15)$ approximant and the approximate jump locations of f_1. (* indicates no relation with the jump.)

The exact Chebyshev coefficients in the Chebyshev expansion of f_1 are given by

$$c_n=\begin{cases}-\frac{2}{3}+\frac{2+\sqrt{3}}{\pi}, & n=0\\ \frac{1+\sqrt{3}}{\pi}-\frac{\sqrt{3}}{4\pi}-\frac{1}{3}, & n=1\\ k, & n\ge 2,\end{cases}$$

where

$$k = \frac{2n\sin\frac{n\pi}{2} - n\sin\frac{2n\pi}{3} - \sqrt{3}\cos\frac{2n\pi}{3} - 2}{\left(n^2-1\right)\pi} + \frac{2}{n\pi}\sin\frac{2n\pi}{3}.$$

The SPC approximant for f_1 is determined by

$$\frac{P(z) + R_1(z)L_1(z) + R_2(z)L_2(z)}{Q(z)},$$

where

$$L_1(z) = \log\left(1 - \frac{z}{i}\right)$$

and

$$L_2(z) = \log\left[1 - \frac{z}{\exp\left(\frac{2\pi i}{3}\right)}\right].$$

Figure 1 shows the Gibbs phenomenon in a PC approximation of f_1. The oscillation caused by the phenomenon is practically eliminated upon the inclusion of the function's singularities into the approximation process as shown in Figure 2. An SPC approximant of f is shown in Figure 2 against the graph of the exact function. The reconstruction is remarkably good that the graph of the exact function is hardly noticeable. As shown in Figure 3, this impressive result by the SPC approximation is clearly marked by an improved convergence of the pointwise error.

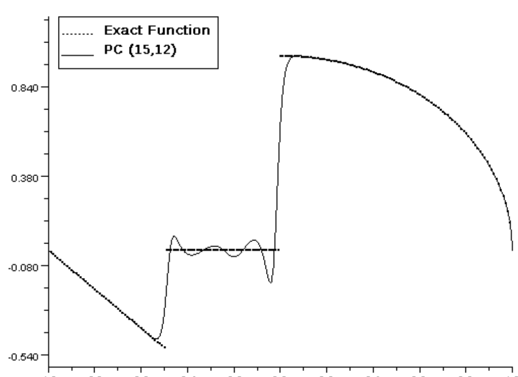

Figure 1: Constrast between the exact f_1 and its PC(15, 12) approximant.

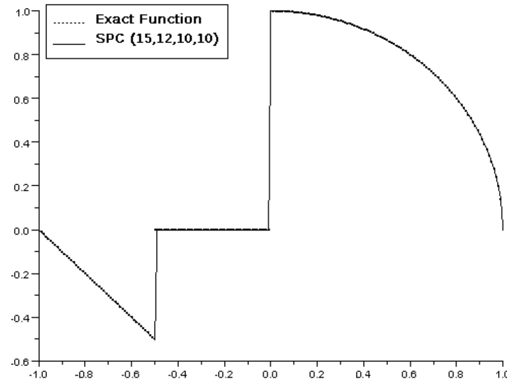

Figure 2: Contrast between the exact f_1 and its SPC (15,12,10,10) approximant.

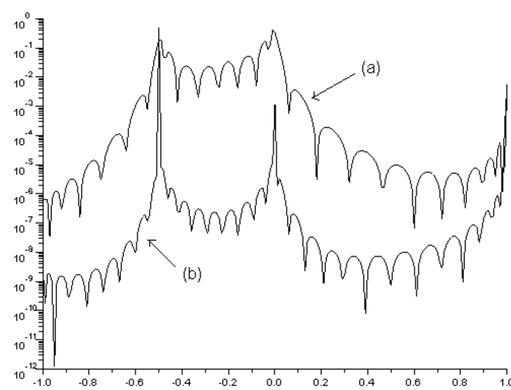

Figure 3: Comparison of the pointwise error convergence in logarithmic scale of (a) PC(15,12) and (b) SPC (15,12,10,10) approximants of f_1.

4.2 Reconstructing f_2

The function f_2 is a test function with a somewhat complicated shape. It is discontinuous at $x = 0.2$ and $x = -\frac{2}{3}$. Table 2 gives information about approximate locations of these singularities obtained using (9) and Proposition 3.

The exact Chebyshev coefficients in the Chebyshev expansion of f_2 are given by

$$c_n = C_1 + C_2 + C_3,$$

Approximating Jump Locations of f_2		
Zeros of $Q_{f_2'}$	Modulus	Jump at
$-0.65871 \pm 0.75255i$	1.00012	$x \approx -2/3$
$0.19454 \pm 0.99007i$	1.00900	$x \approx 0.2$
$0.29769 \pm 1.14446i$	1.18254	*
$0.73156 \pm 1.63159i$	1.78809	*
0.00776	0.00776	*
-1.04728	1.04728	*

Table 2: Zeros of $Q_{f_2'}$ of the PC(20,10) approximant and the approximate jump locations of f_2.

where

$$\begin{aligned} C_1 &= \int_{\cos^{-1}\left(-\frac{2}{3}\right)}^{\pi} -2\sin^{-1}(\cos\theta)\cos n\theta d\theta, \\ C_2 &= \int_{\cos^{-1}(0.2)}^{\cos^{-1}\left(-\frac{2}{3}\right)} \left[\exp\left(-\cos^2\theta\right)+\cos\theta\right]\cos n\theta d\theta, \\ C_3 &= \int_{0}^{\cos^{-1}(0.2)} \left[\cos(3\pi\cos\theta)+2\cos\theta\right]\cos n\theta d\theta. \end{aligned}$$

Using (4), these coefficients are approximated as

$$g(x_k) = \begin{cases} B_1, & -1 \le x \le -\frac{2}{3} \\ B_2, & -\frac{2}{3} \le x < 0.2, \\ B_3, & 0.2 \le x < 1 \end{cases} \tag{13}$$

where

$$\begin{aligned} B_1 &= -2\sin^{-1}(x_k)\cos\left[n\cos^{-1}(x_k)\right] \\ B_2 &= \left\{\exp\left[-\cos^2(x_k)\right]+x_k\right\}\cos\left[n\cos^{-1}(x_k)\right] \\ B_3 &= \left[\cos(3\pi x_k)+2x_k\right]\cos\left[n\cos^{-1}(x_k)\right]. \end{aligned}$$

The SPC approximant of f_2 is given by

$$\frac{P(z)+R_1(z)L_1(z)+R_2(z)L_2(z)}{Q(z)},$$

where

$$L_1(z) = \log\left\{1-\frac{z}{\exp\left[i\cos^{-1}(-2/3)\right]}\right\},$$

and

$$L_2(z) = \log\left\{1-\frac{z}{\exp\left[i\cos^{-1}(0.2)\right]}\right\}.$$

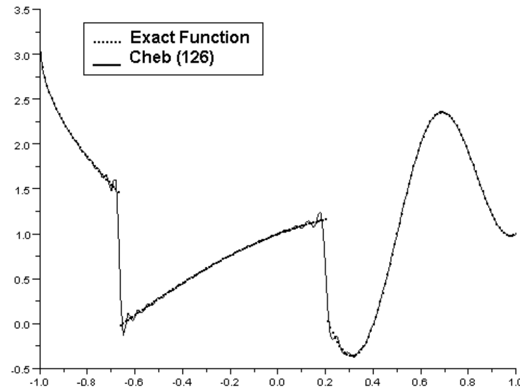

Figure 4: The Gibbs phenomenon in the Cheb (126) approximant of f_2.

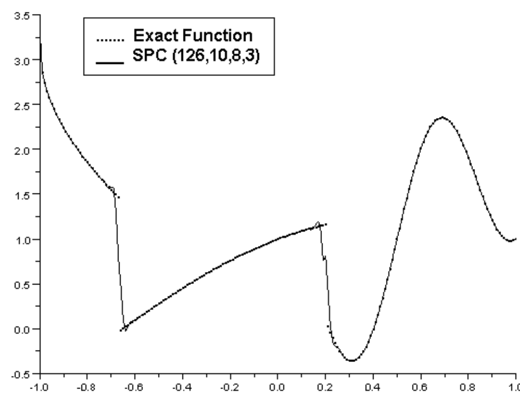

Figure 5: A resolution of the Gibbs phenmenon by the SPC (126,10,8,3) approximant to f_2, shown here against the exact function.

In lieu of the exact form, the approximated Chebyshev coefficients (13) are used in the reconstruction of the function. Using 250 Gauss-Chebyshev quadrature points, the SPC (126,10,8,3) approximant for f_2, shown in Figure 5, almost smoothens the Gibbs phenomenon (Figure 4) seen in the polynomial approximation of the function by Cheb (126). The pointwise errors shown in Figure 6 demonstrate that the reconstruction is practically good.

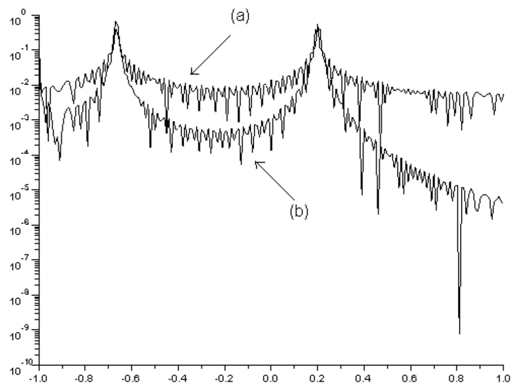

Figure 6: Comparison of the pointwise error convergence in logarithmic scale of (a) Cheb (126) and (b) SPC (126,10,8,3) approximants of f_2.

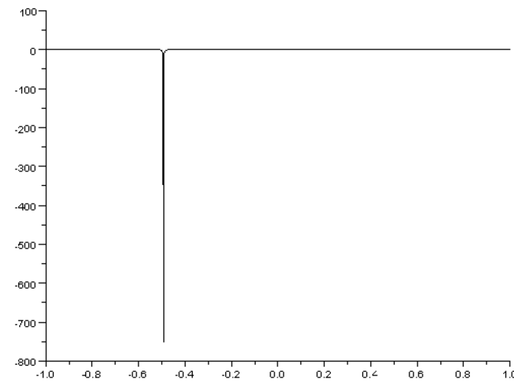

Figure 7: Approximate shock location of u at $x = -0.49321$ when $t = 0$, using PC(3,3) approximant for u'.

4.3 Recovering Solution to Burger's Equation

Numerical solution to the Burgers' equation by spectral method generates a set of computational data that is corrupted by the Gibbs phenomenon in the sense that solutions to such equation are known to develop sharp gradient in time [1]. Here we present some results on the use of the SPC approximation to postprocess or "clean up" the data inorder to recover the solution to the viscous Burger's equation defined in (10)-(12). This equation is a suitable model for testing computational algorithms for flows where steep gradients or shocks are anticipated because it allows exact solutions for many combinations of initial and boundary conditions [1]. It should be noted that the postprocessing needs only to be applied at time levels at which a "clean" solution is desired, and not at every time step [11].

In this case, the transformed Chebyshev series for the solution assumes expansion coefficients that are approximated using (4). The input data are given at 100 Gauss-Chebyshev quadrature points. Working on the assumption that there may be some inherent jump discontinuities or sharp gradient not known or readily observable from the data, we first seek the locations of these possible jumps or shocks in the data by way of the Padé approximation applied to the differentiated expansion that represents the solution u. Incorporating the resulting shock information into the SPC approximation generates a reconstructed u. For illustration, let us consider the case when time $t = 0$ and $t = 0.1$. Under each case, we take as inputs some computed data that serve as values of u at the given Gauss-Chebyshev quadrature points.

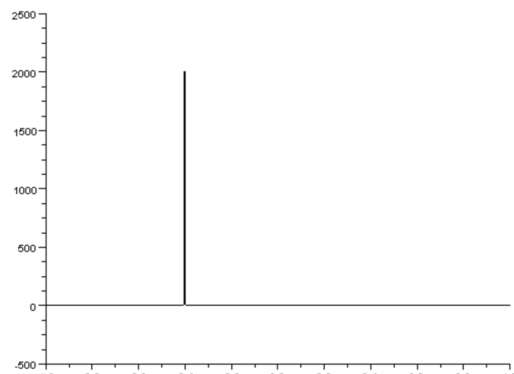

Figure 8: Approximate shock location of u at $x = -0.40666$ when $t = 0.1$, using PC(3,3) approximant for u'.

Determined by the PC(3,3) approximant to the differentiated transformed Chebyshev expansion associated with u, Figure 7 shows that at $t = 0$ a possible jump or shock occurs somewhere very close to $x = -0.5$. The zeros of the denominator of the PC(3,3) approximant are -0.00764 and $-0.49321 \pm 0.86966i$. The complex zero gives a modulus of 0.99978 which strongly indicates that a shock occurs at $x = -0.49321$. This confirms what the plot shows. For the case when $t = 0.1$, Figure 8 indicates that there is a shock very near $x = -0.4$. The zeros of the denominator of the PC(3,3) approximant in this case are $-0.40666{\pm}0.91406i$ and -1.61685. The complex zero gives a modulus of 1.00043 implying that a shock location is at $x = -0.40666$, which is what the plot seems to suggest. In consideration of the two different shock positions at two different points in time, we note that the Burgers' solution involves time evolution of a shock or a sharp gradient.

We present the PC(3,3) and SPC(3,3,3) reconstructions of u in Figures 9 and 11 for the case $t = 0$ and in Figures 10 and 12 for $t = 0.1$. They are plotted against the exact solution. Both approximants in the two cases take the PC approximated jump locations, that is, the jump at $x = -0.49321$ for $t = 0$ and the jump at $x = -0.40666$ for $t = 0.1$. The SPC results are quite impressive notwithstanding the fact the we only use low order approximants to generate them. Comparisons of their respective pointwise error convergence are shown in Figures 13 and 14.

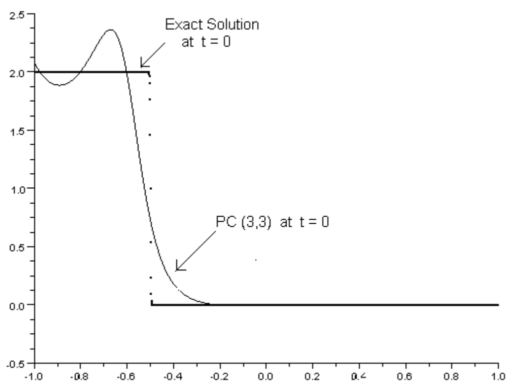

Figure 9: Constrast between the exact solution and its PC(3,3) approximant at $t = 0$.

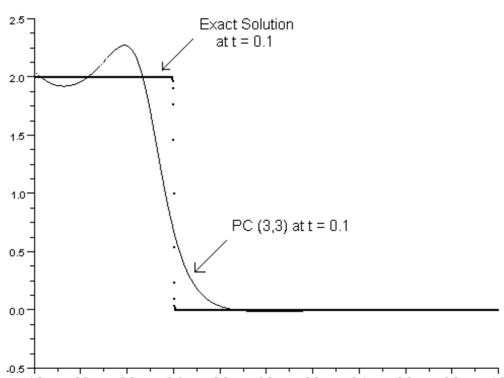

Figure 10: Constrast between the exact solution and its PC(3,3) approximant at $t = 0.1$.

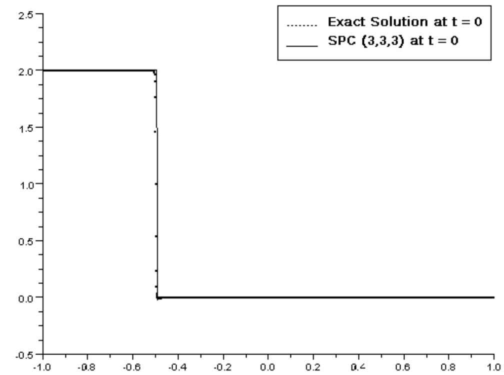

Figure 11: Contrast between exact solution and its SPC(3,3,3) approximant at $t = 0$.

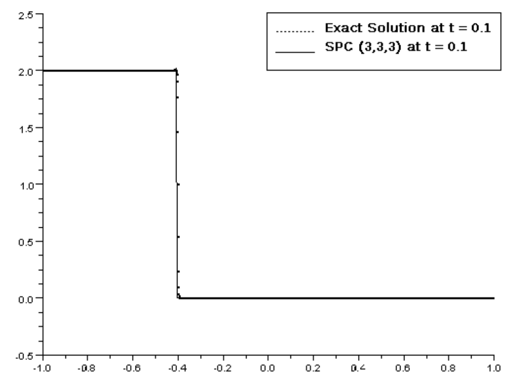

Figure 12: Contrast between exact solution and its SPC(3,3,3) approximant at $t = 0.1$.

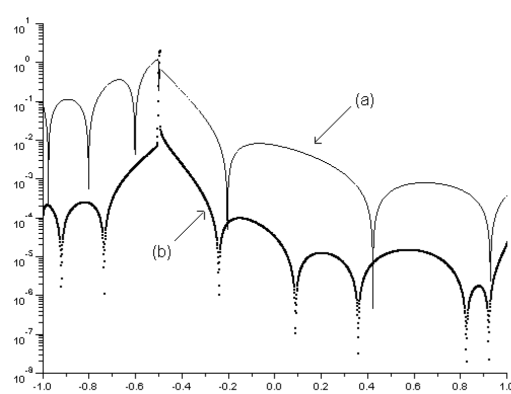

Figure 13: Comparison of pointwise error convergence in logartihmic scale of the (a) PC(3,3) and (b) SPC (3,3,3) approximants at $t = 0$.

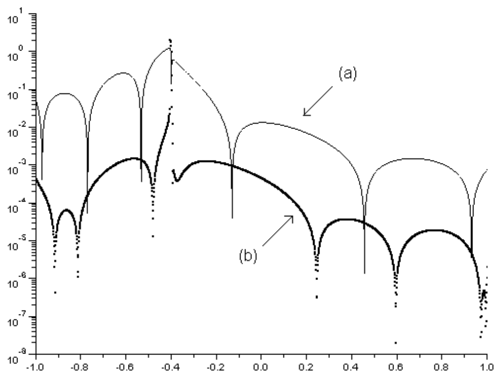

Figure 14: Comparison of pointwise error convergence in logarithmic scale of the (a) PC(3,3) and (b) SPC (3,3,3) approximants at $t = 0.1$.

4.4 Some Remarks on the Degrees of P_N, Q_M, and the R_{V_k}s that Define the SPC Approximant

In the course of implementing the SPC method, choosing the degrees of the polynomials P_N, Q_M, and R_{V_k}s that comprise the SPC approximant can be a crucial issue. In the absence of a rigorous and optimal means of generating these values ([2], [7]), one is left to try several possible combinations. While certain combinations are good which lead to fairly accurate reconstructions, some are obviously bad which either do not remove oscillations or may smoothen out some part but add extraneous oscillations and jumps at the other or cause the jump(s) to steepen further. A simple rule is to compute many approximants with various combinations, select those that are reasonably good, and disregard the bad recontructions [7]; generating an SPC approximant is quick and easy, afterall.

5 Conclusion

The Singular Padé-Chebyshev (SPC) approximation demonstrates how $PC(N,M)$ and $Cheb(N)$ reconstructions of a function with singularities can be enhanced by utilizing its singularities in the approximation process. If the singularities are known, the $SPC\,(N, M, V_1, \ldots, V_k)$ approximant remarkably reconstructs such function. Under restrictive conditions where only the approximated expansion coefficients for the transformed Chebyshev series of the function and the approximated jump locations are used, as in the case of postprocessing computational data, numerical results still reveal that SPC approximant is capable of rectifying the Gibbs phenomenon that occurs in the process of recovering the function. A study on best combinations of the degrees N, M, V_1, ..., V_k in the SPC approximants that yield excellent reconstructions may be pursued as this may not only promote greater ease in approximation but will pave the way as well for a systematic construction of the approximant.

References

[1] W. S. Don, S. M. Kaber, and M. S. Min, *Fourier-Padé approximations and filtering for the spectral simulations of incompressible Boussinesq convection problem*, Mathematics of Computation, 76 (2007), pp. 1275-1290.

[2] T. A. Driscoll and B. Fornberg, *A Pade-based algorithm for overcoming the Gibbs phenomenon*, Numerical Algorithms, 26 (2001), no.1, pp.77 - 92

[3] T. A. Driscoll and B. Fornberg. *Pade-based interpretation and correction of the Gibbs phenomenon*. In Advances in the Gibbs Phenomenon, ed. by A. Jerri, Sigma Sampling Publishing, Potsdam, NY, 2007.

[4] A. Gelb and E. Tadmor, *Detection of edges in spectral data*, Applied and Computational Harmonic Analysis, 7 (1999), pp. 101–135.

[5] D. Gottlieb and C. W. Shu, *On the Gibbs phenomenon and its resolution*, SIAM Review, 39 (1997), pp. 644-668.

[6] J. S. Hesthaven, S. Gottlieb, D. Gottlieb, *Spectral Methods for Time-Dependent Problems*, Cambridge University Press, U.K., 2007

[7] J. S. Hesthaven, S. M. Kaber, L. Lurati, *Padé-Legendre interpolants for Gibbs reconstruction*, Journal of Scientific Computing, 28 (2006), pp. 337-359.

[8] A. J. Jerri, *The Gibbs Phenomenon in Fourier Analysis, Splines and Wavelet Approximation*, Kluwer Dordrecht, 1998

[9] S. M. Kaber and Y. Maday, *Analysis of some Padé-Chebyshev approximants*, SIAM Journal on Numerical Analysis, 43 (2005) no.1, pp. 437-454.

[10] G. K. Kvernadze, *Approximating the jump discontinuities of a function by its Fourier-Jacobi coefficients*, Mathematics of Computations, 73 (2003), pp. 731 – 751.

[11] S. A. Sarra, S.A., *Spectral methods with postprocessing for numerical hyperbolic heat transfer*, Numerical Heat Transfer, Part A, 43, no.7 (2003), pp.717-730

[12] A. L. Tampos and J. E. C. Lope, *Overcoming Gibbs phenomenon in a Padé-Chebyshev approximation*, Matimyas Matimatika: Proceedings of the 7th Taiwan-Philippine Symposium in Mathematics, 25-27 October, 2007, Antipolo, Rizal, Phil., pp.127-135.

[13] A. L. Tampos and J. E. C. Lope, *A Padé-Chebyshev Reconstruction of functions with jump discontinuities*, Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering 2009, WCE 2009, 1 - 3 July, 2009, London, U.K., pp. 1174-1179.

[14] A. L. Tampos, J. E. C. Lope, and J. S. Hesthaven, *A Singularity-based approach in a Padé-Chebyshev resolution of the Gibbs phenomenon*, Lecture Notes in Engineering and Computer Science: Proceedings of the World Congress on Engineering 2012, WCE 2012, 4 - 6 July, 2012, London, U.K., pp. 24-29.