CODIMENSION ONE STABILITY OF THE CATENOID UNDER THE VANISHING MEAN CURVATURE FLOW IN MINKOWSKI SPACE

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ABSTRACT. We study time-like hypersurfaces with vanishing mean curvature in the $(3+1)$ dimensional Minkowski space, which are the hyperbolic counterparts to minimal embeddings of Riemannian manifolds. The catenoid is a stationary solution of the associated Cauchy problem. This solution is linearly unstable, and we show that this instability is the only obstruction to the global nonlinear stability of the catenoid. More precisely, we prove in a certain symmetry class the existence, in the neighborhood of the catenoid initial data, of a co-dimension 1 Lipschitz manifold transverse to the unstable mode consisting of initial data whose solutions exist globally in time and converge asymptotically to the catenoid.

CONTENTS

1. Introduction

We study here *extremal hypersurfaces* embedded in the $(1+3)$-dimensional Minkowski space $\mathbb{R}^{1,3}$. More precisely, we consider for a three-dimensional smooth manifold M the embeddings $\Phi : M \to \mathbb{R}^{1,3}$ such that $\Phi(M)$ has vanishing mean curvature, and such that the pull-back metric has Lorentzian signature[1]. We will consider the associated *Cauchy problem*. Given a two-dimensional smooth manifold Σ and two maps $\Phi_0 : \Sigma \to \mathbb{R}^3$ and $\Phi_1 : \Sigma \to \mathbb{R}^3$, we can ask for the existence and uniqueness of an interval $I = (T_0, T_1) \ni 0$ and a map $\Phi : I \times \Sigma \to \mathbb{R}^{1,3}$ such that $\Phi(I \times \Sigma)$ has vanishing mean curvature, $\Phi : \{t\} \times \Sigma \to \{t\} \times \mathbb{R}^3$, and the initial conditions $\Phi|_{\{0\}\times\Sigma} = (0, \Phi_0)$ and $\partial_t \Phi|_{\{0\}\times\Sigma} = (1, \Phi_1)$ are satisfied. Observe that with the knowledge of Φ_0, Φ_1 it is possible to compute the pullback metric of $I \times \Sigma$ along $\{0\} \times \Sigma$. As it turns out, as long as the pullback metric is Lorentzian, the *quasilinear* system of equations for the extremal hypersurface is second order regularly hyperbolic [7, 31], and local well-posedness for smooth initial data holds (see [18]). It is then natural to consider the large time behavior of the flow.

Note that global existence does not hold in general since there exists explicit finite time blow up solutions. For example, under the assumption of spherical symmetry, so that the initial embedding is that of the round sphere in $\mathbb{R}^3$ with 0 velocity, the equations of motion reduce to an ordinary differential equation for the radius of the sphere

$$RR'' = 2(-1 + (R')^2)$$

which we can integrate to get

$$c_1 R^4 = -1 + (R')^2 .$$

The requirement that the pullback metric is Lorentzian implies that $c_1 < 0$. The equation can then be explicitly solved in terms of the Jacobi elliptic functions, which allows one to check that the radius R must collapse in finite time, forming a singularity. Another example is known in the case where Σ is topologically $\mathbb{S}^1 \times \mathbb{R}$. Let the initial data be given by $\Phi_0(x, y) = (\Phi_0'(x), y)$ and $\Phi_1 = 0$, where $\Phi_0' : \mathbb{S}^1 \to \mathbb{R}^2$ is any embedding. The result of Nguyen-Tian [26] implies that a regular solution cannot exist for all time[2]. By finite speed of propagation this means that there are compactly supported initial data for which the solution blows up in finite time[3].

A particular class of initial data which admits *global* solutions are those for which $\Phi_0 : \Sigma \to \mathbb{R}^3$ is the embedding for a *minimal surface*, and $\Phi_1 = 0$. It is easily checked that the map $\Phi(t, p) = (t, \Phi_0(p))$ embeds $\mathbb{R} \times \Sigma$ into $\mathbb{R}^{1,3}$ with zero mean curvature, and $\partial_t \Phi = (1, 0, 0, 0)$ implies that the pullback metric is

[1] In the case the metric has Riemannian signature, the surface M is usually called a space-like *maximal* hypersurface.

[2] One can see this explicitly in the case Φ_0' is the standard unit circle in $\mathbb{R}^2$, in which case the equation of motion reduces to an ODE and the radius of the cylinder is seen to be given by $\cos(t)$ thus collapsing in finite time.

[3] This is analogous to the blow-up solutions to the focussing semilinear wave equation with power nonlinearity $\Box u = -|u|^{p-1}u$ that are derived from the ODE blow-up mechanism.

Lorentzian. We consider in this paper the problem of *stability* of these stationary solutions. The first consideration of a problem of this sort is due to Brendle (in higher dimensions) [6] and Lindblad [21]. They consider small perturbations of the stationary solution given by a flat hyperplane. One can then write the solution as a graph over the stationary background, and reduce the problem to the small data problem for a *scalar* quasilinear wave equation satisfying both the quadratic and cubic null conditions[4] (following the terminology introduced by Klainerman [16]).

In this paper we will consider the problem of stability for a non trivial stationary background. Our work is in the spirit of recent studies of asymptotic stability of solitary waves for *semilinear* wave equations (see for example [3,4,19,24,25]; see also [13,23,27,29] for finite time blow up regimes which correspond to asymptotic stability in suitable rescaled variables), but in a *quasilinear* setting. The background solution we choose is the *catenoid*, which is an embedded minimal surface in $\mathbb{R}^3$, and is a surface of revolution with topology $\mathbb{S}^1 \times \mathbb{R}$. The induced Riemannian metric on Σ at a fixed time for this stationary solution is asymptotically flat (with two ends). This fact is important in our analysis. Indeed, as it is clear from the study by Brendle and Lindblad, to prove any sort of global existence statement we need to exploit the *dispersive decay* of solutions to the *linearized equation* on our background manifold. In [21] the linearized equation is exactly the linear wave equation on $\mathbb{R}^{1,2}$, and the dispersive decay utilized is the classical one. In our case, the linearized equation is a geometric wave equation on the curved background Σ *with a potential term*. The asymptotic flatness of Σ thus plays an important role in establishing a dispersion mechanism.

As mentioned above, a significant difference with the small data cases considered by Lindblad and Brendle is that[5] the linearized equation is no longer the linear wave equation on the background manifold $\mathbb{R} \times \Sigma$; it also contains a potential term[6]. In addition to introducing complications when applying the vector-field method to obtain decay, the potential term turns out to have the "wrong sign". That is to say, the linearized equation admits an exponentially growing mode. As observed by Krieger-Lindblad [18], if one isolates the perturbation away from the "collar region" (see Figure 1), one can verify that the solution exists "up to the time when the collar begins to move" (due to finite speed of propagation). One should interpret this restriction as when the exponentially growing mode (which is very small initially) *overtakes* the dispersive parts of the perturbation in size. In view of this exponentially growing mode, we cannot obtain stability for arbitrary perturbations.

[4]Note that in the $\mathbb{R}^{1,3}$ case studied by Lindblad [21], this reduction gives rise to a quasilinear wave equation in 2 spatial dimensions, and hence the cubic null condition [1,2] also plays a role.

[5]While we consider the case of embedding a hypersurface in $\mathbb{R}^{1,3}$, the method should easily carry over to the case where the ambient Minkowski space has higher dimensions, as linear dispersive estimates (Section 2.3) improve in higher dimensions, making the nonlinear analysis (Section 2.2.1) simpler. Furthermore the spectral properties of the linearized operators (Section 2.2.2) are qualitatively the same independently of the dimension.

[6]This is related to the fact that the plane is the only complete stable minimal surface in $\mathbb{R}^3$ [12]; the stability here is in the variational sense: there exists small, compactly supported perturbations of the catenoid that further reduces the area locally.

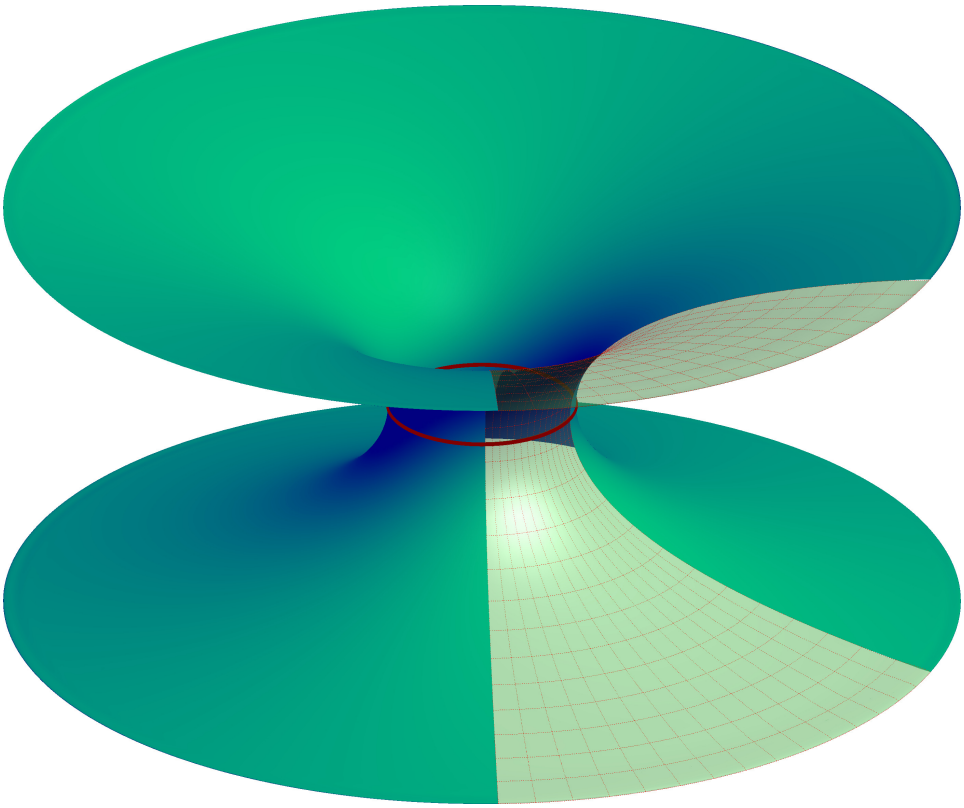

FIGURE 1. The catenoid surface with the "collar" (thinnest part of the surface of revolution) marked out. In the transparent portion we can see the level sets of the angular coordinate ω as well as the "radial" coordinate y.

Similar to the analysis of Krieger-Schlag [19] for the semilinear wave equation, we will show that for any sufficiently small initial perturbation, by adding a suitable multiple of the unstable mode to the Φ_0 component of the initial data, we obtain a new initial data which leads to a global solution converging asymptotically to the catenoid[7].

This paper is organized as follows. In Section 2 we introduce the equation which we will study, discuss some of its main features, describe the linear theory, and state our main theorem. In Section 3 we describe the bootstrap argument which will be used to prove our main theorem. In Sections 4 through 6, we improve on our bootstrap assumptions under the assumption that the projection of our solution

[7]The result in [19] relies on a fixed point argument to solve the problem from infinity. Here we follow instead the approach initiated in [9] (see also [13]) which consists in directly following the flow for any initial data and then using a continuity argument to exhibit the existence of a suitable perturbation of the initial data in the unstable direction such that the unstable mode is extinct for the corresponding solution.

on the unstable mode is under control. In Section 7 we improve our control of the unstable mode. Finally, we prove our main theorem in Section 8.

2. Main Results

2.1. **Formulation of the problem.** As mentioned above, we consider perturbations of the stationary catenoid solution to the extremal surface equation. The catenoid as a surface of revolution can be parametrized by (see also Figure 1)

$$\mathbb{R}\times\mathbb{S}^1 \ni (y,\omega) \mapsto \left(r = \sqrt{1+y^2}, z = \sinh^{-1} y, \theta = \omega\right) \in \mathbb{R}_+ \times \mathbb{R}\times\mathbb{S}^1, \quad (2.1)$$

where we use the standard cylindrical coordinates system on $\mathbb{R}^3$. Throughout we use the notation $\langle y\rangle = \sqrt{1+y^2}$. The parametrization here exposes the catenoid, a surface of revolution, as a warped product manifold with base $\mathbb{R}$ and fibre $\mathbb{S}$; the coordinate y is chosen to be orthogonal to the fibers and to have unit length (note that the parametrization is "by arc length" if we "mod" out the rotational degree of freedom). In this coordinate system we see that the induced Riemannian metric on the catenoid has the line element

$$\mathrm{d}y^2 + \langle y\rangle^2\, \mathrm{d}\omega^2 \,,$$

and that $\langle y\rangle / |y| \to 1$ as $y \to \pm\infty$ captures the asymptotic flatness of this manifold.

In addition to the rotational symmetry, the catenoid also has a reflection symmetry about the plane $z = 0$; in terms of the intrinsic coordinates, this is the mapping $y \mapsto -y$. *For simplicity, we will consider only perturbations that preserve both symmetries.* More precisely, we will consider the case where the perturbed solution is still, at any instance of time, a surface of revolution that is symmetric about the plane $z = 0$. Note that since the induced Riemannian metric on Σ is asymptotically flat with two ends, the Hamiltonian flow on $\mathbb{R}\times\Sigma$ using the pullback metric exhibits trapping, which is manifest in the closed geodesic at the "collar" of Σ (see Figure 1). The rotational symmetry reduces our scenario to the "zero angular momentum case", and hence issues associated with the trapping of the geodesic flow do not appear in our analysis. A treatment of the full problem, without rotational symmetry, will most likely require detailed study of the trapping phenomenon, which usually induces a loss of derivatives. On the other hand, the reflection symmetry is only used to simplify the analysis by effectively fixing the centre of mass; we do not expect there to be obstructions in removing this assumption given finite speed of propagation for nonlinear wave equations.

Given the geometric nature of our problem, there are many different ways of *parametrizing* our solution manifold M (or equivalently, fixing the time parameter t, parametrizing the time slices). To cast the problem as a concrete system of partial differential equations requires choosing a *gauge* (in other words, fixing a preferred parametrization; this problem is typical for geometric equations such as the Ricci flow or Einstein equations). Given the assumed symmetries one may be tempted into a *geometric* gauge choice via intrinsic quantities: for example, the rotational symmetry means that naturally ω is a good candidate coordinate, and we may want to choose the other coordinate y of Σ to be orthogonal to ω and

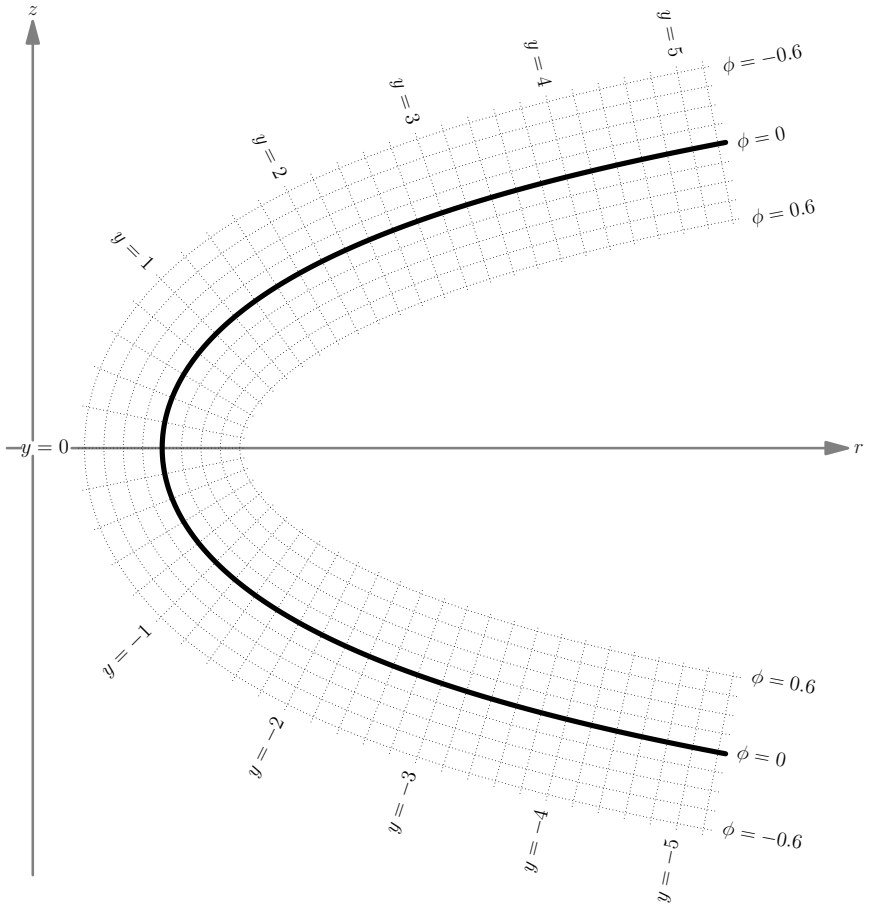

FIGURE 2. The local coordinate system induced by the normal bundle of the catenoid. We show here the cross section for a fixed ω.

of unit length, similar to our parametrization of the catenoid. This choice turns out to be not suitable for studying the stability problem as the equation for the *difference* between our perturbed solution and the stationary catenoid becomes a complicated equation for a vector-valued function with a compatibility constraint (coming from the "unit-length" requirement). By using the compatibility constraint one can convert this to a scalar *non-local* integro-differential equation.

Since we are interested in the stability problem in the rotationally symmetric case, instead we will consider our perturbed solution as a *graph* over the catenoid. More precisely, there is a natural[8] smooth *surjection* from the normal bundle of the catenoid to $\mathbb{R}^3$, given by (see Figure 2)

$$(y,\omega,\phi) \mapsto \left(r = \langle y \rangle + \frac{\phi}{\langle y \rangle}, z = \sinh^{-1} y - \frac{y\phi}{\langle y \rangle}, \theta = \omega \right) . \tag{2.2}$$

[8]Observe that the vector $\langle y \rangle^{-1} (\partial_r - y\partial_z)$ is the *outward pointing unit normal* to the catenoid.

By considering the radius of curvature for the constant ω level curves, we see that restricted to $|\phi| < \langle y\rangle^2$ this mapping is regular and injective[9]. Since we are interested in perturbations of the $\phi = 0$ level set, we make the assumption that our perturbed solution can be written as a graph over $\{\phi = 0\}$ in this coordinate system. That is to say, we will study the *small data problem* for $\phi = \phi(t, y)$. Note that our assumption of reflection symmetry implies that ϕ will be an even function in y, and the lack of ω dependence indicates that the graph is a surface of revolution.

Under this parametrization, we can derive the equation of motion for the extremal surface by formally writing down the Euler-Lagrange equations for the Lagrangian given by the induced volume form on the graph associated to $\phi(t, y)$; this computation is carried out in Appendix A. We find that the equation of motion can be written as a quasilinear wave equation *with potential* for ϕ in the coordinates t, y:

$$-\partial_{tt}^2\phi + \partial_{yy}^2\phi + \frac{y}{\langle y\rangle^2}\partial_y\phi + \frac{2}{\langle y\rangle^4}\phi = Q_2 + Q_3 + Q_4 + S_2 + S_3 + S_4, \tag{2.3}$$

where the quasilinear terms Q_* and semilinear terms S_* are split into those quadratic, cubic, and quartic-or-more in ϕ and its derivatives:

$$Q_2 = -\frac{2\phi}{\langle y\rangle^2}\partial_{tt}^2\phi, \tag{2.4a}$$

$$Q_3 = \frac{\phi^2}{\langle y\rangle^4}\partial_{yy}^2\phi + (\partial_t\phi)^2\partial_{yy}^2\phi - 2\partial_t\phi\partial_y\phi\partial_{ty}^2\phi + (\partial_y\phi)^2\partial_{tt}^2\phi, \tag{2.4b}$$

$$Q_4 = \frac{\phi^2}{\langle y\rangle^4}\left[\left(\frac{2\phi}{\langle y\rangle^2} - \frac{\phi^2}{\langle y\rangle^4} - (\partial_y\phi)^2\right)\partial_{tt}^2\phi + 2\partial_y\phi\partial_t\phi\partial_{ty}^2\phi - (\partial_t\phi)^2\partial_{yy}^2\phi\right], \tag{2.4c}$$

and

$$S_2 = \frac{4\phi^2}{\langle y\rangle^6} + \frac{4y\phi\partial_y\phi}{\langle y\rangle^4} - \frac{(\partial_y\phi)^2}{\langle y\rangle^2}, \tag{2.5a}$$

$$S_3 = \frac{y\phi^2}{\langle y\rangle^6}\partial_y\phi - \frac{2\phi^3}{\langle y\rangle^8} - \left(\frac{3\phi}{\langle y\rangle^4} + \frac{y\partial_y\phi}{\langle y\rangle^2}\right)(\partial_y\phi)^2 + \left(\frac{2\phi}{\langle y\rangle^4} + \frac{y\partial_y\phi}{\langle y\rangle^2}\right)(\partial_t\phi)^2, \tag{2.5b}$$

$$S_4 = -\left(\frac{4y\phi}{\langle y\rangle^4} + \frac{y\phi^2}{\langle y\rangle^6}\right)\partial_y\phi(\partial_t\phi)^2 - \left(\frac{4\phi^2}{\langle y\rangle^6} - \frac{2\phi^3}{\langle y\rangle^8}\right)(\partial_t\phi)^2. \tag{2.5c}$$

We denote by F this nonlinearity

$$F(y, \phi, \nabla\phi, \nabla^2\phi) = Q_2 + Q_3 + Q_4 + S_2 + S_3 + S_4\,. \tag{2.6}$$

[9]On the other hand, at $\phi = \langle y\rangle^2$ the mapping is singular, while at $\phi = -\langle y\rangle^2$ the mapping is not injective (for all ω this maps to the origin).

2.2. **A first look at the structure of the equation.** Let us point out some of the main features of the equations (2.3), (2.4), and (2.5). That our argument can control the nonlinear terms using dispersive estimates is largely due to two special structures: the terms are either *localized* or they exhibit a *null condition*. We comment on these structures in Section 2.2.1. The linear evolution introduces additional difficulties, as it is not purely dispersive: there is an exponentially growing mode. This is discussed in Section 2.2.2.

2.2.1. *Nonlinearities.* The reason that we separated the quadratic, cubic, and quartic-and-higher nonlinearities is that we intend to make use of the *dispersive* effects of the wave equation on a (2+1) dimensional, asymptotically flat space-time to gain decay in the "wave zone", the region where y and t are comparable. The experience with small-data, quasilinear wave equations on $\mathbb{R}^{1,2}$, see [1,2,21], indicates that the most dangerous terms are those which are quadratic and cubic in the nonlinearities, due to the expected linear dispersive decay rate of $1/\sqrt{t}$ for wave equations in 2 spatial dimensions (see also Section 2.3).

On the other hand, in (2.4) and (2.5), almost all the nonlinear terms, in particular all the *quadratic ones*, gain an additional boost in decay from the coefficients of the form $\langle y\rangle^{-k}$ — in the wave zone this term contributes a decay rate of t^{-k} which vastly improves the situation. The term Q_2, for example, has the form $O(t^{-5/2})\cdot\partial_{tt}^2\phi$ with a coefficient which is much better than the integrability threshold of $O(t^{-1})$, if we assume an expected linear decay rate. As we shall see in the analysis, this *localization* of some of the most dangerous nonlinearities plays a crucial role in allowing us to close our decay estimates.

The only exception to this boost in decay occurs in the term Q_3: there we have a non-linearity of the form

$$(\partial_t\phi)^2\partial_{yy}^2\phi - 2\partial_t\phi\partial_y\phi\partial_{ty}^2\phi + (\partial_y\phi)^2\partial_{tt}^2\phi \tag{2.7}$$

which is unweighted. However, as was observed in [21] for the perturbation of the trivial solution, this term carries a *null structure*. One can see this purely at an algebraic level: in terms of the *asymptotically null coordinates* $u = t + y$ and $v = t - y$, the nonlinearity takes the form

$$4(\partial_v\phi)^2\partial_{uu}^2\phi + 4(\partial_u\phi)^2\partial_{vv}^2\phi - 8\partial_u\phi\partial_v\phi\partial_{uv}^2\phi$$

and hence asymptotically[10] verifies the *cubic, quasilinear null condition* [1]. The null condition exhibits in particular a hidden divergence/gradient structure: in the context of elliptic theory it appears in the proof of Wente's inequality [30]; and in the context of wave equations it drives the null form estimates of Klainerman and Machedon [17]. For our explicit nonlinearity above, one can check easily that the

[10]In fact, geometrically if we incorporate the higher order terms we can show that the cubic quasilinear null condition *relative to the Lorentzian metric on* $\mathbb{R}\times\Sigma$, where Σ is the catenoid with the induced Riemannian metric is satisfied *exactly*. That the null condition is always satisfied, even for perturbations of *large data backgrounds*, actually characterizes the extremal surface equation among Lagrangian field theories for scalar fields with certain isotropy assumptions [8, pps.33, 90].

following identity holds

$$(\partial_t\phi)^2\partial_{yy}^2\phi - 2\partial_t\phi\partial_y\phi\partial_{ty}^2\phi + (\partial_y\phi)^2\partial_{tt}^2\phi \\ = \partial_t\big[(\partial_t\phi)^3\big] - 2\partial_y\big[\partial_y\phi(\partial_t\phi)^2\big] + \partial_t\big[(\partial_y\phi)^2\partial_t\phi\big] + 3(\partial_t\phi)^2(\partial_{yy}^2\phi - \partial_{tt}^2\phi).$$

The first three terms of the right-hand side exhibit the hidden divergence structure, while for the last term, we may replace $-\partial_{tt}^2\phi + \partial_{yy}^2\phi$ using our original equation (2.3) and hence obtain terms which are cubic *with sufficient weights* together with quartic and higher terms which have better decay properties.

2.2.2. *Linear spectral analysis.* Having described the difficulties that arise from the "right hand side" of (2.3), we turn our attention to the "left hand side". The linear operator

$$-\partial_{tt}^2\phi + \partial_{yy}^2\phi + \frac{y}{\langle y\rangle^2}\partial_y\phi \tag{2.8}$$

is in fact the coordinate-invariant wave operator $\Box_M\phi$ on the background $\mathbb{R}\times\Sigma$. Indeed, the induced Lorentzian metric on the stationary catenoid solution, as an embedded hypersurface of $\mathbb{R}^{1,3}$, is

$$-\mathrm{d}t^2 + \mathrm{d}y^2 + \langle y\rangle^2\,\mathrm{d}\omega^2\,,$$

and its corresponding Laplace-Beltrami operator can be computed to be exactly (2.8). However, since we are considering the perturbation of a non trivial solution, there is also a lower order correction term generated by the linearization, namely the potential term $2\langle y\rangle^{-4}\phi$ on the left hand side of (2.3). Note that the coefficient $2\langle y\rangle^{-4}$ has a positive sign, which indicates that it is an *attractive* potential, and opens up the possibility of the existence of a negative energy ground state. This is related to the variational instability of the catenoid as a minimal surface [12]. Any corresponding eigenfunction of the linearized operator will generate either non-decaying or exponentially growing modes; clearly this will complicate our estimates based on expectation of linear dispersive decay.

Now, the natural space on which to study our linear operator is the L^2 space adapted to the geometry; that is to say, we should be looking at $L^2(\Sigma)$ where Σ is the catenoid. In the intrinsic coordinates (y,ω) this is $L^2(\langle y\rangle\,\mathrm{d}y\,\mathrm{d}\omega)$. Since we are working with rotationally symmetric functions, we find it convenient to absorb the weight $\langle y\rangle$ onto the function ϕ instead, and work with $L^2(\mathrm{d}y)$. In other words we introduce the notation

$$\tilde{\phi} := \langle y\rangle^{\frac{1}{2}}\,\phi$$

and we obtain in place of (2.3) the following equation:

$$-\partial_t^2\tilde{\phi} + \partial_y^2\tilde{\phi} + \frac{6+y^2}{4\langle y\rangle^4}\tilde{\phi} = \langle y\rangle^{\frac{1}{2}}\,F(\phi,\nabla\phi,\nabla^2\phi). \tag{2.9}$$

Thus, we are now working with the standard $L^2(\mathrm{d}y)$ space and on this space the relevant linear operator

$$\mathcal{L} := -\partial_y^2 - \frac{6+y^2}{4\langle y\rangle^4} \tag{2.10}$$

is a short-range perturbation of the Laplacian. Since the potential term is a bounded multiplier which decays to 0 as $|y| \to \infty$, the operator $\mathcal{L}$ is self-adjoint on $L^2(\mathrm{d}y)$ with domain $\{\partial_{yy}^2 f \in L^2(\mathrm{d}y)\}$, and its essential spectrum is exactly $[0, \infty)$ (this result is classical, see e.g. [14, Sections 13.1 and 14.3]). Due to the $O(\langle y\rangle^{-2})$ decay of the potential term, the solutions to the ordinary differential equation $(\mathcal{L}-\lambda)\eta_\lambda = 0$ for $\lambda > 0$ are given by the Jost solutions [28, Theorem XI.57], and hence there are no L^2 eigenfunctions with positive eigenvalue.

In the case $\lambda = 0$, the equation $\mathcal{L}\eta_0 = 0$ can be solved explicitly: this is simply due to the fact that $\mathcal{L}$ is the natural *linearized* operator for the minimal surface embedding problem, and that after fixing rotational symmetry, the catenoid solutions form a two parameter family due to the freedoms for scaling and translating (along the axis). To be more precise, the standard catenoid we choose in (2.1) is the element of the family

$$(y, \omega) \mapsto \left(r = a\left\langle a^{-1}y\right\rangle, z = b + a\sinh^{-1}(a^{-1}y), \theta = \omega\right) , \tag{2.11}$$

parametrized by $(a, b) \in \mathbb{R}_+ \times \mathbb{R}$, with $a = 1$ and $b = 0$. The two linearly independent solutions to $\mathcal{L}\eta_0 = 0$ correspond to infinitesimal motions in a and b of the above. From this consideration it is clear that the movement in b corresponds to an *odd* solution (and so ruled out by our symmetry assumptions) with a unique root at $y = 0$, while movement in a corresponds to an *even* solution with two roots. We can easily obtain the explicit form of these two solutions by formally taking derivatives relative to a, b after expressing (2.11) in the coordinates (2.2). This yields

$$\eta_0 = \left\langle y\right\rangle^{\frac{1}{2}} \cdot \begin{cases} \frac{y}{\langle y\rangle}\sinh^{-1} y - 1 & \text{(scaling symmetry in } a\text{)}, \\ \frac{y}{\langle y\rangle} & \text{(translation symmetry in } b\text{)}. \end{cases} \tag{2.12}$$

One sees easily from the asymptotic behavior that neither of these functions belong to $L^2(\mathrm{d}y)$.

Remark 2.1. The fact that the solutions η_0 do not belong to $L^2(\mathrm{d}y)$ implies that we do not have to modulate. In other words, the individual elements of our two parameter family (2.11) are "infinitely far" from one another (this can be seen from their asymptotic behavior) and we do not need to track the "motion along the soliton manifold" for our analysis.

We lastly consider the possible discrete spectrum below 0. By testing with bump functions we easily see that there must be a negative eigenvalue. By the Sturm-Picone comparison theorem [5, Section 10.6] and the explicit solutions (2.12) above, we see that the eigenvalue is unique, and its eigenfunction is nowhere vanishing (it is the ground state). We call this eigenfunction $g_d(y)$ and its associated eigenvalue[11] $-k_d^2$. Note that g_d is smooth, and decays exponentially as $|y| \to \infty$.

In the sequel we let P_d denote the projection onto the ground state g_d, and P_c the projection onto the continuous spectrum. Noting that g_d contributes an exponentially growing mode to the linear evolution, we cannot expect to have stability

[11] From numerics $-k_d^2 \sim -0.5857$.

for any perturbation. Instead, we will show that given a sufficiently small initial perturbation $\tilde{\phi}$, we can adjust its projection to the ground state $P_d\tilde{\phi}$ while keeping $P_c\tilde{\phi}$ unchanged so as to guarantee global existence and asymptotic vanishing of the solution. In the analysis we will treat the continuous part and the discrete part of the spectrum separately. We will describe the linear dispersive estimates for the continuous part of the solution in Section 2.3. This will be combined with the analysis of the nonlinear terms (in the spirit of Section 2.2.1) to derive *a priori estimates* assuming that the discrete part of the solution is well behaved. Finally we will close the argument in Section 7 by showing that such a good choice of initial $P_d\tilde{\phi}$ is possible.

2.3. **Dispersive estimates for $\mathcal{L}$.** For the sequel, we shall use the following key energy and dispersive bounds associated with the evolution of the operator $\mathcal{L}$, which is proved in [11]. Recall that we take $P_c = 1 - P_d$ to be the projection to the continuous part of the spectrum of $\mathcal{L}$. In the sequel, we shall frequently use the notations (as well as variations thereof)

$$\|\langle\nabla_{t,y}\rangle^{\alpha}\psi\|_S,\ \|\langle\nabla_{t,y}\rangle^{\alpha}\langle\Gamma\rangle^{\kappa}\psi\|_S$$

for various norms $\|\cdot\|_S$. By these expressions we shall understand the quantities

$$\sum_{0\leqslant|\beta|\leqslant\alpha}\|\nabla_{t,y}^{\beta}\psi\|_S,\quad \sum_{0\leqslant|\beta|\leqslant\alpha}\sum_{0\leqslant\tilde{\kappa}\leqslant\kappa}\|\nabla_{t,y}^{\beta}\Gamma^{\tilde{\kappa}}\psi\|_S.$$

Here Γ stands for either one of the vector fields $\Gamma_1 = t\partial_y + y\partial_t$, $\Gamma_2 = t\partial_t + y\partial_y$.

Proposition 2.1. *For any multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2_{\geqslant 0}$, we have*

$$\|\nabla_{t,y}^{\alpha}P_c e^{it\sqrt{\mathcal{L}}}f\|_{L^2_{dy}} \lesssim \|\langle\partial_y\rangle^{|\alpha|}f\|_{L^2_{dy}} \tag{2.13}$$

with constant depending on $|\alpha| = \alpha_1 + \alpha_2$. Moreover, denoting the scaling vector field

$$\Gamma_2 := t\partial_t + y\partial_y,$$

we have for any $\alpha \in \mathbb{N}_{\geqslant 0}$, $\beta \in \mathbb{N}^2_{\geqslant 0}$ the weighted energy bounds

$$\|\nabla_{t,y}^{\beta}\Gamma_2^{\alpha}P_c e^{it\sqrt{\mathcal{L}}}f\|_{L^2_{dy}} \lesssim \|\langle y\rangle^{\alpha}\langle\partial_y\rangle^{|\beta|+\alpha}f\|_{L^2_{dy}}. \tag{2.14}$$

For the sine evolution, we have the following bounds for $|\alpha| \geqslant 1$:

$$\|\nabla_{t,y}^{\alpha}P_c\frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}f\|_{L^2_{dy}} \lesssim \|\langle\partial_y\rangle^{|\alpha|-1}f\|_{L^2_{dy}} + \|f\|_{L^1_{\langle y\rangle^{\varepsilon}*dy}} \tag{2.15}$$

as well as

$$\|\nabla_{t,y}^{\alpha}\Gamma_2^{\kappa}P_c\frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}f\|_{L^2_{dy}} \lesssim \|\langle\partial_y\rangle^{|\alpha|-1}\langle\Gamma_2\rangle^{\kappa}f\|_{L^2_{dy}} + \|\langle\Gamma_2\rangle^{\kappa}f\|_{L^1_{\langle y\rangle^{\varepsilon}*dy}}. \tag{2.16}$$

As for dispersive bounds, we have the following:

- *Unweighted dispersive bound:*

$$\|\langle y\rangle^{-\frac{1}{2}}P_c e^{it\sqrt{\mathcal{L}}}f\|_{L^{\infty}_{dy}} \lesssim \langle t\rangle^{-\frac{1}{2}}\big[\|\langle y\rangle^{\frac{1}{2}}f\|_{L^1_{dy}} + \|\langle y\rangle^{\frac{1}{2}}f'\|_{L^1_{dy}}\big].$$

- *Weighted dispersive bound:*

$$\|\langle y\rangle^{-1} P_c e^{it\sqrt{\mathcal{L}}} f\|_{L^\infty_{dy}} \lesssim \langle t\rangle^{-1}\big[\|\langle y\rangle f\|_{L^1_{dy}} + \|\langle y\rangle f'\|_{L^1_{dy}}\big].$$

- *Similarly, we get*

$$\|\langle y\rangle^{-\frac{1}{2}} P_c \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} g\|_{L^\infty_{dy}} \lesssim \langle t\rangle^{-\frac{1}{2}} \|\langle y\rangle^{\frac{1}{2}} g\|_{L^1_{dy}},$$

$$\|\langle y\rangle^{-1} P_c \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} g\|_{L^\infty_{dy}} \lesssim \langle t\rangle^{-1} \|\langle y\rangle g\|_{L^1_{dy}},$$

The preceding bounds are still too crude to handle the unweighted cubic interaction terms that shows up in Q_3 of (2.4), and so we complement them with the following.

Proposition 2.2. *For any multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2_{\geqslant 0}$, $|\alpha| \geqslant 1$, we have*

$$\|\nabla^\alpha_{t,y} P_c \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} (\partial_y f)\|_{L^2_{dy}} \lesssim \|\langle \partial_y\rangle^{|\alpha|} f\|_{L^2_{dy}}, \tag{2.17}$$

$$\|\nabla^\alpha_{t,y} \Gamma_2^\kappa P_c \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} (\partial_y f)\|_{L^2_{dy}} \lesssim \|\langle \partial_y\rangle^{|\alpha|} \langle \Gamma_2\rangle^\kappa f\|_{L^2_{dy}}, \tag{2.18}$$

as well as for the inhomogeneous evolution

$$\|\nabla^\alpha_{t,y} P_c \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} (\partial_s F)\|_{L^2_{dy}} \lesssim \|\langle \nabla_{s,y}\rangle^{|\alpha|} F\|_{L^1_s L^2_{dy}}, \tag{2.19}$$

$$\|\nabla^\alpha_{t,y} \Gamma_2^\kappa P_c \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} (\partial_s F)\|_{L^2_{dy}} \lesssim \|\langle \nabla_{s,y}\rangle^{|\alpha|} \langle \Gamma_2\rangle^\kappa F\|_{L^1_s L^2_{dy}}. \tag{2.20}$$

In order to handle the local terms in (2.3), we need a local energy decay result. This is given by the following

Proposition 2.3. *We have the space-time bounds*

$$\begin{aligned}&\|\langle y\rangle^{-1} \nabla^\alpha_{t,y} \Gamma_2^\kappa \cos(t\sqrt{\mathcal{L}}) P_c f\|_{L^2_{t,y}} + \|\langle y\log y\rangle^{-1} \nabla^\alpha_{t,y} \Gamma_2^\kappa \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} P_c g\|_{L^2_{t,y}} \\ &\lesssim \|\langle \partial_y\rangle^{|\alpha|} \langle \Gamma\rangle^\kappa f\|_{L^2_{dy}} + \|\langle \partial_y\rangle^{|\alpha|-1} \langle \Gamma\rangle^\kappa g\|_{L^2_{\langle y\rangle^{1+}dy}}.\end{aligned}$$

The inhomogeneous version with source terms of gradient structure is as follows:

$$\|\nabla^\alpha_{t,y} \Gamma_2^\kappa P_c \int_0^t \frac{\sin((t-s)\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} (\partial_{s,y} F)\|_{L^2_{t,y}} \lesssim \|\langle \nabla_{s,y}\rangle^{|\alpha|} \langle \Gamma_2\rangle^\kappa F\|_{L^1_s L^2_{dy}}.$$

Remark 2.2. Recall that the vectorfields associated to $-\partial_t^2 + \partial_y^2$ are the Lorentz boost generator $\Gamma_1 = t\partial_y + y\partial_t$ and the generator of scaling symmetry $\Gamma_2 = t\partial_t + y\partial_y$. While we will proceed with a variation of the *vector field method* in order to control the nonlinear terms, our weighted linear estimates are derived differently from those commonly used for the small data problem in quasilinear wave equations. In particular, we do not directly estimate the vector field Γ_1, but rely instead on the estimates for Γ_2, the structure of the equation and the behavior of the solution in

the space-time regions $y \ll t$ and $y \gtrsim t$ (see Lemma 4.2). Furthermore, for the vector field Γ_2, our estimate does not follow from commuting against the equation; note that Γ_2 does not commute with the linearized operator $\mathcal{L}$. We instead obtain bounds on Γ_2 by studying its analogue under a distorted Fourier transform. This method, introduced in [11], can be applied to large families of potentials.

2.4. **Main Theorem.** The unstable mode associated with $\mathcal{L}$ should lead in general to exponentially growing solutions for (2.9), even for arbitrarily small initial data. Nonetheless, it is natural to expect the existence of a suitable co-dimension one set of small initial data corresponding to solutions which exist globally in forward time and decay toward zero, i. e. the evolved surface converges to the static catenoid. This is proved in the following theorem which is our main result.

Theorem 2.4 (Codimension one stability of the catenoid)**.** *Let us be given a pair of even functions* $(\tilde{\phi}_1, \tilde{\phi}_2) \in W^{N_0,1}(\mathbb{R}) \cap W^{N_0,2}(\mathbb{R})$ *satisfying the smallness condition*

$$\|\tilde{\phi}\|_{X_0} := \sum_{j=1,2} \|\langle y\rangle^{N_0-j+1}\langle \partial_y\rangle^{N_0-j+1}\tilde{\phi}_j\|_{L^1_{dy}\cap L^2_{dy}} \leqslant \delta_0$$

for $\delta_0 > 0$ *sufficiently small, and* N_0 *sufficiently large. Then there exists a parameter* $a \in \mathbb{R}$ *which depends Lipschitz continuously on* $\tilde{\phi}_{1,2}$ *with respect to* X_0 *such that the solution* $\tilde{\phi}$ *of* (2.9) *corresponding to the initial data*

$$(\tilde{\phi}(0,\cdot), \partial_t\tilde{\phi}(0,\cdot)) = (\tilde{\phi}_1 + a g_d, \tilde{\phi}_2)$$

exists globally in forward time $t > 0$*. Moreover,* $\phi = \langle y\rangle^{-1/2}\tilde{\phi}$ *decays toward zero:*

$$|\phi(t,\cdot)| \lesssim \langle t\rangle^{-\frac{1}{2}}.$$

To our knowledge, this theorem is the *first result of asymptotic stability of a non trivial stationary state in the quasilinear setting*. It is worth mentioning that there has been recently much interest toward questions on stability of non trivial stationary solutions to quasilinear wave equations, especially in the context of the black hole stability problem. In a recent work Holzegel [15] was able to prove decay estimates for the full nonlinear Einstein vacuum equation on the so-called *ultimately Schwarzschildean* backgrounds; the estimates proven, however, are not sufficient to yield stability. Subsequently, Dafermos, Holzegel, and Rodnianski showed [10] the existence of solutions satisfying the ultimately Schwarzschildean condition. Their construction, based on scattering theory, actually implies the *existence* of small perturbations of Schwarzschild (and even Kerr) which are *forward stable*. The solutions which they constructed are however *not* expected to be generic [10, Section 1.3.2] based on the observed decay rates. This stands in contrast[12] to our *generic* construction which is optimal (in terms of the size of the stable manifold) given the linear instability.

[12]Of course, the difficulty faced, even at the level of the linearized problem, in Einstein vacuum equations is very different from those considered here.

An interesting open problem is the description of the flow in the neighborhood of the codimension 1 manifold of Theorem 2.4, and in particular whether this manifold is a threshold between two different types of stable regimes. An analogous problem has been studied in [22] in the case of the L^2 critical nonlinear Schrödinger equation. The initial data corresponding to Bourgain-Wang solutions[13], are shown to lie at the boundary between solutions blowing up in finite time in the log-log regime and solutions scattering to 0 (note that both are known to be stable regimes for that equation). Numerical simulations for the extremal surface equation suggest that a similar behavior might take place here. Indeed, the codimension 1 manifold of Theorem 2.4 seems to be the threshold between two types of regimes: one leading to a collapse of the collar[14], and another leading to the accelerated widening of the collar region[15].

3. Setting up the analysis

The aim of this section is to set up the bootstrap argument.

3.1. Spectral decomposition of the solution.

We decompose our solution $\tilde{\phi}$ as

$$\tilde{\phi} = h(t) g_d + \tilde{\psi}$$

so that $\tilde{\psi}$ satisfies

$$\langle \tilde{\psi}, g_d \rangle = 0.$$

Thus, we have

$$P_d \tilde{\phi} = h(t) g_d, \ P_c \tilde{\phi} = \tilde{\psi}.$$

In particular, $\tilde{\psi}$ satisfies in view of (2.9)

$$\begin{cases} -\partial_t^2 \tilde{\psi} + \partial_y^2 \tilde{\psi} + \frac{1}{2} \frac{3 + \frac{y^2}{2}}{(1+y^2)^2} \tilde{\psi} = P_c((1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)), \\ \tilde{\psi}(0,.) = P_c \tilde{\phi}_1, \ \partial_t \tilde{\psi}(0,.) = P_c \tilde{\phi}_2. \end{cases} \tag{3.1}$$

We derive a formula for $h(t)$ in the following lemma.

Lemma 3.1. *$h(t)$ is given by*

$$\begin{aligned} & h(t) \\ = \ & \frac{1}{2} \left(a + \langle \tilde{\phi}_1, g_d \rangle + \frac{\langle \tilde{\phi}_2, g_d \rangle}{k_d} - \frac{1}{k_d} \int_0^t \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s), g_d \rangle e^{-k_d s} ds \right) e^{k_d t} \\ + \ & \frac{1}{2} \left(a + \langle \tilde{\phi}_1, g_d \rangle - \frac{\langle \tilde{\phi}_2, g_d \rangle}{k_d} + \frac{1}{k_d} \int_0^t \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s), g_d \rangle e^{k_d s} ds \right) e^{-k_d t}. \end{aligned}$$

[13] which are expected to form a co-dimension one manifold, see [20].

[14] More precisely, $\phi \to -\langle y \rangle^2$ for some $|y| \ll 1$. The solution ceases to be a manifold there (see Footnote 9).

[15] Due to the coordinate singularity at $\phi = \langle y \rangle^2$ (see Footnote 9), the long-time behavior in this case is not clear from the simulations.

Proof. $h(t)$ satisfies in view of (2.9) and the fact that g_d is en eigenvector of Ł with eigenvalue $-k_d^2$:

$$-h''(t) + k_d^2 h(t) = \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi), g_d\rangle.$$

Using the variation of constant methods, we deduce

$$\begin{aligned} h(t) &= \left(A_1 - \frac{1}{2k_d}\int_0^t \langle (1+y^2)^{\frac{1}{4}} F(\phi,\nabla\phi,\nabla^2\phi)(s), g_d\rangle e^{-k_d s} ds\right) e^{k_d t} \\ &\quad + \left(A_2 + \frac{1}{2k_d}\int_0^t \langle (1+y^2)^{\frac{1}{4}} F(\phi,\nabla\phi,\nabla^2\phi)(s), g_d\rangle e^{k_d s} ds\right) e^{-k_d t}. \end{aligned}$$

Since we have

$$h(0) = a + \langle \tilde{\phi}_1, g_d\rangle,\ h'(0) = \langle \tilde{\phi}_2, g_d\rangle,$$

we deduce

$$A_1 = \frac{1}{2}\left(a + \langle \tilde{\phi}_1, g_d\rangle + \frac{\langle \tilde{\phi}_2, g_d\rangle}{k_d}\right) \text{ and } A_2 = \frac{1}{2}\left(a + \langle \tilde{\phi}_1, g_d\rangle - \frac{\langle \tilde{\phi}_2, g_d\rangle}{k_d}\right).$$

This concludes the proof of the lemma. □

3.2. **Setting up the bootstrap.** Consider a time $T > 0$ such that the following bootstrap assumptions hold on $[0, T)$:

$$\|\nabla_{t,y}\nabla_{t,y}^{\alpha}\tilde{\phi}\|_{L^2_{dy}} \leqslant \varepsilon\langle t\rangle^{\nu},\ 0 \leqslant |\alpha| \leqslant N_1, \tag{3.2}$$

$$\|\nabla_{t,y}^{\beta}\phi\|_{L^{\infty}_{dy}} \leqslant \varepsilon\langle t\rangle^{-\frac{1}{2}},\ 0 \leqslant |\beta| \leqslant \frac{N_1}{2} + C, \tag{3.3}$$

$$\|\langle y\rangle^{-\frac{1}{2}}\nabla_{t,y}^{\beta}\phi\|_{L^{\infty}_{dy}} \leqslant \varepsilon\langle t\rangle^{-\frac{1}{2}-\delta_1},\ 0 \leqslant |\beta| \leqslant \frac{N_1}{2} + C, \tag{3.4}$$

$$\|\nabla_{t,y}\nabla_{t,y}^{\beta}\Gamma_2^{\gamma}\tilde{\phi}\|_{L^2_{dy}} \leqslant \varepsilon\langle t\rangle^{([\frac{2|\beta|}{N_1}]+1)10^{\gamma}\nu},\ 0 \leqslant |\beta| \leqslant N_1 - \gamma,\ 0 \leqslant \gamma \leqslant 2, \tag{3.5}$$

$$\begin{aligned} \|\langle y\log y\rangle^{-1}\big(\nabla_{t,y}^{\beta}\Gamma_2^{\gamma}\tilde{\phi}\big)\|_{L^2_{t,y}([0,T])} &\leqslant \varepsilon\langle T\rangle^{(\chi_{\gamma>0}[\frac{2|\beta|}{N_1}]+1)10^{\gamma}\nu}, \\ &\qquad 0 \leqslant |\beta| \leqslant 1 + N_1 - \gamma,\ 0 \leqslant \gamma \leqslant 2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\sum_{\beta \leqslant N_1+1} |\partial_t^{\beta} h(t)| \leqslant \varepsilon\langle t\rangle^{-1-2\delta_1}, \\ &\sum_{\beta+\kappa \leqslant N_1+1} |\partial_t^{\beta}(t\partial_t)^{\kappa} h(t)| \leqslant \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])10^{\kappa}\nu},\ \kappa \in \{1,2\}, \\ &\sum_{\beta+\kappa \leqslant N_1+1} \big\|\partial_t^{\beta}(t\partial_t)^{\kappa} h\big\|_{L^2_{[0,T]}} \leqslant \varepsilon\langle T\rangle^{(1+[\frac{2|\beta|}{N_1}])10^{\kappa}\nu},\ \kappa \in \{1,2\}. \end{aligned} \tag{3.7}$$

Our claim is that the above regime is trapped.

Proposition 3.2 (Improvement of the bootstrap assumptions)**.** *There exists an N_1 sufficiently large, such that the following holds: there is N_0 sufficiently large, such that if $N_1 \gg C \geqslant 10$ and given $\varepsilon > 0$, $1 \gg \delta_1 \gg \nu \gg \varepsilon$, there is $\delta_0 = \delta_0(\varepsilon, N_0) > 0$ sufficiently small (as in Theorem 2.4) and*

$$a \in \big[-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}\big]$$

such that $\tilde{\phi}$ satisfies the following bounds

$$\|\nabla_{t,y}\nabla^{\alpha}_{t,y}\tilde{\phi}\|_{L^2_{dy}} \lesssim (\delta_0+\varepsilon^{\frac{3}{2}})\langle t\rangle^{\nu},\ 0\leqslant|\alpha|\leqslant N_1, \tag{3.8}$$

$$\|\nabla^{\beta}_{t,y}\phi\|_{L^{\infty}_{dy}} \lesssim (\delta_0+\varepsilon^{\frac{3}{2}})\langle t\rangle^{-\frac{1}{2}},\ 0\leqslant|\beta|\leqslant\frac{N_1}{2}+C, \tag{3.9}$$

$$\|\langle y\rangle^{-\frac{1}{2}}\nabla^{\beta}_{t,y}\phi\|_{L^{\infty}_{dy}} \lesssim (\delta_0+\varepsilon^{\frac{3}{2}})\langle t\rangle^{-\frac{1}{2}-\delta_1},\ 0\leqslant|\beta|\leqslant\frac{N_1}{2}+C, \tag{3.10}$$

$$\|\nabla_{t,y}\nabla^{\beta}_{t,y}\Gamma^{\gamma}_2\tilde{\phi}\|_{L^2_{dy}} \lesssim (\delta_0+\varepsilon^{\frac{3}{2}})\langle t\rangle^{([\frac{2|\beta|}{N_1}]+1)10^{\gamma}\nu},\ 0\leqslant|\beta|\leqslant N_1-\gamma,\ 0\leqslant\gamma\leqslant 2, \tag{3.11}$$

$$\begin{aligned}\|\langle y\log y\rangle^{-1}\big(\nabla^{\beta}_{t,y}\Gamma^{\gamma}_2\tilde{\phi}\big)\|_{L^2_{t,y}([0,T])} &\lesssim (\delta_0+\varepsilon^{\frac{3}{2}})\langle T\rangle^{(\chi_{\gamma>0}[\frac{2|\beta|}{N_1}]+1)10^{\gamma}\nu},\\ &0\leqslant|\beta|\leqslant 1+N_1-\gamma,\ 0\leqslant\gamma\leqslant 2,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\sum_{\beta\leqslant N_1+1}|\partial^{\beta}_t h(t)| \lesssim (\delta_0+\varepsilon^{\frac{3}{2}})\langle t\rangle^{-1-2\delta_1},\\ &\sum_{\beta+\kappa\leqslant N_1+1}|\partial^{\beta}_t(t\partial_t)^{\kappa}h(t)| \lesssim (\delta_0+\varepsilon^{\frac{3}{2}})\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])10^{\kappa}\nu},\ \kappa\in\{1,2\},\\ &\sum_{\beta+\kappa\leqslant N_1+1}\big\|\partial^{\beta}_t(t\partial_t)^{\kappa}h\big\|_{L^2_{[0,T]}} \lesssim (\delta_0+\varepsilon^{\frac{3}{2}})\langle T\rangle^{(1+[\frac{2|\beta|}{N_1}])10^{\kappa}\nu},\ \kappa\in\{1,2\}.\end{aligned} \tag{3.13}$$

The rest of the paper is as follows. In section 4, we prove the energy bounds (3.8) and (3.11). In section 5, we prove the local energy decay (3.12). In section 6, we prove the dispersive estimates (3.9) and (3.10). In section 7, we prove the existence of a such that (3.13) holds which concludes the proof of Proposition 3.2. Finally, we prove Theorem 2.4 in section 8.

4. Energy bounds

The goal of this section is to prove the estimates (3.8) and (3.11).

4.1. **The proof of the estimate (3.8).** In view of (3.1), we have

$$\tilde{\psi}=\cos(t\sqrt{\mathcal{L}})P_c\tilde{\phi}_1+\frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}P_c\tilde{\phi}_2+\int_0^t\frac{\sin([t-s]\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}P_c\big(G(s,\cdot)\big)\,ds \tag{4.1}$$

where

$$G(s,\cdot)=(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi).$$

In order to derive the desired energy bounds, we can use Proposition 2.1 for the weighted terms without maximum order derivatives, and Proposition 2.2 for the pure cubic terms, as we shall see. In order to deal with the maximum order derivative terms, we have to use a direct integration by parts argument. To begin with, we reveal the gradient structure in the top order cubic terms. One can check easily that the following identity holds

$$\begin{aligned}&\partial_t\big[\phi_t^2\psi_t\big]-2\partial_y\big[\phi_y\phi_t\psi_t\big]+\partial_t\big[\phi_y^2\psi_t\big]+\phi_t^2(\psi_{yy}-\psi_{tt})+2(\phi_{yy}-\phi_{tt})\phi_t\psi_t\\ &\qquad=\phi_t^2\psi_{yy}-2\phi_y\phi_t\psi_{ty}+\phi_y^2\psi_{tt}\end{aligned} \tag{4.2}$$

Denote

$$X_{t,y}(\nabla\phi,\nabla\psi) := \partial_t\big[\phi_t^2\partial_t\psi\big] - 2\partial_y\big[\phi_y\phi_t\partial_t\psi\big] + \partial_t\big[\phi_y^2\partial_t\psi\big].$$

In order to recover the bounds for ψ, we then distinguish between the following three cases:

(1): First order derivatives. Write the equation for ψ as

$$\begin{aligned}&-\partial_t^2\tilde{\psi} + \partial_y^2\tilde{\psi} + \frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}\\ &= \sum_{k=1,2} P_c\big[(1+y^2)^{\frac{1}{4}}F_k(\phi,\nabla\phi,\nabla^2\phi)\big],\ \psi[0] = (P_c\tilde{\phi}_1, P_c\tilde{\phi}_2)\end{aligned}$$

where we define

$$(1+y^2)^{\frac{1}{4}}F_1(\phi,\nabla\phi,\nabla^2\phi) = (1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi) - X_{t,y}(\nabla\phi,\nabla\tilde{\phi}).$$

Then we apply Proposition 2.1, Proposition 2.2 to the source terms F_1, F_2, respectively. The conclusion is that

$$\begin{aligned}\sup_{t\in[0,T]}\|\nabla_{t,y}\tilde{\psi}\|_{L^2_{dy}} \lesssim\ & \|\langle\nabla_y\rangle\tilde{\phi}_1\|_{L^2_{dy}} + \|\tilde{\phi}_2\|_{L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon_*}dy}} + \big\|F_1\big\|_{L^1_tL^2_{\langle y\rangle dy}\cap L^1_tL^1_{\langle y\rangle^{\frac{1}{2}+\varepsilon_*}dy}[0,T]}\\ &+ \sum_{k=1}^{3}\big\|A_k\big\|_{L^1_tL^2_{dy}[0,T]}\end{aligned} \tag{4.3}$$

where

$$A_1 = (\partial_t\phi)^2\partial_t\tilde{\phi},\ A_2 = \partial_y\phi\partial_t\phi\partial_t\tilde{\phi},\ A_3 = (\partial_y\phi)^2\partial_t\tilde{\phi}.$$

¿From our assumptions on $\tilde{\phi}_{1,2}$, we have

$$\|\langle\nabla_y\rangle\tilde{\phi}_1\|_{L^2_{dy}} + \|\tilde{\phi}_2\|_{L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon_*}dy}} \leqslant \delta_0. \tag{4.4}$$

Next, the contributions from the terms A_k are rather straightforward to control. Using the bootstrap assumptions (3.2) and (3.3), we have

$$\begin{aligned}&\big\|(\nabla_{t,y}\phi)^2\nabla_{t,y}\tilde{\phi}\big\|_{L^1_tL^2_{dy}[0,T]} \qquad (4.5)\\ \lesssim\ & \big\|\langle t\rangle^{\nu}|\nabla_{t,y}\phi|^2\big\|_{L^1_tL^\infty_{dy}[0,T]}\big\|\langle t\rangle^{-\nu}\nabla_{t,y}\tilde{\phi}\big\|_{L^\infty_tL^2_{dy}}\\ \lesssim\ & \nu^{-1}\varepsilon^3\langle T\rangle^{\nu} \lesssim \varepsilon^2\langle T\rangle^{\nu}.\end{aligned}$$

It remains to deal with the more complicated source terms F_1, which we do via a separate lemma.

Lemma 4.1. *Under the preceding assumptions, we have*

$$\big\|F_1\big\|_{L^1_tL^2_{\langle y\rangle dy}\cap L^1_tL^1_{\langle y\rangle^{\frac{1}{2}+\varepsilon_*}dy}[0,T]} \lesssim \varepsilon^{\frac{3}{2}}. \tag{4.6}$$

Proof. Observe that by definition of $\langle y\rangle^{\frac{1}{2}}F_1$, this term comprises in addition to the first 4 lines of the right hand side in (2.3) multiplied by $\langle y\rangle^{\frac{1}{2}}$ also the following ones:

$$\begin{aligned}
B_1 &:= (1+y^2)^{\frac{1}{4}}(\frac{\phi}{1+y^2})^2 X_{t,y}(\nabla\phi,\nabla\phi),\\
B_2 &:= (\partial_t\phi)^2\Box(\tilde{\phi}),\ \Box = \partial_y^2-\partial_t^2,\\
B_3 &:= 2\partial_t\phi\Box\phi\partial_t\tilde{\phi},\\
B_4 &:= (1+y^2)^{\frac{1}{4}}\big((\partial_t\phi)^2\phi_{yy}-2\partial_y\phi\partial_t\phi\phi_{ty}+(\partial_y\phi)^2\phi_{tt}\big)\\
&\quad-\big((\partial_t\phi)^2\tilde{\phi}_{yy}-2\partial_y\phi\partial_t\phi\tilde{\phi}_{ty}+(\partial_y\phi)^2\tilde{\phi}_{tt}\big).
\end{aligned}$$

(a): Contribution of B_1. We can write schematically

$$B_1 = (\frac{\phi}{1+y^2})^2 X_{t,y}(\nabla\phi,\nabla\tilde{\phi})+\langle y\rangle^{-\frac{1}{2}}(\frac{\phi}{1+y^2})^2(\nabla_{t,y}\phi)^2\partial_t\tilde{\phi}.$$

Using the bootstrap assumptions (3.2) and (3.3), we conclude that

$$\big\|B_1\big\|_{L_t^1(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}*dy})}\lesssim\varepsilon^5\big\|\langle t\rangle^{-\frac{3}{2}}\big\|_{L_t^1}\lesssim\varepsilon^5.$$

(b): Contribution of B_2. Here we take advantage of the equation satisfied by $\tilde{\phi}$. We obtain the crude bound

$$|\Box\tilde{\phi}|\lesssim\langle y\rangle^{-2}(|\tilde{\phi}|+|\langle\nabla_{t,y}\rangle^2\phi||\langle\nabla_{t,y}\rangle^2\tilde{\phi}|)+|\nabla_{t,y}\phi|^2|\nabla^2_{t,y}\tilde{\phi}|.$$

Then, using the bootstrap assumptions (3.2) and (3.3), we easily infer

$$\big\|B_2\big\|_{L_t^1(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}*dy})}\lesssim\varepsilon^3\big\|\langle t\rangle^{\nu-\frac{3}{2}}\big\|_{L_t^1}\lesssim\varepsilon^3.$$

(c): Contribution of B_3. This is handled like B_2, but using the equation satisfied by ϕ instead.

(d): Contribution of B_4. This term can be schematically written in the form

$$B_4 = (\nabla_{t,y}\phi)^2\big(\langle y\rangle^{-\frac{3}{2}}\phi+\langle y\rangle^{-1}\tilde{\phi}_y\big).$$

Using the bootstrap assumptions (3.2), (3.3) and (3.4), we obtain

$$\big\|B_4\big\|_{L_t^1(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}*dy})}\lesssim\varepsilon^3\big\|\langle t\rangle^{\nu-\frac{\delta_1}{2}-1}\big\|_{L_t^1}\lesssim\varepsilon^2.$$

(e): Contribution of the first 4 lines of the right hand side in (2.3). It is easily verified that this contribution B_5 is bounded in absolute value by

$$|B_5|\lesssim\langle y\rangle^{-\frac{3}{2}}\big[\big|\langle\nabla_{t,y}\rangle\phi\big|^2+\big|\langle\nabla_{t,y}\rangle^2\phi\big|^2\big]$$

and so the bootstrap assumption (3.4) provides the bound

$$\big\|B_5\big\|_{L_t^1(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}*dy})}\lesssim\varepsilon^2\big\|\langle t\rangle^{-\frac{\delta_1}{2}-1}\big\|_{L_t^1}\lesssim\varepsilon^{\frac{3}{2}}.$$

This concludes the proof of the lemma. □

Finally, the estimates (4.3)-(4.6) imply

$$\sup_{t\in[0,T]} \|\nabla_{t,y}\tilde{\psi}\|_{L^2_{dy}} \lesssim \delta_0 + \varepsilon^{\frac{3}{2}} + \varepsilon^2\langle T\rangle^{\nu}. \tag{4.7}$$

(2): Higher order derivatives of degree strictly less than N_1. Here we use induction on the degree of the derivatives, assuming the bound (4.7). Write the equation for $\tilde{\psi}$ schematically in the form

$$-\partial_t^2\tilde{\psi} + \partial_y^2\tilde{\psi} + \frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi} = P_cG.$$

Applying ∂_t^{β} with $1 \leqslant \beta \leqslant N_1 - 1$, and integrating against $\partial_t^{\beta+1}\tilde{\psi}$, we easily infer

$$\begin{aligned}&\Big(\int_{\mathbb{R}} \frac{1}{2}\big[|\partial_t^{\beta+1}\tilde{\psi}|^2 + |\partial_t^{\beta}\partial_y\tilde{\psi}|^2 - \frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}|\partial_t^{\beta}\tilde{\psi}|^2\big]\,dy\Big)\Big|_0^T \\ &= -\int_0^T\int_{\mathbb{R}} (P_c\partial_t^{\beta}G)\partial_t^{\beta+1}\tilde{\psi}\,dtdy.\end{aligned} \tag{4.8}$$

Recall that we have

$$G = (1+y^2)^{\frac{1}{4}}F(\phi, \nabla\phi, \nabla^2\phi).$$

Note that we have the crude bound

$$\big|\partial_t^{\beta}G\big| \lesssim \sum_{\beta_1+\beta_2=\beta} \frac{|\langle\nabla_{t,y}\rangle^2\partial_t^{\beta_1}\phi||\langle\nabla_{t,y}\rangle^2\partial_t^{\beta_2}\tilde{\phi}|}{1+y^2} + \sum_{\beta_1+\beta_2+\beta_3=\beta}\prod_{j=1}^{2}|\nabla_{t,y}\langle\nabla_{t,y}\rangle\partial_t^{\beta_j}\phi||\nabla_{t,y}\langle\nabla_{t,y}\rangle\partial_t^{\beta_3}\tilde{\phi}|$$

where we may assume $\beta_3 \geqslant \beta_2 \geqslant \beta_1$. We use the energy bound (3.2), the local energy decay (3.6) (with $\gamma = 0$), as well as the dispersive bounds (3.3),(3.4), the latter in order to deal with the logarithmic degeneracy in (3.6). It then follows that (using $\beta + 2 \leqslant N_1 + 1$)

$$\begin{aligned}&\langle T\rangle^{-2\nu}\Big|\int_0^T\int_{\mathbb{R}} (P_c\partial_t^{\beta}G)\partial_t^{\beta+1}\tilde{\psi}\,dtdy\Big| \\ &\lesssim \big\|\langle t\rangle^{\frac{1}{2}+}\langle y\rangle^{-\frac{1}{2}}\langle\nabla_{t,y}\rangle^{\frac{N_1+3}{2}}\phi\big\|_{L^{\infty}_{t,y}}\langle T\rangle^{-\nu}\Big\|\frac{\langle\nabla_{t,y}\rangle^{N_1+1}\tilde{\phi}}{1+y^{\frac{3}{2}}}\Big\|_{L^2_{t,y}([0,T])}\big\|\langle t\rangle^{-\nu}\partial_t^{\beta+1}\tilde{\psi}\big\|_{L^{\infty}_t L^2_y([0,T])} \\ &+ \nu^{-1}\big\|\langle t\rangle^{\frac{1}{2}}\langle\nabla_{t,y}\rangle^{\frac{N_1+3}{2}}\phi\big\|^2_{L^{\infty}_{t,y}}\big\|\langle t\rangle^{-\nu}\langle\nabla_{t,y}\rangle^{N_1+1}\tilde{\phi}\big\|_{L^{\infty}_t L^2_y([0,T])}\big\|\langle t\rangle^{-\nu}\partial_t^{\beta+1}\tilde{\psi}\big\|_{L^{\infty}_t L^2_y([0,T])} \\ &\lesssim \varepsilon^3 + \frac{\varepsilon^4}{\nu} \lesssim \varepsilon^3.\end{aligned}$$

This recovers the desired bound (3.8) for $\partial_t^{\beta+1}\tilde{\psi}$. To get control over $\|\nabla_{t,y}^{\beta}\phi\|_{L^2_y}$, $1 \leqslant |\beta| \leqslant N_1$, one uses the pure t-derivative bounds, the equation, and induction on the number of y-derivatives.

(3): Top order derivatives Here we need to perform integration by parts in the top order derivative contributions. Again it suffices to bound the expression

$\partial_t^{N_1+1}\tilde{\psi}$, as the remaining derivatives are controlled directly from the equation. Using (4.8) with $\beta = N_1$, write schematically

$$(P_c\partial_t^{N_1}G)\partial_t^{N_1+1}\tilde{\psi} = \langle y\rangle^{-2}\phi\partial_t^{N_1}\nabla_{t,y}^2\tilde{\phi}\partial_t^{N_1+1}\tilde{\psi} + (\nabla_{t,y}\phi)^2\partial_t^{N_1}\nabla_{t,y}^2\tilde{\phi}\partial_t^{N_1+1}\tilde{\psi} + l.o.t.$$

where the contribution of the lower order terms is treated as in *(2)* above. We conclude that

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}(P_c\partial_t^{N_1}G)\partial_t^{N_1+1}\tilde{\psi}\,dtdy\\
&= \big(\pm\frac{1}{2}\int_{\mathbb{R}}\langle y\rangle^{-2}\phi|\partial_t^{N_1}\nabla_{t,y}\tilde{\psi}|^2\,dy\big)\big|_{t=0}^{t=T}\\
&+ \big(\pm\frac{1}{2}\int_{\mathbb{R}}(\nabla_{t,y}\phi)^2|\partial_t^{N_1}\nabla_{t,y}\tilde{\psi}|^2\,dy\big)\big|_{t=0}^{t=T} + l.o.t.\\
&+ \int_0^T\int_{\mathbb{R}}\big[\langle y\rangle^{-2}\phi + (\nabla_{t,y}\phi)^2\big]\big(\partial_t^{N_1}\nabla_{t,y}^2P_d\tilde{\phi}\big)\partial_t^{N_1+1}\tilde{\psi}\,dydt
\end{aligned}$$

where the terms "l.o.t" can be bounded like in *(2)*. As the first two integral expressions on the right can be bounded by

$$\begin{aligned}
&\big|\big(\pm\frac{1}{2}\int_{\mathbb{R}}\langle y\rangle^{-2}\phi|\partial_t^{N_1}\nabla_{t,y}\tilde{\psi}|^2\,dy\big)\big|_{t=0}^{t=T} + \big(\pm\frac{1}{2}\int_{\mathbb{R}}(\nabla_{t,y}\phi)^2|\partial_t^{N_1}\nabla_{t,y}\tilde{\psi}|^2\,dy\big)\big|_{t=0}^{t=T}\big|\\
&\lesssim \varepsilon\big\|\partial_t^{N_1}\nabla_{t,y}\tilde{\psi}\big\|^2_{L_t^\infty L_y^2([0,T])}.
\end{aligned}$$

It remains to bound the last integral expression, for which we need to control $\partial_t^{N_1}\nabla_{t,y}^2P_d\tilde{\phi}$. We recall that

$$P_d\tilde{\phi} = h(t)g_d(y) \text{ with } h(t) = \langle\tilde{\phi}, g_d\rangle$$

so that

$$\begin{aligned}
\partial_t^{N_1}\nabla_{t,y}^2P_d\tilde{\phi} &= -\partial_t^{N_1-2}\nabla_{t,y}^2P_d\big(\Box\tilde{\phi}\big) - \partial_t^{N_1-2}\nabla_{t,y}^2P_d\big(\partial_y^2\tilde{\phi}\big)\\
&= -\partial_t^{N_1-2}\nabla_{t,y}^2P_d\big(\Box\tilde{\phi}\big) + \partial_t^{N_1-2}\nabla_{t,y}^2\big(\langle\partial_y\tilde{\phi}, g_d\rangle g_d\big)
\end{aligned}$$

where we integrated by parts in y in the second term on the right-hand side. In view of the bootstrap assumption (3.4), the equation for $\Box\tilde{\phi}$ and the decay and smoothness of g_d, we obtain the crude bound

$$\big|\partial_t^{N_1+2}h(t)\big| \lesssim \varepsilon\langle t\rangle^{-\frac{1}{2}-\delta_1}.$$

Using the dispersive bounds (3.3), we then infer

$$\begin{aligned}
&\big|\int_0^T\int_{\mathbb{R}}\big[\langle y\rangle^{-2}\phi + (\nabla_{t,y}\phi)^2\big]\big(\partial_t^{N_1}\nabla_{t,y}^2P_d\tilde{\phi}\big)\partial_t^{N_1+1}\tilde{\psi}\,dydt\big|\\
&\lesssim \varepsilon^3 + \varepsilon\big\|\partial_t^{N_1+1}\tilde{\psi}\big\|^2_{L_t^\infty L_y^2([0,T])}.
\end{aligned}$$

Combining the preceding bounds, one easily infers the improved estimate

$$\big\|\nabla_{t,y}\partial_t^{N_1}\tilde{\psi}\big\|_{L_t^\infty L_y^2([0,T])} \lesssim \delta_0 + \varepsilon^3.$$

The remaining (mixed) derivative terms $\nabla_{t,y}^{\beta}\tilde{\psi}$, $|\beta| = N_1 + 1$, are bounded by induction on the number of y-derivatives, using the equation for $\tilde{\psi}$. This completes the proof of (3.8).

4.2. **The proof of the estimate (3.11).** We next turn to the weighted energy estimates, of the form (3.11). Here we use the weighted bounds in Propositions 2.1, 2.2. The key to control the quadratic nonlinear terms shall be the local energy bounds (3.6). To deal with the cubic terms, we start with the following lemma, which will also be useful later on. It ensures that we get control over the Lorentz boost generator $\Gamma_1 = t\partial_y + y\partial_t$.

Lemma 4.2. *Let $\Gamma_1 := t\partial_y + y\partial_t$. Then, we can infer the bounds*

$$\big\|\nabla_{t,y}\nabla_{t,y}^{\beta}\Gamma_1^{\kappa}\tilde{\psi}\big\|_{L^2_{dy}} \lesssim \varepsilon\langle t\rangle^{|\kappa|-\frac{1}{2}-\delta_1},\ \kappa + |\beta| \leqslant N_1,\ \kappa \in \{1,2\}.$$

$$\big\|\nabla_{t,y}\nabla_{t,y}^{\beta}\Gamma_1\Gamma_2\tilde{\psi}\big\|_{L^2_{dy}} \lesssim \varepsilon\langle t\rangle^{1+(1+[\frac{2|\beta|}{N_1}])10\nu},\ 2 + |\beta| \leqslant N_1.$$

Proof. We start with the first bound of the lemma with $\kappa = 1$.

(1): Proof of the first inequality with $\kappa = 1$. Observe that

$$(\Gamma_1\tilde{\psi})_{t,y} - (\Gamma_2\tilde{\psi})_{t,y} = O\big(\big|(t-y)\nabla_{t,y}^2\tilde{\psi}\big|\big) + O(|\nabla_{t,y}\tilde{\psi}|). \tag{4.9}$$

Further, note

$$(\Gamma_2\tilde{\psi})_t = t\tilde{\psi}_{tt} + y\tilde{\psi}_{ty} + \tilde{\psi}_t,\ (\Gamma_2\tilde{\psi})_y = t\tilde{\psi}_{ty} + y\tilde{\psi}_{yy} + \tilde{\psi}_y. \tag{4.10}$$

We can replace $y\tilde{\psi}_{yy}$ by $y\tilde{\psi}_{tt}$ by using the equation

$$y\tilde{\psi}_{yy} = y\tilde{\psi}_{tt} - \frac{y}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi} + yP_cG.$$

We infer

$$(t-y)\tilde{\psi}_{tt} = \frac{t(\Gamma_2\tilde{\psi})_t - y(\Gamma_2\tilde{\psi})_y + t\psi_t - y\tilde{\psi}_y - y\big(-\frac{y}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi} + yP_cG\big)}{t+y}, \tag{4.11}$$

$$(t-y)\tilde{\psi}_{ty} = \frac{y(\Gamma_2\tilde{\psi})_t - t(\Gamma_2\tilde{\psi})_y + y\psi_t - t\tilde{\psi}_y - t\big(-\frac{y}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi} + yP_cG\big)}{t+y}. \tag{4.12}$$

Using the bootstrap assumption (3.2), we have for $|\beta| + 1 \leqslant N_1$

$$\big\|\nabla_{t,y}^{\beta}\big[yP_cG(t,\cdot)\big]\big\|_{L^2_{dy}} + \big\|\nabla_{t,y}^{\beta}\big[\frac{y}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}(t,\cdot)\big]\big\|_{L^2_{dy}} \lesssim \varepsilon\langle t\rangle^{3\nu}.$$

Together with (4.11), (4.12) and the bootstrap assumptions (3.2) and (3.5), we obtain

$$\big\|(t-y)\nabla_{t,y}^{\beta}\tilde{\psi}_{tt}\big\|_{L^2_{dy}} + \big\|(t-y)\nabla_{t,y}^{\beta}\tilde{\psi}_{ty}\big\|_{L^2_{dy}} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])10\nu},\ |\beta| + 1 \leqslant N_1. \tag{4.13}$$

It remains to bound

$$\big\|(t-y)\nabla_{t,y}^{\beta}\tilde{\psi}_{yy}\big\|_{L^2_{dy}},\ |\beta|+1\leqslant N_1.$$

Here we directly use the equation satisfied by $\tilde{\psi}$. Let $\tilde{\phi}_{1,2}$ be a fundamental system associated with $\mathcal{L}$, with $\tilde{\phi}_1$ given by

$$\frac{-\sqrt{1+y^2}+y\sinh^{-1}(y)}{(1+y^2)^{\frac{1}{4}}}.$$

Note in particular that $|\tilde{\phi}_{1,2}(y)|\lesssim y^{\frac{1}{2}}\log y$ as $y\to\infty$. Then we have the formula

$$\begin{aligned}\tilde{\psi}(t,y)=\ &\tilde{\phi}_2(y)\int_0^y\tilde{\phi}_1(\tilde{y})\big[\tilde{\psi}_{tt}(t,\tilde{y})+P_cG(t,\tilde{y})\big]\,d\tilde{y}\\ &-\tilde{\phi}_1(y)\int_0^y\tilde{\phi}_2(\tilde{y})\big[\tilde{\psi}_{tt}(t,\tilde{y})+P_cG(t,\tilde{y})\big]\,d\tilde{y}\\ &+a(t)\tilde{\phi}_1(y),\end{aligned}\tag{4.14}$$

and the improved local dispersive decay (3.4) implies

$$\big|\partial_t^{\beta}a(t)\big|\lesssim\varepsilon\langle t\rangle^{-\frac{1}{2}-\delta_1},\ \beta\leqslant\frac{N_1}{2}+C.$$

But then (4.13) as well as the precise form of G imply that *restricting to* $y\leqslant t$, we have

$$\big\|(t-y)\nabla_{t,y}^{\beta}\tilde{\psi}_{yy}(t,\cdot)\big\|_{L^2_{dy}(y\leqslant t)}\lesssim\varepsilon\langle t\rangle^{\frac{1}{2}-\delta_1},\ |\beta|\leqslant\frac{N_1}{2}+C,$$

while the bound

$$\big\|(t-y)\nabla_{t,y}^{\beta}\tilde{\psi}_{yy}(t,\cdot)\big\|_{L^2_{dy}(y>t)}\lesssim\varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])10\nu}\tag{4.15}$$

follows directly from the equation satisfied by $\tilde{\psi}$. In fact, replacing $\tilde{\psi}_{yy}$ by $\tilde{\psi}_{tt}$ the bound follows from (4.11), and we can absorb the factor $(t-y)$ in the potential for the linear term (in the region $y\gtrsim t$), while this factor is easily absorbed by the nonlinearity as in the inequality after (4.12). The missing bounds with $|\beta|>\frac{N_1}{2}+C$ are easily obtained directly from the equation (inductively). Together with (4.9), (4.13) and the bootstrap assumption (3.5), we deduce

$$\big\|\nabla_{t,y}\nabla_{t,y}^{\beta}\Gamma_1\tilde{\psi}\big\|_{L^2_{dy}}\lesssim\varepsilon\langle t\rangle^{\frac{1}{2}-\delta_1},\ 1+|\beta|\leqslant N_1.$$

(2): Proof of the first inequality of the lemma with $\kappa=2$. Observe that

$$(\Gamma_1^2-\Gamma_2^2)\tilde{\psi}=(t^2-y^2)(\tilde{\psi}_{yy}-\tilde{\psi}_{tt}),$$

(a): inner region, $y\leqslant t$. We get

$$\big\|\nabla_{t,y}^{\beta}\big[(t^2-y^2)\tilde{\psi}_{tt}(t,\cdot)\big]\big\|_{L^2_{dy}(y\leqslant t)}\lesssim\varepsilon\langle t\rangle^{1+(1+[\frac{2|\beta|}{N_1}])10\nu},\ |\beta|+1\leqslant N_1,$$

on account of (4.13). Using (4.14), we have

$$\big\|\nabla_{t,y}^{\beta}\big[(t^2-y^2)\tilde{\psi}_{yy}\big]\big\|_{L^2_{dy}(y\leqslant t)}\lesssim\varepsilon\langle t\rangle^{\frac{3}{2}-\delta_1},\ |\beta|+1\leqslant N_1,$$

provided $\nabla_{t,y}^{\beta} = \partial_t^{\beta_1}\partial_y^{\beta_2}$ with $\beta_1 \leqslant \frac{N_1}{2} + C$, and the remaining cases are obtained using induction and the equation for $\tilde{\psi}$. It then follows that we have the bounds

$$\big\|\nabla_{t,y}\nabla_{t,y}^{\beta}(\Gamma_1^2 - \Gamma_2^2)\tilde{\psi}\big\|_{L^2_{dy}(y\leqslant t)} \lesssim \varepsilon\langle t\rangle^{\frac{3}{2}-\delta_1},\ |\beta| + 2 \leqslant N_1.$$

(b) For the outer cone region $y > t$, we use

$$(\Gamma_1^2 - \Gamma_2^2)\tilde{\psi} = (t^2 - y^2)\big(P_c G - \frac{1}{2}\frac{3 + \frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}\big).$$

Then use the bound (for suitable $\delta > 0$)

$$\big|P_c G\big| \lesssim \langle y\rangle^{-\frac{3}{2}}\big|\langle\nabla_{t,y}\rangle^2\phi\big|^2 + \langle y\rangle^{\frac{1}{2}}|\nabla_{t,y}\phi|^2\nabla_{t,y}^2\phi + \varepsilon e^{-\delta|y|}.$$

It remains to verify that the weight $t^2 - y^2$ may be absorbed in the cubic terms. Note that for $|\beta| \leqslant N_1 - 1$, we have by the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and bootstrap assumption (3.2)

$$\|\langle y\rangle^{\frac{1}{2}}\nabla_{t,y}^{\beta}\nabla_{t,y}\phi\|_{L^\infty} \lesssim \varepsilon\langle t\rangle^{\nu},$$

while from (4.15), we know that

$$\big\|(t-y)\nabla_{t,y}^{\beta}\nabla_{t,y}^2\tilde{\psi}\big\|_{L^2_{dy}(y>t)} \lesssim \varepsilon\langle t\rangle^{10\nu},\ |\beta| < \frac{N_1}{2}.$$

It then follows that for any $|\beta| \leqslant N_1 - 1$, we have

$$\big\|\nabla_{t,y}^{\beta}\big((t^2-y^2)\langle y\rangle^{\frac{1}{2}}|\nabla_{t,y}\phi|^2\nabla_{t,y}^2\phi\big)\big\|_{L^2_{dy}(y>t)} \lesssim \varepsilon^3\langle t\rangle^{12\nu}.$$

Finally, bootstrap assumption (3.2) yields

$$\big\|\nabla_{t,y}\nabla_{t,y}^{\beta}\big((t^2-y^2)\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}\big)\big\|_{L^2_{dy}(y>t)} \lesssim \varepsilon\langle t\rangle^{\nu},\ |\beta| \leqslant N_1 - 2.$$

It now follows that for $|\beta| + 2 \leqslant N_1$, we have

$$\big\|\nabla_{t,y}\nabla_{t,y}^{\beta}\big((\Gamma_1^2 - \Gamma_2^2)\tilde{\psi}\big)\big\|_{L^2_{dy}(y>t)} \lesssim \varepsilon\langle t\rangle^{12\nu} \lesssim \varepsilon\langle t\rangle^{\frac{3}{2}-\delta_1}.$$

The estimates in *(a), (b)* complete the proof of the first estimate of the lemma for $\kappa = 2$.

(3): Proof of the second inequality of the lemma. We have the following identity

$$\Gamma_1\Gamma_2 - \Gamma_2^2 = (t-y)^2\partial_{ty}^2 - (t-y)^2\partial_t^2 + (ty - y^2)(\partial_y^2 - \partial_t^2) + \Gamma_1 - \Gamma_2.$$

Note that

$$\tilde{\psi}_{ty} - \tilde{\psi}_{tt} = \frac{(\Gamma_2\tilde{\psi})_y - (\Gamma_2\tilde{\psi})_t - \tilde{\psi}_y + \tilde{\psi}_t - y(\tilde{\psi}_{yy} - \tilde{\psi}_{tt})}{t-y}$$

Then, using simple variations of the estimates above, in particular the structure of $\tilde{\psi}_{yy} - \tilde{\psi}_{tt}$, one concludes that

$$\big\|\nabla_{t,y}^{\beta}\big((\Gamma_1\Gamma_2 - \Gamma_2^2)\tilde{\psi}\big)\big\|_{L^2_{dy}(y\lesssim t)} \lesssim \varepsilon\langle t\rangle^{1+(1+[\frac{2|\beta|}{N_1}])10\nu},\ 2 \leqslant |\beta| + 1 \leqslant N_1,$$

which in light of the a priori bound on $\Gamma_2^2\tilde{\psi}$ implies the second estimate of the lemma in the region $y \lesssim t$. In the region $y \gg t$, one uses Lemma B.1 to estimate $\|(t-y)^2\nabla_{t,y}^3\tilde{\psi}\|_{L^2(y\gg t)}$ directly. Note that the proof of the latter actually allows us estimate $\|(t-y)^2\nabla_{t,y}^3\tilde{\psi}\|_{L^2(y\gg t)}$ also in the region $y \sim t$. □

Remark 4.1. The preceding proof reveals that for Γ^κ any product of at most two of the vector fields Γ_1, Γ_2, we have

$$\big\|\nabla_{t,y}\nabla_{t,y}^\beta\Gamma^\kappa\tilde{\psi}\big\|_{L_y^2(y\gtrsim t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ |\beta|+|\kappa| \leqslant N_1,\ \kappa\in\{1,2\}.$$

Lemma 4.3. *We can split* $\Gamma_1\tilde{\psi} = (\Gamma_1\tilde{\psi})_1 + (\Gamma_1\tilde{\psi})_2$*, where we have*

$$\big\|\langle\log y\rangle^{-1}\nabla_{t,y}^\beta(\Gamma_1\tilde{\psi})_1\big\|_{L_y^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{\frac{1}{2}-\delta_1}$$

provided have $0 \leqslant |\beta| \leqslant \frac{N_1}{2} + C$*, while we have*

$$\big\|\nabla_{t,y}\nabla_{t,y}^\beta(\Gamma_1\tilde{\psi})_2\big\|_{L_y^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ |\beta|+2 \leqslant N_1.$$

Moreover, there is a splitting $\Gamma_1^2\tilde{\psi} = (\Gamma_1^2\tilde{\psi})_1 + (\Gamma_1^2\tilde{\psi})_2$*, with*

$$\big\|\langle\log y\rangle^{-1}\nabla_{t,y}^\beta(\Gamma_1^2\tilde{\psi})_1\big\|_{L_y^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{\frac{3}{2}-\delta_1},\ 0 \leqslant |\beta| \leqslant \frac{N_1}{2} + C,$$

as well as

$$\big\|\nabla_{t,y}\nabla_{t,y}^\beta(\Gamma_1^2\tilde{\psi})_2\big\|_{L_y^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ |\beta|+2 \leqslant N_1.$$

Finally, there is a splitting $\Gamma_1\Gamma_2\tilde{\psi} = (\Gamma_1\Gamma_2\tilde{\psi})_1 + (\Gamma_1\Gamma_2\tilde{\psi})_2$*, with*

$$\big\|\langle\log y\rangle^{-1}\nabla_{t,y}^\beta(\Gamma_1\Gamma_2\tilde{\psi})_1\big\|_{L_y^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{1+(1+[\frac{2|\beta|}{N_1}])10\nu},\ 0 \leqslant |\beta| \leqslant \frac{N_1}{2} + C,$$

as well as

$$\big\|\nabla_{t,y}\nabla_{t,y}^\beta(\Gamma_1^2\tilde{\psi})_2\big\|_{L_y^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{\frac{1}{2}-\delta_1},\ |\beta|+2 \leqslant N_1.$$

Proof. In fact, using (4.14), we get

$$\begin{aligned}\Gamma_1\tilde{\psi} = {}& \Gamma_1\big(\tilde{\phi}_2(y)\int_0^y \tilde{\phi}_1(\tilde{y})\big[\tilde{\psi}_{tt}(t,\tilde{y}) + P_cG(t,\tilde{y})\big]\,d\tilde{y}\big)\\ & - \Gamma_1\big(\tilde{\phi}_1(y)\int_0^y \tilde{\phi}_2(\tilde{y})\big[\tilde{\psi}_{tt}(t,\tilde{y}) + P_cG(t,\tilde{y})\big]\,d\tilde{y}\big)\\ & + \Gamma_1\big(a(t)\tilde{\phi}_1(y)\big).\end{aligned}$$

Here we have

$$a'(t) = \partial_t\tilde{\psi}(t,0) = t^{-1}\Gamma_2\tilde{\psi}(t,0),\ a''(t) = \partial_t^2\tilde{\psi}(t,0) = t^{-2}(\Gamma_2^2\tilde{\psi} - \Gamma_2\tilde{\psi})(t,0),$$

and so using the bound (B.3) (proved independently below), we get

$$\big\|\nabla_{t,y}\nabla_{t,y}^\beta\big(ya'(t)\tilde{\phi}_1\big)\big\|_{L_{dy}^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ |\beta|+2 \leqslant N_1,$$

while we have

$$\big\|\langle\log y\rangle^{-1}ta(t)\tilde{\phi}_1'(y)\big\|_{L_y^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{\frac{1}{2}-\delta_1}.$$

Further, write

$$\begin{aligned}
&\Gamma_1\big(\tilde{\phi}_2(y)\int_0^y \tilde{\phi}_1(\tilde{y})\big[\tilde{\psi}_{tt}(t,\tilde{y})+P_cG(t,\tilde{y})\big]\,d\tilde{y}\big)\\
&-\Gamma_1\big(\tilde{\phi}_1(y)\int_0^y \tilde{\phi}_2(\tilde{y})\big[\tilde{\psi}_{tt}(t,\tilde{y})+P_cG(t,\tilde{y})\big]\,d\tilde{y}\big)\\
&=: I+II,
\end{aligned}$$

where

$$\begin{aligned}
I &= t\tilde{\phi}_2'(y)\int_0^y \tilde{\phi}_1(\tilde{y})\big[\tilde{\psi}_{tt}(t,\tilde{y})+P_cG(t,\tilde{y})\big]\,d\tilde{y}\\
&\quad - t\tilde{\phi}_1'(y)\int_0^y \tilde{\phi}_2(\tilde{y})\big[\tilde{\psi}_{tt}(t,\tilde{y})+P_cG(t,\tilde{y})\big]\,d\tilde{y},
\end{aligned}$$

$$\begin{aligned}
II &= y\tilde{\phi}_2(y)\int_0^y \tilde{\phi}_1(\tilde{y})\big[\tilde{\psi}_{ttt}(t,\tilde{y})+(P_cG)_t(t,\tilde{y})\big]\,d\tilde{y}\\
&\quad - y\tilde{\phi}_1(y)\int_0^y \tilde{\phi}_2(\tilde{y})\big[\tilde{\psi}_{ttt}(t,\tilde{y})+(P_cG)_t(t,\tilde{y})\big]\,d\tilde{y}.
\end{aligned}$$

In view of Lemma B.1, we have

$$\big\|\nabla_{t,y}^{\beta}\tilde{\psi}_{ttt}\big\|_{L^2_y(y\ll t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu-2},\ |\beta|+2\leqslant N_1.$$

It then follows that

$$\big\|\nabla_{t,y}\nabla_{t,y}^{\beta}I\big\|_{L^2_{dy}(y\ll t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ |\beta|+2\leqslant N_1$$

For the term II above, observe that

$$\begin{aligned}
II_y &= (y\tilde{\phi}_2)'(y)\int_0^y \tilde{\phi}_1(\tilde{y})\big[\tilde{\psi}_{ttt}(t,\tilde{y})+(P_cG)_t(t,\tilde{y})\big]\,d\tilde{y}\\
&\quad - (y\tilde{\phi}_1)'(y)\int_0^y \tilde{\phi}_2(\tilde{y})\big[\tilde{\psi}_{ttt}(t,\tilde{y})+(P_cG)_t(t,\tilde{y})\big]\,d\tilde{y}
\end{aligned}$$

which can be estimated just like I_t. Finally, we have

$$\begin{aligned}
II_t &= (y\tilde{\phi}_2)(y)\int_0^y \tilde{\phi}_1(\tilde{y})\big[\tilde{\psi}_{tttt}(t,\tilde{y})+(P_cG)_{tt}(t,\tilde{y})\big]\,d\tilde{y}\\
&\quad - (y\tilde{\phi}_1)(y)\int_0^y \tilde{\phi}_2(\tilde{y})\big[\tilde{\psi}_{tttt}(t,\tilde{y})+(P_cG)_{tt}(t,\tilde{y})\big]\,d\tilde{y}
\end{aligned}$$

Then use the equation to write $\tilde{\psi}_{tttt}=\tilde{\psi}_{ttyy}+V\tilde{\psi}_{tt}+l.o.t.$. Performing an integration by parts, this allows us to write

$$\begin{aligned}
II_t &= (y\tilde{\phi}_2)(y)\int_0^y -\tilde{\phi}_1'(\tilde{y})\tilde{\psi}_{tt\tilde{y}}(t,\tilde{y})+\tilde{\phi}_1(\tilde{y})\big[V(\tilde{y})\tilde{\psi}_{tt}+(P_cG)_{tt}(t,\tilde{y})\big]\,d\tilde{y}\\
&\quad - (y\tilde{\phi}_1)(y)\int_0^y -\tilde{\phi}_2'(\tilde{y})\tilde{\psi}_{tt\tilde{y}}(t,\tilde{y})+\tilde{\phi}_2(\tilde{y})\big[V(\tilde{y})\tilde{\psi}_{tt}+(P_cG)_{tt}(t,\tilde{y})\big]\,d\tilde{y}
\end{aligned}$$

Using

$$\big\|\nabla^{\beta}_{t,y}\tilde{\psi}_{tty}\big\|_{L^2_y(y\ll t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu-2},\ |\beta|+2\leqslant N_1,$$

(see Lemma B.1) as well as the identity

$$\tilde{\psi}_{tt} = (t^2-y^2)^{-1}\big[\Gamma_2^2\tilde{\psi} - 2y\partial_y\Gamma_2\tilde{\psi} + 2y\partial_y\tilde{\psi} - \Gamma_2\tilde{\psi} + y^2\Box\tilde{\psi}\big]$$

one gets

$$\big\|\nabla_{t,y}\nabla^{\beta}_{t,y}II\big\|_{L^2_y(y\ll t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ |\beta|+2\leqslant N_1.$$

Next, consider $\Gamma_1^2\tilde{\psi}$. Recall the identity

$$(\Gamma_1^2-\Gamma_2^2)\tilde{\psi} = (t^2-y^2)\Box\tilde{\psi}.$$

This yields for $|\beta|+1\leqslant N_1$:

$$\begin{aligned}\big\|\nabla^{\beta}_{t,y}\big((\Gamma_1^2-\Gamma_2^2)\tilde{\psi}\big)\big\|_{L^2_y(y\ll t)} &\lesssim t^2\|\nabla^{\beta}_{t,y}(V(\cdot)\tilde{\psi})\|_{L^2_y} + l.o.t\\ &\lesssim \varepsilon\langle t\rangle^{\frac{3}{2}-\delta_1},\end{aligned}$$

where we used in particular bootstrap assumption (3.4). Together with the bootstrap assumption (3.5), we conclude that we can split

$$\Gamma_1^2\tilde{\psi} = (\Gamma_1^2\tilde{\psi})_1 + (\Gamma_1^2\tilde{\psi})_2$$

with the desired properties.
The proof of the last assertion of the lemma follows from (3) in the preceding proof. □

We now continue with the proof of (3.11), our main tools being Proposition 2.1, Proposition 2.2. Write the equation for $\tilde{\psi}$ as before in the form

$$-\partial_t^2\tilde{\psi} + \partial_y^2\tilde{\psi} + \frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi} = P_cG,$$

where

$$G = (1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi).$$

We decompose G into its weighted part G_1 (terms with weights at least $\langle y\rangle^{-2}$), as well as the pure cubic part G_2,

$$G = G_1 + G_2.$$

Use the bound

$$\big|\nabla^{\beta}_{t,y}\Gamma_2^{\kappa}G_1\big| \lesssim \sum_{\substack{\kappa_1+\kappa_2\leqslant\kappa\\ |\beta_1|+|\beta_2|\leqslant|\beta|+2}} \big|\frac{\nabla^{\beta_1}_{t,y}\Gamma_2^{\kappa_1}\phi\nabla^{\beta_2}_{t,y}\Gamma_2^{\kappa_2}\phi}{(1+y^2)^{\frac{3}{4}}}\big|.$$

According to (2.16), we need to bound the right-hand side in $\|\cdot\|_{L^1_t(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}dy})}$. Start with the case of less than top-level derivatives, $|\beta|+\kappa\leqslant N_1-1$. When $\kappa=1$, in view of bootstrap assumptions (3.4) and (3.6), the above expression is bounded by

$$\begin{aligned}
&\|\cdot\|_{L^1_t(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}*dy})([0,T])} \qquad (4.16)\\
\lesssim\ & \Big\|\frac{\langle\nabla_{t,y}\rangle^{\frac{N_1}{2}}\phi}{\langle y\rangle^{\frac12-}}\Big\|_{L^2_tL^\infty_y}\big\|\langle y\log y\rangle^{-1}\langle\nabla_{t,y}\rangle^{N_1}\langle\Gamma_2\rangle\tilde\phi\big\|_{L^2_{y,t}([0,T])}\\
&+\sum_{\beta_2<\frac{N_1}{2}}\big\|\langle y\log y\rangle^{-1}\langle\nabla_{t,y}\rangle^{N_1}\tilde\phi\big\|_{L^2_{y,t}([0,T])}\big\|\langle y\log y\rangle^{-1}\nabla^{\beta_2}_{t,y}\langle\Gamma_2\rangle\tilde\phi\big\|_{L^2_{y,t}([0,T])}\\
\lesssim\ & \varepsilon^2\langle T\rangle^{(1+[\frac{2|\beta|}{N_1}])10\nu}+\varepsilon^2\langle T\rangle^{11\nu}\mathbb{1}_{\{|\beta|>\frac{N_1}{2}\}}\\
\lesssim\ & \varepsilon^2\langle T\rangle^{(1+[\frac{2|\beta|}{N_1}])10\nu},
\end{aligned}$$

as required.

The case $\kappa=2$ is estimated, in view of bootstrap assumptions (3.4) and (3.6), as follows

$$\begin{aligned}
&\|\cdot\|_{L^1_t(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}*dy})([0,T])} \qquad (4.17)\\
\lesssim\ & \Big\|\frac{\langle\nabla_{t,y}\rangle^{\frac{N_1}{2}}\phi}{\langle y\rangle^{\frac12-}}\Big\|_{L^2_tL^\infty_y}\big\|\langle y\log y\rangle^{-1}\langle\nabla_{t,y}\rangle^{N_1-1}\langle\Gamma_2\rangle^2\tilde\phi\big\|_{L^2_{y,t}([0,T])}\\
&+\sum_{\beta_2<\frac{N_1}{2}-1}\big\|\langle y\log y\rangle^{-1}\langle\nabla_{t,y}\rangle^{N_1-1}\tilde\phi\big\|_{L^2_{y,t}([0,T])}\big\|\langle y\log y\rangle^{-1}\nabla^{\beta_2}_{t,y}\langle\Gamma_2\rangle^2\tilde\phi\big\|_{L^2_{y,t}([0,T])}\\
&+\sum_{|\beta_1|+|\beta_2|\leqslant N_1-1}\prod_{j=1,2}\big\|\langle y\log y\rangle^{-1}\nabla^{\beta_j}_{t,y}\langle\Gamma_2\rangle\tilde\phi\big\|_{L^2_{y,t}([0,T])}\\
\lesssim\ & \varepsilon^2\langle T\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu}+\varepsilon^2\langle T\rangle^{101\nu}\mathbb{1}_{\{|\beta|>\frac{N_1}{2}\}}+\varepsilon^2\langle T\rangle^{(2+[\frac{2|\beta|}{N_1}])10\nu}\\
\lesssim\ & \varepsilon^2\langle T\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},
\end{aligned}$$

as required.

The case of top level derivatives $|\beta|+\kappa=N_1$ is treated as in *(3)* of Lemma 4.1 via integration by parts and induction on the number of y-derivatives, and omitted.

This leads us to the problem of bounding the contribution of the pure cubic terms G_2. By using the inherent gradient structure (4.2), as well as the estimates (2.16), (2.18) and (2.20), we reduce to bounding the schematic expressions

$$\begin{aligned}
&\big\|\nabla^{\beta}_{t,y}\langle\Gamma_2\rangle^{\kappa}\big(\phi^2_{t,y}\tilde\phi_t\big)\big\|_{L^1_tL^2_{dy}},\ \big\|\langle y\rangle^{-1}\langle\Gamma_2\rangle^{\kappa}\nabla^{\beta}_{t,y}\big(\phi^2_{t,y}\tilde\phi_t\big)\big\|_{L^1_t(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}*dy})},\\
&\big\|\langle y\rangle^{\frac12}\langle\Gamma_2\rangle^{\kappa}\nabla^{\beta}_{t,y}\big(\phi^2_t(\phi_{yy}-\phi_{tt})\big)\big\|_{L^1_t(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon}*dy})}.
\end{aligned}$$

We shall only consider the case of non-top order derivatives, i. e. $|\beta| + \kappa < N_1$, since the remaining case is again handled via the energy identity and the integration by parts trick to reduce to the case of lower order derivatives. We treat the above terms separately:

(1): the bound for $\big\|\nabla_{t,y}^{\beta}\langle\Gamma_2\rangle^{\kappa}\big(\phi_{t,y}^2\tilde{\phi}_t\big)\big\|_{L_t^1L_{dy}^2([0,T])}$. Start with the case $\kappa = 1$, $|\beta| < \frac{N_1}{2}$. We have

$$\begin{aligned}\big\|\nabla_{t,y}^{\beta}\langle\Gamma_2\rangle^{\kappa}\big(\phi_{t,y}^2\tilde{\phi}_t\big)\big\|_{L_t^1L_{dy}^2([0,T])} &\lesssim \sum_{\sum\beta_j=\beta}\big\|\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\phi_{t,y}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\tilde{\phi}_t\big\|_{L_t^1L_{dy}^2([0,T])}\\ &\quad+\sum_{\sum\beta_j=\beta}\big\|\nabla_{t,y}^{\beta_1}\phi_{t,y}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\langle\Gamma_2\rangle\tilde{\phi}_t\big\|_{L_t^1L_{dy}^2([0,T])}\\ &\lesssim \sum_{\sum\beta_j\leqslant\beta}\big\|\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}_{t,y}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_{dy}^2([0,T])}\\ &\quad+\sum_{\sum\beta_j\leqslant\beta}\big\|\langle y\rangle^{-1}\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_{dy}^2([0,T])}.\end{aligned} \tag{4.18}$$

We estimate the first term in the right-hand side of (4.18). Using the bootstrap assumptions (3.3) and (3.5), we have

$$\begin{aligned}&\sum_{\sum\beta_j\leqslant\beta}\big\|\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}_{t,y}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_{dy}^2([0,T])}\\ &\lesssim\sum_{\sum\beta_j\leqslant\beta}\big\|\langle t\rangle^{-10\nu}\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}_{t,y}\big\|_{L_t^\infty L_y^2([0,T])}\big\|\langle t\rangle^{10\nu}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_y^\infty([0,T])}\\ &\lesssim\frac{\varepsilon^3}{\nu}\langle T\rangle^{10\nu}\lesssim\varepsilon^2\langle T\rangle^{10\nu}.\end{aligned}$$

To estimate the second term in (4.18), we use the local energy bound (3.6). Write

$$\begin{aligned}\big|\nabla_{t,y}^{\beta_2}\phi_{t,y}\big| &= \big|\nabla_{t,y}^{\beta_2}\big(\langle y\rangle^{-\frac{3}{2}}\tilde{\phi}\big)\big| + \big|\nabla_{t,y}^{\beta_2}\big(\langle y\rangle^{-\frac{1}{2}}\tilde{\phi}_y\big)\big|\\ &\leqslant\sum_{|\tilde{\beta}_2|\leqslant|\beta_2|}\big[\big|\langle y\rangle^{-\frac{3}{2}}\nabla_{t,y}^{\tilde{\beta}_2}\tilde{\phi}\big| + \big|\langle y\rangle^{-\frac{1}{2}}\nabla_{t,y}^{\tilde{\beta}_2}\tilde{\phi}_y\big|\big],\end{aligned} \tag{4.19}$$

and so

$$\begin{aligned}&\big\|\langle y\rangle^{-1}\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_{dy}^2([0,T])}\\ &\lesssim\big\|\langle y\log y\rangle^{-1}\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}\big\|_{L_{t,y}^2([0,T])}\Big(\sum_{|\tilde{\beta}_2|\leqslant|\beta_2|+1}\big\|\langle t\rangle^{-\nu}\langle\nabla_y\rangle\nabla_{t,y}^{\tilde{\beta}_2}\tilde{\phi}\big\|_{L_t^\infty L_y^\infty([0,T])}\Big)\\ &\qquad\cdot\big\|\langle t\rangle^{\nu}\langle\log y\rangle\langle y\rangle^{-\frac{1}{2}}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^2L_y^\infty}\\ &\lesssim\varepsilon^3\langle T\rangle^{10\nu},\end{aligned}$$

where we used the bootstrap assumption (3.6) for the first term, the bootstrap assumption (3.4) and interpolation for the last term, and the embedding $H^1(\mathbb{R})\subset$

$L^\infty(\mathbb{R})$ and the bootstrap assumption (3.2) for the middle term.

We continue with the case $\kappa = 1, |\beta| \geqslant \frac{N_1}{2}$. Again using (4.18), there may now be terms where only one of the three factors may be bounded in $L^\infty_{t,y}$. Start with the first term, and assume $|\beta_2| > \frac{N_1}{2}$ (as we may by symmetry and since else we can argue as in the previous bounds). Then distinguish between the following two situations:

(a): $y \ll t$. Here the trick is to use the identities

$$\tilde{\phi}_t = \frac{t\Gamma_2\tilde{\phi} - y\Gamma_1\tilde{\phi}}{t^2 - y^2},\ \tilde{\phi}_y = \frac{t\Gamma_1\tilde{\phi} - y\Gamma_2\tilde{\phi}}{t^2 - y^2}$$

which imply

$$\big|\nabla^{\beta_2}_{t,y}\phi_{t,y}\big| \lesssim \langle y\rangle^{-\frac{3}{2}}\big|\tilde{\phi}\big| + t^{-1}\langle y\rangle^{-\frac{1}{2}} \sum_{\substack{|\tilde{\beta}_2|<|\beta_2| \\ \Gamma=\Gamma_{1,2}}} \big|\nabla^{\tilde{\beta}_2}_{t,y}\Gamma\tilde{\phi}\big|. \tag{4.20}$$

To estimate the term $\big|\nabla^{\tilde{\beta}_2}_{t,y}\Gamma\tilde{\phi}\big|$, we observe $\Gamma_1\tilde{\phi}(t,0) = 0$, whence using Lemma 4.2 we get

$$\big|\nabla^{\tilde{\beta}_2}_{t,y}\Gamma_1\tilde{\phi}(t,y)\big| \lesssim \varepsilon\langle y\rangle^{\frac{1}{2}}\langle t\rangle^{\frac{1}{2}-\delta_1}.$$

We also have (see Lemma B.2)

$$\big|\nabla^{\beta}_{t,y}\Gamma_2^{\kappa}\tilde{\psi}\big|(t,y) \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])10^{\kappa}\nu}\langle y\rangle^{\frac{1}{2}},\ |\beta| + \kappa \leqslant N_1.$$

The previous observations imply that

$$\big|\nabla^{\beta_2}_{t,y}\phi_{t,y}\big| \lesssim \langle y\rangle^{-\frac{3}{2}}\big|\tilde{\phi}\big| + \varepsilon t^{-\frac{1}{2}-\delta_1} \lesssim \varepsilon t^{-\frac{1}{2}-\delta_1},\ y \ll t,$$

with a similar bound applying to $\nabla^{\beta_3}_{t,y}\phi_t$. But then we easily get

$$\begin{aligned}
&\big\|\nabla^{\beta_1}_{t,y}\langle\Gamma_2\rangle\tilde{\phi}_{t,y}\nabla^{\beta_2}_{t,y}\phi_{t,y}\nabla^{\beta_3}_{t,y}\phi_t\big\|_{L^1_t L^2_{dy}(y\ll t)} \\
&\lesssim \big\|\langle t\rangle^{-10\nu}\nabla^{\beta_1}_{t,y}\langle\Gamma_2\rangle\tilde{\phi}_{t,y}\big\|_{L^\infty_t L^2_y}\big\|\langle t\rangle^{10\nu}\nabla^{\beta_2}_{t,y}\phi_{t,y}\big\|_{L^2_t L^\infty_y(y\ll t)}\big\|\nabla^{\beta_3}_{t,y}\phi_t\big\|_{L^2_t L^\infty_y(y\ll t)} \\
&\lesssim \frac{\varepsilon^3}{\delta_1 - 10\nu} \lesssim \varepsilon^2.
\end{aligned}$$

The remaining term in (4.18) is treated similarly.

(b): $y \gtrsim t$. Here we may of course assume $y \sim t$, since the case $y \gg t$ is handled just like *(a)*. Note that from (4.19), we get using also the Sobolev embedding $H^1_{ydy}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and the bootstrap assumption (3.2)

$$\big|\nabla^{\beta_2}_{t,y}\phi_{t,y}\big| \lesssim \varepsilon t^{\nu-\frac{1}{2}},\ y \sim t, \tag{4.21}$$

and so the a priori bounds imply

$$\begin{aligned}&\big\|\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}_{t,y}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_{dy}^2(y\sim t),\,t\in[0,T]}\\&\lesssim \big\|\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}_{t,y}\big\|_{L_t^\infty L_y^2[0,T]}\big\|\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_y^\infty([0,T]}\lesssim\frac{\varepsilon^3}{\nu}\langle T\rangle^{11\nu}\leqslant\varepsilon^2\langle T\rangle^{20\nu}\end{aligned}$$

which is the required bound.

For the second term in (4.18), again assuming that $|\beta_2|\geqslant\frac{N_1}{2}$, we get using (4.21) and the bootstrap assumptions (3.3) and (3.6)(and restricting to $y\sim t$)

$$\begin{aligned}&\big\|\langle y\rangle^{-1}\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}\nabla_{t,y}^{\beta_2}\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_{dy}^2([0,T])}\\&\lesssim\big\|\langle\log y\rangle^{-1}\langle y\rangle^{-1}\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\tilde{\phi}\big\|_{L_{t,y}^2([0,T])}\big\|\langle\log y\rangle\nabla_{t,y}^{\beta_2}\phi_{t,y}\big\|_{L_{t,y}^\infty}\big\|\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^2L_y^\infty([0,T])}\\&\lesssim\varepsilon^3\langle\log T\rangle^{\frac12}T^{11\nu},\end{aligned}$$

which is much better than the bound $\varepsilon\langle T\rangle^{20\nu}$ we need.

This completes the case $\kappa=1$ for *(1)*. For the case $\kappa=2$, one proceeds analogously, but now also encounters terms of the form

$$\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\phi_{t,y}\nabla_{t,y}^{\beta_2}\langle\Gamma_2\rangle\phi_{t,y}\nabla_{t,y}^{\beta_3}\tilde{\phi}_t,$$

In the region $y\ll t$ or $y\gg t$, we can proceed for it like in (a) above, applied to the factor $\nabla_{t,y}^{\beta_3}\tilde{\phi}_t$. In the region $y\sim t$, one uses

$$\big\|\nabla_{t,y}^{\beta_2}\langle\Gamma_2\rangle\phi_{t,y}\nabla_{t,y}^{\beta_3}\phi_t\big\|_{L_t^1L_y^\infty([0,T])}\lesssim\varepsilon^2\langle T\rangle^{21\nu};$$

We omit the simple details.

(2): the bound for $\big\|\langle y\rangle^{-1}\langle\Gamma_2\rangle^\kappa\nabla_{t,y}^{\beta}\big(\phi_{t,y}^2\tilde{\phi}_t\big)\big\|_{L_t^1(L_{dy}^2\cap L^1_{\langle y\rangle^{\varepsilon_*}dy})([0,T])}$. The $L_t^1L_y^2$-norm corresponds exactly to the second term in (4.18) (if $\kappa=1$, and analogous with $\kappa=2$), and is easier than the L^1-type bound. Thus consider now the (modified) L_{dy}^1-norm. From (4.20) and a straightforward modification, we get

$$\begin{aligned}&\big|\nabla_{t,y}^{\beta_2}\langle\Gamma_2\rangle\phi_{t,y}\big|\\&\lesssim\langle y\rangle^{-\frac32}\big|\langle\Gamma_2\rangle\tilde{\phi}\big|+(\max\{t,y\})^{-1}\langle y\rangle^{-\frac12}\sum_{\substack{|\tilde{\beta}_2|<|\beta_2|\\\Gamma=\Gamma_{1,2}}}\big|\nabla_{t,y}^{\tilde{\beta}_2}\Gamma\langle\Gamma_2\rangle\tilde{\phi}\big|,\ y\ll t\,\text{or}\,y\gg t,\end{aligned}$$

while from (4.19) we get

$$\big|\nabla_{t,y}^{\beta_2}\langle\Gamma_2\rangle\phi_{t,y}\big|\leqslant\sum_{|\tilde{\beta}_2|\leqslant|\beta_2|}\big[\big|\langle y\rangle^{-\frac32}\nabla_{t,y}^{\tilde{\beta}_2}\langle\Gamma_2\rangle\tilde{\phi}\big|+\big|\langle y\rangle^{-\frac12}\nabla_{t,y}^{\tilde{\beta}_2}\langle\Gamma_2\rangle\tilde{\phi}_y\big|\big]$$

which is useful in the region $y\sim t$. Using Lemma 4.3 and the bootstrap assumption (3.5), we infer

$$\big|\langle\log y\rangle\langle y\rangle^{\frac12+\varepsilon_*}\nabla_{t,y}^{\beta_2}\langle\Gamma_2\rangle\phi_{t,y}\big|\leqslant\varepsilon\langle t\rangle^{\varepsilon_*+(1+[\frac{2|\beta|}{N_1}])10\nu+}$$

If we now write (as usual $\kappa \in \{1,2\}$)

$$\begin{aligned}&\langle y\rangle^{-1}\langle\Gamma_2\rangle^{\kappa}\nabla_{t,y}^{\beta}\big(\phi_{t,y}^2\tilde{\phi}_t\big)\\&=\sum_{\substack{\sum\kappa_j=\kappa,\,\kappa_2\leqslant\min\{1,\kappa_1\}\\ \sum\beta_j=\beta}}\langle y\rangle^{-1}(\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle^{\kappa_1}\phi_{t,y})(\nabla_{t,y}^{\beta_2}\langle\Gamma_2\rangle^{\kappa_2}\phi_{t,y})(\nabla_{t,y}^{\beta_3}\langle\Gamma_2\rangle^{\kappa_3}\tilde{\phi}_t)\end{aligned}$$

then if $\kappa_3=1$, $\kappa_1=1$, we get

$$\begin{aligned}&\big\|\cdot\big\|_{L_t^1L^1_{\langle y\rangle^{\varepsilon_*}dy}([0,T])}\\&\lesssim\big\|\langle y\rangle^{\varepsilon_*+\frac12}\langle\log y\rangle\nabla_{t,y}^{\beta_1}\langle\Gamma_2\rangle\phi_{t,y}\big\|_{L^\infty_{t,y}([0,T])}\Big\|\frac{\nabla_{t,y}^{\beta_2}\phi_{t,y}}{\langle\log y\rangle\langle y\rangle^{\frac12}}\Big\|_{L^2_{t,y}([0,T])}\Big\|\frac{\nabla_{t,y}^{\beta_3}\langle\Gamma_2\rangle\tilde{\phi}_t}{\langle y\rangle\langle\log y\rangle}\Big\|_{L^2_{t,y}([0,T])}\\&\lesssim\frac{\varepsilon^3}{\nu}\langle T\rangle^{\varepsilon_*+42\nu}\lesssim\varepsilon^2\langle T\rangle^{100\nu}\end{aligned}$$

which is as desired; we have used the preceding considerations to bound the first factor. On the other hand, when $\kappa_3=2$, we obtain the bound

$$\begin{aligned}&\big\|\cdot\big\|_{L_t^1L^1_{\langle y\rangle^{\varepsilon_*}dy}([0,T])}\\&\lesssim\big\|\langle y\rangle^{\varepsilon_*-\frac12}\langle\log y\rangle\nabla_{t,y}^{\beta_1}\phi_{t,y}\nabla_{t,y}^{\beta_2}\tilde{\phi}_{t,y}\big\|_{L^2_{t,y}([0,T])}\Big\|\frac{\nabla_{t,y}^{\beta_3}\langle\Gamma_2\rangle^2\tilde{\phi}_t}{\langle y\rangle\langle\log y\rangle}\Big\|_{L^2_{t,y}([0,T])}\\&\lesssim\varepsilon^3\langle T\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu}.\end{aligned}$$

The remaining combinations are handled similarly and this completes the estimate *(2)*.

(3): the bound for $\big\|\langle y\rangle^{\frac12}\langle\Gamma_2\rangle^{\kappa}\nabla_{t,y}^{\beta}\big(\phi_t^2(\phi_{yy}-\phi_{tt})\big)\big\|_{L_t^1(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon_*}dy})}$. Here we use the equation for ϕ. This produces a term just like in *(2)*, as well as a further linear term of the form

$$\langle y\rangle^{-2}\langle\Gamma_2\rangle^{\kappa}\nabla_{t,y}^{\beta}\big(\phi_t^2\tilde{\phi}\big).$$

This term is handled like in *(2)* if we note that

$$\begin{aligned}\big\|\langle y\rangle^{-1}\tilde{\phi}\big\|_{L^2_{dy}}&\lesssim\big\|\langle y\rangle^{-1}\tilde{\phi}(0)\big\|_{L^2_{dy}}+\big\||\langle y\rangle^{-1}[\tilde{\phi}(y)-\tilde{\phi}(0)]\big\|_{L^2_{dy}}\\&\lesssim\big\|\langle y\rangle^{-1}\tilde{\phi}(0)\big\|_{L^2_{dy}}+\big\|\tilde{\phi}_y\big\|_{L^2_{dy}}\lesssim\varepsilon\langle t\rangle^{\nu}.\end{aligned}$$

This then allows us to reduce the above expression to the following crude schematic form

$$\big\|\langle y\rangle^{\frac12}\langle\Gamma_2\rangle^{\kappa}\nabla_{t,y}^{\beta}\big(\phi_t^2[\langle y\rangle^{-2}\phi^2+\phi^3]\big)\big\|_{L_t^1(L^2_{dy}\cap L^1_{\langle y\rangle^{\varepsilon_*}dy})}$$

which is straightforward to estimate by $\lesssim\varepsilon^4$.

This concludes the proof of (3.11).

5. Local energy decay

The goal of this section is to prove the local energy decay (3.12) for which we use Proposition 2.3. This follows essentially along the same lines as the proof of the estimate (3.11), except in the case of top level derivatives, which have to be treated differently.

(1): derivatives below top degree: $|\beta| + \gamma \leqslant N_1$ (referring to (3.12)*).* We follow the same pattern as in the preceding proof, except that now the 'bad norm' $L^1_{\langle y\rangle^{\varepsilon} dy}$ is replaced by $L^2_{\langle y\rangle^{1+} dy}$. Using the equation for $\tilde{\psi}$ as in the preceding proof and splitting the source into

$$G = G_1 + G_2,$$

we see that in order to control the contribution from G_1, we have to bound

$$\sum_{\substack{\kappa_1+\kappa_2\leqslant\kappa \\ |\beta_1|+|\beta_2|\leqslant|\beta|+2}} \Big\| \frac{\nabla^{\beta_1}_{t,y}\Gamma_2^{\kappa_1}\phi \nabla^{\beta_2}_{t,y}\Gamma_2^{\kappa_2}\phi}{(1+y^2)^{\frac{3}{4}}} \Big\|_{L^1_t L^2_{\langle y\rangle^{1+} dy}}.$$

In fact, note that in (4.16) we obtain L^1_{dy}-control by sacrificing one factor $\langle y\rangle^{-\frac{1}{2}}$, and so the $L^2_{\langle y\rangle^{1+} dy}$-norm of the above expressions is bounded exactly by (4.16), (4.17) (corresponding to $\kappa = 1, 2$). The same comment applies to the non-gradient terms constituting G_2, which can hence be estimated just like in *(1) - (3)* of the proof of (3.11) above.

(2): derivatives of top degree: $|\beta| + \gamma = N_1 + 1$ (referring to (3.12)*).* The idea is to again use an inductive argument to reduce to the case of lower order derivatives. This time a simple integration by parts argument seems to no longer work, and we instead use an approximate parametrix to express the top order derivative terms. Specifically, assume $\beta + \gamma = N_1$, and consider the expression $\partial_t^{\beta}\Gamma_2^{\gamma}\tilde{\psi}$. This satisfies the following equation

$$\begin{aligned} & -\partial_t^2(\partial_t^{\beta}\Gamma_2^{\gamma}\tilde{\psi}) + \partial_y^2(\partial_t^{\beta}\Gamma_2^{\gamma}\tilde{\psi}) + \frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}(\partial_t^{\beta}\Gamma_2^{\gamma}\tilde{\psi}) \qquad (5.1) \\ = \;& \partial_t^{\beta}\Gamma_2^{\gamma}(P_c G) + \partial_t^{\beta}[\Box, \Gamma_2^{\kappa}]\tilde{\psi} + \sum_{\tilde{\gamma}<\gamma} V_{\tilde{\gamma}}\partial_t^{\beta}\Gamma_2^{\tilde{\gamma}}\tilde{\psi} \end{aligned}$$

where the potentials $V_{\tilde{\gamma}}$ are of the schematic form

$$V_{\tilde{\gamma}} = (y\partial_y)^{\gamma-\tilde{\gamma}}\big(\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\big).$$

Our goal is to derive an a priori bound for

$$\big\|\langle \log y\rangle^{-1}\langle y\rangle^{-1}\nabla_{t,y}\partial_t^{\beta}\Gamma_2^{\gamma}\tilde{\psi}\big\|_{L^2_{t,y}([0,T])}. \qquad (5.2)$$

To this end, we shall express $\partial_t^\beta \Gamma_2^\gamma \tilde{\psi}$ via an approximate representation formula (a parametrix) based on the method of characteristics (as we are essentially in $1+1$-dimensions), *taking the smaller top order terms in* $\partial_t^\beta \Gamma_2^\gamma (P_c G)$ *into account*. To start with, write

$$\partial_t^\beta \Gamma_2^\gamma (P_c G) = \partial_t^\beta \Gamma_2^\gamma (G) - \partial_t^\beta \Gamma_2^\gamma (P_d G),$$

where the error term $\partial_t^\beta \Gamma_2^\gamma (P_d G)$ is effectively a lower order term. Then collecting all the top order derivative terms contained in

$$\partial_t^\beta \Gamma_2^\gamma (G),$$

we re-cast the equation (5.1) in the form (we normalize the first coefficient to be equal to 1, thereby introducing the factor $\kappa(t,y)$ on the right)

$$-\partial_t^2(\partial_t^\beta \Gamma_2^\gamma)\tilde{\psi} + g_1(\phi,\nabla\phi)\partial_y^2(\partial_t^\beta \Gamma_2^\gamma \tilde{\psi}) + g_2(\phi,\nabla\phi)\partial_{ty}(\partial_t^\beta \Gamma_2^\gamma \tilde{\psi}) = \kappa(t,y)H, \quad (5.3)$$

with

$$\begin{aligned} H &= -\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}(\partial_t^\beta \Gamma_2^\gamma \tilde{\psi}) - \partial_t^\beta \Gamma_2^\gamma (P_d G) + \widetilde{\left(\partial_t^\beta \Gamma_2^\gamma (G)\right)} + \partial_t^\beta [\Box, \Gamma_2^\kappa]\tilde{\psi} \\ &\quad + \sum_{\tilde{\gamma}<\gamma} V_{\tilde{\gamma}} \partial_t^\beta \Gamma_2^{\tilde{\gamma}} \tilde{\psi} \end{aligned}$$

where $\widetilde{\left(\partial_t^\beta \Gamma_2^\gamma (G)\right)}$ denotes all non-top order terms, while the top order terms (i. e. when $\partial_t^\beta \Gamma_2^\gamma$ falls on a second derivative term in G) have been moved to the left. Note in particular that

$$g_1(\phi,\nabla\phi) = 1 + O(\frac{\phi}{1+y^2} + [\nabla_{t,y}\phi]^2),\ g_2(\phi,\nabla\phi) = O(\frac{\phi}{1+y^2} + [\nabla_{t,y}\phi]^2),$$
$$\kappa(t,y) = 1 + O(\frac{\phi}{1+y^2} + [\nabla_{t,y}\phi]^2)$$

Then we approximately factorize the left hand side of (5.3) as follows:

$$\begin{aligned} &-\partial_t^2\tilde{\psi} + g_1(\phi,\nabla\phi)\partial_y^2\tilde{\psi} + g_2(\phi,\nabla\phi)\partial_{ty}\tilde{\psi} \\ &= (-\partial_t - h_1(\phi,\nabla\phi)\partial_y)(\partial_t - h_2(\phi,\nabla\phi)\partial_y)\tilde{\psi} - h_1(\phi,\nabla\phi)\partial_y(h_2(\phi,\nabla\phi))\partial_y\tilde{\psi} \\ &- \partial_t(h_2(\phi,\nabla\phi))\partial_y\tilde{\psi} \end{aligned}$$

where the functions $h_{1,2}$ are chosen to satisfy

$$-h_1 + h_2 = g_2(\phi,\nabla\phi),\ h_1 h_2 = g_1(\phi,\nabla\phi)$$

whence

$$h_{1,2} = 1 + O(\frac{\phi}{1+y^2} + [\nabla_{t,y}\phi]^2).$$

Hence we obtain from (5.3) the relation

$$\begin{aligned} &(-\partial_t - h_1(\phi,\nabla\phi)\partial_y)(\partial_t - h_2(\phi,\nabla\phi)\partial_y)\tilde{\psi} \\ &= h_1(\phi,\nabla\phi)(\partial_y h_2(\phi,\nabla\phi))\partial_y\tilde{\psi} + (\partial_t h_2(\phi,\nabla\phi))\partial_y\tilde{\psi} + H =: H_1. \end{aligned} \quad (5.4)$$

This is the equation we solve approximately via the method of characteristics. Precisely, introduce the functions $\lambda_{1,2}(s;t,y)$ via the ODEs

$$\partial_s\lambda_1(s;t,y) = h_1(\phi,\nabla\phi)(s,\lambda_1(s;t,y)),\ \lambda_1(t;t,y) = y, \tag{5.5}$$

$$\partial_s\lambda_2(s;t,y) = -h_2(\phi,\nabla\phi)(s,\lambda_2(s;t,y)),\ \lambda_2(t;t,y) = y. \tag{5.6}$$

Note that from our a priori bounds, we get the crude asymptotic

$$\lambda_{1,2}(s;t,y) = y \mp (t-s) + O(\varepsilon(t-s)^{\frac{1}{2}-\delta_1})$$

Then we introduce the following approximate parametrix for the problem associated with (5.4):

Lemma 5.1. *Let f, g and $\widetilde{H}$ three given scalar functions. Let $S[f,g,\widetilde{H}]$ be defined by*

$$\begin{aligned} S[f,g,\widetilde{H}](t,y) &= \frac{1}{2}\big[f(\lambda_1(0;t,y)) + f(\lambda_2(0;t,y)\big] \\ &\quad + \int_{\lambda_1(0;t,y)}^{\lambda_2(0;t,y)} \frac{g(\tilde{y})}{(h_1+h_2)(\phi,\nabla\phi)(0,\tilde{y})}\,d\tilde{y} \\ &\quad + \int_0^t \int_{\lambda_1(s;t,y)}^{\lambda_2(s;t,y)} \frac{\widetilde{H}(s,\tilde{y})}{(h_1+h_2)(s,\tilde{y})}\,d\tilde{y}ds. \end{aligned}$$

Then, we have

$$\begin{aligned} S[f,g,\widetilde{H}](0,y) &= f(y), \\ \partial_t S[f,g,\widetilde{H}](0,y) &= \big((h_2-h_1)(\phi,\nabla\phi)\big)(0,y)f'(y) + g(y), \end{aligned}$$

and

$$(-\partial_t - h_1(\phi,\nabla\phi)\partial_y)(\partial_t - h_2(\phi,\nabla\phi)\partial_y)S[f,g,\widetilde{H}](t,y) = \widetilde{H} + E[f,g,\widetilde{H}](t,y),$$

where the error term $E[f,g,\widetilde{H}](t,y)$ is given by

$$\begin{aligned} &E[f,g,\widetilde{H}](t,y) \\ = \ &\big(\partial_t h_2(\phi,\nabla\phi) - h_1(\phi,\nabla\phi)\partial_y h_2(\phi,\nabla\phi) + h_2(\phi,\nabla\phi)\partial_y h_1(\phi,\nabla\phi)\big)(t,y) \\ &\times\partial_y\big(f(\lambda_1(0;t,y))\big) \\ &+\frac{\big(\partial_t h_2(\phi,\nabla\phi) - h_1(\phi,\nabla\phi)\partial_y h_2(\phi,\nabla\phi) + h_2(\phi,\nabla\phi)\partial_y h_1(\phi,\nabla\phi)\big)(t,y)}{(h_1+h_2)(\phi,\nabla\phi)(t,\lambda_1(0;t,y))} \\ &\times\partial_y\lambda_1(0;t,y)g(\lambda_1(0;t,y)) \\ &+\big(\partial_t h_2(\phi,\nabla\phi) - h_1(\phi,\nabla\phi)\partial_y h_2(\phi,\nabla\phi) + h_2(\phi,\nabla\phi)\partial_y h_1(\phi,\nabla\phi)\big)(t,y) \\ &\times\int_0^t \partial_y\lambda_1(s;t,y)\frac{\widetilde{H}(s,\lambda_1(s;t,y))}{(h_1+h_2)(s,\lambda_1(s;t,y))}\,ds. \end{aligned}$$

Proof. First, we trivially have

$$S[f,g,\widetilde{H}](0,y) = f(y),$$

as well as

$$\partial_t S[f,g,\widetilde{H}](0,y) = \frac{1}{2}(\partial_t\lambda_1 + \partial_t\lambda_2)(0;0,y)f'(y) + g(y),$$

where we have exploited the fact that

$$\big(\partial_t + h_1(\phi,\nabla\phi)(t,y)\partial_y\big)\lambda_1(s;t,y) = 0,\ \big(\partial_t - h_2(\phi,\nabla\phi)(t,y)\partial_y\big)\lambda_2(s;t,y) = 0.$$

Together with the fact that

$$\frac{1}{2}(\partial_t\lambda_1 + \partial_t\lambda_2)(0;0,y)f'(y) = \big((h_2 - h_1)(\phi,\nabla\phi)\big)(0,y)f'(y), \tag{5.7}$$

we deduce

$$\partial_t u(0,y) = \big((h_2 - h_1)(\phi,\nabla\phi)\big)(0,y)f'(y) + g(y).$$

Finally, the statement is done by direct check on the definition of $S[f,g,\widetilde{H}]$. This concludes the proof of the lemma. □

Next, we estimate $S[f,g,\widetilde{H}]$ and $E[f,g,\widetilde{H}]$.

Lemma 5.2. *Assume that $\langle y\rangle(f',g) \in L^\infty_y$, and the decomposition*

$$\widetilde{H} = \widetilde{H}^{(1)} + \langle y\rangle^{-2}\widetilde{H}^{(2)},$$

with

$$\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu+1}\|\widetilde{H}^{(1)}(t,\cdot)\|_{L^2_y} + \sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu}\big\|\langle y\log y\rangle^{-1}\widetilde{H}^{(2)}\big\|_{L^2_{t,y}([0,t])} < +\infty.$$

Then, we have the following estimate for $S[f,g,\widetilde{H}]$

$$\begin{aligned}
&\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu}\big\|\langle y\log y\rangle^{-1}S[f,g,\widetilde{H}]\big\|_{L^2_{t,y}([0,t])}\\
\lesssim\ & \|\langle y\rangle(f',g)\|_{L^\infty_y} + \frac{1}{\nu}\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu+1}\|\widetilde{H}^{(1)}(t,\cdot)\|_{L^2_y}\\
&+ \sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu}\big\|\langle y\log y\rangle^{-1}\widetilde{H}^{(2)}\big\|_{L^2_{t,y}([0,t])}.
\end{aligned}$$

Furthermore, $E[f,g,\widetilde{H}]$ satisfies the following decomposition

$$E[f,g,\widetilde{H}] = E^{(1)}[f,g,\widetilde{H}] + \langle y\rangle^{-2}E^{(2)}[f,g,\widetilde{H}],$$

where $E^{(1)}[f,g,\widetilde{H}]$ and $E^{(2)}[f,g,\widetilde{H}]$ satisfy

$$\begin{aligned}
&\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu+1}\|E^{(1)}[f,g,\widetilde{H}](t,\cdot)\|_{L^2_y}\\
&+ \sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu}\big\|\langle y\log y\rangle^{-1}E^{(2)}[f,g,\widetilde{H}]\big\|_{L^2_{t,y}([0,t])}\\
\lesssim\ & \varepsilon\|\langle y\rangle(f',g)\|_{L^\infty_y} + \sqrt{\varepsilon}\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu+1}\|\widetilde{H}^{(1)}(t,\cdot)\|_{L^2_y}\\
&+ \sqrt{\varepsilon}\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu}\big\|\langle y\log y\rangle^{-1}\widetilde{H}^{(2)}\big\|_{L^2_{t,y}([0,t])}.
\end{aligned}$$

Proof. We start by proving the first bound of the lemma. Compute

$$\begin{aligned}\nabla_{t,y} S[f,g,\widetilde{H}](t,y) &= \frac{1}{2}\sum_{j=1,2}\nabla_{t,y}\lambda_j(0;t,y) f'(\lambda_j(0;t,y))\\ &\quad+\sum_{j=1,2}(-1)^j\nabla_{t,y}\lambda_j(0;t,y)\frac{g(\lambda_j(0;t,y))}{(h_1+h_2)(\phi,\nabla\phi)(0,\lambda_j(0;t,y))}\\ &\quad+\sum_{j=1,2}(-1)^j\int_0^t\nabla_{t,y}\lambda_j(s;t,y)\frac{\widetilde{H}(s,\lambda_j(s;t,y))}{(h_1+h_2)(\phi,\nabla\phi)(s,\lambda_j(s;t,y))}\,ds\\ &=: A+B+C.\end{aligned}$$

In order to estimate these terms, we need pointwise bounds on $\nabla_{t,y}\lambda_j(s;t,y)$. By definition, we have the equation

$$\frac{\partial_{ts}\lambda_j(s;t,y)}{\partial_t\lambda_j(s;t,y)} = \pm\partial_y[h(\phi,\nabla\phi)](s,\lambda_j(s;t,y)).$$

Also, we recall the schematic relation

$$\partial_y[h(\phi,\nabla\phi)] = O\big(\partial_y(\frac{\phi}{1+y^2}) + \partial_y([\nabla_{t,y}\phi]^2)\big).$$

We need to check the absolute integrability of this expression with respect to s. First, it is readily verified (since $\partial_s\lambda_j \sim \pm 1$) that

$$\int_0^T \big|\partial_y(\frac{\phi}{1+y^2})\big|(s,\lambda_j(s;t,y))\,ds \lesssim \varepsilon.$$

The expression $\partial_y[\nabla_{t,y}\phi]^2$ is a bit more delicate to control, since it fails logarithmically to be time integrable. In fact, we get

$$\Big|\int_s^t \partial_y[\nabla_{t,y}\phi]^2(s_1,\lambda_j(s_1;t,y))\,ds_1\Big| \lesssim \varepsilon^2\log(\frac{\langle t\rangle}{\langle s\rangle}),$$

and so we obtain the bound

$$\big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{-C\varepsilon^2} \lesssim \big|\nabla_{t,y}\lambda_j(s;t,y)\big| \lesssim \big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}. \tag{5.8}$$

Then using the bound

$$\big|f'(\lambda_j(0;t,y)\big| + \big|g(\lambda_j(0;t,y)\big| \lesssim \varepsilon\sum_{\pm}\langle y\pm t+O(t^{\frac{1}{2}-\delta_1})\rangle^{-1}\|\langle y\rangle(f',g)\|_{L^\infty},$$

it is immediately verified that

$$\big\|\langle\log y\rangle^{-1}\langle y\rangle^{-1}A\big\|_{L^2_{t,y}([0,T])} + \big\|\langle\log y\rangle^{-1}\langle y\rangle^{-1}B\big\|_{L^2_{t,y}([0,T])} \lesssim \|\langle y\rangle(f',g)\|_{L^\infty}.$$

For the term C, first decompose C as

$$C = C^{(1)} + C^{(2)}$$

according to the decomposition $\widetilde{H} = \widetilde{H}^{(1)} + \langle y\rangle^{-2}\widetilde{H}^{(2)}$. We first estimate $C^{(1)}$. Write $\Lambda_j(s;t,y) = \lambda_j(s;t,y)$ if $s \leqslant t$ and

$$\Lambda_j(s;t,y) = y \mp (t-s),\ j=1,2,\ s>t.$$

Then we get (for $t \leqslant T$)

$$|C^{(1)}| \lesssim \sum_{j=1,2} \int_0^T |\nabla_{t,y}\Lambda_j(s;t,y)| \Big| \frac{\widetilde{H}^{(1)}(s,\Lambda_j(s;t,y))}{(h_1+h_2)(\phi,\nabla\phi)(s,\Lambda_j(s;t,y))} \Big| \, ds$$

and by a simple change of variables argument and Minkowski's inequality, one obtains

$$\begin{aligned} \big\| \langle \log y\rangle^{-1}\langle y\rangle^{-1} C^{(1)} \big\|_{L^2_{t,y}([0,T])} &\lesssim \int_0^T \big(\frac{\langle T\rangle}{\langle s\rangle}\big)^{2C\varepsilon^2} \big\|\widetilde{H}^{(1)}(s,\cdot)\big\|_{L^2_{\tilde{y}}} \, dy \\ &\lesssim \frac{\langle T\rangle^{2\cdot 10^\kappa \nu}}{\nu} \sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu+1} \|\widetilde{H}^{(1)}(t,\cdot)\|_{L^2_{\tilde{y}}}. \end{aligned}$$

Next, we estimate $C^{(2)}$. Using the Cauchy-Schwarz inequality, we get

$$|C^{(2)}|(t,y) \lesssim \big(\int_0^T \big(\frac{\langle T\rangle}{\langle s\rangle}\big)^{2C\varepsilon^2} \langle \log \Lambda_j(s;t,y)\rangle^{-2} \langle \Lambda_j(s;t,y)\rangle^{-2} (\widetilde{H}^{(2)}(s,\Lambda_j(s;t,y))^2 \, ds\big)^{\frac{1}{2}}$$

provided $t \in [0,T]$. Using Fubini and a simple change of variables, we conclude

$$\begin{aligned} &\big\| \langle y\log y\rangle^{-1} C^{(2)} \big\|^2_{L^2_{t,y}([0,T])} \\ \lesssim\ & \int_y \langle y\log y\rangle^{-2} \big(\int_0^T \int_0^T \big(\frac{\langle T\rangle}{\langle s\rangle}\big)^{2C\varepsilon^2} \langle \log \Lambda_j(s;t,y)\rangle^{-2} \langle \Lambda_j(s;t,y)\rangle^{-2} (\widetilde{H}^{(2)}(s,\Lambda_j(s;t,y))^2 \, ds dt\big) dy \\ \lesssim\ & \langle T\rangle^{2\cdot 10^\kappa \nu} \big(\int_y \langle y\log y\rangle^{-2} dy\big) \sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu} \big\| \langle y\log y\rangle^{-1} \widetilde{H}^{(2)} \big\|^2_{L^2_{t,y}([0,t])} \\ \lesssim\ & \langle T\rangle^{2\cdot 10^\kappa \nu} \sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu} \|\widetilde{H}^{(2)}(t,\cdot)\|_{L^2_{t,y}([0,t])}. \end{aligned}$$

as desired. This establishes the first bound of the lemma.

Next, we consider the error term $E[f,g,\widetilde{H}]$. As we did for $\nabla_{t,y} S[f,g,\widetilde{H}]$, we decompose $E[f,g,\widetilde{H}]$ in view of its definition as

$$E[f,g,\widetilde{H}] = A + B + C^{(1)} + C^{(2)}$$

where A, B, $C^{(1)}$ and $C^{(2)}$ correspond respectively to the contribution of f, g, $\widetilde{H}^{(1)}$ and $\widetilde{H}^{(2)}$. For A and B, we use the bound

$$\big|f'(\lambda_j(0;t,y)\big| + \big|g(\lambda_j(0;t,y)\big| \lesssim \varepsilon \sum_{\pm} \langle y \pm t + O(t^{\frac{1}{2}-\delta_1})\rangle^{-1} \|\langle y\rangle (f',g)\|_{L^\infty}.$$

Then we infer

$$|A| + |B| \lesssim \varepsilon \big| \big(\big|\nabla_{t,y}\big(\frac{\phi}{\langle y\rangle^2}\big)\big| + \big|\nabla_{t,y}(\phi^2_{t,y})\big| \big) \sum_{\pm} \langle y \pm t + O(t^{\frac{1}{2}-\delta_1})\rangle^{-1} \|\langle y\rangle (f',g)\|_{L^\infty},$$

which together with the bootstrap assumption (3.3) for $\phi_{t,y}$ yields

$$\begin{aligned}
&\big\||A|+|B|\big\|_{L^2_{t,y}([0,T])}\\
\lesssim\ & \varepsilon\|(\langle y\rangle^{-2}+\langle t\rangle^{-1})\sum_{\pm}\langle y\pm t+O(t^{\frac12-\delta_1})\rangle^{-1}\|_{L^2_{t,y}([0,T])}\|\langle y\rangle(f',g)\|_{L^\infty}.\\
\lesssim\ & \varepsilon\|\langle y\rangle(f',g)\|_{L^\infty}.
\end{aligned}$$

Next, we consider the contributions of $C^{(1)}$ and $C^{(2)}$. We have

$$\big|C^{(1)}\big|(t,y)\lesssim\big(\big|\nabla_{t,y}(\frac{\phi}{\langle y\rangle^2})\big|+\big|\nabla_{t,y}(\phi^2_{t,y})\big|\big)\sum_{j=1,2}\int_0^t\big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|\widetilde{H}^{(1)}(s,\Lambda_j(s;t,y))|\,ds$$

and

$$\big|C^{(2)}\big|(t,y)\lesssim\big(\big|\nabla_{t,y}(\frac{\phi}{\langle y\rangle^2})\big|+\big|\nabla_{t,y}(\phi^2_{t,y})\big|\big)\sum_{j=1,2}\int_0^t\big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|\langle y\rangle^{-2}\widetilde{H}^{(2)}(s,\Lambda_j(s;t,y))|\,ds.$$

(a): Contribution of $C^{(1)}$. First, consider the contribution of $\big|\nabla_{t,y}(\phi^2_{t,y})\big|$. Estimating this factor by $\lesssim\varepsilon^2\langle t\rangle^{-1}$ and using a straightforward change of variables (using (5.8)), we obtain

$$\begin{aligned}
&\big\|\big|\nabla_{t,y}(\phi^2_{t,y})\big|(t,\cdot)\sum_{j=1,2}\int_0^t\big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|\widetilde{H}^{(1)}(s,\lambda_j(s;t,y))|\,ds\big\|_{L^2_{dy}}\\
&\lesssim\varepsilon^2\langle t\rangle^{-1}\int_0^t\big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{2C\varepsilon^2}\big\|\widetilde{H}^{(1)}(s,\cdot)\big\|_{L^2_{dy}}\,ds\\
&\lesssim\varepsilon\langle t\rangle^{2\cdot10^\kappa\nu-1}\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot10^\kappa\nu+1}\|\widetilde{H}^{(1)}(t,\cdot)\|_{L^2_y}
\end{aligned}$$

which is as desired. For the contribution of $\nabla_{t,y}(\frac{\phi}{\langle y\rangle^2})$, we estimate

$$\begin{aligned}
&\big\|\langle\log y\rangle^{-1}\langle y\rangle^{-1}\phi(t,y)\int_0^t\big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|\widetilde{H}^{(1)}(s,\lambda_j(s;t,y))|\,ds\big\|_{L^2_{t,y}([0,T])}\\
&\lesssim\big\|\langle\log y\rangle^{-1}\langle y\rangle^{-1}\phi(t,y)\int_0^T\big(\frac{\langle T\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|H_1(s,\Lambda_j(s;t,y))|\,ds\big\|_{L^2_{t,y}([0,T])}\\
&\lesssim\big\|\frac{\phi}{\langle y\rangle^{\frac12}}\big\|_{L^2_tL^\infty_y}\int_0^T\big(\frac{\langle T\rangle}{\langle s\rangle}\big)^{2C\varepsilon^2}\big\|H_1(s,\cdot)\big\|_{L^2_y}\,ds\\
&\lesssim\sqrt{\varepsilon}\langle T\rangle^{2\cdot10^\kappa\nu}\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot10^\kappa\nu+1}\|\widetilde{H}^{(1)}(t,\cdot)\|_{L^2_y},
\end{aligned}$$

again as required.

(b): Contribution of $C^{(2)}$. For the contribution of $\big|\nabla_{t,y}(\phi_{t,y}^2)\big|$, we get

$$\begin{aligned}
&\big\|\big|\nabla_{t,y}(\phi_{t,y}^2)\big|\int_0^t \big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|\langle y\rangle^{-2}\widetilde{H}^{(2)}(s,\lambda_j(s;t,y))|\,ds\big\|_{L^2_{dy}}\\
&\lesssim \big\|\big|\nabla_{t,y}(\phi_{t,y}^2)\big|\big(\int_0^t \big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{2C\varepsilon^2}\big|\frac{\widetilde{H}^{(2)}(s,\cdot)|}{\langle\log(\cdot)\rangle\langle\cdot\rangle}\big|^2\,ds\big)^{\frac12}\big\|_{L^2_{dy}}\\
&\lesssim \varepsilon^2\langle t\rangle^{-1}\left(\int_0^t \big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{3C\varepsilon^2}\big\|\langle y\log y\rangle^{-1}\widetilde{H}^{(2)}(s,.)\big\|^2_{L^2_y}ds\right)^{\frac12},
\end{aligned}$$

where we used Cauchy-Schwartz and a change of variable in y. Integrating by parts in s so that the s derivative falls on $\langle s\rangle^{-3C\varepsilon^2}$, we deduce

$$\begin{aligned}
&\big\|\big|\nabla_{t,y}(\phi_{t,y}^2)\big|\int_0^t \big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|\langle y\rangle^{-2}\widetilde{H}^{(2)}(s,\lambda_j(s;t,y))|\,ds\big\|_{L^2_{dy}}\\
&\lesssim \varepsilon\langle t\rangle^{2\cdot 10^\kappa\nu-1}\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu}\big\|\langle y\log y\rangle^{-1}\widetilde{H}^{(2)}\big\|_{L^2_{t,y}([0,t])}.
\end{aligned}$$

Finally, for the contribution of $\nabla_{t,y}(\frac{\phi}{\langle y\rangle^2})$, we estimate

$$\begin{aligned}
&\big\|\langle\log y\rangle^{-1}\langle y\rangle^{-1}\phi(t,y)\int_0^t \big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|\langle y\rangle^{-2}\widetilde{H}^{(2)}(s,)|\,ds\big\|_{L^2_{t,y}([0,T])}\\
&\lesssim \big\|\langle\log y\rangle^{-1}\langle y\rangle^{-1}\phi(t,y)\big(\int_0^T \big(\frac{\langle T\rangle}{\langle s\rangle}\big)^{2C\varepsilon^2}\big|\frac{\widetilde{H}^{(2)}(s,\lambda_j(s;t,y))}{\langle\log(\lambda_j(s;t,y))\rangle\langle\lambda_j(s;t,y)\rangle}\big|^2\,ds\big)^{\frac12}\big\|_{L^2_{t,y}([0,T])}\\
&\lesssim \varepsilon\left(\int_0^T\int_0^T \big(\frac{\langle T\rangle}{\langle s\rangle}\big)^{2C\varepsilon^2}\big|\frac{\widetilde{H}^{(2)}(s,\lambda_j(s;t,y))}{\langle\log(\lambda_j(s;t,y))\rangle\langle\lambda_j(s;t,y)\rangle}\big|^2\,dsdt\right)^{\frac12}\\
&\lesssim \varepsilon\left(\int_0^T \big(\frac{\langle T\rangle}{\langle s\rangle}\big)^{3C\varepsilon^2}\big\|\langle y\log y\rangle^{-1}\widetilde{H}^{(2)}(s,.)\big\|^2_{L^2_y}\,ds\right)^{\frac12},
\end{aligned}$$

where we used Cauchy-Schwartz, Fubini, and a change of variable in t. Integrating by parts in s so that the s derivative falls on $\langle s\rangle^{-3C\varepsilon^2}$, we deduce

$$\begin{aligned}
&\big\|\langle\log y\rangle^{-1}\langle y\rangle^{-1}\phi(t,y)\int_0^t \big(\frac{\langle t\rangle}{\langle s\rangle}\big)^{C\varepsilon^2}|\langle y\rangle^{-2}\widetilde{H}^{(2)}(s,)|\,ds\big\|_{L^2_{t,y}([0,T])}\\
&\lesssim \sqrt{\varepsilon}\langle T\rangle^{2\cdot 10^\kappa\nu}\sup_{t\in[0,T]}\langle t\rangle^{-2\cdot 10^\kappa\nu}\big\|\langle y\log y\rangle^{-1}\widetilde{H}^{(2)}\big\|_{L^2_{t,y}([0,t])}
\end{aligned}$$

which is again as desired. This completes the proof of the lemma. □

We are now in position to derive the desired bound for (5.2). Let

$$f_1(y)=\partial_t^\beta\gamma_2^\gamma\tilde{\psi}(0,y),\ g_1(y)=\partial_t\partial_t^\beta\gamma_2^\gamma\tilde{\psi}(0,y),\ |\beta|+\gamma=N_1+1,$$

$$f_j(y)=0,\ j\geqslant 2,\ g_j(y)=-\big((h_2-h_1)(\phi,\nabla\phi)\big)(0,y)f'_{j-1}(y),\ j\geqslant 2,$$

H_1 is defined by (5.4), and

$$H_j(t,y)=-E[f_{j-1},g_{j-1},H_{j-1}](t,y),\ \ j\geqslant 2.$$

Note first that f_1 and g_1 satisfy in view of the assumptions on the initial data of $\tilde{\psi}$

$$\|\langle y\rangle(f_1', g_1)\|_{L^\infty} \lesssim \delta_0.$$

Also, H_1 is defined by (5.4) satisfies

$$H_1 = H_1^{(1)} + \langle y\rangle^{-2} H_1^{(2)},$$

where $H_1^{(1)}$ and $H_1^{(2)}$, in view of the bootstrap assumptions on ϕ and the proof of (3.12) for the case of non top order derivatives (i.e. $|\beta| + \gamma \leqslant N_1$), verify

$$\sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu+1} \|H_1^{(1)}(t,\cdot)\|_{L^2_y} + \sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu} \big\|\langle y\log y\rangle^{-1} H_1^{(2)}\big\|_{L^2_{t,y}([0,t])} \lesssim \varepsilon^2.$$

This is clear except for the second term amid the five terms constituting H, and for this it will be an easy consequence of the estimates below used to prove (3.13). Next, we deduce in view of Lemma5.2 that for $j \geqslant 1$, that

$$\|\langle y\rangle(f_j', g_j)\|_{L^\infty_y} \lesssim \delta_0 \varepsilon^{\frac{j-1}{2}},\ j \geqslant 1.$$

Furthermore, we have a decomposition

$$H_j = H_j^{(1)} + \langle y\rangle^{-2} H_j^{(2)},\ j \geqslant 1,$$

where $H_j^{(1)}$ and $H_j^{(2)}$ verify

$$\sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu+1} \|H_j^{(1)}(t,\cdot)\|_{L^2_y} + \sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu} \big\|\langle y\log y\rangle^{-1} H_j^{(2)}\big\|_{L^2_{t,y}([0,t])} \lesssim \varepsilon^{2+\frac{j-1}{2}}.$$

Finally, we have the following estimate for $S[f_j, g_j, H_j]$

$$\sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu} \big\|\langle y\log y\rangle^{-1} S[f_j, g_j, H_j]\big\|_{L^2_{t,y}([0,t])} \lesssim \big(\delta_0 + \frac{\varepsilon^2}{\nu}\big)\varepsilon^{\frac{j-1}{2}},\ j \geqslant 1.$$

We deduce that the sum

$$u_\infty(t,y) = \sum_{j\geqslant 1} S[f_j, g_j, H_j]$$

converges and satisfies

$$\sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu} \big\|\langle y\log y\rangle^{-1} u_\infty\big\|_{L^2_{t,y}([0,t])} \lesssim \varepsilon^{\frac{3}{2}}.$$

Furthermore, in view of Lemma 5.1, we have

$$u_\infty(0,y) = \partial_t^\beta \gamma_2^\gamma \tilde{\psi}(0,y),\ \partial_t u_\infty(0,y) = \partial_t \partial_t^\beta \gamma_2^\gamma \tilde{\psi},(0,y),\ |\beta| + \gamma = N_1 + 1$$

and

$$(-\partial_t - h_1(\phi, \nabla\phi)\partial_y)(\partial_t - h_2(\phi, \nabla\phi)\partial_y) u_\infty(t,y) = H_1(t,y).$$

By uniqueness, we deduce

$$u_\infty = \partial_t^\beta \gamma_2^\gamma \tilde{\psi},\ |\beta| + \gamma = N_1 + 1$$

and hence

$$\sup_{t\in[0,T]} \langle t\rangle^{-2\cdot 10^\kappa \nu} \big\|\langle y\log y\rangle^{-1} \partial_t^\beta \gamma_2^\gamma \tilde{\psi},\ |\beta| + \gamma\big\|_{L^2_{t,y}([0,t])} \lesssim \varepsilon^{\frac{3}{2}},\ |\beta| + \gamma = N_1 + 1.$$

This is the desired bound for the top order derivatives, which concludes the proof of (3.12).

6. Dispersive bounds

The goal of this section is to prove the dispersive estimates (3.9) and (3.10). Our key tool shall be Proposition 2.1. As usual, our point of departure is the schematic equation for $\tilde{\psi}$

$$-\partial_t^2\tilde{\psi} + \partial_y^2\tilde{\psi} + \frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi} = P_c G,$$

$$G = (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi).$$

Using Proposition 2.1 and interpolation, it follows that we need to bound the norms

$$\big\| \langle y\rangle^{\frac{1}{2}+\gamma} \nabla_{t,y}^{\beta} G \big\|_{L_t^1 L_y^1},\ |\beta| \leqslant \frac{N_1}{2} + C$$

for some $\gamma > 0$ *which is sufficiently small but can be chosen independently of* ν. Then $\delta_1 = \delta_1(\gamma)$ will be determined via interpolation from Proposition 2.1. We can write schematically

$$\begin{aligned} \big|\langle y\rangle^{\frac{1}{2}+\gamma} \nabla_{t,y}^{\beta} G\big| \lesssim \langle y\rangle^{\frac{1}{2}+\gamma} & \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2 \\ |\beta_1|\leqslant\frac{N_1}{2}, \beta_2\geqslant 1}} \frac{|\nabla_{t,y}^{\beta_1}\phi||\nabla_{t,y}^{\beta_2}\tilde{\phi}|}{\langle y\rangle^2} \\ & + \langle y\rangle^{\frac{1}{2}+\gamma} \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2 \\ |\beta_1|\leqslant\frac{N_1}{2}}} \frac{|\nabla_{t,y}^{\beta_1}\phi||\nabla_{t,y}^{\beta_2}\tilde{\phi}|}{\langle y\rangle^3} \\ & + \langle y\rangle^{\frac{1}{2}+\gamma} C(\phi, \nabla\phi, \nabla^2\phi) \end{aligned} \tag{6.1}$$

where $C(\phi, \nabla\phi, \nabla^2\phi)$ denotes the cubic nonlinear terms. The first term on the right is straightforward to estimate. Write

$$\begin{aligned} & \big\| \langle y\rangle^{\frac{1}{2}+\gamma} \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2 \\ |\beta_1|\leqslant\frac{N_1}{2}, |\beta_2|\geqslant 1}} \frac{|\nabla_{t,y}^{\beta_1}\phi||\nabla_{t,y}^{\beta_2}\tilde{\phi}|}{\langle y\rangle^2} \big\|_{L_t^1 L_y^1} \\ & \leqslant \big\| \langle y\rangle^{\frac{1}{2}+\gamma} \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2 \\ |\beta_1|\leqslant\frac{N_1}{2}, |\beta_2|\geqslant 1}} \frac{|\nabla_{t,y}^{\beta_1}\phi||\nabla_{t,y}^{\beta_2}\tilde{\phi}|}{\langle y\rangle^2} \big\|_{L_t^1 L_y^1(y\ll t)} \\ & + \big\| \langle y\rangle^{\frac{1}{2}+\gamma} \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2 \\ |\beta_1|\leqslant\frac{N_1}{2}, |\beta_2|\geqslant 1}} \frac{|\nabla_{t,y}^{\beta_1}\phi||\nabla_{t,y}^{\beta_2}\tilde{\phi}|}{\langle y\rangle^2} \big\|_{L_t^1 L_y^1(y\gtrsim t)}. \end{aligned} \tag{6.2}$$

For the first term on the right, use that on $y \ll t$

$$\big|\nabla_{t,y}^{\beta_2}\tilde{\phi}\big| \lesssim \sum_{|\tilde{\beta}_2|<|\beta_2|} t^{-1}\big|\nabla_{t,y}^{\tilde{\beta}_2}\Gamma\tilde{\phi}\big|$$

with Γ comprising both $\Gamma_{1,2}$. Then splitting $\Gamma_1\tilde{\phi} = (\Gamma_1\tilde{\phi})_1 + (\Gamma_1\tilde{\phi})_2$ as in Lemma 4.3, we have

$$\Big\| \sum_{|\tilde{\beta}_2|<|\beta_2|} t^{-1}\big|\nabla_{t,y}^{\tilde{\beta}_2}(\Gamma_1\tilde{\phi})_1\big|\Big\|_{L^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{-\frac{1}{2}-\delta_1}.$$

Furthermore, we get (using also bootstrap assumption (3.5))

$$\Big\| \sum_{|\tilde{\beta}_2|<|\beta_2|} \langle y\rangle^{-1}\big|\nabla_{t,y}^{\tilde{\beta}_2}(\Gamma_1\tilde{\phi})_2\big|\Big\|_{L^2(y\ll t)} + \Big\| \sum_{|\tilde{\beta}_2|<|\beta_2|} \langle y\rangle^{-1}\big|\nabla_{t,y}^{\tilde{\beta}_2}\Gamma_2\tilde{\phi}\big|\Big\|_{L^2(y\ll t)} \lesssim \varepsilon\langle t\rangle^{100\nu}.$$

Then, using also bootstrap assumption (3.6), we estimate the first term on the right of (6.2) by

$$\begin{aligned}
&\Big\|\langle y\rangle^{\frac{1}{2}+\gamma} \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2\\ |\beta_1|\leqslant\frac{N_1}{2},\,|\beta_2|\geqslant 1}} \frac{|\nabla_{t,y}^{\beta_1}\phi||\nabla_{t,y}^{\beta_2}\tilde{\phi}|}{\langle y\rangle^2}\Big\|_{L_t^1L_y^1(y\ll t,\,t\in[0,T])}\\
&\lesssim \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2\\ |\beta_1|\leqslant\frac{N_1}{2},\,|\beta_2|>|\tilde{\beta}_2|}} \sum_{\widetilde{\Gamma\tilde{\phi}}=\Gamma_2\tilde{\phi},(\Gamma_1\tilde{\phi})_2} \Big\|\langle t\rangle^{-2\nu}\frac{\langle\nabla_{t,y}\rangle^{\beta_1}\tilde{\phi}}{\langle y\rangle\langle\log y\rangle}\Big\|_{L^2_{t,y}([0,T])}\Big\|\langle t\rangle^{-100\nu}\langle y\rangle^{-1}\nabla_{t,y}^{\tilde{\beta}_2}\widetilde{\Gamma\tilde{\phi}}\Big\|_{L^\infty L^2_y(y\ll t)}\\
&\qquad\qquad \times \Big\|t^{-1+\gamma+102\nu+}\Big\|_{L^2_t}\\
&+ \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2\\ |\beta_1|\leqslant\frac{N_1}{2},\,|\beta_2|>|\tilde{\beta}_2|}} \Big\|\langle t\rangle^{-2\nu}\frac{\langle\nabla_{t,y}\rangle^{\beta_1}\tilde{\phi}}{\langle y\rangle\langle\log y\rangle}\Big\|_{L^2_{t,y}([0,T])}\Big\|t^{-1+2\nu}\big|\nabla_{t,y}^{\tilde{\beta}_2}(\Gamma_1\tilde{\phi})_1\big|\Big\|_{L^2_{t,y}(y\ll t)}\\
&\lesssim \frac{\varepsilon^2}{\gamma+102\nu} \lesssim \varepsilon^{\frac{3}{2}}.
\end{aligned}$$

For the last term in (6.2) , we estimate it by

$$\begin{aligned}
&\Big\|\langle y\rangle^{\frac{1}{2}+\gamma} \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2\\ |\beta_1|\leqslant\frac{N_1}{2},\,|\beta_2|\geqslant 1}} \frac{|\nabla_{t,y}^{\beta_1}\phi||\nabla_{t,y}^{\beta_2}\tilde{\phi}|}{\langle y\rangle^2}\Big\|_{L_t^1L_y^1(y\gtrsim t)}\\
&\lesssim \sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2\\ |\beta_1|\leqslant\frac{N_1}{2},\,|\beta_2|\geqslant 1}} \Big\|\langle t\rangle^{-1+\gamma+}\|\nabla_{t,y}^{\beta_2}\tilde{\phi}(t,\cdot)\|_{L^2_y}\|\nabla_{t,y}^{\beta_1}\phi(t,\cdot)\|_{L^\infty_y}\Big\|_{L^1_t}\\
&\lesssim \varepsilon^2
\end{aligned}$$

The second term on the right in (6.1) can be handled as follows:

$$\begin{aligned}
&\big\|\langle y\rangle^{\frac{1}{2}+\gamma}\sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2\\ |\beta_1|\leqslant\frac{N_1}{2}}}\frac{|\nabla^{\beta_1}_{t,y}\phi||\nabla^{\beta_2}_{t,y}\tilde{\phi}|}{\langle y\rangle^3}\big\|_{L^1_{t,y}}\\
&\lesssim\sum_{\substack{|\beta_1|+|\beta_2|\leqslant|\beta|+2\\ |\beta_1|\leqslant\frac{N_1}{2}}}\big\|\langle t\rangle^{\nu}\langle\log y\rangle\frac{\nabla^{\beta_1}_{t,y}\phi}{\langle y\rangle^{1-2\gamma}}\big\|_{L^2_tL^\infty_y}\big\|\langle t\rangle^{-2\nu}\frac{\nabla^{\beta_2}_{t,y}\tilde{\phi}}{\langle\log y\rangle\langle y\rangle}\big\|_{L^2_{t,y}}\\
&\lesssim\varepsilon^2.
\end{aligned}$$

It then suffices to consider the pure cubic terms, which we write schematically in the form

$$\langle y\rangle^{1+\gamma}\big((\partial_t\phi)^2\partial_y^2\phi-2\partial_y\phi\partial_t\phi\partial^2_{ty}\phi+(\partial_y\phi)^2\partial_t^2\phi\big).\tag{6.3}$$

This time, we shall have to take advantage of the full inherent null-structure, i. e. cancellations between the various terms. We start by absorbing weights by the factors, i. e. by replacing ϕ by $\tilde{\phi}$. Note that schematically

$$(\partial_t\phi)^2\partial_y^2\phi\sim\langle y\rangle^{-\frac{3}{2}}(\partial_t\tilde{\phi})^2\partial_y^2\tilde{\phi}+\langle y\rangle^{-\frac{5}{2}}(\partial_t\tilde{\phi})^2\partial_y\tilde{\phi}+\langle y\rangle^{-\frac{7}{2}}(\partial_t\tilde{\phi})^2\tilde{\phi}.\tag{6.4}$$

We claim that the contribution of the second and third term are straightforward to handle. In fact, for the second term, write

$$\big|\langle y\rangle^{1+\gamma}\langle y\rangle^{-\frac{5}{2}}(\partial_t\tilde{\phi})^2\partial_y\tilde{\phi}\big|\lesssim\big|\chi_{y\ll t}\langle y\rangle^{-\frac{3}{2}+\gamma}(\partial_t\tilde{\phi})^2\partial_y\tilde{\phi}\big|+\big|\chi_{y\gtrsim t}\langle t\rangle^{-\frac{3}{2}+\gamma}(\partial_t\tilde{\phi})^2\partial_y\tilde{\phi}\big|.$$

We immediately get (assuming $|\beta_1|+|\beta_2|\leqslant\frac{N_1}{2}+C$)

$$\begin{aligned}
\big\|\chi_{y\gtrsim t}\langle t\rangle^{-\frac{3}{2}+\gamma}\nabla^{\beta_1}_{t,y}(\partial_t\tilde{\phi})^2\nabla^{\beta_2}_{t,y}\partial_y\tilde{\phi}\big\|_{L^1_tL^1_y}&\lesssim\big\|\langle t\rangle^{-\frac{3}{2}+\gamma}\|\nabla^{\beta_1}_{t,y}(\partial_t\tilde{\phi})^2\|_{L^2_y}\|\nabla^{\beta_2}_{t,y}\partial_y\tilde{\phi}\|_{L^2_y}\big\|_{L^1_t}\\
&\lesssim\varepsilon^3.
\end{aligned}$$

For the first term above, write

$$\chi_{y\ll t}\tilde{\phi}_t=\chi_{y\ll t}\frac{t\Gamma_2\tilde{\phi}-y\Gamma_1\tilde{\phi}}{t^2-y^2}.$$

The term involving Γ_2 being easier, we focus on the one involving Γ_1. According to Lemma 4.3, we can decompose

$$\Gamma_1\tilde{\phi}=(\Gamma_1\tilde{\phi})_1+(\Gamma_1\tilde{\phi})_2,$$

with

$$\big\|\nabla^{\beta}_{t,y}(\Gamma_1\tilde{\phi})_1(t,\cdot)\big\|_{L^2_y(y\ll t)}\lesssim\varepsilon\langle t\rangle^{\frac{1}{2}-\delta_1},\ |\beta|\leqslant\frac{N_1}{2}+C,$$

while we also get

$$\big\|\nabla^{\beta}_{t,y}(\Gamma_1\tilde{\phi})_2(t,\cdot)\big\|_{L^2_y(y\ll t)}\lesssim\varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ 1\leqslant|\beta|\leqslant N_1-2.$$

Then we reduce to estimating the terms

$$\begin{aligned}
&\big\|\chi_{y\ll t}\langle y\rangle^{-\frac{3}{2}+\gamma}t^{-1}(\Gamma_1\tilde\phi)_1\partial_t\tilde\phi\partial_y\tilde\phi\big\|_{L^1_tL^1_y}\\
&\lesssim \big\|t^{-1}\|\chi_{y\ll t}(\Gamma_1\tilde\phi)_1\|_{L^2_y}\|\partial_t\tilde\phi\|_{L^\infty_y}\|\langle y\rangle^{-\frac{3}{2}+\gamma}\partial_y\tilde\phi\|_{L^2_y}\big\|_{L^1_t}\\
&\lesssim \varepsilon\big\|t^{-1+3\nu}\|\chi_{y\ll t}(\Gamma_1\tilde\phi)_1\|_{L^2_y}\big\|_{L^2_t}\big\|t^{-2\nu}\langle y\rangle^{-\frac{3}{2}+\gamma}\partial_y\tilde\phi\big\|_{L^2_{t,y}}\\
&\lesssim \varepsilon^3,
\end{aligned}$$

$$\begin{aligned}
&\big\|\chi_{y\ll t}\langle y\rangle^{-\frac{3}{2}+\gamma}t^{-1}(\Gamma_1\tilde\phi)_2\partial_t\tilde\phi\partial_y\tilde\phi\big\|_{L^1_tL^1_y}\\
&\lesssim \big\|\chi_{y\ll t}\langle y\rangle^{-\frac{3}{2}+\gamma}t^{-2}(\Gamma_1\tilde\phi)_2(\Gamma_1\tilde\phi)_1\partial_y\tilde\phi\big\|_{L^1_tL^1_y}\\
&+\big\|\chi_{y\ll t}\langle y\rangle^{-\frac{3}{2}+\gamma}t^{-2}(\Gamma_1\tilde\phi)_2(\Gamma_1\tilde\phi)_2\partial_y\tilde\phi\big\|_{L^1_tL^1_y}\\
&\lesssim \big\|t^{-2}\|\langle y\rangle^{-\frac{1}{2}}(\Gamma_1\tilde\phi)_2\|_{L^\infty_y}\|(\Gamma_1\tilde\phi)_1\|_{L^2_y}\|\partial_y\tilde\phi\|_{L^2_y}\big\|_{L^1_t}\\
&+\big\|t^{-2}\|\chi_{y\ll t}\langle y\rangle^{-\frac{1}{2}+\gamma}\|_{L^2_y}\|\langle y\rangle^{-\frac{1}{2}}(\Gamma_1\tilde\phi)_2\|^2_{L^\infty_y}\|\partial_y\tilde\phi\|_{L^2_y}\big\|_{L^1_t}\\
&\lesssim \varepsilon^3\big\|t^{-\frac{3}{2}-\delta_1+201\nu}\big\|_{L^1_t}+\varepsilon^3\big\|t^{-2+\gamma+201\nu}\big\|_{L^1_t}\lesssim\varepsilon^3.
\end{aligned}$$

The estimates with derivatives are analogous and omitted.
The last term in (6.4) is handled similarly, thanks to the fact that

$$\big\|\langle y\rangle^{-1}\tilde\phi\big\|_{L^2_y}\lesssim\varepsilon\langle t\rangle^{\nu}.$$

The remaining terms in (6.3) are treated similarly, and so we now reduce to estimating the following expression

$$\big\|\langle y\rangle^{-\frac{1}{2}+\gamma}\big((\partial_t\tilde\phi)^2\partial_y^2\tilde\phi-2\partial_y\tilde\phi\partial_t\tilde\phi\partial^2_{ty}\tilde\phi+(\partial_y\tilde\phi)^2\partial_t^2\tilde\phi\big)\big\|_{L^1_{t,y}}. \tag{6.5}$$

In fact, if one uses the equation for $\tilde\phi$ to switch $\tilde\phi_{tt},\tilde\phi_{yy}$ and thereby generating error terms at most as bad as the last term in (6.4) (whose contribution we already bounded), it suffices to consider

$$\big\|\langle y\rangle^{-\frac{1}{2}+\gamma}\big((\partial_t\tilde\phi)^2\partial_t^2\tilde\phi-2\partial_y\tilde\phi\partial_t\tilde\phi\partial^2_{ty}\tilde\phi+(\partial_y\tilde\phi)^2\partial_y^2\tilde\phi\big)\big\|_{L^1_{t,y}}. \tag{6.6}$$

Write

$$\begin{aligned}
&(\partial_t\tilde\phi)^2\partial_t^2\tilde\phi-2\partial_y\tilde\phi\partial_t\tilde\phi\partial^2_{ty}\tilde\phi+(\partial_y\tilde\phi)^2\partial_y^2\tilde\phi\\
&=\tilde\phi_t\frac{\Gamma_2\tilde\phi\Gamma_2\tilde\phi_t-\Gamma_1\tilde\phi\Gamma_1\tilde\phi_t}{t^2-y^2}-\tilde\phi_y\frac{\Gamma_2\tilde\phi\Gamma_2\tilde\phi_y-\Gamma_1\tilde\phi\Gamma_1\tilde\phi_y}{t^2-y^2}.
\end{aligned} \tag{6.7}$$

Then we treat a number of different regions, beginning with

(I): interior of the light cone, $y \ll t$. We exploit that the preceding expression is in effect a 'nested double null-structure'. Indeed we can write

$$\begin{aligned}
&\tilde{\phi}_t \frac{\Gamma_2\tilde{\phi}\Gamma_2\tilde{\phi}_t - \Gamma_1\tilde{\phi}\Gamma_1\tilde{\phi}_t}{t^2-y^2} - \tilde{\phi}_y \frac{\Gamma_2\tilde{\phi}\Gamma_2\tilde{\phi}_y - \Gamma_1\tilde{\phi}\Gamma_1\tilde{\phi}_y}{t^2-y^2} \\
= \;& \Gamma_2\tilde{\phi}\frac{\tilde{\phi}_t(\Gamma_2\tilde{\phi})_t - \tilde{\phi}_y(\Gamma_2\tilde{\phi})_y}{t^2-y^2} - \Gamma_1\tilde{\phi}\frac{\tilde{\phi}_t(\Gamma_1\tilde{\phi})_t - \tilde{\phi}_y(\Gamma_1\tilde{\phi})_y}{t^2-y^2} \\
& -\Gamma_2\tilde{\phi}\frac{\tilde{\phi}_t\tilde{\phi}_t - \tilde{\phi}_y\tilde{\phi}_y}{t^2-y^2} \\
= \;& \Gamma_2\tilde{\phi}\frac{\Gamma_2\tilde{\phi}(\Gamma_2^2\tilde{\phi}) - \Gamma_1\tilde{\phi}\Gamma_1\Gamma_2\tilde{\phi}}{[t^2-y^2]^2} - \Gamma_1\tilde{\phi}\frac{\Gamma_2\tilde{\phi}\Gamma_2\Gamma_1\tilde{\phi} - \Gamma_1\tilde{\phi}\Gamma_1^2\tilde{\phi}}{[t^2-y^2]^2} \\
& -\Gamma_2\tilde{\phi}\frac{\Gamma_2\tilde{\phi}\Gamma_2\tilde{\phi} - \Gamma_1\tilde{\phi}\Gamma_1\tilde{\phi}}{[t^2-y^2]^2}. \qquad (6.8)
\end{aligned}$$

Consider the worst term, which is $\frac{(\Gamma_1\tilde{\phi})^2\Gamma_1^2\tilde{\phi}}{[t^2-y^2]^2}$. Our task is to estimate

$$\big\|\chi_{y\ll t}\langle y\rangle^{-\frac{1}{2}+\gamma}\nabla_{t,y}^{\beta}\frac{(\Gamma_1\tilde{\phi})^2\Gamma_1^2\tilde{\phi}}{[t^2-y^2]^2}\big\|_{L^1_{t,y}},\ |\beta| \leqslant \frac{N_1}{2}+C.$$

The most delicate occurs when $|\beta| = 0$, which we deal with here, the other case being similar but simpler. Using Lemma 4.3, we have to estimate the expressions

$$\big\|\chi_{y\ll t}\langle y\rangle^{-\frac{1}{2}+\gamma}\frac{\Gamma_1\tilde{\phi}(\Gamma_1\tilde{\phi})_i(\Gamma_1^2\tilde{\phi})_j}{[t^2-y^2]^2}\big\|_{L^1_{t,y}},\ i,j\in\{1,2\}.$$

Observe that we have by that same lemma

$$\begin{aligned}
\chi_{y\ll t}\big|\Gamma_1\tilde{\phi}\big| &\leqslant \chi_{y\ll t}\big|(\Gamma_1\tilde{\phi})_1\big| + \chi_{y\ll t}\big|(\Gamma_1\tilde{\phi})_2\big| \\
&\leqslant \chi_{y\ll t}\big|(\Gamma_1\tilde{\phi})_1\big| + \chi_{y\ll t}\langle y\rangle^{\frac{1}{2}}\big(\int_0^y |\partial_y(\Gamma_1\tilde{\phi})_2|^2\,d\tilde{y}\big)^{\frac{1}{2}} \\
&\lesssim \varepsilon\langle t\rangle^{\frac{1}{2}+100\nu}.
\end{aligned}$$

Then when $j=2$, we get

$$\begin{aligned}
\big\|\chi_{y\ll t}\langle y\rangle^{-\frac{1}{2}+\gamma}\frac{\Gamma_1\tilde{\phi}(\Gamma_1\tilde{\phi})_i(\Gamma_1^2\tilde{\phi})_2}{[t^2-y^2]^2}\big\|_{L^1_{t,y}} &\lesssim \big\|t^{-3+\gamma}\|\Gamma_1\tilde{\phi}\|_{L^\infty_y}\|(\Gamma_1\tilde{\phi})_i\|_{L^\infty_y}\|\frac{(\Gamma_1^2\tilde{\phi})_2}{\langle y\rangle}\|_{L^2_y}\big\|_{L^1_t} \\
&\lesssim \varepsilon^3\big\|t^{-2+300\nu+\gamma}\big\|_{L^1_t} \lesssim \varepsilon^3.
\end{aligned}$$

On the other hand, if $j=1$, then we obtain

$$\begin{aligned}
\big\|\chi_{y\ll t}\langle y\rangle^{-\frac{1}{2}+\gamma}\frac{\Gamma_1\tilde{\phi}(\Gamma_1\tilde{\phi})_i(\Gamma_1^2\tilde{\phi})_1}{[t^2-y^2]^2}\big\|_{L^1_{t,y}} &\lesssim \big\|t^{-4+\gamma}\|\Gamma_1\tilde{\phi}\|_{L^\infty_y}\|(\Gamma_1\tilde{\phi})_i\|_{L^\infty_y}\|(\Gamma_1^2\tilde{\phi})_1\|_{L^2_y}\big\|_{L^1_t} \\
&\lesssim \varepsilon^3\big\|t^{-\frac{3}{2}+\gamma+200\nu-\delta_1}\big\|_{L^1_t} \lesssim \varepsilon^3.
\end{aligned}$$

The remaining terms above are more of the same.

(II): the region near the light cone; $y \sim t$. We split this into two terms, one restricted to the region very close to the light cone, i. e. $|y-t| < \langle t\rangle^{-\delta_2}$, the other away from the light cone $|y-t| \geqslant \langle t\rangle^{-\delta_2}$. Here $\delta_2 \gg \gamma > 0$ is a small constant to be determined. We start with the latter case

(IIa): The estimate away from the light cone, $|y-t| \geqslant \langle t\rangle^{-\delta_2}$. We further distinguish between a small frequency and a large frequency case. Specifically, write[16]

$$\begin{aligned}
&\tilde{\phi}_t \frac{\Gamma_2\tilde{\phi}\Gamma_2\tilde{\phi}_t - \Gamma_1\tilde{\phi}\Gamma_1\tilde{\phi}_t}{t^2-y^2} - \tilde{\phi}_y \frac{\Gamma_2\tilde{\phi}\Gamma_2\tilde{\phi}_y - \Gamma_1\tilde{\phi}\Gamma_1\tilde{\phi}_y}{t^2-y^2} \\
&= \tilde{\phi}_t \frac{\Gamma_2\tilde{\phi}P_{<t^{-\delta_3}}(\Gamma_2\tilde{\phi}_t) - \Gamma_1\tilde{\phi}P_{<t^{-\delta_3}}(\Gamma_1\tilde{\phi}_t)}{t^2-y^2} \\
&- \tilde{\phi}_y \frac{\Gamma_2\tilde{\phi}P_{<t^{-\delta_3}}(\Gamma_2\tilde{\phi}_y) - \Gamma_1\tilde{\phi}P_{<t^{-\delta_3}}(\Gamma_1\tilde{\phi}_y)}{t^2-y^2} \\
&+ \tilde{\phi}_t \frac{\Gamma_2\tilde{\phi}P_{\geqslant t^{-\delta_3}}(\Gamma_2\tilde{\phi}_t) - \Gamma_1\tilde{\phi}P_{\geqslant t^{-\delta_3}}(\Gamma_1\tilde{\phi}_t)}{t^2-y^2} \\
&- \tilde{\phi}_y \frac{\Gamma_2\tilde{\phi}P_{\geqslant t^{-\delta_3}}(\Gamma_2\tilde{\phi}_y) - \Gamma_1\tilde{\phi}P_{\geqslant t^{-\delta_3}}(\Gamma_1\tilde{\phi}_y)}{t^2-y^2}
\end{aligned} \tag{6.9}$$

where $\delta_3 \gg \delta_2$. We have

$$\begin{aligned}
&\big\|\chi_{t\sim y,|t-y|\geqslant t^{-\delta_2}}\langle y\rangle^{-\frac{1}{2}+\gamma}\tilde{\phi}_{t,y}\frac{\Gamma\tilde{\phi}P_{<t^{-\delta_3}}(\Gamma\tilde{\phi}_{t,y})}{t^2-y^2}\big\|_{L^1_{t,y}} \\
&\lesssim \big\|t^{-1+\gamma}\log t\|\tilde{\phi}_{t,y}\|_{L^\infty_y}\|\chi_{y\sim t}\langle y\rangle^{-\frac{1}{2}}\Gamma\tilde{\phi}\|_{L^\infty_y}\|\chi_{y\sim t}P_{<t^{-\delta_3}}(\Gamma\tilde{\phi}_{t,y})\|_{L^\infty_y}\big\|_{L^1_t}
\end{aligned}$$

where we have used the factor $(t-y)^{-1}$ to control the L^1_y-integral. Also, Γ stands for either Γ_1 or Γ_2. On account of Remark 4.1, we have

$$\|\chi_{y\sim t}\langle y\rangle^{-\frac{1}{2}}\Gamma\tilde{\phi}\|_{L^\infty_y} \lesssim \varepsilon\langle t\rangle^{100\nu}.$$

On the other hand, from Bernstein's inequality, we get

$$\|\chi_{y\sim t}P_{<t^{-\delta_3}}(\Gamma\tilde{\phi}_{t,y})\|_{L^\infty_y} \lesssim \varepsilon\langle t\rangle^{-\frac{\delta_3}{2}+100\nu}$$

It follows that

$$\begin{aligned}
&\big\|\chi_{t\sim y,|t-y|\geqslant t^{-\delta_2}}\langle y\rangle^{-\frac{1}{2}+\gamma}\tilde{\phi}_{t,y}\frac{\Gamma\tilde{\phi}P_{<t^{-\delta_3}}(\Gamma\tilde{\phi}_{t,y})}{t^2-y^2}\big\|_{L^1_{t,y}} \\
&\lesssim \varepsilon^3\big\|t^{-1+\gamma+201\nu-\frac{\delta_3}{2}}\log t\big\|_{L^1_t} \lesssim \varepsilon^2.
\end{aligned}$$

This reduces things to the large frequency case, i. e. the last two expressions in (6.9). Here the idea is to again invoke the 'double null-structure' as in the right-hand side of (6.8). This causes one technical complication as we need to commute

[16]Here the projection operators $P_{<s}$ and $P_{\geqslant s}$ are the *standard Littlewood-Paley projectors* in the spatial variable y, defined via a smooth cut-off function using the standard Fourier transform. They are not to be confused with P_c and P_d which are defined relative to the *distorted* Fourier transform.

frequency localizers and vector fields. Note that

$$[\Gamma_2, P_{\geqslant t^{-\delta_3}}]$$

acts boundedly in the L^2_{dy}-sense. Also, we have

$$\big\|[\Gamma_1, P_{\geqslant t^{-\delta_3}}]\tilde{\phi}\big\|_{L^2_y} \lesssim t^{\delta_3}\big\|\tilde{\phi}_t\big\|_{L^2_y}.$$

It follows that in order to bound the last two terms in (6.9) with respect to $\|\cdot\|_{\langle y\rangle^{-\frac{1}{2}+\gamma}dy}$, we need to bound the following expressions:

$$\begin{aligned}&\big\|\chi_{t\sim y,|t-y|\geqslant t^{-\delta_2}}\langle y\rangle^{-\frac{1}{2}+\gamma}\Gamma\tilde{\phi}\frac{\Gamma\tilde{\phi}P_{\geqslant t^{-\delta_3}}\Gamma^2\tilde{\phi}}{[t^2-y^2]^2}\big\|_{L^1_{t,y}}\\ &\big\|\chi_{t\sim y,|t-y|\geqslant t^{-\delta_2}}\langle y\rangle^{-\frac{1}{2}+\gamma}\Gamma\tilde{\phi}\frac{\Gamma\tilde{\phi}t^{\delta_3}(\Gamma\tilde{\phi})_t}{[t^2-y^2]^2}\big\|_{L^1_{t,y}},\end{aligned} \tag{6.10}$$

where Γ represents either Γ_1 or Γ_2. For the first expression, one writes formally

$$P_{\geqslant t^{-\delta_3}}\Gamma^2\tilde{\phi} \lesssim t^{\delta_3}\partial_y\Gamma^2\tilde{\phi}.$$

Keeping in mind the physical localization due to the cutoff $\chi_{t\sim y,|t-y|\geqslant t^{-\delta_2}}$ as well as Remark 4.1, and the bound

$$\|\chi_{y\sim t}\Gamma\tilde{\phi}\|_{L^\infty_y} \lesssim \varepsilon\langle t\rangle^{\frac{1}{2}+100\nu},$$

we bound the first term in (6.10) by

$$\begin{aligned}&\big\|\chi_{t\sim y,|t-y|\geqslant t^{-\delta_2}}\langle y\rangle^{-\frac{1}{2}+\gamma}\Gamma\tilde{\phi}\frac{\Gamma\tilde{\phi}P_{\geqslant t^{-\delta_3}}\Gamma^2\tilde{\phi}}{[t^2-y^2]^2}\big\|_{L^1_{t,y}}\\ &\lesssim \big\|t^{-\frac{5}{2}+\gamma+\delta_2+\delta_3}\log(t)\|\Gamma\tilde{\phi}\|^2_{L^\infty_y}\|\chi_{y\sim t}\partial_y\Gamma^2\tilde{\phi}\|_{L^\infty_y}\big\|_{L^1_t}\\ &\lesssim \varepsilon^3\big\|t^{-\frac{3}{2}+\gamma+\delta_2+\delta_3+300\nu}\log(t)\big\|_{L^1_t} \lesssim \varepsilon^3\end{aligned}$$

The second term in (6.10) is handled similarly.

(IIb): The estimate near the light cone, $|y-t| < \langle t\rangle^{-\delta_2}$. Here we work again with the 'intermediate null-fom expansion' as in the first line of (6.9). Noting that schematically

$$(\Gamma_1-\Gamma_2)\tilde{\phi} \sim (t-y)\tilde{\phi}_{t,y},$$

we get

$$\begin{aligned}&\tilde{\phi}_t\frac{\Gamma_2\tilde{\phi}\Gamma_2\tilde{\phi}_t-\Gamma_1\tilde{\phi}\Gamma_1\tilde{\phi}_t}{t^2-y^2}-\tilde{\phi}_y\frac{\Gamma_2\tilde{\phi}\Gamma_2\tilde{\phi}_y-\Gamma_1\tilde{\phi}\Gamma_1\tilde{\phi}_y}{t^2-y^2}\\ &\sim \tilde{\phi}_{t,y}\frac{\tilde{\phi}_{t,y}\Gamma\tilde{\phi}_{t,y}}{t+y}+\tilde{\phi}_{t,y}\frac{\Gamma\tilde{\phi}\nabla^2_{t,y}\tilde{\phi}}{t+y}.\end{aligned} \tag{6.11}$$

We then easily get the bound

$$\big\|\chi_{|y-t|<t^{-\delta_2}}\langle y\rangle^{-\frac{1}{2}+\gamma}(6.11)\big\|_{L^1_{t,y}} \lesssim \varepsilon^3\big\|t^{-1+\gamma-\delta_2+102\nu}\big\|_{L^1_t},$$

which is admissible since we may arrange $\max\{\gamma,\nu\} \ll \delta_2$.

(III): exterior of the light cone, $y \gg t$. This is handled analogously to *(I)*.

This finally completes the proof of estimates (3.9) and (3.10).

7. Control over the unstable mode

To complete the proof of Proposition 3.2, we need to prove the existence of a such that the estimates (3.13) are satisfied.

Lemma 7.1. *Let $\underline{\phi}$ be any extension to $t \in [0, +\infty)$ of ϕ which satisfies the bootstrap assumptions* (3.2)-(3.6) *on $t \in [0, +\infty)$. Let $b \in \mathbb{R}$ given by*

$$b = \left(a + \langle \tilde{\phi}_1, g_d \rangle + \frac{\langle \tilde{\phi}_2, g_d \rangle}{k_d} - \frac{1}{k_d} \int_0^{+\infty} \langle (1+y^2)^{\frac{1}{4}} F(\underline{\phi}, \nabla \underline{\phi}, \nabla^2 \underline{\phi})(s), g_d \rangle e^{-k_d s} ds \right) e^{k_d T}.$$

Then, there exists $a \in [-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}]$ such that

$$|b| \lesssim \varepsilon^{\frac{3}{2}} \langle T \rangle^{-2}.$$

Proof. Note in view of the assumptions on the initial data, the bootstrap assumptions for ϕ and the exponential decay of g_d that

$$\left| \langle \tilde{\phi}_1, g_d \rangle + \frac{\langle \tilde{\phi}_2, g_d \rangle}{k_d} - \frac{1}{k_d} \int_0^{+\infty} \langle (1+y^2)^{\frac{1}{4}} F(\underline{\phi}, \nabla \underline{\phi}, \nabla^2 \underline{\phi})(s), g_d \rangle e^{-k_d s} ds \right|$$
$$\lesssim \quad \delta_0 + \varepsilon^2.$$

We infer

$$\left| b e^{-k_d T} - a \right| \lesssim \delta_0 + \varepsilon^2. \tag{7.1}$$

Let us now consider the subsets $\mathcal{I}_\pm$ of $[-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}]$ defined by

$$\begin{aligned} \mathcal{I}_+ &= \{ a \in [-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}] / b > 2\varepsilon^{\frac{3}{2}} \langle T \rangle^{-2} \}, \\ \mathcal{I}_- &= \{ a \in [-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}] / b < -2\varepsilon^{\frac{3}{2}} \langle T \rangle^{-2} \}. \end{aligned}$$

In view of (7.1) and the fact that we may always assume that T satisfies

$$e^{k_d T} > 4 \langle T \rangle^2,$$

we immediately see that $\pm \varepsilon^{\frac{3}{2}} \in \mathcal{I}_\pm$. Furthermore, by the continuity of the flow, $\mathcal{I}_\pm$ are clearly open. Thus, $\mathcal{I}_\pm$ are two open, nonempty and disjoint subsets of $[-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}]$. Hence, there exists $a \in [-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}]$ such that

$$a \in [-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}] \backslash (\mathcal{I}_+ \cup \mathcal{I}_-).$$

This concludes the proof of the lemma. □

For a given by Lemma 7.1, we now prove (3.13). In view of the formula for h of Lemma 3.1 and the definition of b, we have

$$\begin{aligned} h(t) &= b + \frac{1}{2k_d} \left(\int_t^{+\infty} \langle (1+y^2)^{\frac{1}{4}} F(\underline{\phi}, \nabla \underline{\phi}, \nabla^2 \underline{\phi})(s), g_d \rangle e^{-k_d s} ds \right) e^{k_d t} \\ &+ \frac{1}{2} \left(a + \langle \tilde{\phi}_1, g_d \rangle - \frac{\langle \tilde{\phi}_2, g_d \rangle}{k_d} + \frac{1}{k_d} \int_0^t \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla \phi, \nabla^2 \phi)(s), g_d \rangle e^{k_d s} ds \right) e^{-k_d t}. \end{aligned}$$

Let $\underline{\mathrm{h}}$ given by

$$\underline{h}(t) = h(t) - b - \frac{1}{2}\left(a + \langle \tilde{\phi}_1, g_d\rangle - \frac{\langle \tilde{\phi}_2, g_d\rangle}{k_d}\right) e^{-k_d t}.$$

Then, $\underline{h}$ can be also written as

$$\begin{aligned} \underline{h}(t) &= \frac{1}{2k_d}\left(\int_t^{+\infty} \langle (1+y^2)^{\frac{1}{4}} F(\underline{\phi}, \nabla\underline{\phi}, \nabla^2\underline{\phi})(s), g_d\rangle e^{-k_d s} ds\right) e^{k_d t} \\ &\quad + \frac{1}{2k_d}\left(\int_0^t \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s), g_d\rangle e^{k_d s} ds\right) e^{-k_d t}. \end{aligned} \tag{7.2}$$

Remark 7.1. The point of introducing an extension $\underline{\phi}$ of ϕ to $t \in [0, +\infty)$ is to avoid boundary terms at $t = T$ when we will integrate by parts below in the formula (7.2) for $\underline{h}$.

In view of Lemma 7.1 and the assumptions on the initial data, it suffices to prove (3.13) with h replaced with $\underline{h}$. Using that

$$\left|F(\phi, \nabla\phi, \nabla^2\phi)\right| \lesssim \langle y\rangle^{-2} |\langle \nabla_{t,y}\rangle^2 \phi|^2 + |\nabla_{t,y}\phi|^2 |\nabla^2_{t,y}\phi|$$

one immediately infers from (7.2) that

$$|\partial_t^\beta \underline{h}(t)| \lesssim \varepsilon^2 \langle t\rangle^{-1-2\delta_1},\ |\beta| + 1 \leqslant N_1.$$

For the weighted derivatives, we first have

$$\begin{aligned} -t\underline{h}'(t) &= \frac{t}{2} e^{k_d t} \int_t^{+\infty} e^{-k_d s} \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s,\cdot), g_d\rangle\, ds \\ &\quad - \frac{t}{2} e^{-k_d t} \int_0^t e^{k_d s} \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s,\cdot), g_d\rangle\, ds \\ &= \frac{t}{2k_d} e^{k_d t} \int_t^{+\infty} (-\partial_s)(e^{-k_d s}) \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s,\cdot), g_d\rangle\, ds \\ &\quad - \frac{t}{2k_d} e^{-k_d t} \int_0^t \partial_s(e^{k_d s}) \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s,\cdot), g_d\rangle\, ds \\ &= \frac{t}{2k_d} e^{k_d t} \int_t^{+\infty} e^{-k_d s} \partial_s \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s,\cdot), g_d\rangle\, ds \\ &\quad + \frac{t}{2k_d} e^{-k_d t} \int_0^t e^{k_d s} \partial_s \langle (1+y^2)^{\frac{1}{4}} F(\phi, \nabla\phi, \nabla^2\phi)(s,\cdot), g_d\rangle\, ds. \end{aligned}$$

Continuing in this vein, we get

$$\begin{aligned}-(t\partial_t)^2\underline{h}+t\underline{h}'(t) &= \frac{t^2}{2}e^{k_d t}\int_t^{+\infty} e^{-k_d s}\partial_s\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds\\ &\quad-\frac{t^2}{2}e^{-k_d t}\int_0^t e^{k_d s}\partial_s\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds\\ &= \frac{t^2}{2k_d}e^{k_d t}\int_t^{+\infty}(-\partial_s)(e^{-k_d s})\partial_s\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds\\ &\quad-\frac{t^2}{2k_d}e^{-k_d t}\int_0^t \partial_s(e^{k_d s})\partial_s\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds\end{aligned}$$

and performing the integration by parts, we obtain

$$\begin{aligned}-(t\partial_t)^2\underline{h}+t\underline{h}'(t) &= \frac{t^2}{2k_d}e^{k_d t}\int_t^{+\infty} e^{-k_d s}\partial_s^2\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds\\ &\quad+\frac{t^2}{2k_d}e^{-k_d t}\int_0^t e^{k_d s}\partial_s^2\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds.\end{aligned}$$

Then note that

$$\begin{aligned}&\frac{t^2}{2k_d}e^{k_d t}\int_t^{+\infty} e^{-k_d s}\partial_s^2\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds\\ &=\frac{1}{2k_d}e^{k_d t}\int_t^{+\infty}(\frac{t}{s})^2 e^{-k_d s}s^2\partial_s^2\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds,\end{aligned}$$

$$\begin{aligned}&\frac{t^2}{2k_d}e^{-k_d t}\int_0^t e^{k_d s}\partial_s^2\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds\\ &=\frac{1}{2k_d}e^{-k_d t}\int_0^t(\frac{t}{s})^2 e^{k_d s}s^2\partial_s^2\langle(1+y^2)^{\frac{1}{4}}F(\phi,\nabla\phi,\nabla^2\phi)(s,\cdot),g_d\rangle\,ds.\end{aligned}$$

Further, we have the identity

$$s^2\partial_s^2\langle G,g_d\rangle=\langle\Gamma_2^2 G-\Gamma_2 G-2y\partial_y\Gamma_2 G+y^2\partial_y^2 G+2y\partial_y G,g_d\rangle.$$

The bounds (3.13) are now a straightforward consequence of the structure of

$$F(\phi,\nabla\phi,\nabla^2\phi)$$

and the bounds (3.2) - (3.7) for $\tilde{\phi}$. This concludes the proof of (3.13), and hence of Proposition 3.2.

8. Proof of Theorem 2.4

We are now in a position to conclude the proof of Theorem 2.4. In view of the choice of the initial data and by the continuity of the flow, note that the bootstrap assumptions are satisfied for some small $T>0$ and for any

$$a\in[-\varepsilon^{\frac{3}{2}},\,\varepsilon^{\frac{3}{2}}].$$

Then, as a consequence of Proposition 3.2, we have that for any $T > 0$, there exists $\varepsilon > 0$ small enough and $a^{(T)} \in [-\varepsilon^{\frac{3}{2}}, \varepsilon^{\frac{3}{2}}]$ such that the following estimates are satisfied on $t \in [0, T)$

$$\|\nabla^{\beta}_{t,y}\phi\|_{L^{\infty}_{dy}} \lesssim \langle t\rangle^{-\frac{1}{2}},\ 0 \leqslant |\beta| \leqslant \frac{N_1}{2} + C, \tag{8.1}$$

$$\|\langle y\rangle^{-\frac{1}{2}}\nabla^{\beta}_{t,y}\phi\|_{L^{\infty}_{dy}} \lesssim \langle t\rangle^{-\frac{1}{2}-\delta_1},\ 0 \leqslant |\beta| \leqslant \frac{N_1}{2} + C. \tag{8.2}$$

By compactness, we may extract a sequence (t_n, a_n) such that

$$(t_n)_{n\geqslant 0} \text{ is increasing },\ t_n \to +\infty,\ \text{and } a_n \to a \text{ as } n \to +\infty.$$

Then, let us call ϕ_n the solution corresponding to a_n and ϕ the solution corresponding to a. Since the ϕ_n satisfy (8.1) and (8.2) on $[0, t_n)$ with the constants in $\lesssim$ being uniform in n, and since we have chosen $(t_n)_{n\geqslant 0}$ increasing with $t_n \to +\infty$, we deduce that ϕ is a global solution satisfying (8.1) and (8.2) on $[0, +\infty)$ and hence:

$$|\phi(t,.)| \lesssim \langle t\rangle^{-\frac{1}{2}}.$$

This concludes the existence part of the proof of Theorem 2.4.

Consider now the question of the Lipschitz continuity of a with respect to the initial data. Let $\phi^{(1)}$ and $\phi^{(2)}$ two solutions corresponding respectively to parameters $a^{(1)}$ and $a^{(2)}$ and initial data $(\phi_1^{(1)}, \phi_2^{(1)})$ and $(\phi_1^{(2)}, \phi_2^{(2)})$ and let $h^{(1)}$ and $h^{(2)}$ the corresponding projections on g_d. Let us also denote

$$\Delta a = a^{(1)} - a^{(2)},\ (\Delta\phi_1, \Delta\phi_2) = (\phi_1^{(1)} - \phi_1^{(2)}, \phi_2^{(1)} - \phi_2^{(2)}),$$

and

$$\Delta\phi = \phi^{(1)} - \phi^{(2)},\ \Delta h = h^{(1)} - h^{(2)}.$$

$\phi^{(1)}$ and $\phi^{(2)}$ are obtained through the existence part of Theorem 2.4 and are thus global and satisfy estimates (3.8)-(3.13) on $t \in [0, +\infty)$. Using these bounds together with the linear estimates of Propositions 2.1, 2.2 and 2.3, we derive the following estimate for the difference $\Delta\phi$:

$$|\langle\nabla_{t,y}\rangle^2\Delta\phi(t,.)| \lesssim |\Delta a| + \|(\Delta\phi_1, \Delta\phi_2)\|_{X_0},\ t \in [0, +\infty). \tag{8.3}$$

Furthermore, we have in view of Lemma 3.1

$$\begin{aligned}
&\Delta h(t)\\
=\ & \frac{1}{2}\left(\Delta a + \langle\Delta\tilde{\phi}_1, g_d\rangle + \frac{\langle\Delta\tilde{\phi}_2, g_d\rangle}{k_d} - \frac{1}{k_d}\int_0^t \langle(1+y^2)^{\frac{1}{4}}\Delta F(\phi, \nabla\phi, \nabla^2\phi)(s), g_d\rangle e^{-k_d s}ds\right)e^{k_d t}\\
+\ & \frac{1}{2}\left(\Delta a + \langle\Delta\tilde{\phi}_1, g_d\rangle - \frac{\langle\Delta\tilde{\phi}_2, g_d\rangle}{k_d} + \frac{1}{k_d}\int_0^t \langle(1+y^2)^{\frac{1}{4}}\Delta F(\phi, \nabla\phi, \nabla^2\phi)(s), g_d\rangle e^{k_d s}ds\right)e^{-k_d t},
\end{aligned}$$

where

$$\Delta F(\phi, \nabla\phi, \nabla^2\phi) = F(\phi^{(1)}, \nabla\phi^{(1)}, \nabla^2\phi^{(1)}) - F(\phi^{(2)}, \nabla\phi^{(2)}, \nabla^2\phi^{(2)}).$$

Together with the estimates for $\phi^{(1)}$, $\phi^{(2)}$ and (8.3), we deduce

$$|\Delta a| \lesssim (|\Delta a| + \|(\Delta\phi_1, \Delta\phi_2)\|_{X_0})e^{-k_d t} + \|(\Delta\phi_1, \Delta\phi_2)\|_{X_0} + (|\Delta a| + \|(\Delta\phi_1, \Delta\phi_2)\|_{X_0})^2.$$

We let $t \to +\infty$ which yields

$$|\Delta a| \lesssim \|(\Delta\phi_1, \Delta\phi_2)\|_{X_0} + (|\Delta a| + \|(\Delta\phi_1, \Delta\phi_2)\|_{X_0})^2$$

and hence

$$|\Delta a| \lesssim \|(\Delta\phi_1, \Delta\phi_2)\|_{X_0}$$

which implies the Lipschitz continuity of a with respect to the initial data.

This concludes proof of Theorem 2.4.

Appendix A. Derivation of the equation of motion

As discussed in the beginning of Section 2.1, we consider the mapping depending on a scalar function $\phi(t, y)$ satisfying $\phi(t, y) = \phi(t, -y)$:

$$(t, y, \omega) \mapsto \left(t, \langle y\rangle + \frac{\phi(t, y)}{\langle y\rangle}, \sinh^{-1} y - \frac{y}{\langle y\rangle}\phi(t, y), \omega\right)$$

and we ask that this mapping has vanishing mean curvature. We remind our readers that we use the Japanese bracket notation $\langle y\rangle = \sqrt{1 + y^2}$. Using that the mean curvature is the first variation of the volume form, we can derive the equation of motion by considering formally the Euler-Lagrange equation associated to the volume density of the pull-back metric. An elementary computation shows that for the mapping above, the pull-back metric is

$$-(1 - \phi_t^2)\mathrm{d}t^2 + \left(1 - \frac{2\phi}{\langle y\rangle^2} + \frac{\phi^2}{\langle y\rangle^4} + \phi_y^2\right)\mathrm{d}y^2 + 2\phi_t\phi_y \mathrm{d}t\mathrm{d}y + \left(1 + y^2 + 2\phi + \frac{\phi^2}{\langle y\rangle^2}\right)\mathrm{d}\omega^2 \quad \text{(A.1)}$$

whose associated volume element is

$$\underbrace{\left(\langle y\rangle + \frac{\phi}{\langle y\rangle}\right)}_{A} \sqrt{(1 - \phi_t^2)\underbrace{(1 - \frac{\phi}{\langle y\rangle^2})^2}_{B^2} + \phi_y^2}\, \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}\omega\,. \quad \text{(A.2)}$$

Using $L = A\sqrt{B^2(1 - \phi_t^2) + \phi_y^2}$ as the Lagrangian density, we obtain the Euler-Lagrange equations:

$$\frac{\delta L}{\delta\phi} = \frac{\partial}{\partial t}\left(\frac{\delta L}{\delta\phi_t}\right) + \frac{\partial}{\partial y}\left(\frac{\delta L}{\delta\phi_y}\right).$$

Let

$$K = B^2(1 - \phi_t^2) + \phi_y^2.$$

We have

$$\begin{aligned} A &= \langle y\rangle + \frac{\phi}{\langle y\rangle}, & A' &= \frac{1}{\langle y\rangle},\\ B &= 1 - \frac{\phi}{\langle y\rangle^2}, & B' &= -\frac{1}{\langle y\rangle^2},\\ \partial_t A &= A'\phi_t, & \partial_t B &= B'\phi_t,\\ \partial_y A &= A'\phi_y + \frac{y}{\langle y\rangle} - \frac{y\phi}{\langle y\rangle^3} & \partial_y B &= B'\phi_y + \frac{2y\phi}{\langle y\rangle^4}\\ &= A'\phi_y + yA'B, & &= B'\phi_y + 2yB'(B-1), \end{aligned}$$

and also

$$\begin{aligned} K' &= 2BB' - 2BB'\phi_t^2,\\ \partial_t K &= 2\phi_y\phi_{ty} + 2B\partial_t B - 2B\partial_t B\phi_t^2 - 2B^2\phi_t\phi_{tt},\\ \partial_y K &= 2\phi_y\phi_{yy} + 2B\partial_y B - 2B\partial_y B\phi_t^2 - 2B^2\phi_t.\phi_{ty}. \end{aligned}$$

The Euler-Lagrange equations become

$$A'\sqrt{K} + A\frac{K'}{2\sqrt{K}} = -\frac{\partial}{\partial t}\left[\frac{AB^2\phi_t}{\sqrt{K}}\right] + \frac{\partial}{\partial y}\left[\frac{A\phi_y}{\sqrt{K}}\right]$$

which implies

$$\begin{aligned} ABB'\left[1-(\phi_t)^2\right]K + A'K^2 &= K^{\frac{3}{2}}\frac{\partial}{\partial y}\left[\frac{A\phi_y}{\sqrt{K}}\right] - K^{\frac{3}{2}}\frac{\partial}{\partial t}\left[\frac{AB^2\phi_t}{\sqrt{K}}\right]\\ &= K\left[\partial_y A\phi_y + A\phi_{yy}\right] - \frac{1}{2}A\phi_y\partial_y K + \frac{1}{2}AB^2\phi_t\partial_t K\\ &\quad - K\left[\partial_t AB^2\phi_t + 2AB\partial_t B\phi_t + AB^2\phi_{tt}\right]\\ &= K\left[A'(\phi_y)^2 + yA'B\phi_y + A\phi_{yy}\right] - A(\phi_y)^2\phi_{yy}\\ &\quad - AB\partial_y B\phi_y + AB\partial_y B\phi_y(\phi_t)^2 + 2AB^2\phi_y\phi_t\phi_{ty}\\ &\quad + AB^3\partial_t B\phi_t - AB^3\partial_t B(\phi_t)^3 - AB^4(\phi_t)^2\phi_{tt}\\ &\quad - KB\left[AB\phi_{tt} + A'B(\phi_t)^2 + 2AB'(\phi_t)^2\right]. \end{aligned}$$

So we arrive at

$$\begin{aligned} &KABB' + A'K\left[(\phi_y)^2 + B^2 - B^2(\phi_t)^2\right]\\ &\quad = KA'(\phi_y)^2 + KyA'B\phi_y + KA\phi_{yy} - KAB^2\phi_{tt} - KA'B^2(\phi_t)^2\\ &\qquad - KABB'(\phi_t)^2 - A(\phi_y)^2\phi_{yy} + 2AB^2\phi_y\phi_t\phi_{ty} - AB^4(\phi_t)^2\phi_{tt}\\ &\qquad + ABB'\Big[2y(B-1)(\phi_y)(\phi_t)^2 - (\phi_y)^2 - 2y(B-1)\phi_y\\ &\qquad\qquad + \underbrace{(\phi_y)^2(\phi_t)^2 + B^2(\phi_t)^2 - B^2(\phi_t)^4}_{K(\partial_t\phi)^2}\Big] \end{aligned} \tag{A.3}$$

and hence

$$\begin{aligned} KABB' + KA'B^2 &= yKA'B\phi_y + KA\phi_{yy} - KAB^2\phi_{tt} \\ &\quad - A(\phi_y)^2\phi_{yy} + 2AB^2\phi_y\phi_t\phi_{ty} - AB^4(\phi_t)^2\phi_{tt} \\ &\quad + ABB'\left[2y(B-1)(\phi_y)(\phi_t)^2 - (\phi_y)^2 - 2y(B-1)\phi_y\right]. \end{aligned} \tag{A.4}$$

We deduce, after replacing K by its definition

$$\begin{aligned} &\left[(\phi_y)^2 + B^2 - B^2(\phi_t)^2\right]\left[ABB' + A'B^2 - yA'B\phi_y - A\phi_{yy} + AB^2\phi_{tt}\right] \\ &\qquad = -A(\phi_y)^2\phi_{yy} + 2AB^2\phi_y\phi_t\phi_{ty} - AB^4(\phi_t)^2\phi_{tt} \\ &\qquad\quad + ABB'\left[2y(B-1)(\phi_y)(\phi_t)^2 - (\phi_y)^2 - 2y(B-1)\phi_y\right]. \end{aligned}$$

This we can regroup, after collecting all terms depending on the second derivatives, to get

$$\begin{aligned} &AB^2\left[-\phi_{yy} + (\phi_t)^2\phi_{yy} + (\phi_y)^2\phi_{tt} + B^2\phi_{tt} - 2\phi_y\phi_t\phi_{ty}\right] \\ &\qquad = A'B(y\phi_y - B)\left[(\phi_y)^2 + B^2 - B^2(\phi_t)^2\right] \\ &\qquad\quad + ABB'\left[\left(B^2 - \frac{2y\phi}{\langle y\rangle^2}\phi_y\right)\left((\phi_t)^2 - 1\right) - 2(\phi_y)^2\right] \end{aligned}$$

from which we divide through by $\langle y\rangle B$ to obtain

$$\begin{aligned} &\left(1 - \frac{\phi^2}{\langle y\rangle^4}\right)\left[-\phi_{yy} + (\phi_t)^2\phi_{yy} + (\phi_y)^2\phi_{tt} + B^2\phi_{tt} - 2\phi_y\phi_t\phi_{ty}\right] \\ &\qquad = \frac{y\phi_y}{\langle y\rangle^2}\left[(\phi_y)^2 + B^2(1-(\phi_t)^2)\right] - \frac{B}{\langle y\rangle^2}\left[(\phi_y)^2 + B^2(1-(\phi_t)^2)\right] \\ &\qquad\quad - \left(\frac{1}{\langle y\rangle^2} + \frac{\phi}{\langle y\rangle^4}\right)\left[\frac{2y\phi}{\langle y\rangle^2}\phi_y\left(1-(\phi_t)^2\right) - B^2(1-(\phi_t)^2) - 2(\phi_y)^2\right]. \end{aligned}$$

The left hand side we see is precisely

$$\phi_{tt} - \phi_{yy} + Q_2 + Q_3 + Q_4$$

where Q_* are defined in (2.4). The right hand side exhibits some cancellations, and can be rewritten as

$$\begin{aligned} &\frac{y\phi_y}{\langle y\rangle^2}\left[(\phi_y)^2 + B^2(1-(\phi_t)^2)\right] - \frac{1}{\langle y\rangle^2}\left[\frac{2y\phi}{\langle y\rangle^2}\phi_y(1-(\phi_t)^2) - (\phi_y)^2\right] \\ &\qquad - \frac{\phi}{\langle y\rangle^4}\left[\frac{2y\phi}{\langle y\rangle^2}\phi_y\left(1-(\phi_t)^2\right) - 2(B^2-1) - 2 + 2B^2(\phi_t)^2 - 3(\phi_y)^2\right]. \end{aligned}$$

Reorganizing a little bit and picking out the terms, we see that the above expression is equal to

$$\frac{y}{\langle y\rangle^2}\phi_y + \frac{2}{\langle y\rangle^4}\phi - S_2 - S_3 - S_4$$

where the semilinear terms S_* are defined in (2.5). With this we obtain (2.3).

Appendix B. Technical lemmas

Lemma B.1. *We have*

$$\big\|\nabla^{\beta}_{t,y}\tilde{\psi}_{ttt}\big\|_{L^2_{\tilde{y}}(y\ll t)} + \big\|\nabla^{\beta}_{t,y}\tilde{\psi}_{tty}\big\|_{L^2_{\tilde{y}}(y\ll t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu-2},\ |\beta|+2\leqslant N_1.$$

One also gets the bound

$$\big\|\langle y\rangle^2\nabla^{\beta}_{t,y}\nabla^3_{t,y}\tilde{\psi}\big\|_{L^2_{\tilde{y}}(y\gg t)} \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ |\beta|+2\leqslant N_1.$$

Proof. To prove this, write the equation for $\tilde{\psi}$ as usual in the form

$$-\partial_t^2\tilde{\psi}+\partial_y^2\tilde{\psi}+\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}=P_cG.$$

Compute

$$\begin{aligned}\Gamma_2^2\tilde{\psi} &= t^2\tilde{\psi}_{tt}+y^2\tilde{\psi}_{yy}+2ty\tilde{\psi}_{ty}+t\tilde{\psi}_t+y\tilde{\psi}_y\\ &= (t^2+y^2)\tilde{\psi}_{tt}+y^2\big[P_cG-\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}\big]\\ &+2ty\tilde{\psi}_{ty}+\Gamma_2\tilde{\psi}.\end{aligned}$$

By differentiating this equation, we obtain

$$\begin{aligned}(t^2+y^2)\tilde{\psi}_{ttt}+2ty\tilde{\psi}_{tty} = \big(\Gamma_2^2\tilde{\psi}\big)_t - y^2\big[P_cG-\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}\big]_t\\ -2t\tilde{\psi}_{tt}-2y\tilde{\psi}_{ty}-(\Gamma_2\tilde{\psi})_t =: A,\end{aligned}\tag{B.1}$$

$$\begin{aligned}(t^2+y^2)\tilde{\psi}_{tty}+2ty\tilde{\psi}_{tyy} = \big(\Gamma_2^2\tilde{\psi}\big)_y - \big(y^2\big[P_cG-\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}\big]\big)_y\\ -2y\tilde{\psi}_{tt}-2t\tilde{\psi}_{ty}-(\Gamma_2\tilde{\psi})_y =: B.\end{aligned}\tag{B.2}$$

We can turn this into a linear system for the variables $\tilde{\psi}_{ttt}$, $\tilde{\psi}_{tty}$ by observing that

$$2ty\tilde{\psi}_{tyy}-2ty\tilde{\psi}_{ttt}=2ty\big[P_cG-\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}\big]_t =: C.$$

In order to prove the observation above, it now suffices to show that

$$\big\|\nabla^{\beta}_{t,y}A\big\|_{L^2_{\tilde{y}}(y\ll t)}+\big\|\nabla^{\beta}_{t,y}B\big\|_{L^2_{\tilde{y}}(y\ll t)}+\big\|\nabla^{\beta}_{t,y}C\big\|_{L^2_{\tilde{y}}(y\ll t)}\lesssim\varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu},\ |\beta|+2\leqslant N_1.$$

Starting with A, the only delicate term is $y^2[P_cG]_t$, and here we may easily omit the P_c(as the weight y^2 gets absorbed by g_d otherwise). Then the bound

$$\big\|y^2\nabla^{\beta}_{t,y}G_t\big\|_{L^2_{\tilde{y}}}\lesssim\varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu}$$

is clear for all the weighted terms (with weight at least $\langle y\rangle^{-2}$). For the pure cubic terms, we reduce to

$$y^2\big(\phi_t^2\tilde{\psi}_{yy}\big)_t,\ y^2\big(\phi_y\phi_t\tilde{\psi}_{ty}\big)_t,\ y^2\big(\phi_y^2\tilde{\psi}_{tt}\big)_t$$

as well as their derivatives. In each of these we gain one factor t^{-1} by placing $(\phi_{t,y})^2$ into L^∞, and an extra factor t^{-1} by using

$$\|\tilde{\psi}_{tt}\|_{L^2_y(y\ll t)} + \|\tilde{\psi}_{ty}\|_{L^2_y(y\ll t)} \lesssim t^{-1}\|\nabla_{t,y}\langle\Gamma_2\rangle\tilde{\psi}\|_{L^2_y}$$

Finally, the factor $\tilde{\psi}_{yy}$ may be replaced by $\tilde{\psi}_{tt}$ up to easily controllable errors, using the equation for $\tilde{\psi}$.

The same reasoning applies to the term B, except that now we also have the terms

$$y\tilde{\psi}_{tt},\ t\tilde{\psi}_{yt},$$

which are controlled by

$$\|y\tilde{\psi}_{tt}\|_{L^2_y(y\ll t)} + \|t\tilde{\psi}_{yt}\|_{L^2_y(y\ll t)} \lesssim \|\nabla_{t,y}\langle\Gamma_2\rangle\tilde{\psi}\|_{L^2_y}.$$

For term C, the new feature is the expression

$$2ty\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\tilde{\psi}_t = 2y\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\big(\Gamma_2\tilde{\psi} - y\tilde{\psi}_y\big).$$

Then, in view of Lemma B.2, we obtain

$$\begin{aligned}\|\nabla^\beta_{t,y}\big(y\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\big(\Gamma_2\tilde{\psi}\big)\big)\|_{L^2_y(y\ll t)} &\lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])10\nu}\|\langle y\rangle^{-\frac{1}{2}}\|_{L^2_y(y\ll t)}\\ &\lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])100\nu}.\end{aligned}$$

To obtain the second inequality of the lemma, the only new feature is the control of the weighted cubic terms above,

$$y^2\big(\phi_t^2\tilde{\psi}_{yy}\big)_t,\ y^2\big(\phi_y\phi_t\tilde{\psi}_{ty}\big)_t,\ y^2\big(\phi_y^2\tilde{\psi}_{tt}\big)_t,$$

in the region $y \gg t$. But we can schematically write

$$\big|y^2\big(\phi_t^2\tilde{\psi}_{yy}\big)_t\big| \lesssim \langle y\rangle^{-1}\big|\langle\nabla_{t,y}\rangle\Gamma\tilde{\phi}\langle\nabla_{t,y}\rangle\tilde{\phi}_t\langle\nabla_{t,y}\rangle\Gamma\tilde{\psi}_{t,y}\big|,\ y\gg t,$$

where $\Gamma = \Gamma_{1,2}$, and so we get

$$\begin{aligned}\|y^2\big(\phi_t^2\tilde{\psi}_{yy}\big)_t\|_{L^2(y\gg t)} &\lesssim \|\langle y\rangle^{-1}\langle\nabla_{t,y}\rangle\Gamma\tilde{\phi}\|_{L^\infty(y\gg t)}\|\tilde{\phi}_t\|_{L^\infty}\|\langle\nabla_{t,y}\rangle\Gamma\tilde{\psi}_{t,y}\|_{L^2(y\gg t)}\\ &\lesssim \varepsilon^3\langle t\rangle^{21\nu}\end{aligned}$$

The estimate for higher derivatives is similar. This concludes the proof of the lemma. □

Lemma B.2. *We have*

$$\big|\nabla^\beta_{t,y}\Gamma_2^\kappa\tilde{\psi}\big|(t,y) \lesssim \varepsilon\langle t\rangle^{(1+[\frac{2|\beta|}{N_1}])10^\kappa\nu}\langle y\rangle^{\frac{1}{2}},\ |\beta|+\kappa\leqslant N_1.$$

Proof. This follows immediately from the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$(without the factor $\langle y\rangle^{\frac{1}{2}}$), provided $|\beta| > 0$. Hence assume now $|\beta| = 0$. Then the estimate follows from the fundamental theorem of calculus and Cauchy-Schwarz, provided we get a bound of the form

$$\big|\Gamma_2^\kappa\tilde{\psi}(t,y_*)\big| \lesssim \varepsilon\langle t\rangle^{10^\kappa\nu} \tag{B.3}$$

for some $y_* = O(1)$. For this, consider the wave equation satisfied by $\Gamma_2^\kappa\tilde{\psi}$, which is

$$-\partial_t^2\Gamma_2^\kappa\tilde{\psi} + \partial_y^2\Gamma_2^\kappa\tilde{\psi} + \frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\Gamma_2^\kappa\tilde{\psi} + l.o.t. = \Gamma_2^\kappa(P_cG),$$

where we have (pointwise bound)

$$|l.o.t.| \lesssim \sum_{0\leqslant\kappa_1<\kappa} \left|\langle y\rangle^{-2}\Gamma_2^{\kappa_1}\tilde{\psi}\right| + |P_cG|.$$

By a simple calculation, we have

$$\left\|\Gamma_2^\kappa(P_cG)\right\|_{L^2_{dy}} \lesssim \varepsilon\sum_{\tilde{\kappa}\leqslant\kappa}\left\|\langle y\rangle^{-2}\Gamma_2^{\tilde{\kappa}}\tilde{\phi}\right\|_{L^2_{dy}} + \varepsilon\langle t\rangle^{10^\kappa\nu}.$$

Also, in view of the bootstrap assumption (3.5), we have

$$\left\|\partial_{t,y}^2\Gamma_2^\kappa\tilde{\psi}\right\|_{L^2_{dy}} \lesssim \varepsilon\langle t\rangle^{10^\kappa\nu}.$$

Thus, using the previous bound and the wave equation satisfied by $\Gamma_2^\kappa\tilde{\psi}$, we deduce

$$\begin{aligned}\left\|\langle y\rangle^{-2}\Gamma_2^\kappa\tilde{\psi}\right\|_{L^2_{dy}} &\lesssim \left\|\frac{1}{2}\frac{3+\frac{y^2}{2}}{(1+y^2)^2}\Gamma_2^\kappa\tilde{\psi}\right\|_{L^2_{dy}}\\ &\lesssim \left\|\partial_{t,y}^2\Gamma_2^\kappa\tilde{\psi}\right\|_{L^2_{dy}} + \left\|\Gamma_2^\kappa(P_cG)\right\|_{L^2_{dy}} + l.o.t\\ &\lesssim \varepsilon\sum_{\tilde{\kappa}\leqslant\kappa}\left\|\langle y\rangle^{-2}\Gamma_2^{\tilde{\kappa}}\tilde{\phi}\right\|_{L^2_{dy}} + \varepsilon\langle t\rangle^{10^\kappa\nu}.\end{aligned}$$

Using induction on κ, we obtain the bound

$$\left\|\langle y\rangle^{-2}\Gamma_2^\kappa\tilde{\psi}\right\|_{L^2_{dy}} \lesssim \varepsilon\langle t\rangle^{10^\kappa\nu}. \tag{B.4}$$

This implies the existence of a y_* as in (B.3), proving the lemma. □

References

[1] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions I, Invent. Math., **145** (2001), 597–618.

[2] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions II, Amer. J. Math., **123** (2001), 1071–1101.

[3] M. Beceanu, A centre-stable manifold for the energy-critical wave equation in $\mathbb{R}^3$ in the symmetric setting, arXiv:1212.2285 (2012).

[4] I. Bejenaru, J. Krieger, D. Tataru, A codimension two stable manifold of near soliton equivariant wave maps, Anal. PDE, **6**, (2013), 829–857.

[5] G. Birkhoff and G.-C. Rota, Ordinary differential equations, fourth edition, John Wiley and Sons, Inc., New York (1989).

[6] S. Brendle, Hypersurfaces in Minkowski space with vanishing mean curvature, Comm. Pure Appl. Math., **55** (2002), 1249–1279.

[7] D. Christodoulou, The action principle and partial differential equations, Princeton University Press (2000).

[8] D. Christodoulou, The formation of shocks in 3-dimensional fluids, European Mathematical Society (EMS), Zürich (2007).

[9] R. Cote, Y. Martel and F. Merle, N-solitons for the L2-supercritical generalized Korteweg-de Vries and non-linear Schrödinger equations. Revista Mathematica Iberoamericana, **27** (2011), 273–302.
[10] M. Dafermos, G. Holzegel, I. Rodnianski, A scattering theory construction of dynamical vacuum black holes, arXiv:1306.5364.
[11] R. Donninger and J. Krieger, A vector field method on the distorted Fourier side and decay for wave equations with potentials, 59 pp, arXiv:1307.2392 (2013).
[12] D. Fischer-Colbrie and R. Schoen, The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, Comm. Pure Appl. Math., **33** (1980), 199–211.
[13] M. Hillairet and P. Raphaël, Smooth type II blow-up solutions to the four-dimensional energy critical wave equation, Anal. PDE, **5** (2012), 777–829.
[14] P. D. Hislop and I. M. Sigal, Introduction to spectral theory, Springer-Verlag, New York (1996).
[15] G. Holzegel, Ultimately Schwarzschildean spacetimes and the black hole stability problem, arXiv:1010.3216.
[16] S. Klainerman, The null condition and global existence to nonlinear wave equations, Lectures in Appl. Math., Vol. 23 (1986), 293–326.
[17] S. Klainerman and M. Machedon, Smoothing estimates for null forms and applications, Duke Math. J., **81** (1995), 99–133.
[18] J. Krieger and H. Lindblad, On stability of the catenoid under vanishing mean curvature flow on Minkowski space, Dyn. Partial Differ. Equ., **9** (2012), 89–119.
[19] J. Krieger and W. Schlag, On the focussing critical semi-linear wave equation, Amer. J. of Math. **129** (2007), 843–913.
[20] J. Krieger and W. Schlag, Non-generic blow-up solutions for the critical focusing NLS in 1-D, J. Eur. Math. Soc. **11** (2009), 1–125.
[21] H. Lindblad, A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time, Proc. Amer. Math. Soc., **132** (2004), 1095–1102 (electronic).
[22] F. Merle, P. Raphaël and J. Szeftel, On the instability of the pseudoconformal blow up for the L^2 critical NLS. Amer. J. Math. **135** (2013), 967–1017.
[23] F. Merle and H. Zaag, Existence and classification of characteristic points at blow-up for a semilinear wave equation in one space dimension, Amer. J. Math., **134** (2012), 581–648.
[24] K. Nakanishi and W. Schlag, Invariant manifolds and dispersive Hamiltonian evolution equations, Zürich Lectures in Advanced Math., European Mathematical Society, Zurich (2011).
[25] K. Nakanishi and W. Schlag, Invariant manifolds around soliton manifolds for the nonlinear Klein-Gordon equation, SIAM J. Math. Anal., **44** (2012), 1175–1210.
[26] L. Nguyen and G. Tian, On smoothness of timelike maximal cylinders in three-dimensional vacuum spacetimes, Class. Quantum Grav., **30** (2013), 165010.
[27] P. Raphaël and I. Rodnianski, Igor, Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems, Publ. Math. Inst. Hautes Études Sci., (2012), 1–122.
[28] M. Reed and B. Simon, Methods of modern mathematical physics III: Scattering theory, Academic Press, New York–London (1979).
[29] I. Rodnianski and J. Sterbenz, On the formation of singularities in the critical O(3) σ-model, Ann. of Math. (2), **172** (2010), 187–242.
[30] H. Wente, An existence theorem for surfaces of constant mean curvature, J. Math. Anal. Appl., **26** (1969), 318–344.
[31] W. W.-Y. Wong, Regular hyperbolicity, dominant energy condition and causality for Lagrangian theory of maps, Class. Quantum Grav., **28** (2011), 215008.

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