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A remarkable spectral feature of the Schrödinger Hamiltonian of the harmonic oscillator perturbed by an attractive δ'-interaction centred at the origin: double degeneracy and level crossing*

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Abstract

We rigorously define the self-adjoint Hamiltonian of the harmonic oscillator perturbed by an attractive δ'-interaction, of strength β, centred at 0 (the bottom of the confining parabolic potential), by explicitly providing its resolvent. Our approach is based on a 'coupling constant renormalization', related to a technique originated in quantum field theory and implemented in the rigorous mathematical construction of the self-adjoint operator representing the negative Laplacian perturbed by the δ-interaction in two and three dimensions. The way the δ'-interaction enters in our Hamiltonian corresponds to the one originally discussed for the free Hamiltonian (instead of the harmonic oscillator one) by P Sěba. It should not be confused with the δ'-potential perturbation of the harmonic oscillator discussed, e.g., in a recent paper by Gadella, Glasser and Nieto (also introduced by P Sěba as a perturbation of the one-dimensional free Laplacian and recently investigated in that context by Golovaty, Hryniv and Zolotaryuk). We investigate in detail the spectrum of our perturbed harmonic oscillator. The spectral structure differs from that of the one-dimensional harmonic oscillator perturbed by an attractive δ-interaction centred at the origin: the even eigenvalues are not modified at all by the δ'-interaction. Moreover, all the odd eigenvalues, regarded as functions of β, exhibit the rather remarkable phenomenon called 'level crossing' after first producing the double degeneracy of all the even eigenvalues for the value $\beta = \beta_0 = \frac{2\sqrt{\pi}}{B(\frac{3}{4},\frac{1}{2})} \cong 1.47934$($B(\cdot,\cdot)$ being the beta function).

* Dedicated to Professor Gianfausto Dell'Antonio on the occasion of his 80th birthday.

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(Some figures may appear in colour only in the online journal)

1. Introduction

There has recently been renewed interest in the study of the existence of degenerate energy levels in one-dimensional (1D) quantum mechanics, as attested by articles like [1–4] in which the eigenvalue degeneracy exhibited either by quadratic Hamiltonians with various types of potentials (volcano potentials, modified Pösch–Teller potentials) or by particular types of deformed oscillators was investigated. It is worth noting that all the models presented in the aforementioned articles depend on two parameters. However, after reading the following statement by Berry and Mondragon in the final discussion contained in [5]: 'another phenomenon which occurs generically but is forbidden in quadratic Hamiltonians in one dimension is degeneracy of states with different symmetry, which requires only one parameter to be varied', we wondered whether a quadratic Hamiltonian with a strongly singular interaction term (dependent only on one parameter representing the strength of the interaction) instead of the usual potential, could exhibit such a spectral feature.

In particular, we have decided to focus our attention on a strongly singular perturbation of the Hamiltonian of the harmonic oscillator $H_0 = \frac{1}{2}\big(-\frac{d^2}{dx^2} + x^2\big)$ whose entire spectrum consists exclusively of the isolated simple eigenvalues $E_n = n + \frac{1}{2}, \forall n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

From the point of view of physical applications, the Hamiltonian of the harmonic oscillator perturbed by a zero range impurity represented by a δ-interaction has drawn a considerable amount of interest over the last decade both in quantum dot physics (see [6] for the three-dimensional (3D) case) and the physics of Bose–Einstein condensates (see [7–9] for the 1D case). Furthermore, it has also been widely studied in papers pertaining to the mathematical physics literature such as [10–19]. It is certainly worth mentioning that the double degeneracy issue did actually come to the surface in [12, 13], although only in the asymptotical sense: in the limiting case of the 1D harmonic oscillator perturbed by an infinitely repulsive Dirac distribution centred at the origin (equivalent to a Dirichlet boundary condition), the spectrum of the new Hamiltonian consists only of the doubly degenerate odd levels.

In this paper we deal with a different type of 1D attractive point perturbation of the harmonic oscillator, that is to say the δ'-interaction. To the best of our knowledge, this is the first paper treating such perturbations of the harmonic oscillator (the δ'-interaction as a perturbation of $-\frac{1}{2}\frac{d^2}{dx^2}$ was introduced by Šeba in [26]). We are only going to consider the case where such a zero range perturbation is exactly situated at the origin, the bottom of the confining harmonic potential. After rigorously constructing in section 2 the self-adjoint Hamiltonian giving mathematical sense to the merely formal expression

$$H_\beta = H_0 - \beta|\delta'(x)\rangle\langle\delta'(x)|, \beta \geqslant 0 \tag{1.1}$$

by means of the resolvent convergence of Hamiltonians with a suitable energy cut-off, we thoroughly investigate the spectral properties of the perturbed operator in terms of the only parameter of our model, namely the magnitude of the extension parameter, that is to say our coupling constant reciprocal $\alpha = 1/\beta$,

$$\beta > 0, \alpha = +\infty \quad \text{for} \quad \beta = 0.$$

Before proceeding further, we wish to point out that, by using the Fourier transform of the derivative of the Dirac distribution centred at the origin and the representation of the momentum operator using the well-known creation and annihilation operators of the harmonic oscillator, we get the heuristic expression:

$$H_\beta = a^+ a + \frac{1}{2} - \frac{\beta}{2}(a^+ - a)\left|\frac{1}{\sqrt{2\pi}}\right\rangle\left\langle\frac{1}{\sqrt{2\pi}}\right|(a - a^+). \tag{1.1a}$$

As a consequence, our perturbation could be regarded as a momentum dependent interaction, borrowing the terminology used in [20] dealing with many-body systems in one dimension.

Furthermore, as the reader well acquainted with the literature on point interactions will easily realize, our Hamiltonian operator is completely different from the one recently investigated by Gadella *et al* in [21] in which the 1D harmonic oscillator is perturbed by a 'singular potential' of the type

$$-\gamma\delta + \lambda\delta'.$$

By referring to the classification first introduced in [22, 23], it is quite evident that, while the perturbation investigated by Gadella *et al* implies the use of a differential operator with a 'generalized potential', ours involves instead one with a 'singular density'.

Moving to the findings of our paper, they seem rather remarkable to us: whilst the even levels are not affected at all by this type of point interaction, as was to be expected on the basis of the opposite behaviour of the δ-interaction, the odd ones are given by smooth functions of the parameter α. Each function representing an odd eigenvalue does cross the horizontal line representing the adjacent even eigenvalue for the special value

$$\begin{aligned}\alpha_0 &= (H_0+1)^{-1}(0,0) = \sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}}\\ &= \frac{1}{\sqrt{\pi}}\int_0^1 \frac{\xi^{\frac{1}{2}}}{(1-\xi^2)^{1/2}}\,\mathrm{d}\xi = \frac{B\left(\frac{3}{4},\frac{1}{2}\right)}{2\sqrt{\pi}} = \frac{\sqrt{2}\Gamma^2\left(\frac{3}{4}\right)}{\pi} \cong 0.675978,\end{aligned} \tag{1.2}$$

where $(H_0+1)^{-1}(0,0)$ is the value of the resolvent kernel $(H_0+1)^{-1}(x,y)$ at $x = y = 0$, and ψ_n is the normalized eigenfunction associated with the nth eigenvalue E_n of H_0, the Hamiltonian of the above harmonic oscillator. In the second equality we have used Parseval's equality, while the following two are based on the results of [10, 11].

As a consequence, the model exhibits the double degeneracy of all the even levels for that critical value of the parameter and the phenomenon called 'level crossing' in its neighbourhood.

In a final remark we will stress the close analogy between the 1D model investigated here and the 3D isotropic harmonic oscillator perturbed by a Dirac distribution centred at the origin investigated in papers such as [6, 17]: from the point of view of the structure of the spectral curves representing the eigenvalues (energy levels) as functions of the extension parameter and neglecting the degeneracy of the 3D eigenvalues, the harmonic oscillator perturbed by the point interaction considered here seems to be a more legitimate 1D counterpart than the delta distribution, as the latter bears no resemblance to the 3D-level crossing involving eigenstates of different symmetry. Having stated that analogy, an important difference must also be pointed out: whilst in the case of the perturbed isotropic oscillator the eigenenergy of the simple eigenvalue created by the point interaction (emerging out of a degenerate level with an even value of the total angular momentum) can cross the next lower unperturbed level (having an odd value of the total angular momentum) beyond a certain threshold of the key parameter of that model, something of an opposite nature occurs in the 1D model being studied here: each perturbed odd eigenvalue can fall below the next lower unperturbed even eigenvalue beyond a certain threshold. As a consequence, the symmetry of the ground state wavefunction can change in the case of a sufficiently strong δ'-interaction (the ground state wavefunction being given by an odd function discontinuous at the origin).

In a separate paper we will show that (1.1) can be identified with the norm resolvent limit of the Hamiltonian of the 1D harmonic oscillator perturbed by a suitable triple of simple Dirac distributions.

2. The rigorous definition of H_β

The spectral features of the self-adjoint Hamiltonian of the 1D harmonic oscillator perturbed by a single attractive point interaction centred at the origin (resp. centred away from the origin) have been investigated in [10] (resp. [11]; see also [12–15]). The spectrum of the self-adjoint Hamiltonian of the 1D harmonic oscillator perturbed by a pair of such identical point interactions has been studied as well in the recent article [16]. As a further development of such previous work, we consider the formal Hamiltonian H_β, involving a different type of attractive point interaction, that is to say the δ'-interaction given by (1.1), resp. (1.1*a*).

The rigorous definition of H_β as a densely defined self-adjoint operator is bound to be a slightly more challenging task given the fact that such a point interaction is far more singular than its analogue involving the Dirac distribution. This can be immediately understood by adopting the notation and terminology of [23, 24], according to which $\delta\in\mathcal{H}_{-1}$ but $\delta'\in\mathcal{H}_{-2}\backslash\mathcal{H}_{-1}$, δ, δ' being the Dirac measure at 0 and its derivative as a distribution (where, for any positive integer n, the symbol $\mathcal{H}_{-n}$ denotes a negative index Sobolev space associated with $L^2(\mathbb{R})$) and noticing that, whilst in [10] the function $F(E) = F_0(E)$ defined by (1.3) in [23] was given by the convergent series

$$\begin{aligned}\left(\delta, \frac{1}{H_0 - E}\delta\right) &= (H_0 - E)^{-1}(0, 0)\\ &= \sum_{n=0}^{\infty} \frac{\psi_{2n}^2(0)}{2n + \frac{1}{2} - E}, \quad \psi_{2n}^2(0) = \frac{(2n)!}{\sqrt{\pi}2^{2n}(n!)^2}, \quad E \notin \sigma([H_0]_{\text{sym}}) \qquad (2.1)\end{aligned}$$

(where $\sigma([H_0]_{\text{sym}})$ denotes the spectrum of the symmetric part of H_0), its counterpart in the case of (1.1) diverges as $N \to \infty$, since $\forall n \in \mathbb{N}$:

$$\begin{aligned}\left(\delta', \sum_{n=0}^{N} \frac{|\psi_n\rangle\langle\psi_n|}{n + \frac{1}{2} - E}\delta'\right) &= \sum_{n=0}^{N} \frac{[(\delta', \psi_n)]^2}{n + \frac{1}{2} - E} = \sum_{n=0}^{N} \frac{[\psi_{2n+1}'(0)]^2}{2n + \frac{3}{2} - E}\\ &= 2\sum_{n=0}^{N} \frac{(2n + 1)\psi_{2n}^2(0)}{2n + \frac{3}{2} - E}, \quad E \notin \sigma([H_0]_{\text{asym}}) \qquad (2.2)\end{aligned}$$

(where $\sigma([H_0]_{\text{asym}})$ denotes the spectrum of the antisymmetric part of H_0). The divergence is seen by exploiting well-known properties of the eigenfunctions of the harmonic oscillator and their derivatives, by which $\lim_{n\to\infty} n^{1/12}\|\psi_n\|_\infty$ exists and is finite (see, e.g., [25] p 144).

As a consequence of this divergence, the rather straightforward resolvent convergence methods used for the definition of the Hamiltonian in the aforementioned articles can no longer be used. On the other hand, the divergence of (2.2) is reminiscent of that of the 3D counterpart of (2.1). Hence, the strategy used in [17] to deal with the 3D isotropic harmonic oscillator perturbed by an attractive point interaction centred at the origin (Fermi pseudopotential), i.e. the renormalization of the coupling constant, can be exploited with relatively minor modifications (this strategy was already used by Sěba in his renowned article [26] on the Laplacian perturbed by the same point interaction we are considering here, as well as by Albeverio *et al* in [27] dealing with periodic configurations of δ'-interactions; we also refer the reader to [6] for the

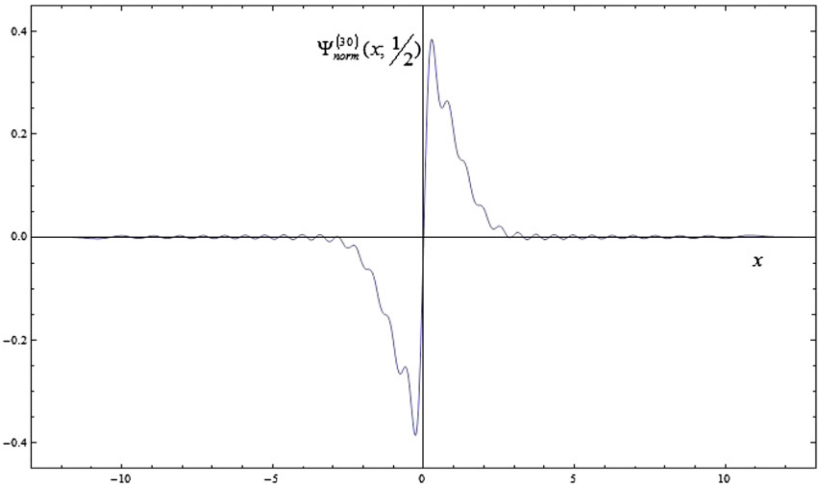

Figure 1. The plot of $\Psi^{(30)}_{\mathrm{norm}}(x; 1/2)$.

more general study of a 3D harmonic oscillator perturbed by a point interaction centred at any point in space).

Essentially, the role played by the function

$$(H_0 - E)^{-1}(\vec{x}) = \sum_{l_1,l_2,l_3=0}^{\infty} \frac{\psi_{2\vec{l}}(\vec{0})}{2|\vec{l}| + \frac{3}{2} - E}\,\psi_{2\vec{l}}(\vec{x}),$$

which belongs to $L^2(\mathbb{R}^3)$, (with $\vec{x} \in \mathbb{R}^3$, $\forall E \notin \sigma(H_0)$, $|\vec{l}| = l_1 + l_2 + l_3$) in [17], will be played here by

$$\Psi(x; E) := \sum_{n=0}^{\infty} \frac{(2n+1)^{1/2}\psi_{2n}(0)}{2n + \frac{3}{2} - E}\,\psi_{2n+1}(x) \tag{2.3}$$

which belongs to $L^2(\mathbb{R})$, (with $x \in \mathbb{R}$, $\forall E \notin \sigma([H_0]_{\mathrm{asym}})$, once again reminding the reader of the above-mentioned crucial property regarding the decay of the uniform norm of the eigenfunctions of the harmonic oscillator already exploited in [10, 11, 16, 17].

Remark 1. By writing for simplicity $\Psi(E) = \Psi(\cdot; E)$, we have that:

$$\begin{aligned}\|\Psi(E)\|_2^2 &= \sum_{n=0}^{\infty} \frac{(2n+1)\psi_{2n}^2(0)}{\left(2n + \frac{3}{2} - E\right)^2} = (H_0 + 1 - E)^{-1}(0,0) + \left(E - \frac{1}{2}\right)(H_0 + 1 - E)^{-2}(0,0)\\ &= \mathrm{Tr}[|(H_0 + 1 - E)^{-1/2}(\cdot, 0)\rangle\langle(H_0 + 1 - E)^{-1/2}(0, \cdot)|]\\ &\quad + \left(E - \frac{1}{2}\right)\mathrm{Tr}[|(H_0 + 1 - E)^{-1}(\cdot, 0)\rangle\langle(H_0 + 1 - E)^{-1}(0, \cdot)|].\end{aligned} \tag{2.3a}$$

By taking for example the value $E = 1/2$, the function (2.3) normalized is given by

$$\Psi_{\mathrm{norm}}(1/2) = \frac{\Psi(1/2)}{\|\Psi(1/2)\|_2} = \frac{\Psi(1/2)}{\sqrt{(H_0 + 1/2)^{-1}(0,0)}}.$$

If we consider the approximating function

$$\Psi^{(30)}_{\mathrm{norm}}(x; 1/2) = \frac{\sum_{n=0}^{30} \frac{\psi_{2n}(0)}{(2n+1)^{1/2}}\psi_{2n+1}(x)}{\sum_{n=0}^{30} \frac{\psi_{2n}^2(0)}{2n+1}},$$

its graph (figure 1) clearly exhibits the vestige of the action of the attractive point interaction of the singular type being considered here: the continuity of the eigenfunction at the origin is bound to be lost.

The latter feature is in perfect agreement with the well-known findings about the Hamiltonian

$$\Xi_{-\beta} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \beta|\delta'(x)\rangle\langle\delta'(x)|,$$

fully investigated in [26–31] (the reader also interested in the literature on the potential related to the derivative of the Dirac distribution is referred to [26, 32, 33]), where the coupling constant β of the interaction also enters in the condition

$$f(0_+) - f(0_-) = -\beta \cdot f'(0)$$

defining the domain of the perturbed operator (the minus sign is due to the fact that we assumed $\beta \geqslant 0$, i.e. we are considering an attractive δ'-interaction).

The definition (2.3) obviously guarantees that $|\Psi(E)\rangle\langle\Psi(E)|$ is a well-defined rank-1 operator trivially vanishing on the subspace spanned by the symmetric eigenfunctions.

We start with the following N-approximation H_β^N for the Hamiltonian H_β:

$$H_\beta^N = H_0 - \mu_\beta(N) \sum_{m,n=0}^{N} |\psi_{2m+1}\rangle[\psi'_{2m+1}(0)\psi'_{2n+1}(0)]\langle\psi_{2n+1}|, \quad \beta \in \mathbb{R}\backslash\{0\}, \tag{2.4}$$

where

$$\frac{1}{\mu_\beta(N)} = \frac{1}{\beta} + 2\sum_{n=0}^{N} \frac{(2n+1)\psi_{2n}^2(0)}{2n+\frac{3}{2}}, \quad \beta \in \mathbb{R}\backslash\{0\}. \tag{2.4a}$$

Its resolvent is explicitly given by:

$$\left(H_\beta^N - E\right)^{-1} = (H_0 - E)^{-1} + \frac{2|\Psi_N(E)\rangle\langle\Psi_N(E)|}{\frac{1}{\mu_\beta(N)} - \sum_{n=0}^{N} \frac{[\psi'_{2n+1}(0)]^2}{2n+\frac{3}{2}-E}}, \quad \operatorname{Im} E > 0, \tag{2.5}$$

($\beta \in \mathbb{R}\backslash\{0\}$), with

$$\Psi_N(E) = \sum_{n=0}^{N} \frac{(2n+1)^{1/2}\psi_{2n}(0)}{2n+\frac{3}{2}-E}\psi_{2n+1} \in L^2(\mathbb{R}) \tag{2.6}$$

$\forall E \notin \sigma([H_0]_{\mathrm{asym}})$.

As a consequence of (2.3) and (2.6), it is not difficult to prove that, in the sense of operator norm convergence

$$|\Psi_N(E)\rangle\langle\Psi_N(E)| \to |\Psi(E)\rangle\langle\Psi(E)|$$

as $N \to \infty$.

This fact and the convergence of the denominator to

$$\beta^{-1} - 2E\left[\sum_{n=0}^{\infty} \frac{(2n+1)\psi_{2n}^2(0)}{\left(2n+\frac{3}{2}\right)\left(2n+\frac{3}{2}-E\right)}\right], \quad \operatorname{Im} E > 0,$$

yield the norm convergence of the second term in (2.5) as $N \to \infty$ (the sum being convergent by the reasons given above).

Hence, the limit in the operator norm of the sequence of resolvents in (2.5) does exist and is explicitly given by:

$$R(\beta, E) = (H_0 - E)^{-1} + \frac{2|\Psi(E)\rangle\langle\Psi(E)|}{\beta^{-1} - 2E\left[\sum_{n=0}^{\infty} \frac{(2n+1)\psi_{2n}^2(0)}{\left(2n+\frac{3}{2}\right)\left(2n+\frac{3}{2}-E\right)}\right]}, \quad \operatorname{Im} E > 0. \tag{2.7}$$

By essentially mimicking the final part of the proof of theorem 2.2 in [17] which was, in turn, a suitable adaptation of the final part of the proof of theorem 1.1.1 in section 2.1 of [27], it is relatively straightforward to prove that (2.7) is indeed the resolvent of a densely defined self-adjoint operator, namely $H_\beta = [R(\beta, E)]^{-1} + E$, $\operatorname{Im} E > 0$.

By means of the algebra of partial fractions the convergent series in the denominator can be easily transformed into:

$$(H_0 - E)^{-1} + \frac{2\beta|\Psi(E)\rangle\langle\Psi(E)|}{1 - 2\beta\left[\frac{1}{2}\sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}} - \left(\frac{1}{2} - E\right)\sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}-E}\right]}, \quad \operatorname{Im} E > 0. \tag{2.8}$$

Taking into account the fact that the two series in the latter formula represent the values of the Krein function of H_0 at the points $E - 1$ and -1, we can rewrite the resolvent of H_β as:

$$\begin{aligned}(H_\beta - E)^{-1} &= (H_0 - E)^{-1} \\ &\quad + \frac{2\beta|\Psi(E)\rangle\langle\Psi(E)|}{1 - 2\beta\left[\frac{1}{2}(H_0+1)^{-1}(0,0) - \left(\frac{1}{2} - E\right)(H_0 + 1 - E)^{-1}(0,0)\right]}, \quad \operatorname{Im} E > 0.\end{aligned} \tag{2.9}$$

The explicit form of the resolvent clearly implies its norm-analyticity in a small neighbourhood of 0 (we have an analytic family in the sense of Kato, see [34, 35]) as a function of β and the analyticity of the eigenvalues.

The results shown so far can be summarized in the following theorem.

Theorem 2.1. *The Hamiltonian making sense of the merely formal expression ((1.1) or (1.1a)) describing the harmonic oscillator perturbed by an attractive point interaction of the δ'-type is the self-adjoint operator whose resolvent is given by:*

$$\begin{aligned}(H_\beta - E)^{-1} &= (H_0 - E)^{-1} + \frac{2\beta|\Psi(E)\rangle\langle\Psi(E)|}{1 - 2\beta\left[\frac{1}{2}(H_0+1)^{-1}(0,0) - \left(\frac{1}{2} - E\right)(H_0 + 1 - E)^{-1}(0,0)\right]}, \\ &\beta \in \mathbb{R}, \quad \operatorname{Im} E > 0.\end{aligned}$$

The latter is the limit as $N \to \infty$ in the norm convergence of the sequence of resolvents (2.5) of the Hamiltonians H_β^N with the energy cut-off defined in (2.4) and (2.4a). Furthermore, H_β regarded as a function of β is an analytic family in the sense of Kato.

Remark 2. If we set $\frac{1}{\mu_{\tilde\beta}(N)} = \frac{1}{\tilde\beta} + 2\sum_{n=0}^{N}\psi_{2n}^2(0)$, $\tilde\beta \in \mathbb{R}\backslash\{0\}$, instead of (2.4a), the resolvent can then be expressed in terms of just one value of the function defined in (2.1), that is to say:

$$(H_{\tilde\beta} - E)^{-1} = (H_0 - E)^{-1} + \frac{2\tilde\beta|\Psi(E)\rangle\langle\Psi(E)|}{1 + 2\tilde\beta\left(\frac{1}{2} - E\right)(H_0 + 1 - E)^{-1}(0,0)}, \quad \operatorname{Im} E > 0. \tag{2.10}$$

The right-hand side of (2.4a) can be written as

$$\frac{1}{\beta} + 2\sum_{n=0}^{N}\psi_{2n}^2(0) - \sum_{n=0}^{N}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}}, \quad \beta \in \mathbb{R}\backslash\{0\}$$

and setting

$$\frac{1}{\tilde\beta} = \frac{1}{\beta} - \sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}},$$

it is straightforward to check that

$$\frac{1}{\mu_\beta(N)} - \frac{1}{\mu_{\tilde\beta}(N)} = \sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}} - \sum_{n=0}^{N}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}} \to 0$$

as $N \to \infty$. By setting $\alpha = 1/\beta, \tilde{\alpha} = 1/\tilde{\beta}, \beta, \tilde{\beta} \in \mathbb{R}\backslash\{0\}$, we can easily express the link between the two choices:

$$\tilde{\alpha} = \alpha - \sum_{n=0}^{\infty} \frac{\psi_{2n}^2(0)}{2n + \frac{3}{2}} = \alpha - \alpha_0,$$

with α_0 defined in (1.2). We will come back to the comparison of the two representations near the end of the next section.

Remark 3. As was explicitly done in [10] (see also [11, 16, 17]) using some basic functional calculus, the series representation of the function $F(E) = F_0(E)$ given by (2.1) can be recast as an integral for any real $E < 1/2$, ($1/2$ being the ground state energy of H_0), that is to say:

$$\sum_{n=0}^{\infty} \frac{\psi_{2n}^2(0)}{2n + \frac{1}{2} - E} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\mathrm{e}^{\left(\frac{1}{2}+E\right)t}}{(\mathrm{e}^{2t} - 1)^{1/2}}\, \mathrm{d}t. \tag{2.11}$$

Hence, the resolvent of H_β given by (2.9) can be rewritten for any $E < 3/2$ (for $\beta > 0$, i.e. in the attractive case) as follows:

$$(H_\beta - E)^{-1} = (H_0 - E)^{-1} + \frac{2\beta|\Psi(E)\rangle\langle\Psi(E)|}{1 - \frac{2\beta}{\sqrt{\pi}}\left[\frac{1}{2}\int_0^{\infty} \frac{\mathrm{e}^{-\frac{1}{2}t}}{(\mathrm{e}^{2t}-1)^{1/2}}\,\mathrm{d}t - \left(\frac{1}{2} - E\right)\int_0^{\infty} \frac{\mathrm{e}^{-\left(\frac{1}{2}-E\right)t}}{(\mathrm{e}^{2t}-1)^{1/2}}\,\mathrm{d}t\right]} \tag{2.12}$$

or equivalently (using the change of variable $\xi = \mathrm{e}^{-t}$),

$$\begin{aligned}(H_\beta - E)^{-1} &= (H_0 - E)^{-1} + \frac{2\beta|\Psi(E)\rangle\langle\Psi(E)|}{1 - \frac{2\beta}{\sqrt{\pi}}\left[\frac{1}{2}\int_0^{1} \frac{\xi^{\frac{1}{2}}}{(1-\xi^2)^{1/2}}\,\mathrm{d}\xi - \left(\frac{1}{2} - E\right)\int_0^{1} \frac{\xi^{\frac{1}{2}-E}}{(1-\xi^2)^{1/2}}\,\mathrm{d}\xi\right]} \\ &= (H_0 - E)^{-1} + \frac{2\beta|\Psi(E)\rangle\langle\Psi(E)|}{1 - \frac{\beta}{2\sqrt{\pi}}\left[B\left(\frac{3}{4}, \frac{1}{2}\right) - (1 - 2E)B\left(\frac{3}{4} - \frac{E}{2}, \frac{1}{2}\right)\right]},\end{aligned} \tag{2.12a}$$

having rewritten the integrals in terms of the well-known beta function $B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ (see [36]). As will be seen in the next section, (2.12a) will be useful in order to study the odd eigenvalues of H_β arising from the point perturbation considered in our paper.

In this context it might be worth reminding the reader that the bound state equation for the Hamiltonian of the 1D harmonic oscillator perturbed by a δ-interaction situated at the origin also involves a ratio of gamma functions; see [6–10, 12].

3. The eigenvalues of the perturbed Hamiltonian

As is clearly shown by the explicit formula of the resolvent of H_β, the even eigenvalues of the harmonic oscillator are not modified at all by the δ'-interaction situated at the origin. This is the opposite of what happens in the case of the δ-interaction centred at 0 which, due to its reflection symmetry, does not affect the odd eigenvalues. As a consequence, our analysis will entirely focus on the new odd eigenvalues which will be given by those Es making the denominator of the second term in (2.9) vanish.

Hence, by using $\alpha = 1/\beta$ and adopting also the notation $H(\alpha) = H_{1/\alpha}$, the following equation will be investigated:

$$\alpha = 2\left[\frac{1}{2}\sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}} - \left(\frac{1}{2}-E\right)\sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}-E}\right], \tag{3.1}$$

with $E \notin 2\mathbb{N}_0 + 3/2$.

It is important to notice that the second series in the latter equation is an increasing multi-branch function of the energy with simple poles given by the odd eigenvalues of the unperturbed Hamiltonian H_0. In particular, it is perfectly defined whenever E is given by any even eigenvalue of H_0.

By using $\alpha_0 = \sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}}$ from (1.2), (3.1) can be rewritten as follows:

$$\alpha_0 - \alpha = (1-2E)\sum_{n=0}^{\infty}\frac{\psi_{2n}^2(0)}{2n+\frac{3}{2}-E}, \tag{3.1a}$$

with $E \notin 2\mathbb{N}_0 + 3/2$.

It is quite straightforward to check that, by using the expression of the series in (3.1a) in terms of the beta function (see 2.12a) and the well-known recurrence relation $\Gamma(z+1) = z\Gamma(z)$, the bound state equation (3.1a) can be written as:

$$\alpha_0 - \alpha = 2\frac{\Gamma\left(\frac{3}{4}-\frac{E}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{E}{2}\right)}. \tag{3.1b}$$

Remark 4. As pointed out at the end of the previous section, the bound state equation for the 1D harmonic oscillator perturbed by a δ-interaction situated at the origin with strength $\lambda \neq 0$ can be written in terms of a ratio of gamma functions, precisely:

$$\frac{2}{\lambda} = \frac{\Gamma\left(\frac{1}{4}-\frac{E}{2}\right)}{\Gamma\left(\frac{3}{4}-\frac{E}{2}\right)}.$$

Remark 5. It may be interesting to realize that, as a consequence of (3.1b), the bound state equation in the case of the alternative renormalization leading to $H(\tilde{\alpha}) = H_{1/\tilde{\alpha}}$ reads:

$$\frac{2}{\tilde{\alpha}} = -\frac{\Gamma\left(\frac{1}{4}-\frac{E}{2}\right)}{\Gamma\left(\frac{3}{4}-\frac{E}{2}\right)}. \tag{3.1c}$$

Let us then consider the following cases, for $\alpha \geqslant 0$ (the case $\alpha < 0$ will be handled in Remark 8 below):

(i) $\alpha = 0$;
(ii) $0 < \alpha < \alpha_0$;
(iii) $\alpha = \alpha_0$;
(iv) $\alpha_0 < \alpha$.

Let us consider (3.1b) in the first case, namely:

$$\alpha_0 = 2\frac{\Gamma\left(\frac{3}{4}-\frac{E}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{E}{2}\right)}, \tag{3.2}$$

recalling that α_0 is given by (1.2).

It is evident from either (3.1) or (3.2) (by using the expression of α_0 in terms of the gamma function and reminding the reader of the well-known identity $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$, see [36]) that $E = 0$ is a solution. Furthermore, by exploiting the well-known properties of the gamma function, it is clear that, whilst the odd eigenvalues of the unperturbed operator H_0

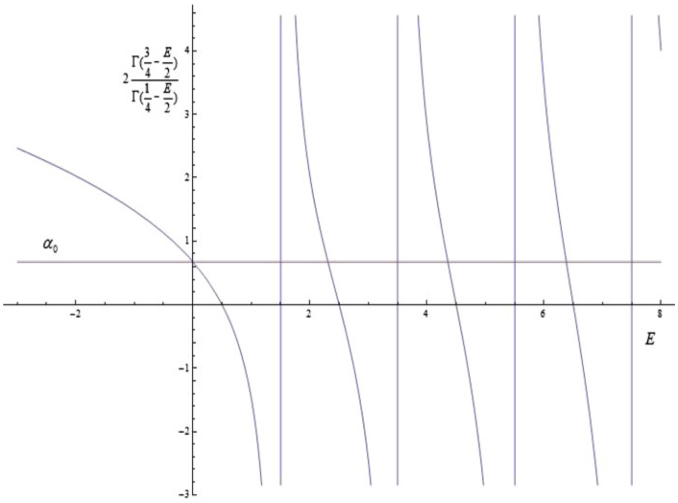

Figure 2. Case (i): the odd eigenvalues of $H(\alpha)$ given by the E-coordinates of the intersections of the horizontal line α_0 with the multi-branch function $2\Gamma\left(\frac{3}{4}-\frac{E}{2}\right)\Big/\Gamma\left(\frac{1}{4}-\frac{E}{2}\right)$ for $\alpha = 0$.

represent the vertical asymptotes of the gamma function in the numerator and, consequently, of the right hand side of the latter equation, the even ones, being the poles of the denominator, represent the zeroes of the entire fraction. The plot in figure 2 shows the location of all the eigenvalues given by the intersections between the graphs of both sides of (3.2).

Remark 6. The spectral feature of having $E = 0$ as an eigenvalue of $H(\alpha)$ for the limiting value $\alpha = 0$, further confirms the link between the renormalization procedure used here to define the attractive δ'-interaction centred at the origin perturbing the 1D harmonic oscillator and the one adopted in [17] to define the attractive δ-interaction perturbing the 3D isotropic harmonic oscillator. This point will be further dealt with in another remark at the end of this section.

The graph (figure 3) illustrating the spectral configuration for the second case (ii) is not too different from the former one. It is crucial to notice that in both cases, due to the positivity of the left hand side of (3.1*b*) for $\alpha_0 > \alpha$, all the intersections occur above the horizontal axis. Hence, all the perturbed antisymmetric eigenvalues lie below the unperturbed symmetric ones (represented by the zeroes of the multi-branch function in figures 2 and 3), namely $E_{2n+1}(\alpha) < E_{2n}(\alpha) = 2n + \frac{1}{2}, \forall n \in \mathbb{N}_0$.

The third case (iii), occurring for the particular value α_0 defined in (1.2) leads to the equation:

$$2\frac{\Gamma\left(\frac{3}{4}-\frac{E}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{E}{2}\right)} = 0. \tag{3.3}$$

As noted earlier, the zeroes of this fraction are exactly given by the unperturbed even eigenvalues, i.e. the even eigenvalues of H_0. Hence, something rather remarkable occurs in this case, namely the odd eigenvalues coincide with the unperturbed even ones, implying their double degeneracy including the ground state energy $E = 1/2$, as shown in the graph in figure 4.

In the case (iv), the horizontal line given by the value $\alpha_0 - \alpha$ lies below the E-axis. Hence, the odd eigenvalues of $H(\alpha)$ are situated to the right of the even eigenvalues of H_0, as shown by the graph drawn for $\alpha = 2$ (figure 5).

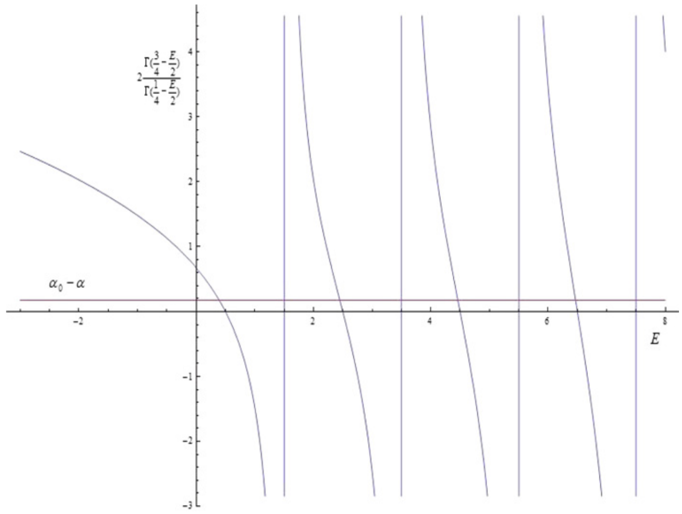

Figure 3. Case (ii): the same as figure 2 but for $\alpha = 1/2$ (with $\alpha_0 \cong 0.675978$).

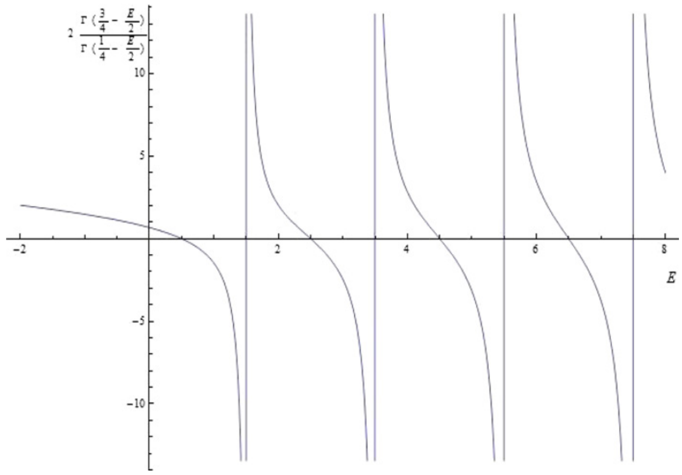

Figure 4. Case (iii): the odd eigenvalues of $H(\alpha)$ coincide with the unperturbed even ones for $\alpha = \alpha_0 = (H_0 + 1)^{-1}(0, 0)$, implying their double degeneracy.

As is quite evident, the limiting case $\alpha = +\infty$ (i.e. $\beta = 0$) corresponds to the Hamiltonian of the unperturbed harmonic oscillator. Hence, summarizing the main points of our analysis, we can state the following.

Theorem 3.1. *The spectrum of the operator $H(\alpha)$ is entirely discrete and its structure is fully described by the following cases:*

(i) for $\alpha \in (\alpha_0, \infty)$ (weak coupling), each odd eigenvalue $E_{2n+1}(\alpha)$ lies above the corresponding even one, that is to say

$$E_{2n+1}(\alpha) > E_{2n}(\alpha) = 2n + \tfrac{1}{2}, \quad \forall n \in \mathbb{N}_0$$

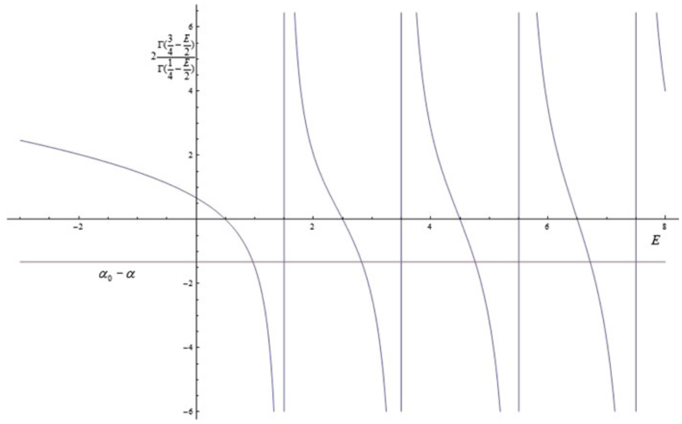

Figure 5. Case (iv): the odd eigenvalues of $H(\alpha)$ are located to the right (i.e. above) of the unperturbed even ones.

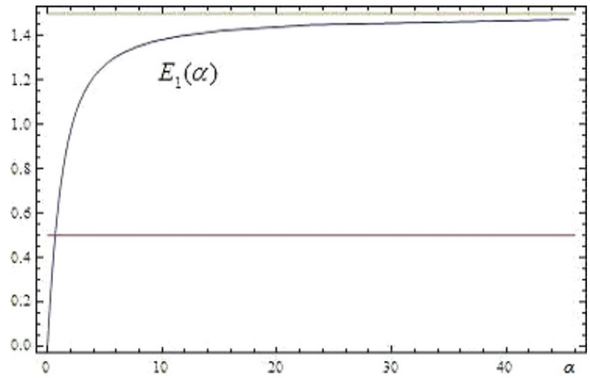

Figure 6. The lowest odd eigenvalue $E_1(\alpha)$ crossing the unperturbed eigenvalue $E_0(\alpha) = 1/2$ exactly at $\alpha = \alpha_0$ and approaching asymptotically $E_1(\infty) = 3/2$,for $\alpha \to +\infty$.

(ii) for the special value $\alpha = \alpha_0 = (H_0+1)^{-1}(0, 0)$ *(critical value coupling), each eigenvalue is doubly degenerate since*

$$E_{2n+1}(\alpha_0) = E_{2n}(\alpha_0) = 2n + \tfrac{1}{2}, \quad \forall n \in \mathbb{N}_0$$

(iii) for $\alpha \in [0, \alpha_0)$ *(strong coupling), each odd eigenvalue* $E_{2n+1}(\alpha)$ *is situated below the corresponding even one, namely*

$$E_{2n+1}(\alpha) < E_{2n}(\alpha) = 2n + \tfrac{1}{2}, \quad \forall n \in \mathbb{N}_0$$

Furthermore, in the limiting case $\alpha = 0$ *the bottom of the spectrum is given by* $E_1(0) = 0$.

Remark 7. In the situation where $\alpha = \alpha_0$, our Hamiltonian $H(\alpha)$ exhibits the phenomenon known as 'level crossing', quite differently from many other Hamiltonians manifesting instead the phenomenon called 'level repulsion' or 'avoided level crossing' (see [37, 38] as well as the renowned original article by von Neumann and Wigner [39] on related issues). As a consequence, the particular value α_0 represents a 'critical point' for this model.

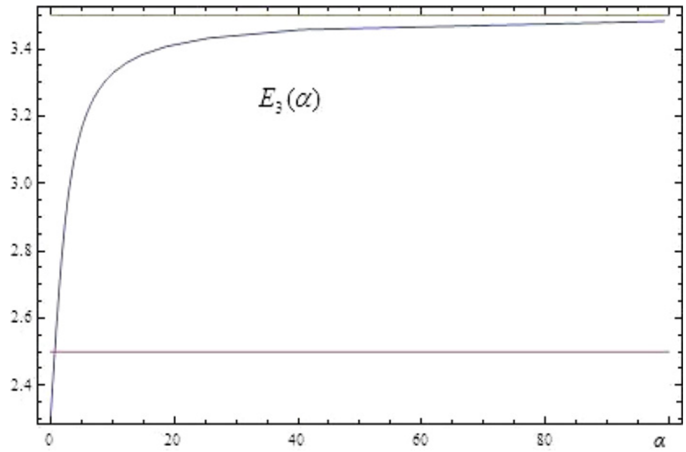

Figure 7. The fourth eigenvalue $E_3(\alpha)$ of $H(\alpha)$ crossing the unperturbed eigenvalue $E_2(\alpha) = 5/2$ exactly at $\alpha = \alpha_0$ and approaching asymptotically $E_3(\infty) = 7/2$ for $\alpha \to +\infty$.

As indicated in the third remark at the end of section 2, the equation determining the eigenvalues can also be recast in integral form for each branch of the right-hand side of (3.1*a*). In particular, the one determining the lowest odd eigenvalue of $H(\alpha)$ reads:

$$\alpha = \frac{2}{\sqrt{\pi}}\left[\frac{1}{2}\int_0^1 \frac{\xi^{\frac{1}{2}}}{(1-\xi^2)^{1/2}}\,\mathrm{d}\xi - \left(\frac{1}{2}-E\right)\int_0^1 \frac{\xi^{\frac{1}{2}-E}}{(1-\xi^2)^{1/2}}\,\mathrm{d}\xi\right] \tag{3.4}$$

or equivalently,

$$\alpha = \frac{1}{2\sqrt{\pi}}\left[B\left(\frac{3}{4},\frac{1}{2}\right) - (1-2E)B\left(\frac{3}{4}-\frac{E}{2},\frac{1}{2}\right)\right]. \tag{3.4a}$$

The plot of the lowest odd eigenvalue $E_1(\alpha)$ is given in figure 6.

Remark 8. Although we have only considered $\alpha \geqslant 0$ so far for the purpose of investigating the double degeneracy and the level crossing at α_0, it is quite clear that our spectral analysis can easily be extended to negative values of α, as can be immediately gathered by looking at the explicit expression of the resolvent

$$[H(\alpha)-E]^{-1} = (H_0-E)^{-1} + \frac{2|\Psi(E)\rangle\langle\Psi(E)|}{\alpha - 2\left[\frac{1}{2}(H_0+1)^{-1}(0,0) - \left(\frac{1}{2}-E\right)(H_0+1-E)^{-1}(0,0)\right]},$$

with $\Psi(E)$ given by (2.3). For example, the equation leading to the lowest odd eigenvalue for negative values of both α and E is:

$$|\alpha| = \frac{2}{\sqrt{\pi}}\left[\left(\frac{1}{2}+|E|\right)\int_0^1 \frac{\xi^{\frac{1}{2}+|E|}}{(1-\xi^2)^{1/2}}\,\mathrm{d}\xi - \frac{1}{2}\int_0^1 \frac{\xi^{\frac{1}{2}}}{(1-\xi^2)^{1/2}}\,\mathrm{d}\xi\right].$$

Remark 9. It is not difficult to guess how the spectral curves representing the odd eigenvalues of $H(\alpha)$ would be modified if we were to adopt the alternative renormalization recipe of the first remark of section 2: we would obviously have $E_1(0) = 1/2$ and all the level crossings would then occur at $\tilde{\alpha} = 0$.

This comparison is actually quite similar to the one regarding the 3D model with the delta distribution centred at the origin. In [17] the following renormalization prescription for the coupling constant was adopted:

$$\frac{1}{\mu_{\alpha^{(3)}}(N)} = \alpha^{(3)} + \sum_{|\vec{l}|\leqslant N} \frac{\psi^2_{2\vec{l}}(0)}{2|\vec{l}| + \frac{3}{2}}, \tag{3.5}$$

(where $\alpha^{(3)}$ denotes the 3D analogue of α) in order to get, for $N \to \infty$, $E_0(0) = 0$ (see figure 1 in that paper). As a consequence of the presence of the point perturbation acting only on states with zero angular momentum, the $\frac{(2l+1)(2l+2)}{2}$-degeneracy of the eigenvalue $E_{2l} = 2l + \frac{3}{2}$ gets lowered by one due to the emergence of the simple eigenvalue generated by the perturbation. Such a simple eigenvalue, regarded as a function of the extension parameter α, does cross the next lower unperturbed eigenvalue $E_{2l-1} = (2l-1) + \frac{3}{2}$. A suitable alternative to (3.5) could have been:

$$\frac{1}{\mu_{\tilde{\alpha}^{(3)}}(N)} = \tilde{\alpha}^{(3)} + \sum_{|\vec{l}|\leqslant N} \frac{\psi^2_{2\vec{l}}(0)}{2|\vec{l}| + 1} \tag{3.6}$$

leading to $E_0 = \frac{1}{2}$ at $\tilde{\alpha}^{(3)} = 0$. By setting

$$\tilde{\alpha}^{(3)} = \alpha^{(3)} - \frac{1}{2}\sum_{|\vec{l}|=0}^{\infty} \frac{\psi^2_{2\vec{l}}(0)}{(2|\vec{l}|+1)\left(2|\vec{l}| + \frac{3}{2}\right)} = \alpha^{(3)} - \alpha_0^{(3)}, \tag{3.7}$$

then:

$$\frac{1}{\mu_{\alpha^{(3)}}(N)} - \frac{1}{\mu_{\tilde{\alpha}^{(3)}}(N)} = \frac{1}{2}\sum_{|\vec{l}|=N+1}^{\infty} \frac{\psi^2_{2\vec{l}}(0)}{(2|\vec{l}|+1)\left(2|\vec{l}| + \frac{3}{2}\right)} \underset{N\to\infty}{\longrightarrow} 0. \tag{3.8}$$

The numerical value of the 3D counterpart $\alpha_0^{(3)}$ of α_0 can be easily computed by converting the series in (3.7) into an integral as was done in [17], that is to say:

$$\begin{aligned}\alpha_0^{(3)} &= \frac{1}{2}\sum_{|\vec{l}|=0}^{\infty} \frac{\psi^2_{2\vec{l}}(0)}{(2|\vec{l}|+1)\left(2|\vec{l}| + \frac{3}{2}\right)} = \frac{1}{\sqrt{\pi^3}}\int_0^{\infty} \frac{e^{\frac{3}{2}t}(e^{\frac{t}{2}}-1)}{(e^{2t}-1)^{3/2}}\,dt \\ &= \frac{1}{\sqrt{\pi^3}}\int_0^1 \frac{1-\xi^{\frac{1}{2}}}{(1-\xi^2)^{3/2}}\,d\xi \cong 0.107585.\end{aligned} \tag{3.9}$$

Although the graphs shown in [17] are those of the inverse function $\alpha^{(3)}(E)$, it can be gathered from equations (4.3) and (4.4) in [17] that both level crossings implied by figures 2 and 3 in that paper take place exactly at the same point $\alpha^{(3)} = \alpha_0^{(3)}$,i.e. $E(\alpha_0^{(3)}) = 5/2, E(\alpha_0^{(3)}) = 9/2$. By using instead the alternative renormalization (3.6), the location of all the level crossings would be exactly $\tilde{\alpha}^{(3)} = 0$, leading to the graph shown in figure 4(*a*) of the aforementioned paper by Brüning *et al* [6].

Going back to the 1D model with the δ'-interaction centred at the origin, if we wish to get the equations determining the higher odd eigenvalues as functions of α in integral form, we need only mimic what was done in [10, 11, 16, 17] to get the corresponding equations in the case of the harmonic oscillator perturbed by attractive δ-interactions in either one or three dimensions and even with point interactions situated away from the origin. For example, in the case of the second odd eigenvalue of $H(\alpha)$, namely the fourth eigenvalue $E_3(\alpha)$, by suitably transforming the integral $\int_0^1 \frac{\xi^{\frac{1}{2}-E}}{(1-\xi^2)^{1/2}}\,d\xi$ in order that (3.4) may be extended to the range $E \in \left(\frac{3}{2}, \frac{7}{2}\right)$, the bound state equation reads:

$$\alpha = \frac{2}{\sqrt{\pi}}\left[\frac{1}{2}\int_0^1 \frac{\xi^{\frac{1}{2}}}{(1-\xi^2)^{1/2}}\,\mathrm{d}\xi - \left(\frac{1}{2} - E\right)\int_0^1 \frac{\xi^{\frac{5}{2}-E}}{(1-\xi^2)^{1/2}[1+(1-\xi^2)^{1/2}]}\,\mathrm{d}\xi - \frac{\frac{1}{2}-E}{\frac{3}{2}-E}\right]. \tag{3.10}$$

The plot of the fourth eigenvalue $E_3(\alpha)$ of $H(\alpha)$ as a function of α is given in figure 7.

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