(Une invitation au) Traitement du Signal sur les Graphes

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Septembre 5, 2013

Symposium GRETSI

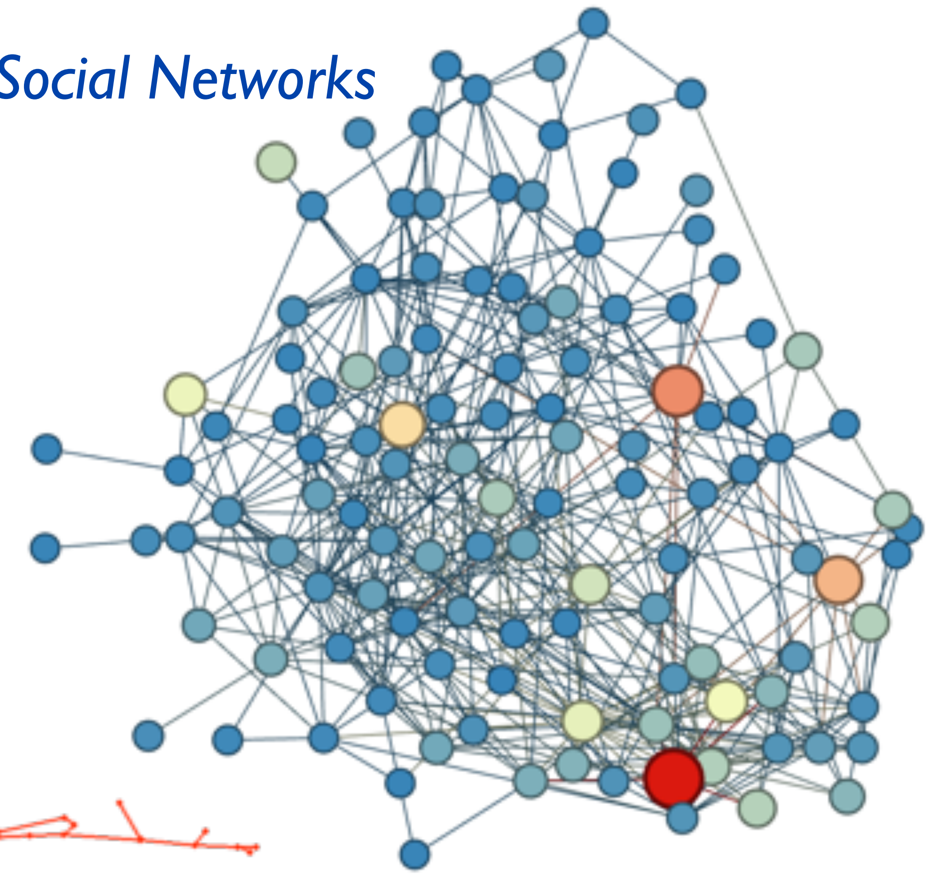
Brest, France

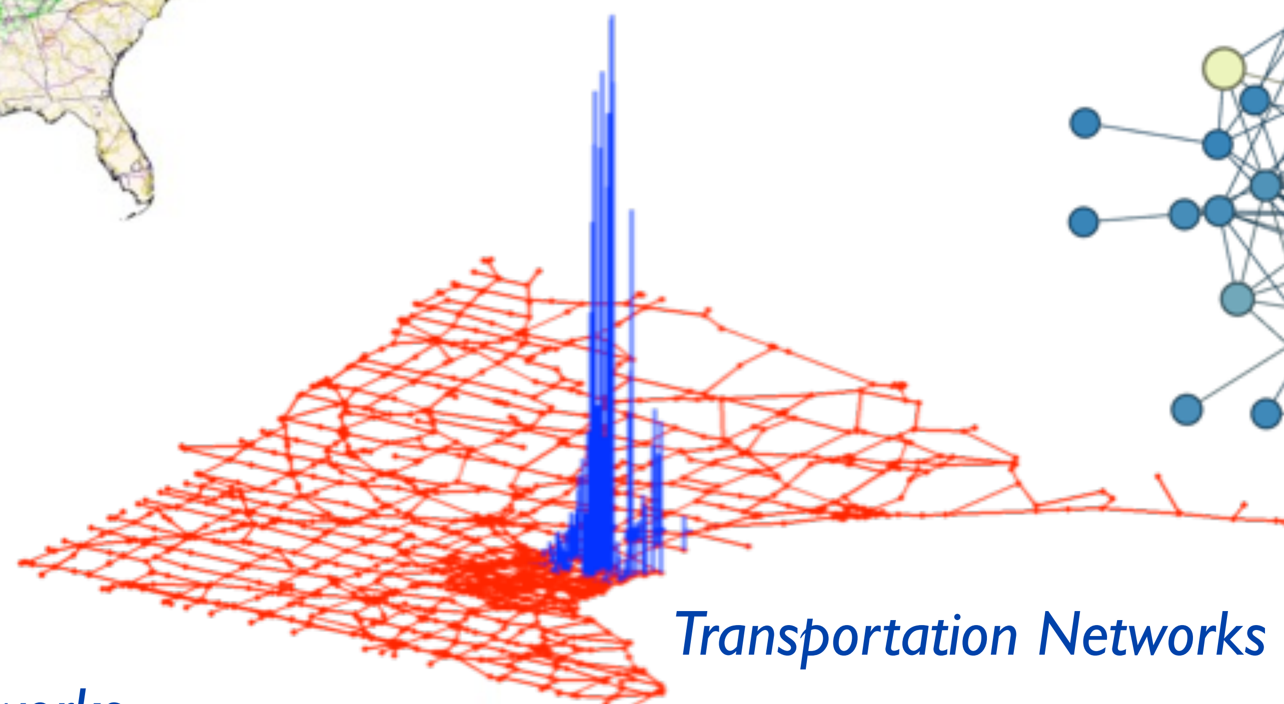
EPFL – Signal Processing Laboratory (LTS2)
http://lts2.epfl.ch

Signal Processing on Graphs

Energy Networks

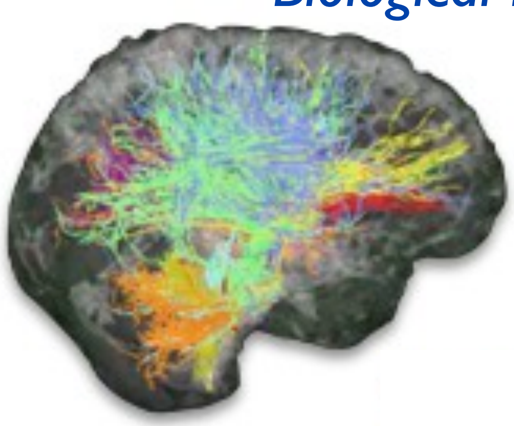
Social Networks

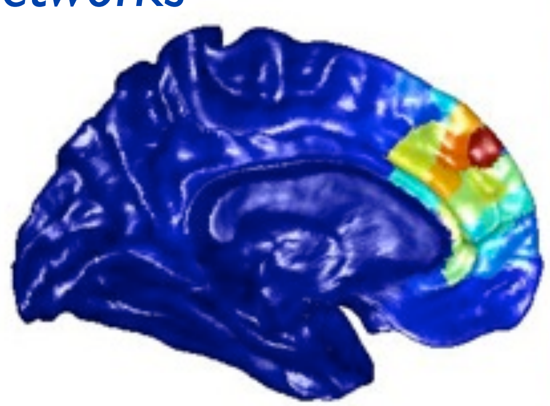




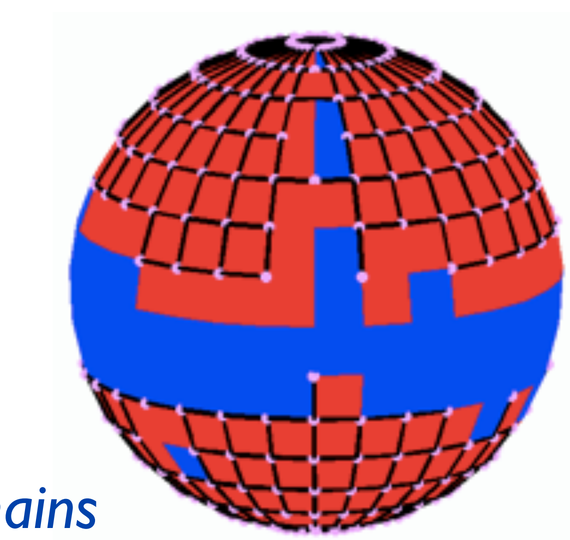
Transportation Networks

Biological Networks



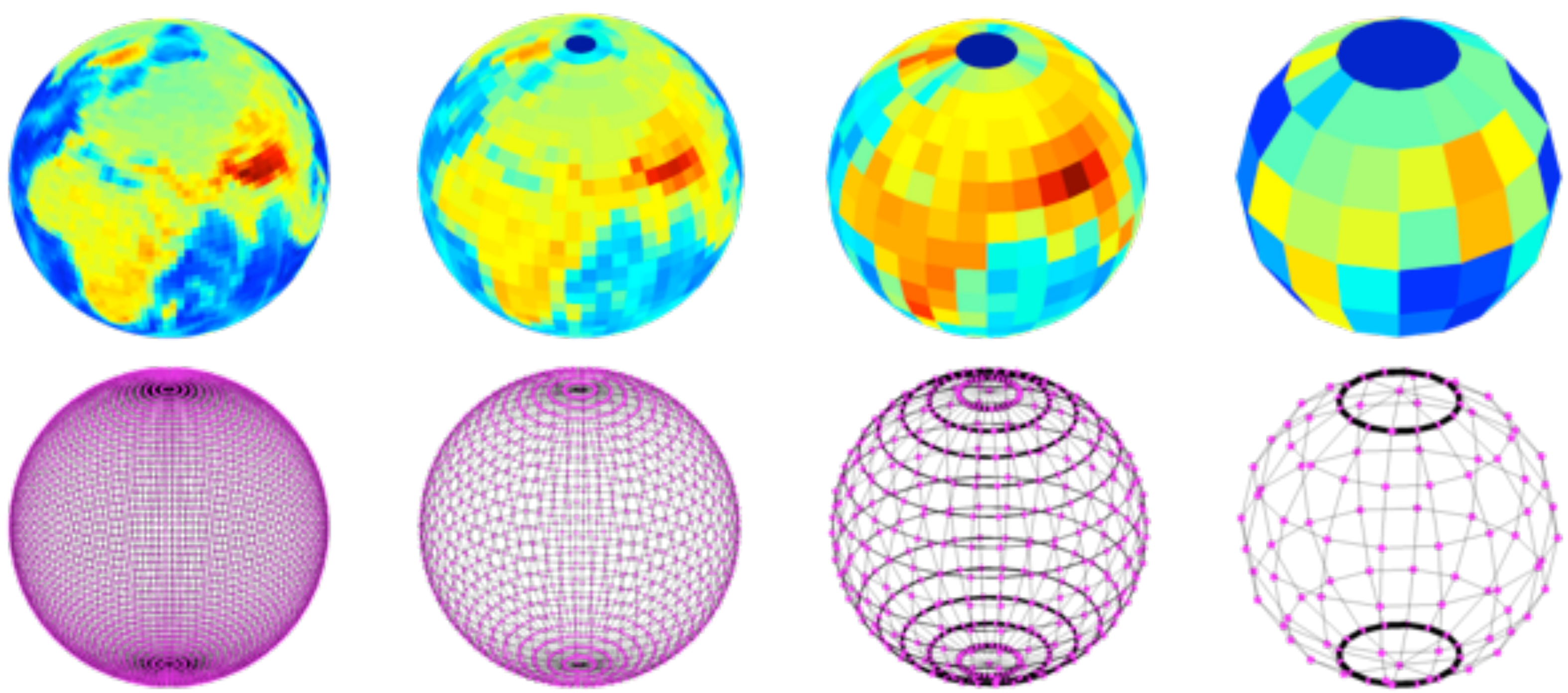


Irregular Data Domains

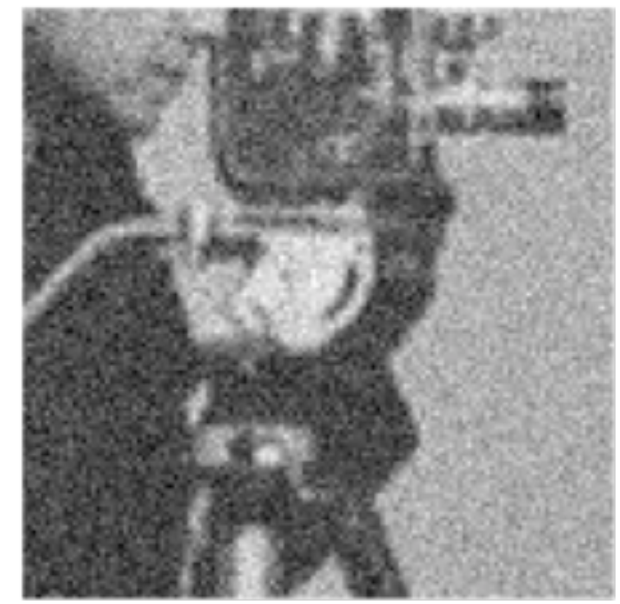


Some Typical Processing Problems

Compression / Visualization

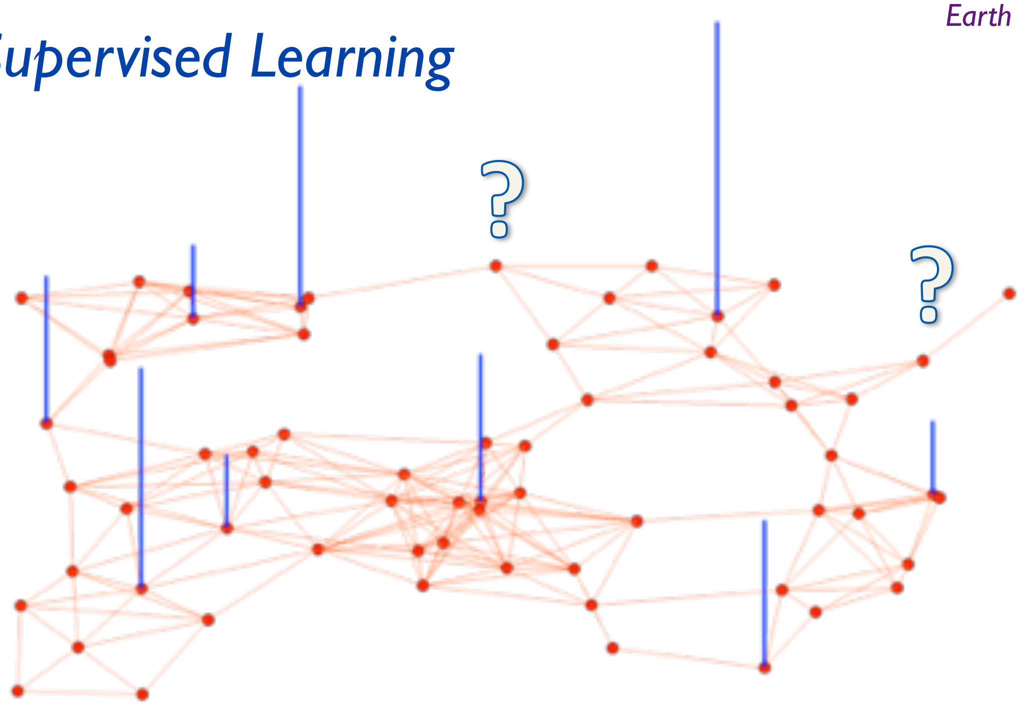


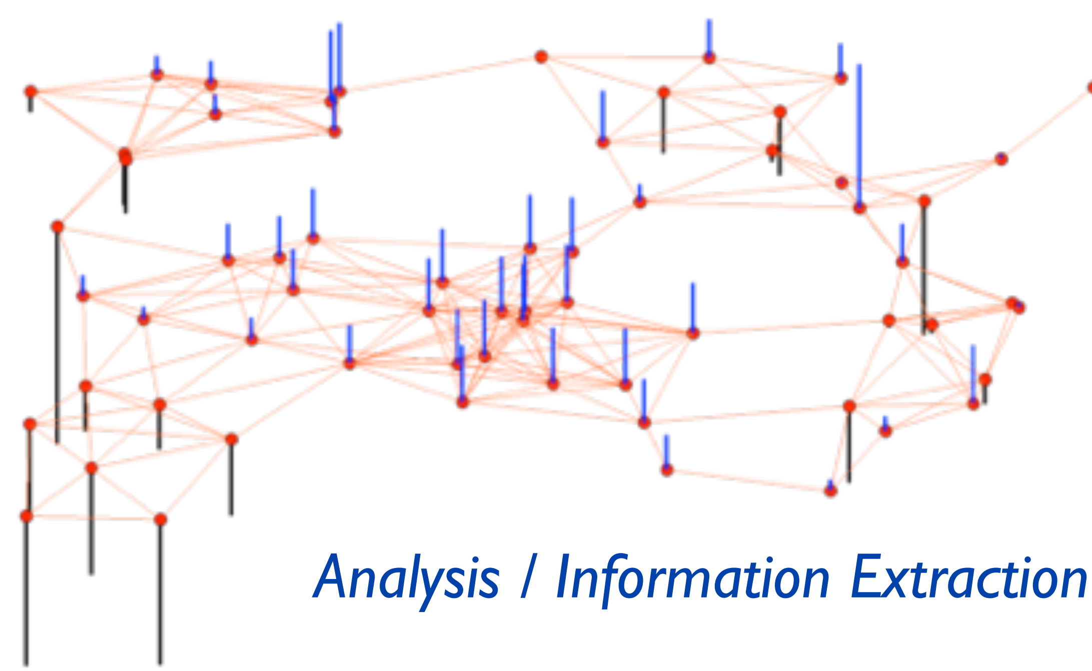
Earth data source: Frederik Simons



Denoising

Semi-Supervised Learning





Analysis / Information Extraction

Fundamentals of DSP

It seems hard to formulate a linear shift-invariant systems theory (LTI) for graphs. But we can try to get close.

The (combinatorial) Laplacian will be our main building block

$$\mathcal{L} = \mathbf{D} - \mathbf{W} \qquad \{(\lambda_\ell, \mathbf{u}_\ell)\}_{\ell=0,1,\ldots,N-1}$$

That particular ortho basis will play the role of the Fourier basis

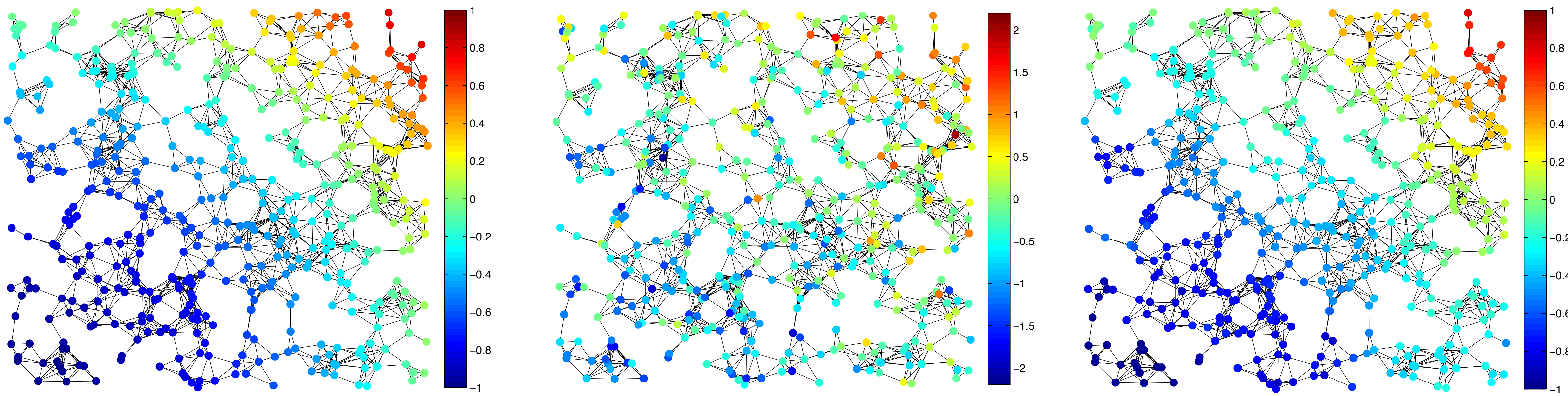
$$\hat{f}(\lambda_\ell) := \langle \mathbf{f}, \mathbf{u}_\ell \rangle = \sum_{i=1}^{N} f(i) u_\ell^*(i)$$

$$\mu := \max_{\ell,i} |\langle \mathbf{u}_\ell, \delta_i \rangle| \in \left[\frac{1}{\sqrt{N}}, 1 \right[$$

Graph Coherence

Simple Motivating Examples

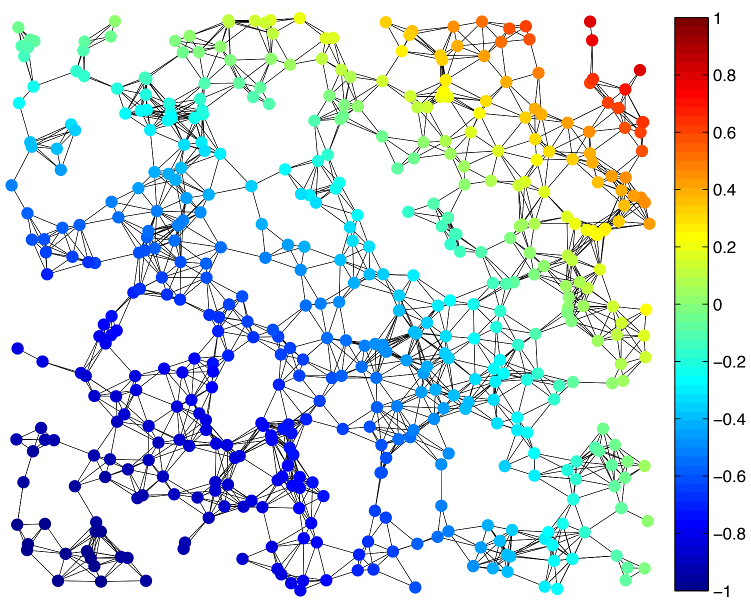
$$\operatorname{argmin}_f \left\{ ||f - y||_2^2 + \gamma f^T \mathcal{L} f \right\}$$



Original   Noisy   Denoised

Simple Motivating Examples

- Tikhonov regularization for denoising: $\text{argmin}_f \left\{ ||f - y||_2^2 + \gamma f^T \mathcal{L} f \right\}$



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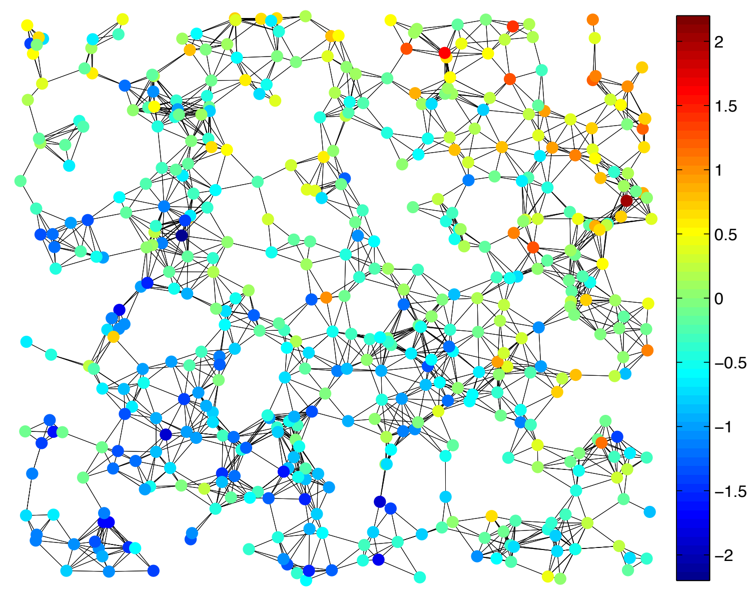

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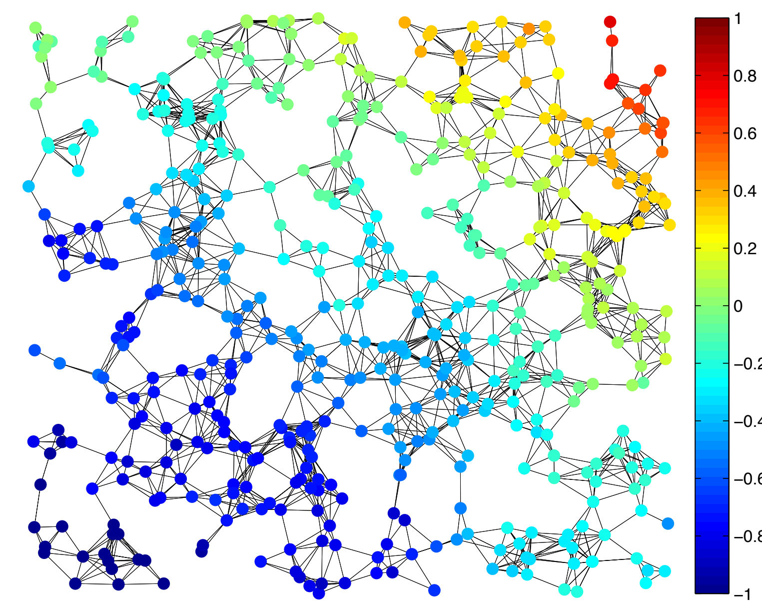

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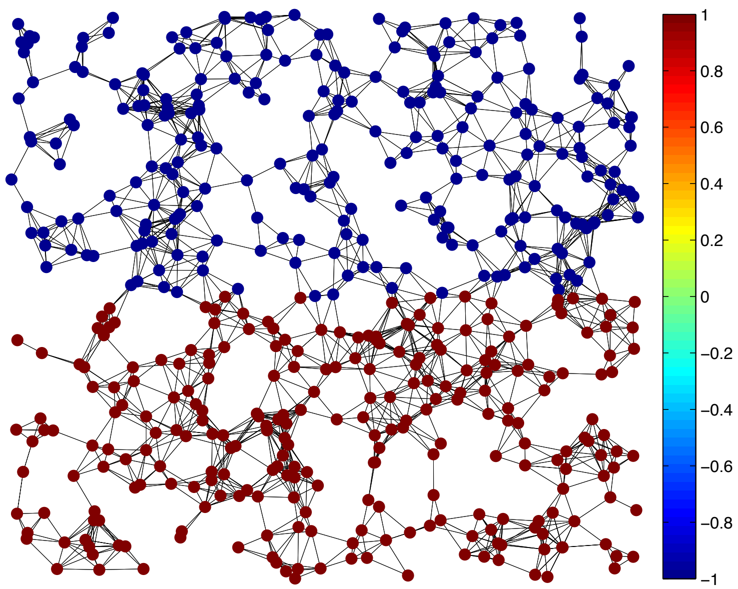

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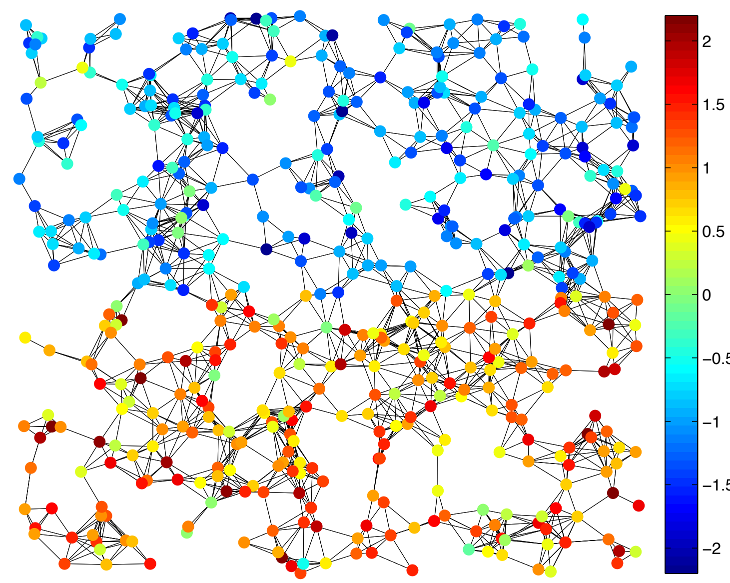

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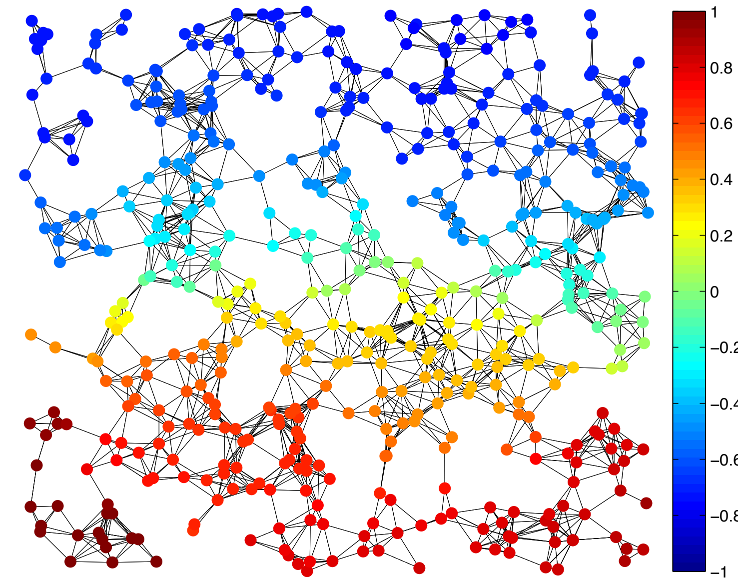

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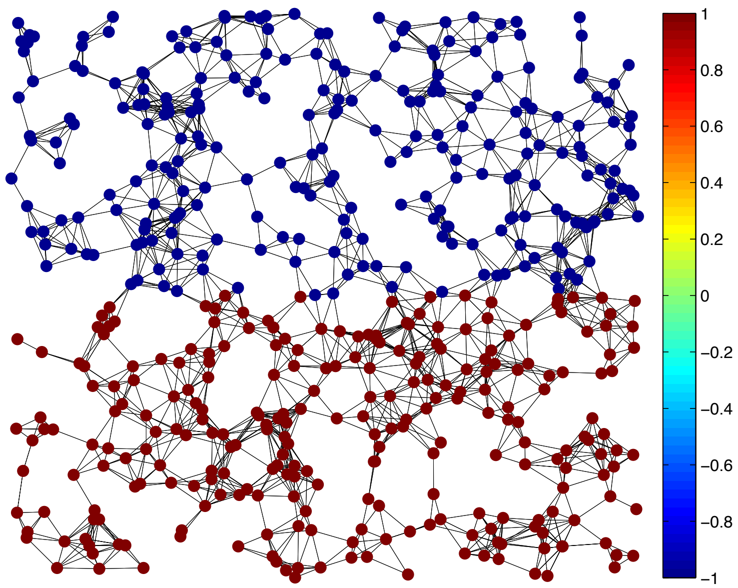

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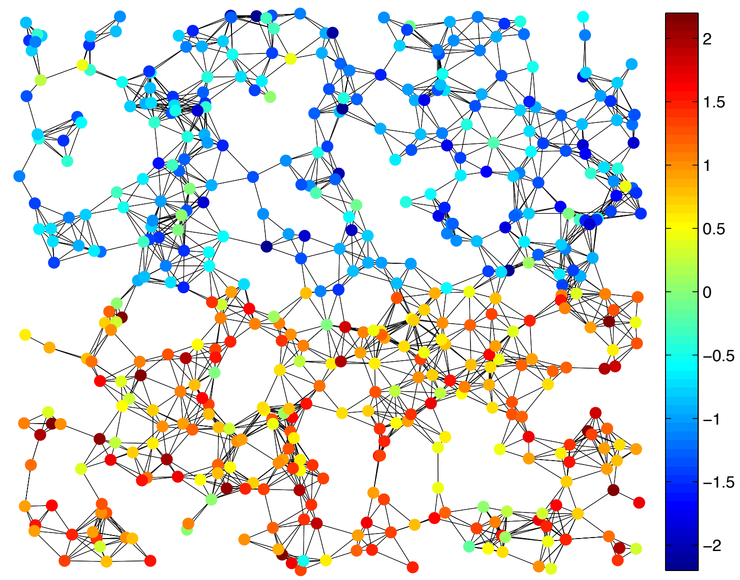

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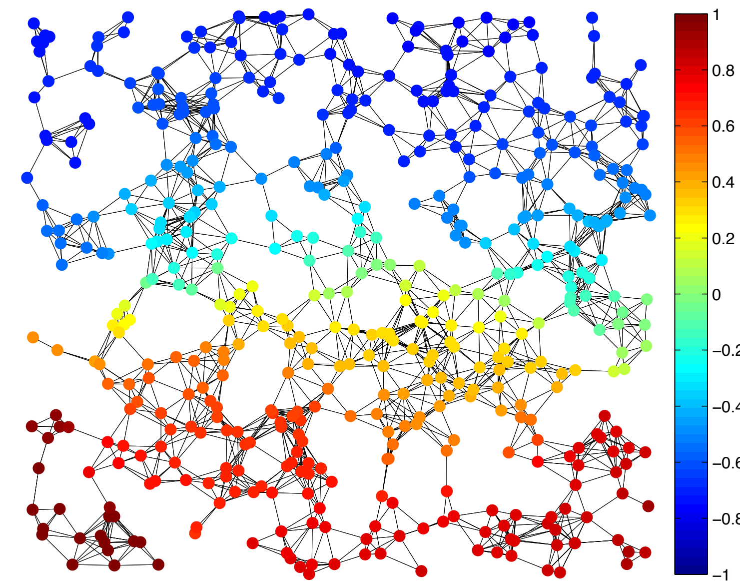
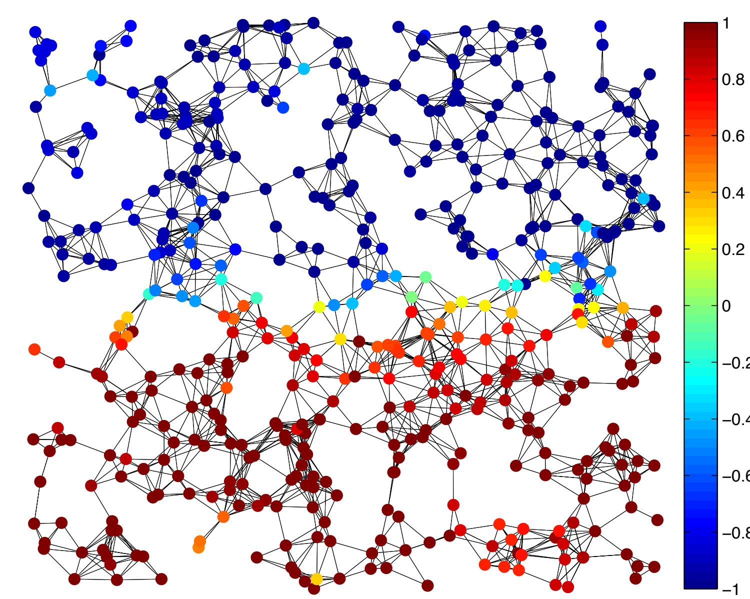


Denoised

- Wavelet denoising: $\text{argmin}_a \{||f - W^* a||_2^2 + \gamma ||a||_{1,\mu}\}$



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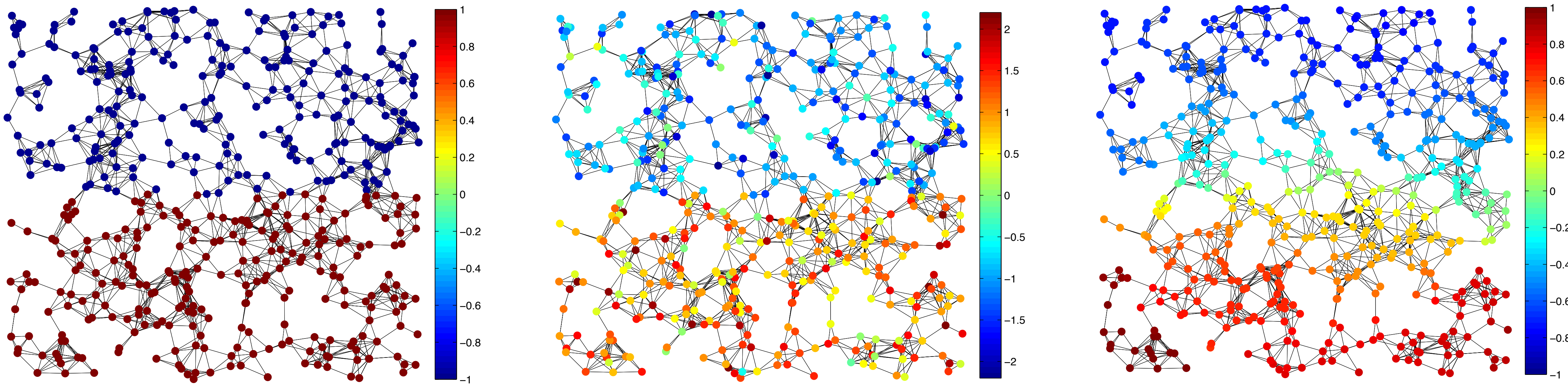

Original Noisy Denoised

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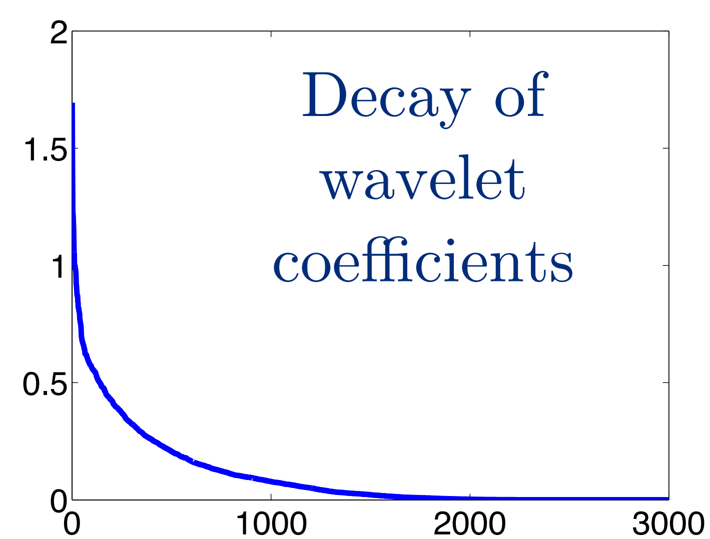

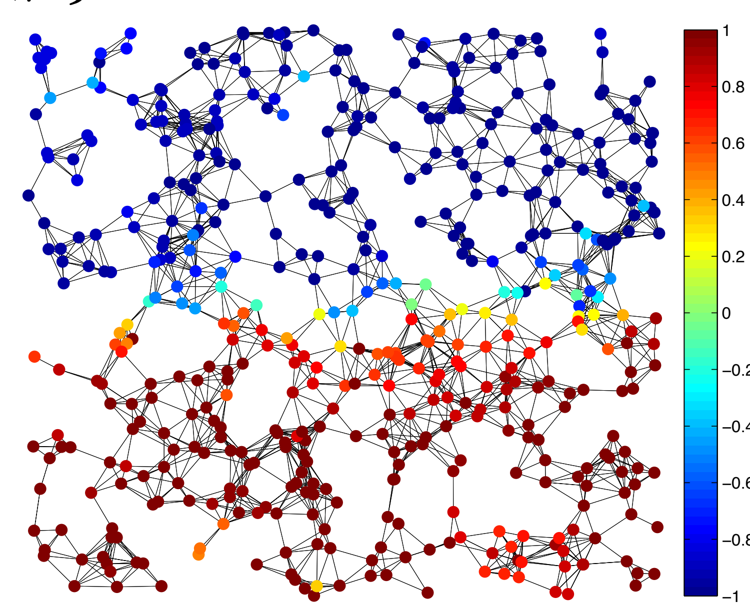

Fundamentals of DSP

Filering: $\hat{f}_{out}(\lambda_\ell) = \hat{f}_{in}(\lambda_\ell)\hat{h}(\lambda_\ell)$ $\quad f_{out}(i) = \sum_{\ell=0}^{N-1} \hat{f}_{in}(\lambda_\ell)\hat{h}(\lambda_\ell)u_\ell(i)$

$$\underset{f}{\operatorname{argmin}} \frac{\tau}{2}\|f - y\|_2^2 + f^{\mathrm{T}}\mathcal{L}^r f$$

See also:
- Smola and Kondor, Kernels and Regularization on Graphs, 2003
- Coifman and Maggioni, Diffusion Wavelets, ACHA, 2006

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Graph Fourier

$$\Rightarrow \quad \widehat{\mathcal{L}^r f_*}(\ell) + \tfrac{\tau}{2}\left(\widehat{f_*}(\ell) - \hat{y}(\ell)\right) = 0, \quad \forall \ell \in \{0, 1, \ldots, N-1\}$$

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$$\Rightarrow \quad \widehat{f_*}(\ell) = \frac{\tau}{\tau + 2\lambda_\ell^r}\hat{y}(\ell)$$ "Low pass" filtering !

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Convolutions and Translations

$$(f * g)(n) := \sum_{\ell=0}^{N-1} \hat{f}(\ell)\hat{g}(\ell)u_\ell(n)$$

Inherits a lot of properties of the usual convolution

associativity, distributivity, diagonalized by GFT

$$g_0(n) := \sum_{\ell=0}^{N-1} u_\ell(n) \quad \Longrightarrow \quad f * g_0 = f$$

$$\mathcal{L}(f * g) = (\mathcal{L}f) * g = f * (\mathcal{L}g)$$

Use convolution to induce translations

$$(T_i f)(n) := \sqrt{N}(f * \delta_i)(n) = \sqrt{N} \sum_{\ell=0}^{N-1} \hat{f}(\ell)u_\ell^*(i)u_\ell(n)$$

Spectral Graph Wavelets

- Hammond et al., Wavelets on graphs via spectral graph theory, ACHA, 2011

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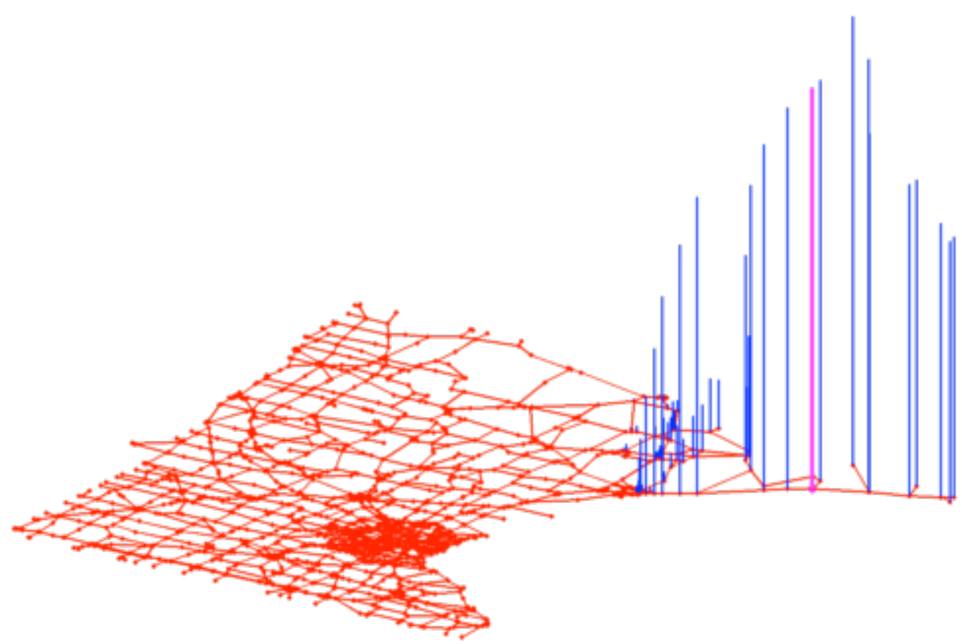

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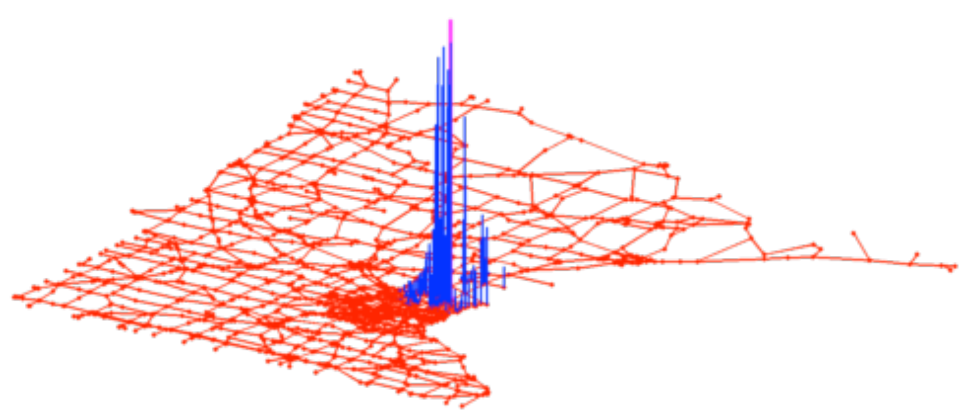

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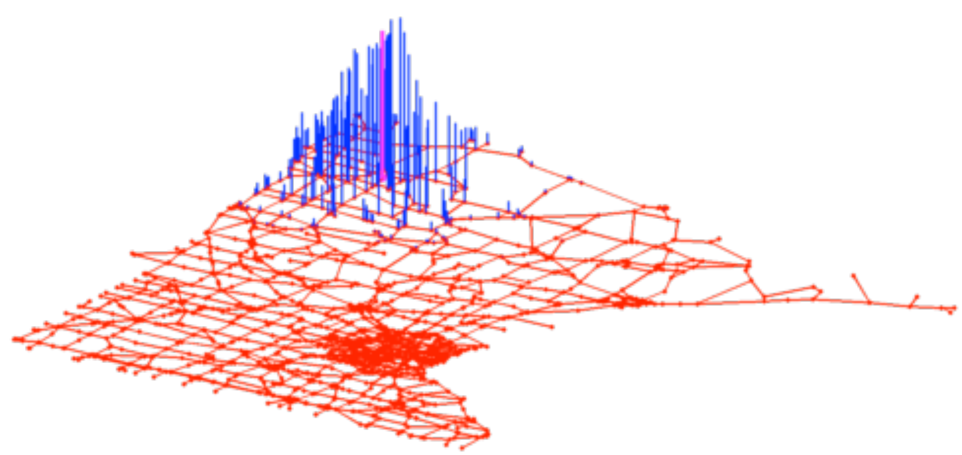

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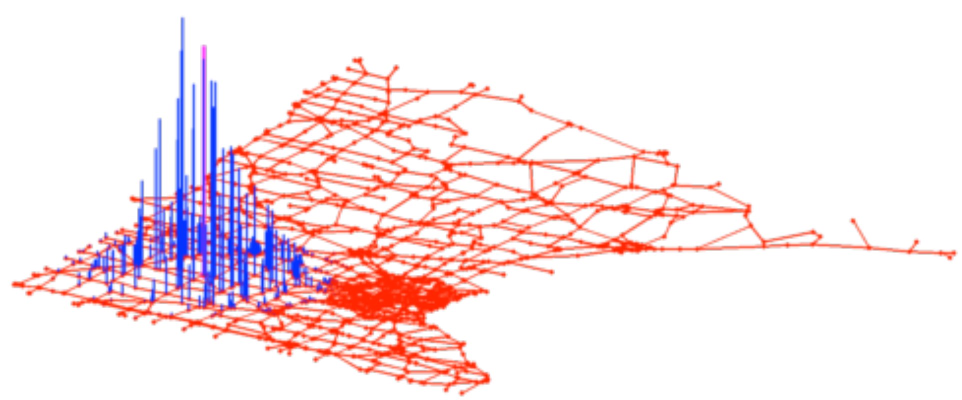

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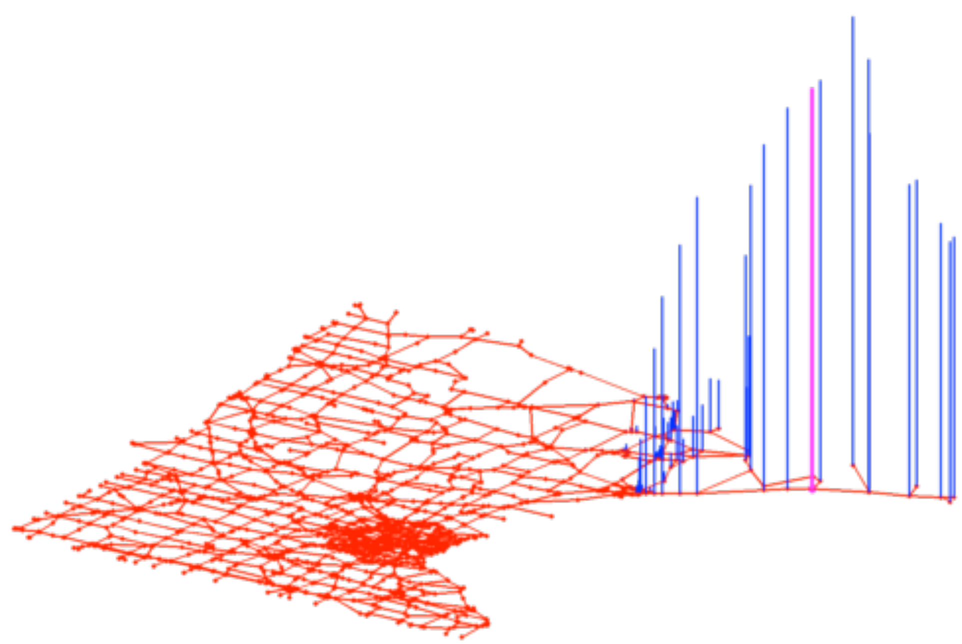

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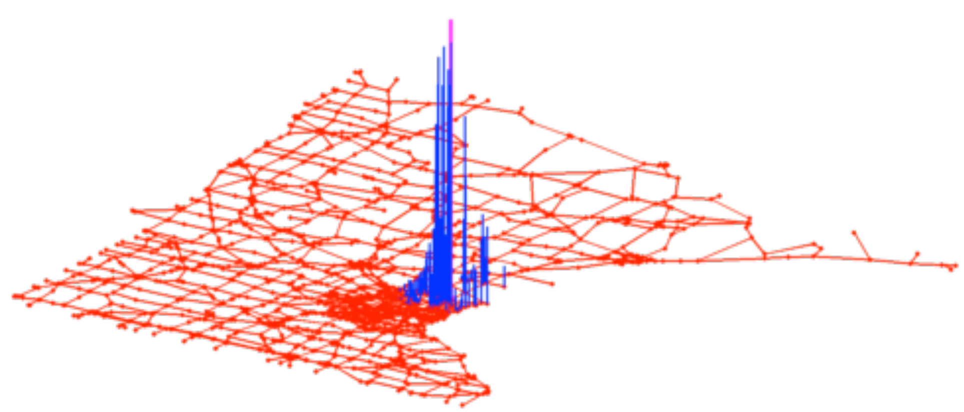

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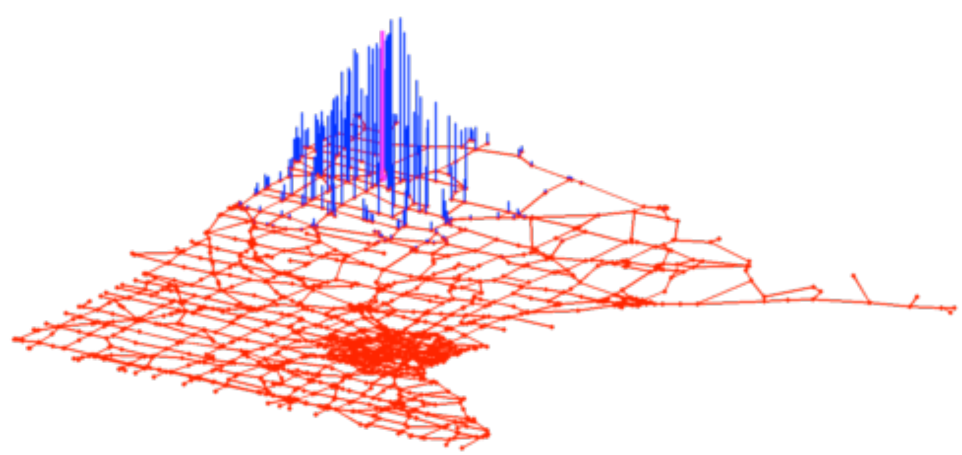

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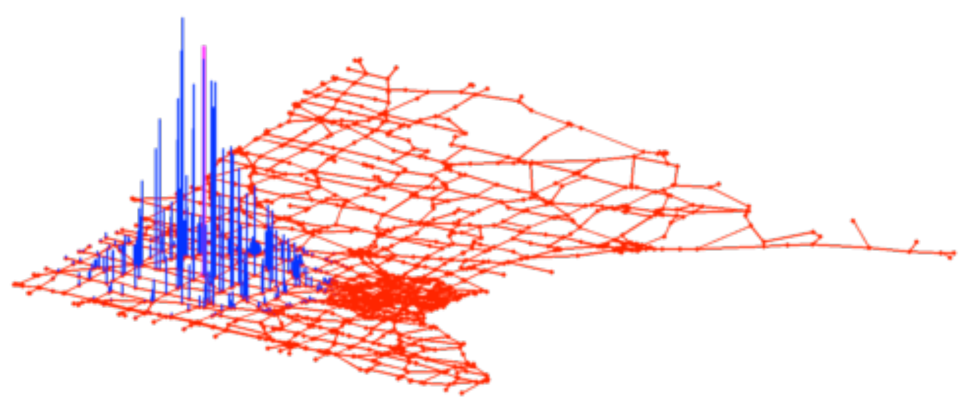

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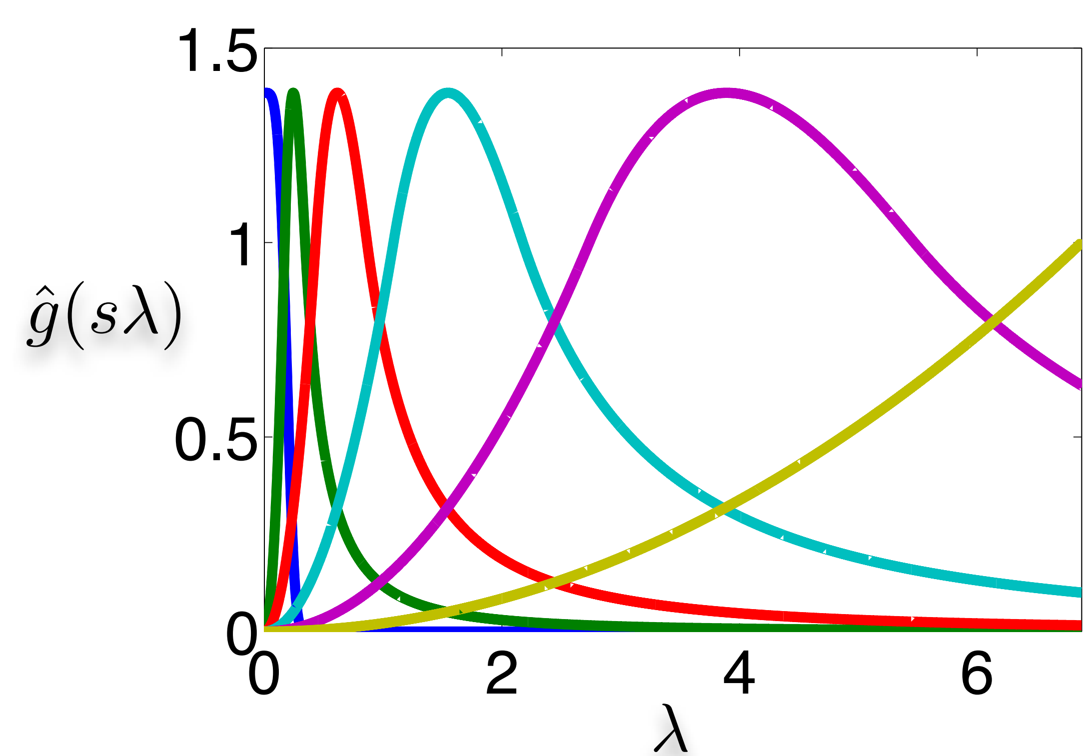

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- Spectral graph wavelet at scale s, centered at vertex n:

$$\psi_{s,n}(i) := (T_n D_s g)(i) = \sum_{\ell=0}^{N=1} \hat{g}(s\lambda_\ell) u_\ell^*(n) u_\ell(i)$$

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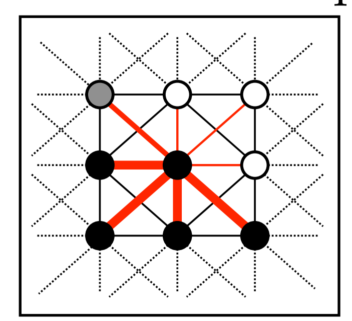
Semi-Local Graph



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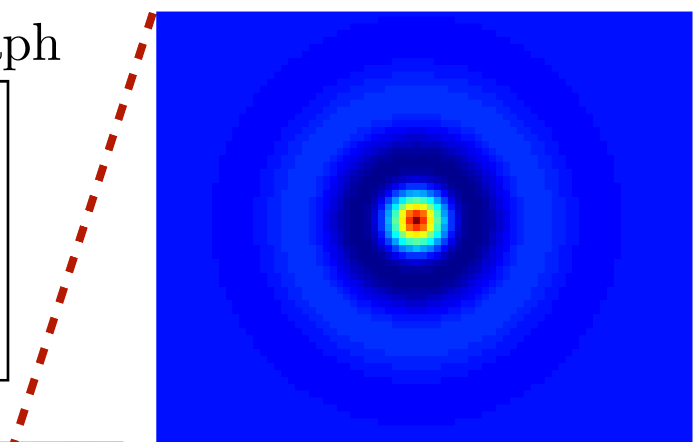

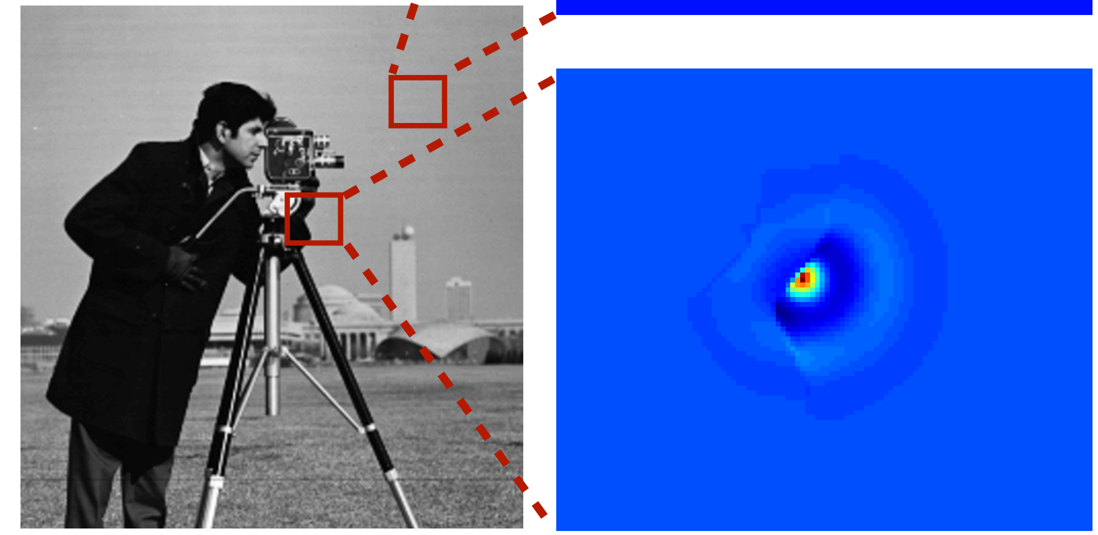

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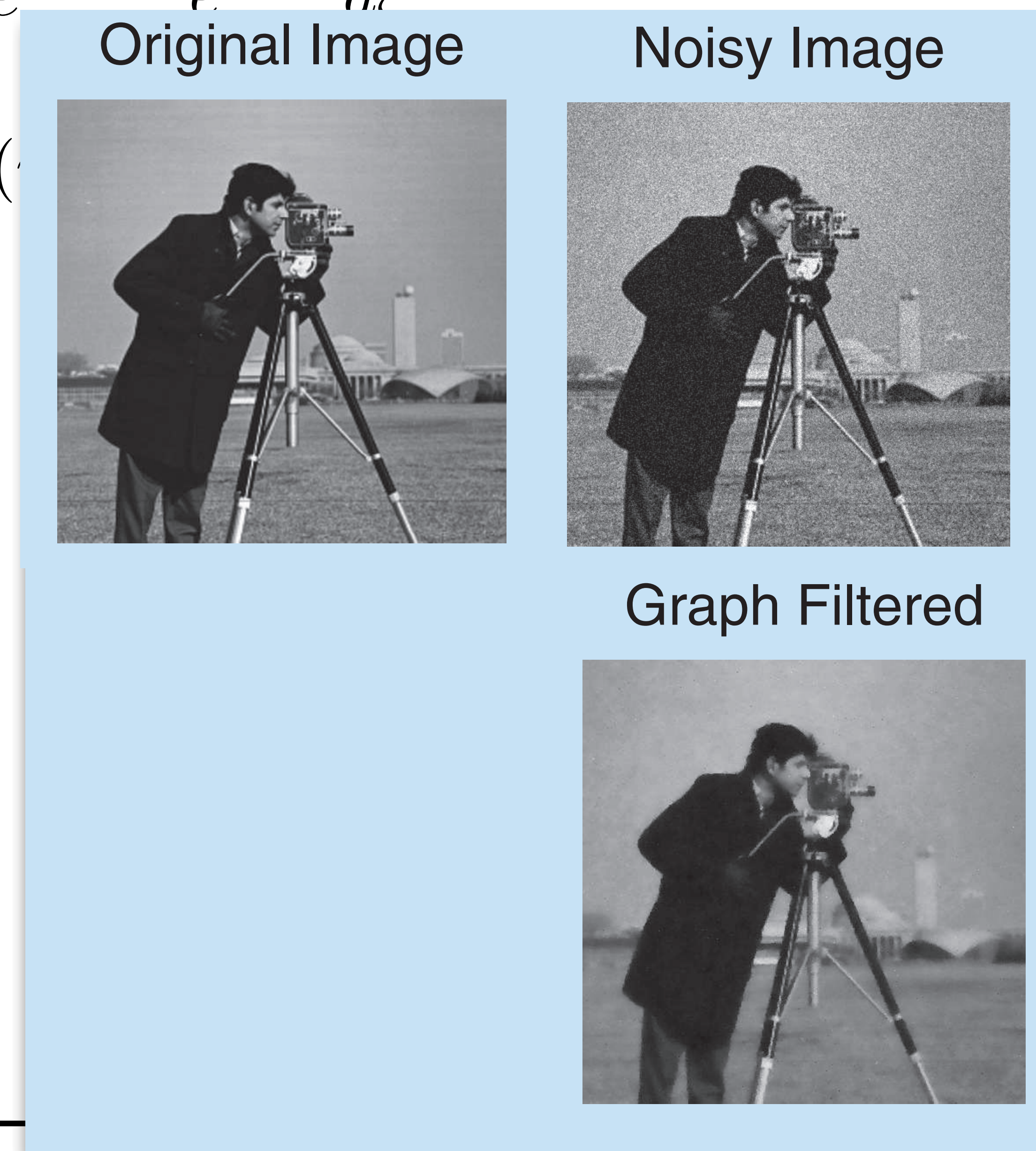

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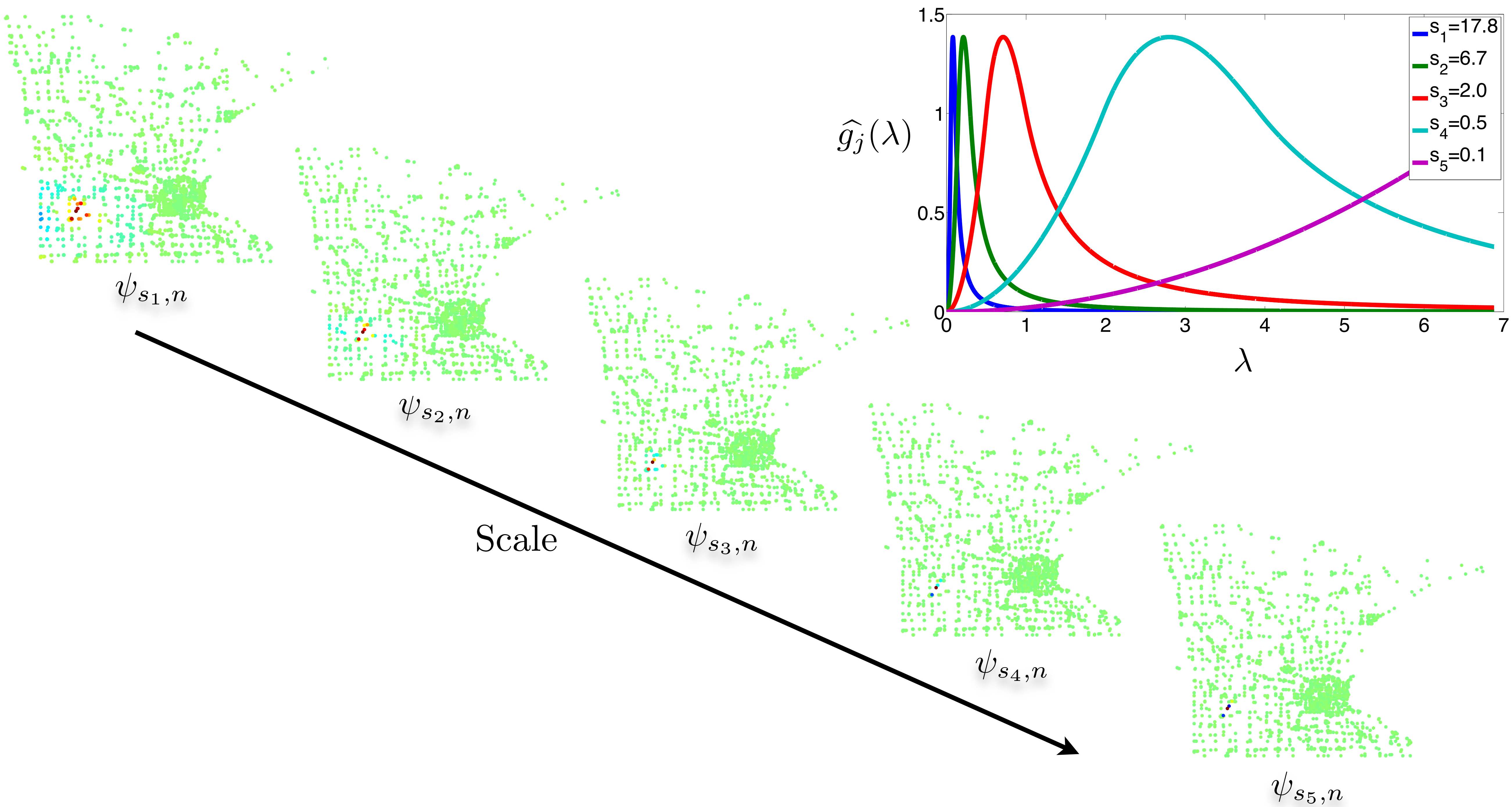
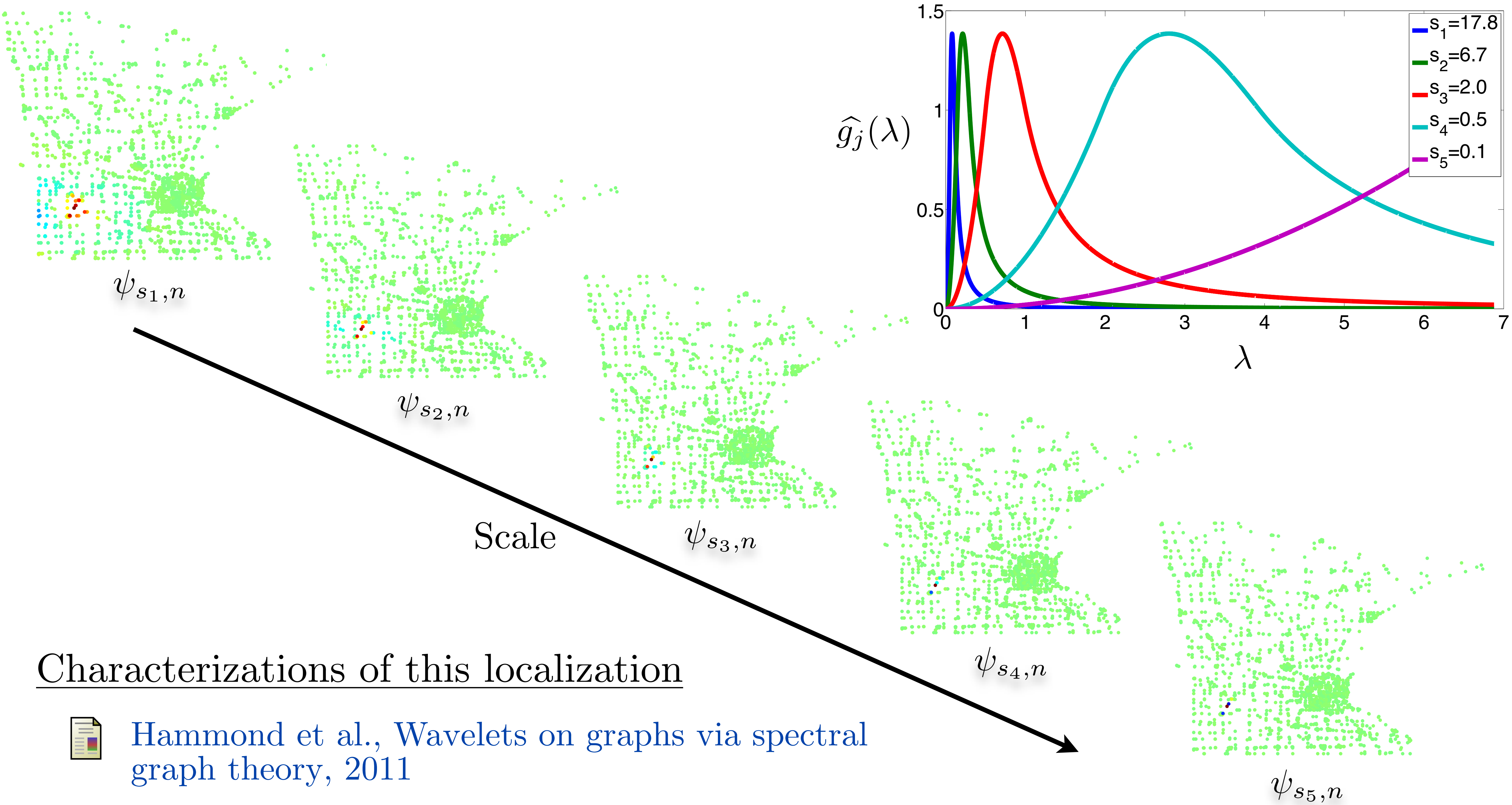


Spectral Graph Wavelet Localization



Characterizations of this localization

- Hammond et al., Wavelets on graphs via spectral graph theory, 2011
- Shuman et al., Vertex-frequency analysis on graphs, 2013

Polynomial Localization

Given a spectral kernel g, construct the family of features:

$$\phi_n(m) = \big(T_n g\big)(m) \qquad \phi_n(m) = \sqrt{N} \sum_{\ell=0}^{N-1} \hat{g}(\ell) u_\ell^*(m) u_\ell(n)$$

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$$\phi'_n(m) = \langle \delta_m, P_K(\mathcal{L}) \delta_n \rangle$$

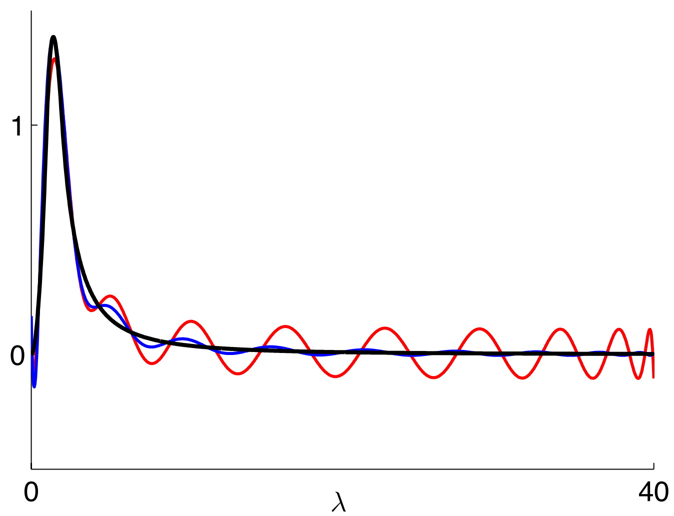
Exactly localized in a K-ball around n

Remark on Implementation

Not necessary to compute spectral decomposition for filtering

Polynomial approximation : $$g(t\omega) \simeq \sum_{k=0}^{K-1} a_k(t) p_k(\omega)$$

ex: Chebyshev, minimax



$$\tilde{W}_f(t_n, j) = \left(\frac{1}{2} c_{n,0} f^{\#} + \sum_{k=1}^{M_n} c_{n,k} \overline{T}_k(\mathcal{L}) f^{\#} \right)_j$$

$$\overline{T}_k(\mathcal{L}) f = \frac{2}{a_1}(\mathcal{L} - a_2 I)\left(\overline{T}_{k-1}(\mathcal{L}) f\right) - \overline{T}_{k-2}(\mathcal{L}) f$$

Computational cost dominated by matrix-vector multiply with (sparse) Laplacian matrix. In particular $O(\sum_{n=1}^{J} M_n |E|)$

http://wiki.epfl.ch/sgwt

LTS EPFL LTS2 ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Uncertainty & Ambiguity

The joint time-frequency localization can be studied via the cross-ambiguity function:

$$A_g f(m,k) = \langle f, M_k T_m g \rangle = \sum_{n=1}^{N} f[n]\bar{g}[n-m]e^{-2\pi i k \frac{n}{N}}$$

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localization in time domain
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localization in time domain via translation

localization in frequency domain via modulation

Uncertainty is a statement about the localization of the ambiguity function

$$\forall f, g \in \mathbb{R}^N \qquad \frac{\|A_g f\|_1}{\|A_g f\|_\infty} \geqslant N.$$

Lieb 1990, Feichtinger et al 2012

Cross-Ambiguity on Graphs

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We will first need a suitable Graph Windowed Fourier Transform (GWFT)
- Translation/Localization

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Spectral localization via generalized modulation ?

Hint: $(M_k u_0)(n) = u_k(n)$

Modulation, Spectral Localization

If a kernel is sufficiently localized around the DC component

$$\sqrt{N}\sum_{l=1}^{N-1}\mu_l|\hat{f}(l)| \leq \frac{|\hat{f}(0)|}{1+\kappa}$$

for some $\kappa > 0$

$$\mu_\ell := \|\mathbf{u}_\ell\|_\infty = \max_i |u_\ell(i)|$$

Then the modulated kernel "peaks" at the right spectral index

$$|\widehat{M_k f}(k)| \geq \kappa|\widehat{M_k f}(\ell)| \ \textit{for all } \ell \neq k$$

Shuman, Ricaud, VDG, Vertex-frequency analysis on graphs, 2013

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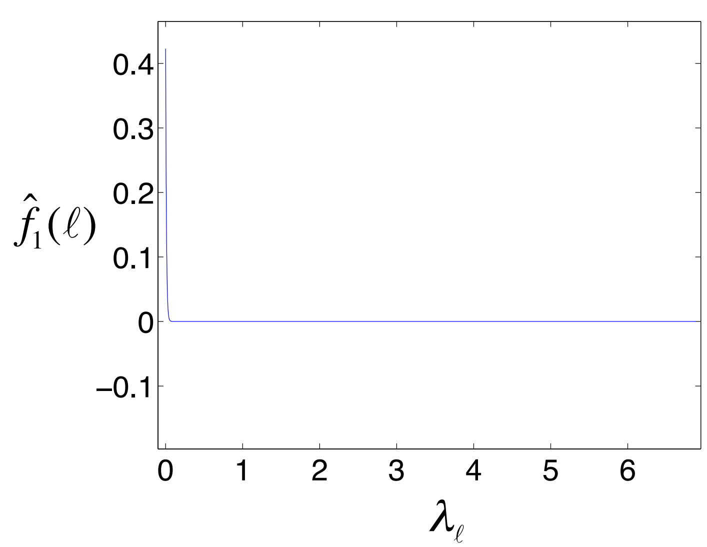

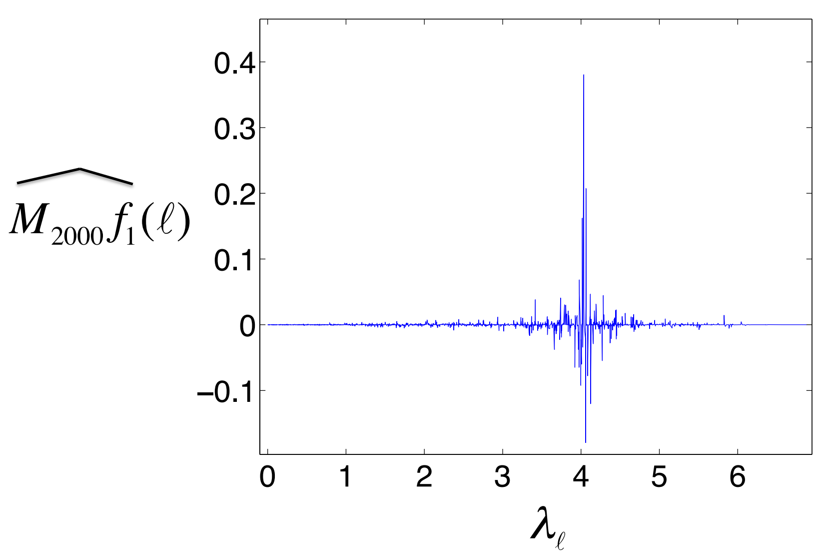

Gabor-like Frames on Graphs

$$g_{i,k}(n) := (M_k T_i g)(n) = \sqrt{N} u_k(n) \sum_{\ell=0}^{N-1} \hat{g}(\ell) u_\ell^*(i) u_\ell(n)$$

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$$A\|f\|_2^2 \leq \sum_{i=1}^{N} \sum_{k=0}^{N-1} |\langle f, g_{i,k} \rangle|^2 \leq B\|f\|_2^2$$

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$\mu \to \frac{1}{\sqrt{N}}$ Tight Frame

Shuman, Ricaud, VDG, Vertex-frequency analysis on graphs, 2013

Ambiguity and Uncertainty

Pick up a *nice* kernel $|\hat{g}(0)| \geqslant |\hat{g}(l)| \geqslant 0 \; for \; l = 1, 2, ... N-1$

Rem: The heat kernel is a good choice

$$\big(T_i g\big)(n) \geq 0$$

$$\frac{\|A_g f\|_1}{\|A_g f\|_\infty} \geq \frac{1}{\mu^2} \qquad \mu \rightarrow \frac{1}{\sqrt{N}}$$

Result of Feichtinger et al.

smaller coherence, bigger uncertainty

Perraudin, Shuman, VDG, 2013

A Laplacian Pyramid on Graphs

A Laplacian Pyramid on Graphs

Single level pyramid

Filtering

Downsampling

A Laplacian Pyramid on Graphs

Single level pyramid

Filtering

Downsampling

Graph reduction

A Laplacian Pyramid on Graphs

Single level pyramid

Graph reduction

Graph sparsification

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Filtering

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Shuman, Faraji, VDG, A framework for multiscale transforms on graphs, 2013

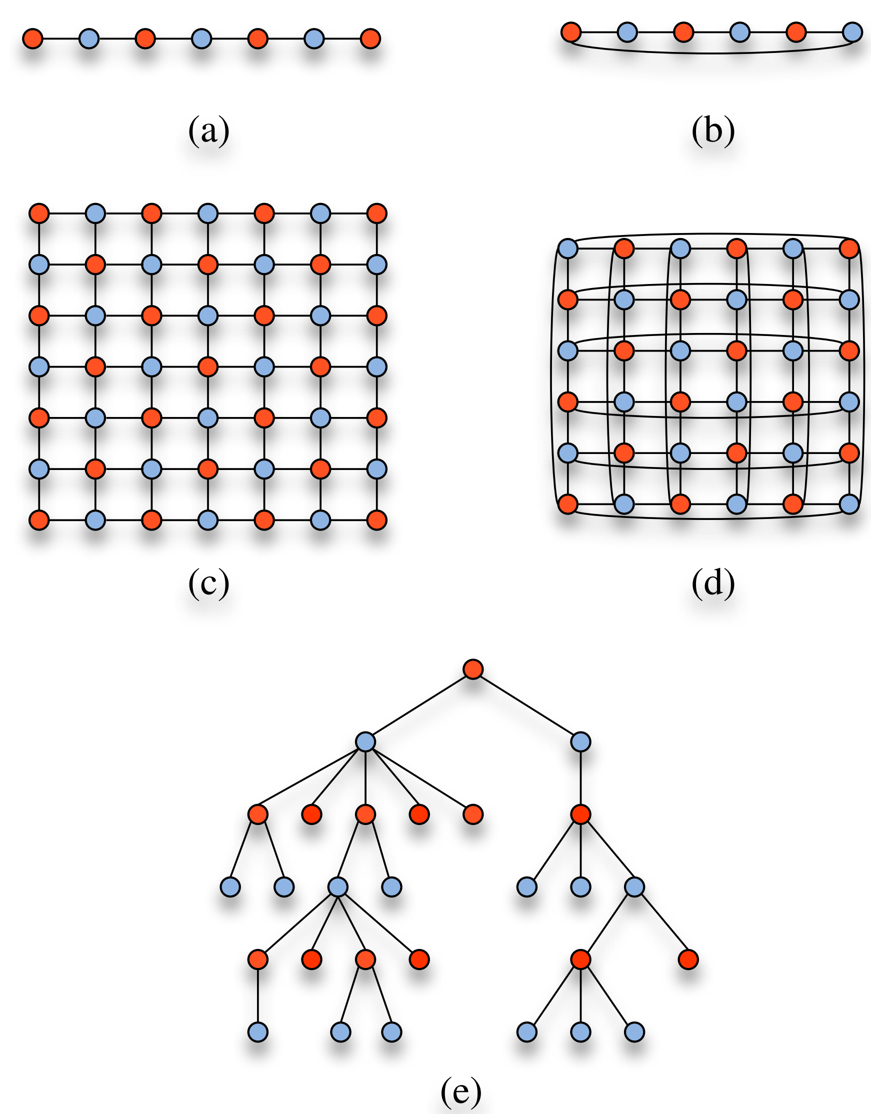
Downsampling

$$\mathcal{V}_1 = \mathcal{V}_+ := \{i \in \mathcal{V} : u_{\max}(i) \geq 0\}$$

Relaxed solution to 2-coloring for regular graphs

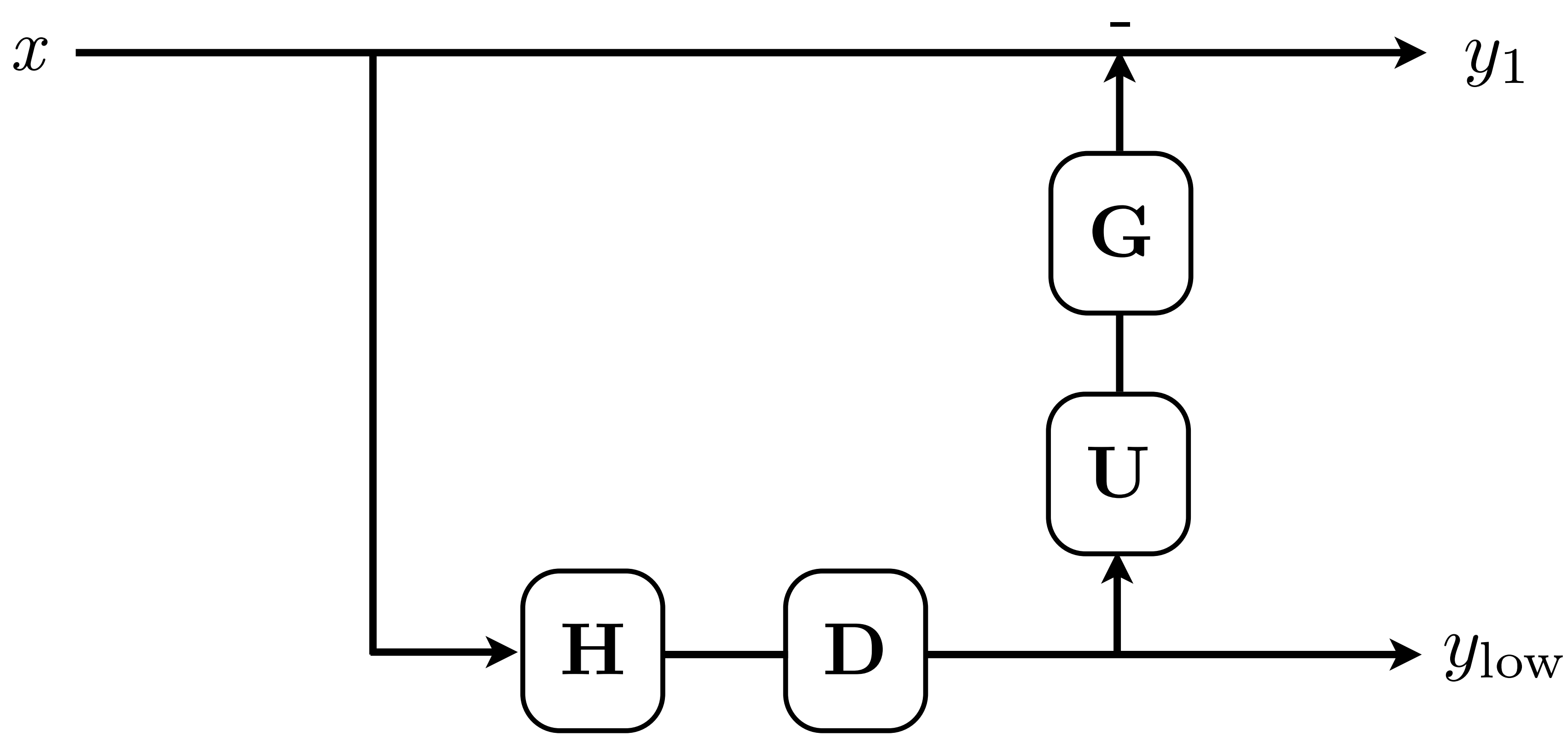
Exact for bipartite graphs

Connections with nodal domains theory for laplacian eigenvectors



The Laplacian Pyramid

Analysis operator



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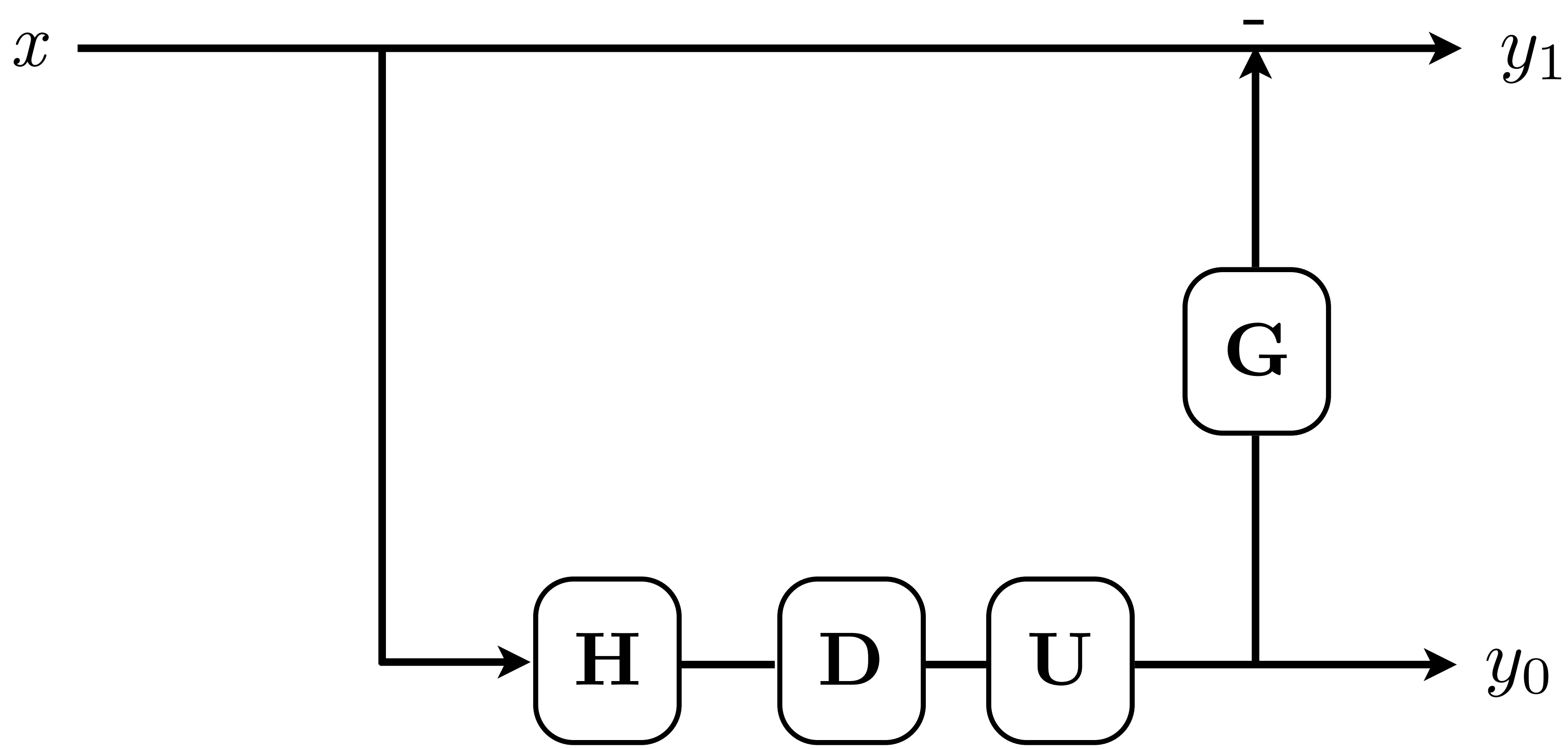

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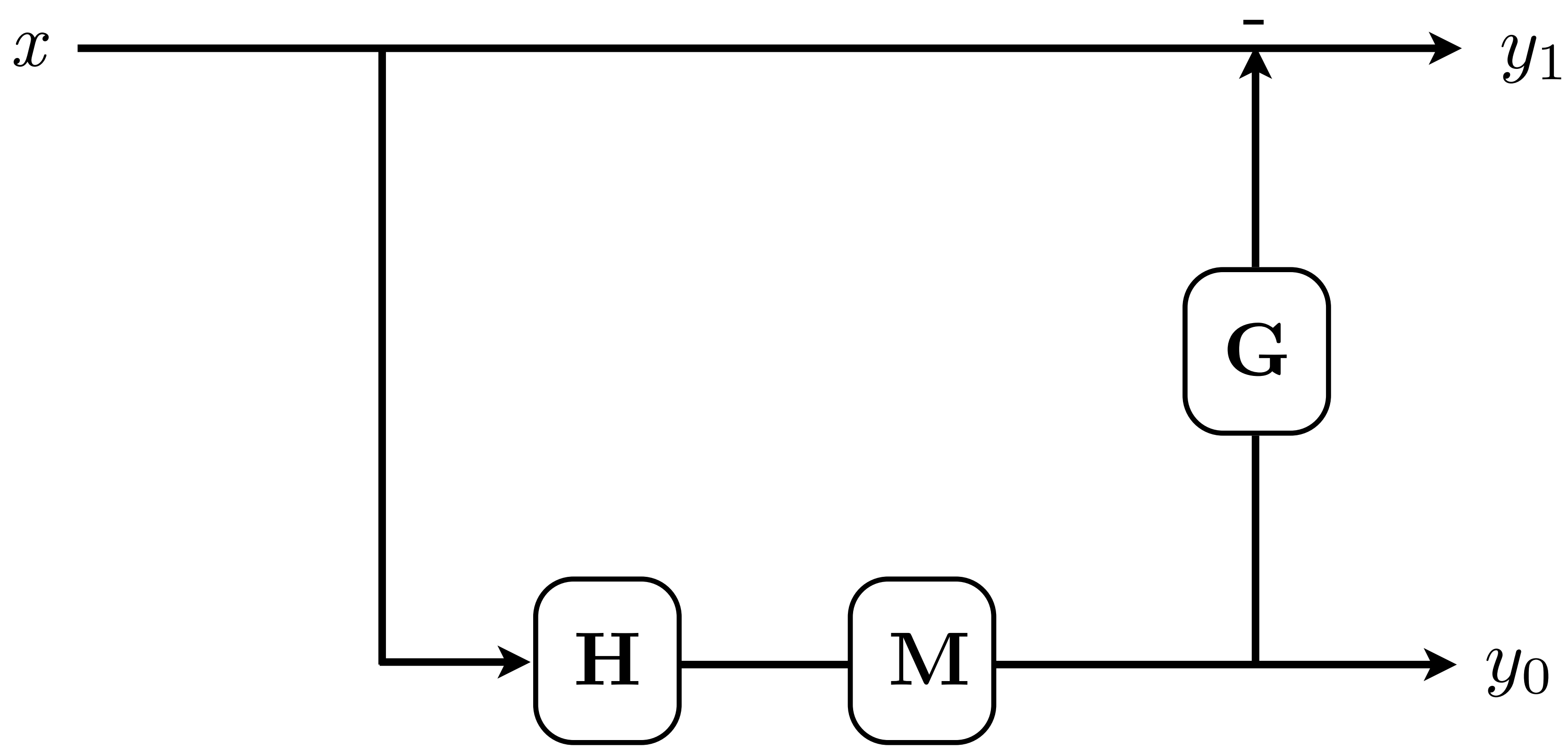

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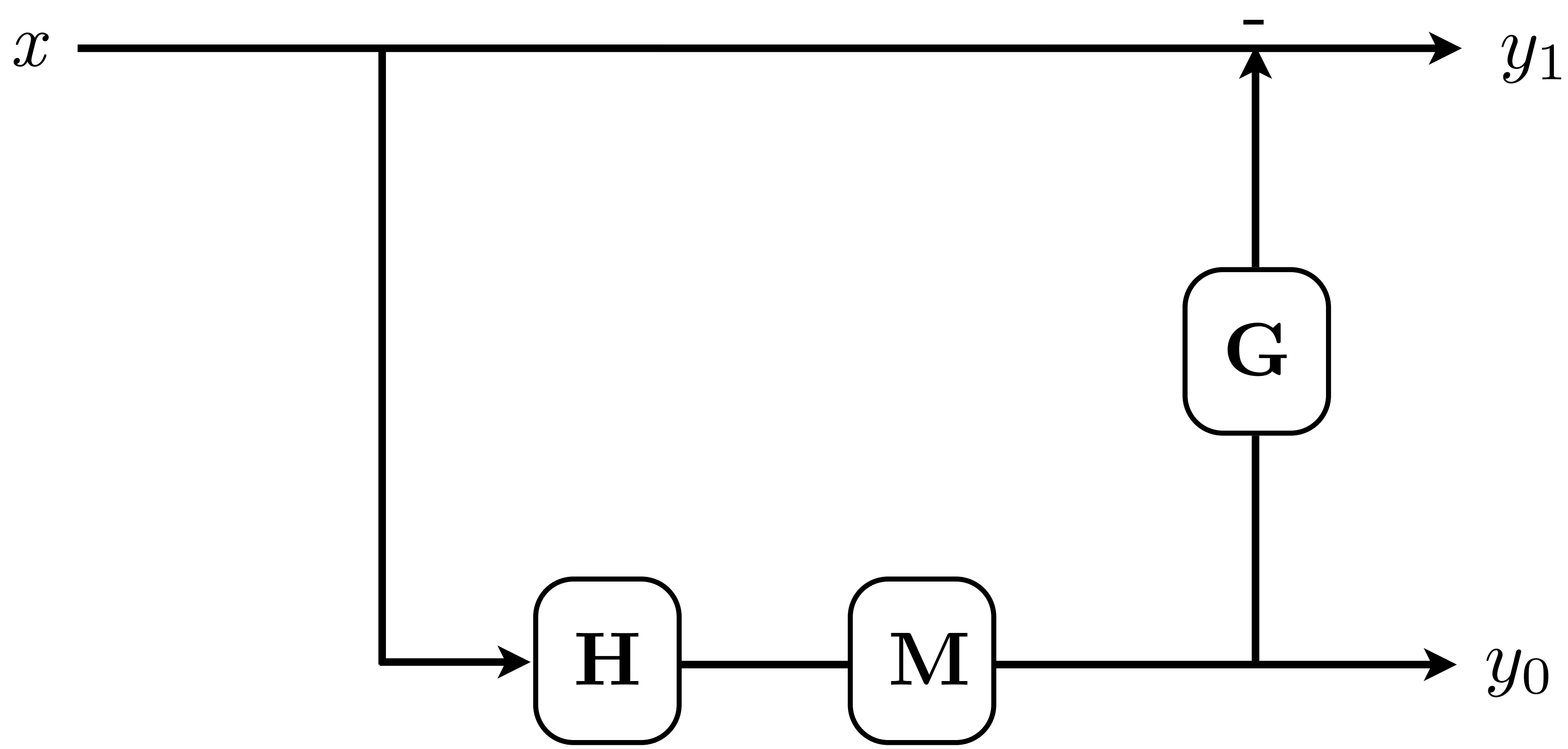
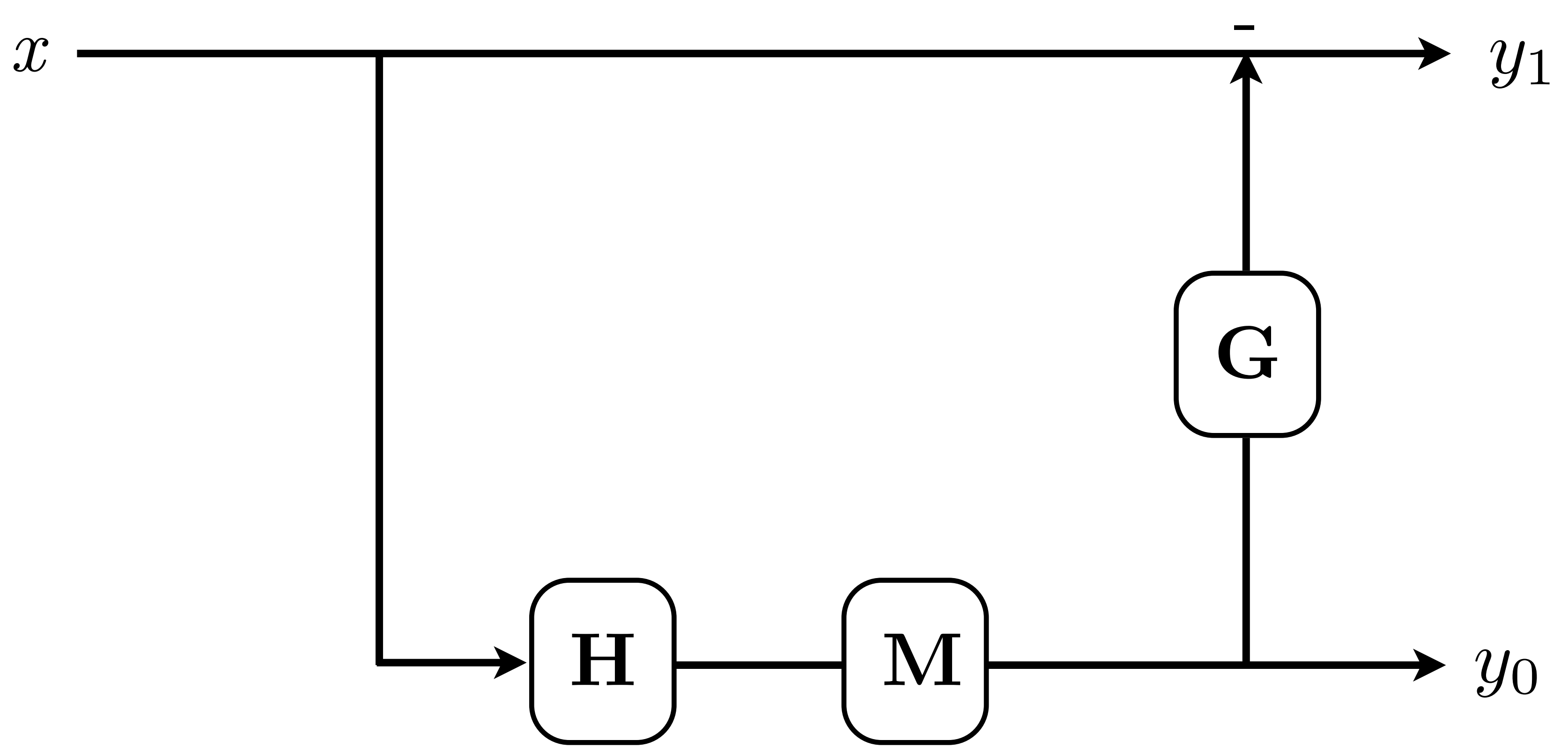


$$y_0 = \mathbf{H_m} x = \mathbf{MH} x \qquad y_1 = x - \mathbf{G} y_0 = x - \mathbf{GH_m} x$$

The Laplacian Pyramid

Analysis operator



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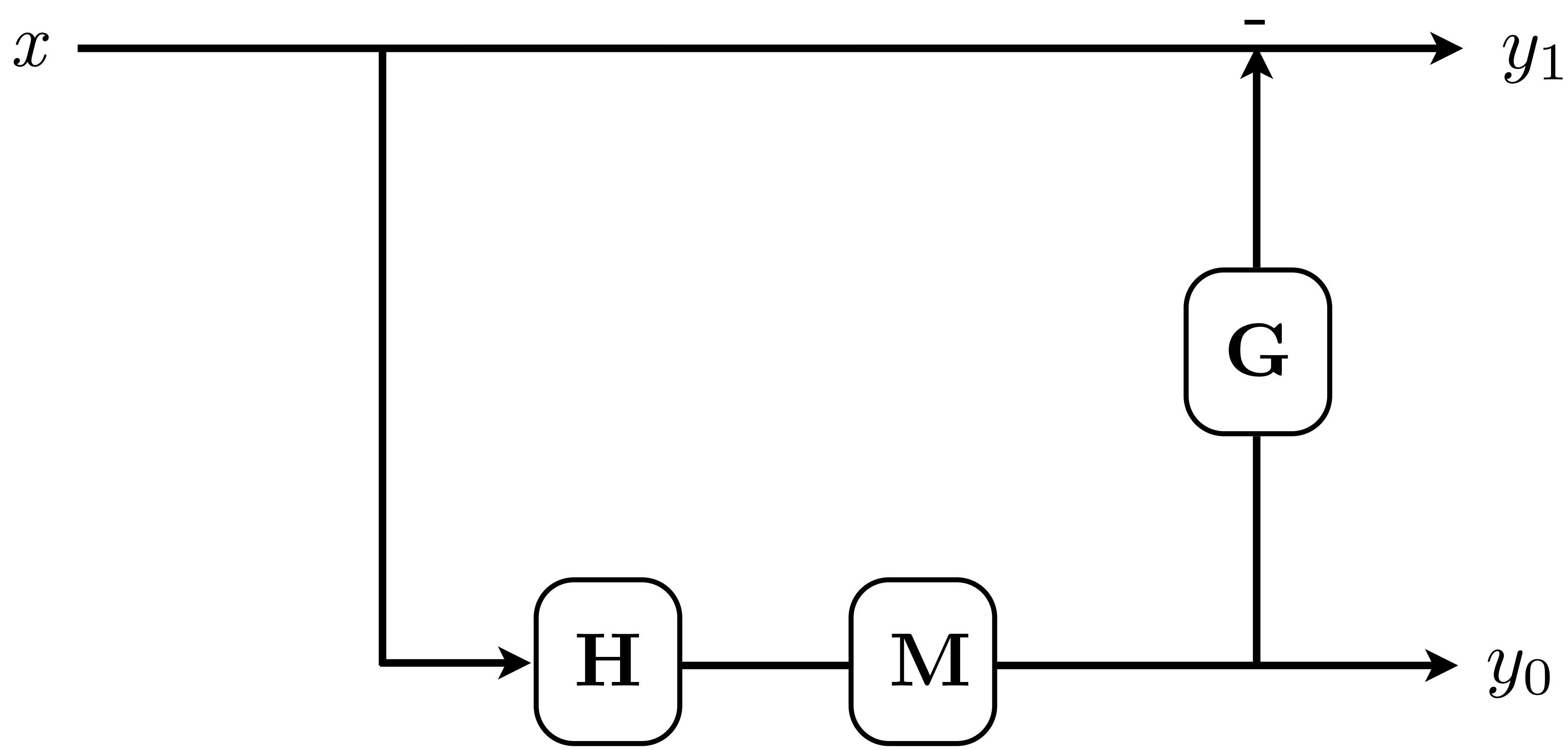

$$\underbrace{\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} \mathbf{H_m} \\ \mathbf{I} - \mathbf{G}\mathbf{H_m} \end{pmatrix}}_{\mathbf{T_a}} x,$$

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Do, Vetterli, Framing Pyramids, IEEE TSP, 2003

The Laplacian Pyramid

Analysis operator

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Simple (traditional) left inverse

$$\hat{x} = \underbrace{\begin{pmatrix} \mathbf{G} & \mathbf{I} \end{pmatrix}}_{\mathbf{T_s}} \underbrace{\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}_{y}$$

$\mathbf{T_s}\mathbf{T_a} = \mathbf{I}$ with no conditions on $\mathbf{H}$ or $\mathbf{G}$

Do, Vetterli, Framing Pyramids, IEEE TSP, 2003

The Laplacian Pyramid

Pseudo Inverse ?

$$\mathbf{T_a}^{\dagger} = \left(\mathbf{T_a}^T \mathbf{T_a}\right)^{-1} \mathbf{T_a}^T$$

Let's try to use only filters

The Laplacian Pyramid

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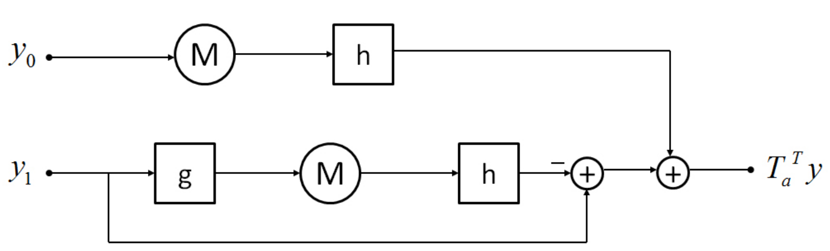
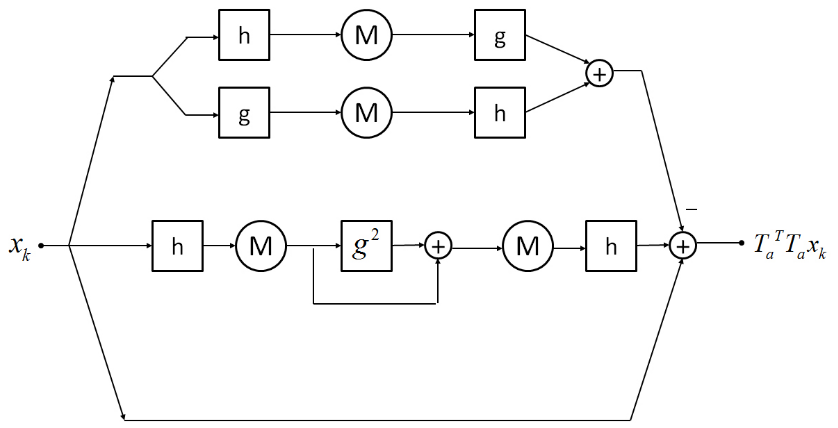
Let's try to use only filters

Landweber iterations involve only filters:

$$\arg\min_x \|\mathbf{T_a} x - y\|_2^2 \longrightarrow \hat{x}_{k+1} = \hat{x}_k + \tau \mathbf{T_a}^T (y - \mathbf{T_a} \hat{x}_k)$$

$$\mathbf{T_a}^T = (\mathbf{H_m}^T \quad \mathbf{I} - \mathbf{H_m}^T \mathbf{G}^T)$$





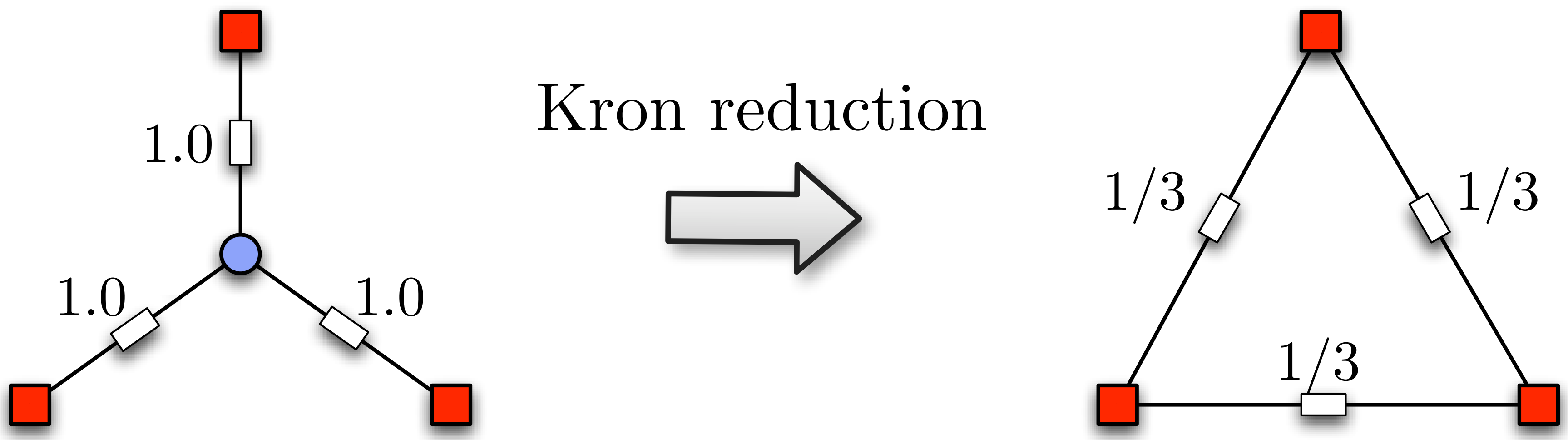
Kron Reduction

In order to iterate the construction, we need to construct a graph on the reduced vertex set.

$$\mathbf{A}_{\mathrm{r}} = \mathbf{A}[\alpha,\alpha] - \mathbf{A}[\alpha,\alpha)\mathbf{A}(\alpha,\alpha)^{-1}\mathbf{A}(\alpha,\alpha]$$

Schur complement

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}[\alpha,\alpha] & \mathbf{A}[\alpha,\alpha) \\ \mathbf{A}(\alpha,\alpha] & \mathbf{A}(\alpha,\alpha) \end{bmatrix}$$



Dorfler et al., ArXiV, 2011

Sparsification

Kron reduction produces denser and denser graphs

After each reduction we use Spielman's sparsification algorithm to obtain an equivallent but sparser graph

Approx preserves Laplacian quadratic form

Explicit control based on effective resistance of edges

Spielman and Srivastava, Graph sparsification by effective resistances, SIAM J. Comp, 2011

Sparsification

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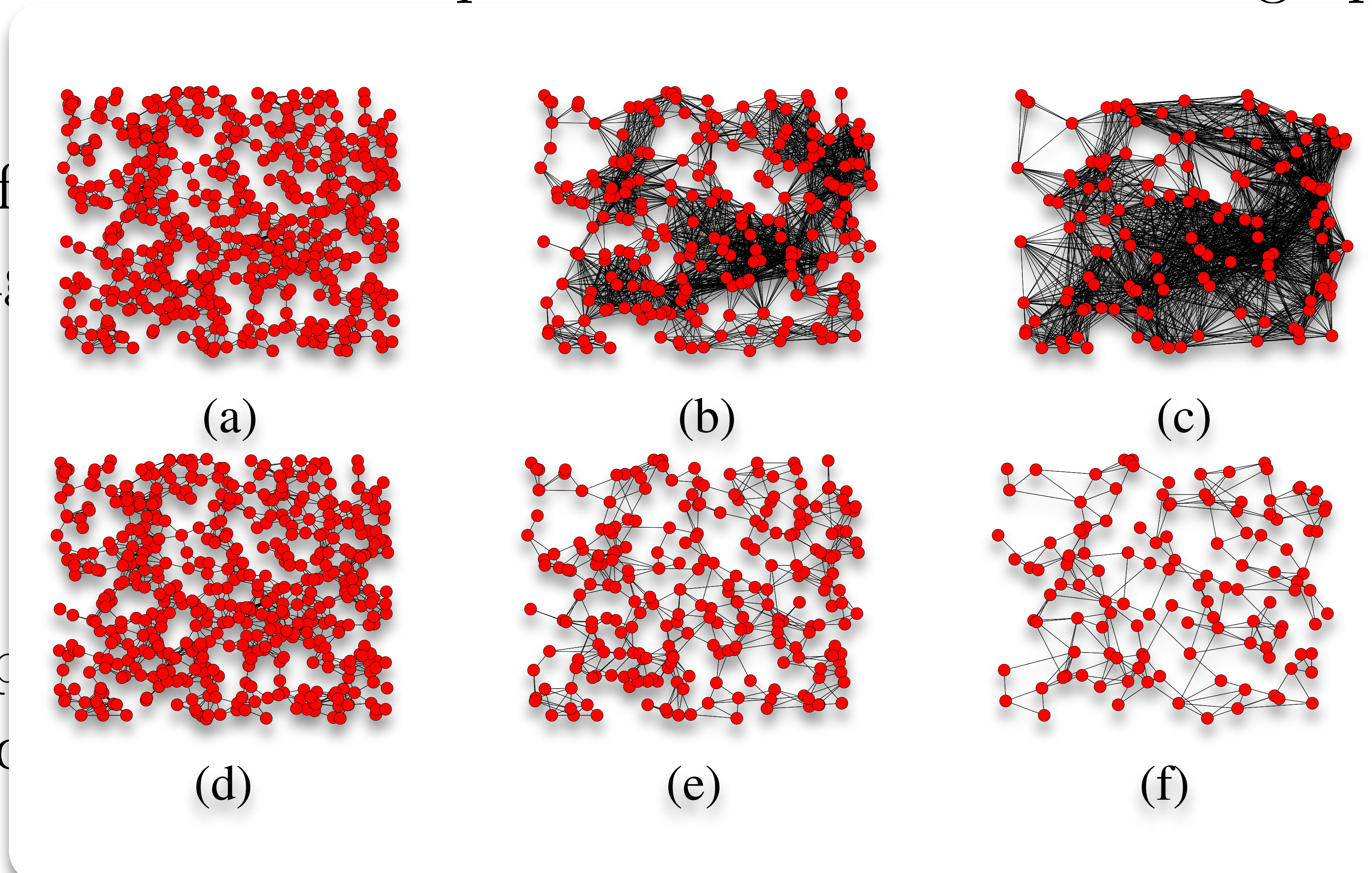

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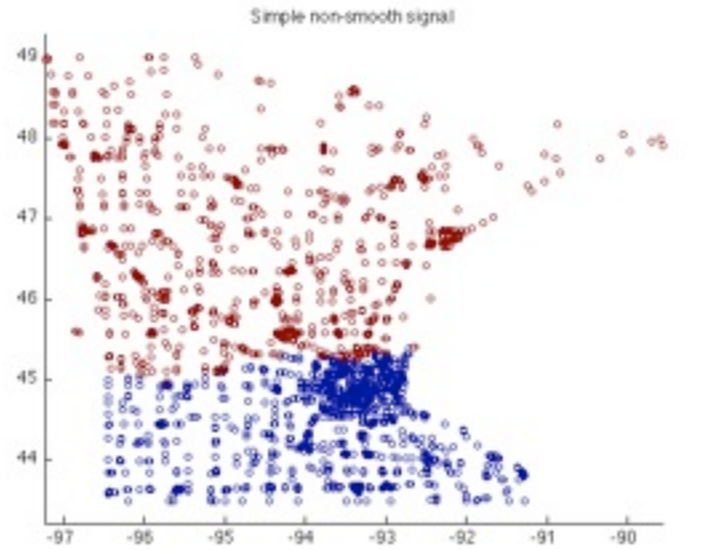
Simple non-smooth signal
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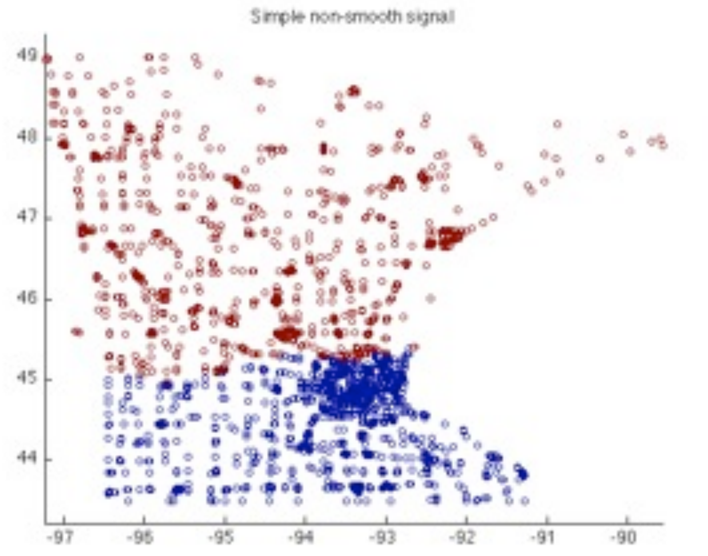
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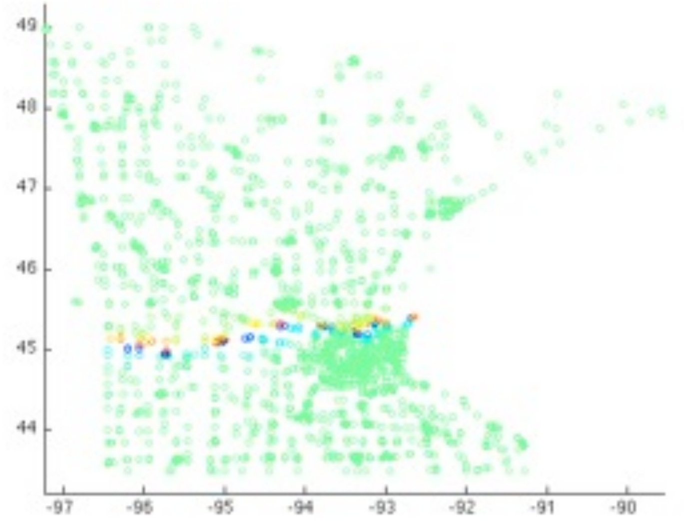
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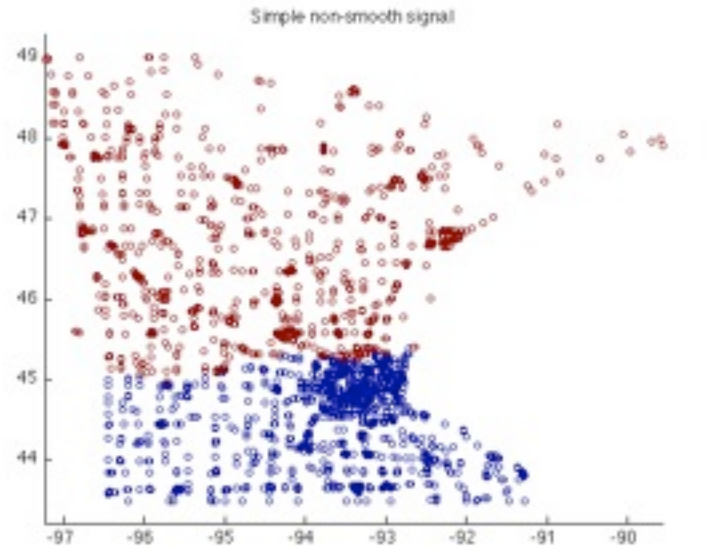
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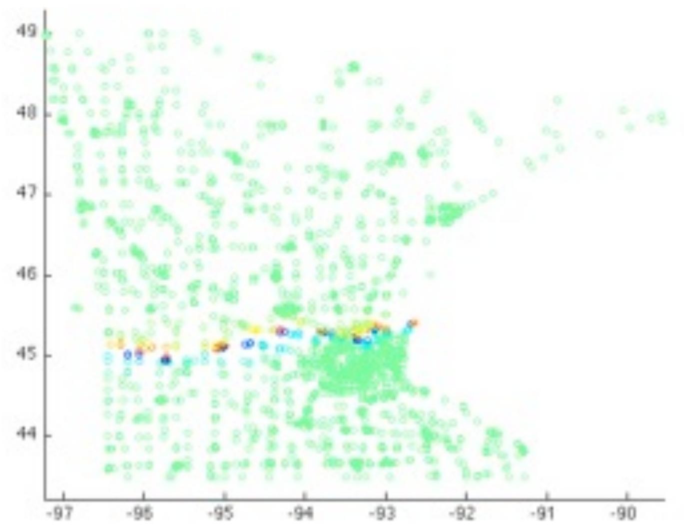

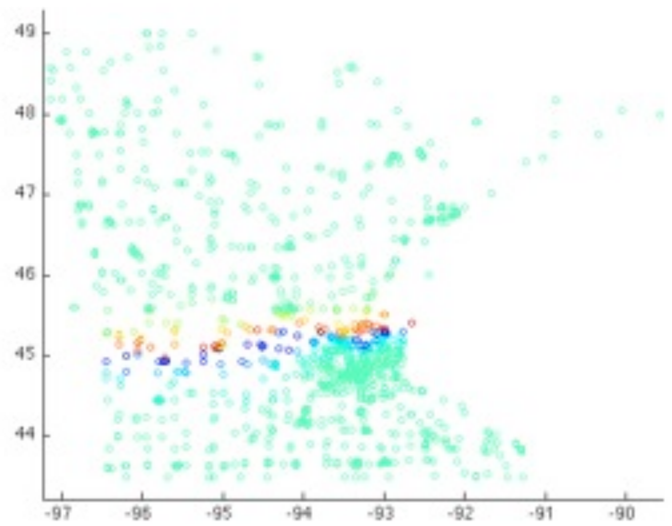

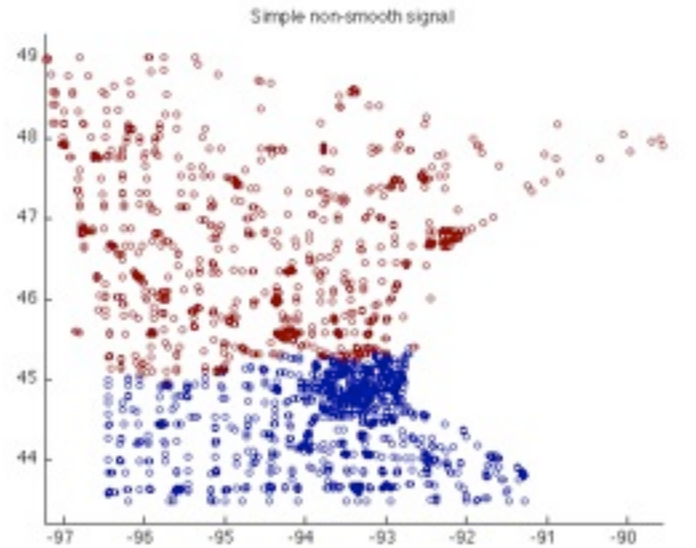
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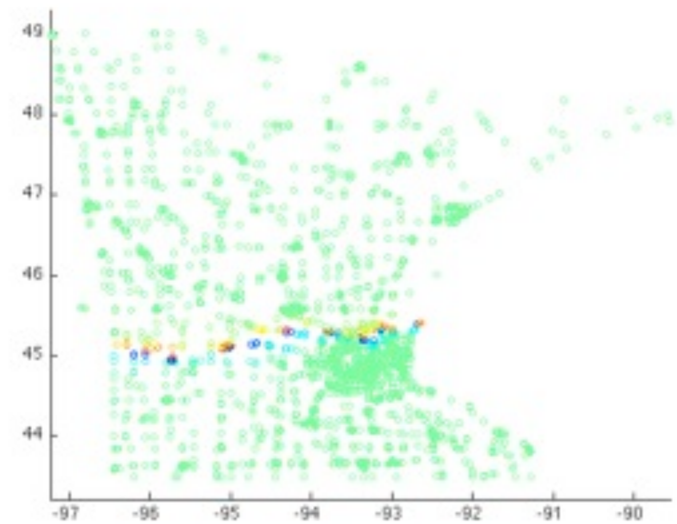

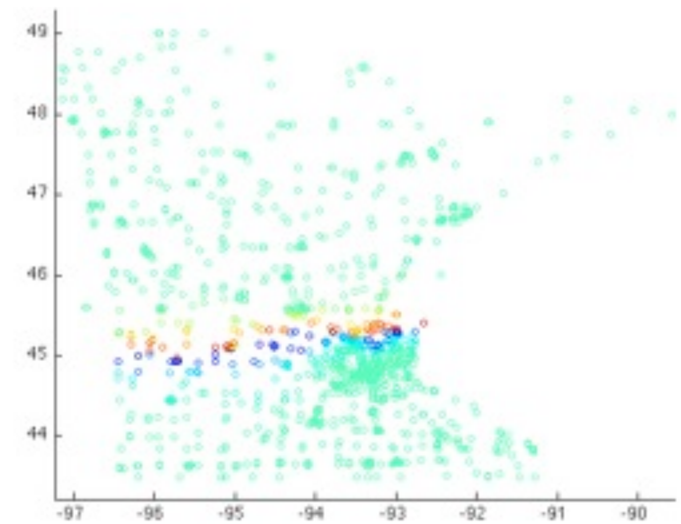

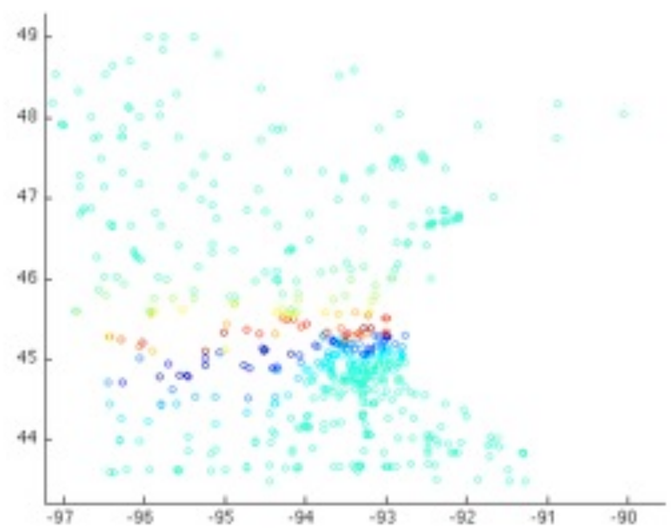

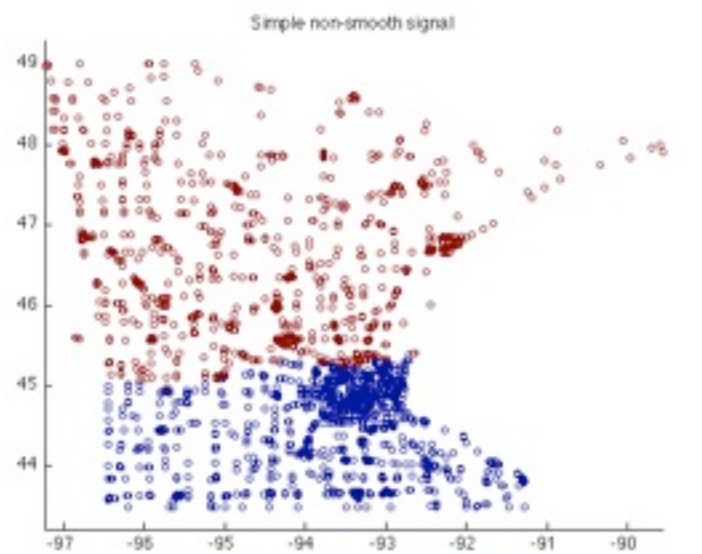
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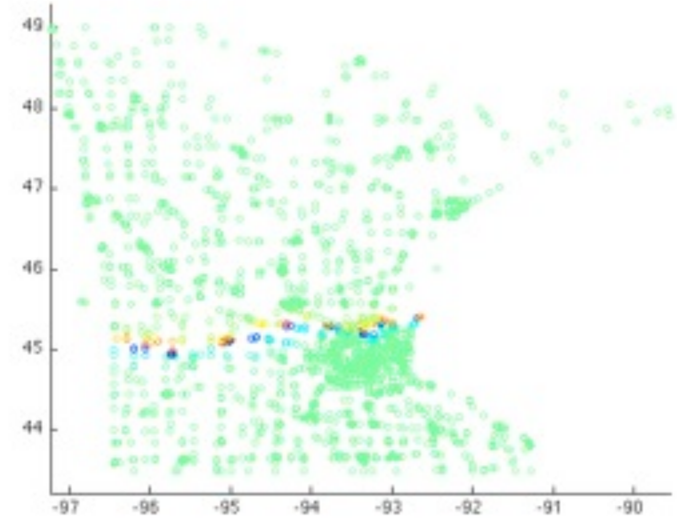

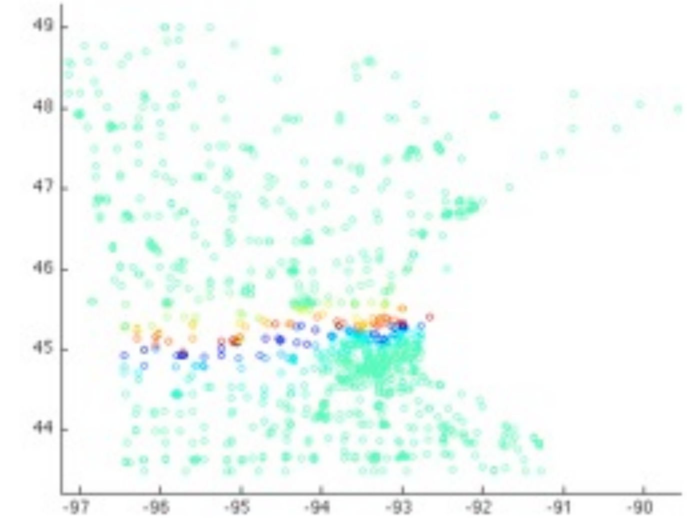

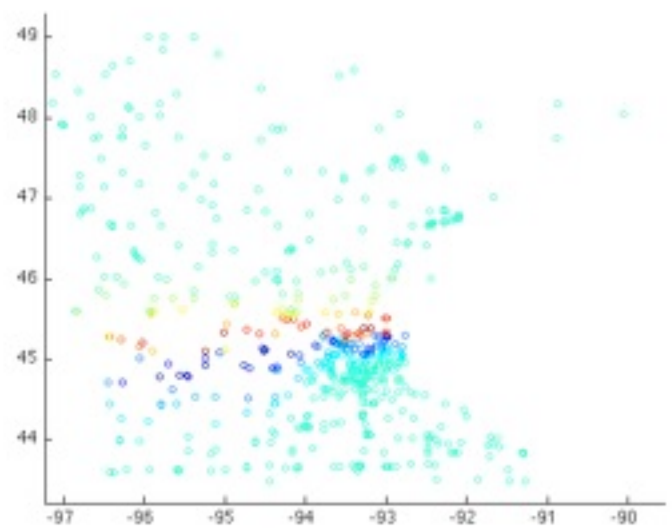

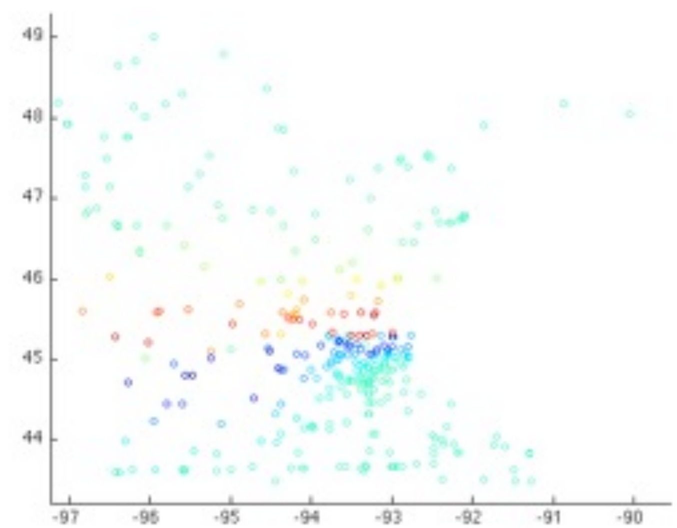

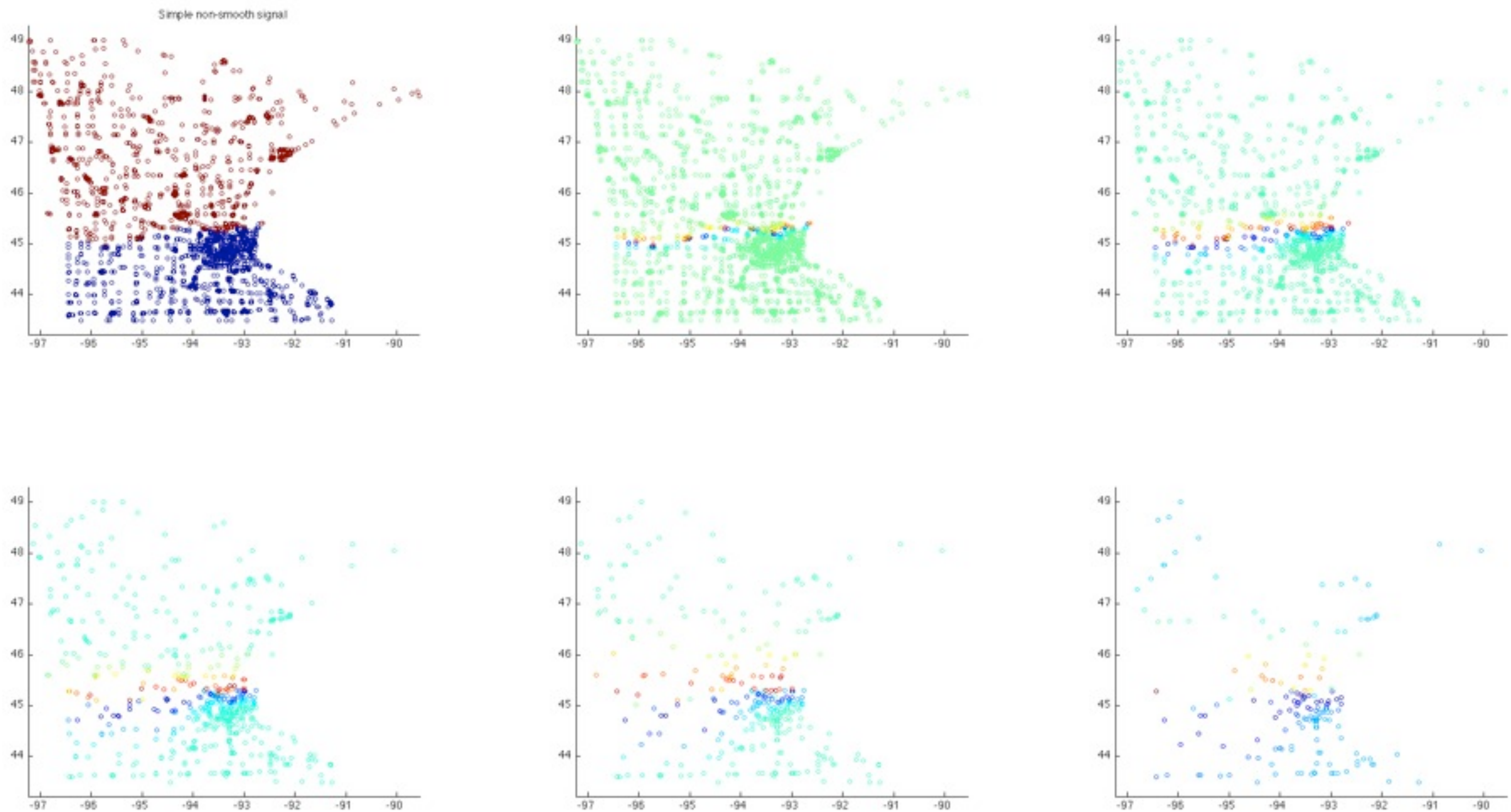
Simple non-smooth signal
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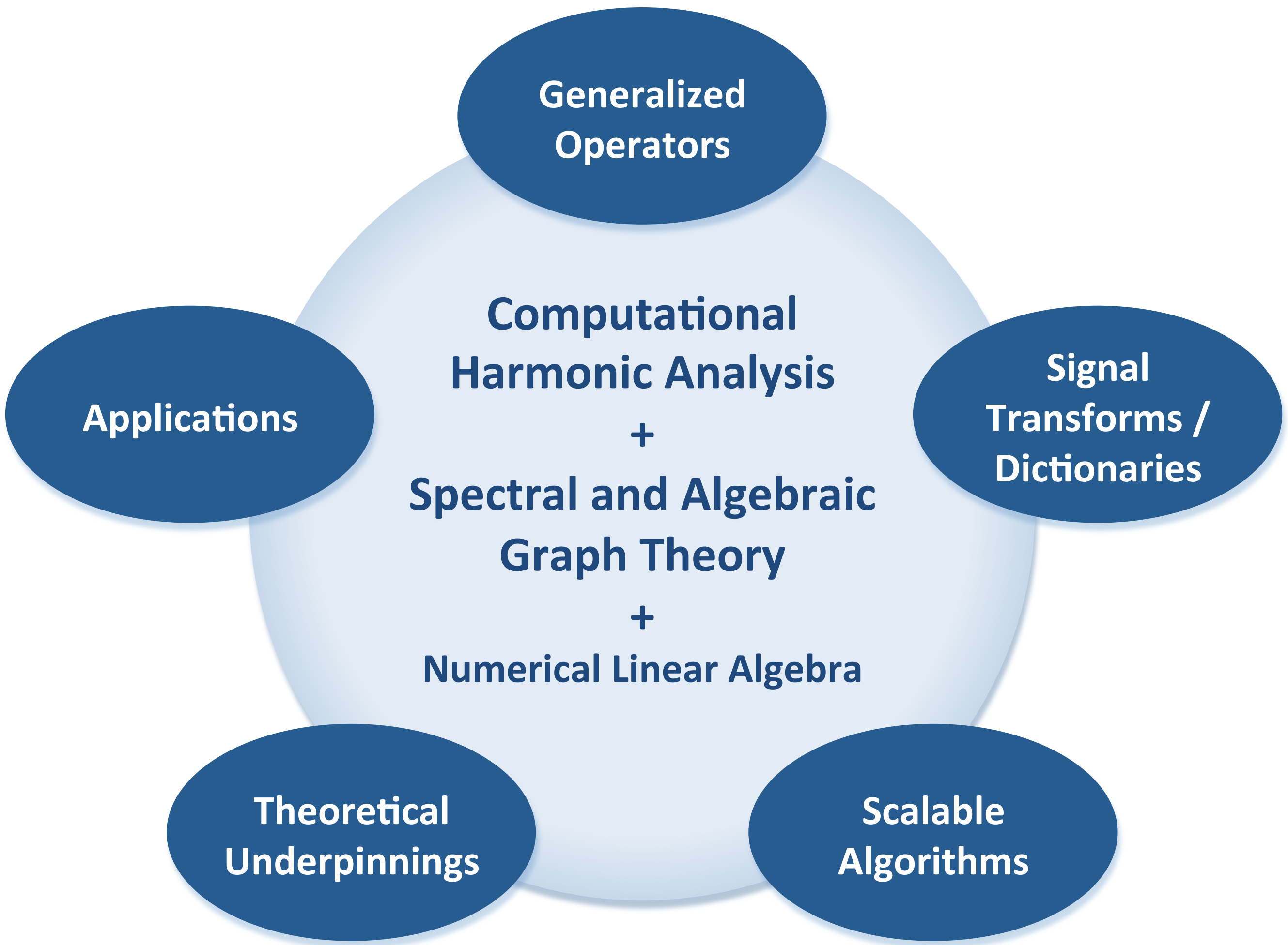
Outlook



- Application of graph signal processing techniques to real science and engineering problems is in its infancy
- Theoretical connections between classes of graph signals, the underlying graph structure, and sparsity of transform coefficients

Conclusions

- Ways to process information at vertices of graphs, inspired by SP
- Importance of algorithms that can scale to very large graphs
- Some counter-intuitive results are expected with respect to traditional SP.
- Many interesting problems/applications

Wavelet Coefficient Decay of Globally Regular Graph Signals

Proposition 1

Let $p \geq 1$, and assume that $C_p := \int_0^\infty |\hat{g}(s)|^2/s^{2p} ds < \infty$. Then

$$\int_0^\infty s^{-2p} \sum_n |\langle f, \psi_{s,n} \rangle|^2 ds = C_p ||f||_{\mathcal{H}^{(2p-1)/2}}.$$

Proposition 2

Assume that $\hat{g}(\lambda) = \sum_{k=p}^{q} a_k \lambda^k$ for some $p \geq 1$ (implying $\hat{g} = 0$)
Then

$$|\Psi f(s,n)| = |\langle f, \psi_{s,n} \rangle| \leq \sum_{k=p}^{q} |a_k| s^k ||f||_{\mathcal{H}^k}.$$

Ongoing Work: Local Regularity and Wavelet Coefficient Decay of Locally Regular Graph Signals

Notions of Local Regularity

Local Variation

$$||\nabla_m \mathbf{f}||_2 = \left[\sum_{n \in \mathcal{N}_m} w(m,n) \left[f(n) - f(m) \right]^2 \right]^{\frac{1}{2}}$$

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Hölder Regularity

A graph signal f is (C, α, r)-Hölder regular with respect to the graph $\mathcal{G}$ at vertex $n \in \mathcal{V}$ if

$$|f(n) - f(m)| \leq C[d_{\mathcal{G}}(m,n)]^{\alpha}, \; \forall m \in \mathcal{N}(n,r)$$

Gavish et al. ICML, 2010

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Laplacian as Derivative

$(\mathcal{L}^k f)(n)$ as a measure of local regularity of f in a neighborhood of radius k around vertex n

- For polynomial kernel:

$$\Psi f(s,n) = \sum_{k=p}^{q} a_k s^k (\mathcal{L}^k f)(n)$$

Wavelet Coefficient Decay of Locally Regular Graph Signals

$|\Psi f(s,n)|$

$\psi_{s,n}$

Wavelet Coefficient Decay of Locally Regular Graph Signals

High-level intuition

- Far away from vertex n, for small scales s, $|\Psi f(s,n)|$ is small because $\psi_{s,n}$ is highly localized around n

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- Close to vertex n, $|\Psi f(s,n)|$ is small because f is locally regular

Proposition 3

Assume that f is (C,α,r)-Hölder regular for some $r \geq 1$, and let $\hat{g}(\lambda) = \sum_{k=r}^{q} a_k \lambda^k$ for some coefficients $\{a_k\}_{k=r,r+1,\ldots,q}$. Then there exist constants C_2 and $\bar{s}$ such that for all $s < \bar{s}$, we have

$$|\Psi f(s,n)| \leq C r^{\alpha} \sum_{m \in \mathcal{N}(n,r)} |\psi_{s,n}(m)| + C_2 s^{r+1} \sum_{m \notin \mathcal{N}(n,r)} |f(m) - f(n)|.$$

Example

Note: For a k-regular bipartite graph

$$\mathbf{L} = \begin{bmatrix} k\mathbf{I}_n & -\mathbf{A} \\ -\mathbf{A}^T & k\mathbf{I}_n \end{bmatrix}$$

Kron-reduced Laplacian: $\mathbf{L}_r = k^2\mathbf{I}_n - \mathbf{A}\mathbf{A}^T$

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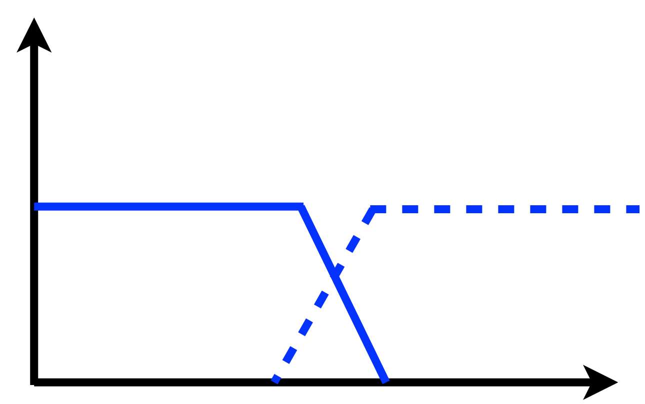
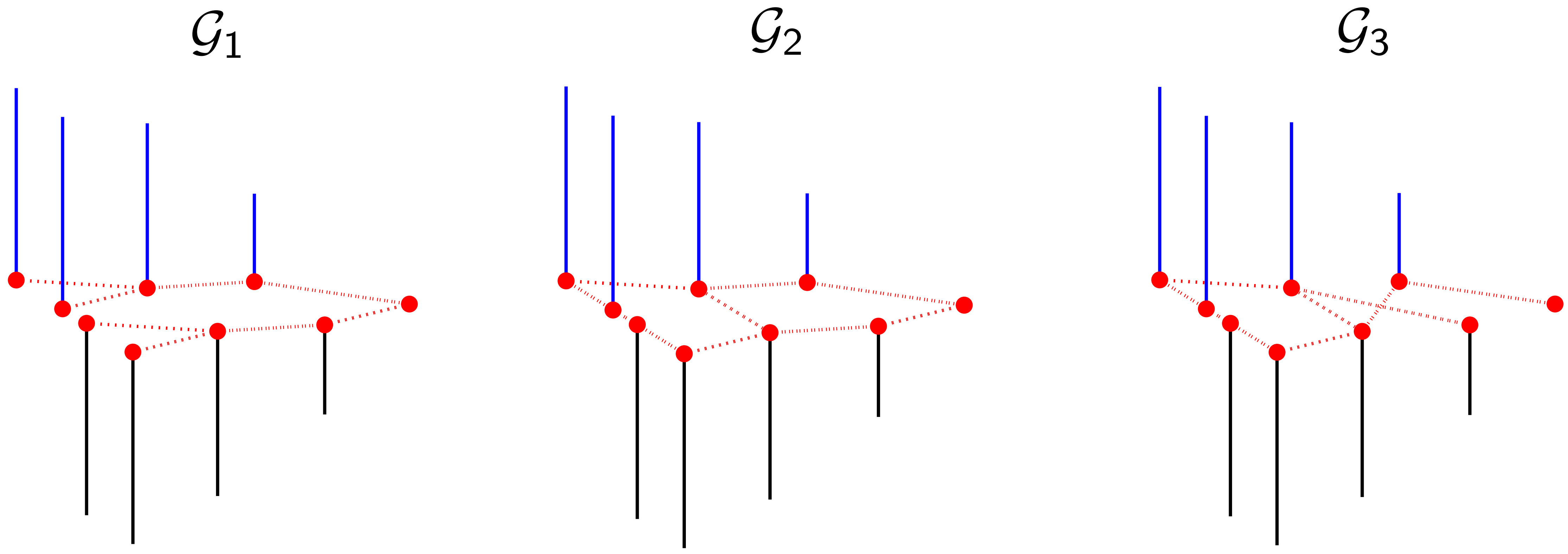
$$\hat{f}_r(i) = \hat{f}(i) + \hat{f}(N-i) \quad i = 1, ..., N/2$$



Smoothness of Graph Signals



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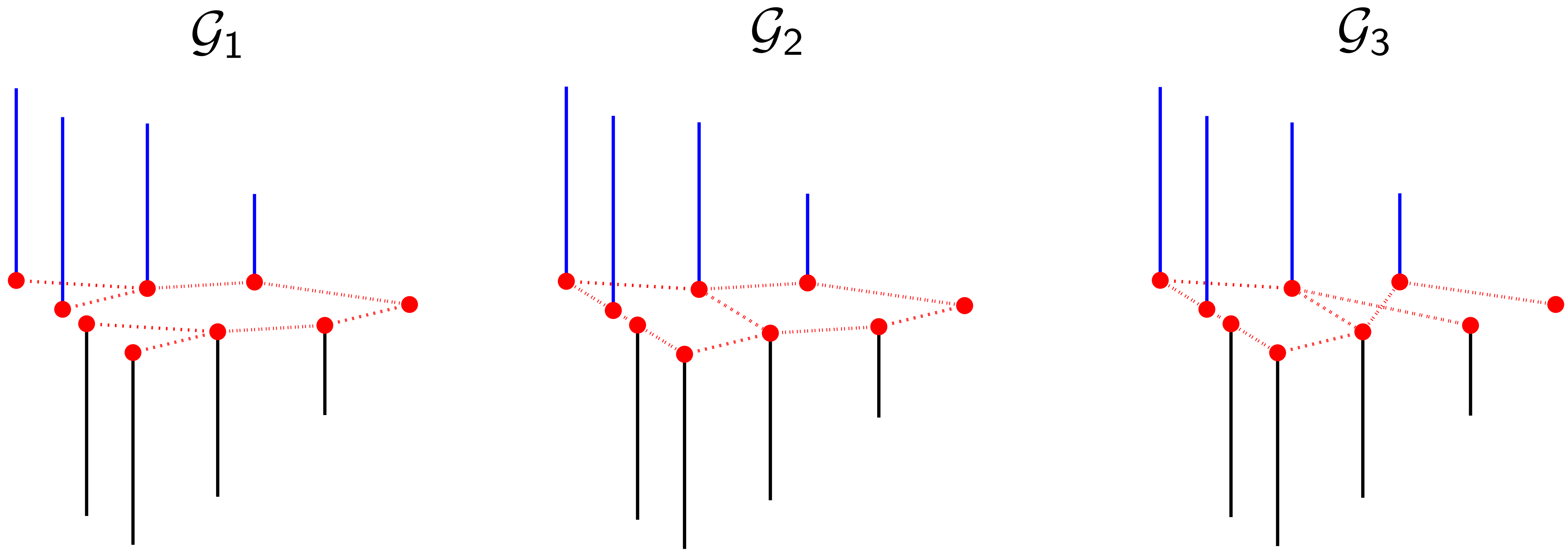


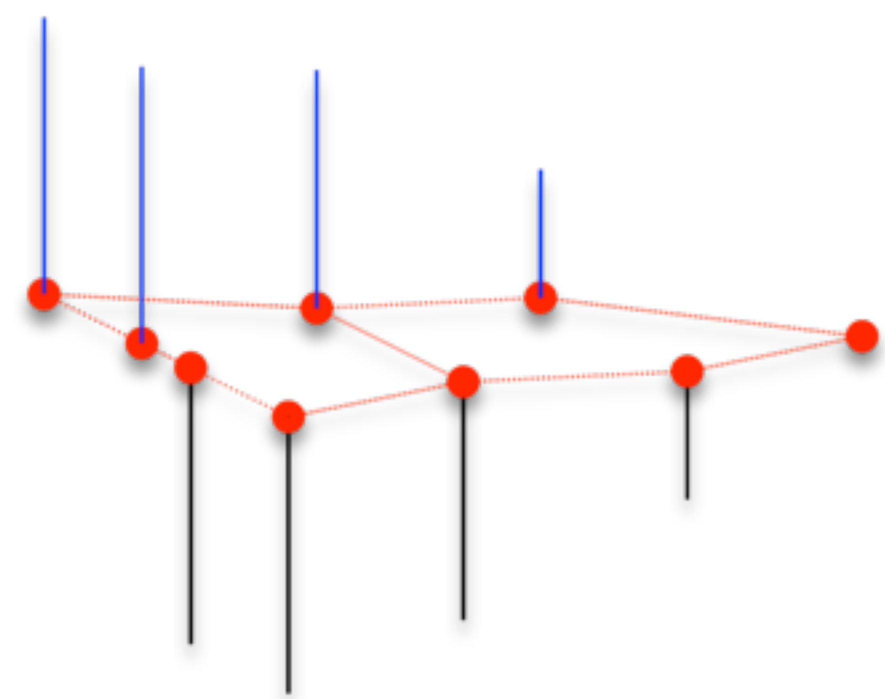
To identify and exploit structure in the data, we need to account for the intrinsic geometric structure of the underlying graph data domain

Notions of Global Regularity for Graph Signals

Discrete Calculus, Grady and Polimeni, 2010

Edge Derivative

$$\left.\frac{\partial \mathbf{f}}{\partial e}\right|_m := \sqrt{w(m,n)}\,[f(n) - f(m)]$$



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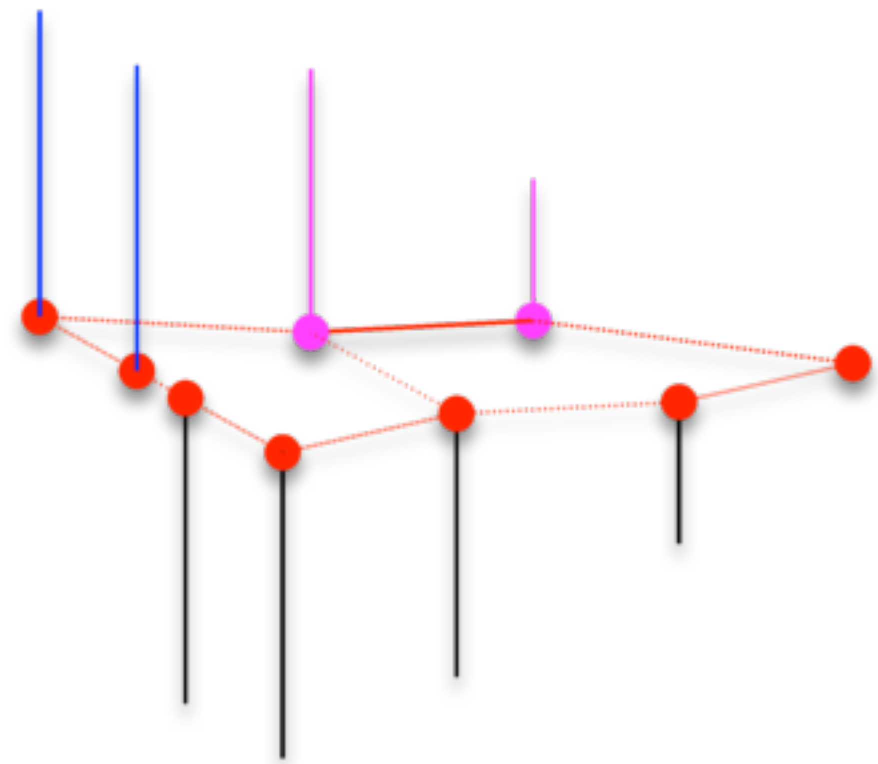

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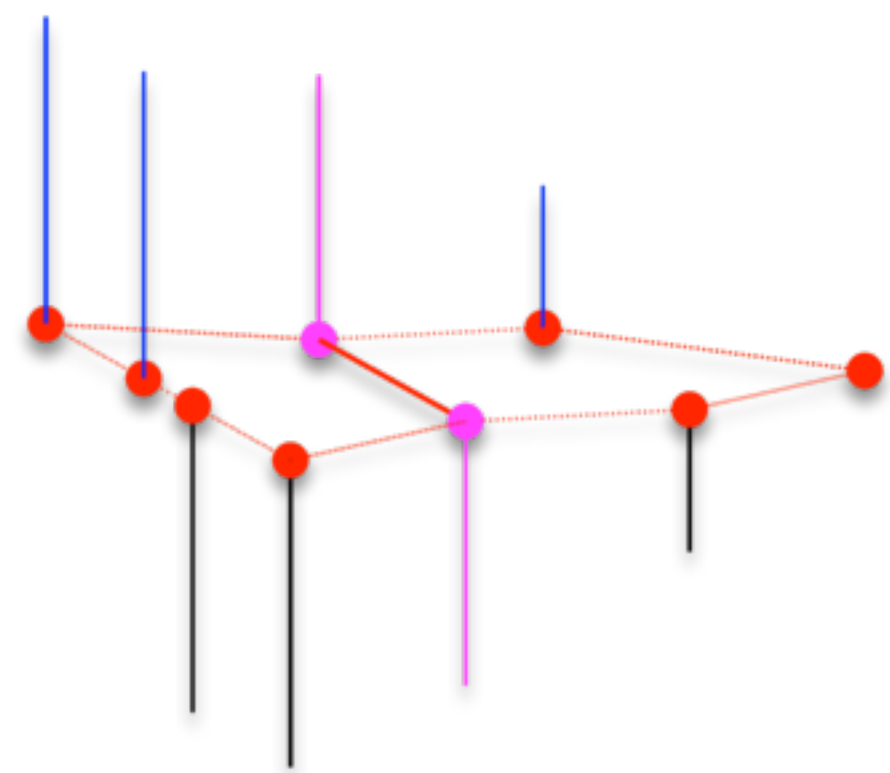

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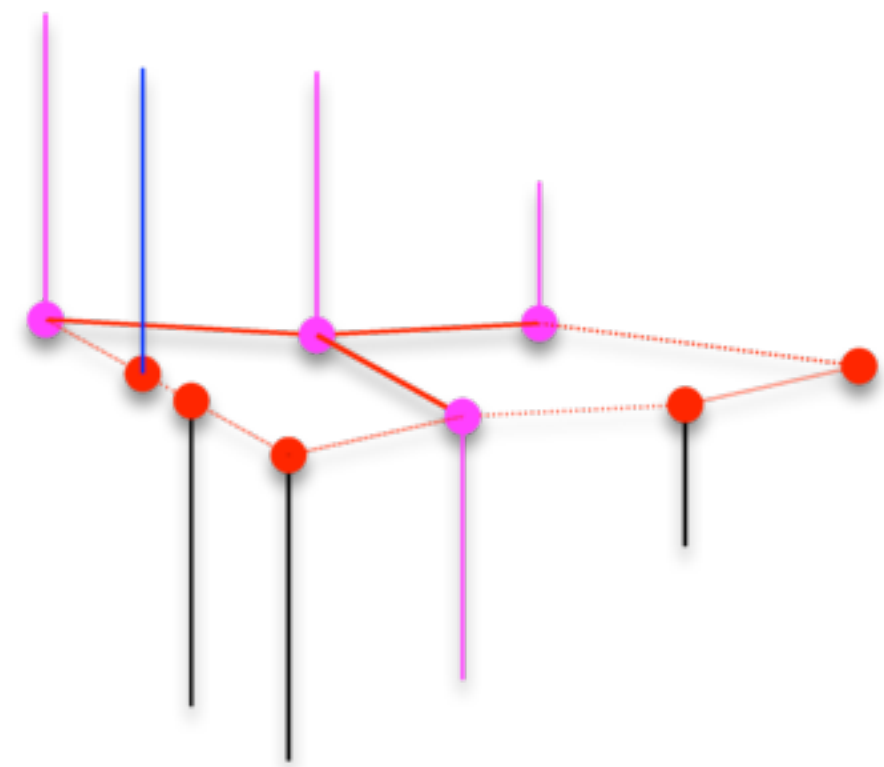

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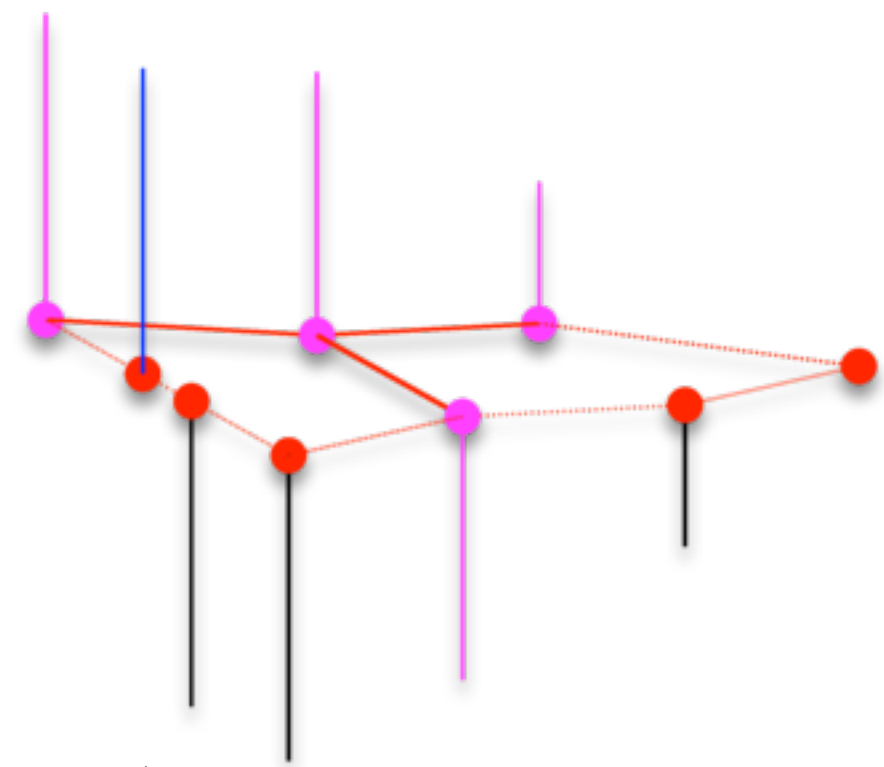

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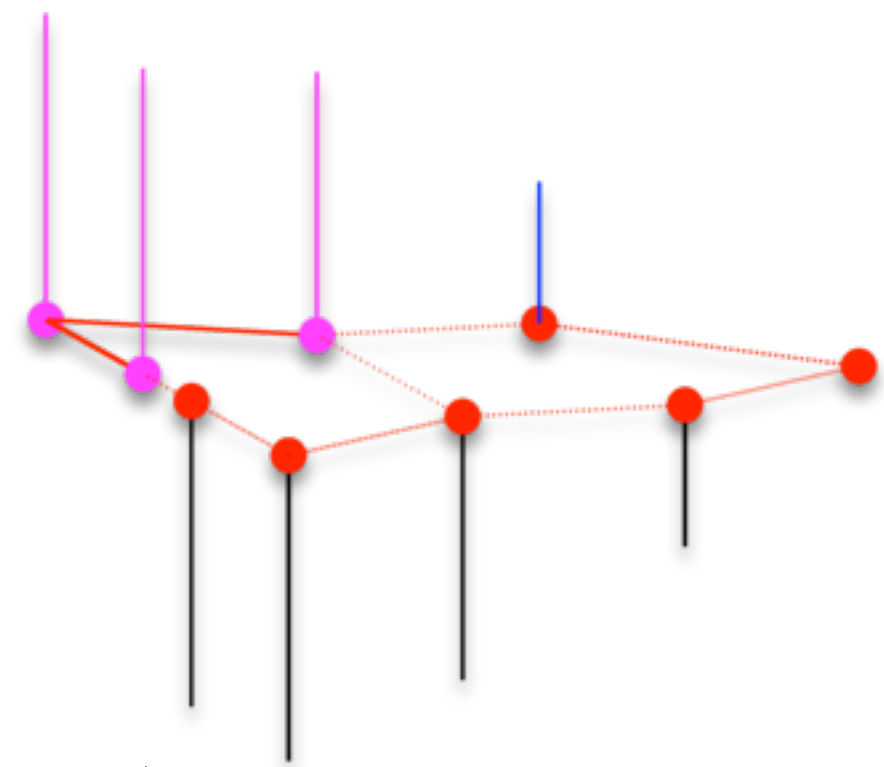

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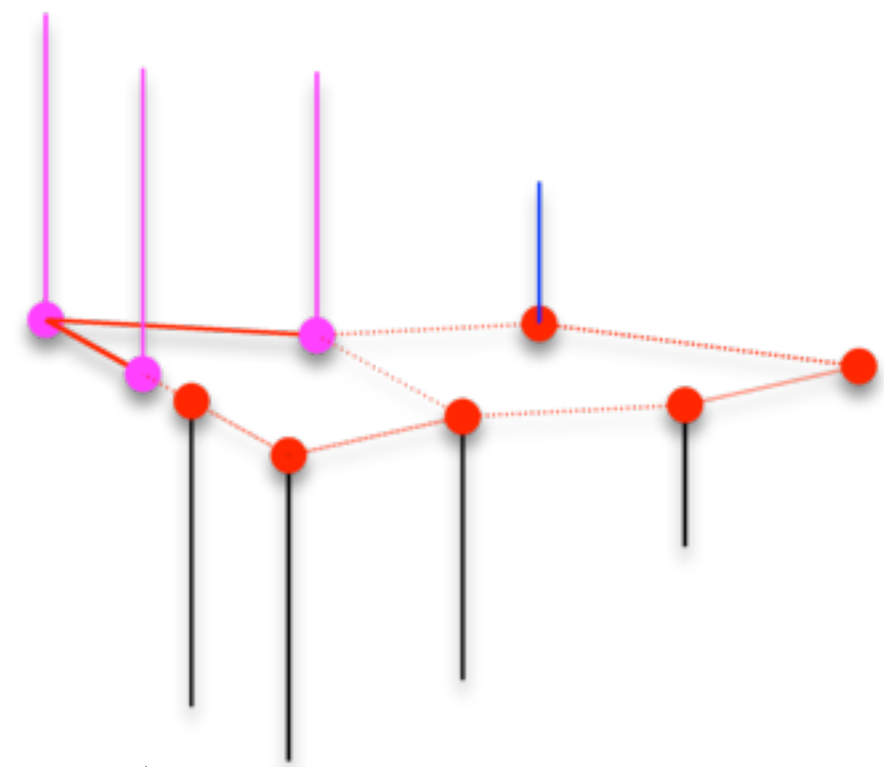

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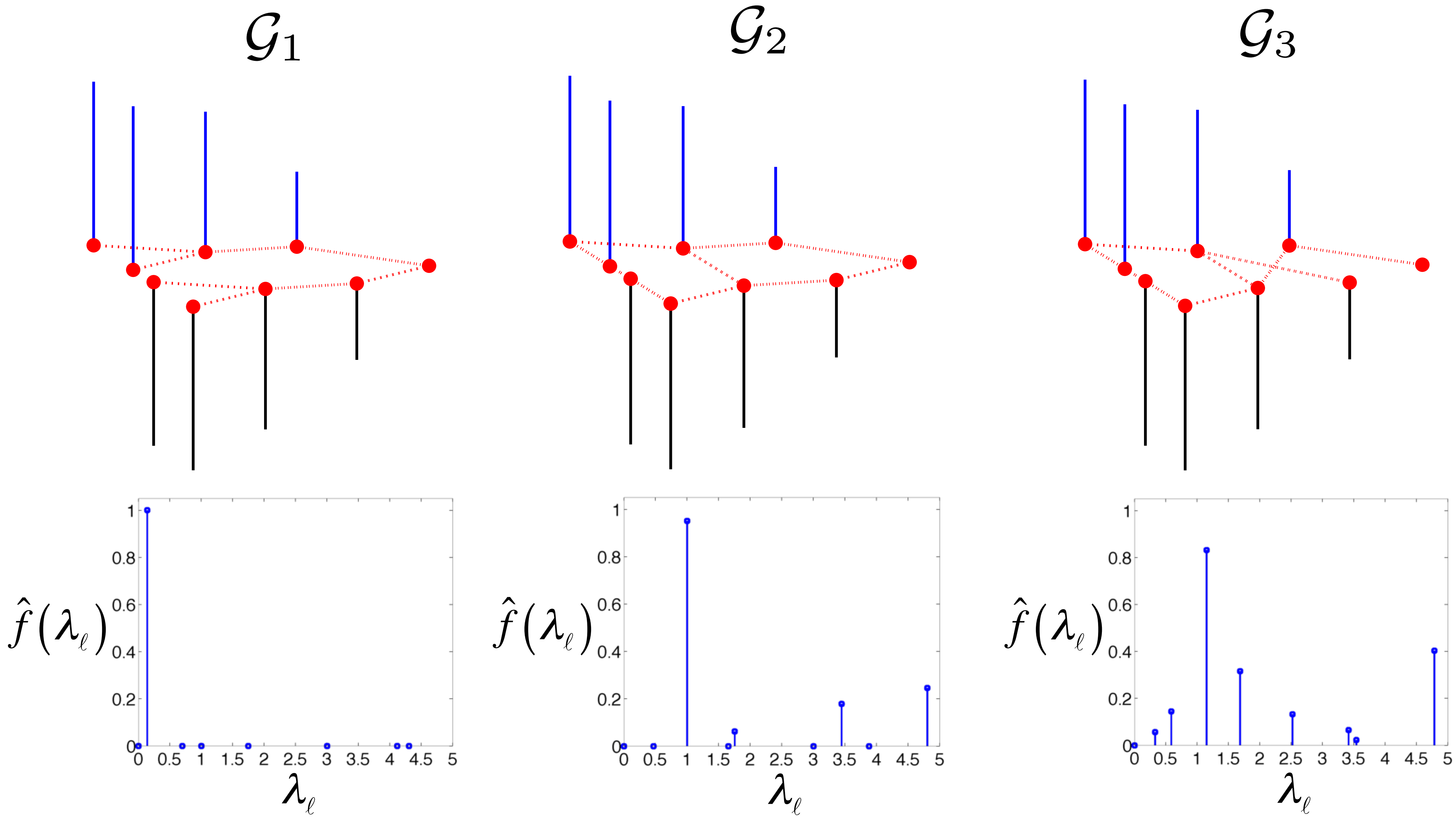
Quadratic Form

$$\frac{1}{2}\sum_{m\in V} ||\nabla_m \mathbf{f}||_2^2 = \sum_{(m,n)\in\mathcal{E}} w(m,n)\,[f(n) - f(m)]^2 = \mathbf{f}^{\mathrm{T}}\mathcal{L}\mathbf{f}$$

Smoothness of Graph Signals Revisited



$$\mathbf{f}^{\mathrm{T}}\mathcal{L}_1\mathbf{f} = 0.14 \qquad \mathbf{f}^{\mathrm{T}}\mathcal{L}_2\mathbf{f} = 1.31 \qquad \mathbf{f}^{\mathrm{T}}\mathcal{L}_3\mathbf{f} = 1.81$$

Notions of Global Regularity for Graph Signals

Generalizations

p-Dirichlet Form
(Elmoataz et al., 2008)

$$\frac{1}{p}\sum_{m\in V}||\nabla_m \mathbf{f}||_2^p = \frac{1}{p}\sum_{m\in\mathcal{V}}\left[\sum_{n\in\mathcal{N}_m} w(m,n)\left[f(n)-f(m)\right]^2\right]^{\frac{p}{2}}$$

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Discrete Sobolev Semi-Norm

$$\|f\|_{\mathcal{H}^p} := \|\mathcal{L}^p f\|_2 = \|\widehat{\mathcal{L}^p f}\|_2 = \sqrt{\sum_{\ell}|\lambda_\ell|^{2p}|\hat{f}(\ell)|^2}$$

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Mallat, 2008, pp. 438-9

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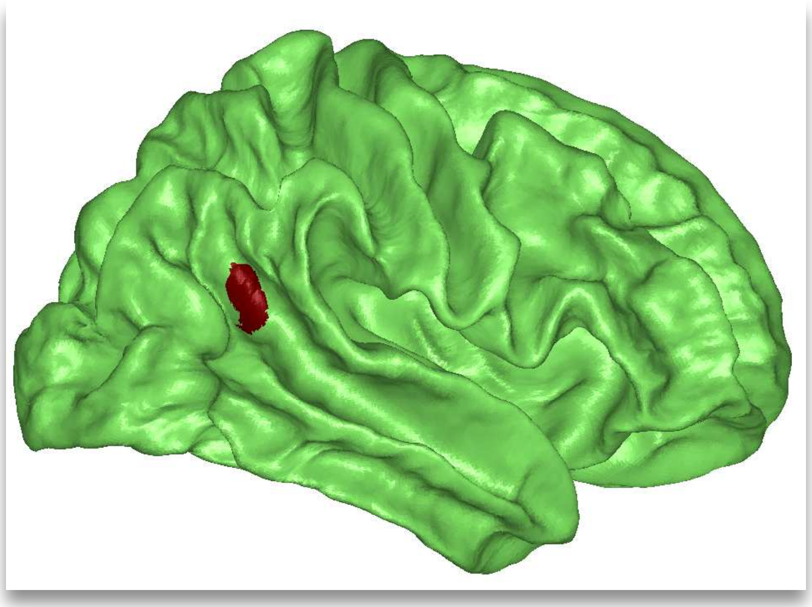
- In the graph setting,

$$\frac{||f||_{\mathcal{H}^p}}{||f||_2} \leq \lambda_{\max}^p \text{ for all } f \in \mathbb{R}^N$$

Example



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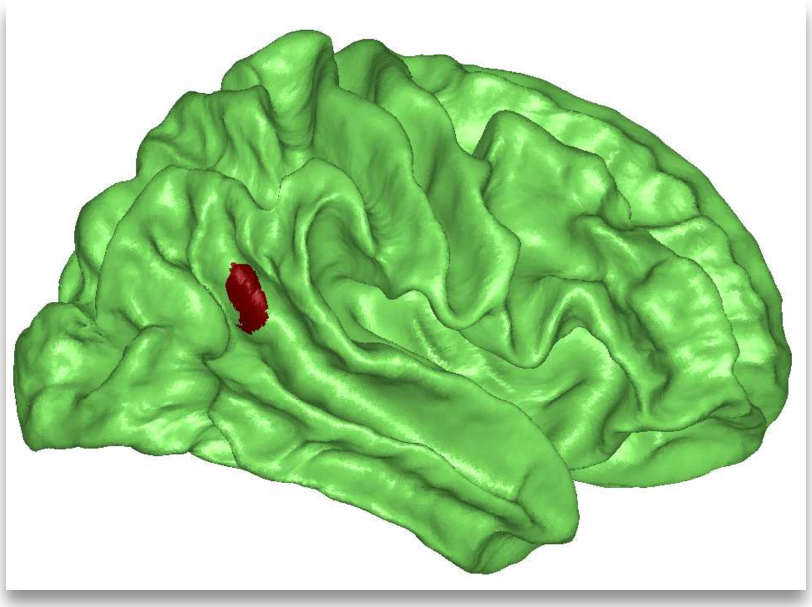

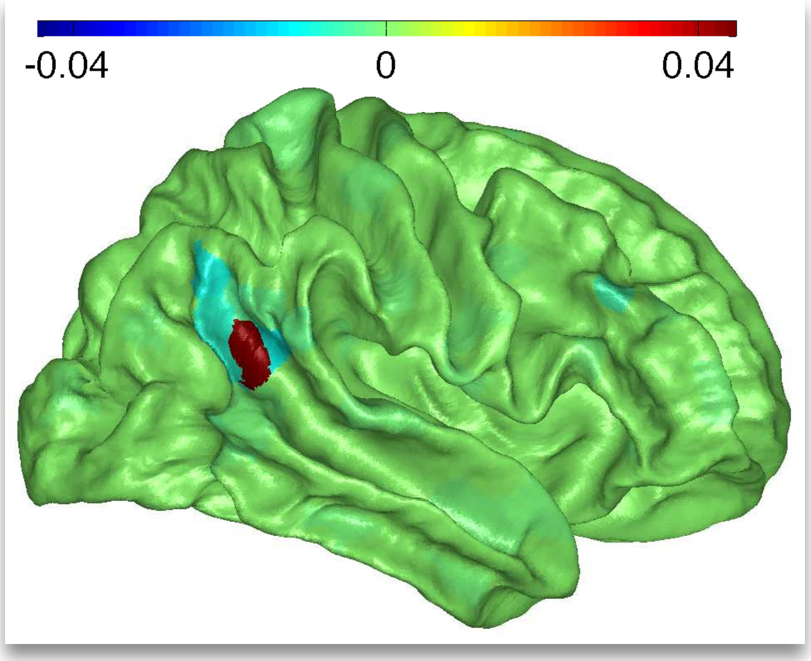

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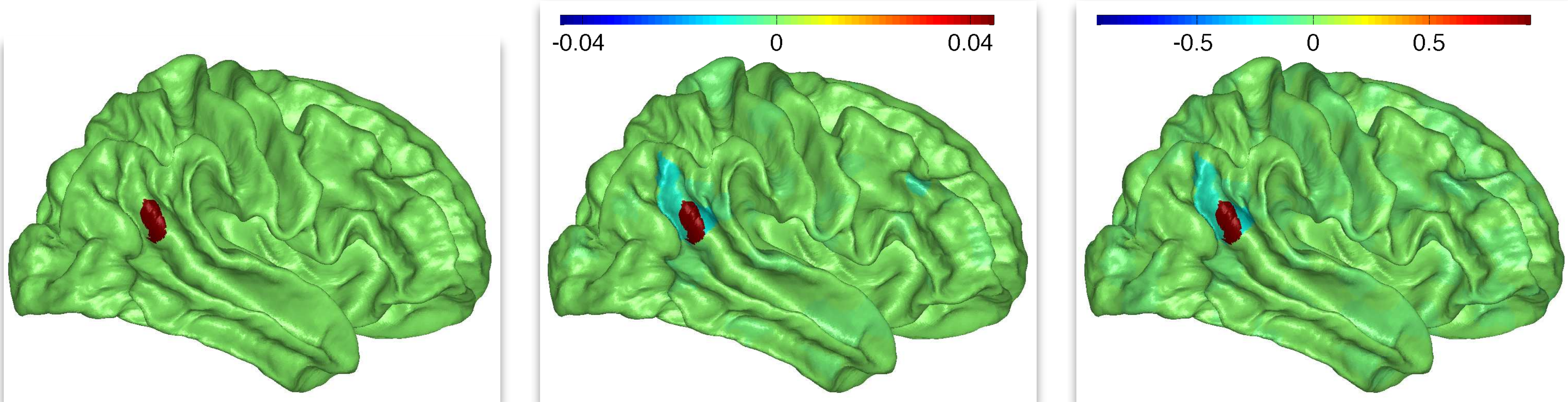

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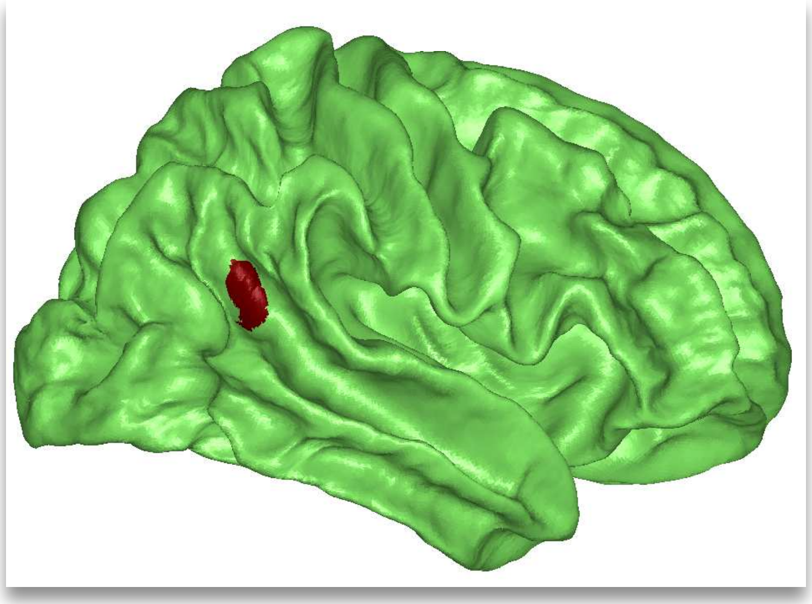

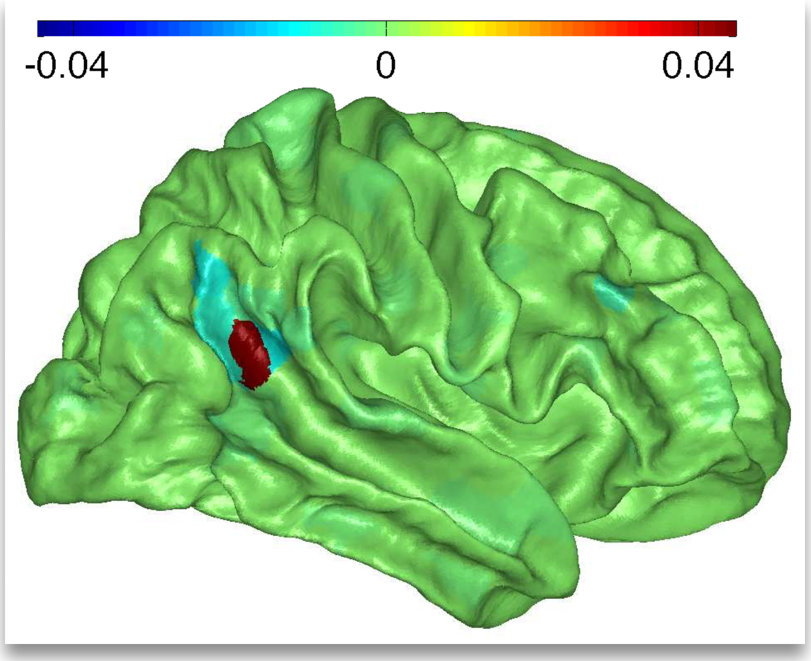

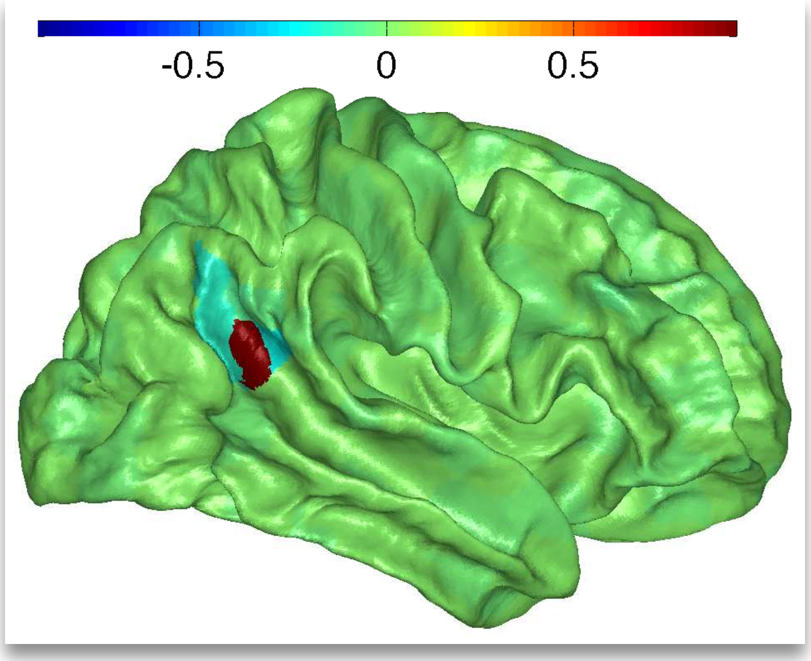

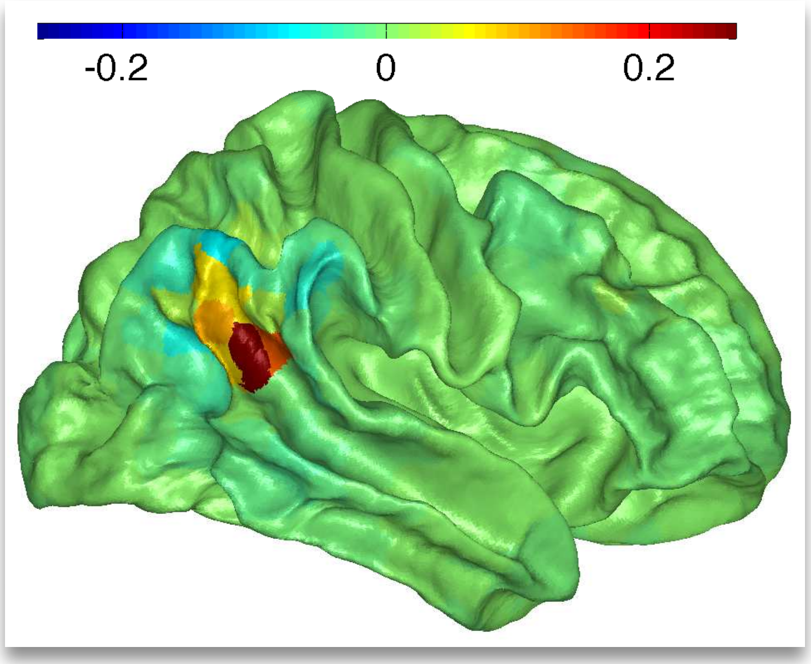

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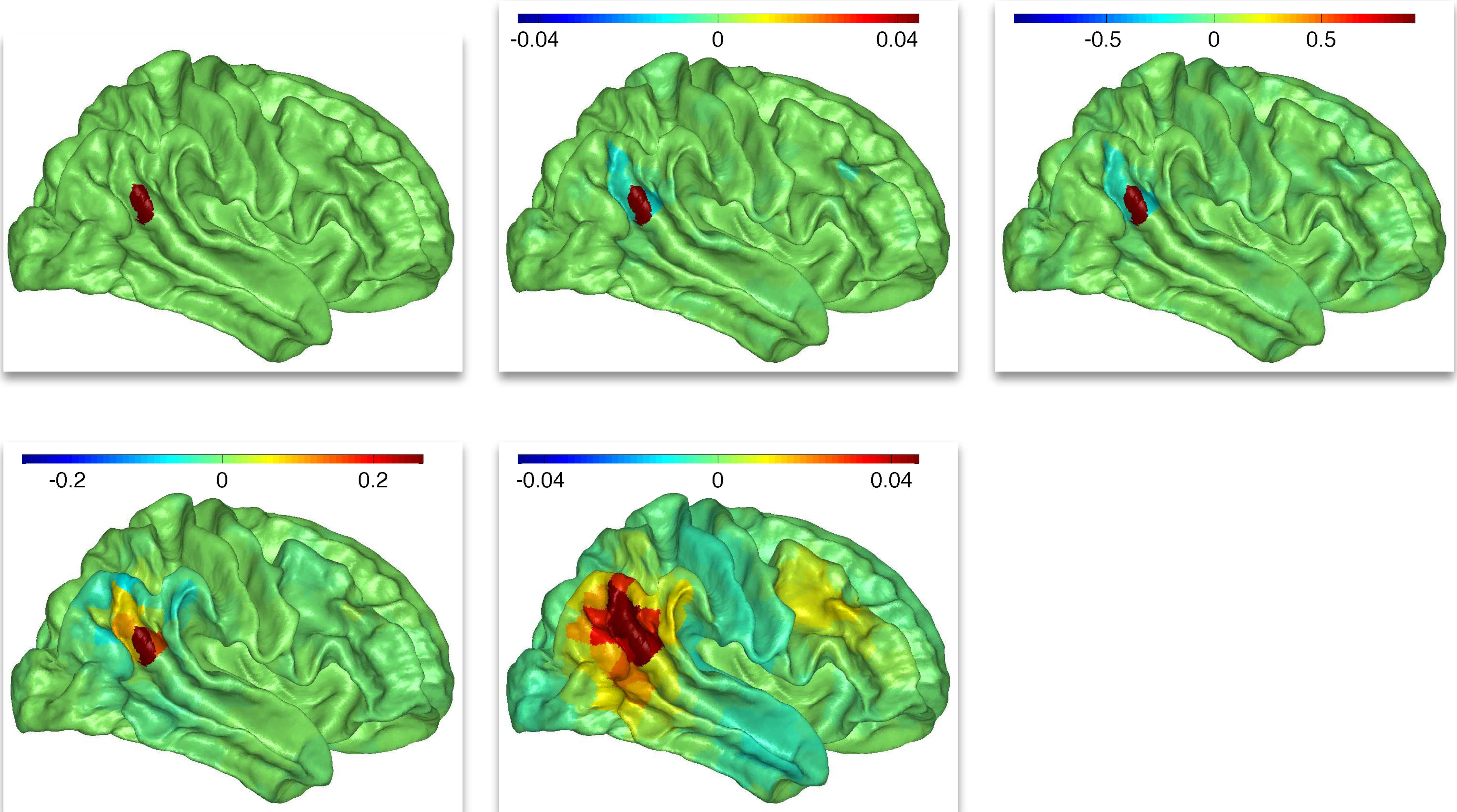

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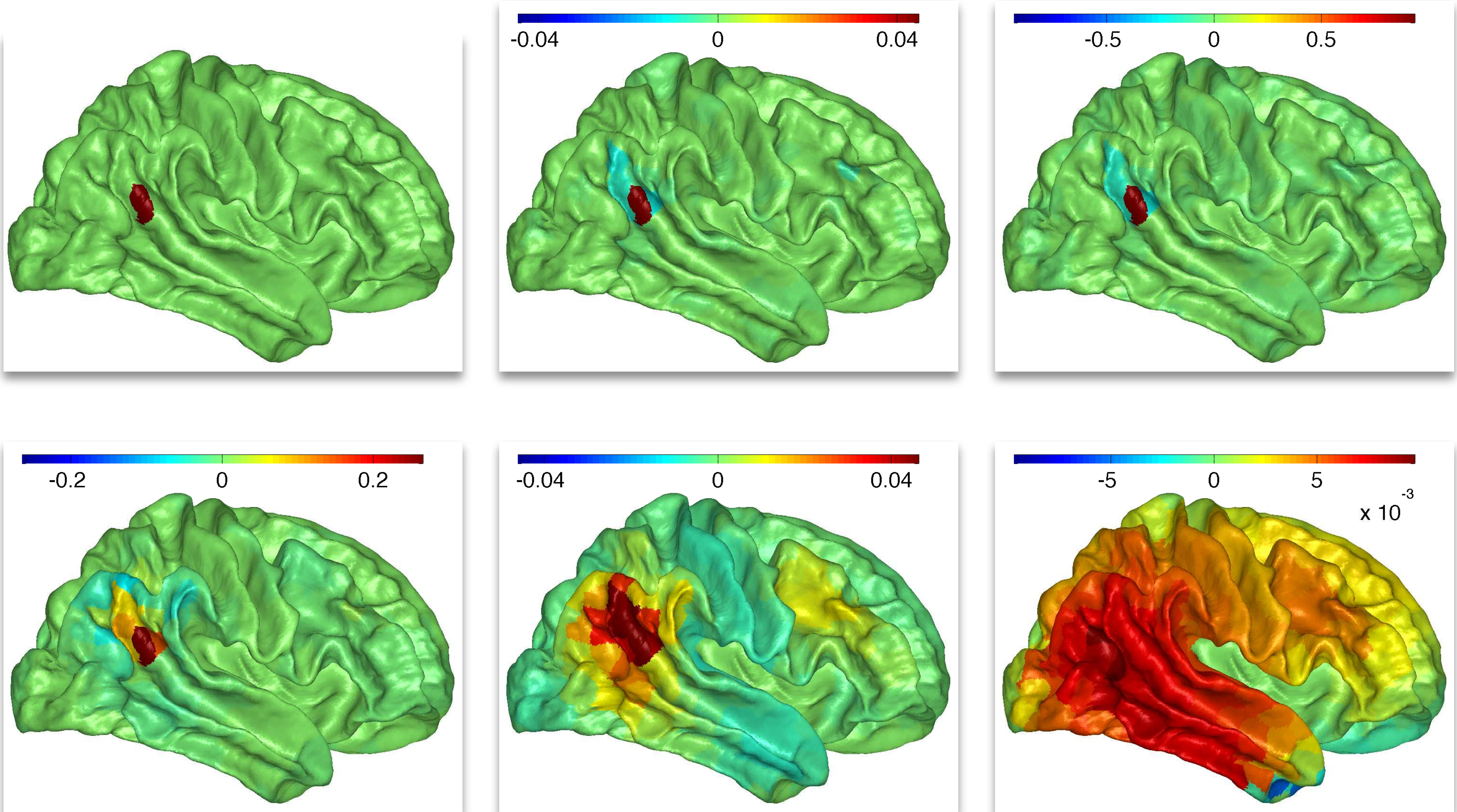

Polynomial Localization

$$\sup_{\ell} |\hat{g}(x) - P_K(x)| \leq \frac{B}{2^K (K+1)!}$$

Now consider:

$$\phi_n(m) = \langle \delta_m, g(\mathcal{L})$$

$$\phi_n'(m) = \langle \delta_m, P_K($$

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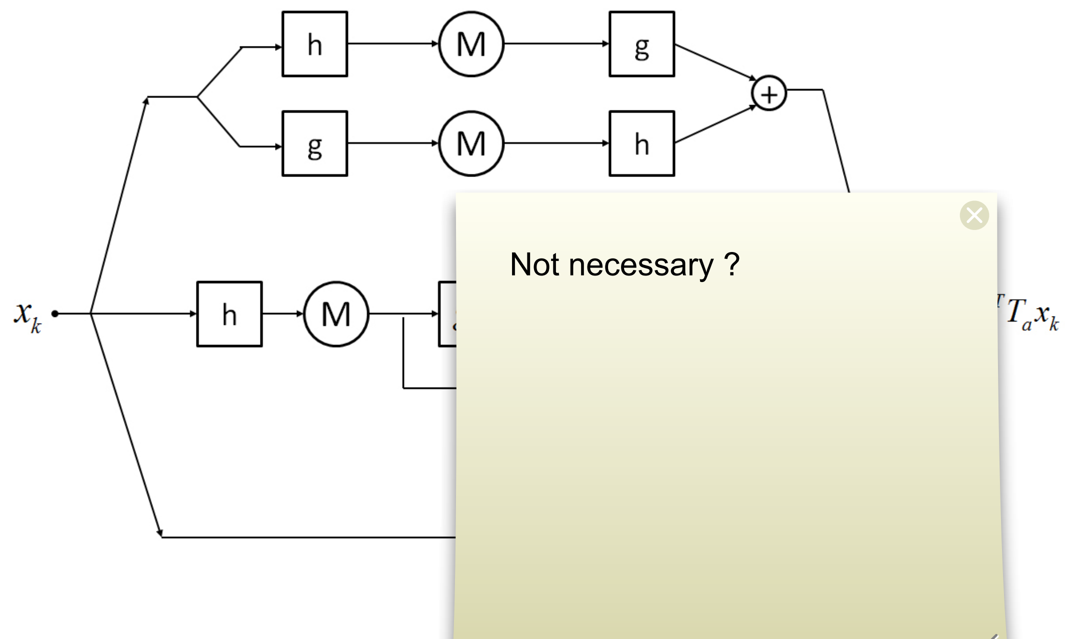
ly localized in a K-ball around n

The original feature is well-localized in a K-ball around n:

$$d_G(m,n) > K \Rightarrow \frac{|\phi_n(m)|}{\|\phi_n\|} \leq \kappa(B,K)$$

The Laplacian Pyramid

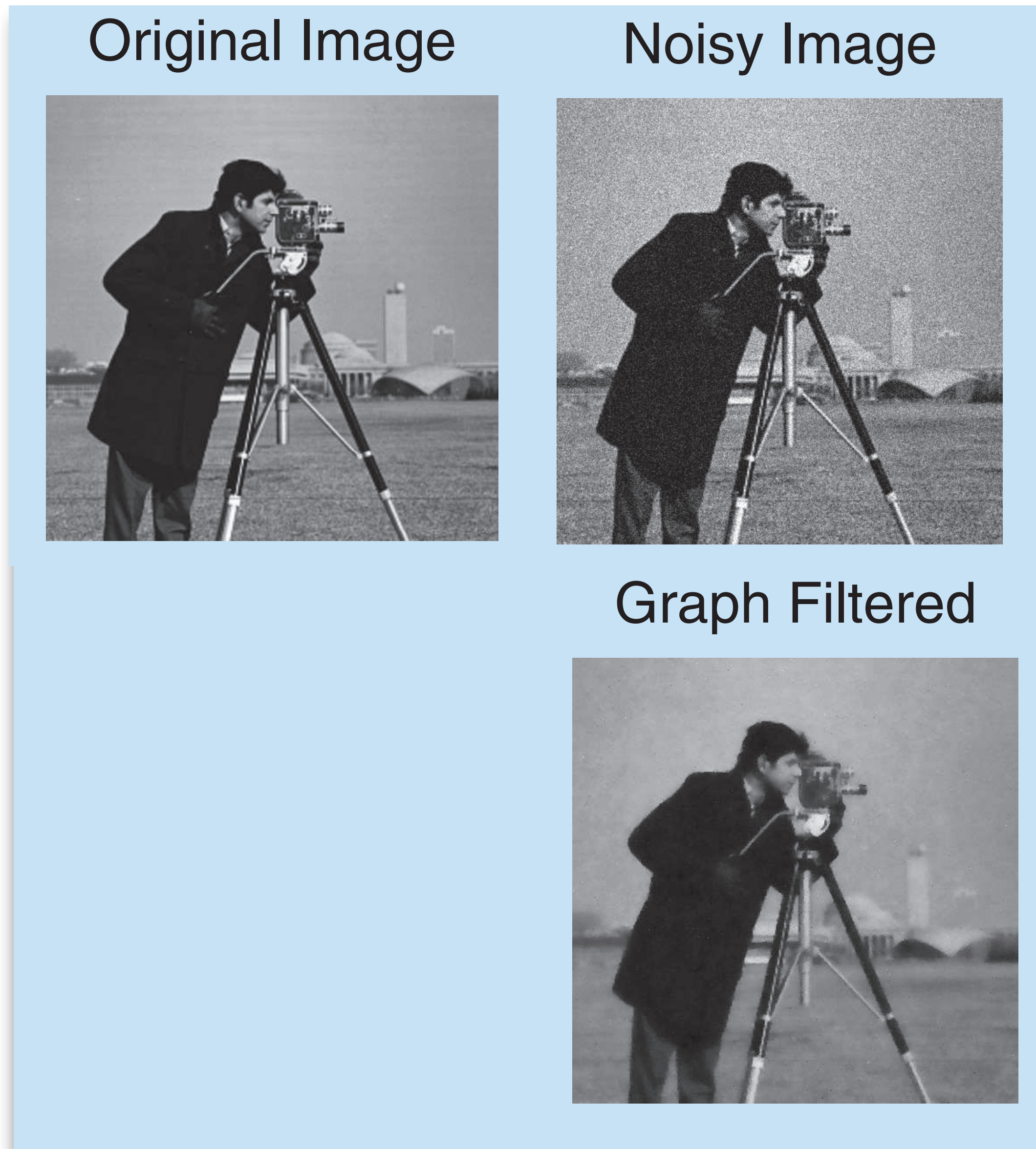
we can easily implement $\mathbf{T_a}^T\mathbf{T_a}$ with filters and masks:

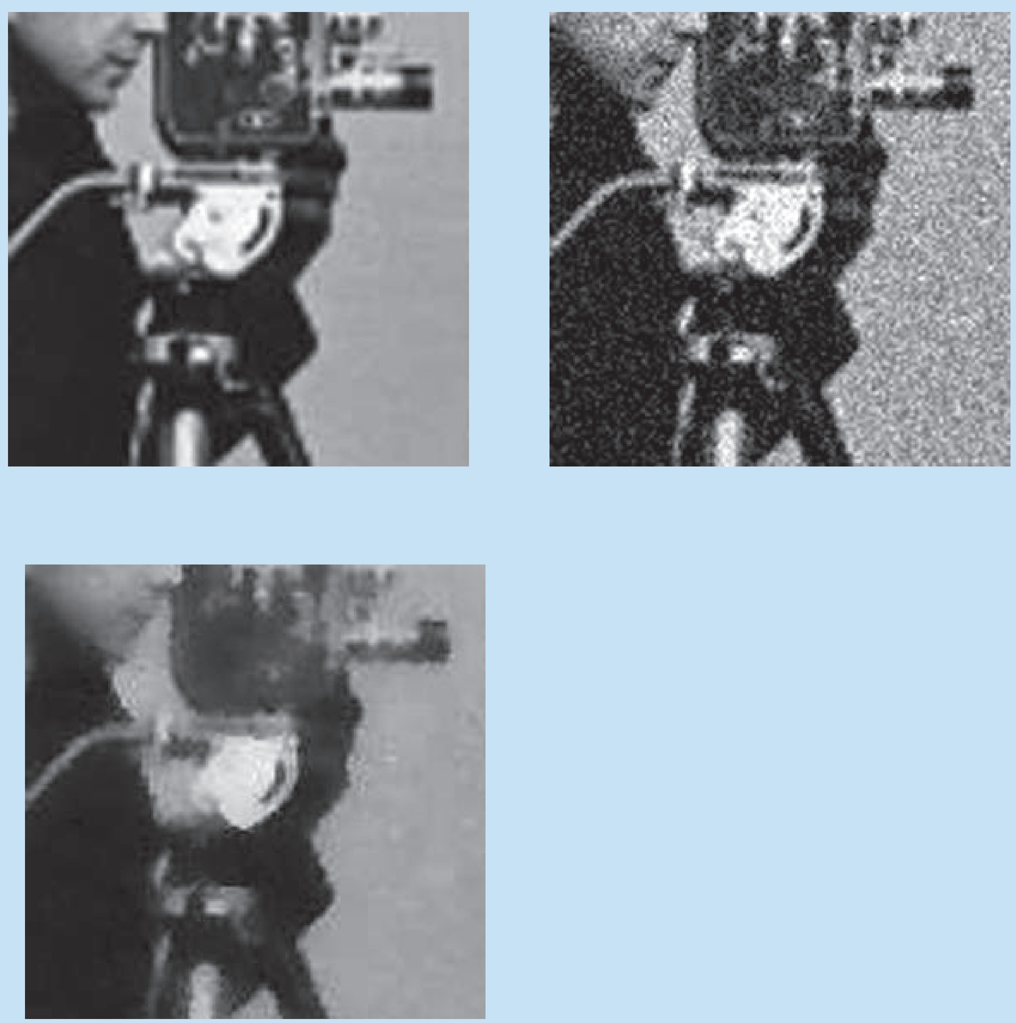


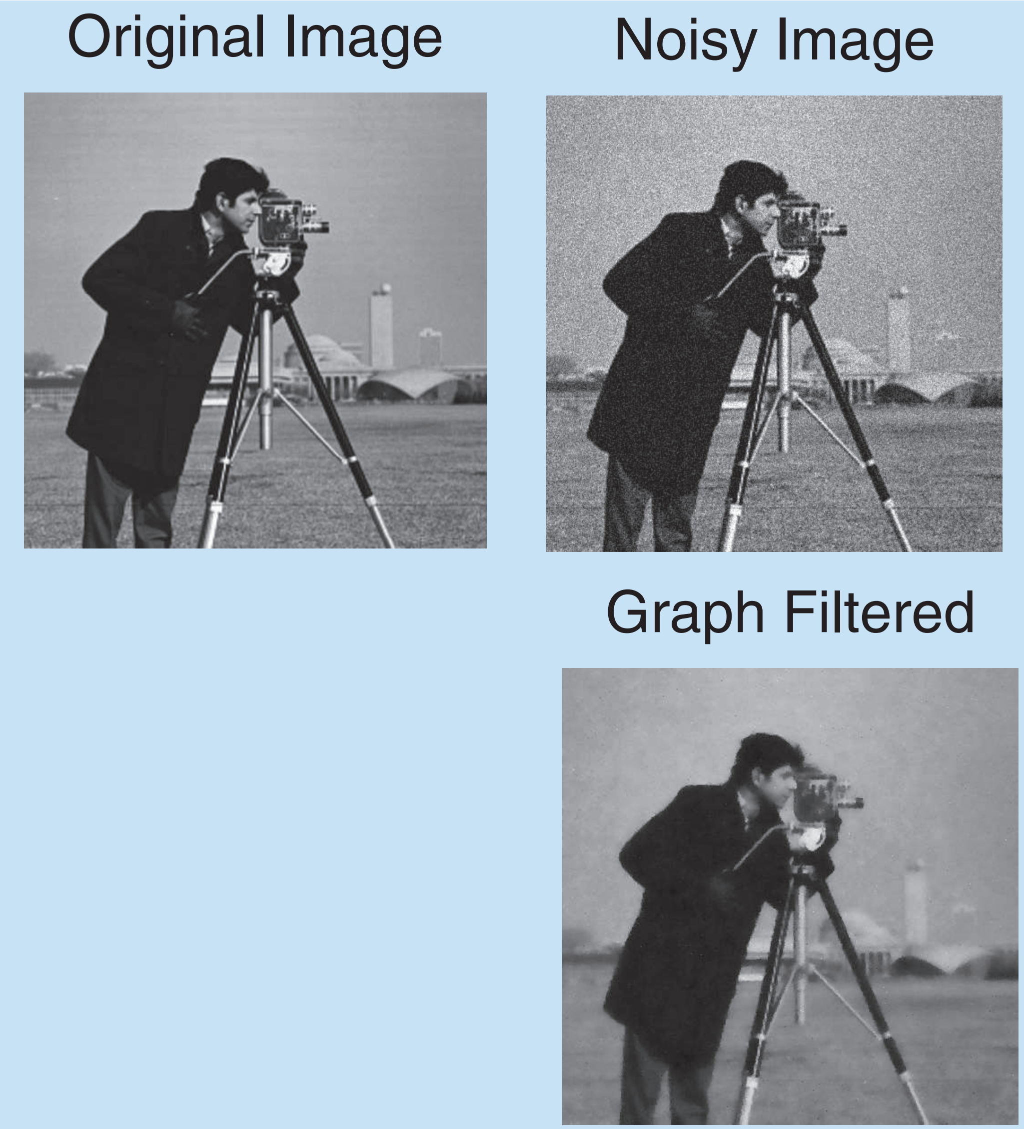
With the real symmetric matrix $\mathbf{Q} = \mathbf{T_a}^T\mathbf{T_a}$ and $b = \mathbf{T_a}^T y$

$$x_N = \tau \sum_{j=0}^{N-1} (\mathbf{I} - \tau\mathbf{Q})^j b$$

Use Chebyshev approximation of: $L(\omega) = \tau \sum_{j=0}^{N-1} (1 - \tau\omega)^j$


Original Image
Noisy Image
Graph Filtered

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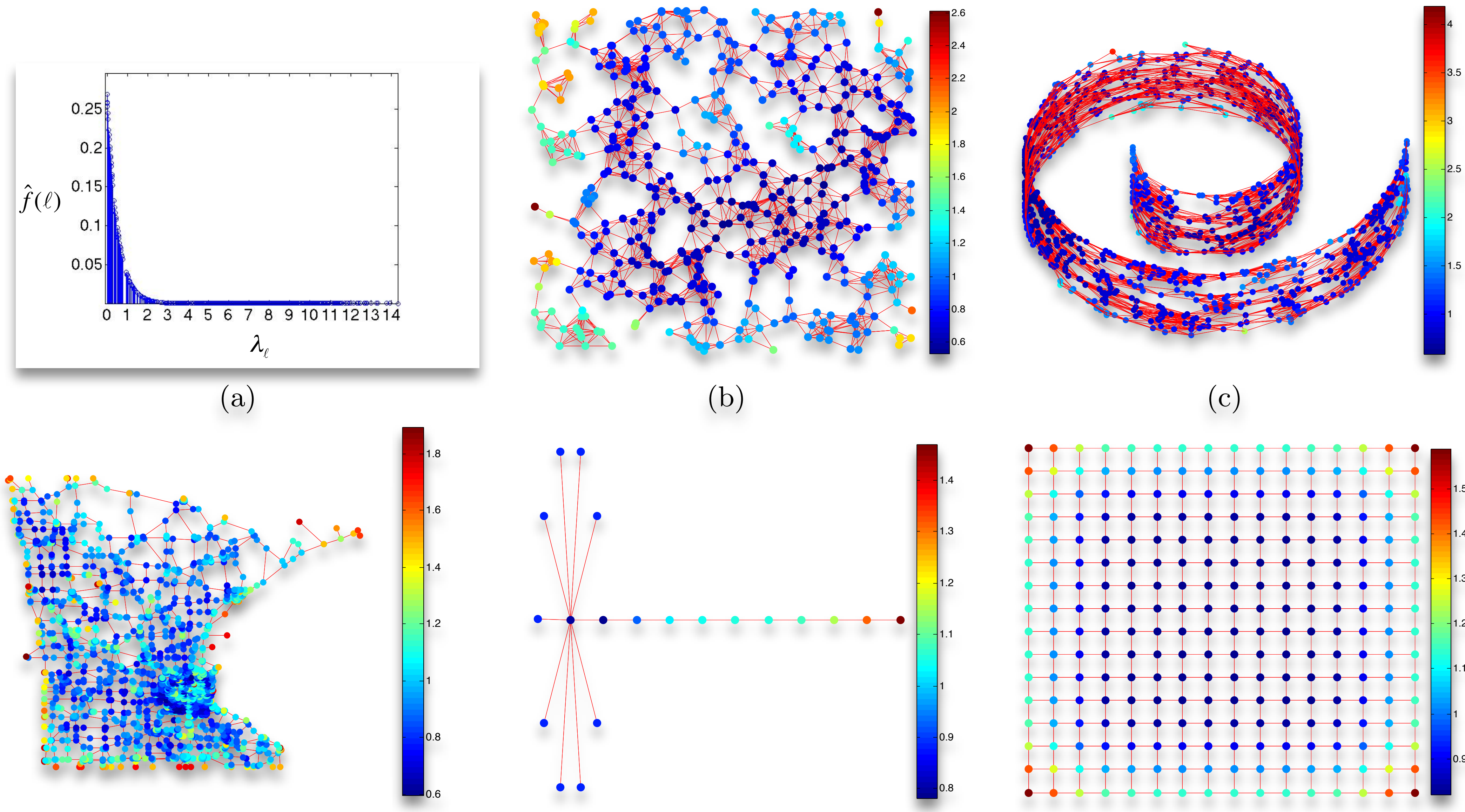

(a) (b) (c)

(d) (e) (f)