Econ Theory (2009) 41:465–494
DOI 10.1007/s00199-008-0406-0

RESEARCH ARTICLE

Fair welfare maximization

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Received: 16 August 2007 / Accepted: 30 July 2008 / Published online: 27 August 2008

Abstract We consider the general problem of finding fair constrained resource allocations. As a criterion for fairness we propose an inequality index, termed "fairness ratio," the maximization of which produces Lorenz-undominated, Pareto-optimal allocations. The fairness ratio does not depend on the choice of any particular social welfare function, and hence it can be used for an a priori evaluation of any given feasible resource allocation. The fairness ratio for an allocation provides a bound on the discrepancy between this allocation and any other feasible allocation with respect to a large class of social welfare functions. We provide a simple representation of the fairness ratio as well as a general method that can be used to directly determine optimal fair allocations. For general convex environments, we provide a fundamental lower bound for the optimal fairness ratio and show that as the population size increases, the

We would like to thank Kenneth Arrow, James Foster, Krishna Pendakur, Clemens Puppe, George Shanthikumar, participants of the 2005 Econometric Society World Congress at University College London, and seminar participants at UC Berkeley for helpful comments. Goel acknowledges NSF CAREER Award 0133968, NSF ITR Grant 0428868, a Terman Engineering Fellowship, a 3-COM Faculty Scholar Award, an Alfred P. Sloan Faculty Fellowship, and gifts from Google, Cisco, and Microsoft. Meyerson acknowledges NSF grant CCR-0122581 and ARO grant DAAG55-98-1-0170. Weber acknowledges a David T. Morgenthaler II Faculty Scholar Award and grant 1094280-1-WAKAB by the Woods Institute for the Environment.

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optimal fairness ratio decreases at most logarithmically in what we call the "inhomogeneity" of the problem. Our method yields a unique and "balanced" fair optimum for an important class of problems with linear budget constraints.

Keywords Fairness · Inequality · Lorenz-dominance · Social justice

JEL Classification C61 · C63 · D63 · I30

Nothing is more fairly distributed than common sense:
No one thinks he needs more of it than he already has.
– René Descartes

1 Introduction

In many economic situations it is desirable to allocate a limited amount of resources in a "fair" manner among a collection of individuals. As a criterion for fairness we propose an inequality index, termed "fairness ratio." The fairness ratio is robust with respect to a large class of social welfare functions, providing a bound on the discrepancy between the social value of a given allocation and any other feasible allocation. The fairness ratio has a simple representation which makes it attractive for practical applications. A maximization of the fairness ratio over all feasible allocations yields Lorenz-undominated, Pareto-optimal allocations which we refer to as optimal fair allocations.

It is often difficult to justify which allocations in particular should be considered fairest among the set of all Pareto-optimal allocations. Judging collective actions by the utility levels achieved, there is a longstanding debate about which of the feasible allocations are most preferable from a social planner's perspective. Since ranking feasible allocations corresponds to evaluating them by a social welfare function (mapping each allocation to a real number), the social planner's question is therefore about which "fair" social welfare function to select. While two prominent social welfare functions are the utilitarian (sum of all utilities) and the egalitarian (minimum of all utilities),[1] it is clear that in fact *any* Pareto-optimal allocation can become the maximizer of an appropriately chosen social welfare function. Hence, choosing a fair allocation can be viewed as essentially equivalent to the problem of choosing an appropriate social welfare function. The latter tends to be thought of as a value judgement in and of itself. We therefore consider an entire class of "canonical" social welfare functions with respect to which we define and characterize the fairness ratio. It minimizes the maximum deviation with respect to *any* social welfare function in the class (of which only a simple finite-dimensional "extremal base" is needed), naturally handling complex feasibility constraints such as limits of the available resources or distributional losses.

[1] Mill (1863, p. 110), in his essay on "utilitarianism," advocates "social utility" to decide the social planner's preference over allocations, while Rawls (1971, pp. 118–123) argues from an egalitarian viewpoint that fair resource allocations should be chosen from behind a "veil of ignorance," maximizing the wealth of the poorest individual.

It is useful to consider a simple example to illustrate these points. When deciding about how much water to allocate to a number of geographically dispersed farmers, a social planner may want to take into account the location of wells, water requirements for different farmers, their pre-existing water rights, their respective crop productivity, return-flow externalities, as well as transportation losses. A fair allocation of the available water resources would need to balance concerns for equality against utilitarian objectives, the latter being tied to the aggregate economic output of the farmers using the water. Even if the farmers' productivities are the same, by just dividing the available water equally among them a substantial amount of water might be wasted in transit, thus potentially leading to an undesirable outcome. To trade off these concerns and achieve a "fair" resource allocation, we construct an inequality index, which we term *fairness ratio* (or *relative fairness*), based on the familiar Lorenz-dominance order. Since the fairness ratio we propose does not depend on any particular social welfare function (including itself), it is possible to obtain an *a priori* preference ordering over feasible allocations, even when their associated Lorenz curves cross. Its maximization leads in many practical cases to a unique most preferred Lorenz-undominated allocation. Furthermore, a fairness ratio that is attained by a particular allocation represents (by construction) in fact a lower bound for the discrepancy between this allocation and *any* other allocation, when evaluated using *any* element of the class of symmetric, increasing, concave social welfare functions.

1.1 Literature

The approach of using indices to measure inequality has a considerable tradition.[2] Lorenz (1905) proposed measuring the *concentration of resources* for any given feasible allocation x by plotting the resources of the poorest k individuals, $P_k(x)$, against the proportion k/n these individuals represent of the total size-n population, resulting in so-called Lorenz curves. He pointed out that

> "[w]ith an unequal distribution, the curves will always begin and end in the same points as with an equal distribution, but they will be bent in the middle; and the rule of interpretation will be, as the bow is bent, concentration increases." (p. 217)

Dalton (1920) showed that a *principle of transfers* holds, in the sense that starting from x an appropriate transfer from a richer individual to a poorer individual reduces the extent of the inequality, so that the Lorenz curve of the new allocation $\hat{x}$ better approximates a perfectly fair allocation represented by the 45-degree line. If the average wealth is the same under both allocations, $\hat{x}$ is said to Lorenz-dominate x, i.e., $\hat{x}Lx$. Kolm (1969) and Atkinson (1970) build on the Lorenz-dominance order, and show that more generally a *principle of progressive transfers* holds: any allocation x, which is Lorenz-dominated by $\hat{x}$, can be obtained from $\hat{x}$ by a finite sequence of

[2] See Foster (1985), Moyes (1999), or Dutta (2002) for an overview.

progressive transfers.[3] This amounts to satisfying the *Pigou–Dalton condition* which is frequently employed as a fundamental test for any meaningful measure of inequality, for instance by Sen (1973/1997).[4]

The Lorenz-dominance relation $\hat{x}Lx$ can be used to establish a *quasi order* over feasible allocations $x, \hat{x}$ with the same total resources (i.e., for which $P_n(x) = P_n(\hat{x})$).[5] Atkinson (1970) realized the drawbacks inherent in a mere quasi order over feasible allocations, as the set of undominated allocations may be potentially large and leaves unanswered the question of how to select any particular undominated allocation over another. He proposes a class of utilitarian (i.e., additively separable) social welfare functions based on a power law, of which Sen (1973/1997) discusses a number of variations to address the inherent difficulties in the utilitarian approach. This method, often referred to as the "Atkinson–Kolm–Sen approach," was somewhat broadened by Dasgupta et al. (1973), who realize that Lorenz-dominance can be characterized using a seminal result by Hardy et al. (1929) in terms of the inequality $V(x) \leq V(\hat{x})$, which needs to hold for *all* symmetric concave functions V. To be able to compare allocations with different population means (i.e., when $P_n(x) \neq P_n(\hat{x})$) we use "weak Lorenz-dominance," denoted by $\hat{x}L_w x$.[6] The latter can be stated equivalently using a result by Tomić (1949) in terms of the same inequality as before, but which now needs to hold only for all nonnegative-valued, symmetric, increasing, concave functions V, which we term *canonical*. For any canonical V and any element x of a set of feasible allocations $\mathscr{X}$, the ratio between $V(x) \geq 0$ and the maximum attainable welfare $V^* = \sup_{\xi \in \mathscr{X}} V(\xi) > 0$ is a measure of the inequality induced by x. A similar ratio has already been suggested by Dalton (1920) to measure inequality for completely symmetric or unconstrained welfare maximization problems (cf. Sen (1973/1997, p. 37) for a brief discussion). Nevertheless, any evaluation of inequality based on such a "Dalton ratio" would heavily depend on the particular social welfare function V that is chosen. Our approach builds on an analogue of the Dalton ratio which accounts for the resource constraints, but achieves a measure of fairness that is essentially *independent* of the particular choice of a social welfare function. Our fairness measure implies a strict partial order over all allocations and turns out to offer much more resolution than weak Lorenz-dominance, which we demonstrate using a simple example.

[3] Allocation x is obtained from $\hat{x}$ by a single progressive transfer if $P_n(x) = P_n(\hat{x})$ and a positive amount of resources is moved from individual j to individual i, such that $\hat{x}_j > x_i > \hat{x}_i$. Atkinson (1970) further showed that (strict) Lorenz-dominance is equivalent to second-order stochastic dominance as introduced by Rothschild and Stiglitz (1970).

[4] Dalton (1920) suggested this condition, which equivalently stated says that a transfer from a wealthy to a poor individual cannot increase inequality. In formulating his condition he referred to an earlier suggestion by Pigou (1912).

[5] A *quasi order* is a binary relationship that is reflexive and transitive (Fishburn 1970).

[6] Shorrocks (1983) uses weak Lorenz-dominance to define "generalized Lorenz curves," which he applies to an inter-country income comparison. Davies and Hoy (1995) consider the ranking of allocations for which the Lorenz curves intersect. Consequently, their ranking based on mean-variance-preserving transformations is somewhat weaker (i.e., more general) than weak Lorenz-dominance, since it can account for multiple intersections of Lorenz curves. However, their approach is quite different from ours in that they do not consider distributional losses and are not concerned with finding constrained-optimal fair allocations.

Remark 1 A number of approaches to fairness reside in the theory of cooperative games. Axiomatic frameworks for cooperative bargaining (Roth 1979; Moulin 1988), for instance, point to particular allocations as a function of the individuals' endowments and the set of feasible allocations. One prominent example is the Shapley value for cooperative games with transferable utility (Shapley 1953), which can be given a probabilistic interpretation. For cooperative games with nontransferable utility, generalizations have been proposed by Harsanyi (1963) and Shapley (1969). The Harsanyi NTU value stresses an egalitarian allocation, whereas the Shapley NTU value tends to produce a utilitarian allocation. In the two-player case they both coincide with Nash's (1950) cooperative bargaining solution.[7] Other approaches to fair allocations (Varian 1974, 1975), e.g., realizing Foley's (1967) notion of envy-freeness, suggest first an appropriate (re-)distribution of the individuals' endowments before employing a Walrasian trading procedure to arrive at "fair" (market) outcomes.

These different approaches not only isolate generally different fair allocations, but also—and at least as importantly—are in most cases not applicable in truly general settings, which would contain nonconvexities in the set of feasible allocations and/or externalities between individuals. Our approach, based on what we term *relative fairness* (as indexed by the *fairness ratio*), can be applied in such general settings, and it is also *robust* with respect to the class of canonical social welfare functions (including the utilitarian and egalitarian, for instance).

1.2 Outline

In this paper, we propose an inequality index $\varphi(x)$ (the "fairness ratio" or "relative fairness"), which we define (roughly stated) as the infimum of the constrained Dalton ratio over the whole class of canonical social welfare functions. We show that $\varphi(\cdot)$ is concave, so that given a compact set of feasible allocations, there exists a most preferred efficient (i.e., Pareto-optimal) fair allocation, x^*. We establish a simple representation of the fairness ratio and provide a general method for computing an optimal fair allocation as well as the optimal fairness ratio. If a solution to a fair welfare maximization problem is known to be "balanced" (satisfying $n-1$ linear optimality conditions), it becomes easier to find an optimal fair allocation. As an important case in point, we show that any solution to a welfare maximization problem with a single linear (cost-ordered) budget constraint is balanced and is unique. For a convex utility possibility set we establish a fundamental lower bound for the optimal fairness ratio, $\varphi^* = \varphi(x^*)$, which scales gracefully with $O(\log_2 I_n)$ as the population size n increases, where I_n denotes the *inhomogeneity* of the problem. Finally we discuss the implementation of fair resource allocations. To illustrate our approach we examine a simple application before summarizing our results and providing directions for further research.

2 The model

Consider a social planner who faces a decision about which allocation $x = (x_1, \ldots, x_n)$ of m transferable resources (such as wealth) for a collection of $n \geq 2$ individuals to

[7] The surveys by Winter (2002) and McLean (2002) provide an overview of the related literature.

choose. We assume that the set of feasible resource allocations $\mathscr{X} \subset \mathbb{R}_+^{mn}$ is non-empty (with at least two elements) and compact. Suppose in addition that individual i's preferences over allocations in $\mathscr{X}$ have a known representation in the form of a continuous, nondecreasing and concave utility function $u_i : \mathbb{R}_+^{mn} \to \mathbb{R}$, for all $i \in \mathscr{N} = \{1, \ldots, n\}$. The vector $u = (u_1, \ldots, u_n)$ could also denote a collection of welfare weights with which the social planner evaluates objective individual resource needs.[8] Let $\mathscr{Y} = u(\mathscr{X}) - u(0)$ be the set of all attainable utility improvements $y = u(x) - u(0)$, which we also term utility possibility set. Since $\mathscr{X}$ is compact and $u(\cdot)$ continuous, $\mathscr{Y}$ is compact. In what follows we will refer to elements of both $\mathscr{X}$ and $\mathscr{Y}$ simply as "allocations." Whenever misunderstandings might arise, we call the former *resource allocations* and the latter *utility allocations*. Throughout this paper, we maintain the assumption of complete information, in which no private information is held by any individuals in the economy.

2.1 The fair welfare maximization problem

If the social planner's preferences over resource allocations can be represented by a continuous and concave social welfare function (SWF) $W : \mathbb{R}_+^n \to \mathbb{R}$, then his problem becomes that of finding

$$y^* \in \arg\max_{y \in \mathscr{Y}} W(y). \tag{1}$$

By the Weierstrass theorem (Magaril-Il'yaev and Tikhomirov 2003, p. 29) a solution of the constrained welfare maximization problem (1) exists, for W is continuous and $\mathscr{Y}$ is compact (Lang 1993, p. 36). For any $y \in \mathscr{Y}$ and $j \in \mathscr{N}$ we denote by $y_{(j)}$ the jth smallest component of y. The function $P_k(y) = \sum_{j=1}^{k} y_{(j)}$ is the k-th *prefix* of y. As outlined in Sect. 1, to determine an optimal resource allocation the social planner uses weak Lorenz-dominance as a concept of fairness for ranking different alternatives.

Definition 1 Allocation $\hat{y}$ *weakly Lorenz-dominates* allocation y, i.e., $\hat{y} L_w y$, if

$$P_k(\hat{y}) \geq P_k(y), \tag{2}$$

for all $k \in \mathscr{N}$.

In other words, if $\hat{y}$ is to be preferred to y, then any k poorest individuals are collectively at least as well off under $\hat{y}$ as under y. Weak Lorenz-dominance implies a

[8] We assume that the social planner is able to make (very limited!) interpersonal comparisons. From d'Aspremont and Gevers (1977) we know that any social welfare functional which does not allow interpersonal comparisons of utility, i.e., which is invariant with respect to independent changes of origin and units across individuals, is necessarily dictatorial, implementing the preferences of one of the individuals in $\mathscr{N}$. As Sen (1973/1997, pp. 12–13) points out, "the attempt to handle social choice without interpersonal comparability or cardinality ha[s] the natural consequence of the social welfare function being defined on the set of individual orderings. And this is precisely what makes this framework so remarkably unsuited to the analysis of distributional questions" (cf. also our discussion preceding Example 2 in Sect. 2). More recently, Fleurbaey and Maniquet (2008) discuss social preferences for the allocation of private goods that depend only on ordinal preferences.

preference quasi order of resource allocations in $\mathscr{Y}$, which is equal to the weak Lorenz-dominance order discussed earlier.[9] Let $\mathscr{V}$ denote the set of all symmetric, increasing, continuous and concave SWFs $V : \mathbb{R}^n_+ \to \mathbb{R}$ with $V(0) = 0$, which we call *admissible*. The following classical result characterizes the weak Lorenz-dominance order in terms of a "simultaneous" representation over the class of admissible SWFs $\mathscr{V}$.

Proposition 1 (Hardy et al. 1929; Tomić 1949)

$$\hat{y} L_w y \quad \Leftrightarrow \quad V(\hat{y}) \geq V(y) \quad \forall\, V \in \mathscr{V}.$$

Since any $V \in \mathscr{V}$ is continuous and the set of feasible utility allocations $\mathscr{Y}$ is compact, the maximum

$$V^* = \max_{y \in \mathscr{Y}} V(y) \tag{3}$$

exists. Since $\mathscr{Y}$ contains at least two elements, we have $V^* > 0$ by monotonicity of V.[10] In other words, the social planner strictly prefers the optimal allocation to the origin. With this, the following inequality index, which we refer to as "fairness ratio" (or "relative fairness"), is well defined.

Definition 2 For any allocation $y \in \mathscr{Y}$ we denote by

$$\varphi(y) = \inf_{V \in \mathscr{V}} \frac{V(y)}{V^*} \tag{4}$$

its *fairness ratio* (or *relative fairness*).

We observe that the fairness ratio $\varphi(y)$ evaluated at any feasible allocation $y \in \mathscr{Y}$ constitutes a lower bound of the Dalton ratio that can be attained for *any* canonical SWF at that allocation. It follows directly from the definition of relative fairness (4) that $\varphi(y) \in [0, 1]$ for all $y \in \mathscr{Y}$. The following result collects other useful properties of φ.

Proposition 2 *(i) The fairness ratio φ is symmetric, increasing, continuous and concave on $\mathscr{Y}$ with $\varphi(0) = 0$, so that $\varphi \in \mathscr{V}$. (ii) Relative fairness is invariant with respect to changes in the origin of individuals' utility functions. (iii) Relative fairness is invariant with respect to common changes of units of individuals' utility functions. (iv) A common increasing concave transformation of individuals' utility functions cannot decrease the fairness ratio, i.e., if $z = (\psi(y_1), \ldots, \psi(y_n))$, with $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(0) = 0$, continuous, increasing and concave, then $\varphi(z) \geq \varphi(y)$.*

Part (i) of the last proposition states that relative fairness is by itself an admissible SWF as it induces an ordering of feasible allocations (i.e., of social states). Since

[9] In the statistics literature weak Lorenz-dominance is generally referred to as "weak majorization," cf. Marshall and Olkin (1979, pp. 9–11).

[10] In fact, only local nonsatiation of V around the origin is needed to guarantee that the fairness index is well defined; away from the origin V may be nondecreasing instead of increasing (cf. also Footnote 22).

we have emphasized initially that the fairness ratio was introduced for its *robustness* with respect to the class $\mathscr{V}$ of all admissible SWFs, of which it turns out to be an element, the resulting seemingly tautological statement requires explanation. Indeed, since $\varphi \in \mathscr{V}$ it figures on *both* sides of its definition (4) and is therefore the single most robust admissible SWF with respect to *all* elements of $\mathscr{V}$. Parts (ii) and (iii) of Proposition 2 imply that when individuals' utilities u_i undergo a positive linear transformation to $\hat{u}_i = \alpha u_i + \beta_i$ for some constants $\alpha > 0$ and $\beta_i \in \mathbb{R}$, correspondingly $\varphi(\hat{y}; \hat{\mathscr{Y}}) = \varphi(y; \mathscr{Y})$ for all $\hat{y} \in \hat{\mathscr{Y}} = \alpha \mathscr{Y}$ and $y \in \mathscr{Y}$. For individuals with homothetic preferences, relative fairness is therefore invariant with respect to different representations in terms of Von Neumann–Morgenstern utility functions. Part (iv) of Proposition 2 implies that if individuals' risk aversion increases by a common concave transformation of their respective utility functions (Pratt 1964), then the relative fairness of *any* feasible allocation (weakly) increases. In other words, if all individuals become more risk averse, their sensitivity for resource differences decreases.

Remark 2 The fairness ratio (in a somewhat different form) and associated optimization techniques for resource allocation and stochastic scheduling were proposed by Goel and Meyerson (2003). They build on a long series of results, most notably those of Kleinberg et al. (2001) and Megiddo (1977). They also provide a detailed list of references to prior research on related algorithmic techniques. Regarding an axiomatic base for the fairness ratio, we show below (cf. Remark 3) that the fairness ratio is related to the Kalai–Smorodinsky bargaining solution and can in some sense be interpreted as a generalization thereof, which in contrast to standard bargaining is applicable even when the utility possibility set is nonconvex.

Based on the definition of relative fairness we now introduce a *strict partial order* of resource allocations.[11]

Definition 3 Let $\hat{y}, y \in \mathscr{Y}$ be two feasible allocations. Allocation $\hat{y}$ is *(relatively) fairer* than allocation y (denoted $\hat{y} \succ y$), if $\varphi(\hat{y}) > \varphi(y)$, i.e.,

$$\hat{y} \succ y \quad \overset{\text{def}}{\Leftrightarrow} \quad \varphi(\hat{y}) > \varphi(y). \tag{5}$$

The strict partial order induced by relative fairness allows us to rank different resource allocations. By Proposition 2 the fairness ratio φ is concave on $\mathscr{Y}$, so that the *fair welfare maximization problem*,

$$y^* \in \arg\max_{y \in \mathscr{Y}} \varphi(y) \equiv f(\mathscr{Y}), \tag{6}$$

has a convex compact solution set $f(\mathscr{Y})$, as a consequence of Berge's (1963) maximum theorem.[12] The set-valued function f maps any compact subset $\mathscr{Y}$ of $\mathbb{R}^n_+$ to a compact

[11] A *strict partial order* is a binary relation that is irreflexive and transitive (Fishburn 1970).

[12] The set $f(\mathscr{Y})$ of all optimal fair allocations does not have to be a singleton. As an example, consider the convex constraint set $\mathscr{Y} = \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4_+ : 2y_1 + y_4 = 3 \text{ and } y_2 + y_3 = 2\}$, which implies (using the representation (8) below) that $y^* = \{y \in \mathbb{R}^4_+ : y = (5/6, y_2, 2 - y_2, 8/6) \text{ and } 5/6 \le y_2 \le 7/6\}$ contains a continuum of elements.

subset $f(\mathscr{Y})$ (in the boundary) of $\mathscr{Y}$. If we denote by $\mathscr{P}$ the Pareto-frontier of the utility possibility set,

$$\mathscr{P} = \left\{ y \in \mathscr{Y} : \nexists\, (\hat{y}, i) \in \mathscr{Y} \times \mathscr{N} \text{ such that } \left(\hat{y} \geq y \text{ and } \hat{y}_i > y_i \right) \right\},$$

and by $\mathscr{L}$ the set of Lorenz-undominated utility allocations,

$$\mathscr{L} = \left\{ y \in \mathscr{Y} : \nexists\, (\hat{y}, k) \in \mathscr{Y} \times \mathscr{N} \text{ such that } P_k(\hat{y}) > P_k(y) \right\},$$

then the relation

$$f(\mathscr{Y}) \subset \mathscr{L} \subset \mathscr{P} \subset \partial \mathscr{Y} \tag{7}$$

expresses the apparent fact that the set of fair allocations is by construction Pareto-optimal and Lorenz-undominated. For generically $f(\mathscr{Y}) \subsetneq \mathscr{L}$, the prescriptive power of the solution of the fair welfare maximization problem (6) is larger than weak Lorenz-dominance, so that we are interested in implementing the social welfare correspondence f.

2.2 Representation of relative fairness

To solve the fair welfare maximization problem (6), the following representation (8) is essential as it transforms the infinite-dimensional optimization problem based on (4) into an equivalent finite-dimensional optimization problem, which can be efficiently solved if the set of constraints is a linear program or satisfies some additional regularity conditions [e.g., existence of a separation oracle, cf. Grötschel et al. (1993)]. The underlying idea of constructing an *extremal base* to solve optimization problems goes back at least to Chebychev (1859); it is frequently employed in the theory of minmax problems (see e.g., Dem'yanov and Malozemov 1974).

Proposition 3 *For any feasible allocation $y \in \mathscr{Y}$, the fairness ratio can be represented in the form*

$$\varphi(y) = \min_{k \in \mathscr{N}} \frac{P_k(y)}{P_k^*}, \tag{8}$$

where $P_k^* = \max_{y \in \mathscr{Y}} P_k(y)$.

The representation (8) is strikingly simple: rather than determining the infimum over uncountably many SWFs $V \in \mathscr{V}$, one needs only to take the minimum over the finitely many prefix functions $P_k \in \mathscr{V}$, $k \in \mathscr{N}$, which constitute an extremal base. Relative fairness corresponds to the k poorest individuals' collective wealth when divided by the maximum wealth that would be feasible for any poorest k individuals. The intuition for one direction of the proof of Proposition 3 is immediate: the right-hand side (RHS) of (8) cannot be smaller than the relative fairness φ, since the infimum is taken over only a (finite) subset of $\mathscr{V}$. On the other hand, if $\Phi(y) = \min_{k \in \mathscr{N}} \frac{P_k(y)}{P_k^*}$, then for any SWF $V \in \mathscr{V}$ one can show that $V(y)/\Phi(y) \geq V(y/\Phi(y)) \geq V^*$; this restricted result is sufficient to conclude that the RHS of (8) cannot be larger than φ.

Example 1 Consider the question of how to allocate a given unit of food among a set of two individuals $\mathscr{N} = \{1, 2\}$. The first individual's cardinal utility for an amount x_1 of food is $u_1(x_1) = \sqrt{x_1}$, while the second individual's cardinal utility for her amount of food x_2 is $u_2(x_2) = \sqrt{x_2}/2$. The total amount of food rations cannot exceed the available supply, i.e., $x_1 + x_2 \leq 1$. The social planner faces the problem of maximizing the relative fairness $\varphi(y)$, where $y = (y_1, y_2)$ with $y_i = u_i(x_i) - u_i(0)$ represents a particular utility allocation in the convex feasible set

$$\mathscr{Y} = \{y \in \mathbb{R}^2_+ : (y_1)^2 + 4(y_2)^2 \leq 1\}.$$

Maximizing the fairness ratio using its simpler representation (8) yields

$$\begin{aligned} y^* &\in \arg\max_{y \in \mathscr{Y}} \left\{ \min\left\{ \frac{P_1(y)}{P_1^*}, \frac{P_2(y)}{P_2^*} \right\} \right\} = \arg\max_{y \in \mathscr{Y}} \left\{ \min\left\{ \frac{y_2}{1/\sqrt{5}}, \frac{y_1 + y_2}{\sqrt{5}/2} \right\} \right\} \\ &= (3/5, 2/5), \end{aligned}$$

so that we obtain the unique optimal fair allocation of food, $x^* = (9/25, 16/25) = (0.36, 0.64)$, which achieves the optimal fairness ratio $\varphi^* = 2/\sqrt{5} \approx 0.89$. This solution compromises between the optimal Rawlsian allocation $x^1 = (1/5, 4/5)$ (maximizing $P_1(y)$) and the utilitarian optimum $x^2 = (4/5, 1/5)$ (maximizing $P_2(y)$).

In his *Nicomachean Ethics* Aristotle suggests that[13]

> "equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences."

For instance, consider the situation in which $\mathscr{N}$ represents a poor country in which some people are close to starvation, whereas $\hat{\mathscr{N}}$ denotes a rich country in which individuals are well-fed. When thinking about how to fairly allocate an additional amount of food, identical for both countries, there is no particular reason to believe—even from a normative point of view—that the optimal fair allocations y and $\hat{y}$ for both countries are the same or should be the same. For instance, if marginal utilities for additional food vary across the two countries, a larger dispersion of marginal utilities in one country means that differences in additional resources matter more, so that the optimal fair resource allocation favors poorer individuals for whom small increments of resources make a larger difference. If on the other hand the dispersion in marginal utilities is low, this prompts the social planner to increase the emphasis on overall utility (i.e., larger prefixes as will become clear below) when maximizing relative fairness, leading to more symmetric allocations. Provided that utility functions are concave, a decrease in the dispersion of marginal utilities is achieved when there is an

13 Here as quoted by Moulin (2003, p. 1); different translations (such as the one by D.P. Chase published by Dover, New York, NY, in 1998) provide different English versions for that passage, which are, however, in the same vein.

increase in the reference resource allocation. The next example illustrates that in the Aristotelian spirit a change in the individuals' circumstances can influence the optimal fair allocation, for the relative differences between individuals also change.

Example 2 Consider the situation in the last example with a change in reference level and thus marginal utility for food. We show that a fair resource allocation in a country where people are well-fed may be quite different from a fair resource allocation in a country where individuals' nourishment is poor. For this assume—in contrast to the preceding result example—that individuals are now well-fed, i.e., that one unit of food is distributed to each individual *before* the social planner decides about how to allocate an additional unit of food: the individuals' respective utilities are thus $\hat{u}_1(x_1) = u(x_1 + 1) = \sqrt{x_1 + 1}$ and $\hat{u}_2(x_2) = u_2(x_2 + 1) = \sqrt{x_2 + 1}/2$, so that the set of attainable utility improvements becomes

$$\hat{\mathscr{Y}} = \left\{ \hat{y} \in \mathbb{R}^2_+ : (\hat{y}_1 + 1)^2 + (2\hat{y}_2 + 1)^2 \leq 3 \right\},$$

resulting in $\hat{P}_1^* = (\sqrt{14} - 3)/5$ with $\hat{x}^1 = ((4\sqrt{14} - 7)/25, (32 - 4\sqrt{14})/25)$, and $\hat{P}_2^* = \sqrt{2} - 1$ with $\hat{x}^2 = (1, 0)$. We thus obtain $\hat{x}^* \approx (0.47, 0.53)$ as the unique optimal fair allocation of food in the rich country (achieving $\hat{\varphi}^* \approx 0.80$). The increase in the reference wealth and concomitant decrease in the difference of marginal utilities allows for a more symmetric allocation to be justifiable in terms of relative fairness.

Remark 3 Maximizing relative fairness can be related to the theory of cooperative bargaining. For instance, in the case of two individuals $i \in \{1, 2\}$ with potential maximum utility gains G_1, G_2 relative to their *status quo*, whereby $0 \leq G_1 \leq G_2$ (without loss of generality), the cooperative bargaining solution $K = (K_1, K_2) \in \mathscr{Y}$ proposed by Kalai and Smorodinsky (1975) is such that $K \in \mathscr{P} \subset \partial\mathscr{Y}$ (Pareto-optimality) and

$$\frac{K_1}{G_1} = \frac{K_2}{G_2}. \tag{9}$$

Maximizing the fairness ratio $\varphi(y)$ over all $y \in \mathscr{Y}$ yields

$$\frac{y_1^*}{P_1^*} = \frac{y_1^* + y_2^*}{P_2^*} \tag{10}$$

and $y^* \in \mathscr{P}$ (cf. also Remark 6 in Sect. 2.5). Hence, the Kalai–Smorodinsky bargaining solution and the optimal fair allocation coincide, i.e., $K = y^*$, if we set the potential utility gains to $G_1 = P_1^*$ and $G_2 = P_2^* - P_1^*$ (which implies $G_1 \leq G_2$), since then relations (9) and (10) are equivalent. In other words, reinterpreting our solution in terms of bargaining amounts to viewing the potential utility gains of "cost-ordered" individuals in terms of first-differences of optimal prefixes.[14] This

[14] We say that individuals are *cost-ordered*, if at any allocation $y^k \in \arg\max_{y \in \mathscr{Y}} P_k(y)$ we have that $y_{j+1}^k \geq y_j^k$ for all $j \in \{1, \ldots, n-1\}$.

observation generalizes to n individuals, as long as the optimal fair allocation y^* is "balanced" (cf. Sect. 2.5).

The last remark suggests that the optimal fair resource allocation in the sense of (6), the definition and representation of which are independent of any convexity assumptions on the utility possibility set $\mathscr{Y}$, provides a natural generalization of the Kalai–Smorodinsky bargaining solution based on the fictitious (because generally not feasible) gains G_i. Clearly, if the fair allocation is balanced, then the interpretation of this allocation coincides with the Kalai–Smorodinsky bargaining solution. The latter had been proposed informally by Raiffa (1953) and Luce and Raiffa (1957, p. 132f) to deal with criticisms of the indifference-of-irrelevant-alternatives axiom in the well-known Nash bargaining solution. An elementary axiomatic base implying our set of fair allocations, $f(\mathscr{Y})$, is subject to further research. Of course, one may trivially require Lorenz-dominance (implying symmetry and Pareto-optimality) and "robustness" in the form of the fair welfare maximization problem (6) as axioms to characterize $f(\mathscr{Y})$.

2.3 Determining optimal fair allocations

Based on the representation (8) of relative fairness, optimal fair allocations can easily be obtained once all the optimal prefixes $P_1^*, \ldots, P_n^*$ are known. Since the maximization of prefixes can be computationally burdensome and may be difficult to implement, we provide here an equivalent formulation which contains a number of adjunct variables. Even though the reformulation increases the dimensionality of the problem, it substantially simplifies the solution.

Proposition 4 *The optimal k-th prefix can be determined as the solution of an equivalent optimization problem,*

$$P_k^* = \max_{(y,z,r)\in\mathscr{Y}\times\mathbb{R}^n\times\mathbb{R}} \left\{ \sum_{i=1}^{n} z_i - (n-k)r \right\}, \tag{11}$$

subject to $z_i \leq y_i$ and $z_i \leq r$ for all $i \in \mathscr{N}$.

Given the optimal prefixes, any optimal fair allocation and associated optimal fairness ratio can then be obtained in a similar manner as the solution of an augmented equivalent optimization problem, where, in analogy to (11), each adjunct variable contains components for each prefix.

Proposition 5 *Any solution y^* to the fair welfare maximization problem* (6) *can be obtained as a solution to the equivalent optimization problem*

$$(y^*, z^*, r^*, \phi^*) \in \arg \max_{(y,z,r,\phi)\in\mathscr{Y}\times\mathbb{R}^{n^2}\times\mathbb{R}^n\times\mathbb{R}} \{\phi\}, \tag{12}$$

subject to

$$\phi \le \frac{1}{P_k^*}\left(\sum_{i=1}^{n} z_{ik} - (n-k)r_k\right),$$

$z_{ik} \le y_i$ *and* $z_{ik} \le r_k$ *for all* $i, k \in \mathscr{N}$*, where the optimal fairness ratio is given by* $\varphi^* = \varphi(y^*) = \phi^*$*.*

Remark 4 If the compact set $\mathscr{Y}$ of feasible utility allocations can be described by $l \ge 1$ linear constraints, i.e., if

$$\mathscr{Y} = \left\{ y \in \mathbb{R}_+^n : Ay \le b \right\},$$

where A is a full-rank $(l \times n)$-matrix and $b \ge 0$ is an l-dimensional vector, then the equivalent formulation (12) of the fair welfare maximization problem (6) becomes a *linear program*, for which computationally efficient algorithms exist. Even when $\mathscr{Y}$ is a general convex set, computationally efficient algorithms exist if an efficient separation oracle can be found (Grötschel et al. 1993).

2.4 A lower bound on the optimal fairness ratio

Let us now turn our attention to the behavior of the optimal fairness ratio as the number n of individuals increases: it is possible to obtain a fundamental lower bound on the optimal fairness ratio. Naturally, depending on the particular resource allocation problem, the set of feasible allocations $\mathscr{Y}_n$ may vary with n in a variety of ways. It could, for instance, be in line with Peter T. Bauer's well-known remark that "for every new mouth to feed, there are two hands to produce," in which case the total available resources would increase in a *linear* fashion. Or, it could be that "although two hands come into the world with every new mouth, it becomes, to use the language of John Stuart Mill, 'harder and harder for the new hands to supply the new mouths' " (George 1879, IV. 2). In the extreme, the total amount of allocable resources may even be fixed (e.g., in a pure division problem when $\mathscr{Y}_n = \{y \in \mathscr{R}_+^n : y_1 + \cdots + y_n \le 1\}$). Without committing to any particular rule for the growth of the feasible set of allocations with the population size n, we provide here a robust lower bound for the optimal fairness ratio φ_n^* which depends only on what we term the *inhomogeneity* I_n of the fair welfare maximization problem.

Definition 4 For any $n \ge 2$ the *inhomogeneity* of the fair welfare maximization problem (6) is given by $I_n = P_n^*/(nP_1^*)$.

The inhomogeneity $I_n \ge 1$ of the problem depends on $\mathscr{Y}_n$. For any fair welfare maximization problem it is minimal (i.e., $I_n = 1$) whenever the maximizer y^n of the n-th prefix P_n is symmetric, i.e., $y_1^n = \cdots = y_n^n$, since then $y^1 = y^n$ and $P_k^*/k = P_n^*/n$ for all $k \in \mathscr{N}$, where we denote the maximizer of the k-th prefix by

$$y^k \in \arg\max_{y \in \mathscr{Y}} P_k(y). \tag{13}$$

Otherwise the inhomogeneity increases with the average wealth of an economically efficient allocation (P_n^*/n) and decreases with individual wealth in a purely Rawlsian allocation.

CONVEXITY ASSUMPTION. *(i) $\mathscr{Y}$ is convex, or (ii) $\mathscr{X}$ is convex and u_i is concave for all $i \in \mathscr{N}$.*

Indeed, if (i) holds, then (1) is a convex optimization problem, so that the set of all y^* is convex. If only (ii) holds, then let $\mathscr{Z} = \mathscr{X} \times \{y \in \mathbb{R}_+^n : y \leq u(x), x \in \mathscr{X}\}$ with elements $z = (x, y)$, and consider the *relaxed* welfare maximization problem of finding

$$z^* \in \arg \max_{(x,y)\in\mathscr{Z}} W(y). \tag{1'}$$

One can readily verify that the relaxed problem (1') is convex[15] and that at the optimal $z^* = (x^*, y^*)$ it is $y^* = u(x^*)$, just as in the original problem (1) with convex $\mathscr{Y}$. In what follows, we require that our convexity assumption holds and limit our attention (without loss of generality) to the case where (i) is satisfied, so that the welfare maximization problem (1) is convex. Furthermore, we concentrate our discussion on (utility) allocations y, knowing that by the monotonicity of u there is a correspondence between utility allocations y and equivalent resource allocations $x \in u^{-1}(y)$.

Proposition 6 *If the convexity assumption is satisfied, then $\varphi(y^*) \geq (2+2\lfloor\log_2 I_n\rfloor)^{-1}$, where $I_n = P_n^*/(nP_1^*)$.*

The main idea of the proof is as follows: we first consider the allocations which maximize the average wealth of the k poorest individuals, $y^k \in \arg\max P_k(y)/k$. Then we take the average over some of these allocations, which by convexity of $\mathscr{Y}$ is feasible. The trick is to carefully select the y^k's to include in the average. Since (as we show in the appendix) the average wealth $\alpha_k = \max P_k(\mathscr{Y})/k$ is nondecreasing in the index k, it is possible to select only indices in $\mathscr{N}$ which increase this ratio by at least a factor of two. This reduces the number of elements to be averaged over to $2+2\lfloor\log_2 I_n\rfloor$, where $I_n = \alpha_n/\alpha_1$ is the inhomogeneity of the fair welfare maximization problem introduced earlier. We note at this point (as will become clear with the example discussed in Sect. 3) that even though the proposed bound may not be tight, its *logarithmic* decrease with the inhomogeneity I_n as n increases cannot be improved upon in general. Our fundamental lower bound in Proposition 6 is also substantially better than the alternative lower bounds $1/n$ or $1/I_n$ which can be easily obtained.

2.5 Balanced solutions

The particular form of the representation (8) suggests that at an optimal fair allocation y^*, which by definition maximizes relative fairness, the corresponding prefix ratios $R_k^* = P_k(y^*)/P_k^*$ are all equal, i.e.,

[15] The constraint set $\mathscr{Z}$ is thereby closed and bounded from above, which (together with the continuity of W) guarantees the existence of z^*. It is also convex, which is implied by the concavity and monotonicity of v.

$$\frac{P_1(y^*)}{P_1^*} = \cdots = \frac{P_n(y^*)}{P_n^*}. \tag{14}$$

If (14) is satisfied, the allocation y^* and the prefix ratios R_k^* are referred to as *balanced*. We note that in general not every fair welfare maximization problem has a balanced solution (cf. Example 3 below), but (14) is satisfied whenever the feasible set $\mathscr{Y}$ can be expressed in terms of a single linear budget constraint.

Proposition 7 *Let the set of feasible allocations be of the form*

$$\mathscr{Y} = \{y \in \mathbb{R}_+^n : \gamma \cdot y \le 1\}, \tag{15}$$

for some strictly positive vector of weights $\gamma = (\gamma_1, \ldots, \gamma_n)$*, where* $\gamma_1 > \cdots > \gamma_n$*. (i) Then the wealth ratios* $P_k(y^*)/P_k^*$ *are balanced at* y^**, i.e., relation* (14) *is satisfied. (ii) Furthermore, the unique optimal fair allocation is given by*

$$y_{(k)}^* = y_k^* = (P_k^* - P_{k-1}^*)\varphi^*, \tag{16}$$

for $k \in \{2, \ldots, n\}$*, and*

$$y_{(1)}^* = y_1^* = \left(1 - \sum_{k=2}^{n} \gamma_k y_k^*\right)/\gamma_1, \tag{17}$$

with optimal fairness ratio

$$\varphi^* = \left(\frac{\gamma_1}{\sum_{i=1}^{n} \gamma_i} + \sum_{k=2}^{n} \gamma_k (P_k^* - P_{k-1}^*)\right)^{-1}. \tag{18}$$

The balancedness condition (14) is important because by introducing $n-1$ optimality conditions it, together with the fact that the optimal fair allocation is Pareto-undominated (i.e., in particular $y^* \in \partial\mathscr{Y}$), simplifies the solution of (6) considerably.[16] We now provide an algorithm that can be used to efficiently compute the optimal prefixes P_k^* for the linear cost-ordered problem described in Proposition 7.

Algorithm. First, define the ith step allocation $z^{(i)}$ to be one where $z_1^{(i)} = z_2^{(i)} = \cdots = z_{i-1}^{(i)} = 0$, and $z_i^{(i)} = z_{i+1}^{(i)} = \cdots = z_n^{(i)} = (\sum_{k=i}^{n} \gamma_k)^{-1}$. Second, note that—as in the proof of Proposition 4—for any $k \in \mathscr{N}$ there exists an index j such that $P_k^* = P_k(z^{(j)})$. Hence, the optimal k-th prefix P_k^* can be efficiently computed, as

$$P_k^* = \max_{i \in \mathscr{N}} \left\{ P_k(z^{(i)}) \right\},$$

for any $k \in \mathscr{N}$.

[16] Note that the constants $P_1^*, \ldots, P_n^*$ still have to be determined by solving n optimization problems (cf. Sect. 2.2).

Remark 5 If in Proposition 7 the vector γ is such that its elements are not strictly cost-ordered, i.e., $\gamma_1 \geq \cdots \geq \gamma_n$, then there exists an optimal fair allocation that is balanced, but there may be other optimal fair allocations that are not balanced.[17]

In the following two examples we show that the balancedness condition (14) does not hold in general. The first example points to problems that can arise through multiple linear constraints.

Example 3 Consider three individuals, A, B, and C, whom we assume to be risk neutral for simplicity, so that $y_i = x_i$ for all $i \in \{A, B, C\}$. Let the set of feasible allocations be given by

$$\mathscr{Y} = \left\{ y \in \mathbb{R}^3_+ : y_A + 2y_C \leq 3 \quad \text{and} \quad y_B \leq 1 \right\}.$$

To determine the relative fairness $\varphi(y)$ of any given allocation $y \in \mathscr{Y}$, let y^k be a maximizer of the k-th prefix as in (13). With this the optimal prefixes become $(P_1^*, P_2^*, P_3^*) = (1, 2, 4)$, and their respective maximizers are $y^1 = y^2 = (1, 1, 1)$ and $y^3 = (3, 1, 0)$. Using the representation (8) of relative fairness, we can determine the unique fair allocation,

$$y^* \in \arg\max_{y \in \mathscr{Y}} \{\min\{P_1(y), P_2(y)/2, P_3(y)/4\}\} = (1.4, 1, 0.8).$$

We note now that $P_1(y^*)/P_1^* = P_3(y^*)/P_3^* = 0.8$, but $P_2(y^*)/P_2^* = 0.9$. The reason for the lack of balancedness in this example is that individual B's allocation is independent of individual A's and individual C's, so that it is impossible to transfer wealth from B without leaving the set of Pareto-undominated allocations.

Even when the Pareto-frontier $\mathscr{P} \subset \partial\mathscr{Y}$ is smooth, the optimal fair allocation y^* lies in the interior of $\mathscr{P}$, and small transfers among all individuals are possible while staying in $\mathscr{P}$, it is possible that balancedness fails, as the following example with a single nonlinear constraint illustrates.

Example 4 Consider the situation when the convex set of feasible utility allocations is given by $\mathscr{Y} = \{(y_1, y_2, y_3) \in \mathbb{R}^3_+ : 4y_1 + y_2^2 + y_3^2 \leq 1\}$. Then maximizing P_2 and P_3 yields $y^2 = y^3 = (0, 1/\sqrt{2}, 1/\sqrt{2})$ with $P_2^* = 1/\sqrt{2}$ and $P_3^* = \sqrt{2}$. To obtain an optimal fair allocation, it is possible to solve the fair welfare maximization problem (6) sequentially, first with respect to (y_2, y_3) (with y_1 as a parameter) and then with respect to y_1. For any given y_1 the Pareto-set containing y^* is given by $\mathscr{P} = \{y \in \mathbb{R}^3_+ : (y_2)^2 + (y_3)^2 = 1 - 4y_1\}$, so that for any given y_1 clearly $y_2^*(y_1) = y_3^*(y_1)$. Furthermore, at any optimal fair allocation $y^* \in \mathscr{P}$ we have that

$$R_2^* = \frac{P_2(y^*)}{P_2^*} = \frac{y_1^* + y_2^*}{1/\sqrt{2}}$$

[17] This can be seen by modifying Step 5 of the proof of Proposition 7 (i) accordingly: with indifference between neighboring cost coefficients it may be possible to transfer resources among individuals without changing the optimal fairness ratio, leading to a multiplicity of solutions. Nevertheless, even when the cost order is not strict, there always exists a balanced optimal fair allocation.

and

$$R_3^* = \frac{P_3(y^*)}{P_3^*} = \frac{y_1^* + y_2^* + y_3^*}{\sqrt{2}}.$$

Using the fact that $y_2^* = y_3^*$ we obtain that $R_2^* > R_3^* = (y_1^*/\sqrt{2}) + y_2^*\sqrt{2}$. The (unique) optimal fair allocation in this example is thus not balanced.

Fair welfare maximization problems with a single linear constraint can easily be cost-ordered (cf. Footnote 14) as in Proposition 7, where we have required that the cost coefficients $\gamma_1, \ldots, \gamma_n$ be labeled in a decreasing order. The corresponding optimal fair allocation y^* is therefore *unique*.[18] By contrast, we would like to stress that the set $\mathscr{L}$ of all undominated allocations with respect to the weak Lorenz-dominance order,

$$\mathscr{L} = \left\{ y \in \mathscr{Y} : (\gamma_i \le \gamma_j \;\Rightarrow\; y_i \ge y_j) \text{ and } \gamma_1 y_1 + \cdots + \gamma_n y_n = 1 \right\} \subset \mathscr{P}, \tag{19}$$

is generally quite large (cf. Fig. 2). By maximizing relative fairness, we obtain a *unique* optimal fair allocation y^*, which substantially strengthens the normative predictions for what constitutes an optimal fair resource allocation in this setting.

Remark 6 The precise characterization of the class of all fair welfare maximization problems with (unique) balanced solutions remains an interesting open research question.

2.6 Implementing fair allocations

We first maintain our assumption of complete information, so that any fair resource allocation x^* can be implemented by direct allocation of quantities. If the resource is a (nonexcludable) public good, a social planner may opt for the use of permits or quotas. Equivalently the planner may choose to impose an appropriate tax/subsidy system to obtain the same fair distribution of resources.[19] The latter may be achieved using a standard nonuniform Pigouvian taxation (following the work of Pigou (1920)), where individuals are generally taxed differently and corresponding to their marginal utility at the optimal fair allocation x^*. In certain special cases it is possible to implement a fair resource allocation via uniform taxation that differentiates only across products but not across individuals (Diamond 1973; Green and Sheshinski 1976).

If we relax the assumption of complete information such that the social planner (principal) may be the only one without complete information about the preferences of the individuals (agents), an implementation of the fair social welfare correspondence f may still be feasible. Since f satisfies Maskin's *monotonicity* axiom (a necessary condition for incentive compatibility in any exact Nash implementation), in the sense that for any two nonempty compact utility possibility sets $\mathscr{Y}$ and $\hat{\mathscr{Y}}$, generated by utilities u and $\hat{u}$ respectively, we must have that $y \in f(\mathscr{Y}) \setminus f(\hat{\mathscr{Y}}) \;\Rightarrow\; \exists(z, i) \in$

18 Cost ordering implies that furthermore $y_{(i)}^* = y_i^*$, for all $i \in \mathscr{N}$.

19 In the presence of uncertainty this equivalence may disappear, as noted by Weitzman (1974).

$\mathbb{R}^n_+ \times \mathcal{N}$ such that $u_i(y) \geq u_i(z)$ and $\hat{u}_i(z) > \hat{u}_i(y)$, as can be seen by choosing any $z \in f(\hat{\mathcal{Y}})$ as a test element. If, in addition, $n \geq 3$ and the individuals' preferences satisfy the *strongly conflicting preferences* assumption, which in our context requires that each Lorenz-undominated alternative $y \in \mathcal{L}$ is the most preferred alternative for at most $n-2$ agents, then Maskin monotonicity and strongly conflicting preferences are together sufficient to guarantee the Nash-implementability of f (see e.g., Palfrey and Srivastava (1993)).

The design of more general mechanisms for a (robust) implementation of the fair social welfare correspondence in the presence of asymmetric information between individuals and social planner remains an interesting open research topic. Subsequent to the authors' work, Cho and Goel (2006) propose a sequential price-quantity mechanism to approximately implement optimal fair allocations. Leroux and Leroux (2004) consider the problem of implementing an envy-free division of a given resource when individuals have (unknown) linear preference representations.

3 An example: fair water distribution

We now look at an application to a stylized water resource system in somewhat more detail to illustrate our results. Consider $n \geq 2$ farmers who own an acre of land each. The land needs to be irrigated via a central aqueduct of capacity $C > 0$ acre-feet. Transporting water from the aqueduct to a farmer may result in losses due to inefficiencies in the transportation system. If x_i units of water are allocated to farmer i, the amount that this farmer actually receives is $y_i = \lambda_i x_i$, where $\lambda_i \in (0, 1]$ for all $i \in \mathcal{N}$. The feasible set describing the amount of water received by each farmer is thus given by

$$\mathcal{Y} = \left\{ y \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \frac{y_i}{\lambda_i} \leq C \right\}.$$

The social planner's goal is to allocate water fairly to all farmers. Note first that the notion of fairness is somewhat nebulous here: for political reasons it might be unacceptable to completely starve any farmer (corresponding e.g., to maximizing minimum output, $P_1(y) = \min\{y_1, \ldots, y_n\}$). On the other hand, for efficiency reasons, farmers close to a water supply should be given more water (corresponding e.g., to maximizing total output, $P_n(y) = y_1 + \cdots + y_n$). At the same time, it might be important to be fair in other ways, for example to ensure that the total amount of water allocated to farmers below the median allocation is reasonable.

This is a fair welfare maximization problem with a linear budget constraint (described in Proposition 7), as can be seen by setting $\gamma_i = 1/(C\lambda_i)$. Hence, the optimal fair allocation y^* which maximizes $\varphi(y)$ is unique and satisfies the balancedness condition (14). Assume, without loss of generality, that the farmers are arranged in nonincreasing order of λ_i. In other words, as in Proposition 7, we assume that the individuals are cost-ordered, so that farmer 1 is the cheapest to serve and farmer n is the most expensive to serve. To solve the fair welfare maximization problem, one first determines the optimal prefixes $P_1^*, \ldots, P_n^*$, which is easily accomplished by

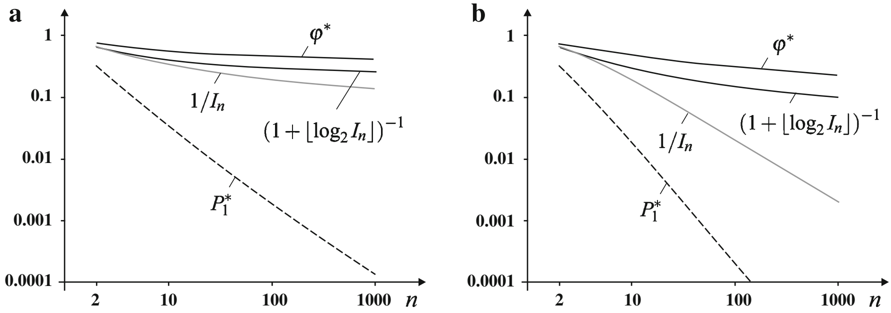


Fig. 1 Behavior of P_1^*, $1/I_n$, φ^*, and the lower bound in Proposition 6 in the fair water allocation problem, for both the linear **(a)** and the hyperbolic **(b)** loss case

using the algorithm provided at the end of Sect. 2.5. Since P_1^* and P_n^* can be obtained analytically, it is possible to explicitly discuss the asymptotic behavior of the optimal fairness ratio for increasing n in terms of the problem inhomogeneity $I_n = P_n^*/(nP_1^*)$ (assuming here that the overall water capacity C stays constant).

To maximize the poorest farmer's utility $y_{(1)} = y_1$, all farmers receive an equal amount of water (after transportation losses), resulting in $P_1^* = C(\sum_{i=1}^n 1/\lambda_i)^{-1}$. On the other hand, maximizing the utilitarian objective $P_n = y_1 + \cdots + y_n$, all the available water should be given to the first farmer, so that $P_n^* = C\lambda_1$. As a result, the inhomogeneity of this problem is (according to Definition 4) given by

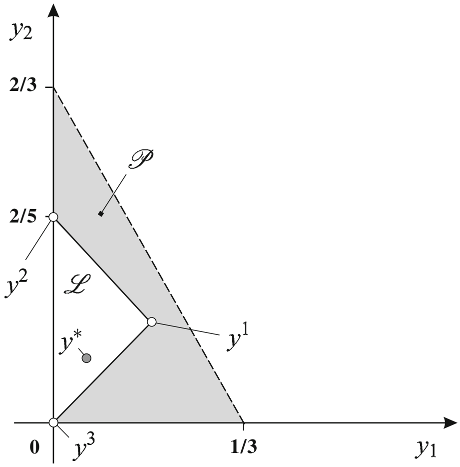
$$I_n = \frac{P_n^*}{nP_1^*} = \frac{1}{n}\sum_{i=1}^{n}\frac{\lambda_n}{\lambda_i}. \tag{20}$$

The allocation y^1 which maximizes P_1 attains thereby a fairness ratio of $\varphi(y^1) = 1/I_n$, while the allocation y^n which maximizes P_n attains a fairness ratio of $\varphi(y^n) = 0$. We note that in both cases the attained relative fairness is significantly below our lower bound on the optimal fairness ratio in Proposition 6. For concreteness, let us illustrate this fact here for two special cases:

1. *Linear Distributional Losses ($\lambda_i = i/n$).* The first farmer receives only a $(1/n)$-fraction of the water shipped to him, whereas the n-th farmer sees no transportation loss. Figure 1a shows the computed value of φ^* as n varies; for comparison, the value of I_n is also shown. Figure 1a illustrates that the optimal fairness ratio degrades gracefully as n increases. Also, the difference between the optimal fairness ratio and the fairness ratio of the allocation that maximizes P_1 (recall that this is just I_n) is quite significant.
   In this scenario, relation (20) simplifies to $I_n = \sum_i(1/i) = h_n$. Here, h_n denotes the n-th harmonic number,[20] satisfying $\ln n < h_n < 1 + \ln n$. Hence, as the population size n increases, our lower bound (from Proposition 6) on the optimal

[20] It is $h_n = \Psi(n+1) + \gamma$, where $\Psi = \Gamma'/\Gamma$ denotes the digamma function and $\gamma \approx 0.5772156649$ is the Euler constant.

Fig. 2 Set of feasible Lorenz-undominated allocations $\mathscr{L} \subset \mathscr{Y}$ in the (y_1, y_2)-plane for the fair water allocation problem (with $n = 3$)



fairness ratio degrades in proportion to $1/\ln(\ln n)$. We expect many practical problems to be similar in the sense that the inhomogeneity grows slowly as the population size increases, resulting in an even slower degradation of the optimal fairness ratio.

To build some numerical intuition and to demonstrate the normative power of our method, we briefly consider the case when $n = 3$. In that case the allocations y^k which maximize the respective prefix P_k, $k \in \mathscr{N}$, are $y^1 = (2/11, 2/11, 2/11)$, $y^2 = (0, 2/5, 2/5)$, and $y^3 = (0, 0, 1)$, giving $P_1^* = 2/11$, $P_2^* = 2/5$, and $P_3^* = 1$. Using the representation (8) of the fairness ratio, we find that the *unique* optimal fair allocation, $y^* = (10/81, 4/27, 11/27)$, compromises among the three prefix-maximizing allocations y^1, y^2, and y^3. The set $\mathscr{L}$ of all Lorenz-undominated allocations [cf. relation (19)] is depicted in Fig. 2.

2. *Hyperbolic Distributional Losses* ($\lambda_i = 1/(n - i + 1)$). This problem has a much higher inhomogeneity. Relation (20) simplifies to $I_n = \sum_{i=1}^{n}(i/n) = (n + 1)/2$. Figure 1b shows the computed value of φ^* as a function of n; it also shows the value of $1/I_n$ for comparison. It is clear that φ^* is proportional to $1/\log_2 I_n$, which indicates that Proposition 6 cannot be improved other than by changing the multiplicative and/or additive constants in the bound. One can show that in this example we have that $\varphi^* \leq 2/(1 + \ln(I_n))$, an inequality which cannot be improved upon.

We note that constraints in this fair water allocation problem can easily be modified to take into account quotas for individual farmers, selective subsidies (e.g., to poor farmers), multiple crops, differential pricing, property rights, multiple water sources, or consumption externalities through return flows. The Pigouvian taxes of $t_i = \lambda_i$ mentioned in Sect. 2.6 can implement the fair resource allocation x^* in this example, albeit in a nonunique fashion, for at this price the farmers, due to the linearity of their utility functions, are indifferent about the amount of water they receive.

4 Discussion

Measures of inequality abound. Most of them either depend heavily on the choice of a particular SWF (e.g., the Atkinson–Kolm–Sen approach) or yield a rather large set of undominated allocations, such as when using the (weak) Lorenz-dominance order as the criterion for social preference. We have introduced a new inequality index, termed *fairness ratio* (or *relative fairness*), which does not depend on any particular SWF and yields—at least when the solution is balanced—a unique prediction as to which allocation should be considered fairest. Our approach can deal with redistributional losses (or gains), for in practice feasible sets rarely admit perfectly symmetric and at the same time economically efficient allocations (in terms of maximizing the sum of all utilities). The approach also very naturally accommodates any constraints on how resources can be allocated and allows an efficient approximate computation, as long as the set of attainable utility vectors stays convex. Such constraints might be introduced through property rights and contractual obligations, transportation losses, capacity constraints, regulation, minimum and maximum resource requirements, to name just a few. We emphasize that our method is not in the least affected by the dimensionality of the resources, provided simply that the "utility image" generated by the set of feasible resource allocations is convex.

Even though originally defined as the infimum over a class of "canonical" (i.e., increasing, symmetric, and concave) SWFs, the representation of relative fairness is strikingly simple (cf. Proposition 3): it depends only on n prefixes, which makes this method readily implementable in practice. Propositions 4 and 5 can be directly used to find optimal fair allocations. In addition, it is possible to reinterpret our approach in terms of robustness: any attainable fairness ratio is in fact a lower bound for the Dalton ratio that can be achieved by any canonical SWF. Thus, when confronted with the question of how fair a particular allocation of resources is, the fairness ratio evaluated at that allocation provides an *a priori* index which guarantees that a certain discrepancy between the considered allocation and an optimal allocation with respect to *any* canonical SWF cannot be exceeded. Furthermore, the inhomogeneity I_n of the problem (cf. Definition 4) serves as an important guide by providing a fundamental lower bound on the optimal fairness ratio that can be obtained. The logarithmic behavior of our general lower bound in Proposition 6 as a function of the inhomogeneity of the problem cannot be improved upon, as the application discussed in Sect. 3 clearly demonstrates.

Following a suggestion by Kenneth Arrow, we would like to note at this point that the fairness ratio can in principle also be used to guide *intertemporal* fair resource allocations, e.g., to achieve intergenerational equity (cf. Arrow et al. 2004). For this, one would need to think of how to allocate a limited resource not to different individuals, but instead to T different time periods or generations, indexed by $t \in \{1, \dots, T\}$. The feasible set might then be implicitly defined by a technological discount rate, which could imply redistributional gains or losses as time goes on. Further work is needed to develop intertemporal fair resource allocation on the basis of the fairness ratio.

Appendix

Proof of Proposition 1 [21] $\Leftarrow$: Since $V(\hat{y}) \geq V(y)$ for all canonical SWFs $V \in \mathscr{V}$ and $P_k \in \mathscr{V}$ for all $k \in \mathscr{N}$, we have that $P_k(\hat{y}) \geq P_k(y)$ for all $k \in \mathscr{N}$. Thus, by Definition 1 allocation $\hat{y}$ weakly Lorenz-dominates allocation y, i.e., $\hat{y} L_w y$. $\Rightarrow$: Suppose now that allocation $\hat{y}$ is at least as fair as y, i.e., $\hat{y} L_w y$. We will use the following lemma, which is essentially due to Muirhead (1903), Hardy et al. (1934/1952, p. 47), Mirsky (1959), and Chong (1976). A similar formulation in terms of T-transformations and doubly superstochastic matrices can be found in Marshall and Olkin (1979, Lemma B.1 and Proposition D.2.a on p. 21, 30 respectively).

Lemma 1 *If $\hat{y} L_w y$, then there exists an allocation $z \in \mathbb{R}^n_+$ (not necessarily feasible), so that zLy and $\hat{y} - z \in \mathbb{R}^n_+$.*

In other words, weak Lorenz-dominance can be decomposed into Lorenz-dominance (which requires that $\hat{y} L_w y$ and $P_n(\hat{y}) = P_n(y)$) and a nonnegative transfer $\delta = \hat{y} - z$, which generalizes the principle of transfers related to (strict) Lorenz-dominance. Now consider any SWF $V \in \mathscr{V}$. Since V is increasing,[22] we have by Lemma 1 that $V(\hat{y}) \geq V(z)$. Moreover, since V is symmetric and concave (i.e., also Schur-concave) we can apply a standard result on Lorenz-dominance, namely that zLy implies that $V(z) \geq V(y)$ (cf. Marshall and Olkin 1979, Proposition C.2, p. 67), which completes our argument.

Proof of Proposition 2 (i) Let $\theta \in (0, 1)$. Then for any two feasible allocations $y, \hat{y} \in \mathscr{Y}$ we have that (using the concavity of any V in $\mathscr{V}$)

$$\varphi(\theta y + (1-\theta)\hat{y}) \geq \inf_{V \in \mathscr{V}} \frac{\theta V(y) + (1-\theta)V(\hat{y})}{V*} \geq \theta\varphi(y) + (1-\theta)\varphi(\hat{y}),$$

which implies that φ is concave. The symmetry, monotonicy and continuity of φ follows directly from the representation (8) in Proposition 3. (ii) This part follows immediately from the definition of $y_i = u_i(x) - u_i(0)$. (iii) Consider a common positive linear transformation of individuals' utility functions, so that $z = \alpha y$ for some constant $\alpha > 0$. If we set $U(y) = V(\alpha y)$, then $U \in \mathscr{V}$ if and only if $V \in \mathscr{V}$. As a result,

$$\varphi(z; \alpha\mathscr{Y}) = \inf_{V \in \mathscr{V}} \frac{V(\alpha y)}{\max V(\alpha\mathscr{Y})} = \inf_{U \in \mathscr{V}} \frac{U(y)}{\max U(\mathscr{Y})} = \varphi(y; \mathscr{Y}).$$

[21] Our statement of Proposition 1 combines a number of well-known results. Its proof is provided here with the intent of showing how the different pieces in the literature fit together to obtain this particular convenient formulation. We also note that a formulation of an analogous result for strict Lorenz-dominance is provided by Sen (1973/1997, p. 64).

[22] Note that it does not matter if V is allowed to be nondecreasing. Adding piecewise constant functions to the class $\mathscr{V}$ of canonical SFWs does not change the fairness, since any symmetric nondecreasing concave function can be approximated by a sequence of (increasing) canonical SWFs, so that the infimum in (4) remains unaffected.

(iv) Consider a common increasing concave transformation of individuals' utilities, so that $z = (\psi(y_1), \ldots, \psi(y_n))$ for some continuous increasing concave function $\psi : \mathbb{R}^n_+ \to \mathbb{R}$ with $\psi(0) = 0$. For any $V \in \mathscr{V}$ the function $U(y) = V(\psi(y_1), \ldots, \psi(y_n))$ is also an element of $\mathscr{V}$.

Proof of Proposition 3 Fix an arbitrary feasible allocation $y \in \mathscr{Y}$ and let

$$\Phi(y) = \min_{k \in \mathscr{N}} P_k(y)/P_k^*.$$

(i) We have that $\varphi(y) \leq \Phi(y)$, since $P_k \in \mathscr{V}$ for all $k \in \mathscr{N}$. (ii) To show that also $\varphi(y) \geq \Phi(y)$, assume without loss of generality that $\Phi(y) > 0$ (if $\Phi(y) = 0$ then $\varphi(y) \geq \Phi(y)$ trivially). Note that for any feasible allocation $\hat{y} \in \mathscr{Y}$ by definition of Φ:

$$P_k(\hat{y}) \leq P_k(y)/\Phi(y). \tag{21}$$

If we set $z = y/\Phi(y)$, then $P_k(z) = P_k(y)/\Phi(y)$ and by (21) we have that $P_k(\hat{y}) \leq P_k(z)$, for all $\hat{y} \in \mathscr{Y}$. Fix any $V \in \mathscr{V}$. By Proposition 1, it is necessarily true that $V(\hat{y}) \leq V(z)$, and thus[23]

$$V^* \leq V(z). \tag{22}$$

Since V is concave and $\Phi(y) \in (0, 1]$, we have by Jensen's inequality that

$$(1 - \Phi(y))V(0) + \Phi(y)V(z) \leq V((1 - \Phi(y)) \cdot 0 + \Phi(y) \cdot z) = V(y),$$

whence using (22),

$$\Phi(y) \leq \frac{V(y)}{V(z)} \leq \frac{V(y)}{V^*}$$

for all $V \in \mathscr{V}$. Taking the infimum over all $V \in \mathscr{V}$ then yields using (4) that $\Phi(y) \leq \varphi(y)$ on $\mathscr{Y}$, which completes our proof.

Proof of Proposition 4 This proof is constructive and a feasible solution to the optimization problem (11) is shown to exist below (in Step 2). Let (y, z, r) be such a solution and let

$$\pi_k = \sum_{i=1}^{n} z_i - (n - k)r$$

be the value of the objective function at the optimum. The remainder of the proof proceeds in two steps.

[23] Note that $z \in \mathbb{R}^n_+$ may not be feasible.

Step 1: $\pi_k \leq P_k^*$. Indeed, since $z_i \leq r$, we have that necessarily

$$\pi_k = \sum_{i=1}^{n} z_i - (n-k)r \leq P_k(z).$$

Furthermore, since $z_i \leq y_i$ it is $P_k(z) \leq P_k(y) \leq P_k^*$, which implies that $\pi_k \leq P_k^*$.

Step 2: $\pi_k \geq P_k^*$. Since larger values of the z_i's can only increase the objective function, the constaints $z_i \leq \min\{y_i, r\}$ must be binding at the optimum, so that $z_i = \min\{y_i, r\}$ for all $i \in \mathscr{N}$. Consider now the feasible solution $(y, z, r) = (y^k, \min\{y, (r, r, \ldots, r)\}, r)$ of (11), where

$$y^k \in \arg\max_{y \in \mathscr{Y}} P_k(y),$$

and $r = y^k_{(k)}$. We note that $z_{(i)} = y^k_{(i)}$, if $i \leq k$ and $z_{(i)} = r$ otherwise. Hence,

$$\begin{aligned}\pi_k \;\geq\; \sum_{i=1}^{n} z_i - (n-k)r &= \sum_{i=1}^{n} z_{(i)} - (n-k)r \\ &= \sum_{i=1}^{k} y^k_{(i)} + \sum_{i=k+1}^{n} r - (n-k)r \\ &= P_k^*.\end{aligned}$$

As a result, $\pi_k \geq P_k^*$.

Steps 1 and 2 together imply that $\pi_k = P_k^*$, i.e., the program (11) produces indeed the optimal k-th prefix, as desired.

Proof of Proposition 5 The equivalence of the program (12) to the fair welfare maximization problem (6) follows directly from our representation of the fairness ratio (8) as well as program (11) in Proposition 4. We omit the details.

Proof of Proposition 6 The proof proceeds in three steps.

Step 1: P_k^*/k *is nondecreasing in* k. Fix $k, j \in \mathscr{N}$ with $k < j$, and let

$$y^k \in \arg\max_{y \in \mathscr{Y}} P_k(y).$$

Arranging the components of y^k in nondecreasing order, we have that $y^k_{(j)} \geq y^k_{(j-1)} \geq \ldots \geq y^k_{(1)}$. Hence, the average share of the k poorest individuals under y^k cannot exceed the average share of the j poorest individuals under the same allocation y^k, so that

$$\frac{P_k^*}{k} = \frac{1}{k}\sum_{i=1}^{k} y^k_{(i)} \leq \frac{1}{j}\sum_{i=1}^{j} y^k_{(i)} \leq \frac{P_j^*}{j}.$$

Step 2: Iteratively construct a set $\mathscr{S}$ *of allocations* $y^k \in \arg\max_{y\in\mathscr{Y}} P_k(y)$, $k \in \mathscr{N}$ *with* $|\mathscr{S}| = 2 + 2\lfloor \log_2 I_n \rfloor$. The set $\mathscr{S}$ contains at most n elements. Put y^1 as first element in the set, and define $\alpha_k = P_k^*/k$ for $k \in \mathscr{N}$. The construction proceeds for $i = 1, 2, \ldots, n-1$ according to the following algorithm. Start with $i = 1$. (a) If y^i is not in $\mathscr{S}$, or if y^i is already in $\mathscr{S}$ *and* $\alpha_n \leq 2\alpha_i$, then stop. (b) Otherwise, if y^i is in $\mathscr{S}$, then find the smallest $j > i$ such that $\alpha_j > 2\alpha_i$ and add x^j to $\mathscr{S}$. Increase the index i by one and continue with (a).

Upon completion of the construction of $\mathscr{S}$, let

$$\bar{y} = \frac{1}{|\mathscr{S}|} \sum_{y\in\mathscr{S}} y$$

be the arithmetic average of all allocations in $\mathscr{S}$. Since $\mathscr{Y}$ is convex by assumption, $\bar{y} \in \mathscr{Y}$ is feasible. Moreover, by construction of $\bar{y}$ we have that

$$\bar{y} \geq \frac{y^k}{|\mathscr{S}|}, \tag{23}$$

for all $k \in \mathscr{N}$. Let $\rho = 1/(2|\mathscr{S}|) = (2 + 2\lfloor \log_2 I_n \rfloor)^{-1}$ with $I_n = \alpha_n/\alpha_1$.

Step 3: For any $k \in \mathscr{N}$: $P_k(\bar{y}) \geq \rho P_k^*$. To show this, it is useful to consider two alternatives. (1) If $y^k \in \mathscr{S}$, then from (23) we obtain that $\bar{y} \geq y^k/|\mathscr{S}|$ and thus $P_k(\bar{y}) \geq \rho P_k^*$. 2. If $y^k \notin S$, then choose the largest index $i \in \mathscr{N}$ such that $i < k$ and $y^i \in \mathscr{S}$. This index exists, since $y^1 \in \mathscr{S}$. By construction of $\mathscr{S}$ it is necessarily true that $\alpha_k \leq 2\alpha_i$, i.e., $P_k^*/k \leq 2P_i^*/i$. In addition,

$$P_i^*/i = P_i(x^i)/i \leq P_k(x^i)/k \leq P_k(\bar{y})|\mathscr{S}|/k,$$

by (23) and a similar reasoning as in Step 1. As a result we obtain that $P_k^*/k \leq 2|\mathscr{S}|P_k(\bar{y})/k$, or in other words $P_k(\bar{y}) \geq \rho P_k^*$.

Our assertion in Proposition 6 then follows directly by applying Proposition 3, which completes our argument.

Proof of Proposition 7 (i) The proof proceeds in five steps. Let $y^* \in \mathscr{Y}$ be an optimal fair allocation and let $R_k^* = P_k(y^*)/P_k^*$ be the k-th *optimal prefix ratio* for any $k \in \mathscr{N}$.

Step 1: $P_k^* - P_{k-1}^*$ *is nondecreasing in* $k \in \{2, \ldots, n\}$. Let $k \in \mathscr{N} \setminus \{1\}$ and y^k be a maximizer of the k-th prefix P_k on $\mathscr{Y}$ as in (13). Then $y^k_{(k)} = P_k(y^k) - P_{k-1}(y^k)$ and $y^k_{(k+1)} = P_{k+1}(y^k) - P_k(y^k)$. Since $y^k_{(k+1)} \geq y^k_{(k)}$, we have that

$$P_{k+1}(y^k) - P_k(y^k) \geq P_k(y^k) - P_{k-1}(y^k),$$

or equivalently

$$P_{k+1}(y^k) + P_{k-1}(y^k) \geq 2P_k(y^k) = 2P_k^*.$$

Hence,

$$P^*_{k+1} + P^*_{k-1} \geq 2P^*_k,$$

so that indeed $P^*_k - P^*_{k-1}$ is nondecreasing in $k \in \{2, \ldots, n\}$.

Since by representation (8) we have that at the optimal fair allocation y^*

$$\varphi^* = \varphi(y^*) = \min_{k \in \mathcal{N}} R^*_k,$$

there must be an index j in $\mathcal{N}$ for which the optimal prefix ratio equals the optimal fairness ratio, i.e., $\varphi^* = R^*_j$. Let j be the largest such "tight" index.[24] Clearly, if $j = n$, then the solution y^* is balanced and relation (14) is satisfied. Let us thus assume in what follows that $j < n$.

Step 2: $y^*_j < y^*_{j+1}$. We show this by induction. For $j = 1$ we have that $R^*_2 > R^*_1 = \varphi^*$. Hence,

$$\frac{y^*_1 + y^*_2}{P^*_2} > \frac{y^*_1}{P^*_1},$$

or equivalently

$$y^*_2 > y^*_1 \left(\frac{P^*_2}{P^*_1} - 1 \right).$$

On the other hand, since P^*_k/k is nondecreasing in k (for a proof see Step 1 of the proof of Proposition 6), it is $P^*_2 \geq 2P^*_1$, so that we can conclude using the previous inequality that $y^*_2 > y^*_1$. Similarly, for $j > 1$ we have that $R^*_{j+1} > R^*_j = \varphi^*$ and at the same time $\varphi^* = R^*_j \leq R^*_{j-1}$. The latter inequality is equivalent to $P_{j-1}(y^*)/P^*_{j-1} \geq P_j(y^*)/P^*_j$ and thus

$$\frac{P_{j-1}(y^*) + y^*_j}{P^*_{j-1} + (P^*_j - P^*_{j-1})} \geq \frac{y^*_j}{P^*_j - P^*_{j-1}}.$$

This implies that[25]

$$\frac{y^*_j}{P^*_j - P^*_{j-1}} \leq \frac{P_{j-1}(y^*)}{P^*_{j-1}}$$

and thus

$$\frac{P_{j-1}(y^*) + 2y^*_j}{P^*_{j-1} + 2(P^*_j - P^*_{j-1})} \leq \frac{P_{j-1}(y^*) + y^*_j}{P^*_{j-1} + (P^*_j - P^*_{j-1})} = R^*_j.$$

24 We refer to an optimal prefix ratio R^*_j or associated index j as *tight* whenever $R^*_j = \varphi^*$.

25 Here we use that for any positive numbers a, b, c, d, e, f it is $((a+c)/(b+d) \leq a/b \Leftrightarrow c/d \leq a/b)$ and $(a/b, c/d \leq e/f \Rightarrow (a+c)/(b+d) \leq (e+c)/(f+d))$.

By Step 1 we have that $P^*_{j+1} - P^*_j \geq P^*_j - P^*_{j-1}$, so that

$$\frac{P_j(y^*) + y^*_j}{P^*_{j+1}} \leq R^*_j.$$

On the other hand, since $R^*_{j+1} > R^*_j$, we obtain

$$\frac{P_j(y^*) + y^*_{j+1}}{P^*_{j+1}} > \frac{P_j(y^*) + y^*_j}{P^*_{j+1}},$$

which implies that $y^*_{j+1} > y^*_j$ as claimed.

Step 3: the nth optimal prefix ratio is tight, i.e., $R^*_n = \varphi^*$. This can be shown by contradiction. Suppose therefore that $R_n > \varphi^*$, so that $j \leq n-1$. From Step 2 we obtain that $y^*_{j+1} > y^*_j$, so that a positive amount can be transferred from the individual with index $j+1$ (under y^*) to all individuals with indices $i \leq j$. More precisely, if we set

$$\varepsilon = \min\left\{ y^*_{j+1} - y^*_j,\ \min_{j+1 \leq i \leq n} \left\{ P_i(y^*) - \varphi^* P^*_i \right\} \right\},$$

it is possible to transfer an amount $\varepsilon/2$ from individual $j+1$ and increase y^*_i by the amount $(\varepsilon/2)\gamma_{j+1}/(\gamma_1 + \cdots + \gamma_j)$ for all $i \in \{1, \ldots, j\}$. Note thereby that $\varepsilon > 0$ by the definition of j and Step 2. The new allocation $\hat{y}^*$ obtained after executing the transfers is feasible, since $\gamma \cdot \hat{y}^* \leq 1$ as can easily be verified. In addition, observe that *all* new prefix ratios $\hat{R}^*_k = P_k(\hat{y}^*)/P^*_k$ are "slack,"[26] which contradicts the optimality of y^*. Hence, the n-th optimal prefix ratio at y^* must be tight, i.e., $j = n$.

As a direct consequence of Step 3 it must be the case that the last several (at least one) optimal prefix ratios are tight. Let $l < n$ be the largest slack index, for which $R^*_l > \varphi^*$.

Step 4: $l \leq n-2 \;\Rightarrow\; y^*_{l+2} > y^*_{l+1}$. We show this by contradiction. Suppose that $y^*_{l+2} = y^*_{l+1}$ (for $y^*_{l+2} < y^*_{l+1}$ is clearly impossible). By assumption we have $R^*_l > \varphi^* = R^*_l$, so that (again using Footnote 25 and Step 1)

$$\begin{aligned} R^*_l > \varphi^* = \frac{P_l(y^*) + y^*_{l+1}}{P^*_l + (P^*_{l+1} - P^*_l)} &> \frac{P_l(y^*) + 2y^*_{l+1}}{P^*_l + 2(P^*_{l+1} - P^*_l)} \\ &\geq \frac{P_l(y^*) + 2y^*_{l+1}}{P^*_l + (P^*_{l+1} - P^*_l) + (P^*_{l+2} - P^*_{l+1})}. \end{aligned}$$

The right-hand side of the last inequality can be rewritten in the form (using our hypothesis that $y^*_{l+2} = y^*_{l+1}$)

[26] We refer to an optimal prefix ratio R^*_l or associated index l as *slack* whenever $R^*_l > \varphi^*$.

$$\frac{P_{l+1}(y^*) + y^*_{l+1}}{P^*_{l+2}} = \frac{P_{l+1}(y^*) + y^*_{l+2}}{P^*_{l+2}} = R^*_{l+2}.$$

Hence, we have shown that $\varphi^* > R^*_{l+2}$, i.e., it implies that R^*_{l+2} is "supertight," clearly in violation of the optimality of y^*.

Step 5: the optimal fair allocation y^* is balanced. If y^* is unbalanced, we know by Step 3 that the optimal prefix ratios must be of the form " …, tight, slack, slack, …, slack, tight, tight, …,tight." With $l < n$ as before denoting the highest slack index, it is possible to make a small positive transfer from l to $l + 1$ and obtain the feasible allocation $\hat{y}^*$ (analogously to the transfer in Step 3). By Step 4, the order $y^*_{l+1} \leq y^*_{l+2} \leq \cdots \leq y^*_n$ remains preserved after the transfer, i.e., $\hat{y}^*_{l+1} \leq \hat{y}^*_{l+2} \leq \cdots \leq \hat{y}^*_n$, since $y^*_{l+2} > y^*_{l+1}$ (for $l = n - 1$ Step 4 is not needed). After the transfer $l + 1$ is now the highest slack index and the transfer process can be repeated until reaching a contradiction (with respect to Step 3) as soon as the n-th prefix ratio becomes slack.[27] We can thus conclude that the optimal fair allocation y^* must be balanced.

(ii) Since $\gamma_1 > \cdots > \gamma_n$ (strict cost ordering), it is clear that $y^*_{(k)} = y^*_k$ for all $k \in \mathcal{N}$. From this and the balancedness of the wealth ratios we obtain immediately that $y^*_k = (P^*_k - P^*_{k-1})\varphi^*$ for all $k \in \{2, \ldots, n\}$ and from (15) that $y^*_1 = (1 - \sum_{i=2}^n \gamma_i y^*_i)/\gamma_1$. After noting that $P^*_1 = (\sum_{i=1}^n \gamma_i)^{-1}$, the expression for φ^* results directly from the balancedness of the wealth rations,

$$\begin{aligned}\varphi^* = \frac{y^*_1}{P^*_1} &= \left(1 - \sum_{k=2}^{n} \gamma_k (P^*_k - P^*_{k-1})\varphi^*\right)\left(\sum_{i=1}^{n} \frac{\gamma_i}{\gamma_1}\right) \\ &= \left(\frac{\gamma_1}{\sum_{i=1}^n \gamma_i} + \sum_{k=2}^{n} \gamma_k (P^*_k - P^*_{k-1})\right)^{-1},\end{aligned}$$

which completes our proof.

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[27] Since by assumption $\gamma_1 > \cdots > \gamma_n$, note the transfers to the right are "superefficient" in the sense that the total cost of serving all individuals actually decreases, producing slack in the feasibility constraint.

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