Harmonic Maps on Smooth Metric Measure Spaces and on Riemannian Polyhedra

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Abstract

This thesis is a study of harmonic maps in two different settings. The first part is concerned with harmonic maps from smooth metric measure spaces to Riemannian manifolds. The second part is study of harmonic maps from Riemannian polyhedra to non-positively curved (locally) geodesic spaces in the sense of Alexandrov.

The first part is organized as follows. We begin by defining a notion of harmonicity, and justifying the definition by checking it against pre-existing definitions and results in special cases. There are two main theorems in this section. The first is Theorem 0.1.1, which is the generalization of the Shoen-Yau theorem [SY76] in our setting. The second is on the convergence of harmonic maps between Riemannian manifolds. Specifically we will show that if $f_i : M_i \to N$ are a sequence of harmonic maps between Riemannian manifolds, and if the manifolds M_i converge to a smooth metric measure space M in the measured Gromov-Hausdorff topology, then the f_i converge to a harmonic map $f : M \to N$. This is the content of Theorem 0.1.2

In the second part, we prove Liouville-type theorems for harmonic maps under two different assumptions on the source space. First we prove the analogue of the Schoen-Yau theorem on a complete (smooth) pseudomanifolds with non-negative Ricci curvature. To this end we generalize some Liouville- type theorems for subharmonic functions from [Yau76]. Then we study 2-parabolic admissible Riemannian polyhedra and prove vanishing results for subharmonic functions and harmonic maps on 2-parabolic pseudomanifolds.

Keywords: Convergence, harmonic maps, Riemannian polyhedra, pseudomanifolds, Liouville-type theorem, non-negative Ricci, smooth metric measure spaces.

Résumé

Cette thèse est consacrée à l'étude de deux classes d'applications harmoniques entre espaces métriques. La première classe d'applications est étudiée dans la première partie de la thèse. Elle concerne les applications harmoniques définies sur un *espace métrique mesuré lisse* (aussi appelé *variété riemannienne à poids*) à valeurs dans une variété riemannienne. La seconde partie de la thèse est consacrée à l'étude des applications harmoniques définies sur un polyèdre riemannien à valeurs dans un espace à courbure (localement) non-positive au sens d'Alexandrov.

La première partie est organisée comme suit. Nous commençons par définir une notion d'harmonicité pour les espaces métriques mesurés lisses, et nous justifions cette définition en la comparant avec les définitions et les résultats classiques dans des cas particuliers. Cette partie contient deux théorèmes principaux. Le premier est le théorème 0.1.1, qui est l'analogue du théorème Schoen-Yau [SY76] dans notre contexte. La deuxième résultat (théorème 0.1.2) est un théorème de convergence pour les suites d'applications harmoniques entre variétés riemanniennes. Plus précisément, nous montrons que si $f_i : M_i \to N$ est une suite d'applications harmoniques entre variétés riemanniennes et si les M_i convergent au sens de Gromov-Hausdorff un espace métrique mesuré lisse M, alors les f_i convergent vers une application harmonique $f : M \to N$.

Dans la deuxième partie, nous obtenons des obstructions (théorèmes de type Liouville) pour les applications harmoniques sous deux hypothèses différentes sur l'espace source. Le premier résultat est un analogue du théoréme de Schoen-Yau sur une pseudo-variété complète à courbure de Ricci positive ou nulle. Pour démontrer ces résultats, nous généralisons certains théorèmes de type Liouville pour les fonctions sous-harmoniques dus à Yau [Yau76]. Ensuite nous étudions le cas des polyèdres riemanniennes admissibles qui sont 2-paraboliques et nous prouvons des obstructions pour les fonctions harmoniques et sous-harmoniques définies sur ces espaces.

Mots clés : Convergence, applications harmoniques, polyèdres riemanniens, pseudo-variétés, théorèmes de type Liouville, Ricci non-négative, espace métrique mesuré lisse

Contents

Introduction

This work is a study of harmonic maps in two different settings. The first part is concerned with harmonic maps from smooth metric measure spaces to Riemannian manifolds. In the second part we study harmonic maps from Riemannian polyhedra to non-positively curved (locally) geodesic spaces in the sense of Alexandrov.

0.1 Part I

A smooth metric measure space is a triple $(M, g, \Phi\,\mathrm{dvol})$, where (M, g) is an n-dimensional Riemannian manifold, dvol denotes the corresponding Riemannian volume element on M, and Φ is a smooth positive function on M. These spaces have been used extensively in geometric analysis and they arise as smooth collapsed measured Gromov-Hausdorff limits in the works of Cheeger-Colding [CC97, CC00a, CC00b], Fukaya [Fuk89] and Gromov [Gro07]. They have been studied recently by [Mor05]. See also [Lot03, Qia97, FLZ09, WW09, Wu10, SZ11, MW11]

Harmonic maps are critical points of the energy functional defined on the space of maps between Riemannian manifolds. Their theory was developed by J. Eells and H. Sampson [ES64] in the 1960s. In [ES64], it was proved that if N is a compact manifold with non-positive sectional curvature, any continuous map from a compact manifold into N is homotopic to a harmonic map. Eells and Sampson also proved that every harmonic map from a compact manifold with positive Ricci curvature to a negatively curved manifold is constant. In [SY76], Schoen and Yau improved this and proved that if M is a complete manifold with non-negative Ricci curvature and N is a compact manifold with non-positive curvature then any smooth map from M to N with finite energy is homotopic to constant on each compact set.

In this part, we are going to study the behavior of harmonic maps under convergence. Let $\mathcal{M}(n, D)$ denote the set of all compact Riemannian manifolds M such that $\dim(M) = n$, $\mathrm{diam}(M) < D$, and the sectional curvature R_M satisfies, $|\mathrm{R}_M| \le 1$, equipped with the measured Gromov-Hausdorff topology. Let $(M_i, g_i, \mathrm{dvol}_i)$ in $\mathcal{M}(n, D)$ be a sequence of manifolds which converge to a smooth metric measure space $(M, g, \Phi\,\mathrm{dvol}_M)$. Suppose $f_i : (M_i, g_i) \to (N, h)$ is a sequence of harmonic maps. We are interested to know under what circumstances the f_i converge to a harmonic map f on the smooth metric measure space (M, g, Φ).

As a first step, we specify what we mean by harmonicity on smooth metric measure spaces. Let $f : M \to N$ be a smooth map where (N, h) is a Riemannian manifold. We say that f is harmonic on (M, g, Φ) if it satisfies

$$\tau(f) + df(\nabla \ln \Phi) = 0, \tag{1}$$

where $\tau(f)$ denotes the tension field of the map f defined in Subsection 1.1. This notion has been introduced by Lichnerowicz in [Lic69]. The stationary points of the energy functional

$$E(f) = E_\Phi(f) = \int_M e(f)\, \Phi \,\mathrm{dvol},$$

are solutions to equation (1). Here $e(f) = \frac{1}{2}|df|^2$ is the energy density of f. Riemannian submersions on smooth metric measure spaces with minimal fibers are examples of harmonic maps in this setting.

We will prove the analogues of the Schoen-Yau theorem for harmonic maps in this setting. As a replacement for the Ricci curvature, we use the Bakry-Émery Ricci tensor which first showed up in the study of diffusion processes. Specifically this tensor is defined as

$$\widetilde{\mathrm{Ric}}_\infty = \mathrm{Ric} - \mathrm{Hess}(\ln \Phi).$$

Geometric, topological and analytical properties of smooth metric measure spaces with Bakry-Émery Ricci tensor bounded below have been subject to intensive study. See for example [Lot03], [WW09] and the references therein. We have the following theorem:

Theorem 0.1.1. *Let (M, g, Φ) be a complete non-compact smooth metric measure space with Φ bounded, and non-negative Bakry-Émery Ricci curvature, N a manifold with non-positive sectional curvature, and $f : M \to N$ a harmonic map of finite energy, $E_\Phi(f) < \infty$. Then f is a constant map.*

We now state our main result of this part, which is a compactness theorem for sequences of harmonic maps.

Theorem 0.1.2. *Let (M_i, g_i) be a sequence of manifolds in $\mathcal{M}(n, D)$ which converges to a metric measure space $(X, g, \Phi\mu_g)$ in the measured Gromov-Hausdorff Topology. Suppose (N, h) is a compact Riemannian manifold. Let $f_i : (M_i, g_i) \to (N, h)$ be a sequence of harmonic maps such that $\|e_{g_i}(f_i)\|_{L^\infty} < C$, where $\|e_{g_i}(f_i)\|_{L^\infty}$ is the L^∞-norm of the energy density of the map f_i and C is a constant independent of i. Then f_i has a subsequence which converges to a map $f : (X, g, \Phi \cdot \mu_g) \to (N, h)$, and this map is a harmonic map in $\mathcal{H}^1((X, \Phi\mu_g), N)$.*

By $\mathcal{H}^1(X, N)$ we mean

$$\{f \in \mathcal{H}^1(X, \mathbb{R}^q) \mid f(x) \in N \text{ for almost all } x \in M\},$$

where $\mathscr{H}^1(M,\mathbb{R}^q)$ is the standard Sobolev space for some isometric embedding N in $\mathbb{R}^q$. We should mention also that in this work we use the notions $\mathscr{H}^1$ and $W^{1,2}$ interchangeably.

This part consists of two chapters. The second chapter contains the main result, and the first sets in place the basic notions and results of the theory of harmonic mappings on a smooth metric measure space. Although I worked independently on this part, I recently learned that some of the results in the first chapter appear in the works of Lichnerowicz and Eells-Lemaire [Lic69, EL78] and some have been proved more recently in [Cou07, WX12, RVar] (see also [CJQ12] for some other results in this subject). Detailed references are given in the first chapter. In the second chapter we prove Theorem 0.1.2. The proof is based on an explicit description of the limit space X as a quotient of a Riemannian manifold by some action of an orthogonal group obtained by Fukaya [Fuk88, Fuk89]. When a sequence of manifolds converges in the setting of Theorem 0.1.2 to a manifold, they form a fiber bundle over the limit manifold. As a main step in the proof, we show that the f_i are almost constant on the fibers of this bundle, see Lemma 2.2.3.

0.2 Part II

In this part, we concentrate on harmonic maps from admissible Riemannian polyhedra to non-positively curved (locally) geodesic spaces in the sense of Alexandrov. Harmonic maps between singular spaces have received considerable attention since the early 1990s. Existence of energy minimizing locally Lipschitz maps from Riemannian manifolds into Bruhat-Tits buildings and Corlette's version of Margulis's super-rigidity theorem were proved in [GS92]. In [KS93] Korevaar and Schoen constructed harmonic maps from domains in Riemannian manifolds into Hadamard spaces as a boundary value problem. [EF01] contains a description of the application of the methods of [KS93] to the study of maps between polyhedra.

In this part we prove Liouville-type theorems for harmonic maps under two different assumptions on the source space. First we prove the analogue of the Schoen-Yau theorem on a complete (smooth) pseudomanifolds with non-negative Ricci curvature. To this end we generalize some Liouville-type theorems for subharmonic functions from [Yau76]. Then we study 2-parabolic admissible Riemannian polyhedra and prove vanishing results for subharmonic functions and harmonic maps on 2-parabolic pseudomanifolds.

The classical Liouville theorem for functions on manifolds states that on a complete Riemannian manifold with non-negative Ricci curvature, any harmonic function bounded from one side must be a constant. In [Yau76], Yau proves that there is no non-constant, smooth, non-negative, L^p, $p>1$, subharmonic function on a complete Riemannian manifold. He also proves that every continuous subharmonic function defined on a complete Riemannian manifold whose local Lipschitz constant is bounded by L^1-function is also harmonic. Furthermore if the L^1-function belongs to L^2 as well, and the manifold has non-negative Ricci curvature, then the subharmonic function is constant. In the smooth setting, there are two types of assumptions that have been studied on the Liouville property of harmonic maps. One is the finiteness of the energy and the other is the smallness of the image. For example, as we mentioned above Schoen and Yau proved that any non-constant harmonic map from a complete manifold of non-negative Ricci curvature to a manifold of non-positive sectional curvature has infinite energy. Hildebrandt-Jost-Widman [HJW81] (see also [Hil82, Hil85]) proved a Liouville-type theorem for harmonic maps into regular geodesic (open) balls in a complete C^3-Riemannian manifold from a simple or compact C^1-Riemannian manifold. For more references for Liouville-type theorems for harmonic maps and functions in both smooth and singular setting see the introduction in [KS08].

A connected locally finite n-dimensional simplicial polyhedron X is called admissible, if X is dimensionally n-homogeneous and X is locally $(n-1)$-chainable. It is called circuit if instead it is $(n-1)$-chainable and every $(n-1)$-simplex is the face of at most two n-simplex and pseudomanifold if it is admissible circuit. A polyhedron X becomes a Riemannian polyhedron when endowed with a Riemannian structure g, defined by giving on each maximal simplex s of X a Riemannian metric g (bounded measurable) equivalent to a Euclidean metric on s (see [EF01]).

There exist slightly different notions of boundedness of Ricci curvature from below on general metric spaces. See for example [Stu06, LV09, Oht07, KS01, KS03] and the references therein. In the following by $\mathrm{Ric}_{N,\mu_g} \geq K$ we mean that (X, g, μ_g) satisfies the measure contraction property. This is the convention adopted in [Oht07, Stu06]. As this definition is somewhat technical we refer the reader to Chapter 4 for a precise statement.

The definition of harmonic maps from admissible Riemannian polyhedra to metric spaces is similar to the one in the smooth setting. However due to lack of smoothness some care is needed in defining the notions of energy density, the energy functional and energy minimizing maps. Precise definitions and related results can be found in Section 3.7.

We can state now the main results which we obtain in this direction.

Theorem 0.2.1. *Suppose (X, g) is a complete, admissible Riemannian polyhedron, and $f \in W^{1,2}_{\mathrm{loc}}(X) \cap L^2(X)$ is a non-negative, weakly subharmonic function. Then f is constant.*

Theorem 0.2.2. *Let (X, g, μ_g) be a complete non-compact pseudomanifold. Let f be continuous, weakly subharmonic and belonging to $W^{1,2}_{\mathrm{loc}}(X)$, such that $\|\nabla f\|_{L^1}$ is finite. Then f is a harmonic function.*

Theorem 0.2.3. *Let (X, g, μ_g) be a complete, smooth n-pseudomanifold. Suppose X has non-negative n-Ricci curvature. Let f be a continuous, weakly subharmonic function belonging to $W^{1,2}_{\mathrm{loc}}(X)$ such that both $\|\nabla f\|_{L^1}$ and $\|\nabla f\|_{L^2}$ are finite and $|\nabla f|$ is locally bounded. Then f is a constant function.*

Here by a smooth pseudomanifold we mean a simplexwise smooth, pseudomanifold which is smooth outside of its singular set. That situation arises for instance when the space is a projective algebraic variety. The difficulty in extending existing results lies in the lack of a differentiable structure on admissible polyhedron in general, and the loss of completeness outside the singular set even in the case of smooth pseudomanifolds. Moreover the classical notion of Laplace operator doesn't exist in the non-smooth setting. To circumvent this latter problem, and following the work of [Gig12], we define the Laplacian of a subharmonic function as a measure for which the Green formula holds, see Theorem 5.1.1.

The following two corollaries are important consequences of theorems 0.2.1 and 0.2.3.

Corollary 0.2.4. *Let (X, g, μ_g) be a complete, smooth n-pseudomanifold. Suppose X has non-negative n-Ricci curvature. Suppose Y is a Riemannian manifold of non-positive curvature, and $u : (X, g) \to (Y, h)$ a continuous harmonic map belonging to $W^{1,2}_{loc}(X, Y)$. If u has finite energy, and $e(u)$ is locally bounded, then it is a constant map.*

Corollary 0.2.5. *Let (X, g, μ_g) be a complete, smooth n-pseudomanifold. Suppose X has non-negative n-Ricci curvature. Let Y be a simply connected, complete geodesic space of non-positive curvature and $u : (X, g) \to Y$ a continuous harmonic map with finite energy, belonging to $W^{1,2}_{\text{loc}}(X, Y)$. If $\int_M \sqrt{e(u)}\, d\mu_g < \infty$, and $e(u)$ is locally bounded, then u is a constant map.*

Our second objective in this thesis is the study of 2-parabolic admissible polyhedra. We say a connected domain Ω in an admissible Riemannian polyhedron is 2-parabolic, if for every compact set in Ω, its relative capacity with respect to Ω is zero. Our main theorem is

Theorem 0.2.6. *Let X be 2-parabolic pseudomanifold. Let f in $W^{1,2}_{\text{loc}}(X)$ be a continuous, weakly subharmonic function, such that $\|\nabla f\|_{L^1}$ and $\|\nabla f\|_{L^2}$ are finite. Then f is constant.*

Just as in the case of complete pseudomanifolds

Corollary 0.2.7. *Let (X, g) be a 2-parabolic pseudomanifold with g simplexwise smooth. Let Y be a simply connected complete geodesic space of non-positive curvature and $u : (X, g) \to Y$ a continuous harmonic map with finite energy belonging to $W^{1,2}_{\text{loc}}(X, Y)$. If we have $\int_X \sqrt{e(u)} d\mu_g < \infty$ then u is a constant map.*

Corollary 0.2.8. *Let (X, g) be a 2-parabolic admissible Riemannian polyhedron with simplexwise smooth metric g. Let Y be a complete geodesic space of non-positive curvature and $u : (X, g) \to Y$ a continuous harmonic map belonging to $W^{1,2}_{\text{loc}}(X, Y)$, with bounded image. Then u is a constant map.*

In order to prove Theorem 0.2.6, we need to generalize some of the results in [Hol90]. This is done in Chapter 6. In particular we will need following propositions.

Proposition 0.2.9. *Let (X, g) be 2-parabolic admissible Riemannian polyhedron. Suppose f in $W^{1,2}_{\text{loc}}(X)$ is a positive, continuous superharmonic function on X. Then f is constant.*

Proposition 0.2.10. *Let X be 2-parabolic admissible Riemannian polyhedron. Let f in $W^{1,2}_{\text{loc}}(X)$ be a harmonic function such that $\|\nabla f\|_{L^2}$ is finite. Then f is constant.*

The proofs of the propositions above follow a similar pattern as their equivalents for Riemannian manifolds. They are based on the fact that admissible Riemannian polyhedra admits an exhaustion by regular domains, and the validity of comparison principle on admissible Riemannian polyhedra. The main new ingredient in the proof of Theorem 0.2.6 is

Proposition 0.2.11. *Let X be a 2-parabolic pseudomanifold. Let f in $W^{1,2}_{\text{loc}}(X)$ be a continuous, weakly subharmonic function, such that $\|\nabla f\|_{L^1}$ and $\|\nabla f\|_{L^2}$ are finite. Then f is harmonic.*

The organization of the second part of our thesis is as follows. In Chapter 3, we give a complete background on Riemannian polyhedron and analysis on them. Most definition and results have been taken directly from [EF01]. In Section 3.2, we compare the L^2 based Sobolev space on admissible Riemannian polyhedra as in [EF01], with the one in [Che99], and show that they are equivalent. As we couldn't find references in the literature we provide a rather detailed explanation of this fact. In Chapter 4, we discuss the definition of two notions of Ricci curvature the curvature dimension condition $CD(K,N)$ and the measure contraction property $MCP(K,N)$ on metric measure spaces. We show that both notions are applicable to Riemannian polyhedra. In Proposition 4.0.14 we show that any non-compact complete n-dimensional Riemannian polyhedron of non-negative Ricci curvature $MCP(0,n)$, has infinite volume. Section 5.1 is devoted to Theorems 0.2.1, 0.2.2, 0.2.3 and Section 5.2 to Corollaries 0.2.4, and 0.2.5. In Chapter 6 we show that as in the smooth case the "approximation by unity" property holds on admissible 2-parabolic polyhedra (Lemma 6.0.2). Moreover, we prove that removing the singular set of a 2-parabolic pseudomanifold yields a 2-parabolic manifold (Lemma 6.0.3). The rest of this Chapter is the detailed proof of Theorem 0.2.6 and its corollaries.

Part I

Harmonic maps on smooth metric measure spaces and their convergence

1 Harmonic Maps on Smooth Metric Measure Spaces

1.1 Definition

In this section we justify the definition of harmonicity on smooth metric measure spaces by showing that harmonic maps are critical points of the energy functional (see also Lemma 1.8 [Cou07]). We first recall the definition of harmonic map. Let $f : (M, g, \Phi) \to N$ be a smooth map where (M, g, Φ) is an n-dimensional smooth metric measure space and (N, h) is a Riemannian manifold. Let $\{e_i\}_{i=1}^n$ denote a local orthonormal frame field on M. The energy density of f is defined

$$e(f) = \frac{1}{2}|df|^2 = \frac{1}{2}\sum_{i=1}^{n}\langle f_*(e_i), f_*(e_i)\rangle$$

and the *energy functional* $E(f)$ with respect to the measure $\Phi\,\mathrm{dvol}$ is defined

$$E(f) = E_\Phi(f) = \int_M e(f)\,\Phi\,\mathrm{dvol}.$$

Its *tension* field $\tau(f)$ is the trace of second fundamental form of f

$$\tau(f) = (\nabla_{e_i} df)(e_i).$$

We say that a map f is *harmonic* on (M, g, Φ) if it satisfies

$$\tau(f) + df(\nabla \ln \Phi) = 0. \tag{1.1}$$

Proposition 1.1.1. *The map f is harmonic if and only if for every compactly supported smooth variation f_t of f, such that $f_0 = f$, we have $\frac{d}{dt}|_{t=0}E(f_t) = 0$.*

Proof. If f_t is a smooth variation of f, then in local coordinates we have

$$E(f_t) = \frac{1}{2}\int g^{ij} h_{\alpha\beta} \frac{\partial f_t^\alpha}{\partial x^i}\frac{\partial f_t^\beta}{\partial x^j}\ \Phi\,\mathrm{dvol}$$

and so

$$\frac{d}{dt}\Big|_{t=0} E(f_t) = \frac{1}{2}\int g^{ij} h_{\alpha\beta} \nabla_{x^i}\left(\frac{\partial}{\partial t}|_{t=0} f_t^\alpha\right)\frac{\partial f_t^\beta}{\partial x^j}\ \Phi\,\mathrm{dvol} \tag{1.2}$$

$$\begin{aligned} &= \frac{1}{2}\int g^{ij} h_{\alpha\beta} \nabla_i V^\alpha \frac{\partial f_t^\beta}{\partial x^j}\ \Phi\,\mathrm{dvol} \\ &= \frac{1}{2}\int \langle \nabla V, df\rangle\ \Phi\,\mathrm{dvol}\end{aligned}$$

where V is the variation vector of f_t. We have $V \in \Gamma(f^{-1}TN)$ and $\nabla V \in \Gamma(TM^* \otimes f^{-1}TN)$ and $df \in \Gamma(TM^* \otimes f^{-1}TN)$. Here $\Gamma(E)$ for a vector bundle E over M denotes the infinite dimensional vector space consisting of all sections of E. Equation (1.2) is called first variation formula. We define a vector X on M as

$$X = g^{ij} h_{\alpha\beta} V^\alpha \frac{\partial f^\beta}{\partial x^i}\frac{\partial}{\partial x^j}$$

If we calculate the $\mathrm{div}(X) = \nabla_i X^i$, here $\nabla X = \sum \nabla_i X^j dx^i \otimes \frac{\partial}{\partial x^j}$, we have

$$\mathrm{div}(X) = \langle \nabla V, df\rangle + \langle V, \tau(f)\rangle$$

since f_t has compact support on M, so does X. By using Green formula

$$\int \mathrm{div}(\Phi X)\ \mathrm{dvol} = \int \Phi\cdot \mathrm{div}(X) + \langle \nabla\Phi, X\rangle\ \mathrm{dvol} = 0$$

and so

$$\int \langle \nabla V, df\rangle\ \Phi\,\mathrm{dvol} + \int \langle V, \tau(f)\rangle\ \Phi\,\mathrm{dvol} + \int \langle \nabla\Phi, X\rangle\ \mathrm{dvol} = 0.$$

We can easily show

$$\langle \nabla\Phi, X\rangle = \langle V, df(\nabla\Phi)\rangle$$

By using the variation formula (1.2), we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}E(f_t) &= \frac{1}{2}\int \langle \nabla V, df\rangle\, \Phi\, \mathrm{dvol} \\ &= -\frac{1}{2}\int \langle V, \tau(f) + df(\nabla \ln \Phi)\rangle\, \Phi\, \mathrm{dvol}\end{aligned}$$

now the proof is complete. □

1.2 Minimal Immersion in Smooth Metric Measure Spaces

We begin first by definition of minimal immersions in smooth metric measure spaces.

Definition 1.2.1. *Let $(\overline{M}, \overline{g}, \phi)$ be a smooth metric measure space. Suppose (M, g) is a Riemannian manifold and $f : M \to \overline{M}$ be an immersion with mean curvature vector* H. *We define*

$$\mathrm{H}_\Phi = \mathrm{H} - \frac{1}{n}(\nabla \ln \Phi)^N.$$

where $(\nabla \ln \Phi)^N$ denotes the normal part of $\nabla \ln \Phi$ to the tangent space on $f(M)$. We call H_Φ the Φ-mean curvature vector of M in the smooth metric measure space $(\overline{M}, \Phi)$. The immersion f is called minimal if $\mathrm{H}_\Phi \equiv 0$.

To justify our definition of minimality we prove a variation formula as in classical literature. Let vol_Φ denotes the volume with respect to the measure $\Phi\, \mathrm{dvol}_{\overline{M}}$. Consider $\mathscr{I}(M, \overline{M})$, the space of all immersions $f : M \to \overline{M}$. Then $\mathrm{vol}_\Phi(f(M))$, is a functional on this space. The critical points of the volume functional are minimal submanifolds by the following proposition.

Proposition 1.2.2. *Let M and $\overline{M}$ be as in definition above and $f : M \to \overline{M}$ be an isometric immersion with mean curvature vector* H. *Let f_t, $|t| < \epsilon$, $f_0 = f$ be a smooth family of immersion satisfying $f_t|_{\partial M} = f|_{\partial M}$ and denote $V = \frac{\partial f_t}{\partial t}|_{t=0}$ to be the variational vector field along f. Then*

$$\begin{aligned}\frac{d}{dt}|_{t=0}\, \mathrm{vol}_\Phi(f_t(M)) &= \int_M \langle (\nabla \ln \Phi)^N - n\mathrm{H}, V\rangle\, \Phi\, \mathrm{dvol}_M \\ &= -n\int_M \langle \mathrm{H}_\Phi, V\rangle\, \Phi\, \mathrm{dvol}_M.\end{aligned}$$

Proof. Suppose g_t to be the induced metric of the immersion f_t and dvol_t its corresponding volume element. Choose a local orthonormal frame field $e_1, \ldots, e_n$ in M with respect to metric g_0. In this coordinate, $g_{ij}(t) = \langle f_{t*}e_i, f_{t*}e_j\rangle$ and $g(t) = \det(g_{ij}(t))$. Thus we have

$$\mathrm{vol}_\Phi(f_t(M)) = \int_M \sqrt{g(t)}\, \Phi\, \mathrm{dvol}_M.$$

We have

$$\frac{d}{dt}|_{t=0}\operatorname{vol}_\Phi(f_t(M)) = \frac{1}{2}\int_M g'(0)\;\Phi\,\mathrm{dvol}_M + \int_M \langle \nabla \ln\Phi, V\rangle\;\Phi\,\mathrm{dvol}_M,$$

and so

$$\frac{d}{dt}|_{t=0}\,\mathrm{dvol}_t = \frac{1}{2}\sum_{k=1}^{n} g'_{kk}(0)\,\mathrm{dvol}.$$

After some calculation ([Xin03] Theorem 1.2.2), we find

$$\frac{1}{2}\sum_{k=1}^{n} g'_{kk}(0) = \operatorname{div}(V^T) - \langle n\mathrm{H}, V\rangle,$$

and finally,

$$\begin{aligned}
\frac{d}{dt}|_{t=0}\operatorname{vol}_\Phi(f_t(M)) &= \int_M \operatorname{div}(V^T) - \langle n\mathrm{H}, V\rangle\;\Phi\,\mathrm{dvol}_M + \int_M \langle \nabla\ln\Phi, V\rangle\;\Phi\,\mathrm{dvol}_M \\
&= \int_M \operatorname{div}(\Phi\cdot V^T)\;\mathrm{dvol} - \int_M \langle\nabla\Phi, V^T\rangle\;\mathrm{dvol}_M - \int_M \langle n\mathrm{H}, V\rangle\;\Phi\,\mathrm{dvol}_M \\
&+ \int_M \langle\nabla\ln\Phi, V\rangle\;\Phi\,\mathrm{dvol}_M \\
&= \int_M \langle\nabla\ln\Phi, V^N\rangle - \langle n\mathrm{H}, V\rangle\;\Phi\,\mathrm{dvol}_M \\
&= \int_M \langle(\nabla\ln\Phi)^N - n\mathrm{H}, V\rangle\;\Phi\,\mathrm{dvol}_M.
\end{aligned}$$

□

The calculations show that this notion of mean curvature of an isometric embedding into a metric measure space is consistent with the one introduced in [WW09] for geodesic spheres and with the one in [Mor05] for hypersurfaces.

1.3 Examples of Harmonic Maps

1. Weighted curves: A map $u : (\mathbb{R}, \Phi) \to N$ is harmonic if it satisfies

$$\nabla_{\dot u}\Phi\cdot\dot u = 0.$$

In fact every solution of the equation above is reparametrization of a geodesic in the manifold (N,h), $u(t) = \gamma(\rho(t))$, where $\rho'(t) = c\frac{1}{\Phi(t)}$, and c is an arbitrary constant.

2. Riemannian Submersions: Let $u : (M, g, \Phi) \to (N, h)$ be a Riemannian submersion. Then a necessary and sufficient condition for u to be a harmonic map is that the fibers be minimal as Riemannian submanifolds of (M,g,Φ). We choose a point x in M and an orthonormal frame $\{e'_i\}_{i=1}^n$ around $u(x)$. Denote by $\{e_i\}_{i=1}^n$ its horizontal lift at point x. We extend $\{e_i\}$ to the orthonormal basis $\{e_1,\dots,e_n,e_{n+1},\dots,e_m\}$ at x. u is harmonic if and only if

$$\begin{aligned}&\tau(u)+du(\nabla\ln\Phi)=\\&\sum_{i=1}^{m}({}'\nabla_{e_i}du(e_i)-du(\nabla_{e_i}e_i))+du(\nabla\ln\Phi)=0.\end{aligned}$$

Name ${}'\nabla$ the induced connection on N. Also for $1\le i\le n$ we have

$${}'\nabla_{e_i}du(e_i)=du(\nabla_{e_i}e_i).$$

For $n+1\le i\le m$, we have ${}'\nabla_{e_i}du(e_i)=0$. Finally we have

$$-du(\sum_{i=n+1}^{m}\nabla_{e_i}e_i)+du(\nabla\ln\Phi)=0,$$

and so

$$\sum_{i=n+1}^{m}\nabla_{e_i}e_i-\nabla\ln\Phi\in V_x,$$

where V_x is tangent to the fiber $u^{-1}(u(x))$. This means that u is a harmonic map if and only if its fibers are minimal submanifolds in (M,g,Φ).

1.4 Co-differential and Bochner's Formula

In this section we are going to prove a Bochner formula for harmonic maps in the smooth metric measure space setting. This formula has also appeared in [Lic69] page 188 and has been proved in [RVar] explicitly. First we set in place some definitions. Let E be a Riemannian vector bundle over the Riemannian manifold (M,g) and $\Phi\,\mathrm{dvol}$ denote the density on M. For ω, ν in $\Gamma(\Lambda^pTM^*\otimes E)$ and θ in $\Gamma(\Lambda^{p+1}TM^*\otimes E)$, we define the following inner products

$$\begin{aligned}(\omega\mid\nu)&=\int\langle\omega,\nu\rangle\ \mathrm{dvol}\\(\omega\mid\nu)_\Phi&=\int\langle\omega,\nu\rangle\ \Phi\,\mathrm{dvol}.\end{aligned}$$

We define the exterior differential and co-differential of a form ω on M as follow,

$$\begin{aligned} d\omega(X_0, X_1, \dots, X_p) &= (-1)^k (\nabla_{X_k}\omega)(X_0, \dots, \widehat{X_k}, \dots, X_p) \\ \delta\omega(X_1, \dots, X_{p-1}) &= (\nabla_{e_i}\omega)(e_i, X_1, \dots, X_{p-1}) \end{aligned}$$

where $\{e_i\}$ is a local orthonormal frame filed on M and

$$(\nabla_X\omega)(X_1, \dots, X_p) = \nabla_X(\omega(X_1, \dots, X_p)) - \Sigma_j \omega(X_1, \dots, X_j, \dots, X_p)$$

Now let us compute the formal adjoint of d, $\tilde{\delta}$, with respect to inner product $(\cdot \mid \cdot)_\Phi$. We know on a compact manifold M (without boundary), d and $-\delta$ are adjoint.

$$(d\omega \mid \theta) = (\omega \mid -\delta\theta)$$

here $\omega \in \Gamma(\Lambda^p TM^* \otimes E)$ and $\theta \in \Gamma(\Lambda^{p+1} TM^* \otimes E)$. We have

$$\begin{aligned} (d\omega \mid \theta)_\Phi &= \int \langle d\omega, \Phi \cdot \theta \rangle \text{ dvol} \\ &= \int \langle \omega, -\delta(\Phi \cdot \theta) \rangle \text{ dvol} \\ &= \int \langle \omega, -(d\Phi(e_i) \cdot i_{e_i}\theta + \Phi \cdot \delta\theta) \rangle \text{ dvol} \\ &= (\omega \mid -(d \ln \Phi(e_i) \cdot i_{e_i} + \delta)\theta)_\Phi. \end{aligned}$$

We put

$$\tilde{\delta}\omega = \delta\omega + i_{\nabla \ln \Phi}\omega.$$

Therefore we can introduce a new notion of the Laplace-Beltrami operator $\widetilde{\Delta}$ as

$$\begin{aligned} \widetilde{\Delta} &= d\tilde{\delta} + \tilde{\delta} d \\ &= \Delta + (d \circ i_{\nabla \ln \Phi} + i_{\nabla \ln \Phi} \circ d). \end{aligned}$$

where $\Delta = d\delta + \delta d$. It is easy to show that on a compact manifold, $\tilde{\Delta}$ is a self-adjoint semi-negative elliptic operator.

Lemma 1.4.1. *A map $f : (M, g, \Phi) \to N$ is harmonic on (M, g, Φ) if and only if df is co-closed. ($\tilde{\delta} df = 0$)*

Proof.

$$\delta(df) = (\nabla_{e_i} df)(e_i) = \tau(f)$$

and

$$\tilde{\delta} df = df(\nabla \ln \Phi) + \tau(f) = 0$$

□

To obtain our Bochner type formula we use a Weitzenbock formula on a Riemannain manifold which we represent in the following proposition (see [Xin96] Proposition 1.3.4).

Proposition 1.4.2. *Let ω be a p-form with values in a vector bundle. Then*

$$\Delta\omega = \nabla^2\omega - \mathrm{S} \tag{1.3}$$

where ∇^2 denotes the trace-Laplace operator

$$\nabla^2\omega = \nabla_{e_i}\nabla_{e_i}\omega$$

where $\nabla_{e_i} e_j = 0$ and for any $X_1, \dots, X_p$ in $\Gamma(TM)$

$$S(X_1, X_2, \dots, X_p) = (-1)^k (\mathrm{R}(e_i, X_k)\omega)(e_i, X_1, \dots, \hat{X}_k, \dots, X_p)$$

here R *denotes the curvature connection.*

For a given smooth map f, $f : M \to N$, we have

$$\mathrm{S}(X) = -\mathrm{R}^N(f_* e_i, f_* X) f_* e_i + f_*(\mathrm{Ric}^M X)$$

Proposition 1.4.3. *Let $f : (M, g, \Phi) \to (N, h)$ be a harmonic map, then the following formula is satisfied*

$$\widetilde{\Delta} e(f) = |B(f)|^2 - \langle \mathrm{R}^N(f_*(e_i), f_*(e_j)) f_* e_i, f_* e_j \rangle + \langle f_*(\widetilde{\mathrm{Ric}}_\infty e_i), f_* e_i \rangle \tag{1.4}$$

Proof. If we put $\omega = df \in \Gamma(TM^* \otimes f^{-1}TN)$ in formula (1.3) and for $T = \nabla \ln \Phi$we have

$$\begin{aligned}
\widetilde{\Delta} e(f) &= \Delta e(f) + i_T \circ d\ e(f) \\
&= \langle \nabla^2 df, df \rangle + |B(f)|^2 + i_T \circ d\ e(f) \\
&= \langle \Delta df + S, df \rangle + |B(f)|^2 + i_T \circ d\ e(f)
\end{aligned}$$

The first equality comes from $\Delta e(f) = \langle \nabla^2 df, df\rangle + |B(f)|^2$, so

$$\begin{array}{rcl} \widetilde{\Delta} e(f) & = & \langle \widetilde{\Delta} df - (d\circ i_T + i_T \circ d) df, df\rangle \\ & + & |B(f)|^2 + i_T \circ d\; e(f) \\ & - & \langle R^N(f_*(e_i), f_*(e_j)) f_* e_i, f_* e_j\rangle + \langle f_* Ric^M e_i, f_* e_i\rangle \end{array}$$

Since f is harmonic then $\tilde{\Delta} df = 0$ and $(i_T\circ d)df = 0$, and therefore

$$\begin{array}{rcl} \widetilde{\Delta} e(f) & = & \langle -d\circ i_T\; df, df\rangle + i_T\circ d\; e(f) \\ & + & |B(f)|^2 - \langle \mathrm{R}^N(f_*(e_i), f_*(e_j)) f_* e_i, f_* e_j\rangle + \langle f_*(\widetilde{\mathrm{Ric}}_\infty e_i), f_* e_i\rangle \\ & + & \langle f_*(\mathrm{Hess}(\ln\Phi)(e_i)), f_* e_i\rangle. \end{array}$$

We have

$$\begin{array}{rcl} \langle d\circ i_T\; df, df\rangle & = & \langle \nabla_{e_i}(df(T)), df(e_i)\rangle \\ i_T\circ d\; e(f) & = & \dfrac{1}{2} T.\langle df, df\rangle = \langle \nabla_T df, df\rangle = \langle B(f)(e_i, T), df(e_i)\rangle \\ \langle f_*(\mathrm{Hess}(\ln\Phi))(e_i), f_*(e_i)\rangle & = & \langle df(\nabla_{e_i} T), f_*(e_i)\rangle. \end{array}$$

On the other hand we have

$$\langle B(f)(e_i, T), df(e_i)\rangle = \langle \nabla_{e_i}(df(T)), df(e_i)\rangle - \langle df(\nabla_{e_i} T), df(e_i)\rangle$$

Finally we get the Bochner-type formula (1.4). □

Using a Bochner-type formula we have the following results:

Theorem 1.4.4. *Let (M, g, Φ) be a compact smooth metric-measure space with non-negative Bakry-Émery Ricci curvature, (N, h) a Riemannian manifold with non-positive sectional curvature and $f : M \to N$ a harmonic map. Then f is a totally geodesic map. Furthermore,*

(a) If the Bakry-Émery Ricci curvature of the domain manifold M is positive somewhere, then f is a constant map.

(b) If the sectional curvature of the target manifold N is negative then f is either a constant map or $f(M)$ is a closed geodesic in N.

Proof. We integrate formula (1.4) on M with respect to the measure $\Phi\,\mathrm{dvol}_M$, to get

$$-\int_M |B(f)|^2 \Phi\,\mathrm{dvol} = \int_M -\langle \mathrm{R}^N(f_*(e_i), f_*(e_j)) f_* e_i, f_* e_j\rangle + \langle f_*(\widetilde{\mathrm{Ric}}_\infty e_i), f_* e_i\rangle\; \Phi\,\mathrm{dvol}$$

The right hand side of the equality is positive and the left hand side is negative so we have

$$|B(f)|^2 \equiv 0$$

therefore f is totally geodesic.
(a) From the above we also have

$$-\langle \mathrm{R}^N(f_*(e_i), f_*(e_j))f_*e_i, f_*e_j\rangle + \langle f_*(\widetilde{\mathrm{Ric}}_\infty^M e_i), f_*e_i\rangle \equiv 0$$

Suppose that the Bakry-Émery Ricci curvature is positive at some point x. Choose a local frame field $\{e_i\}$ at point x which coincides with the principle direction of $\widetilde{\mathrm{Ric}}_\infty$ ($\widetilde{\mathrm{Ric}}_\infty$ is symmetric). We have $\widetilde{\mathrm{Ric}}_\infty(e_i) = \lambda_i e_i$ and $\lambda_i > 0$ for each i at point x.

$$\langle f_*(\widetilde{\mathrm{Ric}}_\infty(e_i)), f_*(e_i)\rangle = \lambda_i \langle f_*(e_i), f_*(e_i)\rangle$$

and we have

$$\min \lambda_i |df|^2 \leq \lambda_i \langle f_*(e_i), f_*(e_i)\rangle = 0$$

so $|df|^2 = 0$ at point x. On the other hand $\widetilde{\Delta}$ is an elliptic operator. So according to the maximum principle $e(f)$ must be constant and so f is a constant map.
(b) Note that we have

$$-\langle \mathrm{R}^N(f_*(e_i), f_*(e_j))f_*e_i, f_*e_j\rangle \equiv 0 \tag{1.5}$$

Here, there is no summation over the indices i and j. If there exists a point x such that at least two vectors f_*e_1 and f_*e_2 among f_*e_i, are linearly independent at x, then when the sectional curvature of N is negative,

$$\langle \mathrm{R}^N(f_*(e_1), f_*(e_2))f_*e_1, f_*e_2\rangle < 0$$

This contradicts (1.5). Hence the rank of f is at most one. If the rank of f is zero at one point, then from the above discussion $e(f)$ is constant and the rank is zero everywhere. Otherwise the rank of f is equal to one everywhere and $f(M)$ is a closed geodesic in N. □

Theorem 1.4.5. *For every smooth map $f : (M, g, \Phi) \to (N, h)$, where M is a compact manifold and N is a negatively curved compact manifold, there is a harmonic map homotopic to f.*

For the proof of the above theorem, one can follow the heat flow method for deforming a given map to a harmonic map (see the proof in [Nis02] for the proof in classical case when $\Phi \equiv 1$

). The classical case have been proved first by Eells and Sampson in [ES64]. In the classical case, for a given map $f:(M,g)\to(N,h)$, the following initial value problem of the parabolic equation for harmonic maps is considered,

$$\begin{cases} \frac{\partial u}{\partial t}(x,t)=\tau(u(x,t)) & (x,t)\in M\times(0,T) \\ u(x,0)=f(x) \end{cases}$$

the solution is called the local time dependent solution. It is proved that the above equation has solution on $M\times(0,\infty)$ and thus there is a subsequence t_i such that $u(\cdot,t_i)$ converges to a harmonic map $u(x,\infty)$ which is homotopic to f. Now if we consider the smooth metric measure space (M,g,Φ) and the initial value f, we should solve

$$\begin{cases} \frac{\partial u}{\partial t}(x,t)=\tau(u(x,t))+du(\nabla\ln(\Phi)) & (x,t)\in M\times(0,T) \\ u(x,0)=f(x). \end{cases} \tag{1.6}$$

Since M is compact, we still have a nonlinear parabolic equation with the same nonlinearity and the classical proof works here. If we follow the same procedure as in [Nis02], we will see there are not many changes in the proof. We will leave the proof to the section Appendix. There we explain very briefly why each step in the original proof works in our case.

For a Riemannian manifold M of $m=\dim(M)\geq 3$, a map $f:(M,g,\Phi)\to(N,h)$ is harmonic if and only if the map $f:(M,\Phi^{\frac{2}{m-2}}g)\to(N,h)$ is harmonic. Lichnerowicz in [Lic69] and Eells-Lemaire in [EL78] used this fact and proved the above two theorems.

We state now the Theorem 0.1.1 and we prove it. See also [WX12, RVar] for similar result.

Theorem. *Let (M,g,Φ) be a complete non-compact, smooth metric-measure space with Φ bounded, and non-negative Bakry-Émery Ricci curvature, N a manifold with non-positive sectional curvature, and $f:M\to N$ a harmonic map of finite energy, $E_\Phi(f)<\infty$. Then f is a constant map.*

Proof. By Bochner formula we have

$$\widetilde{\Delta}e(f)\geq|B(f)|^2$$

and by Schwarz inequality we have

$$|\nabla e(f)|^2\leq 2e(f)|B(f)|^2.$$

From the equations above we conclude

$$\widetilde{\Delta}\sqrt{e(f)}\geq 0$$

or

$$\int_M \langle \nabla \sqrt{e(f)}, \nabla \eta \rangle \, \Phi \, \mathrm{dvol} \leq 0$$

for any $\eta \in C_0^\infty(M)$. This means that $\sqrt{e(f)}$ is Φ-subharmonic and it is in L^2. Therefore by lemma below we have $\sqrt{e(f)}$ is constant. According to theorem 1.3 in [WW09], M has at least linear Φ-volume growth and so $e(f) \equiv 0$ on M. Therefore f is a constant map. □

Lemma 1.4.6. *Let h be a non-negative smooth Φ-subharmonic function on M. Then $\int_M h^p \Phi \cdot \mathrm{dvol} = \infty$ for $p > 1$, unless h is a constant function.*

By Stokes formula for complete manifold (see Lemma 1 in [Yau76]), one can prove every Φ-subharmonic function h with $\int_M |dh| \, \Phi \, \mathrm{dvol} < \infty$ is Φ-harmonic. Then it is enough to repeat the argument in Theorem 3 in [Yau76], this time for Φ-subharmonic function and $\widetilde{\Delta}$ to prove the lemma.

Remark 1. *In Theorem 0.1.1, it is sufficient to have $\int_M \Phi \, \mathrm{dvol} \, M = \infty$.*

2 Harmonic Maps and Convergence

2.1 Preliminaries

2.1.1 Weakly harmonic maps

In this subsection, we first recall the definition of a weakly harmonic map on smooth metric measure spaces. We then briefly review this concept on Riemannian polyhedron. Let (N,h) be a compact Riemannian manifold and i be an isometric embedding $i : N \to \mathbb{R}^q$. Since $i(N)$ is a smooth, compact submanifold, there exists a number $\kappa > 0$ such that the neighborhood

$$U_\kappa(N) = \{y \in \mathbb{R}^q : \mathrm{dist}(y,N) < \kappa\}$$

has the following property: for every $y \in U_\kappa(N)$ there exists a unique point $\pi_N(y) \in N$ such that

$$|y - \pi_N(y)| = \mathrm{dist}(y,N)$$

The map $\pi_N : U_\kappa(N) \to N$ defined as above is called the *nearest point projection* onto N.

The $\mathrm{Hess}\,\pi_N$ defines an element in $\Gamma(TN^* \otimes TN^* \otimes TN^\perp)$ which coincides with the second fundamental form of $i : N \to \mathbb{R}^q$ up to a negative sign

$$\langle \mathrm{Hess}\,\pi_N(y)(X,Y),\eta\rangle = -\langle \nabla_Y \eta, X\rangle \tag{2.1}$$

where X and Y are in TN, y in N and η in $TN^\perp$ (see [Mos05]).

Before we define a weakly harmonic map on a Riemannian manifold (M,g), we define the Sobolev space $W^{k,q}(M)$. The space $W^{k,q}(M)$ is the set of $f \in L^q(M)$, such that f is k-times weakly differentiable and $|\nabla^j f| \in L^q(M)$ for $j \le k$. Define the Sobolev norm on $W^{k,q}(M)$ to be

$$\|f\|_{W^{k,q}} = \left(\sum_{j=0}^{j=k} \int_M |\nabla^j f|^q \ \mathrm{dvol}_g\right)^{1/q}.$$

Then $W^{k,q}$ is a Banach space with the above norm. Furthermore $\mathscr{H}^1 := W^{1,2}$ is a Hilbert space.

A map $f : (M, g, \Phi) \to (N, h)$, belonging to $\mathscr{H}^1_{loc}((M, \Phi\,\mathrm{dvol}), N)$ is called weakly harmonic map if and only if

$$\Delta i \circ f - \Pi(f)(df, df) + di \circ f(\nabla \ln(\Phi)) = 0 \tag{2.2}$$

in the weak sense. Here

$$\Pi(f)(df, df) = \text{trace Hess}(\pi_N)(i \circ f)(di \circ f, di \circ f) \tag{2.3}$$

and in coordinate

$$\Pi(f)(df, df) = \sum g^{ij} \frac{\partial^2 \pi_N^A}{\partial z^B \partial z^C} \frac{\partial f^B}{\partial x^i} \frac{\partial f^C}{\partial x^j}$$

For the map $f : (M^n, g) \to (N^m, h)$ and $\eta : M \to \mathbb{R}^q$, we define

$$\Xi_g(f, \eta) = \langle di \circ f, d\eta \rangle - \langle \Pi(f)(df, df), \eta \rangle. \tag{2.4}$$

We explain now the definition of harmonic maps on Riemannian polyhedra. We refer the reader to the Subsection 3.7 for complete review on this subject. Following [EF01], on an admissible Riemannian polyhedron X, a continuous weakly harmonic map $u : (X, g, \mu_g) \to (N, h)$ is of class $\mathscr{H}^1_{loc}(X, N)$ and satisfies: For any chart $\eta : V \to \mathbb{R}^n$ on N and any open set $U \subset u^{-1}(V)$ of compact closure in X, the equation

$$\int_U g(\nabla \lambda, \nabla u^k)\, d\mu_g = \int_U \lambda(\Gamma^k_{\alpha\beta} \circ u) g(\nabla u^\alpha, \nabla u^\beta)\, d\mu_g \tag{2.5}$$

holds for every $k = 1, \ldots, n$ and every bounded function $\lambda \in \mathscr{H}^1_0(U)$. Here $\Gamma^k_{\alpha\beta}$ denotes the Christoffel symbols on N. Similarly on a polyhedron X with a measure $\Phi\mu_g$ a continuous weakly harmonic map is a map in $\mathscr{H}^1_{loc}((X, \Phi\mu_g), N)$ which satisfies equation (3.9) with $\Phi d\mu_g$ in place of $d\mu_g$.

We present some theorems and lemmas that we need in this chapter.

Theorem 2.1.1. *[Mos05] Let $f \in \mathscr{H}^1(U, N) \cap C^0(U, N)$ be a weakly harmonic map, where U is an open domain in $\mathbb{R}^n$. Then $f \in C^\infty(U, N)$.*

The energy functional is lower semi continuous, and we have

Lemma 2.1.2. *[Xin96] Let $S \subset \mathscr{H}^1(M, N)$ on which the energy is bounded and S is closed under weak limit, then S is sequentially compact.*

The following lemma shows a formula on the second fundamental form of the composition of two maps.

Lemma 2.1.3. *Let M, N and $\bar{N}$ be Riemannian manifolds, $f : M \to N$ and $\bar{f} : N \to \bar{N}$ smooth maps. For the composite $\bar{f} \circ f : M \to \bar{N}$, we have the following composition formula. For X and Y in $\Gamma(TM)$*

$$B_{X,Y}(\bar{f} \circ f) = B_{f_* X, f_* Y}(\bar{f}) + d\bar{f}(B_{X,Y}(f)).$$

Taking its trace we have,

$$\tau(\bar{f} \circ f) = B_{f_*(e_i), f_*(e_i)}(\bar{f}) + d\bar{f}(\tau(f)).$$

Here we present a theorem which shows the relation between the tension field of equivariant harmonic maps under Riemannian submersions.

Theorem 2.1.4. *[Reduction Theorem][Xin96] Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ be Riemannian submersions, H_1 the mean curvature vector of the submanifold F_1 in E_1 and B_2 the second fundamental form of the fiber submanifold F_2 in E_2. Let $f : E_1 \to E_2$ be a horizontal equivalent map, $\bar{f}$ its induced map from M_1 to M_2 with tension field $\tau(\bar{f})$. $f^{\perp}$ denotes the restriction of f on the fibers F_1. Then we have following formula,*

$$\tau(f) = \tau^*(\bar{f}) + B_2(f_*(e_t), f_*(e_t)) - f_*(H_1) + \tau(f^{\perp})$$

where $\{e_t\}$, $t = n_1 + 1, \ldots, m_1$ is local orthonormal frame field of fibers F_1 and $\tau^*(\bar{f})$ denotes the horizontal lift of $\tau(\bar{f})$.

2.1.2 Hölder spaces on manifolds

Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection on M. Let $V \to M$ be a vector bundle on M equipped with the Euclidean metric on its fibers. Let $\hat{\nabla}$ be a connection on V preserving these metrics. Let $C^k(M)$ be the space of continuous, bounded function f that have k continuous, bounded derivatives, and define the norm $\|\cdot\|_{C^k}$ on $C^k(M)$ by, $\|f\|_{C^k} = \sum_{j=0}^{k} \sup_M |\nabla^j f|$.

Now we define the Hölder space $C^{0,\alpha}(M)$, for $\alpha \in (0,1)$. The function f on M is said to be Hölder continuous with exponent α, if

$$[f]_\alpha = \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

is finite. The vector space $C^{0,\alpha}(M)$ is the set of continuous, bounded functions on M which are Hölder continuous with exponent α and the norm $C^{0,\alpha}(M)$ is $\|f\|_{C^{0,\alpha}} = \|f\|_{C^0} + [f]_\alpha$.

In the same way, we shall define Hölder norms on spaces of sections v of a vector bundle V over M, equipped with Euclidean metrics in the fibres as above. Let $\delta(g)$ be the injectivity radius of the metric g on M, which we suppose to be positive and set

$$[v]_\alpha = \sup_{\substack{x\neq y\in M\\ d(x,y)<\delta(g)}} \frac{|v(x)-v(y)|}{d(x,y)^\alpha} \tag{2.6}$$

We now interpret $|v(x)-v(y)|$. When $x\neq y\in M$, and $d(x,y)\le\delta(g)$, there is unique geodesic γ of length $d(x,y)$ joining x and y in M. Parallel translation along γ using $\hat{\nabla}$ identifies the fibres of V over x and y and the metrics on the fibres. With this understanding, the expression $|v(x)-v(y)|$ is well defined.

So define $C^{k,\alpha}(M)$ to be the set of f in $C^k(M)$ for which the supremum $[\nabla^k f]_\alpha$ defined by 2.6 exists, working in the vector bundle $\bigotimes^k T^*M$ with its natural metric and connection. The Hölder norm on $C^{k,\alpha}(M)$ is $\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\nabla^k f]_\alpha$.

Lemma 2.1.5. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain. Suppose that $F:\Omega\to\mathbb{R}^q$ is bounded and Hölder continuous. Let $Q:\mathbb{R}^q\to\mathbb{R}^p$ be a quadratic function, then $Q\circ F:\Omega\to\mathbb{R}^p$ is also Hölder continuous and*

$$[Q\circ F]_\alpha \le A\sup_\Omega \|F\|_{\mathbb{R}^q}[\|F\|_{\mathbb{R}^q}]_\alpha$$

where A is a constant.

In the above lemma by a Quadratic function we mean

$$Q(y) = \sum_{i,j=1}^{q} Q_{ij}y_iy_j \qquad Q_{ij}\in C^1(\overline{\Omega}).$$

We have

Corollary 2.1.6. *Let $f\in C^{1,\alpha}(M,N)$, then*

$$[\Pi(f)(df,df)]_{C^\alpha} \le A\cdot\|d(f)\|_{L^\infty}\cdot[df]_{C^\alpha}$$

Proof. Let $\{\Omega_j\}$ be an atlas of M, such that $\operatorname{diam}(\Omega_j)\le\operatorname{injrad}(M)$ and set $F_j = df|_{\Omega_j}$ and $Q = \operatorname{Hess}\pi_N(X,X)$, for an smooth vector field X. Then using the previous lemma and an appropriate partition of unity we will have the result. □

Schauder Estimate. In this part, we give a quick review on the Schauder estimate of solutions to linear elliptic partial differential equations. Suppose (M,g) is compact and L is an elliptic operator, $L = a^{ij}\nabla_i\nabla_j + b_i\nabla_i + c$, where a symmetric and positive definite tensor, b is a $C^{0,\alpha}$ vector field on M and $c\in C^{0,\alpha}(M)$, and that L satisfies the conditions

$$\|a\|_{C^{0,\alpha}} + \|b\|_{C^{0,\alpha}} + \|c\|_{C^{0,\alpha}} \le \Lambda$$
$$\lambda\|\xi\|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda\|\xi\|^2, \text{ for all } x\in M, \text{ and } \xi\in\mathbb{R}^n.$$

Consider the following problem,

$$Lu = f \qquad \text{in } M,$$

if $\partial M = \emptyset$ and

$$\begin{cases} Lu = f & \text{in } M \\ u = g & \text{on } \partial M. \end{cases}$$

if $\partial M \neq \emptyset$. Then we have

Theorem 2.1.7. *[Schauder Estimate] [GT83] If $f \in C^{0,\alpha}(M)$ and $u \in C^2(M)$, then $u \in C^{2,\alpha}(M)$ and we have*

$$\|u\|_{C^{1,\alpha}} \leq C(\|f\|_{L^\infty} + \|u\|_{L^\infty}),$$
$$\|u\|_{C^{2,\alpha}} \leq C(\|f\|_{C^{0,\alpha}} + \|u\|_{L^\infty}),$$

where C depends on M, λ, Λ.

Hereafter we present an introduction to the convergence and collapsing theory. Most of the materials in this part was gathered from [Ron10].

2.1.3 Gromov-Hausdorff distance

Let X and Y be two compact metric spaces. The Gromov-Hausdorff distance between X and Y is defined as

$$d_{GH}(X,Y) = \inf_Z \{d_H^Z(\phi(X), \psi(Y)) : \exists \text{ isometric embedding } \phi : X \hookrightarrow Z,\ \psi : Y \hookrightarrow Z\}$$

where Z runs over all such metric spaces and ϕ and ψ runs over all possible isometric embedding and d_H is Hausdorff distance.

Let $\mathcal{MET}$ denote the set of all isometry class of nonempty compact metric spaces, then $(\mathcal{MET}, d_{GH})$ is a complete metric space.

An alternative definition of Gromov-Hausdorff distance. Let X and Y be two elements of $\mathcal{MET}$, a map $\phi : X \to Y$ is said to be an ϵ-Hausdorff approximation, if the following two conditions satisfied,
i) ϵ-onto: $B_\epsilon(\phi(X)) = Y$.
ii) ϵ-isometry: $|d(\phi(x), \phi(y)) - d(x,y)| < \epsilon$ for all $x, y \in X$.

The Gromov-Hausdorff distance $\hat{d}_{GH}(X,Y)$, between X and Y defined to be the infimum of the positive number ϵ such that there exist $\epsilon-$Hausdorff approximation from X to Y and form

Y to X. In fact $\hat{d}_{GH}$ doesn't satisfy triangle inequality and $\hat{d}_{GH} \neq d_{GH}$ but we can show that

$$\frac{2}{3} d_{GH} \leq \hat{d}_{GH} \leq 2 d_{GH}$$

because a sequence in $\mathcal{MET}$ converges with respect to d_{GH} if and only if it converges with respect to $\hat{d}_{GH}$, we will not distinguish $\hat{d}_{GH}$ from d_{GH}.

Lipschitz distance. Let X, Y in $\mathcal{MET}$ and $f : X \to Y$ be a Lipschitz map. We put

$$\text{dil } f = \sup_{x,y \in X} \frac{d(f(x), f(y))}{d(x,y)}$$

We define the Lipschitz distance, $d_L(X,Y)$, between X and Y by

$$\log(d_L(X,Y)) = \inf\{max\{\text{dil } f, \text{dil } f^{-1}\} | \, f : X \to Y \text{is a Lipschitz homeomorphism}\}$$

Lemma 2.1.8. *[Fuk88] Let X_i, Y_i, X, Y be compact metric spaces. Suppose that*

$$\lim_{i \to \infty} d_{GH}(X_i, X) = 0, \qquad \lim_{i \to \infty} d_{GH}(Y_i, Y) = 0$$

Let for any i, $d_L(X_i, Y_i) \leq \epsilon$. Then $d_L(X,Y) \leq \epsilon$.

Pointed Gromov-Hausdorff Convergence. The goal here is to extend the notion related to GH-convergence to complete metric spaces that are primarily non-compact or their diameters go to infinity.

let (X,p) and (Y,q) be pointed metric space. A pointed map $f : (X,p) \to (Y,q)$ is called ϵ-pointed Gromov-Hausdorff approximation (ϵ-PGHA) , if
i) $B_{\frac{1}{\epsilon}}(q) \subseteq B_\epsilon(\Phi(B_{\frac{1}{\epsilon}}(p)))$.
ii) $|d(\phi(x), \phi(y)) - d(x,y)| < \epsilon$ for all $x, y \in B_{\frac{1}{\epsilon}}(p)$.

The pointed Gromov-Hausdorff distance

$$d^p_{GH}((X,p),(Y,q)) = \inf_\epsilon \left\{\epsilon, \, \exists \, \epsilon - \text{PGHA from } (X,p) \text{ to } (Y,q) \text{ and from } (Y,q) \text{ to } (X,p)\right\}.$$

We say that a sequence $\{(X_i, p_i)\}$ in $\mathcal{MET}^p$, (the isometric classes of all pointed complete metric spaces such that any closed bounded subsets are compact), converges to $\{(X,p)\}$ if there is a sequence of ϵ_i-PGHA, $f_i : (X_i, p_i) \to (X,p)$, such that $\epsilon_i \to 0$.

Equivariant Gromov-Hausdorff Convergence. Let X (resp. Y) be a compact metric space on which a compact group G (resp. H) acts isometrically. For $\epsilon > 0$, a triple of maps (f, ϕ, ψ) is called an ϵ-equivariant GHA (briefly, ϵ-EGHA), if, for $x \in X$ and $y \in Y$, $t \in G$ and $s \in H$, the following applies,

i) $f : X \to Y$ is an ϵ-GHA,
ii) $\phi : G \to H$ is a map such that $d(f(t(x)), \phi(t)f(x)) < \epsilon$.
iii) $\psi : H \to G$ is a map such that $d(f(\psi(s)(x)), s(f(x))) < \epsilon$. We define the equivariant GH-distance (briefly GHD) by

$$d_{eqGH}((X,G),(Y,H)) = \inf\{\epsilon,\ \exists\, \epsilon - \text{EGHA } f : (X,G) \to (Y,H) \text{ and } h : (Y,H) \to (X,G)\}$$

Lemma 2.1.9. *Let $X_i \overset{d_{GH}}{\to} X$. Assume that for any i, a compact group G_i acts isometrically on X_i. Then there is a closed group G of isometries on X and a subsequence such that $(X_{i_k}, G_{i_k}) \overset{d_{eqGH}}{\to} (X,G)$.*

The following lemma determines the relation between equivariant GH-convergence and the convergence of obtained quotient spaces.

Lemma 2.1.10. *If $(X_i, G_i) \overset{d_{eqGH}}{\to} (X,G)$, then $X_i/G_i \overset{d_{GH}}{\to} X/G$.*

Convergence of maps. Let (X_i, p_i), (X, p), (Y_i, q_i) and (Y, q) be pointed metric spaces, such that (X_i, p_i) converges to (X, p) in the pointed Gromov-Hausdorff topology (resp. (Y_i, q_i) converges to (Y, q)). We say that a sequence of maps $f_i : (X_i, p_i) \to (Y_i, q_i)$ converges to a map $f : (X, p) \to (Y, q)$, if there exists a subsequence X_{i_k} such that if $x_{i_k} \in X_{i_k}$ and x_{i_k} converges to x (in $\coprod X_{i_k} \coprod X$ with the admissible metric), then $f_{i_k}(x_{i_k})$ converges to $f(x)$ and we have

Lemma 2.1.11. *i) If f_is are equicontinuous, then there is uniformly continuous map f and a converging subsequence X_{i_k} such that $f_i \to f$.*
ii) If f_is are isometries, then the limit map $f : (X, p) \to (Y, q)$ is also an isometry.

For the proof of the above lemma see [Ron10] Lemma 1.6.12.

Measured Gromov-Hausdorff Convergence. Let $\mathcal{M}\mathcal{M}$ denotes the class of all pairs (X, μ) of compact metric space X and a Borel measure μ on it such that $\mu(X) = 1$. Let (X_i, μ_i) be a sequence in $\mathcal{M}\mathcal{M}$. We say that (X_i, μ_i) converges to an element (X, μ) in $\mathcal{M}\mathcal{M}$ with respect to measured Gromov-Hausdorff topology if there exist Borel measurable ϵ-Hausdorff approximation $f_i : (X_i, \mu_i) \to (X, \mu)$ and $f_{i*}(\mu_i)$ converges to μ with respect to weak* topology. when M is a Riemannian manifold with finite volume, we put $\mu_M = \frac{\mathrm{dvol}_M}{\mathrm{vol}(M)}$, where dvol_M denotes the volume element of M and regard (M, μ_M) as an element in $\mathcal{M}\mathcal{M}$.

2.1.4 Convergence Theorems, Non-Collapsing

This subsection is devoted to the theory of convergence of manifolds in non-collapsing case. A sequence of n-manifolds M_i converging to a metric space X is called non-collapsing, if $\mathrm{vol}(M_i) \geq v > 0$. Otherwise it is called collapsing. For a non-collapsing sequence of manifolds, there is a uniform lower bound on the injectivity radius of M_i and thus M_is are diffeomorphic for large i. The next result is due to Cheeger-Gromov in [Che70, Pet84, GW88] and is formulated as following.

Theorem 2.1.12. *Let M_i be a sequence of closed n-manifolds such that* $|\sec_{M_i}| \leq 1$ *and* $\mathrm{vol}(M_i) > v > 0$, *and M_i converges to the metric space X. Then X is homeomorphic to manifold M, and for large i, there are diffeomorphisms, $\phi_i : M \to M_i$ such that the pullback metric converges to a $C^{1,\alpha}$-metric on M in the $C^{1,\alpha}$-norm.*

The following smoothing result concerns the approximation of Riemannian manifolds uniformly by the smooth one.

Theorem 2.1.13. *[BMOR84] Let (M, g) be a compact n-manifold with* $|\sec_M| < 1$. *For any* $\epsilon > 0$, *there is a metric g_ϵ such that*

$$|g_\epsilon - g|_{C^1} < \epsilon, \quad |\sec_{(M,g_\epsilon)}| \leq 1, \quad |\nabla^k \mathrm{R}_{g_\epsilon}| \leq C(n,k)\cdot \epsilon^k.$$

In particular

$$\begin{cases} \mathrm{e}^{-\epsilon}\,\mathrm{injrad}(M,g) \leq \mathrm{injrad}(M,g_\epsilon) \leq \mathrm{e}^{\epsilon}\,\mathrm{injrad}(M,g) \\ \mathrm{e}^{-\epsilon}\,\mathrm{diam}(M,g) \leq \mathrm{diam}(M,g_\epsilon) \leq \mathrm{e}^{\epsilon}\,\mathrm{diam}(M,g) \\ \mathrm{e}^{-\epsilon}\,\mathrm{vol}(M,g) \leq \mathrm{vol}(M,g_\epsilon) \leq \mathrm{e}^{\epsilon}\,\mathrm{vol}(M,g) \end{cases}$$

2.1.5 Convergence Theorems-Collapsing

This subsection is devoted to the theory of convergence of manifolds, in the collapsing case. Here we state some of the main results in this context.

Theorem 2.1.14. *[Fibration theorem] [Fuk87b] Let M^n and N^m be compact manifolds satisfying*

$$\sec_{M^n} \geq -1, \quad |\sec_{N^m}| \leq 1 \ (m \geq 2), \quad \mathrm{injrad}(N^m) \geq i_0 > 0$$

There exist a constant $\epsilon(n, i_0)$ such that if $d_{GH}(M^n, N^m) < \epsilon \leq \epsilon(n, i_0)$, then there is a C^1-fibration map $f : M^n \to N^m$ with connected fibre such that

i) The diameter of any f-fibres is at most $c \cdot \epsilon$, where $c = c(n, \epsilon)$ is such that $c \to 1$ as $\epsilon \to 0$.
ii) f is an almost Riemannian submersion , that is for any vector $\xi \in TM$ orthogonal to a fibre,

$$\mathrm{e}^{-\tau(\epsilon)} \leq \frac{|df(\xi)|}{|\xi|} \leq \mathrm{e}^{\tau(\epsilon)},$$

where $\tau(\epsilon) \to 0$ as $\epsilon \to 0$.
iii) If in addition, $\sec_{M^n} \leq 1$, *then f is smooth and the second fundamental form of any fiber satisfies $|II_{f^{-1}(\bar{x})}| \leq c(n)$.*

We have the following complement of the previous theorem.

Theorem 2.1.15. *[Equivariant affine fibration theorem] Let M^n and N^m be compact manifolds satisfying*

$$\sec_{M^n} \geq -1, \quad |\sec_{N^m}| \leq 1 \ (m \geq 2), \quad \operatorname{injrad}(N^m) \geq i_0 > 0.$$

Assume M^n and N^m admit isometric compact Lie group G-actions. There exist a constant $\epsilon(n, i_0) > 0$ such that if $d_{eqGH}((M^n, G), (N^m, G)) < \epsilon \leq \epsilon(n, i_0)$, then there is a C^1-fibration G-map, $f : M^n \to N^m$ satisfies i-iii above and the following.

iv) The fibers are diffeomorphic to an infranilmanifold, $\Gamma\backslash N$, where N is simply connected nilpotent group, $\Gamma \subset N \ltimes \mathrm{Aut}(N)$, such that $[\Gamma, N \cap \Gamma] \leq \omega(n)$.
v) There are canonical flat connections on fibres that vary continuously and the G-action preserves flat connections.
vi) The structure group of the fibration is contained in $\frac{\mathrm{cent}(N)}{\mathrm{cent}(N)\cap\Gamma} \ltimes \mathrm{Aut}(N \cap \Gamma)$.

By the Fibration Theorem, each fiber is an almost flat manifold and so is diffeomorphic to an infranilmanifold $\Gamma\backslash N$. First any Lie group has a canonical flat connection ∇^{can}, for which left invariant fields are parallel. The N-action on $\Gamma\backslash N$ from the right generate the left invariant fields on $\Gamma\backslash N$ and thus defines the canonical flat connection. By Malcev's rigidity theorem, any affine structure on $\Gamma\backslash N$ are affine equivalent to $(\Gamma\backslash N, \nabla^{\mathrm{can}})$. If the flat connection on the fibre can be chosen smoothly, then the structure group of the fibration reduces to a subgroup of the affine transformation on $\Gamma\backslash N$. The flat connection on $\Gamma\backslash N$ may depend on the choice of base point on the fiber. By averaging among all flat connections from the various choices of base points, a continuous family of flat connections on each fibers are constructed. In this way we can construct a canonical invariant metric for the left action of N.

When a sequence of n-manifolds with bounded curvature collapses, the limit space can be a singular space. We have

Theorem 2.1.16. *[Singular fibration theorem][Fuk88] Let M_i be a sequence of closed n-manifolds with $|\sec_{M_i}| \leq 1$ and $\mathrm{diam}(M_i) \leq D$, which converges to the closed metric space (X, d) in $\mathcal{MET}$. Then*

i) The frame bundles equipped with canonical metrics converge, $(F(M_i), O(n)) \to (Y, O(n))$, where Y is a manifold.
ii) There is an $O(n)$-invariant fibration $\tilde{f}_i : F(M_i) \to Y$ satisfying the conditions in Theorem 2.1.14 which becomes for $\epsilon > 0$, a nilpotent Killing structure with respect to an ϵ C^1-closed metric. Moreover each fibre on M_i has positive dimension.
iii) For any $\bar{x} \in X$, a fibre $f_i^{-1}(\bar{x})$ is singular if and only if $p^{-1}(\bar{x})$ is a singular $O(n)$-orbit in Y.

In the above theorem, the fibration map $\tilde{f}_i$ descends to a (singular) fibration map $f_i : M_i \to X = Y/O(n)$ such that the following diagram commutes
A *pure nilpotent Killing structure* on M^n, is a fibration $N \to M^n \to N^m$, with fibre N a nilpotent manifold (equipped with flat connection) on which parallel fields are Killing fields and the

$$\begin{tikzcd} F(M_i) \arrow[r, "\tilde{f}_i"] \arrow[d, "p_i"] & Y \arrow[d, "p"] \\ M_i \arrow[r, "f_i"] & X \end{tikzcd}$$

G-action preserves affine fibration. The underlying G-invariant affine bundle structure is called a pure N-structure. When we have a pure nilpotent Killing structure on M^n as before, we can construct an invariant metric (invariant under the left action of N), and therefore the fibration map f is a Riemannian submersion considering the induced metric on N^m. Let $\langle\ ,\ \rangle$ denotes the original metric and $(\ ,\ )$, the invariant one and suppose M^n and N^m are A-regular, which means that for some sequence $A = \{A_k\}$ of real non-negative numbers, we have

$$|\nabla^k \mathrm{R}| \leq A_k, \tag{2.7}$$

then we have

$$|\nabla^k(\langle\ ,\ \rangle - (\ ,\ ))| \leq c(n, A) \cdot \epsilon \cdot \mathrm{injrad}(N)^{-(k+1)}, \tag{2.8}$$

where $c(n, A)$ is a generic constant depending to finitely many A_k and n (see Proposition 4.9 in [CFG92]). Here the GH-distance M^n and N^m is less than ϵ.

In the following $\mathcal{M}(n, D)$ denotes the set of all compact Riemannian manifold M such that, $\dim(M) = n$, $\mathrm{diam}(M) < D$ and sectional curvature $|\sec_M| \leq 1$, and $\mathcal{M}(n, D, v)$ the set of manifolds in $\mathcal{M}(n, D)$, with volume $\geq v$. In the following remark we collect the main points that we need from the theorems above and explain the classification in the proof of Theorem 0.1.2.

Remark 2. *When a sequence of manifolds M_i converges in $\mathcal{M}(n, D)$ to a metric space X, then according to Theorem 2.1.16, the frame bundles over M_i equipped with canonical metrics $\tilde{g}_i$ converge to a manifold Y, and $\tilde{f}_i : (F(M_i), \tilde{g}_i, O(n)) \to (Y, O(n))$ is an $O(n)$ invariant fibration map. To see this, let $\tilde{g}_{i_\epsilon}$ be the smooth metric as it appeared in Theorem 2.1.13, then $(F(M_i), \tilde{g}_{i_\epsilon})$ converges to smooth manifold Y_ϵ. For a small fixed ϵ_0 and $\epsilon < \epsilon_0$, the sectional curvature on $(F(M_i), \tilde{g}_{i_\epsilon})$ is uniformly bounded and we can apply Theorem 2.1.15, to conclude that there exists an $O(n)$-invariant smooth fibration map $\tilde{f}_{i_\epsilon}$. By the continuity $(F(M_i), \tilde{g}_{i_\epsilon})$ is conjugate to $(F(M_i), \tilde{g}_{i_{\epsilon_0}})$. This implies that $Y_\epsilon \to Y$ is equivalent to a convergent sequence of metrics on*

Y_{ϵ_0} and so $(Y, O(n))$ conjugate $(Y_{\epsilon_0}, O(n))$

$$(F(M_i), O(n)) \simeq (F(M_i), \tilde{g}_{i_{\epsilon_0}}, O(n)) \xrightarrow{\tilde{f}_{i_{\epsilon_0}}} (Y_{\epsilon_0}, O(n)) \simeq (Y, O(n)),$$

and it induces a fibration map $(F(M_i), \tilde{g}_i, O(n)) \xrightarrow{\tilde{f}_i} (Y, O(n))$ (see proof of Theorem 4.1.3 *in [Ron10]). Note that $(F(M_i), \tilde{g}_{i_\epsilon}, O(n))$ is a pure nilpotent Killing structure and so there exists an invariant Riemannian metric close to $\tilde{g}_{i_\epsilon}$ which satisfies the inequality (2.8). Consequently the fibration map $\tilde{f}_{i_\epsilon}$ is a Riemannian submersion (considering the induced Riemannian metric on Y_ϵ by this map).*

2.1.6 Density function

Let $\mathscr{DM}(n, D)$ denote the closure $\mathscr{M}(n, D)$ in $\mathscr{MM}$ with respect to the measured Hausdorff topology. Then $\mathscr{DM}(n, D)$ is compact with respect to measured Hausdorff topology. Let $(M_i, g_i, \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}) \in \mathscr{M}(n, D)$, be a sequence of manifolds which converges to a manifold (M, g, μ) in measured Gromov-Hausdorff topology. Then there exists a fibration map $\psi_i : M_i \to M$ satisfying the following: for $x \in M$, we put

$$\Phi'_i(x) = \mathrm{vol}(\psi_i^{-1}(x)), \qquad \Phi_i = \frac{\Phi'_i}{\mathrm{vol}(M_i)}.$$

Then there exists Φ such that $\Phi = \lim_{i \to \infty} \Phi_i$ and μ is absolutely continuous with respect to dvol_M, $\mu = \Phi \cdot \mathrm{dvol}_M$ (see §3 in [Fuk87a]).
In the general case, for $(X, \mu) \in \mathscr{DM}(n, D)$, we recall first a remark on quotient spaces. Here $S(B)$ denotes the singular part of B.

Remark 3. *[Bes08] Let (M, g) be a Riemannian manifold and G a closed subgroup of isometries of M. Assume that the projection $\pi : M \to M/G$ is a smooth submersion. Then there exist one and only one Riemannian metric $\check{g}$ on $B = M/G$ such that π is a Riemannian submersion.*
We recall that using the general theory of slices for the action of a group of isometries on a manifold, one may show that there always exists an open dense submanifold U of M (the union of the principle orbits), such that the restriction $\pi|_U : U \to U/G$ is a smooth submersion.
Considering now M/G as a Riemannian polyhedron and μ_g as its Riemannian volume, the restriction of μ_g on U/G is equal to $\mathrm{dvol}_{U/G} = \mathrm{dvol}_{B - S(B)}$.

Suppose M_i in $\mathscr{M}(n, D)$ converges to the metric space X. We may assume that FM_i with the induced $O(n)$-invariant metric $\tilde{g}_i$, converges to $(Y, g, \Phi_Y \cdot \mathrm{dvol}_Y)$ with respect to the $O(n)$-measured Hausdorff topology, g , Φ_Y are $C^{1,\alpha}$-regular. Moreover, since $\pi_i : F(M_i) \to M_i$ is a Riemannian submersion with totally geodesic fibres and since the fibres are isometric to each other, it follows that $(FM_i, \mathrm{dvol}_{FM_i})/O(n) = (M_i, \mathrm{dvol}_{M_i})$. Hence by equivariant Gromov-Hausdorff convergence M_i converges to $(X, \nu) = (Y, \Phi_Y \cdot \mathrm{dvol}_Y)/O(n)$ (see Theorem 0.6 in

[Fuk89]) and by Remark (3)

$$\nu(S(X)) = 0$$

For all x in X, we put

$$\Phi_X(x) = \int_{y \in p^{-1}(x)} \Phi_Y(y) \ \mathrm{dvol}_{p^{-1}(x)}$$

where $\pi : Y \to X$ is the natural projection. For each open set U

$$\nu(U) = \int_U \Phi_X(x) \ \mathrm{dvol}_{X-S(X)}$$

Now we are ready to start the proof of Theorem 0.1.2.

2.2 Proof of the Convergence Theorem

In this section we are going to prove Theorem 0.1.2. First we recall the statement.

Theorem. *Let (M_i, g_i) be a sequence of manifolds in $\mathscr{M}(n, D)$ which converges to a metric measure space $(X, g, \Phi\mu_g)$ in the measured Gromov-Hausdorff Topology. Suppose (N, h) is a compact Riemannian manifold. Let $f_i : (M_i, g_i) \to (N, h)$ be a sequence of harmonic maps such that $\|e_{g_i}(f_i)\|_{L^\infty} < C$, where $\|e_{g_i}(f_i)\|_{L^\infty}$ is the L^∞-norm of the energy density of the map f_i and C is a constant independent of i. Then f_i has a subsequence which converges to a map $f : (X, g, \Phi \cdot \mu_g) \to (N, h)$, and this map is a harmonic map in $\mathscr{H}^1((X, \Phi\mu_g), N)$.*

We split the proof in three cases:

Case 1 [non-collapsing]. M_i converge to M in $\mathscr{M}(n, D, v)$. We first consider the situation where $M_i = M$ and g_i converges to a metric g in $\mathscr{M}(n, D, v)$. Then we study the problem in the general case using Theorem 2.1.12.

Case 2 [collapsing to a manifold]. (M_i, g_i) converge to manifold (M, g) in $\mathscr{M}(n, D)$ with g a $C^{1,\alpha}$-metric. We first consider the situation when (M_i, g_i) satisfies some regularity assumption (see Assumption 1 below). Then we discuss the general case using the fact that there is always a sequence of metric $g_i(\epsilon)$ on M_i, C^1-close to the the metric g_i which satisfies Assumption 1 as it is explained in Remark 2.

Case 3 [collapsing to a singular space]. M_i converge to a metric space (X, d) in $\mathscr{M}(n, D)$. When a sequence of manifolds M_i converges in $\mathscr{M}(n, D)$ to a metric space X, the frame bundles over M_i converge to a Riemannian manifold Y, with a $C^{1,\alpha}$-metric and $X = Y/O(n)$. The harmonic maps over M_i, induce harmonic maps over $F(M_i)$ and this case followed from the study of harmonic maps on quotient spaces.

We fix an isometric embedding $i: N \to R^q$ and we often denote the composition $i \circ f$ simply by f, unless we need to explicitly distinguish these two maps

Case 1. M_i converge to M in $\mathscr{M}(n, D, v)$.

To go through the proof in this case, we consider first the situation when a sequence of metrics g_i on a manifold M, converges to a metric g.

Lemma 2.2.1. *Let g_i be a sequence of metrics on smooth Riemannian manifold M and (M, g_i) converge to (M, g) in $\mathscr{M}(n, D, v)$. Suppose $f_i : M \to N$ is a sequence of harmonic maps such that*

$$\|e_{g_i}(f_i)\|_{L^\infty} < C$$

where C is a constant independent of i. Then there exists a subsequence of f_i which converges to some f in C^k-topology for $k \geq 0$ and f is also harmonic.

Proof. By Theorem 2.1.12, the metric g_i converges to g in $\mathscr{M}(n, D, v)$ in $C^{1,\alpha}$-topology. Using Schauder estimate, f_is have bounded norm in $C^k(M, g)$ for every $k \geq 0$ and so they are converging to a map $f \in C^k(M, g)$. We have

$$\lim_{i\to\infty} \Delta_{g_i} f_i = \Delta_g f$$

and

$$\lim_{i\to\infty} \Pi(f_i)(df_i, df_i) = \Pi(f)(df, df)$$

The above limits lead to harmoniciity f. □

When M_i converges to M in $\mathscr{M}(n, D, v)$, by Theorem 2.1.12 there is a $C^{1,\beta}$-diffeomorphism $\phi_n : M_n \to M$, such that the push forward of $\phi_{n*}(g_n)$ of the metrics g_n on M_n converges to the metric g (see [Kas89]). Since the map $\Phi_n : (M_n, g_n) \to (M, \phi_{n*}(g_n))$ is an isometry

$$e_{g_n}(f_n) = e_{\phi_{n*}(g_n)}(\bar{f}_n) \tag{2.9}$$

where $\bar{f}_n$ is the map $f_n \circ \phi_n^{-1}$. The map f_n is harmonic and so $\bar{f}_n$. So all the assumptions of Lemma 2.2.1 are satisfied here and the proof of theorem 0.1.2 in this case is complete.

Lemma 2.2.1 is not true if we assume only uniform bound on the energy, $E_{g_i}(f_i) < C$. To explain the reason we recall some of the results in [Sch84] and [Lin99]. Suppose $f_i : (M, g) \to (N, h)$ are smooth harmonic maps and M and N are compact manifolds. We first define

$$\mathscr{F}_\Lambda = \{u \in C^\infty(M, N) : \ u \text{ is harmonic}, E(u) \leq \Lambda\}.$$

We have the following theorem.

Theorem. *[Sch84] Let M and N be compact manifolds. Any map u in the weak closure $\mathscr{F}_\Lambda$ is smooth and harmonic outside a relatively closed singular set of locally finite Hausdorff $(n-2)$-dimensional measure.*

Let u_i be a sequence in $\mathscr{F}_\Lambda$, then there exists a subsequence which converges weakly to some u in $\mathscr{H}^1(M,N)$. Define

$$\Sigma = \bigcap_{r>0} \left\{ x \in M,\ \liminf_{i\to\infty} r^{2-n} \int_{B_r(x)} e(u_i) \geq \epsilon_0 \right\}$$

where $\epsilon_0 = \epsilon_0(n,N) > 0$ is a constant independent of u_i as in Theorem 2.2 in [Sch84]. If we consider a sequence of Radon measure $\mu_i = |du_i|^2 dx$, without loss of generality we may assume $\mu_i \rightharpoonup \mu$ weakly as Radon measures. By Fatou's lemma, we may write

$$\mu = |du|^2 dx + \nu$$

for some non-negative Radon measure ν. We can show that $\Sigma = \operatorname{spt} \nu \cup \operatorname{sing} u$ and ν is absolute continuous with respect to $H^{n-2}|_\Sigma$. Therefore u_i strongly converges in $\mathscr{H}^1(M,N)$ to u if and only if $|df_i|^2 dx \rightharpoonup |df|^2 dx$ weakly if and only if $\nu = 0$ if and only if $H^{n-2}(\Sigma) = 0$ if and only if there is no smooth non-constant harmonic map from S^2 (2-sphere) into N (e.g. negatively curved manifolds). See [Lin99] for the complete discussion on the mentioned results.

When we have a sequence of manifolds (M_i, g_i) converges in $\mathscr{M}(n,d,v)$, then the injectivity radius is bounded from below and dvol_{g_i} converges to dvol_g weakly. Therefore if $E_{g_i}(f_i) < C$, C independent of i, then $E_g(f_i)$ is uniformly bounded and we have the following lemma,

Lemma 2.2.2. *Let (M_i, g_i) be a sequence of manifolds in $\mathscr{M}(n,D,v)$ which converges to a Riemannian manifold (M,g) in the measured Gromov-Hausdorff tpology. Suppose (N,h) is a compact Riemannian manifold which doesn't carry any harmonic 2-sphere S^2. Let $f_i : (M_i, g_i) \to (N,h)$ be a sequence of harmonic maps such that $\|E_{g_i}(f_i)\| < C$, where C is a constant independent of i. Then f_i has a subsequence which converges to a map $f : (M,g) \to (N,h)$, and this map is a weakly harmonic map.*

By the above discussion, we know that f_i converges strongly in $\mathscr{H}^1$ to the map f. Also $\mathrm{Hess}(\pi_N)$ restricted to a neighborhood of N is Lipschitz and by Lemma 6.4 in [Tay00], $\mathrm{Hess}(\pi_N) \circ f_i$ converges to $\mathrm{Hess}(\pi_N) \circ f$ in $\mathscr{H}^1$-norm and so we have $\Pi(f_i)(df_i, df_i) \rightharpoonup \Pi(f)(df, df)$ weakly. We have the same for Δf_i and so f is a weakly harmonic map.

Remark 4. *In Lemma 2.2.2, if we consider that N is only a non-positively curved manifold, using Bochner-type formula we can prove that f is strongly harmonic. Also if we only assume that there is no strictly convex bounded function on $f(M)$, then one can show that f doesn't have any singular point and so f is harmonic (see proposition 2.1 and corollary 2.4 in [Sch84]).*

Case 2. (M_i, g_i) converge to manifold (M,g) in $\mathscr{M}(n,D)$ with g a $C^{1,\alpha}$-metric.

We first prove the Theorem 0.1.2, under the following assumption:

Assumption 1. *We assume first that there exists a sequence $C = \{C_k\}$ of positive number C_k independent of i, such that*

$$|\nabla_{g_i}^k \mathrm{R}(M, g_i)| < C_k. \tag{2.10}$$

By the above extra regularity assumption, on $(M_i, g_i, \frac{\mathrm{dvol}_{M_i}}{vol(M_i)})$ converges to a smooth Riemannian manifold (M, g, Φ), with the smooth pair (g, Φ) (see Lemma 2.1 in [Fuk89]). By Theorem 2.1.14, we know that for i large enough there is a fibration map, $\psi_i : M_i \to M$. Let g_i be the invariant metric as in Remark 2, therefore there exist metrics g_i^M on M such that the maps $\psi_i : (M_i, g_i) \to (M, g_i^M)$ is a Riemannian submersion.

We know that (M, g_i^M) converges to (M, g) in $C^{1,\alpha}$-topology. Before we continue, we recall some results from [Fuk89] and [Fuk88] in the following remarks.

Remark 5. *Under the assumption 2.10, we have Φ is of class C^∞ .*

Remark 6. *Take an arbitrary point p_0 in M and choose $p_i \in \psi_i^{-1}(p_0)$. By $|\sec_{M_i}| \leq 1$, we know at point p_i on M_i, the conjugate radius is greater than some constant name it ρ. If we consider the pullback metric by exponential map at p_i, $\exp_{p_i}$, on the conjugate domain on the tangent space at p_i, then the injectivity radius at 0 is at least the conjugate radius at p_i (see Corollary 2.2.3 in [Ron10]).*

Consider the ball $B = B(0, \rho)$ in $T_{p_i} M_i$ with the metric $\tilde{g}_i$ induced by the exponential map. By virtue of assumption (2.10), $\tilde{g}_i$ will converge to some g_0 in the C^∞-topology. There are local groups G_i converging to a Lie group germ G such that

1. G_i acts by isometries on the pointed metric space $((B, \tilde{g}_i), 0)$.

2. $((B, \tilde{g}_i), 0)/G_i$ is isometric to a neighborhood of p_i in M_i.

3. G acts by isometries on the pointed metric space $((B, g_0), 0)$.

4. $((B, g_0), 0)/G$ is isometric to a neighborhood of p_0 in M and the action of G is free.

It follows that there is a neighborhood U of p_0 in M and a C^∞ map $s : U \to B$ such that

1. $s(p_0) = 0$.

2. $P \circ s = Id$, where P denotes the composition of the projection map and the above mentioned isometry in 4.

3. $d_{(B, g_0)}(s(q), 0) = d_N(q, p_0)$ holds for $q \in N$.

Therefore there is some ρ independent of i such that, $M = \bigcup_{j=1}^m B_\rho(x_j, M)$ and $B_\rho(x_j, M)$ satisfies the preceding conditions. Also we can construct a C^∞ section $s_{i,j} : B_{\frac{\rho}{2}}(x_j, M) \to M_i$ of ψ_i, such *that*

$$\frac{|(s_{i,j})_*(v)|}{|v|} < C \tag{2.11}$$

for each $v \in TB_{\frac{\rho}{2}}(x_j, M)$ *and* C *is a constant independent of* i*. Hereafter we put* $p_{i,j} = \psi_i^{-1}(x_j)$ *and by* $B(p_{i,j})$ *we mean a ball centered at* $p_{i,j}$*, with radius* ρ *in* $T_{p_{i,j}}M_i$*.*

Now we show that f_i's are almost constant on the fibers of M_i. The following lemma is similar to Lemma 4.3 in [Fuk87a]. Recall that if v is a tangent vector to M_i and h is a function on M_i, then $v \cdot h = dh(v)$ denotes the derivative of h in the direction of v.

Lemma 2.2.3. *Let* $h_i : M_i \to i(N) \subset \mathbb{R}^q$ *be a smooth maps which satisfies the Euler-Lagrange equation (2.2). Suppose* $v_i \in T_p(M_i)$*, satisfies* $(\psi_i)_*(v_i) = 0$*, and* $v_i', v_i'' \in T_p(M_i)$ $(p \in B_{2\rho/3}(p_{i,j}, M_i))$*. Then we have*

$$\begin{aligned} |v_i \cdot h_i| &\leq C_1 \cdot \epsilon_i' \cdot |v_i| \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}) && (2.12)\\ |v_i' \cdot v_i'' \cdot h_i| &\leq C_2 \cdot |v_i'| \cdot |v_i''| \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}) && (2.13)\end{aligned}$$

where C_1 *and* C_2 *are some constants independent of* i *and* ϵ_i' *is a sequence converging to zero.*

Proof. We put $\Phi_{i,j} = \exp_{p_{i,j}} : B(p_{i,j}) \to M_i$, $\tilde{g}_{i,j} = {\Phi_{i,j}}_*(g_i)$ and $a = \Phi_{i,j}^{-1}(p)$. We also denote $h_i \circ \Phi_{i,j}$ by $h_{i,j}$.

From the Schauder estimate for elliptic equations (see Theorem 2.1.7) we have,

$$\|h_{i,j}\|_{C^{1,\alpha}} \leq C' \cdot (\|\Delta h_{i,j}\|_{L^\infty} + \|h_{i,j}\|_{L^\infty}) \tag{2.14}$$

and hence

$$\|v_i' \cdot h_{i,j}\|_{C^\alpha} \leq C' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}) \tag{2.15}$$

where C' depends on the metric $\tilde{g}_{i,j}$. Since $\Phi_{i,j}$ is an isometry, by Lemma 2.1.3, we have $\Delta h_{i,j}(x) = \Delta h_i(\Phi_{i,j}(x))$. Also from (2.14), and the fact that $\tilde{g}_{i,j}$ converges in C^∞,

$$\|\Pi(h_{i,j})(dh_{i,j}, dh_{i,j})\|_{C^\alpha} \leq C'' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty})$$

where C'' is a constant independent of i. By equation (2.2), we have

$$\|\Delta h_{i,j}\|_{C^\alpha} \leq C'' \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty})$$

Using the Schauder estimate for second derivative, we have

$$\|h_{i,j}\|_{C^{2,\alpha}} \leq C \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}) \tag{2.16}$$

for some C independent of i and we have (2.13).

Now we will prove (2.12) by contradiction. Assume $|v_i| = 1$. Let $\sigma^i(t) = \exp_p^{F_i}(tv_i)$ be a geodesic in the fiber containing p, $F_i \subset M_i$ such that $\frac{d}{dt}|_{t=0}\sigma^i(t) = v_i$. For $0 \leq t \leq \frac{\rho}{5}$ this curve has a lift $l^i(t) \subset B(p_{i,j})$ such that $\Phi_{i,j}(l^i(t)) = \sigma^i(t)$. We have

$$d(\sigma_i(t), p) \leq \mathrm{diam}(F_i) \leq \epsilon_i$$

By contradiction we assume that there is subsequence of h_i and a positive number A such that

$$|v_i \cdot h_{i,j}| > A \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty})$$

We know that

$$v_i \cdot h_i = v_i \cdot h_{i,j} = \frac{d}{dt}\bigg|_{t=0} h_{i,j} \circ l^i(t).$$

There exist $\beta > 0$ and $\delta > 0$ independent of i such that for any $t < \delta$, we have

$$|h_{i,j} \circ l^i(t) - h_{i,j}(a)| > \beta \cdot t \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}). \tag{2.17}$$

To explain this, let $h_{i,j} \circ l^i(t) = q_{i,j}(t)$. We know from (2.16) that

$$|\frac{d}{dt}\bigg|_{t=0} q'_{i,j}(t)| \leq C(\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),$$

so for some fix δ and $0 < t < \delta$ we have

$$|q'_{i,j}(t) - q'_{i,j}(0)| \leq C' \cdot t \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

On the other hand we have

$$|q'_{i,j}(0)| > A \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),$$

so for δ small enough and $t < \delta$ we have

$$|q'_{i,j}(t)| > \beta \cdot (\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

Therefore

$$|q_{i,j}(t) - q_{i,j}(0)| = |q'_{i,j}(\theta_i)\cdot t| > \beta\cdot t\cdot(\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

from which (2.17) follows.

There exists $b \in B(p_{i,j})$, such that $d(a,b) < \epsilon_i$ and $\Phi_{i,j}(l_i(\delta')) = b$, for a fixed $\delta' < \delta$ we have

$$|h_{i,j}(b) - h_{i,j}(a)| > \beta\cdot\delta'\cdot(\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}).$$

If we fix $\{\xi_k\}_{k=0}^{k=n}$ as a coordinate system at the point $a \in B(p_{i,j})$, for some $b' \in B(p_{i,j})$ we have

$$\sum_{k=0}^{k=n}\frac{\partial h_{i,j}}{\partial\xi^k} > C\cdot\beta\cdot\frac{\delta'}{\epsilon_i}\cdot(\|\Delta h_i\|_{L^\infty} + \|h_i\|_{L^\infty}),$$

and this contradicts (2.15). □

As we assumed $\|e(f_i)\|_{L^\infty} < c$ and by the Euler-Lagrange equation and Corollary 2.1.6, we have that $\|\Delta f_i\|_{L^\infty}$ is uniformly bounded. Moreover, $\|f_i\|_{L^\infty}$ is uniformly bounded, therefore by the above lemma (2.12), the maps f_is are equicontinuous. By Lemma 2.1.11, there is a limit map $f : M \to N$ which is continuous.

We consider the following maps on M,

$$\tilde{f}_i = \sum\beta_j\cdot(i\circ f_i)\circ s_{i,j}, \tag{2.18}$$

β_j is an arbitrary C^∞ partition of unity associated to $B_{\frac{\varrho}{2}}(x_j, M)$ and $s_{i,j}$ is the section associated to ψ_i as mentioned in Remark 6 and $i : N \to \mathbb{R}^q$ is an an isometric embedding . Along a subsequence which we again denote by f_i, we have

$$\lim_{i\to\infty} f_i(s_{i,j}(x)) = f(x) \qquad \text{for } x \in B_{\frac{\varrho}{2}}(x_j, M),$$

and also

$$\lim_{i\to\infty}\tilde{f}_i(x) = i\circ f(x) \qquad \text{for } x \in B_{\frac{\varrho}{2}}(x_j, M).$$

Since the energy density of f_i is bounded and also $s_{i,j}$ satisfies in (2.11), we have $\|e(\tilde{f}_i)\|_{L^\infty}$ is uniformly bounded. Indeed, we have $\|\tilde{f}_i\|_{C^1}$ is bounded, $\tilde{f}_i$ converge uniformly to $i\circ f$. Moreover ψ_i has bounded second fundamental form (see Theorem 2.6 in [CFG92]) and the same is true for $s_{i,j}$. So $\tilde{f}_i$ has bounded C^2-norm and there is a subsequence of $\tilde{f}_i$ which converges to $i\circ f$ in C^1-topology.

Choose a local orthonormal frame $\{\bar{e}_k\}_{k=1}^m$ on (M, g_i^M). Denote its horizontal lift on (M_i, g_i) by $\{e_k\}_{k=1}^m$. Suppose $\{e_t\}_{t=m+1}^n$ is a local orthonormal frame field of the fiber F_i in M_i. Thus $\{e_k, e_t\}$ form a local orthonormal frame field in M_i. (Note that we omit put the index i for the orthonormal frame fields on (M_i, g_i) and (M, g_i^M).) Our aim is to show that f is also weakly harmonic.

Lemma 2.2.4. *We have*

$$\lim_{i\to\infty} |\langle d\, i\circ f_i, d\eta_i\rangle(p) - \langle d\tilde{f}_i, d\tilde{\eta}\rangle(\psi_i(p))| = 0,$$

where $\tilde{\eta}: M \to \mathbb{R}^q$, is a C^∞-map and $\eta_i = \tilde{\eta}\circ\psi_i$ and p in M_i.

Proof. By inequality (2.12),

$$|\langle d\, i\circ f_i, d\eta_i\rangle(p) - \sum_{k=1}^m \langle d\, i\circ f_i(e_k), d\eta_i(e_k)\rangle(p)| \le C\cdot\epsilon_i$$

for i large enough and C, a constant. Let F_i denote the fibre containing p and choose a point q in F_i. By 2.13, and $\operatorname{diam}(F_i) \le \epsilon_i$

$$|d\, i\circ f_i(e_k)(p) - d\, i\circ f_i(e_k)(q)| \le C\cdot\epsilon_i,$$

and so

$$|d\, i\circ f_i(e_k)(p) - d\, i\circ f_i(e_k)(s_{i,j}\circ\psi_i(p))| \le C\cdot\epsilon_i.$$

Because $\psi_i\circ s_{i,j} = \mathrm{Id}$, we have for $x\in M$

$$\psi_{i*}\big(e_k(s_{i,j}(x)) - s_{i,j*}(\bar{e}_k(x))\big) = 0.$$

By the inequality 2.11, we have

$$|e_k(s_{i,j}(x)) - s_{i,j*}(\bar{e}_k(x))| \le C,$$

for some constant C, therefore by (2.12),

$$|d\, i\circ f_i(e_k)(p) - d(i\circ f_i)\circ s_{i,j*}(\bar{e}_k)(\psi_i(p))| \le C\cdot\epsilon_i.$$

From the convergence of $f_i\circ s_{i,j}$ to f, we have

$$\lim_{i\to\infty} |\sum d\beta_j\cdot(i\circ f_i)\circ s_{i,j} - \sum d\beta_j\cdot(i\circ f)| = 0,$$

So

$$\lim_{i\to\infty} |d\tilde{f}_i - \sum \beta_j\cdot d((i\circ f_i)\circ s_{i,j})| = 0,$$

since $\sum_j \beta_j = 1$, we finally have

$$\lim_{i\to} |\langle d i \circ f_i, d\eta_i \rangle (p) - \langle d\tilde{f}_i, d\tilde{\eta} \rangle (\psi_i(p))| = 0.$$

□

Lemma 2.2.5. *We have*

$$\lim_{\imath\to\infty} \left| \Pi(f_i)(p)(d i \circ f_i, d i \circ f_i) - \Pi(\tilde{f}_i)(\psi_i(p))(d\tilde{f}_i, d\tilde{f}_i) \right| = 0,$$

Proof. By the above Lemma, we have

$$\lim_{i\to\infty} |d f_i(p) - d\tilde{f}_i(\psi_i(p))| = 0$$

By the same argument as Lemma **??**, we can conclude

$$\begin{aligned} &\left| \Pi(f_i)(p)(d i \circ f_i, d i \circ f_i) - \Pi(\tilde{f}_i)(\psi_i(p))(d\tilde{f}_i, d\tilde{f}_i) \right| \\ &\qquad \le C \cdot \left| d f_i(p) - d\tilde{f}_i(\psi_i(p)) \right| \end{aligned}$$

and so we have the result □

The map $\tilde{f}_i : (M, g_i^M, \mathrm{dvol}_{g_i^M}) \to \mathbb{R}^q$, converges in C^1 to the map $i \circ f$, and Φ_i converges to Φ in C^∞ topology. Also (M, g_i^M) converges to (M, g) in $\mathscr{M}(n, D, v)$. Therefore, we have

$$\left| \int_M \Xi_{g_i^M}(\eta, \tilde{f}_i)\ \Phi_i\, \mathrm{dvol}^{g_i^M} - \int_M \Xi(\eta, f)\ \Phi\, \mathrm{dvol}_g \right| \le C \cdot \epsilon_i.$$

where $\Xi(\cdot,\cdot)$ is defined in 2.4. By Lemma 2.2.4 and 2.2.5, we have

$$\lim_{i\to\infty} \left| \int_{M_i} \Xi_{g_i}(\eta_i, f_i) \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} - \int_M \Xi_{g_i^M}(\eta, \tilde{f}_i) \psi_{i*} \left(\frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} \right) \right| = 0.$$

and so we have

$$\lim_{i\to\infty} \int_{M_i} \Xi_{g_i}(\eta_i, f_i) \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} = \int_M \Xi_g(\eta, f)\ \Phi\, \mathrm{dvol}_M.$$

This prove the result in Case 2 under assumption 1.

For the general metric, by Theorem 2.1.16 we can obtain C^1-close metric $g_i(\epsilon)$ which satisfies 2.10 and such that the map $\psi_i : (M_i, g_i(\epsilon)) \to (M, \psi_{i*}(g_i(\epsilon)))$ is a Riemannian submersion.

For small ϵ, let $M(\epsilon)$ be the Gromov-Hausdorff limit of some subsequence $(M_i, g_i(\epsilon))$. By Lemma 2.1.8, $(M_i, g_i(\epsilon))$ and $(M(\epsilon), g(\epsilon))$ converge to (M_i, g_i) and (M, g) in $\mathcal{M}(n, D, v)$ respectively.

$f_i : (M_i, g_i) \to (N, h)$ is harmonic and since $g_i(\epsilon)$ is C^1-close to g, we have

$$|\Xi_{g_i}(f_i, \eta_i) - \Xi_{g_i(\epsilon)}(f_i, \eta_i)| \leq C \cdot \epsilon_i.$$

By (2.19), we have

$$\lim_{i\to\infty} \left| \int_{M_i} \Xi_{g_i(\epsilon)}(f_i, \eta_i) \frac{\mathrm{dvol}_{(M_i, g_i(\epsilon))}}{\mathrm{vol}((M_i, g_i(\epsilon)))} - \int_{M(\epsilon)} \Xi_{g(\epsilon)}(f, \eta) \cdot \Phi(\epsilon)\, \mathrm{dvol}_{M(\epsilon)} \right| = 0,$$

and finally since $g(\epsilon)$ converges to g in $C^{1,\alpha}$-topology, we have the desired result.

Case 3. M_i Converge to a Metric Space (X, d) in $\mathcal{M}(n, D)$.

Now we are going to investigate the general case when the sequence converges to a singular space. This means that $M_i \in \mathcal{M}(n, D)$ converges to some metric space (X, d). First we recall the following remark.

Remark 7 ([Fuk87a], §7)**.** *Let Y be a Riemannian manifold on which $O(n)$ acts as isometries, and $\theta : Y \to [0, \infty)$ be an $O(n)$-invariant smooth function. Put $X = Y/O(n)$. Let $\pi : Y \to X$ be the natural projection, $\bar{\theta} : X \to [0, \infty)$ the function induced from θ, and $S(X)$ the set of all singular points of X. The set $S(X) \subset X$ has a well defined normal bundle on the codimension 2 strata ($X = Y/O(n)$ is a Riemannian polyhedra and $S(X)$ is a subset of $n-2$-skeleton of X). Set*

$$\mathrm{Lip}(X, S(X)) = \{u \in \mathrm{Lip}(X) \mid v \cdot u = 0 \text{ if } v \text{ is perpendicular to } S(X)\}.$$

Define $Q_1 : \mathrm{Lip}(Y) \times \mathrm{Lip}(Y) \to [0, \infty)$ *and* $Q_2 : \mathrm{Lip}(X, S(X)) \times \mathrm{Lip}(X, S(X)) \to [0, 1)$ *by*

$$\begin{aligned} Q_1(\tilde{k}, \tilde{h}) &= \int_Y \theta \cdot \langle \nabla \tilde{k}, \nabla \tilde{h} \rangle \ \mathrm{dvol}_Y \\ Q_2(k, h) &= \int_X \bar{\theta} \cdot \langle \nabla k, \nabla h \rangle \ d\mu_g \end{aligned}$$

It is easy to see that $f \circ \pi \in \mathrm{Lip}(Y)$ for each f contained in $\mathrm{Lip}(X, S(X))$. Define $\pi^* : \mathrm{Lip}(X, S(X)) \to \mathrm{Lip}(Y)$ by $\pi^*(f) = f \circ \pi$. Let $\mathrm{Lip}_{O(n)}(Y)$ be the set of all $O(n)$- invariant elements of $\mathrm{Lip}(Y)$. Then, we can easily prove the following:

Lemma 2.2.6. *π^* is a bijection between $\mathrm{Lip}(X, S(X))$ and $\mathrm{Lip}_{O(n)}(Y)$. For elements f , k of $\mathrm{Lip}(X, S(X))$, we have*

$$Q_1(f, k) = Q_2(\pi^*(f), \pi^*(k)), \tag{2.19}$$

and

$$\int_Y \theta \cdot \pi^*(f)\pi^*(k)\ \mathrm{dvol}_Y = \int_X \bar{\theta} \cdot fk\ d\mu_g. \tag{2.20}$$

The frame bundle $\pi_i : F(M_i) \to M_i$ is a Riemannian submersion with totally geodesic fibres. So using the reduction formula (2.1.4), the map $\tilde{f}_i = f_i \circ \pi_i$ is harmonic on $F(M_i)$ and it is invariant under the action of $O(n)$. Furthermore $\|e(\tilde{f}_i)\|_\infty$ is bounded (π_i is a Riemannian submersion). Using Case 2, $\tilde{f}_i$ converge to some map $\tilde{f}$ on $(Y, g, \Phi_Y\,\mathrm{dvol}_Y)$. The map $\tilde{f}$ satisfies

$$\int_Y \Xi_g(\tilde{f}, \eta)\ \Phi_Y\,\mathrm{dvol}_Y = 0.$$

where η is a test function. The map $\tilde{f}$ is also $O(n)$ invariant and continuous. Consider the quotient map f such that $\tilde{f} = \pi^*(f)$. First we show that f is in $\mathcal{H}^1((X,\nu), N)$. By the argument in Case 2, $\tilde{f}$ is in $\mathcal{H}^1((Y, \Phi_Y\,\mathrm{dvol}_Y), N)$ and so by equation (2.19), f is of finite energy.
Now, we show that f is weakly harmonic on (X,ν). By equation (2.19), for η in $\mathrm{Lip}(X, S(X))$

$$\int_Y \langle \nabla i \circ \tilde{f}, \nabla \pi^*(\eta)\rangle\ \Phi_Y\,\mathrm{dvol}_Y = \int_X \langle \nabla i \circ f, \nabla \eta\rangle\ \Phi_X d\mu_g.$$

We have

$$\int_Y \langle \Pi(\tilde{f})(\nabla^g(i \circ \tilde{f}), \nabla^g(i \circ \tilde{f})), \pi_*(\eta)\rangle\ \Phi_Y\,\mathrm{dvol}_Y$$
$$= \int_X \langle \Pi(f)(\nabla(i \circ f), \nabla(i \circ f)), \pi(\eta)\rangle\ \Phi_X d\mu_g$$

and since $\Phi_Y = \pi^*(\Phi_X)$

$$\int_Y \Xi_g(\tilde{f}, \pi_*(\eta))\ \Phi_Y\,\mathrm{dvol}_Y = \int_X \Xi(f,\eta)\ \Phi_X d\mu_g.$$

which shows that $f : X \to N$ is a weakly harmonic map.

2.2.1 Further Discussion

In this subsection we are going to give a second proof on Theorem 0.1.2 under the following assumptions:
1. We consider the Assumption 1.

2. We consider the sections $s_{i,j}$ is almost harmonic,

$$|\tau(s_{i,j})| \leq C \cdot \epsilon_i, \tag{2.21}$$

and also

$$|\nabla_{\bar{X}} ds_{i,j}(X)| \leq C \cdot \epsilon_i. \tag{2.22}$$

where X is a smooth vector field on M and $\bar{X}$ is its horizontal lift.

This proof has the advantage of showing how the term $df(\nabla \ln \Phi)$ appears in the Euler-Lagrange equation for harmonic maps on weighted manifold (M, g, Φ).

Let $\{e_k, e_t\}$ and $\bar{e}_k$ be as in Case 2. We compute the tension field of the map f_i in these coordinates.

$$\begin{array}{rcl} \tau(f_i) & = & (\nabla_{e_k} df_i)e_k + (\nabla_{e_t} df_i)e_t \\ & = & (\nabla_{e_k} df_i)e_k + \nabla_{f_{i*}(e_t)} f_{i*}(e_t) \\ & - & f_{i*}(\nabla_{e_t} e_t)^H - f_{i*}(\nabla_{e_t} e_t)^V \\ & = & (\nabla_{e_k} df_i)e_k - f_{i*}(\mathrm{H}_i) + \tau(f_i{}^{\perp}) \end{array}$$

where $f_i{}^{\perp}$ denotes the restriction of f_i to the fibers F_i, and H_i is the mean curvature vector of the submanifold F_i.

By the discussion in Case 2, we know $\tilde{f}_i$ converges to f in C^1-topology. Now we will investigate how each term of the equation above behave as $f_i \to f$.

Lemma 2.2.7. *We have*

$$\lim_{i\to\infty} \left| di(\nabla_{e_k} df_i)e_k(p) - \left(\Delta^{g_i^M} \tilde{f}_i - \Pi(\tilde{f}_i)(d\tilde{f}_i, d\tilde{f}_i)\right)(\psi_i(p))\right| = 0 \tag{2.23}$$

Proof. By Lemma 2.1.3, we have

$$di(Bf_i(X_1, X_2)) = B(i \circ f_i)(X_1, X_2) - B(\pi_N)(d(i \circ f_i)(X_1), d(i \circ f_i)(X_2))$$

and so for $k = 1, \ldots, n$

$$di((\nabla_{e_k} df_i)e_k) = (\nabla_{e_k} d(i \circ f_i))e_k - B(\pi_N)(d(i \circ f_i)(e_k), d(i \circ f_i)(e_k))$$

First we show that

$$|\nabla_{e_k} d(i \circ f_i) e_k(p) - \Delta^{g_i^M} \tilde{f}_i(\psi_i(p))| \leq o(\epsilon_i)$$

By definition of $\tilde{f}_i$,

$$\begin{aligned}(\nabla_{\bar{e}_k} d\tilde{f}_i)\bar{e}_k &= \sum \big(d\beta_j(\bar{e}_k)\cdot df_i(s_{i,j*}(\bar{e}_k)) \\ &+ \beta_j \cdot (\nabla_{\bar{e}_k} d(f_i \circ s_{i,j}))\bar{e}_k + \triangle\beta_j \cdot f_i \circ s_{i,j}\big).\end{aligned}$$

and again by Lemma 2.1.3, we have,

$$\tau(f_i \circ s_{i,j}) = B_{s_{i,j*}(\bar{e}_k), s_{i,j*}(\bar{e}_k)} f_i + df_i(\tau(s_{i,j})). \tag{2.24}$$

Since $f_i \circ s_{i,j}$ converges in C^1 to f, we have

$$\lim_{i\to\infty} |\sum d\beta_j(\bar{e}_k)\cdot df_i(s_{i,j*}(\bar{e}_k))| = 0,$$
$$\lim_{i\to\infty} \sum \Delta\beta_j \cdot f_i \circ s_{i,j}(x) = \sum \triangle\beta_j \cdot f(x) = 0.$$

Also, $\psi_{i*}(e_k - s_{i,j*}(\bar{e}_k)) = 0$ and so $e_k - s_{i,j*}(\bar{e}_k)$ is vertical. On the other hand we have

$$|e_k - s_{i,j*}(\bar{e}_k)| \leq \epsilon_i.$$

By inequality (2.12) and assumption (2.21), the second term on the right hand side 2.24 converges to zero. Again by inequality (2.13) and the assumption (2.22), we have

$$\lim_{i\to\infty} |(\nabla_{e_k} df_i)(e_k - s_{i,j*}(\bar{e}_k))| = 0,$$
$$\lim_{i\to\infty} |(\nabla_{(e_k - s_{i,j*}(\bar{e}_k))} df_i) e_k| = 0.$$

Finally

$$\lim_{\to\infty} |(\nabla_{e_k} d(i\circ f_i)) e_k(p) - (\nabla_{\bar{e}_k} d\tilde{f}_i)\bar{e}_k(\psi_(p))| = 0.$$

We have the same for the second term

$$\lim_{i\to\infty} |\Pi(f_i)(p)(df_i, df_i) - \Pi(\tilde{f}_i)(\psi_i(p))(d\tilde{f}_i, d\tilde{f}_i)| = 0.$$

□

By the above lemma and $\psi_{i*}(\frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}) = \Phi_i \,\mathrm{dvol}^{g_i^M}$, we have

$$\lim_{i\to\infty} \left| \int_{M_i} \langle di((\nabla_{e_k} df_i)e_k), \eta_i\rangle \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} - \int_M \langle \Delta^{g_i^M} \tilde{f}_i - \Pi(\tilde{f}_i)(d\tilde{f}_i, d\tilde{f}_i), \eta\rangle \,\Phi_i \,\mathrm{dvol}^{g_i^M} \right| = 0$$

and we conclude

$$\begin{aligned}&\lim_{i\to\infty}\int_{M_i}\langle di((\nabla_{e_k}df_i)e_k),\eta_i\rangle\,\tfrac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}\\&=\int_M\left[\langle df,d\eta\rangle+\langle df(\nabla\ln\Phi)-\Pi(f)(df,df),\eta\rangle\right]\,\Phi\,\mathrm{dvol}_M\,.\end{aligned}\tag{2.25}$$

Now we will consider the second and third term in the decomposition of $\tau(f_i)$.

Lemma 2.2.8. *With the same assumptions as above, we have*

$$(a)\quad \lim_{i\to\infty}\int_{M_i}\langle df_i(\mathrm{H}_i^x),\eta_i\rangle\,\frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}=-\int_M\langle df(\nabla\ln\Phi),\eta\rangle\,\Phi\,\mathrm{dvol}_M\tag{2.26}$$

$$(b)\quad \lim_{i\to\infty}\|\tau(f_i{}^{\perp})\|=0\tag{2.27}$$

here H_i^x *denotes the mean curvature vector of the fiber* $F_i^x=\psi_i^{-1}(x)$*. As before* η *is a test map on* M *and* $\eta_i=\eta\circ\psi_i$*.*

Proof. To prove equality (2.26), we need the following.

Sublemma 1. *We have*

$$\nabla\ln\Phi(x)=-\lim_{i\to\infty}\psi_{i\,*}(\mathrm{H}_i^x)\qquad\textit{weakly.}\tag{2.28}$$

proof. Suppose X is a smooth vector field on M and X_i its horizontal lift on M_i. The flow θ_t^i of X_i sends fibers to fibers diffeomorphically. By the first variation formula

$$\left.\frac{d}{dt}\right|_{t=0}\theta_t^{i\,*}(\mathrm{dvol}_{F_i^x})=-\int_{F_i^x}\langle X_i,\mathrm{H}_i^x\rangle\,\mathrm{dvol}_{F_i^x}\tag{2.29}$$

Also

$$\Phi_i(x)=\frac{\mathrm{vol}(\psi_i^{-1}(x))}{\mathrm{vol}(M_i)}$$

and by (2.29),

$$d\Phi_i(X)(x)=-\int_{F_i^x}\langle X_i,\mathrm{H}_i^x\rangle\,\frac{\mathrm{dvol}_{F_i^x}}{\mathrm{vol}(M_i)},$$

For an arbitrary η in $C^\infty(M)$, we prove

$$\int_M \eta d\Phi_i(X)\ \mathrm{dvol}^{g_i^M} = -\int_{M_i} \eta_i \langle X_i, \mathrm{H}_i\rangle \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}. \tag{2.30}$$

If we consider (U_γ, h_γ) as a local trivialization of the the fibration ψ_i, then

$$\int_M \chi_{U_\gamma} d\Phi_i(X)\cdot \mathrm{dvol}^{g_i^M} = -\int_{U_\gamma}\int_{F_i^x} \chi_{U_\gamma} \langle X_i, \mathrm{H}_i^x\rangle \frac{\mathrm{dvol}_{F_i^x}}{\mathrm{vol}(M_i)} \mathrm{dvol}^{g_i^M}$$

and so

$$\int_M \chi_{U_\gamma} d\Phi_i(X)\ \mathrm{dvol}_M^{g_i^M} = -\int_{\psi_i^{-1}(U_\gamma)} \langle X_i, \mathrm{H}_i\rangle \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}$$

where χ_{U_γ} denotes the characteristic function on U_γ and so we have (2.30). Φ_i tends to Φ in C^∞ and also $\mathrm{dvol}^{g_i^M}$ tends to dvol_M as i goes to infinity. Letting $i\to\infty$ on the both hand side of (2.30) and by the definition of weak derivatives

$$\int_M \eta \nabla \ln\Phi(X)\ \Phi\,\mathrm{dvol}_M = -\lim_{i\to\infty}\int_M \eta\langle X, \psi_{i*}(\mathrm{H}_i)\rangle\ \Phi_i\,\mathrm{dvol}^{g_i^M}.$$

This proves the Sublemma 1.

Now recall that by definition of $\tilde{f}_i$, we have

$$d\tilde{f}_i(x) = \sum d\beta_j(x)\cdot f_i\circ s_{i,j}(x) + \sum \beta_j(x)\cdot d(f_i\circ s_{i,j})(x)$$

and so

$$\begin{aligned}
&\lim_{i\to\infty}\textstyle\int_{M_i}\langle d\tilde{f}_i(\psi_{i*}(\mathrm{H}_i^x))\circ\psi_i, \eta_i\rangle\ \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}\\
&= \lim_{i\to\infty}\textstyle\int_{M_i}\langle\sum[d\beta_j(\psi_{i*}(\mathrm{H}_i^x))\cdot(f_i\circ s_{i,j})]\circ\psi_i, \eta_i\rangle\ \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}\\
&+ \lim_{i\to\infty}\textstyle\int_{M_i}\langle\sum[\beta_j\cdot d(f_i\circ s_{i,j})(\psi_{i*}(\mathrm{H}_i^x)]\circ\psi_i, \eta_i\rangle\ \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}\\
&= \lim_{i\to\infty}\textstyle\int_{M_i}\langle df_i(\mathrm{H}_i^x), \eta_i\rangle\ \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)}.
\end{aligned}$$

We have

$$\lim_{i\to\infty}\sum d\beta_j(\psi_{i*}(\mathrm{H}_i^x))\cdot(f_i\circ s_{i,j}) = 0,$$

and by inequality (2.12) and the fact that $s_{i,j*}(\psi_{i*}(\mathrm{H}_i^x)) - \mathrm{H}_i^x$ is vertical, we obtain the last equality. By the above sublemma,

$$\lim_{i\to\infty}\int_{M_i}\langle df_i(\mathrm{H}_i^x), \eta_i\rangle\ \frac{\mathrm{dvol}_{M_i}}{\mathrm{vol}(M_i)} = -\int_M \langle df(\nabla\ln\Phi), \eta\rangle\cdot\Phi\,\mathrm{dvol}_M$$

To prove (b) start with

$$\tau(f_i{}^{\perp}) = \nabla_{f_{i*}(e_t)} f_{i*}(e_t) - f_{i*}(\nabla_{e_t} e_t)^V$$

From (2.12) and (2.13)

$$|\nabla_{f_{i*}(e_t)} f_{i*}(e_t)| < C \cdot \epsilon_i \tag{2.31}$$

where C is a constant independent of i. Again from (2.12), we have

$$\| f_{i*}(\nabla_{e_t} e_t)^V \|_{L^\infty} < C \cdot \epsilon_i |(\nabla_{e_t} e_t)^V|$$

and so we see that

$$\lim_{i\to\infty} \|\tau(f_i{}^{\perp})\| = 0.$$

□

The limits (2.25), (2.26) and (2.27), shows that f satisfies Euler-Lagrange equation in weak sense.

$$\int_M \langle df, d\eta\rangle - \langle \Pi(f)(df, df), \eta\rangle \cdot \Phi \,\mathrm{dvol}_M = 0. \tag{2.32}$$

When $e(f_i)$ are uniformly bounded, then $\|f_i\|_{C^k}$ are uniformly bounded, for $k \geq 0$. By Lemma 1.6 in [Fuk89] or Theorem 6.2 in [CFG92], we have that the map ψ_i are $\{B_k\}$-regular, where B_k are positive number and so we have $\tilde{f}_i$ have uniformly bounded C^k-norm. Therefore under the assumption in the beginning of this subsection, we have proved that $\tau(f_i) \to \tau(f) + df(\nabla \ln \Phi)$.

Part II

Riemannian polyhedra and Liouville-type theorems on them

3 Preliminaries

3.1 Riemannian polyhedra

In this section we recall the definitions and results about Riemannian polyhedra which will be used in the rest of the manuscript.

Definition 3.1.1. *[EF01] A countable locally finite simplicial complex K, consists of a countable set $\{v\}$ of elements, called vertices, and a set $\{s\}$ of finite non void subsets of vertices, called simplexes, such that*

- *any set consisting of exactly one vertex is a simplex,*
- *any non void subset of a simplex is a simplex,*
- *every vertex belongs to only finitely many simplexes (the local finiteness condition).*

To the simplicial complex K, we associate a metric space $|K|$ defined as follows. The space $|K|$ of K is the set of all formal finite linear combinations $\alpha = \sum_{v\in K} \alpha(v)v$ of vertices of K such that $0 \le \alpha(v) \le 1$, $\sum_{v\in K} \alpha(v) = 1$ and $\{v : \alpha(v) > 0\}$ is a simplex of K. $|K|$ is made into a metric space with barycentric distance $\rho(\alpha,\beta)$ between two points $\alpha = \sum \alpha(v)v$ and, $\beta = \sum \beta(v)v$ of $|K|$ given by the finite sum

$$\rho(\alpha,\beta)^2 = \sum_{v\in K} (\alpha(v) - \beta(v))^2.$$

With this metric $|K|$ is locally compact and separable. The metric ρ is not intrinsic. We denote by $d(\alpha,\beta)$ the length metric associated to ρ by the standard procedure [BBI01].

Lemma 3.1.2. *[EF01] Let K be a countable, locally finite simplicial complex of finite dimension n, and V a Euclidean space of dimension $2n+1$. There exists an affine Lipschitz homeomorphism i of $|K|$ onto a closed subset of V.*

Definition 3.1.3. *[EF01] We shall use the term polyhedron to mean a connected locally compact separable Hausdorff space X for which there exists a simplicial complex K and a homeomorphism θ of $|K|$ onto X. Any such pair $T = (K, \theta)$ is called a triangulation of X.*

Definition 3.1.4. *[EF01] A Lipschitz polyhedron is a metric space X which is the image of the metric space $|K|$ of some complex K under a Lipschitz homeomorphism $\theta : |K| \to X$. The pair (K, θ) is then called a Lipscitz triangulation (or briefly a triangulation) of the Lipschitz polyhedron X.*

A null set in a Lipschitz polyhedron X is understood a set $Z \subset X$ such that Z meets every maximal simplex s (relative to some, and hence any triangulation $T = (K, \theta)$ of X)) in a set whose pre-image under θ has p-dimensional Lebesgue measure 0, $p = \dim s$.

From Lemma 3.1.2 follows that every Lipschitz polyhedron (X, d_X) can be mapped Lipschitz homeomorphically and (simplexwise) affinely onto a closed subset of a Euclidean space.

Riemannian Structure on a polyhedron. The class of domains that we consider for our harmonic maps are Riemannian polyhedra. A Riemannian polyhedron is a Lipschitz polyhedron (X, d) such that for some triangulation $T = (K, \theta)$, there exist a measurable Riemannian metric $g^s = g_{ij} dx^i dx^j$ on each maximal simplex s of $i(|K|)$ (i as in Lemma 3.1.2), which satisfies

$$\Lambda^{-2} \|\xi\|^2 \le g_{ij}(x) \xi^i \xi^j \le \Lambda^2 \|\xi\|^2 \tag{3.1}$$

almost everywhere in standard coordinate in the simplex s. Here the constant Λ is independent of a given simplex. The distance d_X^g on X is an intrinsic distance with respect to the metric g, meaning that $d^g = d_X^g$ is the infimal length of admissible path joining x to y. Actually (X, d^g) is a length space. The detailed definition is somewhat subtle and we refer to [EF01], for a careful discussion of Riemannian polyhedra.

A Riemannian metric g on a polyhedron X is said to be continuous, if relative to some (hence any) triangulation, g_s is continuous up to the boundary on each maximal simplex s, and for any two maximal simplexes s and s' sharing a face t, g_s and $g_{s'}$ induce the same Riemannian metric on t. There is a similar notion of a Lipschitz continuous Riemannian metric.

A Riemannian polyhedron has a well defined volume element given simplexwise by

$$d\mu_g = \sqrt{\det(g_{ij}(x))}\, dx_1 dx_2 \dots dx_n,$$

this measure coincide with Hausdorff measure.

Further definitions.

Definition 3.1.5. *[EF01] A polyhedron X will be called admissible if in some (hence in any) triangulation,* *i) X is dimensionally homogeneous, i.e. all maximal simplexes have the same dimension $n(=\dim X)$, or equivalently every simplex is a face of some n-simplex and*

ii) X is locally $(n-1)$-chainable, i.e. for every connected open set $U \subset X$, the open set $U \backslash X^{n-2}$ is connected.

iii) The boundary ∂X of a polyhedron X is the union of all non maximal simplexes contained in only one maximal simplex.

In this work we always assume that (X, g) satisfies $\partial X = \emptyset$.

Definition 3.1.6. *[EF01] By an n-circuit we mean a polyhedron X of homogeneous dimension n such that in some, (and hence any) triangulation,*

i) every $(n-1)$-simplex is a face of at most two n-simplexes (exactly two if $\partial X = \emptyset$), and

ii) X is $(n-1)$-chainable, i.e. $X \backslash X^{n-2}$ is connected, or equivalently any two n-simplexes can be joined by a chain of contiguous $(n-1)$- and n-simplexes.

Let $S = S(X)$ denote the singular set of an n-circuit X, i.e. the complement of the set of all points of X having a neighborhood which is a topological n-manifold (possibly with boundary). S is a closed triangulable subspace of X of codimension ≥ 2, and $X \backslash S$ is a topological n-manifold which is dense in X. An admissible circuit is called a pseudomanifold. We call a pseudomanifold (X, g, d_X) a Lipschitz pseudomanifold, if g is Lipschitz continuous. If g is simplexwise smooth such that $(X \backslash S, g|_{X \backslash S})$ has the structure of a smooth Riemannian manifold, we call (X, g, d_X) a smooth pseudomanifold. [1]

3.2 The Sobolev space $W^{1,2}(X)$

Let (X, g, d_X) denote an admissible Riemannian polyhedron of dimension n. We denote by $\mathrm{Lip}^{1,2}(X)$ the linear space of all Lipschitz continuous functions $u : (X, d_X) \to \mathbb{R}$ for which the Sobolev $(1,2)$-norm $\|u\|$ defined by

$$\|u\|_{1,2}^2 = \int_X (u^2 + |\nabla u|^2)\, d\mu_g = \sum_{s \in S^{(n)}(X)} \int_s (u^2 + |\nabla u|^2)\, d\mu_g$$

is finite, $S^{(n)}(X)$ denoting the collection of all n-simplexes s of X, and $|\nabla u|$ the Riemannian norm of the Riemannian gradient on each s. (The Riemannian gradient is defined a.e. in X or a.e. in each $s \in S^n(X)$, by Rademacher's theorem for Lipschitz functions on Euclidean domains.)

The Lebesgue space $L^2(X)$ is likewise defined with respect to the volume measure.

[1] In many texts the term pseudomanifold is used for what we called a circuit.

The Sobolev space $W^{1,2}(X)$ is defined as the completion of $\mathrm{Lip}^{1,2}(X)$ with respect to the Sobolev norm $\|\cdot\|_{1,2}$. We use the notations $\mathrm{Lip}_c(X)$, $W_0^{1,2}(X)$ and $W_{\mathrm{loc}}^{1,2}(X)$, for the linear space of functions in $\mathrm{Lip}(X)$ with compact support, the closure of $\mathrm{Lip}_c(X)$ in $W^{1,2}(X)$ and all $u \in L^2_{\mathrm{loc}}(X)$ such that $u \in W^{1,2}(U)$ for all relatively compact subdomains U in X.

Sobolev spaces on metric spaces. Here we recall a few basic notions on analysis on metric spaces. For the sake of completeness, we compare the L^2 based Sobolev space on admissible Riemannian polyhedra as in [EF01], with the one in [Che99], and show that they are equivalent. We use [Che99] as our main reference. See also [Sha00, HK98, Haj96, HK00] and [BB11] for further references.

Let (Y, d, μ) be a metric measure space, μ Borel regular. Assume also the measure of balls of finite and positive radius are finite and positive. Fix a set $A \subset Y$. Let f be a function on A with values in the extended real numbers.

Definition 3.2.1. *An upper gradient, for f is an extended real valued Borel function, $g : A \to [0, \infty]$, such that for all points, $y_1, y_2 \in A$ and all continuous rectifiable curves, $c : [0, l] \to A$, parameterized by arc length s, with $c(0) = y_1$, $c(l) = y_2$, we have*

$$|f(y_2) - f(y_1)| \leq \int_0^l g(c(s))\, ds$$

Note that in above definition the left-hand side is interpreted as ∞, if either $f(y_1) = \pm\infty$ or $f(y_2) = \pm\infty$. If on the other hand, the right-hand side is finite then it follows that $f(c(s))$ is a continuous function of s. For a Lipschitz function f we define the lower pointwise Lipschitz constant of f at x as

$$\mathrm{lip}\, f(x) = \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r}$$

$\mathrm{lip}\, f$ is Borel, finite and bounded by the Lipschitz constant. Also $\mathrm{lip}\, f$ is an upper gradient for f. Similarly for Lipschitz function f, the upper pointwise Lipschitz constant f, $\mathrm{Lip}\, f$, is the Borel function

$$\mathrm{Lip}\, f(x) = \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(y) - f(x)|}{r}.$$

For any Lipschitz function f we have $\mathrm{lip}\, f(x) \leq \mathrm{Lip}\, f$. In the special case $Y = \mathbb{R}^n$, if x is a point of differentiability of f, we observe that $\mathrm{lip}\, f(x) = \mathrm{Lip}\, f(x) = |\nabla f(x)|$. We now define the Soblolev space $H^{1,p}$, for $1 \leq p < \infty$.

Definition 3.2.2. *Whenever $f \in L^p(Y)$, let*

$$\|f\|_{1,p} = \|f\|_{L^p} + \inf_{g_i} \liminf_{i\to\infty} \|g_i\|_{L^p},$$

where the infimum is taken over all sequence $\{g_i\}$, for which there exists a sequence $f_i \xrightarrow{L^p} f$, such that g_i is an upper gradient for f_i, for all i.

For $p \geq 1$, the Sobolev space, $H^{1,p}$, is the subspace of L^p consisting of functions, f, for which $\|f\|_{1,p} < \infty$, equipped with the norm $\|\cdot\|_{1,p}$. The space $H^{1,p}$ is complete.

We define now the notions of generalized upper and minimal upper gradients. This will allow us to give a nice interpretation of the $H^{1,p}$ norm of Sobolev functions.

Definition 3.2.3. *i) The function, $g \in L^p$ is a generalized upper gradient for $f \in L^p$, if there exist sequences, $f_i \xrightarrow{L^p} f$, $g_i \xrightarrow{L^p} g$, such that g_i is an upper gradient for f_i, for all i.*
ii) For fixed p, a minimal generalized upper gradient for f is a generalized upper gradient g_f, such that $\|f\|_{1,p} = \|f\|_{L^p} + \|g_f\|_{L^p}$.

The following theorem ensures the existence of minimal generalized upper gradient for any Sobolev function.

Theorem 3.2.4. *[Che99] For all $1 < p < \infty$ and $f \in H^{1,p}$ there exists a minimal generalized upper gradient, g_f, which is unique up to modification on subsets of measure zero.*

We will discuss two important properties of metric spaces called the *ball doubling property* and the *Poincaré inequality* for functions on them. These are essential assumptions to get a richer theory on metric spaces.

Definition 3.2.5. *Let (Y, d, μ) be a metric measure space. The measure μ is said to be locally doubling if for all r' there exists $\kappa = \kappa(r')$ such that for all $y \in Y$ and $0 < r < r'$*

$$0 < \mu(B_r(y)) \leq 2^\kappa \mu(B_{r/2}(y)). \tag{3.2}$$

Definition 3.2.6. *Let $q \geq 1$. We say that Y supports a weak Poincaré inequality of type (q, p), if for all $r' > 0$, there exist constants $1 \leq \lambda < \infty$ and $C = C(p, r') > 0$ such that for all $r \leq r'$, and all upper gradients g of f,*

$$\left(\int_{B_r(x)} |f - f_{x,r}|^q \, d\mu\right)^{1/q} \leq Cr \left(\int_{\lambda B_r(x)} |g|^p \, d\mu\right)^{1/p}, \tag{3.3}$$

where $f_{x,r} := \int_{B_r(x)} f \, d\mu$. If $\lambda = 1$, then we say that X supports a strong (q, p)-Poincaré inequality.

For every admissible Riemannian polyhedron (X, g, μ_g), μ_g is locally doubling. Moreover X supports a weak $(2,2)$-Poincaré inequality and by Hölder's inequality $(1,2)$-Poincaré inequality (see Corollary 4.1 and Theorem (5.1) in [EF01]). In the sequel, the words "Poincaré inequality" refer to $(2,2)$-Poincaré inequality.

By Theorem 4.24 in [Che99], for any metric space which satisfies (3.2) and (3.3), for some $1 \leq p < \infty$ and $q = 1$, the subspace of locally Lipschitz functions is dense in $H^{1,p}$. Furthermore on a locally complete metric space with the mentioned properties, we have for some $1 < p < \infty$ and for any f locally lipschitz, $g_f = \mathrm{Lip}\, f$, μ-almost everywhere (see [Che99] Theorem 6.1). Therefore, on a Riemannian polyhedron (X, g, μ_g), for any $f \in H^{1,2}$, $g_f(y) = |\nabla f(y)|$ for a.e. y and it follows that $H^{1,2}$ is equivalent to $W^{1,2}$. In the following, we always consider $X = (X, g, \mu_g)$ to be an admissible Riemannian polyhedron. Some of the concepts below are defined on metric spaces in general but for simplicity we present them only on Riemannian polyhedron and for $p = 2$. For more information on metric spaces we refer the reader to [BB11].

3.3 Capacities

In this section we recall some of the definitions in potential theory. First we define Sobolev capacity.

Definition 3.3.1. *[BB11] The Sobolev capacity of a set $E \subset X$ is the number*

$$C(E) = \inf \|u\|^2_{W^{1,2}(X)},$$

where the infimum is taken over all $u \in W^{1,2}(X)$ such that $u \geq 1$ on E.

The variational capacity is defined as follow,

Definition 3.3.2. *[BB11] Assume $\Omega \subset X$ is bounded. Let $E \subset \Omega$. We define the variational capacity*

$$\mathrm{cap}(E, \Omega) = \inf_u \int_\Omega |\nabla u|^2 \, d\mu_g, \tag{3.4}$$

where the infimum is taken over all $u \in W^{1,2}_0(\Omega)$ such that $u \geq 1$ on E.

In the above definitions the infimum can be taken only over $u \leq 1$ such that it is equal 1 on a neighborhood of E. Also we write $\mathrm{cap}(E) = \mathrm{cap}(E, X)$.

Definition 3.3.3. *[EF01] A set $U \subset X$ is quasi open if there are open sets ω of arbitrarily small capacity such that $U \backslash \omega$ is open relative to $X \backslash \omega$.*

and a quasicontinuous map is

Definition 3.3.4. *[EF01] A map $\phi : U \to Y$ from a quasiopen set U to a topological space Y with a countable base of open sets is quasicontinuous if there are open sets ω of arbitrarily small capacity such that $\phi|_{U\setminus\omega}$ is continuous*

Clearly this amounts to $\phi^{-1}(V)$ being quasiopen for every open subset V of Y.

3.4 Weakly harmonic and weakly sub/super harmonic functions

Definition 3.4.1. *A function $u \in W^{1,2}_{loc}(X)$ is said to be weakly harmonic if*

$$\int_X \langle \nabla u, \nabla \rho \rangle \, d\mu_g = 0 \qquad \textit{for every } \rho \in \mathrm{Lip}_c(X).$$

A function $u \in W^{1,2}_{loc}(X)$ is said to be weakly subharmonic, resp. weakly superharmonic, if

$$\int_X \langle \nabla u, \nabla \rho \rangle \, d\mu_g \leq 0, \ \textit{resp.} \ \geq 0 \qquad \textit{for every } \rho \in \mathrm{Lip}_c(X).$$

Now we have,

Proposition 3.4.2. *[EF01] A function $u \in W^{1,2}(X)$ is weakly harmonic if and only if u minimizes the energy $E(v)$ among all functions $v \in W^{1,2}(X)$ such that $v - u \in W^{1,2}_0(X)$.*

[EF01] In the following we discuss on the existence of minimizer under specific assumption on the Riemannian polyhedra.

Theorem 3.4.3. *Suppose the following Poincaré inequality holds:*

$$\int_X |u|^2 \, d\mu_g \leq c \int_X |\nabla u|^2 \, d\mu_g \qquad \textit{for all } u \in W^{1,2}_0(X), \tag{3.5}$$

with c depending only on the admissible Riemannian polyhedron X. For any $f \in W^{1,2}(X)$ the class of competing maps

$$W^{1,2}_f(X) = \{v \in W^{1,2}(X) : v - f \in W^{1,2}_0(X)\}, \tag{3.6}$$

contains a unique weakly harmonic function u. That function is the unique minimizer of

$E(u) = E_0$, where

$$\begin{aligned} E_0 &:= \inf\{E(v) : v \in W^{1,2}(X), v - f \in \mathrm{Lip}_c(X)\} \\ &= \min\{E(v) : v \in W^{1,2}_f(X)\}. \end{aligned}$$

As a corollary of the above theorem we have,

Corollary 3.4.4. *Assume that the domain $\Omega \subset X$ is bounded and such that the Sobolev capacity $C(X\backslash\Omega) > 0$. For any $f \in W^{1,2}(\Omega)$, the class of functions*

$$W^{1,2}_f(\Omega) = \{v \in W^{1,2}(\Omega) : v - f \in W^{1,2}_0(\Omega)\}$$

has a unique solution u of the equation $E(u) = E_\Omega$, where

$$E_\Omega := \inf\{E(v) : v \in W^{1,2}(\Omega), v - f \in W^{1,2}_0(\Omega)\}.$$

Proof. Since X satisfies the Poincaré inequality and using Theorem 5.54 in [BB11], Ω satisfies the inequality (3.5). By the above theorem, there is a unique minimizer which is weakly harmonic. □

We have the following theorem on the regularity of weakly harmonic functions on Riemannian polyhedra.

Theorem 3.4.5. *[EF01] Every weakly harmonic function on X is Hölder continuous (after correction on a null set).*

A continuous weakly harmonic function on X is called harmonic.

Remark 8. *From the discussion above one can see in the definition of variational capacity that there is a harmonic function u which takes the minimum in (3.4). This function is not necessarily continuous on the boundary of $\Omega\backslash E$.*

3.5 Polar sets

Definition 3.5.1. *A set $S \subset X$ is said to be a polar set for the capacity if for every pair of relatively compact open sets $U_1 \Subset U_2 \subset X$ such that $d(U_1, X\backslash U_2) > 0$ we have*

$$\mathrm{cap}(S \cap \overline{U_1}, U_2) = 0.$$

According to Theorem 9.52 in [BB11] (see also section 3 in [GT02]), S is a polar set if and only if every point of X has an open neighborhood U on which there is a superharmonic function which equals $+\infty$ at every point of $S \cap U$.

Lemma 3.5.2. *A closed set $S \subset X$ is a polar set if and only if for every neighborhood U of S and every $\epsilon > 0$, there exists a function $\varphi \in \mathrm{Lip}(X)$ such that*
i) the support of φ is contained in $X \backslash S$,
ii) $0 \leq \varphi \leq 1$,
iii) $\varphi \equiv 1$ on $X \backslash U$.,
iv) $\int_X |\nabla\varphi|^2 < \epsilon$.

Proof. The proof is based on the definition of polar set and it is completely the same as the case of Riemannian manifolds. See Proposition 3.1 in [Tro99] for the proof of the equivalence on Riemannian manifolds. □

3.6 The Dirichlet space $L_0^{1,2}(X)$

In this section we introduce the Drichlet space $L_0^{1,2}(X)$ on an admissible Riemannian polyhedron X (see [EF01]). The Drichlet space $L_0^{1,2}(X)$ determines a Brelot harmonic structure on X. Using this fact we can show, X has a symmetric Green function which gives us information on the singularities of X.

Proposition 3.6.1. *[EF01] Suppose that, for every compact set $K \subset X$,*

$$\left(\int_K |u| \, d\mu_g\right)^2 \leq c(K)E(u) \qquad \text{for all } x \in \mathrm{Lip}_c(X), \tag{3.7}$$

with $c(K)$ depending only on X and K. In particular, X is non-compact. The completion $L_0^{1,2}(X)$ of space $\mathrm{Lip}_c(X)$ within $L^1_{\mathrm{loc}}(X)$ with respect to the norm $E(u)^{1/2}$ is then a regular Dirichlet space of strongly local type. $L_0^{1,2}(X)$ is a subset of $W^{1,2}_{\mathrm{loc}}(X)$.

Note that $W_0^{1,2}(X) \subset L_0^{1,2}(X) \subset W^{1,2}_{\mathrm{loc}}(X)$. According to the above proposition, $(L_0^{1,2}(X), E)$ is a strongly local regular Drichlet form. Let

$$\Delta : L_0^{1,2}(X) \supset D(\Delta) \to L^2(X)$$

denote the generator induced from $(L_0^{1,2}(X), E)$, which is a densely defined non-positive definite self-adjoint operator satisfying $E(u,v) = (\Delta u, v)_{L^2}$. Here $D(\Delta)$ denotes the domain of operator Δ. We have

Theorem 3.6.2. *[EF01] Let (X, g, μ_g) be an admissible Riemannian polyhedra such that the inequality (3.7) holds. Then X has a unique symmetric Green kernel*

$$G : X \times X \to (0, \infty]$$

which is finite and Hölder continuous off the diagonal $X \times X$.

For local questions, condition (3.7) is not required (it is automatically satisfied with X replaced by the open star of a point a of X relative to a sufficiently fine triangulation and in view of inequality (3.3)). As a consequence of Theorem 3.6.2, we have

Proposition 3.6.3. *The $(n-2)$-skeleton $X^{(n-2)}$ of an admissible Riemannian n-polyhedron is a polar set.*

We should note that being polar is independent of the Riemannian structure on the polyhedron.

Remark 9. *Every closed polar subset F of X is removable for Sobolev $(1,2)$- functions, i.e. $W^{1,2}(X \backslash F) = W^{1,2}(X)$. A larger class of removable sets in this sense is that of all (closed) sets of $(n-1)$-dimensional Hausdorff measure zero (see Proposition 7.7 in [EF01]).*

3.7 Harmonic maps on Riemannian polyhedra

The energy of a map from a Riemannian domain to an arbitrary metric space was defined and investigated by Korevaar and Schoen [KS93]. Here, we give an introduction to the concept of energy of maps, energy minimizing maps and harmonic maps on a Riemannian polyherdron. In the case that the target Y is a Riemannian C^1-manifold the energy of the map is given by the usuall expression (similarly when the target is a Riemannian polyhedron with continuous Riemannian metric).

Let (X, g) be an admissible n-dimensional Riemannian polyhedron with simplexwise smooth Riemannian metric. We do not require that g is continuous across lower dimensional simplexes. Let Y be an arbitrary metric space. Denote by $L^2_{\text{loc}}(X, Y)$ the space of all μ_g-measurable maps $\varphi : X \to Y$ having separable essential range (The essential range of a map φ is a closed set of points $q \in Y$ such that for any neighborhood V of q, $\varphi^{-1}(V)$ has positive measure.), and for which $d_Y(\varphi(\cdot), q) \in L^2_{\text{loc}}(X, \mu_g)$ for some point q (and therefore by the triangle inequality for any $q \in Y$). For $\varphi, \psi \in L^2_{\text{loc}}(X, Y)$ define their distance

$$D(\varphi, \psi) = \left(\int_X d_Y^2(\varphi(x), \psi(x))\, d\mu_g(x) \right)^{1/2}.$$

The approximate energy density of a map $\varphi \in L^2_{\text{loc}}(X, Y)$ is defined for $\varepsilon > 0$ by

$$e_\varepsilon(\varphi)(x) = \int_{B(x,\varepsilon)} \frac{d_Y^2(\varphi(x), \varphi(x'))}{\varepsilon^{n+2}} \, d\mu_g(x')$$

The function $e_\varepsilon(\varphi)$ is of class $L^1_{\text{loc}}(X, \mu_g)$ (see [KS93]).

Definition 3.7.1. *The energy $E(\varphi)$ of a map φ of class $L^2_{\text{loc}}(X, Y)$ is defined as*

$$E(\varphi) = \sup_{f \in C_c(X,[0,1])} \left(\limsup_{\varepsilon \to 0} \int_X f e_\varepsilon(\varphi) \, d\mu_g \right).$$

We say that φ is locally of finite energy, and write $\varphi \in W^{1,2}_{\text{loc}}(X, Y)$, if $E(\varphi|_U) < \infty$ for every relatively compact domain $U \subset X$. For example every Lipschitz continuous map $\varphi : X \to Y$ is in $W^{1,2}_{\text{loc}}(X, Y)$. Now we give a necessary and sufficient condition for a map φ to be in $W^{1,2}_{\text{loc}}(X, Y)$.

Lemma 3.7.2. *Let (X, g) be an admissible n-dimensional Riemannian polyhedron with simplexwise smooth Riemannian metric, and (Y, d_Y) a metric space. A map $\varphi \in L^2_{\text{loc}}(X, Y)$ is locally of finite energy if and only if there is a function $e(\varphi) \in L^1_{\text{loc}}(X)$ such that $e_\varepsilon(\varphi) \to e(\varphi)$ as $\varepsilon \to 0$, in the sense of weak convergence of measures:*

$$\lim_{\varepsilon \to 0} \int_X f e_\varepsilon(\varphi) \, d\mu_g = \int_X f e(\varphi) \, d\mu_g \qquad f \in C_c(X)$$

Energy of maps into Riemannian manifolds. Let the domain be an arbitrary admissible Riemannian polyhedron (X, g) (g is only measurable with local elliptic bounds, unless otherwise specified), and the target is a Riemannian C^1-manifold (N, h) without boundary, X of dimension n and Y of dimension m.

A chart η of N, $\eta : V \to \mathbb{R}^m$ is bi-Lipschitz if the components $h_{\alpha\beta}$ of $h|_V$ have elliptic bounds:

$$\Lambda_V^{-2} \sum_{\alpha=1}^{m} (\eta^\alpha)^2 \le h_{\alpha\beta} \eta^\alpha \eta^\beta \le \Lambda_V^2 \sum_{\alpha=1}^{m} (\eta^\alpha)^2. \tag{3.8}$$

Definition 3.7.3. *Relative to a given countable atlas on a Riemannian C^1-manifold (N, h), a map $\varphi : (X, g) \to (N, h)$ is of class $W^{1,2}_{\text{loc}}(X, N)$, or locally of finite energy, if*

i) φ is a quasicontinuous (after correction on a set of measure zero),

ii) its components $\varphi_1, \dots, \varphi_m$ in charts $\eta : V \to \mathbb{R}^m$ are of class $W^{1,2}(U)$ for every quasiopen set

$U \subset \varphi^{-1}(V)$ of compact closure in X, and

iii) the energy density $e(\varphi)$ of φ, defined a.e. in each of the quasiopen sets $\phi^{-1}(V)$ covering X by

$$e(\varphi) = (h_{\alpha\beta} \circ \varphi) g(\nabla\varphi^{\alpha}, \nabla\varphi^{\beta}),$$

is locally integrable over (X, μ_g).

The energy of $\varphi \in W^{1,2}_{\text{loc}}(X, N)$ is defined by $E(\varphi) = \int_X e(\varphi)\, d\mu_g$.
There is also corresponding definition for the energy of maps into Riemannian polyhedra. There, (X, g) is admissible, $\dim X = n$, and g is measurable with elliptic bounds on each n-simplex of X. The polyhedron Y is not required to be admissible, but its Riemannian metric h is assumed to be continuous.

Energy minimizing maps. We suppose that (X, g), n-dimensional admissible Riemannian polyhedra with g simplexwise smooth and Y any metric space.

Definition 3.7.4. *A map $\varphi \in W^{1,2}_{\text{loc}}(X, Y)$ is said to be locally energy minimizing if X can be covered by relatively compact domains $U \subset X$ for which $E(\phi|_U) \leq E(\psi|_U)$ for every map $\psi \in W^{1,2}_{\text{loc}}(X, Y)$ such that $\varphi = \psi$ a.e. in $X \backslash U$.*

Some of the results concerning energy minimizing maps on Riemannian manifolds, extend to the case of Riemannian polyhedra with some restrictions on the geometry of the target.

Theorem 3.7.5. *[EF01] If Y is a simply connected complete Riemannian polyhedron of non-positive curvature, every locally energy minimizing map $\varphi : X \to Y$ is Hölder continuous.*

There is a harmonic map in each homotopy class of every continuous map between Riemannian manifolds. This result can be generalized for the map between Riemannian polyhedra as following.

Theorem 3.7.6. *[EF01] Let X and Y be compact Riemannian polyhedra. Assume that*

(i) X is admissible, and

(ii) Y has non-positive curvature.

Then every homotopy class $\mathscr{H}$ of continuous maps $X \to Y$ has an Energy minimizer relative to $\mathscr{H}$, and any such is Holder continuous.

Harmonic maps. Consider an admissible Riemannian polyhedron (X, g), of dimension n, and a metric space (Y, d_Y),

Definition 3.7.7. *A harmonic map $\varphi : X \to Y$ is a continuous map of class $\varphi \in W^{1,2}_{\mathrm{loc}}(X,Y)$, which is locally energy minimizing in the sense that X can be covered by relatively compact subdomains U, for each of which there is an open set $V \supset \varphi(U)$ in Y such that*

$$E(\varphi|_U) \leq E(\psi|_U)$$

for every continuous map $\psi \in W^{1,2}_{\mathrm{loc}}(X,Y)$ with $\psi(U) \subset V$ and $\varphi = \psi$ in $X \backslash U$.

Every continuous, locally energy minimizing map $\varphi : X \to Y$ is harmonic. Also if Y is a simply connected complete Riemannian polyhedron of non-positive curvature, then a harmonic map $\varphi : X \to Y$ is the same as a continuous locally energy minimizing map. For the definition of the energy of a map, we consider the case when (X,g) is an arbitrary admissible Riemannian polyhedron and g just bounded measurable with local elliptic bounds, X of dimension n, and (N,h) a smooth Riemannian manifold without boundary, and the dimension of N is m. We denote by $\Gamma^k_{\alpha\beta}$ the Christoffel symbols on N.

Definition 3.7.8. *A weakly harmonic map $\varphi : X \to N$ is a quasicontinuous map of class $W^{1,2}_{\mathrm{loc}}(X,N)$ with the following property: for any chart $\eta : V \to \mathbb{R}^n$ on N and any quasiopen set $U \subset \varphi^{-1}(V)$ of compact closure in X, the equation*

$$\int_U \langle \nabla\lambda, \nabla\varphi^k \rangle \, d\mu_g = \int_U \lambda \cdot (\Gamma^k_{\alpha\beta} \circ \varphi) \langle \nabla\varphi^\alpha, \nabla\varphi^\beta \rangle \, d\mu_g \tag{3.9}$$

holds for every $k = 1, \ldots, m$ and every bounded function $\lambda \in W^{1,2}_0(U)$.

According to [EF01], a continuous map $\varphi \in W^{1,2}_{\mathrm{loc}}(X,N)$ is harmonic (Definition 3.7.7) if and only if it is weakly harmonic (Definition 3.7.8).

4 Ricci Curvature on Riemannian Polyhedra

In the past few years, several notions of boundedness of Ricci curvature from below on general metric spaces have appeared. Sturm [Stu06] and Lott-Villani [LV09] independently introduced the so called curvature-dimension condition on a metric measure space denoted by $CD(K,N)$. The curvature dimension condition implies the generalized Brunn-Minkowski inequality (hence the Bishop-Gromov comparison and Bonnet-Myer's theorem) and a Poincaré inequality (see [Stu06, LV07, LV09]). Meanwhile, Sturm and Ohta introduced a measure contraction property denoted as $MCP(K,N)$ in Ohta's work. The condition $MCP(K,N)$ also implies the Bishop-Gromov comparison, Bonnet-Myer's theorem and a Poincaré inequality (see [Stu06, Oht07]). Note that all of these generalized notions of Ricci curvature bounded below are equivalent to the classical one on smooth Riemannian manifolds. Here we define both conditions and show that on a Riemannian polyhedron we can use both of them. In the following definitions, we always assume that (X,d) is a seprable length space , $P(X)$ is the set of all Borel probability measures μ satisfying $\int_X d_X(x,y)^2\, d\mu(y) < \infty$ for some $x \in X$. $P_2(X)$ is the set $P(X)$ equipped with the L^2-Wasserstein distance W_2 defined as

$$W_2(\mu_0,\mu_1)^2 = \inf_{\pi} \int_{X\times X} d(x_0,x_1)^2 d\pi(x_0,x_1),$$

For μ_0, μ_1 in $P_2(X)$ and π in $P(X\times X)$ ranges between all transference plan between μ_0 and μ_1 which defined as

$$p_{0\,*}(\pi) = \mu_0, \qquad p_{1\,*}(\pi) = \mu_1$$

$p_0, p_1 : X\times X \to X$ are projection to the first and second factores, respectively.

Curvature Dimension Condition: We now define the notion of spaces satisfying $CD(K,N)$ condition following [LV09]. Suppose (X,d) is a compact length space. Let $U:[0,\infty) \to \mathbb{R}$ be a continuous convex function with $U(0) = 0$. We define the non-negative function

$$p(r) = rU'_{+}(r) - U(r)$$

with $p(0) = 0$. Given a reference probability measure $\nu \in P_2(X)$, define the function $U_\nu : P_2(X) \to \mathbb{R} \cup \{\infty\}$ by

$$U_\nu(\mu) = \int_X U(\rho(x))d\nu(x) + U'(\infty)\mu_s(X),$$

where

$$\mu = \rho\nu + \mu_s$$

is the Lebesgue decomposition of μ with respect to ν into an absolutely continuous part $\rho\nu$ and a singular part μ_s, and

$$U'(\infty) = \lim_{r\to\infty} \frac{U(r)}{r}.$$

If $N \in [1,\infty)$ then we define $\mathscr{DC}_N$ to be the set of such functions U so that

$$\psi(\lambda) = \lambda^N U(\lambda^{-N})$$

is convex on $(0,\infty)$. We further define $\mathscr{DC}_\infty$ to be the set of such functions U so that the function

$$\psi(\lambda) = e^{\lambda} U(e^{-\lambda})$$

is convex on $(-\infty,\infty)$. A relevant example of an element in $\mathscr{DC}_N$ is given by

$$H_{N,\nu} = \begin{cases} Nr(1-r^{-1/N}) & \text{if } 1 < N < \infty, \\ r\log r & \text{if } N = \infty. \end{cases} \tag{4.1}$$

Definition 4.0.9. *i) Given $N \in [1,\infty]$, we say that a compact measured length space (X,d,ν) has non-negative N-Ricci curvature if for all $\mu_0,\mu_1 \in P_2(X)$ with $\operatorname{supp}(\mu_0) \subset \operatorname{supp}(\nu)$ and $\operatorname{supp}(\mu_1) \subset \operatorname{supp}(\nu)$, there is some Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ from μ_0 to μ_1 so that for all $U \in \mathscr{DC}_N$ and all $t \in [0,1]$,*

$$U_\nu(\mu_t) \le tU_\nu(\mu_1) + (1-t)U_\nu(\mu_0). \tag{4.2}$$

ii) Given $K \in \mathbb{R}$, we say that (X, d, ν) has ∞-Ricci curvature bounded below by K if for all $\mu_0, \mu_1 \in P_2(X)$ with $\operatorname{supp}(\mu_0) \subset \operatorname{supp}(\nu)$ and $\operatorname{supp}(\mu_1) \subset \operatorname{supp}(\nu)$, there is some Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ from μ_0 to μ_1 so that for all $U \in \mathscr{DC}_\infty$ and all $t \in [0,1]$,

$$U_\nu(\mu_t) \le tU_\nu(\mu_1) + (1-t)U_\nu(\mu_0) - \frac{1}{2}\lambda(U)t(1-t)W_2(\mu_0,\mu_1)^2, \tag{4.3}$$

where $\lambda : \mathscr{DC}_\infty \to \mathbb{R} \cup \{-\infty\}$ is defined as,

$$\lambda(U) = \inf_{r>0} K\frac{p(r)}{r} = \begin{cases} K\lim_{r\to 0^+} \frac{p(r)}{r} & \text{if } K > 0, \\ 0 & \text{if } K = 0, \\ K\lim_{r\to\infty} \frac{p(r)}{r} & \text{if } K < 0. \end{cases}$$

Note that inequalities (4.2) and (4.3) are only assumed to hold along some Wasserstein geodesic from μ_0 to μ_1, and not necessarily along all such geodesics. This is what is called weak displacement convexity.

Proposition 4.0.10. *If a compact measured length space (X, d, ν) has non-negative N-Ricci curvature for some $N \in [1,\infty)$, then for all $x \in \operatorname{supp}(\nu)$ and all $0 < r_1 \le r_2$ the following inequality holds.*

$$\nu(B_{r_2}(x)) \le \left(\frac{r_2}{r_1}\right)^N \nu(B_{r_1}(x)).$$

To generalize the notion of N-Ricci curvature to the non-compact case, we always consider a complete pointed locally compact metric measure space $(X, \star, \nu)$. Also for U_ν to be a well-defined functional on $P_2(X)$, we impose the restriction $\nu \in M_{-2(N-1)}$, where $M_{-2(N-1)}$ is the space of all non-negative Radon measures ν on X such that

$$\int_X (1 + d(\star, x)^2)^{-(N-1)}\, d\nu(x) < \infty.$$

We define $M_{-\infty}$, by the condition $\int_X e^{-cd(x,\star)^2}\, d\nu(x) < \infty$, where c is a fixed positive constant. We should mention that most of the results for compact case (for example the Bishop-Gromov comparison) are valid for the non-compact case.

Measure Contraction Property We define now the notion of measure contraction property $MCP(K,N)$ following [Oht07]. Let (X, d_X) be a length space, and μ a Borel measure on X such that $0 < \mu(B(x,r)) < \infty$ for every $x \in X$ and $r > 0$, where $B(x,r)$ denotes the open ball with center $x \in X$ and radius $r > 0$.

Let Γ be the set of minimal geodesics, $\gamma : [0,1] \to X$, and define the evaluation map e_t by $e_t(\gamma) := \gamma(t)$ for each $t \in [0,1]$. We regard Γ as a subset of the set of Lipschitz maps $\mathrm{Lip}([0,1], X)$ with the uniform topology. A dynamical transference plan Π is a Borel probability measure on Γ, and a path $\{\mu_t\}_{t\in[0,1]} \subset P_2(X)$ given by $\mu_t = (e_t)_*\Pi$ is called a displacement interpolation associated to Π. For the exact definition of dynamical transference plan and displacement interpolation we refer the reader to [LV09]. For $K \in \mathbb{R}$, we define the function s_K on $[0,\infty)$ (on $[0, \pi/\sqrt{K})$ if $K > 0$) by

$$s_K(t) := \begin{cases} (1/\sqrt{K})\sin(\sqrt{K}t) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ (1/\sqrt{-K})\sinh(\sqrt{-K}t) & \text{if } K < 0. \end{cases} \tag{4.4}$$

Definition 4.0.11. *For $K, N \in \mathbb{R}$ with $N > 1$, or with $K \leq 0$ and $N = 1$, a metric measure space (X, d, μ) is said to satisfy the (K,N)-measure contraction property, the $MCP(K,N)$, if for every point $x \in X$ and measurable set $A \subset X$ (provided that $A \subset B(x, \pi\sqrt{(N-1)/K})$ if $K > 0$) with $0 < \mu(A) < \infty$, there exists a displacement interpolation $\{\mu_t\}_{t\in[0,1]} \subset P_2(X)$ associated to a dynamical transference plan $\Pi = \Pi_{x,A}$ satisfying:*
(1) We have $\mu_0 = \delta_x$ and $\mu_1 = (\mu|_A)^-$ as measures, where we denote by $(\mu|_A)^-$ the normalization of $\mu|_A$, i.e., $(\mu|_A)^- := \mu(A)^{-1} \cdot \mu|_A$;
(2) For every $t \in [0,1]$,

$$d\mu \geq (e_t)_* \left(t \left\{ \frac{s_K(t\, l(\gamma)/\sqrt{N-1})}{s_K(l(\gamma)/\sqrt{N-1})} \right\}^{N-1} \mu(A) d\Pi(\gamma) \right)$$

holds as measures on X, where we set $0/0 = 1$ and, by convention, we read

$$\left\{ \frac{s_K(t\, l(\gamma)/\sqrt{N-1})}{s_K(l(\gamma)/\sqrt{N-1})} \right\}^{N-1} = 1$$

if $K \leq 0$ and $N = 1$.

Here we state two results that we are going to use in the sequel.

Proposition 4.0.12. *Let (M, g) be an n-dimensional, complete Riemannian manifold without boundary with $n \geq 2$. Then a metric measure space (M, d_g, ν_g) satisfies the $MCP(K, n)$ if and only if $\mathrm{Ric}_g \geq K$ holds. Here d_g and ν_g denote the Riemannian distance and Riemannian volume element.*

In the following theorem we state Bishop volume comparison theorem for the space satisfying $MCP(K,N)$.

Proposition 4.0.13. *Let (X, μ) be a metric space satisfying the $MCP(K,N)$. Then, for any $x \in X$,* *the function*

$$\mu(B(x,r))\cdot\left\{\int_0^r s_K\left(\frac{s}{\sqrt{N-1}}\right)^{N-1} d_s\right\}^{-1}$$

is monotone non-increasing in $r\in(0,\infty)$ *(*$r\in(0,\pi\sqrt{\frac{N-1}{K}})$ *if* $K>0$*).*

In the following we show that we can apply both measure contraction property and curvature dimension condition to a complete Riemannian polyhedra (X,g,μ_g). By previous section, a Riemannian polyhedron (X,g,μ_g) with the metric $d_X=d_X^g$ is a length space. Also for any $x,y\in X$ we have

$$e(x,y)\le d_X^e(x,y).$$

It is easy then to show that μ_g is in $M_{-2(N-1)}$ and so on a complete Riemannian polyhedron we can use the notion of $CD(K,N)$. Also μ_g is Borel and by Lemma 4.4 in [EF01], for any r there exist a constant $c(r)$ such that

$$c(r)^{-1}\Lambda^{-2n}r^n\le\mu_g(B(x,r))\le c(r)\Lambda^{2n}r^n$$

for all $x\in X$. Therefore $0<\mu_g(B(x,r))<\infty$ and the notion of $MCP(K,N)$ is also applicable here, for $N\ge n$. (By Theorem 2.4.3 in [AT04], we have the Hausdorff dimension is n and by Corollary 2.7 in [Oht07] N should be greater than n.)

In the rest of this work by $\mathrm{Ric}_{N,\mu_g}\ge K$ we mean that (X,g,μ_g) satisfies the $MCP(K,N)$. In the following Lemma we show that any complete Riemannian polyhedron with non-negative Ricci curvature has infinite volume.

Lemma 4.0.14. *Let* (X,g,μ_g) *be a complete, non-compact Riemannian polyhedron. If* $\mathrm{Ric}_{N,\mu_g}(X)\ge 0$*, for* $N\ge n$*, then* X *has infinite volume.*

Proof. By the Bishop comparison theorem, Theorem 4.0.13, for $x\in X$ and all $0<r_1\le r_2$,

$$\mu_g(B_{r_2}(x))\le\left(\frac{r_2}{r_1}\right)^N\mu_g(B_{r_1}(x))$$

By Proposition 10.1.1 in [Pap05], for every point in X, there exist a geodesic ray from that point. Consider a geodesic ray $\gamma(t)$, $0\le t<\infty$, such that $\gamma(0)=x$. We construct the balls $B(\gamma(t),t-1)$

and $B(\gamma(t), t+1)$ centered at $\gamma(t)$ with radius $t-1$ and $t+1$. We have

$$\frac{\mu_g(B(\gamma(0),1)) + \mu_g(B(\gamma(t),t-1))}{\mu_g(B(\gamma(t),t-1))} \leq \frac{\mu_g(B(\gamma(t),t+1))}{\mu_g(B(\gamma(t),t-1))} \leq \left(\frac{t+1}{t-1}\right)^N,$$

and so

$$1 + \frac{\mu_g(B(\gamma(0),1))}{\mu_g(B(\gamma(t),t-1))} \leq \left(\frac{t+1}{t-1}\right)^N.$$

Letting $t \to \infty$, we get $\mu_g(B(\gamma(t), t-1)) \to \infty$ and therefore X has infinite volume. □

By Theorem 4.0.10, and since X is a complete locally compact length space, the above theorem is still valid for the case when X satisfies the non-negative N-Ricci curvature condition $CD(0,N)$, for $N \in (1,\infty)$.

Remark 10. *By Remark* 5.8 *in [Stu06] if* (X,d,μ) *satisfy* $MCP(K,N)$ *so does any convex set* $A \subset X$. *When* X *is a smooth pseudomanifold, for any point* $x \in X \backslash S$, *there exist a closed totally convex neighborhood* V *around* x *(for every point in a Riemannian manifold there is a geodesic ball which is totally convex). Therefore if* X *satisfies* $\mathrm{Ric}_{N,\mu_g} \geq K$, *so does* $X\backslash S$.

5 Some Function Theoretic Properties On Complete Riemannain Polyhedra

5.1 Liouville-Type Theorems for Functions

The aim of this chapter is to generalize some of the results in [Yau76] in order to prove some vanishing theorems for harmonic maps on Riemannian polyhedra. In [Yau76], Yau used the Gaffney's Stokes theorem on complete Riemannian manifolds to prove that every smooth subharmonic function with bounded $\|\nabla f\|_{L_1}$ is harmonic. He uses this fact to prove that there is no non-constant L^p, $p > 1$, non-negative subharmonic function on a complete manifold. We will prove this theorem on admissible polyhedra for $p = 2$.

Theorem. *Suppose (X, g) is a complete, admissible Riemannian polyhedron, and $f \in W^{1,2}_{\text{loc}}(X) \cap L^2(X)$ is a non-negative, weakly subharmonic function. Then f is constant.*

Proof. Fix a base point $x_0 \in X$ and define $\rho : X \to \mathbb{R}$ as

$$\rho(x) = \max\{0, \min\{1, 2 - \frac{1}{R} d(x, x_0)\}\}$$

Observe that ρ is $\frac{1}{R}$-Lipschitz and $\rho = 0$ on $X \backslash B(x_0, 2R)$ and $\rho = 1$ on $B(x_0, R)$.

Since f is subharmonic,

$$\begin{aligned}
0 &\geq \int_X \langle \nabla(\rho^2 f), \nabla f \rangle \, d\mu_g \\
&= \int_X \langle (\nabla \rho^2) f + (\nabla f)\rho^2, \nabla f \rangle \, d\mu_g \\
&= \frac{1}{2} \int_X \langle \nabla \rho^2, \nabla f^2 \rangle \, d\mu_g + \int_X \rho^2 |\nabla f|^2 \, d\mu_g \\
&= 2 \int_X \langle \rho \nabla \rho, f \nabla f \rangle \, d\mu_g + \int_X \rho^2 |\nabla f|^2 \, d\mu_g,
\end{aligned}$$

From Cauchy-Schwarz we have now

$$\begin{aligned}\int_X \langle \rho\nabla\rho, f\nabla f\rangle \, d\mu_g &= \int_X \langle f\nabla\rho, \rho\nabla f\rangle \, d\mu_g \\ &\geq -\Big(\int_X |f\nabla\rho|^2 \, d\mu_g\Big)^{\frac{1}{2}}\Big(\int_X |\rho\nabla f|^2 \, d\mu_g\Big)^{\frac{1}{2}}.\end{aligned}$$

Combining the two previous inequality, we obtain

$$\begin{aligned} 0 &\geq 2\int_X \langle \rho\nabla\rho, f\nabla f\rangle d\mu_g + \int_X \rho^2|\nabla f|^2 d\mu_g \\ &\geq \int_{B_{2R}\setminus B_R} |\rho\nabla f|^2 \, d\mu_g - 2\Big(\int_{B_{2R}\setminus B_R} |f\nabla\rho|^2 \, d\mu_g\Big)^{\frac{1}{2}}\Big(\int_{B_{2R}\setminus B_R} |\rho\nabla f|^2 \, d\mu_g\Big)^{\frac{1}{2}} + \int_{B_R} |\nabla f|^2 d\mu_g.\end{aligned}$$

The last line is a polynomial, $P(\psi) = \psi^2 - 2b\psi + c$, where ψ is

$$\Big(\int_{B_{2R}\setminus B_R} |\rho\nabla f|^2 \, d\mu_g\Big)^{\frac{1}{2}}$$

and it has non-positive value only if $b^2 \geq c$, which means that

$$\int_{B_R} |\nabla f|^2 d\mu_g \leq \int_{B_{2R}\setminus B_R} f^2|\nabla\rho|^2 \leq \frac{c^2}{R^2}\int_{B_{2R}} f^2 d\mu_g,$$

and so

$$\int_{B_R} |\nabla f|^2 d\mu_g \leq \frac{c^2}{R^2}\int_X f^2 d\mu_g. \tag{5.1}$$

Sending R to infinity and using the fact that f has finite L^2-norm, we conclude that

$$\int_X |\nabla f|^2 d\mu_g = 0.$$

Since X is admissible, f is constant on X. (First we prove that f is constant on each maximal n-simplex S and then using the $n-1$-chainability of X, we prove this in the star of any vertex p of X and then by connectedness on X.) □

In the following theorem, we show that the Laplacian of a weakly subharmonic function $f \in W^{1,2}_{\text{loc}}(X)$ on a pseudomanifold in the distributional sense is a locally finite Borel measure. This gives us a verifying of Green's formula on these spaces. We then use this theorem, to prove that a continuous weakly subharmonic function with $\|\nabla f\|_{L^1} < \infty$ on a complete normal circuit is harmonic.

Theorem 5.1.1. *Let (X, g, μ_g) be an n-pseudomanifold. Let f be a weakly subharmonic function in $W^{1,2}_{\text{loc}}(X)$, such that $\|\nabla f\|_{L^1}$ is finite. Then there exist a unique locally finite Borel measure $\boldsymbol{m}_f$ on X such that*

$$\int_X h\, \boldsymbol{m}_f = -\int_X \langle \nabla f, \nabla h\rangle\, d\mu_g \qquad \text{for all } h \in \mathrm{Lip}_c(X).$$

Proof. We consider the Lipschitz manifold $M = X\backslash S$ and the chart $\{(U_\alpha, \psi_\alpha)\}$ on M. We show that

$$\Lambda_\alpha(h) = -\int_{U_\alpha} \langle \nabla f, \nabla h \circ \psi_\alpha\rangle\, d\mu_g,$$

is a linear continuous functional on $D_\alpha = \mathrm{Lip}_c(\psi_\alpha(U_\alpha))$ with respect to the topology of uniform convergence on compact sets. The linearity is obvious. We have

$$\Lambda_\alpha(h) = -\int_{U_\alpha} \langle \nabla f, \nabla h \circ \psi_\alpha\rangle\, d\mu_g \le \sup_{x\in U_\alpha} |\nabla h(x)| \cdot \|\nabla f\|_{L^1(U_\alpha)},$$

and so Λ_α is continuous. Since $\mathrm{Lip}_c(U)$ is dense in $C_c(U)$ for a locally compact domain U, see Proposition 1.11 in [BB11], then Λ_α is also continuous on $C_c(\psi_\alpha(U_\alpha))$. By assumption f is subharmonic and so Λ_α is positive. By Riesz representation theorem, Λ_α is a unique positive Radon measure. It follows that there is a positive Radon measure m_α such that

$$\Lambda_\alpha(h) = \int_{U_\alpha} h\, dm_\alpha.$$

Now we consider the partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$. We put $m = \sum_\alpha \rho_\alpha \psi_*(m_\alpha)$ and we define $\mathbf{m}_f(U) = m(U\backslash S)$ on each Borel set U. Obviously $\mathbf{m}_f$ is positive and locally finite. The uniqueness comes from the uniqueness of m_α. □

We recall a remark concerning the above theorem.

Remark 11. *Gigli introduced the notion of Laplacian as a set of locally finite Borel measure (see Definition* 4.4 *in [Gig12]). There he proved that on infinitesimally Hilbertian spaces this set contains only one element* [1]. *Admissible Riemannian polyhedra are the examples of infintesimally Hilbertian space.*

In the smooth setting, as a corollary of Gaffney's Stokes theorem, we have that on a complete Riemannian manifold every smooth subharmonic function f with bounded $\|\nabla f\|_{L^1}$ is harmonic. We can generalize this theorem on pseudomanifolds.

[1] see Definition 4.18 in [Gig12] for the definition of infinitesimally Hilbertian

Theorem. *Let (X, g, μ_g) be a complete non-compact n-pseudomanifold. Let f be a continuous weakly subharmonic belonging to $W^{1,2}_{\text{loc}}(X)$ such that $A_1 = \|\nabla f\|_{L^1}$ is finite. Then f is a harmonic function.*

Proof. We consider a sequence of cut-off functions ρ_n for fixed $q \in X$ such that ρ_n is $\frac{1}{n}$-Lipschitz and such that ρ_n is equal to 1 on $B(q, R)$ and its support is in $B(q, R+n)$. f is a subharmonic function which satisfies the condition of previous lemma, so there is a unique Borel measure $\mathbf{m}_f$ such that

$$0 \leq \int_X \rho_n \, d\mathbf{m}_f = -\int_X \langle \nabla \rho_n, \nabla f \rangle \, d\mu_g \leq \int_X |\nabla \rho_n| |\nabla f| \, d\mu_g \leq \frac{1}{n} A_1,$$

and

$$0 \leq \int_{B(q,R)} d\mathbf{m}_f \leq \int_X \rho_n \, d\mathbf{m}_f \leq \frac{1}{n} A_1.$$

Let h be any function in $\text{Lip}_c(X)$ with support in $B(q, R)$. We have

$$0 \leq \int_X h \, d\mathbf{m}_f \leq (\sup_X h) \frac{1}{n} A_1,$$

and tending n to infinity, we have

$$\int_X h \, d\mathbf{m}_f = -\int_X \langle \nabla h, \nabla f \rangle \, d\mu_g = 0,$$

and implying that f is harmonic. □

Now we prove a generalization of Proposition 2 in [Yau76]. We give here another proof of the theorem above for smooth pseudomanifolds under the extra assumption that f should have finite energy. Instead of Theorem 5.1.1, we goal Cheeger's Green formula on compact smooth pseudomanifolds in the proof.

Theorem. *Let (X, g, μ_g) be a complete non-compact smooth pseudomanifold. Suppose X has non-negative n-Ricci curvature. Let f be a continuous weakly subharmonic function belonging to $W^{1,2}_{\text{loc}}(X)$, such that both $A_1 = \|\nabla f\|_{L^1}$ and $A_2 = \|\nabla f\|_{L^2}$ are finite and $|\nabla f|$ is locally bounded. Then f is constant.*

Proof. We present the proof in several steps.

Step 1. We consider a sequence of cut-off functions ρ_n as above such that the support of ρ_n is in $B(q, R+n)$ for fixed $q \in X \backslash X^{n-2}$ and some R and ρ_n is equal to 1 on $B(q, R)$ and ρ_n is $\frac{1}{n}$-Lipschitz.

Step 2. The $(n-2)$-skeleton in X, X^{n-2}, is a polar set. We consider, shrinking bounded neighborhoods U_j of X^{n-2} in $B(q, R+j)$, such that in each $B(q, R+j)$, we have

$$U_j \supset U_{j+1} \supset \ldots \supset \bigcap_{j=1}^{\infty} U_j.$$

By the definition of polar set, for the open domains U_j and U_{j-1}, we have $\operatorname{cap}(X^{n-2} \cap U_j, U_{j-1}) = 0$. This means that for every j, there exists a function $\varphi_j \in \operatorname{Lip}(X)$ such that $\varphi_j \equiv 1$ in a neighborhood of $X^{n-2} \cap U_j$ and φ_j is zero outside U_{j-1} and $\int_X |\nabla \varphi_j|^2 < \frac{1}{j}$. Moreover we have $0 \le \varphi_j \le 1$.

We put $\eta_j = 1 - \varphi_j$. The function η_j has the property that the closure of its support, $\overline{\operatorname{supp} \eta_j}$, is contained in $X \setminus X^{n-2}$ and the set $K_j = \overline{\operatorname{supp} \eta_j} \cap \overline{B(q, R+j)}$ is compact. Furthermore K_js make an exhaustion of $M = X \setminus X^{n-2}$.

Step 3. According to Theorem 2 in [GW79], for any j, there exist a smooth subharmonic function f_j on M such that $\sup_{x \in K_j} |f_j(x) - f(x)| < \frac{1}{j}$ and $|\nabla f_j(x)| \le |\nabla f(x)|$ on K_j.

Step 4. In this step we prove

$$\int_M \Delta f_j \cdot \xi_j \, d\mu_g \quad = \quad -\int_M \langle \nabla f_j, \nabla \xi_j \rangle \, d\mu_g.$$

where $\xi_j = \rho_j \cdot \eta_j$. To prove the above equality, first we recall a Remark from [Che80].

Remark 12. *Let (Y, h) be a closed n-dimensional admissible Riemannian polyhedron, then for $\zeta, \psi \in \operatorname{Dom}(\Delta)$ we have the following Stokes theorem on $Y \setminus Y^{n-2}$ (see Theorem 5.1 in [Che80]),*

$$\int_{Y \setminus Y^{n-2}} \Delta \zeta \cdot \psi \, d\mu_h = -\int_{Y \setminus Y^{n-2}} \langle \nabla \zeta, \nabla \psi \rangle \, d\mu_h. \tag{5.2}$$

Also, every closed smooth pseudomanifold (Y, h) such that h is equivalent to some piecewise flat metric is admissible (in the sense of Cheeger).

Now we construct the closed Riemannian polyhedron $\overline{Y}_j \subset X$ as following: Let Y_j be an arbitrary Riemannian polyhedron containing $B(q, R+j)$. We consider its double $\tilde{Y}_j$ and equip it with a Riemannian metric $\tilde{g}_j$, which is the same as Riemannian metric on Y_j. The Riemannian polyhedron $\overline{Y}_j = Y_j \cup \tilde{Y}_j$ with the metric $\overline{g}_j$ is an admissible closed Riemannian pseudomanifold. (The metric g_j on Y_j is equivalent to piecewise flat metric g^e (see [EF01], Chapter 4) and so $\bar{Y}_j$ is admissible.)

We extend ρ_j to $\overline{Y}_j$ such that it is zero on the copy of Y_j and f_j, η_j such that they are the same functions on the copy of Y_j. The function f_j is in $W^{1,2}_{\mathrm{loc}}(\overline{Y}_j)$ (see Theorem 1.12.3. in [KS93]).

By applying formula (5.2) on $\overline{Y}_j$, for the functions f_j and ξ_j, we obtain

$$\int_{M_j} \Delta f_j \cdot \xi_j \, d\mu_{g_j} = -\int_{M_j} \langle \nabla f_j, \nabla \xi_j \rangle \, d\mu_{g_j},$$

where $M_j = \overline{Y}_j \backslash \overline{Y}_j^{n-2}$. Since $\xi_j \in \mathrm{Lip}_c(M) \cap Y_j$, we can write the above Stokes formula as follows

$$\int_M \Delta f_j \cdot \xi_j \, d\mu_g = -\int_M \langle \nabla f_j, \nabla \xi_j \rangle \, d\mu_g.$$

Step 5. In this step, we prove that f is harmonic on M. From the fact that $\mathrm{supp}(\xi_j) \subset K_j$ we have

$$\begin{aligned}
\int_M \Delta f_j \cdot \xi_j \, d\mu_g &= -\int_M \langle \nabla f_j, \nabla(\rho_j \cdot \eta_j) \rangle \, d\mu_g \\
&= -\int_M |\langle \nabla f_j, \eta_j \cdot (\nabla \rho_j) \rangle \, d\mu_g - \int_M \langle \nabla f_j, \rho_j \cdot (\nabla \eta_j) \rangle \, d\mu_g \\
&\leq \int_{K_j} |\nabla f_j||\nabla \rho_j| \, d\mu_g + \int_{K_j} |\nabla f_j|^2 \, d\mu_g \cdot \int_{K_j} |\nabla \eta_j|^2 \, d\mu_g \\
&\leq \frac{1}{j} \int_M |\nabla f| \, d\mu_g + \frac{1}{j} \int_M |\nabla f|^2 \, d\mu_g,
\end{aligned}$$

so we have

$$0 \leq \int_M \Delta f_j \cdot \xi_j \, d\mu_g \leq \frac{1}{j}(A_2 + A_1). \tag{5.3}$$

Let h be any smooth function with compact support in $M \cap B(q, R)$. Then there is a K_m such that the support of h is in $B(q, R) \cap K_m$. For j large enough we will have $\xi_j \equiv 1$ on K_m and so we have

$$0 \leq \int_{B(q,R) \cap K_m} \Delta f_j \, d\mu_g \leq \frac{1}{j}(A_2 + A_1).$$

considering the formula (5.2) as above, for j large enough we have

$$\begin{aligned}
0 \leq \int_M \Delta h \cdot f_j \, d\mu_g &= \int_M h \cdot \Delta f_j \, d\mu_g \\
&\leq (\sup h) \cdot \frac{1}{j}(A_2 + A_1).
\end{aligned}$$

Letting j go to infinity, we got $\int_M \Delta h \cdot f \, d\mu_g = 0$. By use of Weyl's lemma f is a smooth harmonic function on M.

Step 6. Now we show f is constant. Since M has non-negative Ricci curvature, by the Bochner formula $|\nabla f|$ is subharmonic on M and since $|\nabla f|$ is locally bounded on X, by Theorem 12.2 in [BB11], $|\nabla f|$ is subharmonic on X. By Lemma 0.2.1, $|\nabla f|$ is constant. Since the L^2-norm of $|\nabla f|$ is finite we have $|\nabla f| \equiv 0$. By Lemma 4.0.14, f should be constant. □

5.2 Vanishing Results for Harmonic Maps on Complete Smooth Pseudomanifolds

In this section we prove Corollaries 0.2.4 and 0.2.5.

Corollary. *Let (X, g, μ_g) be a complete smooth n-pseudomanifold. Suppose X has non-negative n-Ricci curvature. Suppose Y is a Riemannian manifold of non-positive curvature and $u : (X, g) \to (Y, h)$ a continuous harmonic map belonging to $W^{1,2}_{\mathrm{loc}}(X, Y)$. If u has finite energy and $e(u)$ is locally bounded, then u is a constant map.*

Proof. By Remark 10, we know that on the Riemannian manifold $M = X\backslash S$ we have non-negative Ricci curvature. We show that for $\epsilon > 0$, $\sqrt{e(u)+\epsilon}$ is weakly subharmonic on X. As the restriction maps $u = u|_M : (M, g) \to Y$ is harmonic, we have a Bochner type formula for harmonic map on M and

$$\Delta e(u) > |B(u)|^2,$$

where $B(u)$ is the second fundamental form of the map u. Also by Cauchy-Schwarz we have,

$$|\nabla e(u)|^2 \leq 2e(u)|B(u)|^2,$$

and so for $\epsilon > 0$, on $X\backslash S$

$$\Delta\sqrt{e(u)+\epsilon} \geq 0.$$

See e.g. the calculation in [Xin96] Theorem 1.3.8. Thus $\sqrt{e(u)+\epsilon}$ is subharmonic on $X\backslash S$ and locally bounded, subharmonicity on X follows. Therefore,

$$\int_X \langle \nabla\sqrt{e(u)+\epsilon}, \nabla\rho\rangle \, d\mu_g \leq 0 \qquad \rho \in \mathrm{Lip}_c(X).$$

As in the proof of Theorem 0.2.1,

$$\int_{B_R} |\nabla\sqrt{e(u)+\epsilon}|^2 \, d\mu_g \leq \frac{1}{R^2}\int_{B_{2R}} e(u)+\epsilon \, d\mu_g. \tag{5.4}$$

Note that $\sqrt{e(u)+\epsilon}$ satisfies all the assumptions of the Theorem 0.2.1, except the finiteness of L^2-norm which we do not need in this step.

Set $B'_R = B_R \backslash \{x \in B_R, e(u)(x) = 0\}$. Then

$$\int_{B'_R} \frac{|\nabla(e(u)+\epsilon)|^2}{4(e(u)+\epsilon)} d\mu_g \leq \frac{1}{R^2} \int_{B_{2R}} e(u) + \epsilon d\mu_g. \tag{5.5}$$

Letting $\epsilon \to 0$ gives

$$\int_{B'_R} \frac{|\nabla e(u)|^2}{4e(u)} \, d\mu_g \leq \frac{1}{R^2} \int_{B_{2R}} e(u) \, d\mu_g, \tag{5.6}$$

and letting $R \to \infty$ and by finiteness of the energy we have

$$\int_{B'_R} \frac{|\nabla e(u)|^2}{4e(u)} d\mu_g \leq 0, \tag{5.7}$$

which implies that $e(u)$ is constant. If $e(u)$ is not zero everywhere this means that the volume of X is finite. By Lemma 4.0.14, this is impossible and so u is constant. □

Now we extend this theorem to the case when the target Y is a metric space. By the following lemma, the function $d(u(\cdot), q)$, where q is an arbitrary point in Y, is subharmonic under suitable assumption on the curvature of Y. We refer the reader to [EF01] Lemma 10.2, for the proof.

Lemma 5.2.1. *Let (X, g) be an admissible Riemannian polyhedron, g simplexwise smooth. Let (Y, d_Y) be a simply connected complete geodesic space of non-positive curvature, and let $u \in W^{1,2}_{\text{loc}}(X, Y)$ be a locally energy minimizing map. Then u is a locally essentially bounded map and for any $q \in Y$, the function $d(u(\cdot), q)$ of class $W^{1,2}_{\text{loc}}(X, Y)$ is weakly subharmonic and in particular essentially locally bounded.*

Now we have

Corollary. *Let (X, g, μ_g) be a complete non-compact smooth n-pseudomanifold. Suppose X has non-negative n-Ricci curvature. Let Y be a simply connected complete geodesic space of non-positive curvature and $u : (X, g) \to Y$ a continuous harmonic map with finite energy belonging to $W^{1,2}_{\text{loc}}(X, Y)$. If $\int_M \sqrt{e(u)} d\mu_g < \infty$ and $e(u)$ is locally bounded, then u is a constant map.*

Proof. According the lemma above the function $v(x) = d(u(x), u(x_0))$ for some $x_0 \in X$, is weakly subharmonic. We know that $|\nabla v|^2 \leq ce(u)$, where c is a constant. v is a continuous subharmonic function whose gradient is bounded by an L^1 and L^2 integrable function and also it is locally bounded. According to Lemma 0.2.3, v is a constant function and so u is a constant map. □

Remark 13. *In Theorem 0.2.3, we hope that we can remove the assumption locally boundedness on ∇f. On complete Riemannian manifold and on Alexandrov spaces, every harmonic function is locally Lipschitz (see [GKO13]). Also using Cheng-Yau's gradient estimate (see [CY75, Yau75]) one can prove that there is no positive harmonic function on complete Riemannian manifold with non-negative Ricci curvature. We have the same result on Alexandrov spaces with non-negative Ricci curvature (for some specific notion of Ricci curvature bound, see [ZZ12]). Therefore in Theorem 0.2.5, we expect that we can remove also the assumption of finiteness of energy.*

6 2-Parabolic Riemannian Polyhedra

In this last chapter we prove Liouville-type theorems for harmonic maps defined on a Riemannian polyhedra X without any completeness or Ricci curvature bound assumption. We assume instead X to be 2-parabolic. Some of these results extend known results for the case of Riemannian manifolds. As for Riemannian manifolds, we say that a domain $\Omega \subset X$ in an admissible Riemannian polyhedra X is 2-parabolic, if $\operatorname{cap}(D, \Omega) = 0$ for every compact set D in Ω, otherwise 2-hyperbolic. A reference on this subject is [GT02], where the notion is discussed for general metric measure spaces. The following two Corollaries are the main results of this chapter:

Corollary. *Let (X, g) be a 2-parabolic smooth pseudomanifold. Let Y be a simply connected, complete, geodesic space of non-positive curvature and $u : (X, g) \to Y$ a continuous, harmonic map with finite energy, belonging to $W^{1,2}_{\mathrm{loc}}(X, Y)$. If we have $\int_X \sqrt{e(u)}\, d\mu_g < \infty$ then u is a constant map.*

and

Corollary. *Let (X, g) be a 2-parabolic admissible Riemannian polyhedron with simplexwise smooth metric g. Let Y be a complete, geodesic space of non-positive curvature and $u : (X, g) \to Y$ a continuous, harmonic map belonging to $W^{1,2}_{\mathrm{loc}}(X, Y)$ with bounded image. Then u is a constant map.*

We will need the following characterization of 2-parabolicity.

Lemma 6.0.2. *The domain Ω is 2-parabolic if and only if there exists a sequence of functions $\rho_j \in \mathrm{Lip}_c(\Omega)$ such that $0 \le \rho_j \le 1$, ρ_j converges to 1 uniformly on every compact subset of Ω and*

$$\int_\Omega |\nabla \rho_j|^2 \, d\mu_g \to 0.$$

Proof. First suppose Ω is 2-parabolic. Then every compact set $D \subset \Omega$, with nonempty interior satisfies $\operatorname{cap}(D, \Omega) = 0$. We choose an exhaustion $D \subset D_1 \subset D_2 \subset \ldots \subset \Omega$ of Ω by compact

subsets such that $\mathrm{cap}(D_j,\Omega) = 0$ for all j. Hence we can find the function $\rho_j \in \mathrm{Lip}_c(\Omega)$ (using the fact that $\mathrm{Lip}_c(\Omega)$ is dense in $W_0^{1,2}(\Omega)$) such that $\rho_j \equiv 1$ on D_j and $\int_\Omega |\nabla \rho_j|^2 \, d\mu_g \leq 1/j^2$. We have constructed the desired sequence ρ_j.
Conversely, suppose there exists, a sequence $\rho_j \in \mathrm{Lip}_c(\Omega)$ with the stated properties. Then we can find a compact subset $B \subset \Omega$ and j_0 such that $\rho_j \geq 1/2$ for every $j \geq j_0$. It follows that $\mathrm{cap}(B,\Omega) = 0$ □

The following lemma shows that the 2-parabolicity remains after removing the singular set of a Riemannian polyhedron.

Lemma 6.0.3. *If X is a 2-parabolic admissible Riemannian polyhedron and $E \subset X$ is a polar set, then $\Omega := X \backslash E$ is 2-parabolic.*

Proof. X is 2-parabolic, so by Lemma 6.0.2, there are an exhaustion of X and a sequence of function $\rho_j \in \mathrm{Lip}_c(X)$ such that $0 \leq \rho_j \leq 1$ and $\rho_j \to 1$ uniformly on each compact set, and $\int_X |\nabla \rho_j|^2 \, d\mu_g \to 0$. Also by Lemma 3.5.2, there exist another sequence of functions φ_j with support in $X \backslash E$ such that $\varphi_j \to 1$ on each compact set of $X \backslash E$ and $\int_X |\nabla \varphi_j|^2 \, d\mu_g \to 0$. The functions $\rho_j \varphi_j$ on Ω provides the condition for 2-parabolicity in Lemma 6.0.2. □

The following result is an extension of Theorem 5.2 in [Hol90] to admissible Riemannian polyhedra.

Proposition. *Let (X, g) be 2-parabolic admissible Riemannian polyhedron. Suppose f in $W^{1,2}_{\mathrm{loc}}(X)$ is a positive, continuous superharmonic function on X. Then f is constant.*

Proof. Since f is continuous, for any ϵ and at any point x_0 in X there exist a relatively compact neighborhood B_0 of x_0 such that $f(x) > f(x_0) - \epsilon$ on $\overline{B_0}$. X is 2-parabolic, so $\mathrm{cap}(B_0, X) = 0$. Consider an exhaustion of X by regular domains U_i such that $B_0 \Subset U_1 \Subset U_2 \Subset \ldots \Subset X$. By Corollary 11.25 in [BB11], such exhaustion exists.

There exist functions u_i which are harmonic on $U_i \backslash \overline{B_0}$, $u_i \equiv 1$ on B_0 and $u_i \equiv 0$ on $X \backslash U_i$ (See [GT01] and also Lemma 11.17 and 11.19 in [BB11]). The maximum principle (see Theorem 5.3 in [EF01] or Lemma 10.2 in [BB11] for the comparison principle) implies that

$$\begin{cases} 0 \leq u_i \leq 1 \\ u_{i+1} \geq u_i \quad \text{on } U_i. \end{cases}$$

Define the function $h_i = (f(x_0) - \epsilon) u_i$, we have $\lim_{i\to\infty} h_i = f(x_0) - \epsilon$. On the other hand $f \geq h_i$ on the boundary of $U_i \backslash \overline{B_0}$. By the comparison principle $f \geq h_i$ in $U_i \backslash \overline{B_0}$, so $f \geq f(x_0) - \epsilon$ on X. Letting $\epsilon \to 0$, we obtain $f \geq f(x_0)$ on X. If f is non-constant, there exist $x_1 \in X$ with $f(x_1) > f(x_0)$. By the same argument we obtain $f > f(x_1)$. This is a contradiction and thus f is constant. □

We prove the analogue of Theorem 0.2.3, for 2-parabolic admissible Riemannian polyhedra.

Proposition. *Let X be 2-parabolic pseudomanifold. Let f in $W^{1,2}_{\mathrm{loc}}(X)$ be a continuous weakly subharmonic function such that $\|\nabla f\|_{L^1}$ and $\|\nabla f\|_{L^2}$ are finite. Then f is harmonic.*

Proof. since X is 2-parabolic, by Lemma 6.0.2, for every compact set $D \subset X$, and an arbitrary exhaustion $D \subset D_1 \subset D_2 \subset \ldots \subset X$ of X by compact subsets, there exist a sequence of functions $\rho_j \in \mathrm{Lip}_c(X)$ such that $\rho_j \equiv 1$ on D_j and $\int_X |\nabla \rho_j|^2 \, d\mu_g \leq 1/j^2$.

$$\begin{aligned} 0 \leq -\int_X \langle \nabla \rho_j, \nabla f \rangle \, d\mu_g &\leq \left(\int_X |\nabla \rho_j|^2 \, d\mu_g \right)^{\frac{1}{2}} \left(\int_X |\nabla f|^2 \, d\mu_g \right)^{\frac{1}{2}} \\ &\leq \frac{1}{j} \|\nabla f\|^2_{L^2}. \end{aligned}$$

By Lemma 5.1.1, there is a locally finite Borel measure $\mathbf{m}_f$ such that

$$0 < \int_D \mathbf{m}_f \leq \int_X \rho_j \, \mathbf{m}_f \leq |\int_X \langle \nabla \rho_j, \nabla f \rangle \, d\mu_g| \leq \frac{1}{j} \|\nabla f\|^2_{L^2}.$$

Now let h be an arbitrary test function in $\mathrm{Lip}_c(X)$ where its support is in D. We have

$$0 \leq \int_D h \, \mathbf{m}_f \leq (\sup_X h) \frac{\|\nabla f\|^2_{L^2}}{j}.$$

and so f is harmonic on X. □

Similarly we have the following result generalizing Theorem 5.9 in [Hol90].

Proposition. *Let X be 2-parabolic admissible Riemannian polyhedron. Let f in $W^{1,2}_{\mathrm{loc}}(X)$ be a harmonic function such that $\|\nabla f\|_{L^2}$ is finite. Then f is constant function.*

Proof. Set

$$f_i = \max(-i, \min(i, f)).$$

Let U_j be an exhaustion of X by regular domains $U_j \subset U_{j+1} \Subset X$. There is a continuous function $u_{i,j}$ such that $u_{i,j}$ are harmonic on U_j and $u_{i,j} = f_i$ in $X \setminus U_i$. Also $u_{i,j}$ is continuous on X and $\|\nabla u_{i,j}\|_{L^2}$ is finite. We have $-i \leq u_{i,j} \leq i$. According to Theorem 6.2 in [EF01], $u_{i,j}$ are Hölder continuous (after correction on a null set), and since they are uniformly bounded, by Theorem 6.3 in [EF01], they are locally uniformly Hölder equicontinuous and by Theorem 9.37 in [BB11], there is a subsequence which converges locally uniformly to some u_i as $j \to \infty$. (Note that the definition of harmonicity as in [BB11] is consistent with our definition.) The function u_i is bounded and harmonic and hence is constant (see Proposition **??**.). Moreover $u_{i,j} - f_i \in L_0^{1,2}$ and so $f_i \in L_0^{1,2}$. Therefore

$$\int_X |\nabla f|^2 \, d\mu_g = \lim_{i\to\infty} \int_X \langle \nabla f, \nabla f_i \rangle \, d\mu_g = 0,$$

and f is constant.

□

By use of Lemma 5.2.1 and the above propositions, the proofs of Corollaries 0.2.7 and 0.2.8 are straightforward.

7 Appendix

In this section we give the sketch of the proof of Theorem 1.4.5. First we recall the statement.

Theorem. *For every smooth map $f:(M,g,\Phi)\to(N,h)$, where M is a compact manifold and N is a negatively curved compact manifold, there is a harmonic map homotopic to f.*

Using the same method of the proof of the Bochner-type formula for harmonic maps we have,

Proposition 7.0.4. *Let $u\in C^0(M\times[0,T),N)\cap C^\infty(M\times(0,T),N)$ be a solution to the equation (1.6) and let $u_t(x)=u(x,t)$. We have in $M\times(0,T)$,*

(a) Weitzenbock formula for $e(u_t)$:

$$\begin{aligned}\frac{\partial e(u_t)}{\partial t}&=\widetilde{\Delta}e(u_t)-|B(u_t)|^2\\&\quad+\langle R^N(u_{t*}(e_i),u_{t*}(e_j))u_{t*}e_i,u_{t*}e_j\rangle-\langle u_{t*}(\widetilde{Ric}_\infty e_i),u_{t*}e_i\rangle\end{aligned}\tag{7.1}$$

(b) Weitzenbock formula for $k(u_t)$:

$$\frac{\partial k(u_t)}{\partial t}=\widetilde{\Delta}k(u_t)-\left|\nabla\frac{\partial u_t}{\partial t}\right|^2+\langle R^N(u_{t*}(e_i),u_{t*}(e_j))u_{t*}e_i,u_{t*}e_j\rangle\tag{7.2}$$

where $k(u_t)=\frac{1}{2}\left|\frac{\partial u_t}{\partial t}\right|^2$.

Theorem 7.0.5 (Existence of time-dependent local solutions)**.** *Let (M,g,Φ) be a compact smooth metric measure space and (N,h) be compact Riemannian manifold. For a given map $f\in C^{2,\alpha}(M,N)$, there exist a positive number $T=T(M,N,\Phi,f,\alpha)>0$ and $u\in C^{2+\alpha,1+\alpha/2}(M\times[0,T),N)\cap C^\infty(M\times(0,T),N)$ such that*

$$\begin{cases}\frac{\partial u}{\partial t}(x,t)=\tau(u(x,t))+du(\nabla\ln(\Phi)) & (x,t)\in M\times(0,T)\\ u(x,0)=f(x).\end{cases}\tag{7.3}$$

For the definition of $C^{2+\alpha,1+\alpha/2}(M\times[0,T))$ we refer the reader to the page 135 of the book [Nis02].

Proof. To prove this theorem, we reduce equation (7.3) to a system of parabolic differential equation for vector valued functions. Let i be an isometric embedding of N into an Euclidean space $\mathbb{R}^q$, for q large enough. Let $\tilde{N}$ be a tubular neighborhood of N

$$\tilde{N}=\left\{(x,V)|x\in i(N), v\in T_x i(N)^{\perp}, |v|<\epsilon\right\},$$

for sufficiently small ϵ. For $u: M\times[0,T)\to\tilde{N}$, we consider the following initial value problem.

$$\begin{cases} \frac{\partial u}{\partial t}=\widetilde{\Delta}u(x,t)-\Pi(u)(du,du) & (x,t)\in M\times(0,T) \\ u(x,0)=i\circ f(x). \end{cases} \tag{7.4}$$

(see formula (2.2), for the definition of $\Pi(u)(du,du)$.) One can prove that if u is the solution to the equation (7.4) and $u(M\times[0,T))\subset i(N)$ holds, then u is a solution to the equation (7.4). The converse also holds true.

The proof of existence of a time-dependent local solution (Theorem 4.7 in [Nis02]) is based on three steps. We explain why each step works in our situation. The proof uses the inverse function theorem in Banach spaces. The idea of the inverse function theorem is to reduce solvability of a nonlinear differential equation to solvability of a linearized equation.

Step 1 (Construction of an approximate solution). By identifying f with $i\circ f$, we consider the following system of linear parabolic equation:

$$\begin{cases} \frac{\partial v}{\partial t}=\widetilde{\Delta}v(x,t)-\Pi(f)(df,df) & (x,t)\in M\times(0,1) \\ v(x,0)=f(x). \end{cases}$$

By the assumptions on f, there is unique solution v in $C^{2+\alpha,1+\alpha/2}(M\times[0,1],\mathbb{R}^q)$. Then v approximate u as t goes to zero.

Step 2 (Application of the inverse function theorem). Consider the following differential operator on $C^{2+\alpha,1+\alpha/2}(M\times[0,\epsilon),N)$:

$$P(u)=\widetilde{\Delta}u-\partial_t u-\Pi(u)(du,du)$$

for some $0<\alpha'<1$. We define the (Banach) subspaces X and Y in $C^{2+\alpha',1+\alpha'/2}(M\times[0,T),N)$ and $C^{\alpha',\alpha'/2}(M\times[0,T),N)$ respectively (see page 138 in [Nis02] for the exact definition). For a given $z\in X$ if we set

$$\mathscr{P}(z)=P(v+z)-P(v),$$

then $\mathscr{P}$ is a map from X into Y and $\mathscr{P}(0)=0$. By application of the inverse function theorem,

$\mathscr{P}$ is homeomorphism in a neighborhood of 0 (see page 139 in [Nis02]). Therefore there exists a small $\delta = \delta(M, N, f) > 0$ such that for k in Y with $\|k\|_{\alpha,\alpha/2} < \delta$, there exists unique z in X such that

$$\mathscr{P}(z) = k, \qquad z(x,0) = 0, \quad \text{and} \quad \partial_t z(x,0) = 0$$

Set $P(v) = w$, $u = v + z$ satisfies

$$\begin{cases} P(u)(x,t) = (w+k)(x,t) & (x,t) \in M \times (0,1) \\ u(x,0) = f(x) \end{cases}$$

Step 3. With appropriate choice of a function $\zeta : \mathbb{R} \to \mathbb{R}$ (page 139 in [Nis02]), and setting $k = -\zeta w$, for sufficiently small T there exist some u (constructed as in step 2) which satisfies in equation (7.4). □

Theorem 7.0.6 (Existence of time-dependent global solutions)**.** *Let (M, g, Φ) be a compact smooth metric measure space and (N, h) a compact Riemannian manifold with non-positive sectional curvature. For a given map $f \in C^{2,\alpha}(M,N)$, there exists a unique $u \in C^{2+\alpha,1+\alpha/2}(M \times [0,\infty), N) \cap C^{\infty}(M \times (0,\infty), N)$ such that*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \tau(u(x,t)) + du(\nabla \ln(\Phi)) & (x,t) \in M \times (0,\infty) \\ u(x,0) = f(x). \end{cases} \tag{7.5}$$

Proof. The following propositions are the main steps in the proof of this theorem.

Proposition 7.0.7. *Let $u \in C^{2,1}(M \times [0,T), N) \cap C^{\infty}(M \times (0,T), N)$ be a solution to the equation (7.3) and let $u_t(x) = u(x,t)$. Assume N has non-positive curvature and $\widetilde{Ric}^M \geq -cg$ for a constant $c \in R$. Let $0 < \epsilon < T$. Then the following holds for the energy density $e(u)$ of u,*

$$e(u_t)(x) \leq e^{2ct} \sup_{x \in M} e(f)(x) \qquad (x,t) \in M \times [\epsilon, T),$$

and

$$e(u_t)(x) \leq C(M,\epsilon) E_{\Phi}(f) \qquad (x,t) \in M \times [\epsilon, T).$$

$C(M,\epsilon)$ is a constant depending only on M and ϵ.

Proposition 7.0.8. *Let $u \in C^{2,1}(M \times [0,T), N) \cap C^{\infty}(M \times (0,T), N)$ be a solution to the equation (7.3). If N is of non-positive curvature, then for any $0 < \alpha < 1$, there exists a positive number $C = C(M, N, \Phi, f, \alpha)$ such that*

$$|u(\cdot,t)|_{C^{2+\alpha}(M,N)} + \left|\frac{\partial u}{\partial t}\right|_{C^{\alpha}(M,N)} \leq C \tag{7.6}$$

at any $t \in [0,T)$.

The techniques used for the proofs of the propositions above in the classical case, such as the maximum principle for elliptic and linear parabolic equation, their solutions and their heat kernels are valid in our setting. These properties have been completely studied in the book [Gri09] in chapters 4-7.

The proof of the existence of a time-dependant global solution in Theorem 7.0.6, follows from inequality (7.6) and a contradiction argument. The uniqueness comes from the maximum principle for parabolic equations. □

Finally we verify that u as a solution to the equation (7.5) converges to a harmonic map which is free homotopic to f. Let $\{t_i\}$ be a sequence of times which tends to infinity. By inequality (7.6), the sequences $u(\cdot, t_i)$ and $\partial_t u(\cdot, t_i)$ converge uniformly to u_∞ and $\partial_t u_\infty$. In view of equation (7.2), $\partial_t u_\infty = 0$ and so u_∞ is a harmonic map which is freely homotopic to $f(x) = u(x, 0)$.

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