Finite-Time Blowup and Existence of Global Positive Solutions of a Semi-Linear SPDE with Fractional Noise

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Abstract

We consider stochastic equations of the prototype

$$du(t,x) = \left(\Delta u(t,x) + \gamma u(t,x) + u(t,x)^{1+\beta}\right) dt + \kappa u(t,x)\, dB_t^H$$

on a smooth domain $D \subset \mathbb{R}^d$, with Dirichlet boundary condition, where $\beta > 0$, γ and κ are constants and $\{B_t^H, t \geq 0\}$ is a real-valued fractional Brownian motion with Hurst index $H > 1/2$. By means of an associated random partial differential equation we estimate the probability of existence of non-trivial positive global solutions.

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1 Introduction and background

In a classical paper [7], Fujita proved that for a bounded smooth domain $D \subset \mathbb{R}^d$, the equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u^{1+\beta}(t,x), \quad x \in D,$$

with Dirichlet boundary condition, where $\beta > 0$ is a constant, explodes in finite time for all nonnegative initial values $u(0,x) \in L^2(D)$ satisfying

$$\int_D u(0,x)\psi(x)\, dx > \lambda_1^{1/\beta}. \tag{1}$$

Here $\lambda_1 > 0$ is the first eigenvalue of the Laplacian on D and ψ the corresponding eigenfunction normalized so that $\|\psi\|_{L^1} = 1$.

In this paper we consider a stochastic analog of the above equation, namely we investigate the semi-linear SPDE

$$\begin{aligned} du(t,x) &= (\Delta u(t,x) + \gamma u(t,x) + G(u(t,x)))\,dt + \kappa u(t,x)\,dB_t^H, \quad t>0, \\ u(0,x) &= f(x) \geqq 0, \quad x\in D, \\ u(t,x) &= 0, \quad t\geqq 0, \quad x\in\partial D, \end{aligned} \tag{2}$$

where $G:\mathbb{R}\to\mathbb{R}_+$ is locally Lipschitz and satisfies

$$G(z)\geq Cz^{1+\beta} \quad \text{for all} \quad z>0, \tag{3}$$

$C>0, \gamma,\ \beta>0$ and κ are given numbers, $\{B_t^H,\ t\geq 0\}$ is a one-dimensional fractional Brownian motion with Hurst index $H>1/2$ on a stochastic basis $(\Omega,\mathcal{F},P)$, and $f:D\to\mathbb{R}_+$ is of class C^2 and not identically zero. We assume (3) in sections 1 to 3 only; it is replaced by (13) in sections 4 and 5.

The results on global solutions of parabolic equations perturbed by an additive or multiplicative time or space-time fractional noise established up to now are sufficient to state the existence and uniqueness of the variational (weak) and of the mild solution of (2) and the equivalence of both; see Maslowski and Nualart [8], Nualart and Vuillermot [11], and Sanz and Vuillermot [16], where the integral with respect to B^H is understood in the sense of fractional calculus (see e.g. Zähle [18], [19]). Let us recall the notions of variational and mild solutions we are going to use here; see [11], [16]. Let $\alpha\in(1-H,\frac{1}{2})$, $t>0$, and let $\mathcal{B}^{\alpha,2}([0,t],L^2(D))$ be the Banach space of all measurable mappings $u:[0,t]\to L^2(D)$ endowed with the norm $\|\cdot\|_{\alpha,2}$, defined by

$$\|u\|_{\alpha,2}^2 = \left(\operatorname*{ess\,sup}_{s\in[0,t]}\|u(s,\cdot)\|_2\right)^2 + \int_0^t ds\left(\int_0^s dr\frac{\|u(s,\cdot)-u(r,\cdot)\|_2}{(s-r)^{\alpha+1}}\right)^2 < \infty$$

where $\|\cdot\|_2$ is the usual norm in $L^2(D)$. An $L^2(D)-$valued random field $u=\{u(t,\cdot),\, t\geq 0\}$ is a *variational solution* of (2) on the interval $]0,\varrho[$ if, a.s.,

$$u\in L^2([0,t],H^1(D))\cap\mathcal{B}^{\alpha,2}([0,t],L^2(D)) \tag{4}$$

for all $t<\varrho$ and if, for every $\varphi\in H^1(D)$ vanishing on ∂D,

$$\begin{aligned} \int_D u(t,x)\varphi(x)\,dx &= \int_D f(x)\varphi(x)\,dx \\ &+ \int_0^t\int_D [\langle\nabla u(s,x),\nabla\varphi(x)\rangle_{\mathbb{R}^d} + \gamma u(t,x)\varphi(x) + G(u(s,x))\varphi(x)]\,dx\,ds \\ &+ \kappa\int_0^t\int_D u(s,x)\varphi(x)\,dx\,dB_s^H \quad P-\text{a.s.} \end{aligned}$$

for all $t\in[0,\varrho[$. The requirement for u to belong to the $\mathcal{B}^{\alpha,2}$ spaces implies that the integral with respect to B^H exists as a generalized Stieltjes integral in the sense of [19], see

Proposition 1 in [11]. Let $\{S_t,\, t \geqq 0\}$ be the semigroup of d-dimensional Brownian motion with variance parameter 2, killed at the boundary of D. An $L^2(D)-$valued random field $u = \{u(t,\cdot),\, t \geqq 0\}$ is a *mild solution* of (2) on the interval $]0,\tau[$ if (4) holds a.s. for all $t < \tau$ and if

$$\begin{aligned} u(t,x) &= S_t f(x) + \int_0^t [\gamma S_{t-r}(u(r,\cdot))(x) + S_{t-r}(G(u(r,\cdot))(x)]\; dr + \kappa S_{t-r}(u(r,\cdot))(x)\, dB_r^H \\ &\quad P\text{-a.s. and } x\text{-a.e. in } D \end{aligned}$$

for all $t \in]0,\tau[$ (see e.g. [14], Chapter IV). Let us remark that the proof of the uniqueness of the mild solution and the equivalence of the variational and the mild solutions are carried out in [16] under the conditions $H \in (\frac{4d+1}{4d+2}, 1)$ and $\alpha \in (1-H, \frac{1}{4d+2})$, and for the more general case where B^H is a space-dependent fractional Brownian motion. For an approach based on stochastic integrals in the Wick sense we refer to [12]. The positivity of the solution of (2) will be addressed in the next section.

Our aim in this communication is to study the blowup behaviour of u by means of the random partial differential equation of section 2 (see (6) below). The case of $H = 1/2$, in which $\{B_t^H\}$ is a standard one-dimensional Brownian motion, was investigated in [4]. There we obtained estimates of the probability of blowup and conditions for the existence of a global solutions of (2) with $H = 1/2$ and $\gamma = 0$. Following closely the approach in [4], here we are going to derive the same kind of bounds for the positive solutions of Equation (2), in the case $H > 1/2$ and with a constant drift in the non-random linear part. Moreover, we obtain useful lower and upper bounds τ_*, τ^* for the explosion time ϱ of (2). We remark that both, the estimates we obtain and the distributions of the random times τ_*, τ^*, are given in terms of exponential functionals of B^H of the form

$$\int_0^t e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\, ds \quad \text{and} \quad \int_0^\infty e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\, ds. \tag{5}$$

When $H = 1/2$ the distribution of the integrals above can be obtained, respectively, from Dufresne's and Yor's formuale [5, 17]. However, to our knowledge such precise results are not presently available for $H \neq 1/2$. It remains a challenge to obtain more accurate information on the explosion times of (2).

We describe in sections 3 and 4 the blowup behaviour of the solution v of this random partial differential equation in terms of the first eigenvalue and the first eigenfunction of the Laplace operator on D. This is done by solving explicitly a stochastic equation in the time variable which is obtained from the weak form of (6). The solution of this differential equation can be written in terms of integrals of the exponential of fractional Brownian motion with drift. Near the end of the paper, sufficient conditions for v to be a global solution are given in terms of the semigroup of the Laplace operator using recent sharp results on its transition density. These conditions show in particular that the initial condition f has to be small enough in order to avoid for a given G the blowup of v, as well as a sufficiently small $|\gamma|$ and a sufficiently big β. The results presented here can be used to investigate the blowup behaviour of u for non-linearities satisfying (3) or (13).

2 Weak solutions of a random PDE

In this section we investigate the random partial differential equation

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x) &= \Delta v(t,x)+\gamma v(t,x)+e^{-\kappa B_t^H}G(e^{\kappa B_t^H}v(t,x)), \quad t>0, \quad x\in D,\\ v(0,x) &= f(x), \quad x\in D, \\ v(t,x) &= 0, \quad x\in\partial D.\end{aligned} \tag{6}$$

This equation is understood trajectorywise and classical results for partial differential equations of parabolic type apply to show existence and uniqueness of a solution $v(t,x)$ up to eventual blowup (see e.g. Friedman [6] Chapter 7, Theorem 9). Moreover,

$$v(t,x)=e^{\gamma t}S_t f(x)+\int_0^t e^{\gamma(t-s)}S_{t-s}\left(e^{-\kappa B_s^H}G(e^{\kappa B_s^H}v(s,x)\right)\,ds, \tag{7}$$

and therefore $v(t,x)\geq e^{\gamma t}S_t f(x)\geq 0$.

Proposition 1 *Let u be a weak solution of (2). Then the function v defined by*

$$v(t,x)=e^{-\kappa B_t^H}u(t,x), \quad t\geq 0, \quad x\in D,$$

solves (6).

Remark 2 Proposition 1 implies in particular that Eq. (2) possesses a strong local solution $u(t,x)$. Moreover, $u(t,x)\geq 0$ due to (7).

Proof. By Itô's formula for B^H (see e.g. [10], Lemma 2.7.1)

$$e^{-\kappa B_t^H}=1-\kappa\int_0^t e^{-\kappa B_s^H}dB_s^H.$$

We notice that the last integral can be defined as a Riemann-Stieltjes integral. Let us write $u(t,\varphi)\equiv\int_D u(t,x)\varphi(x)\,dx$. Then the weak solution of (2) can be written as

$$u(t,\varphi)=u(0,\varphi)+\int_0^t u(s,\Delta\varphi)\,ds+\int_0^t[\gamma u(s,\varphi)+G(u)(s,\varphi)]\,ds+\kappa\int_0^t u(s,\varphi)\,dB_s^H.$$

By applying the integration by parts formula, which is a special case of the 2-dimensional Itô's formula (see [10], p. 184), we get

$$\begin{aligned}v(t,\varphi):= & \int_D v(t,x)\varphi(x)\,dx\\ = & \ v(0,\varphi)+\int_0^t e^{-\kappa B_s^H}\,du(s,\varphi)+\int_0^t u(s,\varphi)\left(-\kappa e^{-\kappa B_s^H}\,dB_s^H\right).\end{aligned}$$

Therefore,

$$\begin{aligned} v(t,\varphi) &= v(0,\varphi) + \int_0^t e^{-\kappa B_s^H} \left[u(s,\Delta\varphi) + \gamma u(s,\varphi) + G(u)(s,\varphi)\right] ds + \kappa \int_0^t e^{-\kappa B_s^H} u(s,\varphi)\, dB_s^H \\ &\quad -\kappa \int_0^t e^{-\kappa B_s^H} u(s,\varphi)\, dB_s^H \\ &= v(0,\varphi) + \int_0^t \left[v(s,\Delta\varphi) + \gamma v(s,\varphi) + e^{-\kappa B_s^H} G(e^{\kappa B_\cdot^H} v)(s,\varphi)\right] ds. \end{aligned}$$

Moreover, by self-adjointness of the Laplacian, and the fact that $\varphi(x) = 0$ for $x \in \partial D$,

$$v(s,\Delta\varphi) = \int_D v(s,x)\Delta\varphi(x)\, dx = \int_D \Delta v(s,x)\varphi(x)\, dx = \Delta v(s,\varphi).$$

■

In what follows ϱ denotes the blowup time of Eq. (6). Due to Proposition 1 and to the a.s. continuity of $B_\cdot^H$, ϱ is also the explosion time of Eq. (2).

3 An upper bound for ϱ

Without loss of generality, let us assume that $C = 1$ in (3). Let ψ be the eigenfunction corresponding to the first eigenvalue λ_1 of the Laplacian on D, normalized by $\int_D \psi(x)\, dx = 1$. It is well-known that ψ is strictly positive on D. Due to Proposition 1 we have that

$$v(t,\psi) = v(0,\psi) + \int_0^t \left[v(s,\Delta\psi) + \gamma v(s,\psi)\right] ds + \int_0^t e^{-\kappa B_s^H} G(e^{\kappa B_\cdot^H} v)(s,\psi)\, ds.$$

Moreover,

$$v(s,\Delta\psi) = -\lambda_1 v(s,\psi),$$

and, due to (3),

$$\int_D e^{-\kappa B_s^H} G(e^{\kappa B_s^H} v(s,x))\psi(x)\, dx \ \geq\ e^{\kappa\beta B_s^H} \int_D v(s,x)^{1+\beta}\psi(x)\, dx.$$

By Jensen's inequality

$$\int_D v(s,x)^{1+\beta}\psi(x)\, dx \ \geq\ \left[\int_D v(s,x)\psi(x)\, dx\right]^{1+\beta} \ =\ v(s,\psi)^{1+\beta},$$

and therefore

$$\frac{d}{dt} v(t,\psi) \geq (-\lambda_1 + \gamma) v(t,\psi) + e^{\kappa\beta B_t^H} v(t,\psi)^{1+\beta}. \qquad (*)$$

Hence $v(t,\psi) \geq I(t)$ for all $t \geq 0$, where $I(\cdot)$ solves

$$\frac{d}{dt}I(t) = (-\lambda_1+\gamma)I(t) \, + e^{\kappa\beta B_s^H} I(t)^{1+\beta}\,, \quad I(0) = v(0,\psi),$$

and is given by

$$I(t) \;=\; e^{(-\lambda_1+\gamma)t}\left[v(0,\psi)^{-\beta} - \beta\int_0^t e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,ds\right]^{-\frac{1}{\beta}}, \quad 0 \leq t < \tau^*,$$

with

$$\tau^* := \inf\left\{t \geq 0 \,\middle|\, e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,ds \geq \frac{1}{\beta}v(0,\psi)^{-\beta}\right\}. \tag{8}$$

It follows that I exhibits finite time blowup on the event $[\tau^* < \infty]$. Due to $I \leqq v(\cdot,\psi)$, τ^* is an upper bound for the blowup time of $v(\cdot,\psi)$. Since by assumption $\int_D \psi(x)\,dx = 1$, $v(t,x)$ cannot be bounded on $[\tau^* < \infty]$. Hence τ^* is also an upper bound for the blowup times of v and u.

We subsume the above argumentation into the following corollary.

Corollary 3 *The function $v(t,\psi) = \int_D v(t,x)\psi(x)\,dx$ explodes in finite time on the event $[\tau^* < \infty]$, hence $u(t,x) = e^{\kappa B_t^H}v(t,x)$ also explodes in finite time if $\tau^* < \infty$, and the blowup times of u and v are the same.*

Remark 4 Notice that, from (8),

$$\begin{aligned} P[\tau^* = +\infty] &= P\left[\int_0^t e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,ds < \frac{1}{\beta}v(0,\psi)^{-\beta} \text{ for all } t > 0\right] \\ &= P\left[\int_0^\infty e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,ds \leqq \frac{1}{\beta}v(0,\psi)^{-\beta}\right]. \end{aligned} \tag{9}$$

Assume now that $\gamma > \lambda_1$, and recall the law of the iterated logarithm for B^H (ref: Arc) :

$$\lim\inf_{t\to+\infty}\frac{B_t^H}{t^H\sqrt{\log\log t}} = -1, \;\; \lim\sup_{t\to+\infty}\frac{B_t^H}{t^H\sqrt{\log\log t}} = +1.$$

It follows that the integral in eqref diverges. Therefore $P[\tau^* = +\infty] = 0$ and any nontrivial positive solution of Eq. (2) explodes in finite time a.s.. If $\gamma < \lambda_1$ this is not true anymore, and it would be interesting to estimate this probability. As mentioned in the introduction, the law of these integrals is known only in the case $H = \frac{1}{2}$, i.e. for Brownian motion. After the following remark we consider this case in more detail.

Remark 5 By putting $\kappa = \gamma = 0$ we get $v = u$ and, moreover, in (9) we obtain that $P[\tau^* = +\infty] = 0$ or 1 according to $\int_D f(x)\psi(x)\,dx > \lambda_1^{1/\beta}$ or $\int_D f(x)\psi(x)\,dx \leq \lambda_1^{1/\beta}$, which is a probabilistic counterpart to condition (1).

For $H = \frac{1}{2}$ Itô's formula contains a second order term and the associated random PDE therefore reads (we write W instead of $B^{1/2}$)

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \Delta v(t,x) + (\gamma - \frac{\kappa^2}{2})v(t,x) + e^{-\kappa W_t} G(e^{\kappa W_t} v(t,x)), \quad t>0, \quad x \in D, \\ v(0,x) &= f(x), \quad x \in D, \\ v(t,x) &= 0, \quad x \in \partial D. \end{aligned} \tag{10}$$

We get again a differential inequality for $v(t,\psi)$, and the blowup time of the associated differential equation for I is

$$\tilde{\tau}^* = \inf\left\{ t \geq 0 \ \int_0^t e^{-(\lambda_1+\kappa^2/2-\gamma)\beta s+\kappa\beta W_s}\, ds \geq \frac{1}{\beta} v(0,\psi)^{-\beta} \right\}. \tag{**}$$

Now

$$\begin{aligned} P[\tilde{\tau}^* = +\infty] &= P\left[\int_0^\infty e^{-(\lambda_1+\kappa^2/2-\gamma)\beta s+\kappa\beta W_s}\, ds \leqq \frac{1}{\beta} v(0,\psi)^{-\beta} \right] \\ &= P\left[\int_0^\infty e^{2\hat{\beta} W_s^{(\mu)}}\, ds \leqq \frac{1}{\beta} v(0,\psi)^{-\beta} \right], \end{aligned} \tag{11}$$

where $W_s^{(\mu)} := \mu s + W_s$, $\mu := -(\lambda_1 - \gamma + \kappa^2/2)/\kappa$, and $\hat{\beta} := \kappa\beta/2$. Setting $\widehat{\mu} = \mu/\widehat{\beta}$ we get by performing the time change $s \longmapsto s(\hat{\beta})^2$,

$$P[\tilde{\tau}^* = +\infty] = P\left[\frac{4}{\kappa^2\beta^2} \int_0^\infty \exp(2W_s^{(\widehat{\mu})})\, ds \leqq \frac{1}{\beta} v(0,\psi)^{-\beta} \right]. \tag{12}$$

If $\hat{\mu} = -(\lambda_1 - \gamma + \kappa^2/2)/\kappa\beta > 0$, it follows again that $P[\tilde{\tau}^* = +\infty] = 0$ and any nontrivial positive solution of Eq. (2) with B^H replaced by W explodes in finite time a.s., see also [9], Proposition 6.4, or [15], Section 2. If $\hat{\mu} < 0$, it follows from [17] (Chapter 6, Corollary 1.2) that

$$\int_0^\infty \exp(2W_s^{(\widehat{\mu})})\, ds \ = \ \frac{1}{2Z_{-\widehat{\mu}}}$$

in distribution, where $Z_{-\widehat{\mu}}$ is a random variable with law $\Gamma(-\widehat{\mu})$, i.e. $P(Z_{-\widehat{\mu}} \in dy) = \frac{1}{\Gamma(-\widehat{\mu})} e^{-y} y^{-\widehat{\mu}-1} dy$. We get therefore (see also formula 1.10.4(1) in [3])

$$P[\tilde{\tau}^* = +\infty] = \int_0^{\frac{1}{\beta}v(0,\psi)^{-\beta}} h(y)dy,$$

where

$$h(y) \ = \ \frac{(\kappa^2\beta^2 y/2)^{(2(\lambda_1-\gamma)+\kappa^2)/\kappa^2\beta}}{y\Gamma((2(\lambda_1-\gamma)+\kappa^2)/(\kappa^2\beta))} \exp\left(-\frac{2}{\kappa^2\beta^2 y} \right).$$

In this way we have proved the following

Proposition. The probability that the solution of (2) (ref:) with B^H replaced by W blows up in finite time is lower bounded by $\int_{\frac{1}{\beta}v(0,\psi)^{-\beta}}^{+\infty} h(y)\,dy$.

We end this section by reviewing another method to find upper estimates of the blowup time of the solution of (2). In cite: BDS it is shown that the formula (**) for $\tilde{\tau}^*$ can also be found by replacing the random differential inequality (*) by a stochastic differential inequality, whose associated equality can be solved explicitely. A comparison theorem for stochastic differential inequalities is needed for this, and since no such theorem seems to be known at present for inequalities with fractional Brownian motion, we have to restrict ourselves to Brownian motion where these theorems are classical.

Proceeding with the variational solution in section 1 in the same way as with the random PDE ref : Eq3 at the beginning of this section, we get the following stochastic differential inequality

$$u(t,\psi) \geqq u(0,\psi) + \int_0^t [(\gamma-\lambda_1)u(s,\psi) + u(s,\psi)^{1+\beta}]\,ds + \kappa\int_0^t u(s,\psi)\,dW_s.$$

The corresponding stochastic differential equation

$$X_t = u(0,\psi) + \int_0^t [(\gamma-\lambda_1)X_s + X_s^{1+\beta}]\,ds + \kappa\int_0^t X_s\,dW_s$$

can be solved explicitely. In fact, by the ansatz $Y_t = h(X_t)$ and by Itô's formula, we then get

$$Y_t = Y_0 + \int_0^t [h'(Y_s)((\gamma-\lambda_1)Y_s + Y_s^{1+\beta}) + \frac{\kappa^2}{2}h''(Y_s)Y_s^2]\,ds + \kappa\int_0^t h'(Y_s)Y_s\,dW_s.$$

The function h can now be chosen in such a way that Y satisfies the linear stochastic differential equation

$$Y_t = Y_0 + \int_0^t (a+bY_s)ds + \int_0^t (c+dY_s)dW_s.$$

for suitable constants $a,b,c,d \in \mathbb{R}$. In fact, a comparison of the martingale parts of both representations of Y gives a differential equation for h whose solution is given by $h(Y_t) = kY_t^{d/\kappa} - \frac{c}{d}$ for any constant $k \in \mathbb{R}$. By comparing the finite variation parts of the representations of Y we get

$$\frac{kd}{\kappa}Y_t^{d/\kappa+\beta} + \frac{kd}{\kappa}(\gamma-\lambda_1)Y_t^{d/\kappa} + \frac{1}{2}kd(d-\kappa)Y_t^{d/\kappa} = a + bkY_t^{d/\kappa} - \frac{bc}{d}.$$

We choose $d = -\beta\kappa$, $c = 0$, $b = \beta(\frac{(1+\beta)\kappa^2}{2} - \gamma + \lambda_1)$, $a = -k\beta$ and get $Y_t = X_t^{-\beta}$. With the explicit formula for the solution of the linear equation for Y we get

$$X_t = Y_t^{-1/\beta} = [u(0,\psi)^{-\beta} Y_t^0 - \beta Y_t^1]^{-1/\beta}, \tag{***}$$

where

$$Y_t^0 = \exp[(\lambda_1 + \kappa^2/2 - \gamma)\beta t - \kappa\beta W_t], \quad Y_t^1 = Y_t^0 \int_0^t \exp[-(\lambda_1 + \kappa^2/2 - \gamma)\beta s + \kappa\beta W_s] ds.$$

It can easily be seen that (***) yields the same formula for the blowup time of X as (**) for I.

4 A lower bound for ϱ

We consider again equation (6), but we assume that $\kappa \neq 0$ and that $G : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $G(0) = 0$, $G(z)/z$ is increasing and

$$G(z) \leq \Lambda z^{1+\beta} \quad \text{for all} \quad z > 0, \tag{13}$$

where Λ and β are certain positive numbers. Let $\{S_t, t \geq 0\}$ again denote the semigroup of d-dimensional Brownian motion killed at the boundary of D. Recall the integral form (7) of Equation (6). We define

$$F(t) = \left(1 - \Lambda\beta \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta \, dr\right)^{-\frac{1}{\beta}}, \quad 0 \leq t < \tau_*, \tag{14}$$

where

$$\tau_* = \inf\left\{t > 0 : \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta \, dr \geq (\Lambda\beta)^{-1}\right\}. \tag{15}$$

Hence $F(0) = 1$ and

$$\frac{dF}{dt}(t) = \Lambda e^{\kappa\beta B_t^H} \|e^{\gamma t} S_t f\|_\infty^\beta F^{1+\beta}(t),$$

which implies

$$F(t) = 1 + \Lambda \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta F^{1+\beta}(r) \, dr.$$

Let

$$R(V)(t,x) := e^{\gamma t} S_t f(x) + \int_0^t e^{-\kappa B_r^H} e^{\gamma(t-r)} S_{t-r} \left(G(e^{\kappa B_r^H} V_r(\cdot))\right)(x) \, dr, \quad x \in D, t \geq 0,$$

where $(t,x) \mapsto V_t(x)$ is any nonnegative continuous function such that $V_t(\cdot) \in C_0(D)$, $t \geq 0$, and

$$V_t(x) \leq e^{\gamma t} S_t f(x) F(t), \quad 0 \leq t < \tau_*, \quad x \in D. \tag{16}$$

Then $e^{\gamma t}S_tf(x) \le \; R(V)(t,x)$ and

$$\begin{aligned}
&R(V)(t,x)\\
&= \; e^{\gamma t}S_tf(x) + \int_0^t e^{-\kappa B_r^H} e^{\gamma(t-r)} S_{t-r}\left(\frac{G(e^{\kappa B_r^H}V_r(\cdot))}{V_r(\cdot)}V_r(\cdot)\right)(x)\,dr\\
&\le \; e^{\gamma t}S_tf(x) + \int_0^t e^{-\kappa B_r^H} e^{\gamma(t-r)} S_{t-r}\left(\frac{G(e^{\kappa B_r^H}F(r)\|e^{\gamma r}S_rf\|_\infty)}{F(r)\|e^{\gamma r}S_rf\|_\infty}V_r(\cdot)\right)(x)\,dr\\
&\le \; e^{\gamma t}S_tf(x) + \Lambda\int_0^t e^{\kappa\beta B_r^H}F^{1+\beta}(r)\|e^{\gamma r}S_rf\|_\infty^\beta e^{\gamma(t-r)}S_{t-r}(e^{\gamma r}S_rf)(x)\,dr\\
&= \; e^{\gamma t}S_tf(x)\left[1+\Lambda\int_0^t e^{\kappa\beta B_r^H}F^{1+\beta}(r)\|e^{\gamma r}S_rf\|_\infty^\beta\,dr\right] = e^{\gamma t}S_tf(x)F(t), \qquad (17)
\end{aligned}$$

where to obtain the first inequality we used (16) and the fact that $G(z)/z$ is increasing, and to obtain the second inequality we used (13). Consequently,

$$e^{\gamma t}S_tf(x) \le \; R(V)(t,x) \le e^{\gamma t}S_tf(x)F(t), \quad 0 \le t < \tau_*, \quad x \in D.$$

Let

$$v_t^0(x) := e^{\gamma t}S_tf(x) \quad \text{and} \quad v_t^{n+1}(x) = \; R(v^n)(t,x), \quad n = 0,1,2,\ldots.$$

Using induction, one can easily prove that the sequence $\{v^n\}$ is increasing, and therefore the limit

$$v_t(x) = \lim_{n\to\infty} v_t^{(n)}(x)$$

exists for all $x \in D$ and all $0 \le t < \tau_*$. The monotone convergence theorem implies

$$v_t(x) = Rv_t(x) \text{ for } x \in D \text{ and } 0 \le t < \tau_*,$$

i.e. the function $v_t(x)$ solves (7) on $[0,\tau_*) \times D$. Moreover, because of (17) and (14),

$$v_t(x) \;\le\; \frac{e^{\gamma t}S_tf(x)}{\left(1-\Lambda\beta\int_0^t e^{\kappa\beta B_r^H}\|e^{\gamma r}S_rf\|_\infty^\beta\,dr\right)^{1/\beta}} \;<\; \infty$$

as long as

$$\int_0^t e^{\kappa\beta B_r^H}\|e^{\gamma r}S_rf\|_\infty^\beta\,dr < (\Lambda\beta)^{-1}.$$

In this way we have proved the following proposition.

Proposition 6 *The blowup time of (7) is bounded from below by the random variable* τ_* *defined in (15).*

5 Non explosion of v

An immediate consequence of the discussion in the preceding section is the following result.

Theorem 7 *Assume that f satisfies*

$$\Lambda\beta \int_0^\infty e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta \, dr < 1. \tag{18}$$

Then Equation (6) admits a global solution $v(t,x)$ that satisfies

$$0 \le v(t,x) \le \frac{e^{-\gamma t} S_t f(x)}{\left(1 - \Lambda\beta \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta \, dr\right)^{\frac{1}{\beta}}}, \qquad t \ge 0. \tag{19}$$

When the boundary of D is sufficiently smooth, it is possible to derive a sufficient condition for (18) in terms of the transition kernels $\{p_t(x,y),\, t > 0\}$ of $\{S_t,\, t \ge 0\}$ and the first eigenvalue λ_1 and corresponding eigenfunction ψ. We recall the following sharp bounds for $\{p_t(x,y),\, t > 0\}$, which we borrowed from Ouhabaz and Wang [13].

Theorem 8 *Let $\psi > 0$ be the first Dirichlet eigenfunction on a connected bounded $C^{1,\alpha}$-domain in $\mathbb{R}^d$, where $\alpha > 0$ and $d \ge 1$, and let $p_t(x,y)$ be the corresponding Dirichlet heat kernel. There exists a constant $c > 0$ such that, for any $t > 0$,*

$$\max\left\{1, \frac{1}{c} t^{-(d+2)/2}\right\} \le e^{\lambda_1 t} \sup_{x,y} \frac{p_t(x,y)}{\psi(x)\psi(y)} \le 1 + c(1 \wedge t)^{-(d+2)/2} e^{-(\lambda_2 - \lambda_1)t},$$

where $\lambda_2 > \lambda_1$ are the first two Dirichlet eigenvalues. This estimate is sharp for both short and long times.

The above theorem is useful in verifying condition (18). Let the domain D satisfy the assumptions in Theorem 8, and let the initial value $f \ge 0$ be chosen so that

$$f(y) \le K S_\eta \psi(y), \quad y \in D, \tag{20}$$

where $\eta \ge 1$ is fixed and $K > 0$ is a sufficiently small constant to be specified later on. Arguing as in [4] we obtain that condition (18) is satisfied provided that

$$\Lambda\beta \left[K(1+c)e^{-\lambda_1\eta} \left(\sup_{x\in D} \psi(x)\right)^2 \int_D \psi(y)\, dy\right]^\beta \int_0^\infty dr\, e^{\kappa\beta B_r^H + (-\lambda_1+\gamma)\beta r} < 1,$$

or

$$\int_0^\infty dr\, e^{\kappa\beta W_r + (-\lambda_1+\gamma)\beta r} \; < \; \frac{e^{\lambda_1\beta\eta}}{\Lambda\beta \left[K(1+c) \left(\sup_{x\in D} \psi(x)\right)^2 \int_D \psi(y)\, dy\right]^\beta}, \tag{21}$$

which holds if K in (20) is sufficiently small. In this way we get the following

Theorem 9 *Let G satisfy (13), and let D be a connected, bounded $C^{1,\alpha}$-domain in $\mathbb{R}^d$, where $\alpha > 0$. If (20) and (21) hold for some $\eta > 0$ and $K > 0$, then the solution of Equation (7) is global.*

Remark 10 The integral on the left side of (21) coincides with the corresponding integral in Section 3. If $G(z) = \Lambda z^{1+\beta}$, the results of this section can be applied also to the solution u of equation (2) because $v(t,x) = e^{-\kappa B_t^H} u(t,x)$, $t \geq 0$, $x \in D$.

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