Problème de sous-convexité pour GL2xGL1

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Les charmes enchanteux de cette sublime science
ne se décèlent dans toute leur beauté
qu'à ceux qui ont le courage de l'approfondir.
— Carl Friedrich Gauss

To my parents…

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Abstract

Let F be a number field, π an irreducible automorphic representation of $GL_2(F)\backslash GL_2(\mathbb{A}_F)$ with unitary central character, and χ a Hecke character of analytic conductor Q. We are interested in bounding $L(1/2,\pi\otimes\chi)$ in terms of Q.
If π is cuspidal, then we get a Burgess-like bound as $L(1/2,\pi\otimes\chi) \ll Q^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\epsilon}$, where $0 \leq \theta \leq 1/2$ is any exponent towards the Ramanujan-Petersson conjecture. The implicit constant depends polynomially on the analytic conductor of π.
If π is the unitary Eisenstein series representation induced by the trivial character, then we get $L(1/2,\pi\otimes\chi) \ll Q^{\frac{1}{2}-\frac{1}{12}(1-2\theta)+\epsilon}$. As a consequence, we get a subconvex bound for the L-function $L(1/2,\chi) \ll Q^{\frac{1}{4}-\frac{1}{24}(1-2\theta)+\epsilon}$.
The proof is based on an idea of unipotent translation originated from P.Sarnak then developped by P.Michel and A.Venkatesh, combined with the method of amplification.

Key Words: Automorphic Representation, Automorphic L-Function, Subconvexity

Résumé

Soient F un corps de nombres, π une représentation automorphe irréductible de $GL_2(F)\backslash GL_2(\mathbb{A}_F)$ admettant un caractère central unitaire, et χ un caractère de Hecke avec conducteur analytique Q. Nous nous inérèssons à borner $L(1/2, \pi \otimes \chi)$ en fonction de Q.

Si π est cuspidale, alors nous obtenons une borne du type Burgess comme $L(1/2, \pi \otimes \chi) \ll Q^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\epsilon}$, où $0 \leq \theta \leq 1/2$ est un exposant vers la conjecture de Ramanujan-Petersson. La constante implicite depend polynomialement en le conducteur analytique de π.

Si π est une représentation de séries d'Eisenstein unitaire induite par les caractères triviaux, alors nous avons $L(1/2, \pi \otimes \chi) \ll Q^{\frac{1}{2}-\frac{1}{12}(1-2\theta)+\epsilon}$. Par conséquent, nous obtenons une borne de sousconvexité pour la fontion-L : $L(1/2, \chi) \ll Q^{\frac{1}{4}-\frac{1}{24}(1-2\theta)+\epsilon}$.

La preuve est basée sur une idée de translation unipotente provenant originalement de P.Sarnak, ensuite développée par P.Michel et A.Venkatesh, combinée avec la méthode d'amplification.

Mots-Clés : Représentation Automorphe, Fonction-L Automorphe, Sous-convexité

Contents

Introduction

0.1 A Brief Introduction to the Problem of Subconvexity

Generally speaking, the type of problem we are interested in is the subconvexity of Ranking-Selberg L-functions, particularly for $\mathrm{GL}_2 \times \mathrm{GL}_1$. More precisely, let F be a number field with its ring of adeles $\mathbb{A}$. Fix an automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$. We know that $\pi = \otimes_v' \pi_v$ as a restricted tensor product of unitary irreducible representations π_v of $\mathrm{GL}_2(F_v)$. Let χ vary over Hecke characters of $F^\times \backslash \mathbb{A}^\times$. The analytic conductor of χ is denoted by $C(\chi) = C_\infty(\chi) C_f(\chi)$, where $C_f(\chi)$ is the usual conductor of χ, and $C_\infty(\chi)$ is formed from the parameters of χ at infinite places. We can form the Rankin-Selberg L-function $L(s, \pi \times \chi)$ initially defined for $\Re s \gg 1$ as an Euler product. Similar properties hold for $L(s, \chi)$. They can be analytically continued to all $s \in \mathbb{C}$. Furthermore there is a functional equation linking $L(s, \pi \times \chi)$ (resp. $L(s, \chi)$) to $L(1 - s, \pi \times \bar{\chi})$ (resp. $L(1 - s, \bar{\chi})$), which implies the convex bounds

$$L(1/2, \pi \times \chi) \ll_{\epsilon, F, \pi} C(\chi)^{1/2+\epsilon}, L(1/2, \chi) \ll_{\epsilon, F} C(\chi)^{1/4+\epsilon}, \forall \epsilon > 0.$$

The generalized Lindelöf hypothesis, which is a consequence of the generalised Riemann hypothesis, implies that we can replace 1/2 by 0 in the above estimation. Any bound in between as

$$L(1/2, \pi \times \chi) \ll_{\epsilon, F, \pi} C(\chi)^{1/2-\delta+\epsilon}, \forall \epsilon > 0 \tag{0.1.1}$$

for some constant $0 < \delta < 1/2$ is a subconvex bound.

When $F = \mathbb{Q}$, π is cupidal with trivial central character, and π_∞ is a discrete series representation of $\mathrm{GL}_2(\mathbb{R})$, there is a unique modular form $f \in S_k(N)$ for some $k, N \in \mathbb{N}$, which is also a newform and Hecke eigenform, generating π and having Fourier expansion at infinity as

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \lambda_f(1) = 1, z \in \mathbb{C}, \Im(s) > 0.$$

Assume that the image of χ is finite, thus χ comes from a Dirichlet character still denoted by $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^1$ for some integer $d > 0$. Then $C(\chi) = C_f(\chi) = d$, and the L-functions are defined by

$$L(s,\pi\times\chi)=L(s,f\times\chi)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)\chi(n)}{n^s},L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},\Re(s)\gg 1.$$

The analytic continuation and functional equation are then classical. Historically, certain bounds are critical. When $\chi=\chi_0$ i.e. $d=1$, Weyl obtained in [34]

$$L(1/2+i\tau,\chi_0)=\zeta(1/2+i\tau)\ll_\epsilon |\tau|^{1/4-1/12+\epsilon},$$

i.e. $\delta=1/6$ in (0.1.1). When $d\neq 1$, Burgess in [3] obtained

$$L(1/2+i\tau,\chi)\ll_{\epsilon,\tau}|d|^{1/4-1/16+\epsilon}, \tag{0.1.2}$$

i.e. $\delta=1/8$ in the level aspect of (0.1.1). Similar results for $L(s,f\times\chi)$ can be found in [16]

$$L(1/2+i\tau,f\times\chi)\ll_{\epsilon,f}|\tau|^2|d|^{1/2-1/22+\epsilon},$$

and in [5]

$$L(1/2+i\tau,f\times\chi)\ll_{\epsilon,f}|\tau|^{1/2+\epsilon}|d|^{1/2-1/8+\epsilon}. \tag{0.1.3}$$

The generalisation to number fields in the level aspect of the above results for π cuspidal was obtained by Venkatesh in [33]

$$L(1/2,\pi\times\chi)\ll_{\epsilon,F,\pi,C_\infty(\chi)}C_f(\chi)^{\frac{1}{2}-\frac{(1-2\theta)^2}{14-12\theta}+\epsilon},$$

and by Blomer and Harcos for totally real F in [6]

$$L(1/2,\pi\times\chi)\ll_{\epsilon,F,\pi,C_\infty(\chi)}C_f(\chi)^{\frac{1}{2}-\frac{1}{8}(1-2\theta)+\epsilon}.$$

Here and in all the following, θ is any constant towards the conjecture of Ramanujan-Petersson ($\theta=0$). We can take $\theta=7/64$ thanks to [4]. In their joint paper [28], Michel and Vankatesh gave a hybrid bound as (0.1.1), with δ uniform in all aspects.

0.2 An Application

Besides of being a consequence of the (generalized) Riemann hypothesis, the subconvexity problem is intimately related to various problems of equidistribution among which we mention one problem of Linnik. Let q be a ternary positive definite quadratic form with coefficients in $\mathbb{Z}$. For a positive integer d, define

$$\mathscr{R}_q(d)=\left\{(a,b,c)\in\mathbb{Z}^3\mid q(a,b,c)=d\right\},V_{q,1}(\mathbb{R})=\left\{(x,y,z)\in\mathbb{R}^3\mid q(x,y,z)=1\right\}.$$

We say that d is representable by q if $\mathscr{R}_q(d) \neq \emptyset$. We ask if $\frac{1}{\sqrt{|d|}}\mathscr{R}_q(d)$ is equidistributed as $d \to \infty$ among representable integers. We adelize this problem by introducing the quaternion algebra B associated with q. We then define the algebraic group $G = PB^{\times}$. To each d, we can associate a torus $T = T_d \subset G$ such that the above Linnik's problem is translated into the equidistribution of $T_d(\mathbb{Q})\backslash T_d(\mathbb{A})$ inside $G(\mathbb{Q})\backslash G(\mathbb{A})$ all projected to some quotient by some compact subgroup of $G(\mathbb{A})$. The Waldspurger's formula

$$|\int_{T(F)\backslash T(\mathbb{A})} e(t.g)dt|^2 = \frac{L(1/2,\pi')L(1/2,\pi' \times \chi_d)}{C_F} \prod_v \alpha_v(W_{e,v}, T_v, g_v)$$

then relates the problem to the subconvexity of $L(1/2,\pi' \times \chi_d)$, where π' is the Jacquet-Langlands lifting of the automorphic representation π of $G(\mathbb{A})$ which contains the pure tensor e, and χ_d is the quadratic Hecke character associated with the quadratic extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. In particular $C(\chi_d) = C_f(\chi_d) \asymp d$. The product of local terms $\alpha_v(W_{e,v}, T_v, g_v)$ decreases as $|d|^{-1/2}$. Therefore the convex bound is not sufficient for this application, and the subconvex bound as in [33] is already good enough. For more details, please consult [15].

0.3 Statement of the Main Results

Let π, π_1, π_2 be generic automorphic representations of $G(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A})$, where at least one of π_1, π_2 is cuspidal. Let χ be a Hecke character. Denote by $C(\pi)$ (resp. $C(\chi)$) the analytic conductor of π (resp. χ).

Ph. Michel and A.Venkatesh in [28] solved the subconvexity problem for GL_2. In fact, the main result of that paper is the existence of some $\delta > 0$ such that

$$L(1/2,\pi_1 \times \pi_2) \ll_{F,\epsilon,\pi_1} C(\pi_2)^{1/4-\delta+\epsilon}, \forall \epsilon > 0.$$

That is to say, if one fixes π_1, then we have subconvex bound for $L(1/2,\pi_1 \otimes \pi_2)$ as $C(\pi_2)$ tends to infinity. As a preliminary result, they also obtained the following subconvex bound

$$L(1/2,\pi \times \chi) \ll_{F,\epsilon,\pi} C(\chi)^{1/2-\delta+\epsilon}, \forall \epsilon > 0.$$

The main result is to give an explicit value for δ.

Theorem 0.3.1. *For any cuspidal automorphic representation π of $G(\mathbb{A})$ and any Hecke character χ of analytic conductor $C(\chi) = Q$, we have*

$$L(1/2,\pi \otimes \chi) \ll_{F,\epsilon,\pi} Q^{1/2-\delta+\epsilon}, \forall \epsilon > 0$$

with

$$\delta = \frac{1-2\theta}{8}.$$

Note that under the Ramanujan-Petersson conjecture ($\theta = 0$), $\delta = 1/8$.

Remark 0.3.2. *This bound, when $\theta = 0$, is called a Burgess-like bound in view of (0.1.2). The best known value $\theta = 7/64$ is due to Kim and Sarnak in [25] over $\mathbb{Q}$, and to Blomer and Brumley in [4] over an arbitrary number field.*

Remark 0.3.3. *In [7], Blomer, Harcos and Michel first established such a Burgess-like bound in the level aspect for $F = \mathbb{Q}$. It was then generalized in [5] and [6] by Blomer and Harcos to any totally real number field F. Theorem 2 of [5] ($\delta = 1/8$) is the best bound for $F = \mathbb{Q}$ in the level aspect. In the case $F = \mathbb{Q}$ and χ is quadratic, $\delta = 1/6$ was obtained by Conrey and Iwaniec as Corollary 1.2 of [14].*

Another result is

Theorem 0.3.4. *Let χ be a Hecke character of $\mathbb{A}^\times$ with analytic conductor $C(\chi) = Q$, we have*

$$L(1/2,\chi) \ll_{F,\epsilon} Q^{1/4-\delta+\epsilon}, \forall \epsilon > 0$$

with

$$\delta = \frac{1-2\theta}{24}.$$

0.4 Plan of the Thesis

Chapter 1 is concerned with some technical but fundamental aspects of the proof of Theorem 0.3.1:
In Section 1.1 we provide notations and conventions. In Sections 1.2 to 1.4 we recall how Hecke's theory can be extended from K-finite vectors to smooth vectors. In Section 1.5 we discuss Whittaker models and their norms. In Sections 1.6 and 1.7, we discuss various forms of the spectral decomposition of automorphic functions. In Section 1.8 we use results from the Section 1.5 to construct and study local test vectors to be used in the sequel. In Section 1.9 we discuss the decay of matrix coefficients of automorphic representations.

Chapter 2 complements the technical aspects needed for the the proof of Theorem 0.3.4:
In Section 2.1 we recall Hecke-Jacquet-Langlands theory of L-functions in the case of the Eisenstein series which concerns us. In Section 2.2 we generalize results from Section 1.7 to all compact reductive groups. This is practical for giving Sobolev inequalities on homogeneous spaces of such compact groups. In Section 2.3 we study the intertwining operator in unitary principal series representations. This will be used to control the constant term of our degenerate Eisenstein series.

The proofs of the main results are outlined in Chapter 3. In Section 3.1 we start the proof of Theorem 0.3.1 by first giving the intuition, then setting up the amplification method. We split to two sorts of arguments: local ones and the global one. Then we give more details for the global argument by stating all intermediate lemmas. Note that most of the lemmas are stated without proof. The details of proof follow in Chapter 4,5. In Section 3.2 we extend our method to treat the case of the Eisenstein series from $\pi(1,1)$. This follows exactly the discussion in Section 5.1.7 of [28]. In particular, it uses twice the argument given in Section 3.1.

Chapter 4&5 complement the discussion in Section 3.1. In Chapter 4 we deal with the local arguments and prove Proposition 3.1.1. In Chapter 5 we conclude the proof by putting local estimations into the global arguments.

Chapter 6 complements the discussion in Section 3.2. Section 6.1 presents our method of smoothly truncating an Eisenstein series and gives a bound of the Sobolev norms of the truncated function in terms of the height of truncation. Section 6.3 generalizes the machinery of treating contribution from constant terms of Section 5.2.

A suggested order of reading the paper is as follows: Chapter 0 - Chapter 1 - Section 3.1 - Chapter 4 - Chapter 5 - Chapter 2 - Chapter 6 - Section 3.2.

The difference in methods between this paper and [28] is explained in Remark 3.1.11.

1 Preliminaries: Cuspidal Case

1.1 Notations and Conventions

From now on, F is a number field of degree $r = [F:\mathbb{Q}] = r_1 + 2r_2$, where r_1 is the number of real places and r_2 is the number of complex places. V_F is the set of all places of F. For any $v \in V_F$, F_v is the completion of F at the place v. $\mathbb{A} = \mathbb{A}_F$ is the adele ring of F. $\mathbb{A}^\times$ is the idele group. We fix once for all an isometric section $\mathbb{R}_+ \to \mathbb{A}^\times$ of the adelic norm map $|\cdot| : \mathbb{A}^\times \to \mathbb{R}_+$, thus identify $\mathbb{A}^\times$ with $\mathbb{R}_+ \times \mathbb{A}^{(1)}$ where $\mathbb{A}^{(1)}$ is the kernel of the adelic norm map. We'll constantly identify $\mathbb{R}_+$ with its image under the section map. Let $F_\infty = \prod_{v|\infty} F_v$ and $F_\infty^{(1)}$ be the subgroup of $F_\infty^\times$ of adelic norm 1. $\mathbb{A}_f$ is the subring of finite adeles. $\mathbb{A}_f^\times$ is the unit group of $\mathbb{A}_f$.

We denote by $\psi = \prod_v \psi_v$ the additive character $\psi = \psi_{\mathbb{Q}} \circ Tr_{F/\mathbb{Q}}$, where $\psi_{\mathbb{Q}}$ is the additive character of $\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}$ taking $e^{2\pi i x}$ on $\mathbb{R}$. At each place $v \in V_F$, dx_v denotes a self-dual measure w.r.t. ψ_v. Note if $v < \infty$, then dx_v is the measure which gives the ring of integers $\mathcal{O}_v$ of F_v the measure $q_v^{-d_v/2}$, where q_v is the cardinal of the residue field of F_v, and $\prod_{v<\infty} q_v^{d_v}$ is the discriminant of F. We set $v(\psi) = -d_v$. Define $dx = \prod_{v \in V_F} dx_v$ on $\mathbb{A}$. The quotient measure on $F\backslash\mathbb{A}$ has total volume 1 (c.f. Proposition 7 [26] Chapter XIV). Define for $s \in \mathbb{C}$, if $v < \infty$ $\zeta_v(s) = (1 - q_v^{-s})^{-1}$, if v is real $\zeta_v(s) = \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, if v is complex $\zeta_v(s) = \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Take $d^\times x_v = \zeta_v(1)\dfrac{dx_v}{|x_v|}$ as the Haar measure on the multiplicative group $F_v^\times$, and $d^\times x = \prod_v d^\times x_v$ as the Haar measure on the idele group $\mathbb{A}^\times$. If χ is a character of $F_v^\times$ (resp. $\mathbb{A}^\times$), define the real part of χ to be the real number $\Re(\chi)$ satisfying $|\chi(t)| = |t|_v^{\Re(\chi)}, \forall t \in F_v^\times$ (resp. $|\chi(t)| = |t|^{\Re(\chi)}, \forall t \in \mathbb{A}^\times$).

Unless otherwise specified, $G = \mathrm{GL}_2$ as an algebraic group defined over F. Hence $G_v = \mathrm{GL}_2(F_v)$. If v is a complex place, then $K_v = \mathrm{SU}_2(\mathbb{C})$; if v is a real place, then $K_v = \mathrm{SO}_2(\mathbb{R})$; if $v < \infty$ then $K_v = G(\mathcal{O}_v)$. $Z_v = \left\{ z(u) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} : u \in F_v^\times \right\}$, $N_v = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F_v \right\}$,

$A_v = \left\{ a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} : y \in F_v^\times \right\}$. If $v|\infty$, we denote their Lie algebras by $\mathfrak{z}_v, \mathfrak{n}_v, \mathfrak{a}_v$, the Lie algebra of K_v by $\mathfrak{k}_v$ and the Lie algebra of G_v by $\mathfrak{g}_v$. We also write $B_v = Z_v N_v A_v$. If $g_v \in Z_v N_v a(y) K_v$, define the local height function $H_v(g_v) = |y|_v$. For $g = \prod'_v g_v$, define the global height function $H(g) = \prod_v H_v(g_v)$. The probability Haar measure on K_v is dk_v. Z_v(resp. N_v resp. A_v) is equipped with the measure $d^\times u$ (resp. dx resp. $d^\times y$). Consider the Iwasawa decomposition $G_v = Z_v N_v A_v K_v$, a Haar measure of G_v is given by $dg_v = d^\times u dx d^\times y/|y|_v dk_v$, which in fact gives $K_v \subset G_v$ the measure $q_v^{-d_v}$ for $v < \infty$. View $Z_v \backslash G_v$ as $N_v A_v K_v$, equipped with the measure $d\bar{g}_v = dx d^\times y/|y|_v dk_v$. The center of $G(\mathbb{A})$ is $Z = \prod_{v \in V_F} Z_v$. Denote $A = \prod_v A_v$. The quotient group $Z\backslash G(\mathbb{A})$ is equipped with the product measure $d\bar{g} = \prod_{v\in V_F} d\bar{g}_v$. The quotient measure on $X(F) = ZG(F)\backslash G(\mathbb{A})$ is also denoted by $d\bar{g}$, with total mass $\mathrm{Vol}(X(F))$. $K = \prod_{v \in V_F} K_v$ is equipped with the product measure $dk = \prod_v dk_v$. Write $K_\infty = \prod_{v|\infty} K_v$ and $K_f = \prod_{v<\infty} K_v$.

Given a Hecke character ω, $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ is the space of Borel functions φ satisfying

$$\forall \gamma \in G(F), \varphi(\gamma g) = \varphi(g); \forall z \in Z, \varphi(zg) = \omega(z)\varphi(g);$$

$$\|\varphi\|^2_{X(F)} = \int_{X(F)} |\varphi(\bar{g})|^2 d\bar{g} < \infty.$$

Let $L^2_0(G(F)\backslash G(\mathbb{A}), \omega)$ be the (closed) subspace of cusp forms $\varphi \in L^2(G(F)\backslash G(\mathbb{A}), \omega)$, satisfying

$$\int_{F\backslash \mathbb{A}} \varphi(n(x)g)dx = 0, a.e. g \in G(\mathbb{A}).$$

Denote by R the right regular representation of $G(\mathbb{A})$ on $L^2(G(F)\backslash G(\mathbb{A}), \omega)$, and by R_0 the subrepresentation $L^2_0(G(F)\backslash G(\mathbb{A}), \omega)$. We know that each irreducible component π of R decomposes into $\pi = \hat{\otimes}'_v \pi_v$ where π_v are irreducible unitary representations of G_v. $R = R_0 \oplus R_{res} \oplus R_c$ is the spectral decomposition. R_0 decomposes as a direct sum of irreducible $G(\mathbb{A})$-representations, whose components are called cuspidal representations. R_{res} is the sum of all one dimensional subrepresentations. R_c is a direct integral of irreducible $G(\mathbb{A})$-representations, expressed via Eisenstein series. Components of R_0 and R_c are the generic automorphic representations. Let θ be such that no complementary series representation with parameter $> \theta$ appears as a local component of a cuspidal representation. Recall that a principal series representation $\pi(\mu_1, \mu_2) = \mathrm{Ind}_{B(F_v)}^{G(F_v)}(\mu_1, \mu_2)$ with $|\mu_1\mu_2^{-1}(t)| = |t|_v^s, \forall t \in F_v$ is a complementary series if s is a non-zero real number in the interval $(-1, 1)$. $|s|/2$ is called its parameter.

A compact open subgroup $K'_f \subset G(\mathbb{A}_f)$ is said to be of (congruence) type 0 if for every finite place v, there is an integer m_v such that the local component

$$K_v' = K_v^0[m_v] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathscr{O}_v) \mid c \equiv 0 \mod \varpi_v^{m_v} \right\},$$

where ϖ_v is a uniformiser of the local field F_v. Let $\varphi \in \pi$ be a pure tensor vector in an automorphic representation. Suppose for every $v < \infty$, φ is invariant by $K_v^0[m_v]$ but not by $K_v^0[m_v - 1]$, then define $m_v = v(\varphi)$. Define $v(\pi) = v(\pi_v) = \min_{\varphi \in \pi_v} v(\varphi)$. The local conductor $C(\pi_v) = \varpi_v^{v(\pi_v)}$. We similarly define the principal congruence subgroups $K_v[n] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathscr{O}_v) \mid a-1, b, c, d-1 \equiv 0 \mod \varpi_v^n \right\}$.

For any semisimple (real) Lie group G, denote by $\mathscr{C}_G$ the Casimir element. In our case, $G = \mathrm{GL}_2$. At each place $v \mid \infty$, $Z_v \backslash G_v$ is semisimple, and $\Delta_v = -\mathscr{C}_{Z_v \backslash G_v} - 2\mathscr{C}_{K_v}$ is an elliptic operator on $Z_v \backslash G_v$. Note that here we calculate $\mathscr{C}_{K_v}$ by using the Killing form of $\mathrm{Lie}(Z_v \backslash G_v)$ instead of its own Killing form.

If A is a positive semi-definite operator on a Hilbert space V, we denote by A^s the operator which acts on the eigenspace V_λ for the eigenvalue λ of A by the multiplication by λ^s for $\forall s \geq 0$. Note that A need not have discrete spectrum, i.e. V_λ need not be a subspace of V. This notation concerns in particular $\Delta_v, \mathscr{C}_{K_v}$ etc.

1.2 L-function Theory for K-finite Vectors: Cuspidal Case

The proof of the fact that the representation of $G(\mathbb{A})$ on $L_0^2(G(F)\backslash G(\mathbb{A}), \omega)$ decomposes as a discrete direct sum of irreducible representations, as in Lemma 5.2 of [18], actually gives important information on K-finite vectors in an irreducible component π. They consequently have representatives in the space of smooth functions on the automorphic quotient, and are rapidly decreasing in any Siegel domain (Lemma 5.6 of [18]). Let the superscript "fin" mean "K-finite". The rapid decay is important, because it adds to the description of W_π^{fin}, the image of $\pi^{\mathrm{fin}} \subset \pi \subset L_0^2(G(F)\backslash G(\mathbb{A}), \omega)$ under the Whittaker intertwiner

$$\varphi \mapsto W_\varphi(g) = \int_{F\backslash \mathbb{A}} \varphi(n(x)g)\psi(-x)dx \tag{1.2.1}$$

the important growth property, which is essential for the uniqueness of Whittaker model at archimedean places (Section 2.8 and 4.4 of [2] for local uniqueness, Section 3.5 of [2] for global uniqueness). If φ has a prescribed K-type and is a pure tensor, i.e. $W_\varphi(g) = \prod_v W_{\varphi,v}(g_v)$ splits, $W_{\varphi,v}(a(y)k)$ is forced to have rapid decay at ∞, thus has nice behavior around 0

$$|W_{\varphi,v}(a(y)k)| \ll_\epsilon |y|_v^{1/2-\theta+\epsilon}, \forall \epsilon > 0. \tag{1.2.2}$$

Now let χ be a character of $F^\times \backslash \mathbb{A}^\times$ and $s \in \mathbb{C}$. Jacquet-Langlands [23] defined a functional on

π^{fin}, called the (global) zeta-functional :

$$\zeta(s,\varphi,\chi) = \int_{F^\times\backslash\mathbb{A}^\times} \varphi(a(y))\chi(y)|y|^{s-1/2}d^\times y, \forall \varphi\in\pi, a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.2.3}$$

Since $\varphi(a(y))$ is rapidly decreasing at ∞, it is also rapidly decreasing at 0 since

$$\varphi(a(y)) = \varphi(wa(y)) = \omega(y)\cdot w.\varphi(a(y^{-1})), w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Thus $\zeta(s,\varphi,\chi)$ is well defined for all s, and the functional equation characterizes the left invariance by w of φ

$$\zeta(s,\varphi,\chi) = \zeta(1-s,w.\varphi,\omega^{-1}\chi^{-1}). \tag{1.2.4}$$

If φ is a pure tensor in $\pi^{\mathrm{fin}} \simeq \otimes'_v \pi_v^{\mathrm{fin}}$, i.e. W_φ factorizes, then since

$$\varphi(g) = \sum_{t\in F^\times} W_\varphi(a(t)g), \tag{1.2.5}$$

we can see

$$\zeta(s,\varphi,\chi) = \prod_v \zeta(s,W_{\varphi,v},\chi_v,\psi_v), \Re(s) > 1+\theta$$

with

$$\zeta(s,W_{\varphi,v},\chi_v,\psi_v) = \int_{F_v^\times} W_{\varphi,v}(a(y))\chi(y)|y|^{s-1/2}d^\times y.$$

The convergence is justified by the above local growth property of $W_{\varphi,v}$ and the fact that on an unramified finite place v, the local zeta-function equals

$$\zeta(s,W_{\varphi,v},\chi_v,\psi_v) = L(s,\pi_v\otimes\chi_v) = (1-\mu_v\chi_v(\varpi_v)q_v^{-s})^{-1}(1-\nu_v\chi_v(\varpi_v)q_v^{-s})^{-1}$$

where $\pi_v = \mathrm{Ind}_{B(F_v)}^{G(F_v)}(\mu_v,\nu_v)$ determines μ_v,ν_v.

The analysis of local zeta-functions shows that $\zeta(s,W_{\varphi,v},\chi_v,\psi_v)$, as $W_{\varphi,v}$ varies over $W_{\pi,v}^{\mathrm{fin}}$, have a "common divisor" $L(s,\pi_v\otimes\chi_v)$, which is a meromorphic function on s such that $\dfrac{\zeta(s,W_{\varphi,v},\chi_v,\psi_v)}{L(s,\pi_v\otimes\chi_v)}$, originally defined for $\Re(s)>\theta$, can be analytically continued into an entire function on $s\in\mathbb{C}$, and equals 1 for almost all places v. Furthermore, there is a functional equation

$$\frac{\zeta(s,W_{\varphi,v},\chi_v,\psi_v)}{L(s,\pi_v\otimes\chi_v)}\epsilon(s,\pi_v,\chi_v,\psi_v) = \frac{\zeta(1-s,wW_{\varphi,v},\omega_v^{-1}\chi_v^{-1},\psi_v)}{L(1-s,\pi_v\otimes\omega_v^{-1}\chi_v^{-1})} \tag{1.2.6}$$

where $\epsilon(s,\pi_v,\chi_v,\psi_v)$ is an entire function of exponential type. Define usual and completed L-functions as

$$L(s,\pi\otimes\chi) = \prod_{v<\infty} L(s,\pi_v\otimes\chi_v), \Lambda(s,\pi\otimes\chi) = \prod_v L(s,\pi_v\otimes\chi_v), \Re(s)>1+\theta$$

then the analytic continuations and functional equations of these L-functions follow from the well-definedness of $\zeta(s,\varphi,\chi)$ and (1.2.4), (1.2.6). The identity

$$\zeta(s,\varphi,\chi) = L(s,\pi\otimes\chi)\prod_{v\mid\infty}\zeta(s,W_{\varphi,v},\chi_v,\psi_v)\prod_{v<\infty}\frac{\zeta(s,W_{\varphi,v},\chi_v,\psi_v)}{L(s,\pi_v\otimes\chi_v)}$$

can be evaluated at $s=1/2$ without analytic continuation of any integral. Thus

$$L(1/2,\pi\otimes\chi) = \prod_{v\mid\infty}\zeta(1/2,W_{\varphi,v},\chi_v,\psi_v)^{-1}\cdot\prod_{v<\infty}\frac{L(1/2,\pi_v\otimes\chi_v)}{\zeta(1/2,W_{\varphi,v},\chi_v,\psi_v)}\cdot\zeta(1/2,\varphi,\chi). \qquad (1.2.7)$$

1.3 Smooth Vectors in Different Models

For any Lie group G and a unitary representation (ρ,V) of G, let ρ^∞ be the subspace of smooth vectors in V. This is naturally a Fréchet space, defined by the semi-norms $\|X.v\|, X\in U(\mathfrak{g})$. If $V\subset L^2(M)$ is realized as a space of functions on a orientable real manifold M equipped with a smooth (right) G-action, and with a G-invariant volume form, then we can talk about Sobolev functions for the action. Note that the action $\rho: G\to U(V)$ need not coincide with the regular representation on $L^2(M)$ induced by the action of G on M. One may think about $\rho=\pi(\mu_1,\mu_2)$ in the principal unitary series of $G=\mathrm{GL}_2(\mathbb{R})$.

Definition 1.3.1. *With the above notations, a function f on M is called Sobolev (for the G-action), if it is smooth for the differential structure of M, and if its class $[f]$ in $V\subset L^2(M)$ is a smooth vector. We write V^∞ or $\rho^{\mathrm{nam},\infty}$, if* nam *is the name of the model, or just ρ^∞ if the underlying model is clear, for the space of Sobolev functions.*

We obviously have $[\rho^{\mathrm{nam},\infty}]\subset\rho^\infty$. Reciprocally

Lemma 1.3.2. *Assume that:*
1. For any $p\in M$, the map $s_p: G\to M, g\mapsto p.g$ is a submersion at the identity $e\in G$.
2. The action of any element $X\in\mathfrak{g}$ on $V\cap C^\infty(M)$ corresponds to a smooth vector field $v(X)$ on M.
Then every vector $v\in\rho^\infty\subset L^2(M)$ has a representative in $C^\infty(M)$.

Definition 1.3.3. *Fix a basis $\mathscr{B}$ of $\mathfrak{g}$, for any positive integer $d>0$, one can define a Sobolev norm on ρ^∞ by*

$$S_d^\rho(v) = \max_{X_i\in\mathscr{B}, l\le d}\|X_1\ldots X_l.v\|.$$

In fact, since the condition and the conclusion are of local nature, one may interpret everything on the open set C_p of some euclidean space, diffeomorphic to some open neighborhood U_p of some point $p\in M$. The assumptions 1,2 ensures that the Sobolev norms S_d^ρ are equivalent to the usual Sobolev norms on C_p in the underlying euclidean space. One can apply the classical Sobolev embedding theorem.

Corollary 1.3.4. *Under the assumptions of the above lemma, for any $p \in M$, there is an integer d s.t. $\forall f \in L^2(M) \cap \rho^\infty$,*

$$\sup_{q \in U_p} |f(q)| \ll_{p,U_p} S_d^\rho([f]).$$

The assumptions of the above lemma apply in the following situations:
– $\rho \subset R$ is a subrepresentation of the right regular representation on automorphic quotient space. In such a situation, we say that ρ is realized in the automorphic model: "aut".
– $\rho = \pi$ is locally in principal unitary series with induced model, we say that ρ is realized in the induced model: "ind".
– $\rho = W_\pi$ is the Whittaker model of a generic automorphic representation π. We say it is realized in the Whittaker model.
– $\rho = K_\pi$ is the Kirillov model of a generic automorphic representation π. We say it is realized in the Kirillov model.

Definition 1.3.5. *If G is a totally disconnected group, acting on a totally disconnected space M, then a function f on M is said to be smooth, if it is locally constant on M and K-finite for any maximal compact subgroup K of G.*

1.4 Smooth Vectors and Extended L-function Theory

We generalize the theory of L-function to smooth vectors. Use Corollary 1.3.4 and compactness of $F\backslash\mathbb{A}$, one may easily see (Corollary I.1.5 [12]) that the Whittaker functional

$$l : R^\infty \to \mathbb{C}, \varphi \mapsto W_\varphi(1)$$

is in the continuous dual space of R^∞ verifying

$$l(R(n(x)\varphi)) = \psi(x) l(\varphi)$$

and is related to the Whittaker intertwiner (1.2.1) by

$$W_\varphi(g) = l(R(g).\varphi).$$

When we restrict to an irreducible component π of R, or more precisely to $\otimes'_v \pi_v^\infty \subset \pi^\infty$, it splits as

$$l = \otimes'_v l_v$$

where l_v are local (continuous) Whittaker functionals of π_v^∞ verifying

$$l(n(x)w) = \psi_v(x) l(w), w \in \pi_v^\infty.$$

The study of $l_v, v < \infty$ is the same as in the K_v-finite case. So the uniqueness, the local functional equation (1.2.6), the rapid decay and the controlled behavior at 0 (1.2.2) remain valid. For a $v|\infty$, the uniqueness of l_v is established by Shalika [31]. So one can define the smooth Whittaker model associated with a unitary irreducible representation π_v by

$$W^{\infty}_{\pi_v} = \{W_w(g) = l_v(\pi_v(g)w); w \in \pi^{\infty}_v\} \tag{1.4.1}$$

as well as its smooth Kirillov model

$$K^{\infty}_{\pi_v} = \{K_w(y) = W_w(a(y)); w \in \pi^{\infty}_v\}. \tag{1.4.2}$$

The rapid decay at infinity of the local Whittaker functions $W_w(g)$ can be found in Lemma I.1.2 [12]. Note that here, the rapid decay property is derived from the continuity of l_v. In fact, much more information is obtained by Jacquet, as a special case in Proposition 3.6 [11], where the behavior of $W_w(g)$ is completely characterized, which implies rapid decay and (1.2.2) in this situation. Consequently the rapid decay of $\varphi \in \otimes'_v \pi^{\infty}_v \subset \pi^{\infty} \subset R^{\infty}_0$ follows, by using (1.2.5). Furthermore, local functional equations (1.2.6) are obtained by Jacquet [22] with absolute convergence for $\Re(s) > \theta$ as in K_v-finite case.

Remark 1.4.1. *For a proof that rapid decay at infinity and local functional equation imply the controlled behavior at* 0*, see Proposition 3.2.3 [28].*

1.5 An Identification of Norms

A by-product of the above theory, already known in the K-finite case, is the identification of the norm on $\pi \subset R_0$ and the natural norm we put on global Whittaker models.

Lemma 1.5.1. *If* $\pi = \hat{\otimes}'_v \pi_v \subset R_0$ *and* $\varphi \in \otimes'_v \pi^{\infty}_v$ *is a pure tensor, then*

$$\|\varphi\|^2_{X(F)} = \frac{(\mathrm{disc}F)^{3/2}\Lambda^*(1,\pi\times\bar{\pi})}{\Lambda_F(2)} \prod_{v\in V_F} \frac{\zeta_v(2)\int_{F_v^{\times}\times K_v} |W_{\varphi,v}(a(y)k)|^2 d^{\times}y\,dk}{L(1,\pi_v\times\bar{\pi}_v)}$$

where Λ_F *is the complete Dedekind zeta-function,* $\Lambda(s,\pi\times\bar{\pi}) = \prod_{v\in V_F} L(s,\pi_v\times\bar{\pi}_v)$ *is the completed* L*-function associated with* $\pi\times\bar{\pi}$ *and* $\Lambda^*(1,\pi\times\bar{\pi})$ *is its residue at* 1*.*

Remark 1.5.2. *By [21],* $C(\pi)^{-\epsilon} \ll L^*(1,\pi\times\bar{\pi}) \ll C(\pi)^{\epsilon}$*. Here* $C(\pi) = C_{\infty}(\pi)C_f(\pi)$ *is the ananlytic conductor of* π*.* $C_f(\pi)$ *is the normal conductor.* $L(s,\pi\times\bar{\pi}) = \prod_{v<\infty} L(s,\pi_v\times\bar{\pi}_v)$ *is the incomplete Rankin-Selberg* L*-function.*

The proof of Lemma 1.5.1 is a standard use of Rankin-Selberg's method (c.f. [28] 4.4.2) : Unfold, for $\Re s \gg 1$

$$\int_{ZG(F)\backslash G(\mathbb{A})} \varphi(g)\bar{\varphi}(g)E(s,f)(g)\,d\bar{g}$$

to get

$$\int_{\mathbb{A}^{\times}\times K} |W_{\varphi}(a(y)k)|^2 f_s(a(y)k)|y|^{-1} d^{\times}y\,dk$$

$$= \int_{\mathbb{A}^\times \times K} |W_\varphi(a(y)k)|^2 |y|^{s-1} d^\times y dk$$

where $f_s \in \pi(|\cdot|^{s-1/2}, |\cdot|^{1/2-s})$ is a spherical flat section, and

$$E(s,f)(g) = \sum_{\gamma \in B(F)\backslash G(F)} f_s(\gamma g). \tag{1.5.1}$$

Then take the residue at $s = 1$. In fact, $E(s,f)$ converges for $\Re(s) > 1$, has a meromorphic continuation to all $s \in \mathbb{C}$, and is of moderate growth for any given s (see for example section 3.7 of [2]). On an unramified place v, for a spherical $W_{\varphi,v}$, one has

$$\frac{\zeta_v(2s) \int_{F_v^\times \times K_v} |W_{\varphi,v}(a(y)k)|^2 |y|_v^{s-1} d^\times y dk}{L(s, \pi_v \times \bar{\pi}_v)} = |W_{\varphi,v}(1)|^2 \tag{1.5.2}$$

which is 1 for a.e.v. The product $\prod_{v \in V_F} L(s, \pi_v \times \bar{\pi}_v)$ converges for $\Re(s) > 1$. Thus

$$\int_{ZG(F)\backslash G(\mathbb{A})} \varphi(g)\bar{\varphi}(g) E(s,f)(g) d\bar{g} = \frac{(\mathrm{disc} F)^{3/2}}{\Lambda_F(2s)} \Lambda(s, \pi, Ad)$$

$$\cdot \prod_{v \in V_F} \frac{\zeta_v(2s) \int_{F_v^\times \times K_v} |W_{\varphi,v}(a(y)k)|^2 |y|_v^{s-1} d^\times y dk}{L(s, \pi_v \times \bar{\pi}_v)}, \Re s > 1.$$

By the local behavior (1.2.2), one can evaluate the integrals on the right of $s = 1$. We can do more by taking into account the theory of Kirillov model.

Define $B_1(F_v) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in F_v^\times, b \in F_v \right\}$.

Proposition 1.5.3. *There are only two types of unitary irreducible representations of $B_1(F_v)$:*
1. A character of $F_v^\times \simeq B_1(F_v)/N_v$.
2. The representation of $B_1(F_v)$ on $L^2(F_v^\times)$ defined by the formula : $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f(x) = \psi(bx) f(ax)$, where ψ is a nontrivial character of F_v. Different ψ give equivalent representations. In particular, there is only one non one-dimensional unitary irreducible representation of $B_1(F_v)$. Let's denote this model by $\sigma(\psi)$.

A riguous proof of this proposition, in the case of an archimedean field, can be found in [27] Page 34 (29); and in the case of a non archimedean filed, can be found in [8] Chapter 8.

We finally deduce:

Proposition 1.5.4. *Let π be the local component on v of a generic automorphic representation. For a $W \in W_\pi^\infty$ in the Whittaker model one actually has*

$$\int_{F_v^\times \times K_v} |W(a(y)k)|^2 d^\times y dk = \int_{F_v^\times} |W(a(y))|^2 d^\times y.$$

As a consequence, the formula in Lemma 1.5.1 becomes

$$\|\varphi\|_{X(F)}^2 = \frac{(\mathrm{disc}F)^{1/2}\Lambda^*(1,\pi,Ad)}{\Lambda_F(2)} \prod_{v\in V_F} \frac{\zeta_v(2)\int_{F_v^\times}|W_{\varphi,v}(a(y))|^2 d^\times y}{L(1,\pi_v\times\bar{\pi}_v)}.$$

Remark 1.5.5. *The norm identifications actually justify the notations $W_{\pi_v}^\infty$ and $K_{\pi_v}^\infty$ as smooth vectors in their completions W_{π_v} and K_{π_v}.*

We have similar relation for Eisenstein series.

Proposition 1.5.6. *If $\pi = \pi(\chi_1,\chi_2)$ is (unitary) Eisenstein, and $\varphi(g) = E(0,f)(g)$ with $E(s,f)(g)$ defined as in (1.5.1), for some $f = \prod_v f_v \in \pi^{ind,fin}$ in the induced model, then one can define the Eisenstein norm of φ by*

$$\|\varphi\|_{Eis}^2 = \int_K |f(k)|^2 dk.$$

The following relation holds

$$(\mathrm{disc}F)^{1/2}\prod_{v\in V_F}\frac{\zeta_v(2)}{\zeta_v(1)^2}\int_{F_v^\times}|W_{\varphi,v}(a(y))|^2 d^\times y = \|\varphi\|_{Eis}^2,$$

and the local data are defined as the analytic continuation in (χ_1,χ_2) of

$$W_{\varphi,v}(g) = W_{f,v}(g) = \int_{F_v} f_v(wn(x)g)\psi_v(-x)dx.$$

Remark 1.5.7. *One can interpret $W_{\varphi,v}(a(y))\chi_{2,v}(y)^{-1}|y|^{-1/2}$ as the Fourier transform of $x \mapsto f(wn(x))$. The above norm identification is then a formal consequence of Plancherel formula as discussed in 3.1.6 of [28]. One can also use Theorem 4.6.5 of [2].*

1.6 Spectral Decomposition

The spectral decomposition, in the L^2 sense, is established in the first 4 sections of [20], which gives

$$R = \bigoplus_{\pi \text{ cuspidal}} \pi \oplus \int_{-i\infty}^{i\infty} \bigoplus_{\xi\in\widehat{F^\times\backslash\mathbb{A}^{(1)}}} \pi_{s,\xi}\frac{ds}{4\pi i} \oplus \bigoplus_{\chi\in\widehat{F^\times\backslash\mathbb{A}^\times},\chi^2=\omega} \chi\circ\det \tag{1.6.1}$$

where, $\pi_{s,\xi} = \pi(\xi|\cdot|^s, \omega\xi^{-1}|\cdot|^{-s})$. Note that $\pi_{s,\xi} \simeq \pi_{-s,\omega\xi^{-1}}$. According to Proposition I.1.4 of [11], the above spectral decomposition has an analogy for smooth vectors, namely

$$R^\infty = \bigoplus_{\pi \text{ cuspidal}} \pi^\infty \oplus \int_{-i\infty}^{i\infty} \bigoplus_{\xi\in\widehat{F^\times\backslash\mathbb{A}^{(1)}}} \pi_{s,\xi}^\infty\frac{ds}{4\pi i} \oplus \bigoplus_{\chi\in\widehat{F^\times\backslash\mathbb{A}^\times},\chi^2=\omega} \chi\circ\det \tag{1.6.2}$$

with convergence in the topology of R^∞. We are going to establish

Theorem 1.6.1. *Suppose $\varphi \in R^\infty$, viewed as a function on $G(\mathbb{A})$, then the following decomposition*

$$\varphi(g) = \sum_{\chi \in \widehat{F^\times \backslash \mathbb{A}^\times}, \chi^2=\omega} \frac{\langle \varphi, \chi \circ \det \rangle}{\mathrm{Vol}(X(F))} \chi \circ \det(g) + \sum_{\pi \ cuspidal} \sum_{e \in \mathscr{B}(\pi)} \langle \varphi, e \rangle e(g)$$

$$+ \sum_{\xi \in \widehat{F^\times \backslash \mathbb{A}^{(1)}}} \sum_{\Phi \in \mathscr{B}(\pi_{s,\xi})} \int_{-i\infty}^{i\infty} \langle \varphi, E(s,\Phi) \rangle E(s,\Phi)(g) \frac{ds}{4\pi i}$$

converges absolutely and uniformly on any compact subset, where $\mathscr{B}(?)$ means taking a basis of $?$ consisting of K-isotypical pure tensors. We may assume that if φ is $K_v[n_v]$-invariant, then every function appearing at the right hand side is $K_v[n_v]$-invariant for any finite place v. K_v need not be the standard maximal compact subgroup of G_v.

Remark 1.6.2. *Therefore, the sum $\sum_{\xi \in \widehat{F^\times \backslash \mathbb{A}^{(1)}}}$ is actually finite and the number depends only on F and n_v's.*

If we consider the theory of Whittaker model as a theory of spectral decomposition with respect to the left action of $N(\mathbb{A})$, then we further have

Theorem 1.6.3. *Conditions are the same as in the above theorem. $\varphi \in R^\infty$, as functions on $G(\mathbb{A})$:*

$$\varphi(g) = \varphi_N(g) + \sum_{\pi \ cuspidal} \sum_{e \in \mathscr{B}(\pi)} \langle \varphi, e \rangle \sum_{\alpha \in F^\times} W_e(a(\alpha)g) +$$

$$\sum_{\xi \in \widehat{F^\times \backslash \mathbb{A}^{(1)}}} \sum_{\Phi \in \mathscr{B}(\pi_{s,\xi})} \int_{-i\infty}^{i\infty} \langle \varphi, E(s,\Phi) \rangle \sum_{\alpha \in F^\times} W_{\Phi,s}(a(\alpha)g) \frac{ds}{4\pi i}$$

converges absolutely and uniformly on any given Siegel domain.

Remark 1.6.4. *In practice, the basis $\mathscr{B}(?)$ will be chosen so that the components of its elements at some archimedean place v are K_v-isotypic where K_v is the standard maximal compact subgroup of G_v.*

We begin with some local Sobolev type analysis.

1.6.1 Local Bounds of K-isotypical Functions

Lemma 1.6.5. *Let v be a finite place, and π be a unitary irreducible representation of G_v. Suppose $W \in W_\pi^\infty$, the smooth Whittaker model of π w.r.t. ψ_v, is invariant by $K_v[m]$, then we have the following Sobolev inequality*

$$|W(na(y)k)|^2 Vol(1+\varpi_v^m \mathcal{O}_v) \leq \|W\|^2 1_{v(y) \geq v(\psi)-m}, n \in N_v, y \in F_v^\times, k \in K_v$$

with the convention $1+\varpi_v^0\mathcal{O}_v=\mathcal{O}_v^\times$*. On an unramified place* ($m=0$)*, recall that*

$$W(na(\varpi_v^l)k)=q_v^{-l/2}\frac{\alpha_1^{l+1}-\alpha_2^{l+1}}{\alpha_1-\alpha_2}1_{l\geq 0}$$

for some $|\alpha_1\alpha_2|=1, q_v^{-\theta}\leq|\alpha_1|\leq q_v^\theta$

We leave the proof to the reader.

Lemma 1.6.6. *Let* v *be a real place, and* π *be a unitary irreducible representation of* G_v *with central character* ω*. If* $W\in W_\pi^\infty$*, then*

$$\forall n\in N(\mathbb{R}), y\in\mathbb{R}^\times, k\in SO_2(\mathbb{R}), N\equiv 1 \pmod 2, N>0$$

$$|W(na(y)k)|\ll_{N,\epsilon}|y|^{-N}\max(|y|^\epsilon,|y|^{-1})S_{N+1}^\pi(W).$$

Suppose further $W\in W_\pi^{\mathrm{fin}}$ *transforms under the action of* $K_v=SO_2(\mathbb{R})$ *accroding to the character*

$$\kappa_\alpha=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\mapsto e^{im\alpha}.$$

Then we have the following Sobolev inequality, uniform in m*,*

$$|W(na(y)k)|\ll_{N,\omega,\theta}|y|^{-N}\max(|y|^\epsilon,|y|^{-1})\lambda_W^{N'}\|W\|$$

where λ_W *is the eigenvalue for* W *of the elliptic operator* $\Delta_v=-\mathscr{C}_{G_v}+2\mathscr{C}_{K_v}$*, and* N' *depends only on* N *and* ω*.*

Let $U=\begin{pmatrix}1 & \\ & 0\end{pmatrix}$, $T=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ be elements in the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$, then

$$T.W(a(y))=-2\pi iyW(a(y)), U.W(a(y))=y\frac{\partial}{\partial y}W(a(y)).$$

We may only consider the case $y\in\mathbb{R}_+^\times$. Then for $\forall x,y\in\mathbb{R}_+^\times$, we have

$$(-2\pi iy)^NW(a(y))=T^N.W(a(x))+\int_x^y UT^N.W(a(u))d^\times u.$$

Note that

$$|\int_x^y UT^N.W(a(u))d^\times u|\leq(\int_x^y|UT^N.W(a(u))|^2d^\times u)^{1/2}(\int_x^y d^\times u)^{1/2}$$

$$\leq\|UT^N.W\||\log(y/x)|^{1/2}.$$

Thus

$$|(-2\pi iy)^NW(a(y))|\leq|T^N.W(a(x))|+\|UT^N.W\||\log(y/x)|^{1/2}.$$

Integrating against $\min(x, 1/x)dx/x$ for $0 < x < \infty$, using Cauchy-Schwarz and $\sqrt{1/2}(\sqrt{a}+\sqrt{b}) \leq \sqrt{a+b}$, we get

$$2|(-2\pi i y)^N W(a(y))| \leq \|T^N.W\| + \|UT^N.W\| \int_0^\infty \min(x, 1/x)|\log(y/x)|^{1/2} d^\times x.$$

Using the bound $|\log t| \ll_\epsilon \max(t^\epsilon, t^{-\epsilon})$, we get

$$2|(-2\pi i y)^N W(a(y))| \ll_\epsilon \|T^N.W\| + \|UT^N.W\| \max(|y|^\epsilon, |y|^{-1}).$$

Thus the first inequality follows for $k = 1$. The general case follows by noting $S^\pi_{N+1}(k.W) \ll_N S^\pi_{N+1}(W)$, since the adjoint action of K on $\mathfrak{g}$ has bounded coefficients.

The second follows from the equivalence of two system of Sobolev norms, one is S^π_d's, the other is defined with Δ_v and $I \in Z(\mathfrak{g})$. The proof is technical. We give it in the next section (Theorem 1.7.1).

Before proceding to the complex place case, let's first recall that the irreducible representations of $\mathrm{SU}_2(\mathbb{C})$ are parametrized by $m \in \mathbb{N}$, denoted by (ρ_m, V_m). Here V_m is the space of homogeneous polynomials in $\mathbb{C}[z_1, z_2]$ of degree $m+1$, equipped with the inner product

$$\langle P_1, P_2 \rangle = \int_{|z_1|^2+|z_2|^2 \leq 1} P_1(z_1, z_2)\overline{P_2(z_1, z_2)} dz_1 dz_2.$$

The action of $\mathrm{SU}_2(\mathbb{C})$ is given by

$$u.P(z_1, z_2) = P((z_1, z_2).u).$$

Let $P_{m,k}(z_1, z_2)$ be a multiple of $z_1^{m-k} z_2^k$, normalized s.t. they form an othonormal basis of V_m. Now let π be a unitary irreducible representation of $G(\mathbb{C})$. Let $W_{m,k} \in W^{\mathrm{fin}}_\pi$ span the ρ_m-isotypical subspace, with $W_{m,k}$ corresponding to $P_{m,k}$. Since ρ_m is unitary, we have the following relation

$$\sum_{k=0}^m |W_{m,k}(gu)|^2 = \sum_{k=0}^m |W_{m,k}(g)|^2, \forall u \in \mathrm{SU}_2(\mathbb{C}).$$

Therefore, we only need to bound $W_{m,k}(a(y))$ in order to bound $W_{m,k}(g)$. This works exactly as in the real place case. We omit the proof.

Lemma 1.6.7. *Let v be a complex place, and π be a unitary irreducible representation of G_v with central character ω. If $W \in W^\infty_\pi$, then*

$$\forall n \in N(\mathbb{C}), y \in \mathbb{C}^\times, k \in SU_2(\mathbb{C}), N \in \mathbb{N}$$

$$|W(na(y)k)| \ll_{N,\epsilon} |y|_v^{-N} \max(|y|_v^\epsilon, |y|_v^{-1/2}) S^\pi_{2N+2}(W).$$

Suppose further $W \in W^{\mathrm{fin}}_\pi$ transforms under the action of $K_v = SU_2(\mathbb{C})$ accroding to ρ_m and *corresponds to some $P_{m,k}$. Then we have the following Sobolev inequality, uniformly in m,*

$$|W(na(y)k)| \ll_{N,\omega,\theta} |y|_v^{-N} \max(|y|_v^{\epsilon}, |y|_v^{-1/2}) \lambda_W^{N'} \|W\|$$

where λ_W is the eigenvalue for W of the elliptic operator $\Delta_v = -\mathscr{C}_{G_v} + 2\mathscr{C}_{K_v}$, and N' depends only on N and ω.

1.6.2 Proof of Theorems 1.6.1, 1.6.3

We first deal with the cuspidal parts in the equations of Theorems 1.6.1, 1.6.3.

Let $e \in \pi \subset R_0$ be a K-isotypic vector, with local Whittaker model $W_{e,v}$. Denote by n_v the K_v-type of $W_{e,v}$ (or the weight & level), i.e.

– if $v < \infty$, then $W_{e,v}$ is $K_v[n_v]$-invariant. For a.e. v, $n_v = 0$.

– if v is a real place, then $W_{e,v}$ transforms under $\mathrm{SO}_2(\mathbb{R})$ as $e^{in_v\alpha}$.

– if v is a complex place, then $W_{e,v}$ transforms under $\mathrm{SU}_2(\mathbb{C})$ as some $P_{n_v,k}$.

Collecting all the estimations in the previous subsection, using Lemma 1.5.1 or Proposition 1.5.4 with $\|e\| = 1$ and $C_\infty(\pi) \ll \lambda_{e,\infty} = \prod_{v|\infty} \lambda_{e,v}, C_f(\pi) \leq \prod_{v<\infty} q_v^{n_v}$ we obtain

$$W_e(na(y)k) \ll_{F,N,\epsilon} |y|_\infty^{-N} \lambda_{e,\infty}^{N'} (\prod_{v<\infty} q_v^{n_v})^{\epsilon} \prod_{v<\infty, n_v\neq 0} L(1, \pi_v \times \bar{\pi}_v) \mathrm{Vol}(1 + \varpi_v^{n_v} \mathscr{O}_v)^{-1}$$

$$\cdot \prod_{v<\infty} 1_{v(y)\geq v(\psi)-n_v}, \text{where } |y|_\infty = \prod_{v|\infty} |y|_v.$$

The term $\prod_{v<\infty, n_v\neq 0} L(1, \pi_v \times \bar{\pi}_v) \mathrm{Vol}(1 + \varpi_v^{n_v} \mathscr{O}_v)^{-1}$ can be bounded from above by a constant depending only on $n_v, v < \infty$, we thus get

$$W_e(na(y)k) \ll_{F,N,\epsilon,(n_v)_{v<\infty}} \lambda_{e,\infty}^{N'} |y|_\infty^{-N} \prod_{v<\infty} 1_{v(y)\geq v(\psi)-n_v}.$$

Now since

$$e(na(y)k) = \sum_{\alpha\in F^\times} W_e(a(\alpha)na(y)k) = \sum_{\alpha\in F^\times} W_e(n'a(\alpha y)k), n' = a(\alpha)na(\alpha)^{-1}$$

we have

$$\sum_{\alpha\in F^\times} |W_e(a(\alpha)na(y)k)| \ll_{F,N,\epsilon} C(n_v, v<\infty) \lambda_{e,\infty}^{N'} \sum_{\alpha\in F^\times} |\alpha y|_\infty^{-N} \prod_{v<\infty} 1_{v(\alpha y)\geq v(\psi)-n_v}.$$

Consider the splitting $\mathbb{A}^\times \simeq \mathbb{A}^1 \times \mathbb{R}_+$ and write $y = y_1 t$ s.t. $y_1 \in \mathbb{A}^1$ and $t \in \mathbb{R}_+ \hookrightarrow \mathbb{A}^\times$ with trivial finite components. We need only consider y_1 in a fundamental domain of $F^\times \backslash \mathbb{A}^1$. Since the quotient $F^\times \backslash \mathbb{A}^1$ is compact, we may assume that there exist $0 < c < C$ s.t. for any place v,

$c \le |y_{1,v}|_v \le C$ and for a.e. v, say $\forall v > v_0$, $|y_{1,v}|_v = 1$. So the condition imposed in $\prod_{v<\infty}$ implies $|\alpha|_v \le c^{-1} q_v^{n_v - v(\psi)}$ and $|\alpha|_v \le 1, \forall v > v_0$ (one may choose v_0 big enough depending only on n_v's) in oder to get a non zero contribution. Thus, α runs over the non zero elements in a lattice of F_∞ depending only on n_v's. Therefore

$$\sum_{\alpha \in F^\times} |\alpha y|_\infty^{-N} \prod_{v<\infty} 1_{v(\alpha y) \ge v(\psi) - n_v} \ll_{n_v, v<\infty} |y|_\infty^{-N} \ll_{F,N} |y|^{-N}.$$

We conclude

$$\sum_{\alpha \in F^\times} |W_e(a(\alpha) n a(y) k)| \ll_{F,N,n_v,v<\infty} \lambda_{e,\infty}^{N'} |y|^{-N}. \tag{1.6.3}$$

Now let's turn to the Eisenstein parts of Theorems 1.6.1, 1.6.3.

Using Lemma 1.5.6 instead of 1.5.1 in the above argument, we get

$$\sum_{\alpha \in F^\times} |W_{\Phi,s}(a(\alpha) n a(y) k)| \ll_{F,N,n_v,v<\infty} \lambda_{\Phi,s,\infty}^{N'} |y|^{-N}. \tag{1.6.4}$$

We have an expression for the constant term

$$E(s,\Phi)_N(g) = \Phi_s(g) + M(s)\Phi_s(g).$$

$\Phi_s|_K$ belongs to some irreducible component σ of $Res_K^{G(\mathbb{A})} \pi_{s,\xi} = \mathrm{Ind}_{K \cap B(\mathbb{A})}^K (\xi, \omega\xi^{-1})$. From basic representation theory, it is easy to see that

$$\Phi_s(k) = \sqrt{\dim \sigma} < \sigma(k).v, v_0 >_\sigma, \ v, v_0 \in \sigma \text{ of norm } 1, \sigma(b).v_0 = (\xi, \omega\xi^{-1})(b).v_0.$$

Thus follows the bound ($\Re(s) = 0$)

$$|\Phi_s(n a(y) k)| = |y|^{1/2} |\Phi_s(k)| \le |y|^{1/2} \sqrt{\dim \sigma} \ll_{n_v, v<\infty} |y|^{1/2} \lambda_{K_\infty}(\Phi)^{1/2}$$

where $\lambda_{K_\infty}(\Phi)$ is the eigenvalue of Φ for the Casimir of K_∞. Note that $M(s)$ is unitary for $s \in i\mathbb{R}$ and doesn't change the K-type, thus

$$|M(s)\Phi_s(n a(y) k)| \ll_{n_v, v<\infty} |y|^{1/2} \lambda_{K_\infty}(\Phi)^{1/2}.$$

Hence

$$|E(s,\Phi)_N(n a(y) k)| \ll_{n_v, v<\infty} |y|^{1/2} \lambda_{K_\infty}(\Phi)^{1/2} \le |y|^{1/2} \lambda_{\Phi,s,\infty}^{1/2}. \tag{1.6.5}$$

Theorems 1.6.1 & 1.6.3 will be established using the following generalized Weyl's law

Theorem 1.6.8. *Given a sequence of non-negative integers $\bar{n} = (n_v)_{v<\infty}$ with $n_v = 0$ for a.e. v. Define*

$$K_{fin}[\bar{n}] = \prod_{v<\infty} K_v[n_v]$$

and consider the space $R^{K_{fin}[\bar{n}]} = L^2(G(F)\backslash G(\mathbb{A}),\omega)^{K_{fin}[\bar{n}]}$. *It is actually a representation of* $G(F_\infty)\times K_f$. *The operator* $\Delta_\infty = \prod_{v<\infty}\Delta_v$ *is self-dual and commutes with the action of* K. *We have* $\Delta_\infty^{-1-\epsilon}$ *is of trace class in* $R^{K_{fin}[\bar{n}]}$. *More precisely,*

$$\sum_{\pi'}\sum_{e}|\lambda_{e,\infty}|^{-1-\epsilon}+\sum_{\xi}\sum_{\Phi}\int_{-\infty}^{\infty}|\lambda_{\Phi_{i\tau},\infty}|^{-1-\epsilon}\frac{d\tau}{2\pi}$$

$$= O_\epsilon(Vol(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})/K_{fin}[\bar{n}])).$$

Here $\lambda_{e,\infty}$ *runs over the discrete spectrum of* Δ_∞, *and* $\lambda_{\Phi_{i\tau},\infty}$ *runs over the continuous spectrum of* Δ_∞.

Remark 1.6.9. *We only need a weaker version here. Namely, we only need* Δ_∞^{-N} *to be of trace class for some* $N>0$.

Remark 1.6.10. *If instead of* $K_{fin}[\bar{n}]$ *we consider* $K_\infty\times K_{fin}[\bar{n}]$, *the above theorem would coincide with the traditional geometrical Weyl's law. Note that this kind of Weyl's law was already used to establish theorems like 1.6.1 for* K_∞*-fixed case, e.g. [15]. Weyl's law is at the heart of the theory of analytical spectral decomposition.*

Remark 1.6.11. *This theorem will appear as a part of the Ph.D thesis of Marc R. Palm at Göttingen.*

Definition 1.6.12. *(c.f.page 292 [9]) The Schwartz function space* R^s *is the space of smooth functions* φ *in* $\mathrm{Ind}_{Z(\mathbb{A})G(F)}^{G(\mathbb{A})}\omega$, *which are rapidly decreasing in any given Siegel domain, as well as* $X.\varphi$ *for any* $X\in U(\mathfrak{g})$.

The above argument also gives

Corollary 1.6.13. *We have* $R_0^\infty\subset R^s\subset R^\infty$.

Remark 1.6.14. *If we take into account the central character, namely, if we write* R_ω *instead of* R, *we have* $R_\omega^s R_{\omega'}^s\subset R_{\omega\omega'}^s$. *In particular, if the central character is the trivial character* ω_0, $R_{\omega_0}^s$ *is a ring for the pointwise multiplication.*

1.7 Two Sobolev Norm Systems

Let v be an archimedean place, and π be a unitary irreducible representation of G_v with a fixed central character ω. Let $\{I_1,...,I_r\}$ be a basis of $Z(\mathfrak{g}_v)$. In our case, $r=1$ if v is a real place, and $r=2$ if v is a complex place. We define the Sobolev norm system

$$H_d^\pi(v)=\max_{i_1+\cdots+i_r+2j=d}\|I_1^{i_1}\cdots I_r^{i_r}\Delta_v^j v\|.$$

Theorem 1.7.1. *The Sobolev norm system* H_d^π *is equivalent to the Sobolev system* S_d^π *for* π *a local component of an automorphic representation. If the parameter* s *of* π *belongs to* $i\mathbb{R}\cup[-\theta,\theta]$ *with* $\theta<1/2$, *then the implicit constants in the above equivalence depend only on* θ.

The rest of this section is devoted to the proof of Theorem 1.7.1.

1.7.1 v is a real place

The Hecke algebra $\mathscr{H}_v = U(\mathfrak{g}) \oplus \underline{\epsilon} * U(\mathfrak{g})$, where $\underline{\epsilon}$ is the Dirac measure at $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. There is a classification of unitarizable irreducible $(\mathscr{H}_v, K_v)$ modules (c.f. for example 4.A [18]). Each of them $\pi(\mu_1,\mu_2)$ is parametrized by $s_1, s_2 \in \mathbb{C}, m_1, m_2 \in \{0,1\}$. Put $s = s_1 - s_2$, $t = s_1 + s_2 \in i\mathbb{R}$, $m = m_1 - m_2$. There are 3 different series: 1. $s \in i\mathbb{R}$; 2. $0 < s < 1$ but only $s < 2\theta$ is possible for the local component of an automorphic representation; 3. $0 < s = p \in \mathbb{Z}$, $s - m$ is an odd integer. In each case, there is an orthogonal, not necessarily normalized, basis consisting of K_v-isotypical vectors, $\{e_k\}$. In cases 1 and 2, k runs through $k \equiv m \pmod 2$, and in the case 3, $|k| \geq p+1$, $k \equiv p+1 \pmod 2$. There is a basis of $\mathfrak{g}_{\mathbb{C}}$, $\left\{ H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, V_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, V_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, J = id \right\}$ with explicit action given as

$$H.e_k = ike_k; V_+.e_k = (s+1+k)e_{k+2}; V_-.e_k = (s+1-k)e_{k-2}; J.e_k = te_k$$

$$\Delta_v.e_k = (\frac{1-s^2}{8} + \frac{k^2}{4})e_k.$$

Consider a general vector $v = \sum_k a_k e_k, a_k \in \mathbb{C}$. In the case 1, Theorem 2.6.2 of [2] implies $\|e_k\| = 1$. We easily deduce

$$\|H.v\|^2, \|V_+.v\|^2, \|V_-.v\|^2 \leq 16\|\Delta_v^{1/2}.v\|^2.$$

In the case 2, $\|e_k\|^2 = |\sqrt{\pi}\frac{\Gamma((s+1)/2)\Gamma(s/2)}{\Gamma((s+1+k)/2)\Gamma((s+1-k)/2)}|$ according to Theorem 2.6.4 of [2]. As a consequence

$$\frac{\|e_{k+2}\|^2}{\|e_k\|^2} = |\frac{s-1-k}{s+1+k}| \ll_\theta 1, \frac{\|e_{k-2}\|^2}{\|e_k\|^2} = |\frac{s-1+k}{s+1-k}| \ll_\theta 1.$$

We get

$$\|H.v\|^2, \|V_+.v\|^2, \|V_-.v\|^2 \ll_\theta 16\|\Delta_v^{1/2}.v\|^2.$$

In the case 3, it can be inferred from Theorem 2.6.5 of [2] that $\pi(\mu_1,\mu_2)$ has the following model: Let $\mathbb{H}^+$ be the Poincaré half plane, and $\mathbb{H}^-$ be its opposite. The space is, with the coordinates $z = x + iy$

$$L^2(\mathbb{H}^\pm) = \left\{ f : \mathbb{H}^\pm \to \mathbb{C}, \text{holomorphic} : \int_{y\neq 0} |f(z)|^2 y^{p+1} \frac{dxdy}{|y|^2} < \infty \right\}.$$

Therefore one may take, for $|k| \geq p+1$

$$e_k(z) = (z-i)^{-(k+p+1)/2}(z+i)^{(k-p-1)/2} 1_{\mathrm{sgn}(k)\mathrm{sgn}(y)<0}.$$

Change to the Poincaré disk model, one calculates easily, with $B(\cdot,\cdot)$ the Beta function

$$\|e_k\|^2 = \pi 4^{-p} B((|k|-p-1)/2+1, p).$$

Consequently

$$\frac{\|e_{k+2}\|^2}{\|e_k\|^2} \ll k^2, \frac{\|e_{k-2}\|^2}{\|e_k\|^2} \ll k^2$$
$$\|H.v\|^2, \|V_+.v\|^2, \|V_-.v\|^2 \ll 16\|\Delta_v.v\|^2.$$

We conclude that in all cases

$$\|H.v\|^2, \|V_+.v\|^2, \|V_-.v\|^2 \ll_\theta \|\Delta_v.v\|^2$$

thus $S_d^\pi \ll_{\theta,d} H_d^\pi \ll S_{2d}^\pi$, and the two systems are equivalent.

1.7.2 v is a complex place

The unitary irreducible series $\pi(\mu_1,\mu_2)$ are parametrized by $s_1, s_2 \in \mathbb{C}, k_1, k_2 \in \mathbb{Z}$ with $t = s_1+s_2 \in i\mathbb{R}, s = s_1 - s_2 \in i\mathbb{R}$ and $\mu_j(\rho e^{i\alpha}) = \rho^{2s_j} e^{ik_j\theta}, j = 1,2$. Or $t = s_1+s_2 \in i\mathbb{R}, 0 < s = s_1 - s_2 < 2\theta, k_1 = k_2$. Let $n_0 = k_1 - k_2$. We may suppose $n_0 \geq 0$ after exchange μ_1 and μ_2 if necessary. The representation $\pi(\mu_1,\mu_2)$ has an orthogonal basis $\left\{e_{n,k}^{(n_0)} : 0 \leq k \leq n, n \geq |n_0|, n \equiv |n_0| \pmod 2\right\}$ determined by

$$e_{n,k}^{(n_0)}(\begin{pmatrix} y_1 & x \\ 0 & y_2 \end{pmatrix} g) = \mu_1(y_1)\mu_2(y_2)|y_1/y_2| e_{n,k}^{(n_0)}(g), \forall g \in G_v$$

$$e_{n,k}^{(n_0)}(\begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{-i\alpha_1} \end{pmatrix} u \begin{pmatrix} e^{i\alpha_2} & 0 \\ 0 & e^{-i\alpha_2} \end{pmatrix}) = e^{in_0\alpha_1} e^{i(n-2k)\alpha_2}, \forall u \in K_v = \mathrm{SU}_2(\mathbb{C})$$

$$e_{n,k}^{(n_0)}(\begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}) = (\cos\beta)^{\frac{n+n_0}{2}-k} (\sin\beta)^{k-\frac{n-n_0}{2}} P_{\frac{n-n_0}{2}}^{(\frac{n_0-n}{2}+k, \frac{n_0+n}{2}-k)}(\cos 2\beta)$$

where $P_k^{(\alpha,\beta)}$ are the Jacobi polynomials. Alternatively,

$$e_{n,k}^{(n_0)} = \frac{\langle \rho_n(u) z_1^{n-k} z_2^k, z_1^{n-k_0} z_2^{k_0}\rangle_{\rho_n}}{\langle z_1^{n-k_0} z_2^{k_0}, z_1^{n-k_0} z_2^{k_0}\rangle_{\rho_n}}, n - 2k_0 = n_0.$$

It will also be convenient to extend by 0 to all integers n, k. The (complexified) Lie algebra SU_2 has a basis

$$H_2 = \begin{pmatrix} i & \\ & -i \end{pmatrix}, X_\pm = \pm\begin{pmatrix} & -1/2 \\ 1/2 & \end{pmatrix} - i \begin{pmatrix} & i/2 \\ i/2 & \end{pmatrix}$$

which act as

$$H_2.e_{n,k}^{(n_0)} = i(n-2k)e_{n,k}^{(n_0)}, X_+.e_{n,k}^{(n_0)} = (n-k)e_{n,k+1}^{(n_0)}, X_-.e_{n,k}^{(n_0)} = ke_{n,k-1}^{(n_0)}$$

$$\Delta_v.e_{n,k}^{(n_0)} = ((1-s^2-n_0^2)/8+n(n+2)/4)e_{n,k}^{(n_0)}.$$

It is obvious then that $\Delta_v^{-1-\epsilon}$ is of trace class in $\pi(\mu_1,\mu_2)$. A standard argument then shows that it suffices to prove Theorem 1.7.1 for v running over an orthonormal basis. The Cartan complement $\mathfrak{p}$ of SU_2 has a basis (We ignore the center)

$$H_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, Y_+ = ad(X_+)(H_1), Y_- = ad(X_-)(H_1).$$

Using the recurrence relations of Jacobi polynomials (c.f. [1]), we can find

$$H_1.e_{n,k}^{(n_0)} = \frac{2(k+1)(k+n_0+1)(2k+n_0+2+2s)}{(2k+n_0+1)(2k+n_0+2)}e_{n+2,k+1}^{(n_0)}+$$

$$\frac{2sn_0(n-2k)}{(2k+n_0)(2k+n_0+2)}e_{n,k}^{(n_0)}+\frac{(n+n_0)(4k-n+n_0)(2s-2k+n_0)}{2(2k+n_0)(2k+n_0+1)}e_{n-2,k-1}^{(n_0)}.$$

Since

$$Y_+.e_{n,k}^{(n_0)} = X_+H_1.e_{n,k}^{(n_0)} - H_1X_+.e_{n,k}^{(n_0)}, Y_-.e_{n,k}^{(n_0)} = X_-H_1.e_{n,k}^{(n_0)} - H_1X_-.e_{n,k}^{(n_0)}$$

we can only consider the actions of H_1, H_2, X_+, X_- if we don't want to optimize.

Case 1: $s \in i\mathbb{R}$. Then we are in the unitary principal series case and the norm structure is the standard L^2 norm on $\mathrm{SU}_2(\mathbb{C})$.

$$\| e_{n,k}^{(n_0)} \|^2 = \frac{(n-k)!k!}{(\frac{n-n_0}{2})!(\frac{n+n_0}{2})!(n+1)}.$$

One easily verifies $\|X.e_{n,k}^{(n_0)}\| \ll \|\Delta_v^{3/2}.e_{n,k}^{(n_0)}\|, X = H_1, H_2, X_+, X_-$, hence

$$\|X.v\| \ll \|\Delta_v^4.v\|, \forall v \in \pi^\infty, X = H_1, H_2, X_\pm, Y_\pm.$$

Case 2: $0 < s < 2\theta < 1$. Then $n_0 = 0$, thus $n \equiv 0 \pmod 2$. Let's write $e_{n,k}^{(s,0)} = e_{n,k}^{(0)}$ to emphasize the dependence on s. The norm structure is defined via the intertwining operator (with analytic continuation for $s < 0$)

$$M(s)e_{n,k}^{(s,0)}(g) = \int_{\mathbb{C}} e_{n,k}^{(s,0)}(n(x)g)dx = \lambda_{n,k}(s)e_{n,k}^{(-s,0)}(g).$$

Lemma 1.7.2. $\lambda_{n,k}(s) = (-1)^{n/2}\pi\frac{(s-1)\cdots(s-n/2)}{s(s+1)\cdots(s+n/2)}$ *Therefore,*

$$\|e_{n,k}^{(s,0)}\|^2 = (-1)^{n/2}\pi\frac{(s-1)\cdots(s-n/2)}{s(s+1)\cdots(s+n/2)}\frac{(n-k)!k!}{(\frac{n}{2})!(\frac{n}{2})!(n+1)}$$

With this, we easily see

$$\|X.v\| \ll_\theta \|\Delta_v^4.v\|, \forall v \in \pi^\infty, X = H_1, H_2, X_\pm, Y_\pm.$$

To prove the lemma, first consider $n = 2k$. We know $e_{2k,k}^{(s,0)}(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}) = P_k^{(0,0)}(1) = 1$, so

$$\lambda_{2k,k}(s) = M(s)e_{n,k}^{(s,0)}(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}) = \frac{\pi}{2}\int_{-1}^{1}(\frac{1-t}{2})^{s-1}P_k^{(0,0)}(t)dt.$$

Now we can use the recurrence relation of Legendre polynomials to establish

$$\lambda_{2k+2,k+1}(s) = \frac{2(2k+1)}{s}\lambda_{2k,k}(s+1) + \lambda_{2k-2,k-1}(s).$$

The first two values are easy to obtain $\lambda_{0,0}(s) = \pi/s, \lambda_{2,1}(s) = -\frac{\pi(s-1)}{s(s+1)}$. By induction, we get

$$\lambda_{2k,k}(s) = (-1)^k\pi\frac{(s-1)\cdots(s-k)}{s(s+1)\cdots(s+k)}.$$

Since $M(s)$ commutes with the action of G_v, it commutes with the action of X_+, X_-. It follows that $\lambda_{n,k}(s) = \lambda_{n,n/2}(s), \forall k$. This proves the above lemma and concludes the proof of Theorem 1.7.1.

1.8 Construction of Automorphic Forms from Local Kirillov Models

Suppose π is cuspidal. The norm identifications tell us that, given a pure tensor $\varphi \in \otimes_v'\pi_v^\infty$, resulting from (1.2.1), the $W_{\varphi,v}$ or the $K_{\varphi,v}$ must be a smooth vector in W_{π_v} or K_{π_v}. Conversely, if we are given $K_v \in K_{\pi_v}^\infty$, which uniquely determine corresponding $W_v \in W_{\pi_v}^\infty$, and form $W(g) = \prod_v W_v(g_v)$, and φ by (1.2.5), are we sure to get an element in π^∞? The converse theorem, as is discussed in the section 5.2 of [11], gives an affirmative answer. Note that, to determine W_v from K_v for an archimedean place v, a concrete way is to apply the Casimir element $\mathscr{C}$ of $\mathrm{GL}_2(\mathbb{R})$ in the real case, or the two embedded Casimir elements of $\mathrm{GL}_2(\mathbb{R})$ in $\mathrm{GL}_2(\mathbb{C})$ to get partial differential equations, since these elements should act as scalars depending only on π_v, then solve the corresponding Dirichlet problems.

Alternatively, maybe also more naturally and directly, if one wants to avoid the converse theorem, one may decompose W as an infinite sum of K-isotypical Whittaker functions, then change the order of summation to show that φ is a convergent (thanks to the local and global estimations in the above sections) infinite sum of K-isotypical functions in π, with rapidly decreasing spectral parameter for K, thus is itself in π^∞.

If π is Eisenstein, the situation is simpler. In fact, $W_{\varphi,v}$ determines φ at v in the induced model on a dense open subset, thus determines the corresponding function in the induced model. Then we apply (1.5.1).

1.9 Decay of Matrix Coefficients

Consider a place v, let π_λ be the complementary series representation of G_v with parameter $\lambda/2$ and with trivial central character. It has a unique K_v invariant unit vector w^0. The elementary spherical function associated with π_λ is defined to be (following Harish-Chandra's notation)

$$\varphi_{v,\lambda}(g) = \langle \pi_\lambda(g) w^0, w^0 \rangle.$$

Its limit when $\lambda \to 0$, denoted by $\varphi_{v,0} = \Xi_v$, is the Harish-Chandra function. They are all positive and bi-K_v-invariant.

Theorem 1.9.1. *Let π be any unitary irreducible representation of G_v. Let x_1, x_2 be 2 K_v-finite vectors in π. Then*

1 *If π is tempered, then*

$$\langle \pi(g)x_1, x_2 \rangle \leq \dim(K_v x_1)^{1/2} \dim(K_v x_2)^{1/2} \|x_1\| \cdot \|x_2\| \Xi_v(g).$$

2 *If π is in the complementary series with parameter $\lambda/2$, then for any $\epsilon > 0$, there is a $A_v(\epsilon) > 0$*

$$\langle \pi(g)x_1, x_2 \rangle \leq A_v(\epsilon) \dim(K_v x_1)^{1/2} \dim(K_v x_2)^{1/2} \|x_1\| \cdot \|x_2\| \Xi_v(g)^{1-\lambda-\epsilon}.$$

Here $\dim(K_v x) = \dim span(K_v \cdot x)$ *is the dimension of the span of x by K_v action.*

The tempered case is well known in [13]. The non-tempered case, first proved in Theorem 2.11 [32] for real case, then recaptured in Lemma 9.1 [33], essentially is based on the following estimation

$$A_v(\epsilon)^{-1} \varphi_{v,0}^{1-\lambda+\epsilon} \leq \varphi_{v,\lambda} \leq \varphi_{v,0}^{1-\lambda}. \tag{1.9.1}$$

This is an elementary exercise in analysis, we leave it to the reader.

2 Preliminaries: Eisenstein Case

2.1 L-function Theory for K-finite Vectors: Eisenstein Case

Before generalizing, let's first consider a very general but motivating question. Given a group G, a subgroup H, and a (unitary) irreducible representation π of G. A general question is to determine which irreducibles of H occur in the restriction of π to H, and with what multiplicity. This is called the question of "Branching Rules". If we take $G = \mathrm{GL}_2(\mathbb{A})$, $H = ZA$ with notations as in section 2.1, and let σ be an irreducible representation of H which coincides with π when restricting to Z, then with obvious notations, one has

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_v}(\mathrm{Res}_{H_v}^{G_v} \pi_v, \sigma_v) = 1, \forall v \in V_F,$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_H(\mathrm{Res}_H^G \pi, \sigma) \leq 1.$$

Here, σ need not even be unitary. If the restriction of σ to A is given by the (quasi-)character $\chi^{-1}|\cdot|^{-s+1/2}$, then the local zeta-functional $\zeta(s,\cdot,\chi_v,\psi_v)$ introduced in the previous section is an element in $\mathrm{Hom}_{H_v}(\mathrm{Res}_{H_v}^{G_v}\pi_v,\sigma_v)$. Their converging product

$$\prod_{v|\infty} \zeta(s,\cdot,\chi_v,\psi_v) \prod_{v<\infty} \frac{\zeta(s,\cdot,\chi_v,\psi_v)}{L(s,\pi_v \otimes \chi_v)}$$

is an element in $\mathrm{Hom}_H(\mathrm{Res}_H^G\pi,\sigma)$. When π is a cuspidal representation realized in the space of automorphic functions, the global zeta-functional $\zeta(s,\cdot,\chi)$ is another element in $\mathrm{Hom}_H(\mathrm{Res}_H^G\pi,\sigma)$. As we have seen, their proportionality is given by $L(s,\pi\otimes\chi)$.

When π is an Eisenstein series representation, the local candidates of $\mathrm{Hom}_{H_v}(\mathrm{Res}_{H_v}^{G_v}\pi_v,\sigma_v)$ and their converging factors still make sense, as is shown in the sections 4.5, 4.6, 4.7 and 3.7 of [2]. But the global one defined in (1.2.3) is no longer a good candidate since it does not converge any more. A remedy is provided by

$$\zeta(s,\varphi,\chi) = \int_{F^\times\backslash\mathbb{A}^\times} (\varphi-\varphi_N)(a(y))\chi(y)|y|^{s-1/2} d^\times y, \tag{2.1.1}$$

where φ_N is the constant term defined by

$$\varphi_N(g) = \int_{F\backslash\mathbb{A}} \varphi(n(x)g)dx.$$

However, $\zeta(s,\varphi,\chi)$ is still not defined for all s but only for $\Re s > 1$. In the range $\Re s > 1$, it is easy to see that (1.2.7) continues to hold. As we shall see soon, the global zeta-functional can be analytically continued. This is reminiscent of Tate's thesis: the global zeta-functional in the GL_1 case there is also an analytically continued one. In order to apply the Poisson formula, one should single out the contribution at the origin and analytically continue it.

For our purpose, we are particularly interested in the principal series $\pi = \pi(1,1)$ induced from trivial characters. We restrict ourselves to this case, even though the following treatment is applicable to a wider situation. If $f_\tau \in \mathrm{Ind}_B^G(|\cdot|^\tau, |\cdot|^{-\tau})$ is a flat section with $f_0 \in \pi^{\mathrm{fin}}$, we define the normalized Eisenstein series as

$$\varphi(g) = \Lambda_F(1+2\tau)E(f_\tau)(g)\,|_{\tau=0}, \tag{2.1.2}$$

where $E(f_\tau) = E(\tau, f_0)$ is the usual Eisenstein intertwiner defined as in (1.5.1), and $\Lambda_F(s) = \Lambda(s,1)$ is the completed zeta-function. As (4.11) of [28] shows,

$$\varphi_N(a(y)g) = |y|^{1/2}\varphi_N(g) + |y|^{1/2}\log|y|\Lambda_F^*(1)f_0(g), \tag{2.1.3}$$

where $\Lambda_F^*(1)$ is the residue of Λ_F at 1. The global zeta-functional can be rewritten as

$$\begin{aligned}\zeta(s,\varphi,\chi) = &\int_{|y|\geq 1} (\varphi - \varphi_N)(a(y))\chi(y)|y|^{s-1/2}d^\times y \\ &+ \int_{|y|\leq 1} (\varphi(a(y)) - \varphi_N(wa(y)))\chi(y)|y|^{s-1/2}d^\times y \\ &- \int_{F^\times\backslash\mathbb{A}^\times} \varphi_N(a(y))1_{|y|\leq 1}\chi(y)|y|^{s-1/2}d^\times y \\ &+ \int_{F^\times\backslash\mathbb{A}^\times} \varphi_N(wa(y))1_{|y|\leq 1}\chi(y)|y|^{s-1/2}d^\times y.\end{aligned}$$

$\varphi-\varphi_N$ is rapidly decaying in any Siegel domain. In particular $(\varphi-\varphi_N)(a(y))$ is rapidly decaying as $|y| \to \infty$. Note that

$$\varphi(a(y)) - \varphi_N(wa(y)) = (w\varphi - w\varphi_N)(a(y^{-1})),$$

is also rapidly decaying as $|y| \to 0$, since $w\varphi$ is also K-finite. We deduce that the first two integrals are convergent for all $s \in \mathbb{C}$. The last two integrals can be calculated explicitly. They are 0 unless $\chi(y) = |y|^{i\sigma}$ for some $\sigma \in \mathbb{R}$, in which case they converge for $\Re s > 1$ and are equal to $\mathrm{Vol}(F^\times\backslash\mathbb{A}^{(1)})$ times

$$-\frac{\varphi_N(1)}{s+i\sigma} + \frac{f_0(1)}{(s+i\sigma)^2}\text{resp.} -\frac{\varphi_N(w)}{1-s-i\sigma} + \frac{f_0(w)}{(1-s-i\sigma)^2}.$$

The above terms obviously admit analytic continuation to $s \in \mathbb{C}$. We can also see the obvious functional equation from it. Let's sumarize:

Proposition 2.1.1. *Define $C = Vol(F^\times \backslash \mathbb{A}^{(1)})$ if π is Eisenstein and $\chi(y) = |y|^{i\sigma}$, and $C = 0$ otherwise. We define the global zeta-functional as*

$$\zeta(s,\varphi,\chi) = \int_{|y|\geq 1} (\varphi - \varphi_N)(a(y))\chi(y)|y|^{s-1/2}d^\times y$$

$$+\int_{|y|\leq 1} (\varphi(a(y)) - \varphi_N(wa(y)))\chi(y)|y|^{s-1/2}d^\times y$$

$$+C(-\frac{\varphi_N(1)}{s+i\sigma} + \frac{f_0(1)}{(s+i\sigma)^2} - \frac{\varphi_N(w)}{1-s-i\sigma} + \frac{f_0(w)}{(1-s-i\sigma)^2}).$$

Then (1.2.7) holds. It satisfies the functional equation (1.2.4).

Remark 2.1.2. *In fact, the above theory in the last 2 sections is valid for smooth (not necessarily K-finite) vectors as we explain in the next sections.*

2.2 Two Sobolev Norm Systems (continued)

We may need Theorem 1.7.1 for more groups, namely for a general compact reductive group. In this case, the unitary irreducible representations of G are classified by the theory of highest weight (c.f. IV.7 [24]).

Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Recall $\mathfrak{h}_\mathbb{R} = i\mathfrak{h}$. Let $\Phi = \Phi(\mathfrak{h}_\mathbb{C} : \mathfrak{g}_\mathbb{C})$ be the set of roots. Choose an ordering which determines Φ_+ as the set of positive roots. Denote by Π the resulting set of simple roots. Then there is a one-one correspondence between the set of irreducible representations and the set of dominant analytically integral linear functionals λ on $\mathfrak{h}_\mathbb{C}$. If Φ_λ is the representation corresponding to λ in the above correspondence, then the induced representation ϕ_λ on $\mathfrak{g}_\mathbb{C}$ is given by the Verma model $L(\lambda+\delta)$, where δ is half the sum of all positive roots (c.f. IV.8 [24]). We also denote by ϕ_λ its extension to the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Let T be the torus corresponding to $\mathfrak{h}$. Let $\tilde{T}$ be the torus in the universal covering group $\tilde{G}$ of G corresponding to $\mathfrak{h}$. Then the character χ_λ of Φ_λ is given on $\tilde{T}$ by (Theorem 4.46 [24])

$$\chi_\lambda(t) = D(t)^{-1}\sum_{w\in W}(\det w)\xi_{w(\lambda+\delta)}(t). \tag{2.2.1}$$

Here, W is the Weyl group. $\xi_\lambda(e^H) = e^{\lambda(H)}, \forall H \in \mathfrak{h}, \forall \lambda \in \mathfrak{h}'_\mathbb{R}$, and $H \mapsto e^H$ is the exponential map from $\mathfrak{h}$ to $\tilde{T}$. $\det w$ is computed for the linear action of W on $\mathfrak{h}_\mathbb{R}$. $D(t)$ is the Weyl's denominator

$$D(t) = \xi_\delta(t)\prod_{\alpha\in\Phi_+}(1-\xi_{-\alpha}(t)) = \sum_{w\in W}(\det w)\xi_{w\delta}(t). \tag{2.2.2}$$

It should be understood that χ_λ descends to T when λ is analytically integral. If $\{e_i\}_{i=1}^{d_\lambda}$ is an orthonormal basis of Φ_λ, then we have the dimension formula (Theorem 4.48 [24])

$$d_\lambda = \frac{\prod_{\alpha\in\Phi_+} <\lambda+\delta,\alpha>}{\prod_{\alpha\in\Phi_+} <\delta,\alpha>}. \tag{2.2.3}$$

By definition, $\chi_\lambda(g) = \sum_{i=i}^{d_\lambda} <\Phi_\lambda(g)e_i, e_i>$. We deduce that

$$\chi_\lambda(e^{(s-t)X}) = \sum_{i=i}^{d_\lambda} <\Phi_\lambda(e^{sX})e_i, \Phi_\lambda(e^{tX})e_i>, \forall X\in\mathfrak{g}, \forall s,t\in\mathbb{R},$$

$$\frac{\partial^2}{\partial s\partial t}|_{s=t=0}\,\chi_\lambda(e^{(s-t)X}) = \sum_{i=i}^{d_\lambda}\|\phi_\lambda(X)e_i\|^2.$$

But $\frac{\partial^2}{\partial s\partial t}|_{s=t=0}\,\chi_\lambda(e^{(s-t)X}) = -\frac{d^2}{ds^2}|_{s=0}\,\chi_\lambda(e^{sX})$, we get the inequality

$$\|\phi_\lambda(X)e_i\|^2 \le -\frac{d^2}{ds^2}|_{s=0}\,\chi_\lambda(e^{sX}). \tag{2.2.4}$$

We shall apply (2.2.4) to a basis of $\mathfrak{g}$ and compare the right hand side with $\|\phi_\lambda(\Delta)e_i\|^2$.

Lemma 2.2.1. *The Laplacian Δ acts on Φ_λ as multiplication by*

$$c_\lambda = <\lambda,\lambda> + 2<\lambda,\delta>.$$

In fact, we only need to consider the effect of Δ on the highest vector v_λ in $L(\lambda+\delta)$. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha\in\Phi}\mathfrak{g}_\alpha$ be the root decomposition. Let $H_\alpha \in \mathfrak{h}_{\mathbb{C}}$ be such that $B(H_\alpha, H) = \alpha(H), \forall H \in \mathfrak{h}_{\mathbb{C}}$, here $B(\cdot,\cdot)$ is the Killing form. Choose $E_\alpha \in \mathfrak{g}_\alpha$ s.t. $B(E_\alpha, E_{-\alpha}) = 1$, then $[E_\alpha, E_{-\alpha}] = H_\alpha$. $\mathscr{B}_{\mathbb{C}} = \{H_\beta, E_\alpha : \beta\in\Pi, \alpha\in\Phi\}$ is a basis of $\mathfrak{g}_{\mathbb{C}}$. Write the dual basis with respect to $B(\cdot,\cdot)$ by $\mathscr{B}'_{\mathbb{C}} = \{H'_\beta, E'_\alpha : \beta\in\Pi, \alpha\in\Phi\}$. It is easy to see $H'_\beta \in \mathfrak{h}_{\mathbb{R}}, E'_\alpha = E_{-\alpha}$. We thus get

$$\Delta = \sum_{\beta\in\Pi} H'_\beta H_\beta + \sum_{\alpha\in\Phi} E_{-\alpha}E_\alpha = \sum_{\beta\in\Pi} H'_\beta H_\beta + \sum_{\alpha\in\Phi_+} H_\alpha + 2\sum_{\alpha\in\Phi_+} E_{-\alpha}E_\alpha.$$

If v_λ is a highest vector of Φ_λ, it is killed by $E_\alpha, \alpha\in\Phi_+$, hence

$$\phi_\lambda(\Delta).v_\lambda = \left(\sum_{\beta\in\Pi}\lambda(H'_\beta)\lambda(H_\beta) + \sum_{\alpha\in\Phi_+}\lambda(H_\alpha)\right)v_\lambda = c_\lambda v_\lambda.$$

Now if $E_\alpha = X_\alpha + iY_\alpha$ with $X_\alpha, Y_\alpha \in \mathfrak{g}$, then $\bar{E}_\alpha = X_\alpha - iY_\alpha \in \mathbb{C}E_{-\alpha}$. One may adjust s.t. $E_{-\alpha} = -\bar{E}_\alpha$. Consequently, $[X_\alpha, Y_\alpha] = -\frac{i}{2}H_\alpha$. $\mathscr{B} = \{iH_\beta, X_\alpha, Y_\alpha : \beta\in\Pi, \alpha\in\Phi_+\}$ is a basis of $\mathfrak{g}$ w.r.t. which the Sobolev system S_d can be defined. Furthermore, the conjugation relations

$Ade^{X_\alpha}(e^{sY_\alpha}) = e^{-\frac{is}{2}H_\alpha}, Ade^{Y_\alpha}(e^{sX_\alpha}) = e^{\frac{is}{2}H_\alpha}$ tell us

$$\chi_\lambda(e^{sY_\alpha}) = \chi_\lambda(e^{-\frac{is}{2}H_\alpha}), \chi_\lambda(e^{sX_\alpha}) = \chi_\lambda(e^{\frac{is}{2}H_\alpha}),$$

since χ_λ is central. We therefore only need to consider $-\frac{d^2}{ds^2}|_{s=0}\,\chi_\lambda(e^{isH_\alpha})$ for $\alpha \in \Phi_+$. From Weyl's formulas (2.2.1),(2.2.2) and (2.2.3), we know that for any regular $H \in \mathfrak{h}$, as functions of s, the numerator and the denominator of $\chi_\lambda(e^{sH})$ vanishes at $s=0$ up to order $r_+ = |\Phi_+|$. Define

$$a_n(\lambda, H) = \sum_{w\in W} (\det w)(w(\lambda+\delta)(H))^n.$$

We then get the Taylor expansion of $\chi_\lambda(e^{sH})$ at $s=0$

$$(\sum_{n=r_+}^{\infty} a_n(\lambda, H)\frac{s^{n-r_+}}{n!})(\sum_{n=r_+}^{\infty} a_n(0, H)\frac{s^{n-r_+}}{n!})^{-1}.$$

Note the easy trivial bounds

$$|a_n(\lambda, iH_\alpha)| \le |W|\|\lambda+\delta\|^n\|\alpha\|^n,$$

and the fact that $a_n(0, iH_\alpha), \alpha \in \Phi_+, n \in \mathbb{N}$ are constants depending only on the structure of G, we deduce

$$\begin{aligned}-\frac{d^2}{ds^2}|_{s=0}\,\chi_\lambda(e^{isH_\alpha}) &\ll_G |a_{r_+}(\lambda, iH_\alpha)| + |a_{r_++1}(\lambda, iH_\alpha)| + |a_{r_++2}(\lambda, iH_\alpha)| \\ &\ll_G \|\lambda+\delta\|^{r_++2}.\end{aligned}$$

But by Lemma 2.2.1, $c_\lambda \simeq \|\lambda+\delta\|^2$, we thus get

$$\|\phi_\lambda(iH_\alpha)e_i\| \ll (-\frac{d^2}{ds^2}|_{s=0}\,\chi_\lambda(e^{isH_\alpha}))^{1/2} \ll_G c_\lambda^{(r_++2)/2}.$$

Finally, for a general vector $v \in \Phi_\lambda$, write $v = \sum_{i=1}^{d_\lambda} v_i$ with $v_i \in \mathbb{C}e_i$, then

$$\|\phi_\lambda(iH_\alpha)v\| \le \sum_{i=1}^{d_\lambda}\|\phi_\lambda(iH_\alpha)v_i\| \ll_G c_\lambda^{(r_++2)/2}\sum_{i=1}^{d_\lambda}\|v_i\| \ll c_\lambda^{(r_++2)/2} d_\lambda\|v\|.$$

We observe from (2.2.3) that

$$d_\lambda \ll_G \|\lambda+\delta\|^{r_+} \ll c_\lambda^{r_+/2}. \tag{2.2.5}$$

Therefore

$$\|\phi_\lambda(iH_\alpha)v\| \ll_G c_\lambda^{r_++1}\|v\| = \|\phi_\lambda(\Delta^{r_++1})v\|,$$

which concludes Theorem 1.7.1 for G compact.

2.3 Unitary Principal Series and Eisenstein Intertwiner

Take the unramified principal series representation $\pi_\tau = \pi(|\cdot|^\tau, |\cdot|^{-\tau})$. Let $e_{\vec{n}}^{(\tau)}$ denote the unitary vector of weight and level $\vec{n} = (n_v)_v$ in π, with n_v being defined as in Section 1.6.2. Denote by $M(\tau)$ the standard intertwining operator from π_τ to π_{tau}. Recall from the section 4 of [20] that, after analytical continuation we can write

$$M(\tau) = \frac{\Lambda_F(1-2\tau)}{\Lambda_F(1+2\tau)} R(\tau)$$

where $R(\tau)$ takes the spherical function taking value 1 on K to the spherical one taking value 1 on K, and is unitary if $\tau \in i\mathbb{R}$.

Proposition 2.3.1. *We have* $R(\tau)e_{\vec{n}}^{(\tau)} = \lambda_{\vec{n}}(\tau)e_{\vec{n}}^{(-\tau)}$ *with*

$$\lambda_{\vec{n}}(\tau) = \prod_{v \text{ real}} \left(\prod_{k=0,2|k}^{n_v-2} \frac{k+1-2\tau}{k+1+2\tau} \right) \prod_{v \text{ complex}} \left(\prod_{k=1}^{n_v/2} \frac{k-2\tau}{k+2\tau} \right) \prod_{v<\infty, n_v>0} \left(q_v^{-2n_v\tau} \frac{1-q_v^{-(1-2\tau)}}{1-q_v^{-(1+2\tau)}} \right).$$

If we write $M(\tau) \prod_v M_v(\tau), R(\tau) = \prod_v R_v(\tau), \lambda_{\vec{n}}(\tau) = \prod_v \lambda_{n_v}(\tau)$ with notations of obvious meaning, then in the proof of Theorem 1.7.1 we have already obtained $\lambda_{n_v}(\tau) = \prod_{2|k,k=0}^{n_v-2} \frac{k+1-2\tau}{k+1+2\tau}$ if v is a real place, $\lambda_{n_v}(\tau) = \prod_{k=1}^{n_v/2} \frac{k-2\tau}{k+2\tau}$ if v is a complex place.

It is a classical result on conductors that, for $\pi_v = \pi_{\tau,v}$, $\pi_v^{K_v^0[m]}$ is of dimension $m+1$. It is also easy to see that $K_v^0[m]\backslash K_v/K_v^0[m]$ has a full set of representatives $\{1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, n_-(\varpi_v^l), 1 \le l \le m-1\}$, such that the double cosets are invariant under the anti-involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & b \\ c & a \end{pmatrix}$. Thus $(K_v, K_v^0[m])$ is a Gelfand pair. Therefore $\pi_v^{K_v^0[m]}$ is spanned by the $K_v^0[m]$-invariant vector of every unitary irreducible representation of K_v occurring in $\mathrm{Res}_{K_v}^{G_v} \pi_v = \mathrm{Ind}_{B(\mathcal{O}_v)}^{K_v}(1,1)$. For simplicity of notations, let's suppress the index v from now on, and denote the unitary vector invariant by $K^0[l]$, orthogonal to $\pi^{K^0[l-1]}$, by $e_l^{(\tau)}$. Each $e_l^{(\tau)}$ generates a K-subrepresentation which are distinct one from the other, in $\mathrm{Ind}_{B(\mathcal{O}_v)}^{K_v}(1,1)$, and $e_l^{(\tau)} = e_l^{(0)}$ is fixed as upon restriction to K. Since $M(\tau)$ intertwines the action of G thus K, we must have that $M(\tau)e_l^{(\tau)}$ is proportional to $e_l^{(-\tau)}$, which explains the existence of $\lambda_{\vec{n}}(\tau)$.

We are going to characterize $e_l^{(\tau)}$ explicitly. In fact, if we write $\mathcal{O}_n = \varpi^n \mathcal{O} - \varpi^{n+1}\mathcal{O}, n \ge 1$, then

$$D_0 = B(\mathcal{O})wN(\mathcal{O}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : c \in \mathcal{O}^\times \right\},$$

$$D_n = B(\mathscr{O})N_-(\mathscr{O}_n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K : c \in \mathscr{O}_n \right\}, 1 \leq n \leq m,$$

$$D'_m = K^0[m] = \cup_{n=m}^{\infty} D_n$$

are the double cosets of K w.r.t. $B(\mathscr{O})$ and $K^0[m]$. For any $f \in \pi^{K^0[m]}$, there is a sequence of complex numbers $f_n, 0 \leq n \leq m$ s.t. $f \mid_{D_n} = f_n$ and $f_n = f_m, \forall n \geq m$. We denote the sequence associated with $e_l^{(\tau)}$ by $a(l,n)$. It is elementary to calculate the mass d_n of D_n assuming the mass of K is 1. We have

$$d_0 = \frac{q}{q+1}, d_n = \frac{q^{-(n-1)}}{q+1}(1-q^{-1}), n \geq 1.$$

Then the $a(l,n)$ satisfy

1 $a(l,n) = a(l,l)$ for all $n \geq l$ and $a(0,n) = 1$ for all $n \geq 0$;

2 $\sum_{n=0}^{\infty} a(l,n)\overline{a(l',n)} d_n = 0$ and $\sum_{n=0}^{\infty} |a(l,n)|^2 d_n = 1$.

A solution is given by

1 $a(0,n) = 1; a(1,0) = q^{-1/2}, a(1,1) = -q^{1/2}$;

2 $a(l,l-1) = q^{\frac{l-2}{2}}\sqrt{\frac{q+1}{q-1}}, a(l,l) = -(q-1)q^{\frac{l-2}{2}}\sqrt{\frac{q+1}{q-1}}, a(l,n) = 0, 0 \leq n \leq l-2, l \geq 2.$

Since we have

$$M(\tau)e_l^{(\tau)}(n_-(\varpi^n)) = \int_{\mathscr{O}} e_l^{(\tau)}\left(\begin{pmatrix} -\varpi^n & -1 \\ 1+\varpi^n x & x \end{pmatrix}\right) dx + \int_{|x|>1} |x|^{-2\tau-1} e_l^{(\tau)}\left(\begin{pmatrix} 1 & 0 \\ 1/x+\varpi^n & 1 \end{pmatrix}\right) dx,$$

by taking $n = l-1$ if $l > 0$ or $n = 0$ if $l = 0$, we easily obtain

$$M(\tau)e_0^{(\tau)} = \mathrm{Vol}(\mathscr{O})\frac{1-q^{-(1+2\tau)}}{1-q^{-2\tau}} e_0^{(-\tau)}, M(\tau)e_l^{(\tau)} = \mathrm{Vol}(\mathscr{O})q^{-2l\tau}\frac{1-q^{-(1-2\tau)}}{1-q^{-2\tau}} e_l^{(-\tau)}, l \geq 1.$$

Proposition 2.3.1 follows.

Corollary 2.3.2. *If we write*

$$\varphi_{\vec{n}}^0 = \Lambda_F(1+2\tau)E(e_{\vec{n}}^{(\tau)})(g)\mid_{\tau=0},$$

then its constant term is

$$\varphi_{\vec{n},N}^0 = \left(2\gamma_F + h_F\left(\sum_{v \text{ real}}\sum_{2|k,k=0}^{n_v-2}\frac{2}{k+1} + \sum_{v \text{ complex}}\sum_{k=1}^{n_v/2}\frac{2}{k} + \sum_{v<\infty, n_v>0}\left(-n_v - \frac{2}{q-1}\right)\log q_v\right)e_{\vec{n}}^{(0)}\right),$$

where γ_F, h_F are defined by the Taylor expansion of Λ_F as

$$\Lambda_F(1+s) = \frac{h_F}{s} + \gamma_F + o(s).$$

Corollary 2.3.3. *If $\varphi = \Lambda_F(1+2\tau)E(f_s)\mid_{\tau=0}$ is associated with $f_0 \in \pi(1,1)^\infty$ in the induced model, with level $\vec{n}_f = (n_v = 0)_{v<\infty}$, then*

$$\sup_{k\in K} |\varphi_N(k)| \ll_{F,\epsilon} \|\mathscr{C}_{K_\infty}^{3/4+\epsilon}.f_0\|,$$

where $\mathscr{C}_{K_\infty} = \prod_{v\mid\infty} \mathscr{C}_{K_v}$ is the product of Casimir elements of maximal compact subgroups at infinite places. Together with (2.1.3) we get

$$|\varphi_N(g)| \ll_F H(g)^{1/2} \log H(g) \|\mathscr{C}_{K_\infty}^{3/4+\epsilon}.f_0\|.$$

Consequently, given $c_0 > 0$,

$$\varphi(g) - \varphi_N(g) \ll_{F,N,c_0} S_d(f_0)H(g)^{-N}, \forall N \in \mathbb{N}, \forall H(g) \geq c_0,$$

$$\varphi(g) \ll_{F,c_0} S_d(f_0)H(g)^{1/2} \log H(g), \forall H(g) \geq c_0,$$

The first d depends on N and the degree of $F/\mathbb{Q}$. The second d depends only on the degree of $F/\mathbb{Q}$. Norms of f_0 are calculated in the induced model.

In fact, we decompose f_0 as

$$f_0 = \sum_{\vec{n}} a_{\vec{n}}(f_0)e_{\vec{n}}^{(0)}$$

use the previous corollary, bounds from the previous section especially (2.2.5) and the obvious Wely's law for $\mathrm{Ind}_{B_\infty}^{K_\infty}(1,1)$. Furthermore, the argument in Section 1.6.2 gives,

$$\varphi(g) - \varphi_N(g) \ll_{F,N,c_0} S_d(f_0)H(g)^{-N}, \forall H(g) \geq c_0.$$

Remark 2.3.4. *Corollary 2.3.3 is valid for any unitary smooth Eisenstein series.*

3 Outline of the Proof

3.1 Cuspidal Case

The departure point of the proof is Jacquet-Langlands' generalization of Hecke's integral representation of L-function, namely equation (1.2.7) that we copy here

$$L(1/2,\pi\otimes\chi)=\prod_{v|\infty}\zeta(1/2,W_{\varphi,v},\chi_v,\psi_v)^{-1}\cdot\prod_{v<\infty}\frac{L(1/2,\pi_v\otimes\chi_v)}{\zeta(1/2,W_{\varphi,v},\chi_v,\psi_v)}\cdot\zeta(1/2,\varphi,\chi).$$

with $\varphi\in\pi^\infty$ a pure tensor and smooth vector.

Then based on P.Sarnak's idea in [30], we consider the following family of test functions

$$\varphi=n(t).\varphi_0,\varphi_0\in\pi^\infty \text{ is a fixed pure tensor },t\in\mathbb{A}.$$

With this choice, the study of local zeta-functions shows that, under some technical conditions on φ_0, each local integral reaches its natural asymptotic lower bound for some $t_v=T_v$ with $|T_v|_v\asymp_\epsilon C(\chi_v)^{1\pm\epsilon}$. Take $\varphi=n(T).\varphi_0$ with $T=(T_v)_v$ chosen above, then we get the estimation of the product of local terms in (1.2.7):

Proposition 3.1.1. *There is a pure tensor $\varphi\in\otimes'_v\pi_v^\infty$ such that $\forall\epsilon>0$*

$$\prod_{v|\infty}\zeta(1/2,W_{\varphi,v},\chi_v,\psi_v)^{-1}\cdot\prod_{v<\infty}\frac{L(1/2,\pi_v\otimes\chi_v)}{\zeta(1/2,W_{\varphi,v},\chi_v,\psi_v)}\ll_{\epsilon,F}Q^{1/2+\epsilon}\tag{3.1.1}$$

where $Q=C(\chi)$ is the analytic conductor of χ.

Recall the global zeta-function defined by

$$\zeta(1/2,\varphi,\chi)=\int_{\mathbb{A}^\times}(\varphi-\varphi_N)(a(y))\chi(y)d^\times y, a(y)=\begin{pmatrix} y & \\ & 1\end{pmatrix},$$

where the constant term $\varphi_N=0$ if π is cuspidal. For simplicity, let's focus on this case. We want to bound the global zeta-function by some negative power of $C(\chi)$. To deal with the fact that $F^\times\backslash\mathbb{A}^\times$ is non compact, we then truncate the integral $\int_{F^\times\backslash\mathbb{A}^\times}$ as $\int^*_{F^\times\backslash\mathbb{A}^\times} = \int_{F^\times\backslash\mathbb{A}^\times} h(|y|)$ and remark that

$$\int^*_{F^\times\backslash\mathbb{A}^\times} \varphi(a(y))\chi(y)d^\times y \leq \left(\int^*_{F^\times\backslash\mathbb{A}^\times} 1d^\times y\right)^{1/2} \left(\int^*_{F^\times\backslash\mathbb{A}^\times} \left(n(T).|\varphi_0|^2\right)(a(y))d^\times y\right)^{1/2}.$$

The translation $n(t)$ on $|\varphi_0|^2$ is the same as translating the domain of integration $a(F^\times\backslash\mathbb{A}^\times)$ into $a(F^\times\backslash\mathbb{A}^\times)n(t)$. In the classical case ($F=\mathbb{Q}$ and φ_0 is spherical), the translated domain is the same as the semi straight-line $\{yt+yi : y>0\}$. As $t\to\infty$, the slope of the line tends to 0. The line becomes equidistributed on the modular surface $SL_2(\mathbb{Z})\backslash\mathbb{H}$. As a consequence the $n(t)$ (or $n(T)$) translation "kills" the portion of $|\varphi_0|^2$ orthogonal to the 1-dimensional representations. Intuitively,

$$\int^*_{a(F^\times\backslash\mathbb{A}^\times)n(T)} |\varphi_0(t)|^2 dt \to \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} |\varphi_0(g)|^2 dg = \langle\varphi_0,\varphi_0\rangle. \tag{3.1.2}$$

In order to diminish the right hand side, we amplify φ_0 by defining, for E equal to some positive power of Q to be chosen, Consider the following average of Dirac measures :

$$\sigma = 1/M_E^2 \sum_{v,v'\in I_E} \delta_{a(|\varpi_v|_v|\varpi_{v'}^{-1}|_{v'})}$$

with

$$I_E = \left\{v \mid q_v\in[E,2E], T_v=0\right\}, M_E = |I_E| \gg E/\log E,$$

and take, with ϖ_v denoting a uniformiser at the place v,

$$\varphi_0' = 1/M_E^2 \sum_{v,v'\in I_E} \chi(\varpi_v\varpi_{v'}^{-1})a(\varpi_v\varpi_{v'}^{-1}).\varphi_0 = \sigma_\chi' * \varphi_0,$$

where $\sigma_\chi' = 1/M_E^2 \sum_{v,v'\in I_E} \chi(\varpi_v\varpi_{v'}^{-1})\delta_{a(\varpi_v\varpi_{v'}^{-1})}$ is the adjoint measure of σ, i.e.

$$\int_{F^\times\backslash\mathbb{A}^\times} h(|y|)n(T).\varphi_0'(a(y))\chi(y)d^\times y = \int_{F^\times\backslash\mathbb{A}^\times} \sigma * h(|y|)n(T).\varphi_0(a(y))\chi(y)d^\times y.$$

Instead of φ_0, we put φ_0' into the above argument. This modification does not change the quality of truncation on integral. But in (3.1.2), we get $\langle\varphi_0',\varphi_0'\rangle$ instead, which is some weighted average of

$$\langle a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0, a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\varphi_0\rangle, v_1, v_1', v_2, v_2' \in I_E. \tag{3.1.3}$$

Since the decay of matrix coefficients is of local nature, when v_1, v_1', v_2, v_2' are distinct, the above term must have size some negative power of E. When v_1, v_1', v_2, v_2' are not distinct, they are bounded by $O(1)$, and killed by the big denominator M_E^2. Of course this modification will increase the contribution of non dimension 1 parts of $|\varphi_0|^2$ by some positive power of E as a factor.

Finally, we optimize the choice of E and the truncation on integral to get

Proposition 3.1.2. *Suppose π is cuspidal. For the same φ, there is an absolute constant $\delta > 0$ such that $\forall \epsilon > 0$*

$$\zeta(1/2, \varphi, \chi) \ll_{\epsilon,\pi} Q^{-\delta+\epsilon}. \tag{3.1.4}$$

We may choose $\frac{1-2\theta}{8}$, or $\frac{25}{256}$ using the best known result of [4] i.e. $\theta = 7/64$.

which together with (3.1.1) gives Theorem 0.3.1.

We discuss Proposition 3.1.2 in more detail. In order to simplify notations and for further convenience, we introduce a functional on automorphic representations:

$$\varphi \mapsto l^{\chi|\cdot|^s}(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi(a(y))\chi(y)|y|^s d^\times y.$$

Note that for π cuspidal one has

$$l^{\chi|\cdot|^s}(\varphi) = \zeta(s + 1/2, \varphi, \chi).$$

So (3.1.4) is equivalent to

$$l^{\chi}(\varphi) \ll_{\epsilon,\pi} Q^{-\delta+\epsilon}.$$

There is a local analogue of this functional :

$$W_{\varphi,v} \mapsto l^{\chi_v|\cdot|^s}(W_{\varphi,v}) = \int_{F_v^\times} W_{\varphi,v}(a(y))\chi(y)|y|^s d^\times y.$$

The truncation function $h \in C_c^\infty(\mathbb{R}_+)$ is made from a fixed function h_0 such that h is supported in $[Q^{-\kappa-1}, Q^{\kappa-1}]$. Here, $\kappa \in (0,1)$ is a parameter to be chosen later.

Lemma 3.1.3. *We have*

$$l^{\chi}(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \sigma * h(|y|)\varphi(a(y))\chi(y) d^\times y + O_{h_0,\varphi_0,\epsilon}(Q^{-\kappa/2+\epsilon}).$$

Define another functional:

$$\varphi \mapsto l^{\chi,h}(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} h(|y|)\varphi(a(y))\chi(y) d^\times y.$$

We are reduced to examine :

$$l^{\chi,\sigma*h}(\varphi) = l^{\chi,h}(\sigma'_\chi * \varphi) = \int_{F^\times \backslash \mathbb{A}^\times} h(|y|)\sigma'_\chi * \varphi(a(y))\chi(y) d^\times y$$

The inequality of Cauchy-Schwarz gives

$$|l^{\chi,h}(\sigma'_\chi * \varphi)|^2 \leq \int_{F^\times\backslash\mathbb{A}^\times} h(|y|)d^\times y \int_{F^\times\backslash\mathbb{A}^\times} |\sigma'_\chi * \varphi(a(y))|^2 h(|y|)d^\times y. \tag{3.1.5}$$

We then spectrally decompose $|\sigma'_\chi * \varphi_0|^2$ in $L^2(G(F)\backslash G(\mathbb{A}), 1)$ as in Theorem 1.6.3, which is possible because $\varphi_0 \in R^s$. Setting $l^h = l^{1,h}$ we have

$$l^h(n(T)|\sigma'_\chi * \varphi_0|^2) = l^h(n(T)|\sigma'_\chi * \varphi_0|^2_N) \tag{3.1.6}$$

$$+ \sum_{\pi' \text{cuspidal}} l^h(n(T)P_{\pi'}(|\sigma'_\chi * \varphi_0|^2)) \tag{3.1.7}$$

$$+ \frac{1}{4\pi} \sum_{\xi\in\widehat{F^\times\backslash\mathbb{A}^{(1)}}} \int_{-\infty}^{\infty} l^h(n(T)(P_{\xi,i\tau}(|\sigma'_\chi * \varphi_0|^2) - P_{\xi,i\tau}(|\sigma'_\chi * \varphi_0|^2)_N))d\tau \tag{3.1.8}$$

interchanging integrals being verified by Theorem 1.6.3. In every summand of (3.1.7) (resp. (3.1.8)) $P_{\pi'}$(resp. $P_{\xi,i\tau}$) denotes the projection on the space of π'(resp. $\pi(\xi|\cdot|^{i\tau}, \xi^{-1}|\cdot|^{-i\tau})$). The function

$$|\sigma'_\chi * \varphi_0|^2 = \frac{1}{M_E^4} \sum_{v_1,v'_1,v_2,v'_2\in I_E} \chi(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\chi^{-1}(\frac{\varpi_{v_2}}{\varpi_{v'_2}})a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0$$

Let's write

$$S_{cusp}(v_1, v'_1, v_2, v'_2) = \sum_{\pi' \text{cuspidal}} l^h(n(T)P_{\pi'}(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)),$$

hence

$$(3.1.7) = \frac{1}{M_E^4} \sum_{v_1,v'_1,v_2,v'_2\in I_E} \chi(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\chi^{-1}(\frac{\varpi_{v_2}}{\varpi_{v'_2}})S_{cusp}(v_1, v'_1, v_2, v'_2).$$

Define

$$S_{cst}(v_1, v'_1, v_2, v'_2) = l^h(n(T)(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)_N) = l^h((a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)_N),$$

$$S_{Eis}(v_1, v'_1, v_2, v'_2) = \sum_{\xi\in\widehat{F^\times\backslash\mathbb{A}^{(1)}}} \int_{-\infty}^{\infty} l^h(n(T)P_{\xi,i\tau}(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)$$
$$-P_{\xi,i\tau}(a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0)_N)d\tau.$$

Therefore,

$$(3.1.6) = \frac{1}{M_E^4} \sum_{v_1,v'_1,v_2,v'_2\in I_E} \chi(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\chi^{-1}(\frac{\varpi_{v_2}}{\varpi_{v'_2}})S_{cst}(v_1, v'_1, v_2, v'_2), \tag{3.1.9}$$

$$(3.1.8) = \frac{1}{4\pi M_E^4} \sum_{v_1,v'_1,v_2,v'_2\in I_E} \chi(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\chi^{-1}(\frac{\varpi_{v_2}}{\varpi_{v'_2}})S_{Eis}(v_1, v'_1, v_2, v'_2).$$

Remark 3.1.4. *Not every cuspidal representation π' (resp. not every character ξ) has a non trivial contribution in this decomposition. Only the ones which have less conductors than*

$\sigma'_\chi * \varphi_0$ on every place v has. The exact choice of the base for spectral decomposition is a subtle matter. It will be described in Section 5.3. Similarly, the number of ξ's with non zero contribution is also finite and depends on F and φ_0.

Lemma 3.1.5. *We have*

$$(3.1.6) \ll_{\epsilon,F,\pi} \kappa E^{\epsilon-2} Q^{(2+\kappa)\epsilon}.$$

Recall that, θ is such that no complementary series representation with parameter $> \theta$ appears as a local component of a cuspidal representation. Let $\lambda_{e,\infty}$ (resp. $\lambda_{\Phi_{i\tau},\infty}$) be the eigenvalue for e (resp. $E(\Phi, i\tau)$) with respect to Δ_∞, for e (resp. Φ) runsing through an orthonormal base $\mathscr{B}(\pi')$ (resp. $\mathscr{B}(\pi(\xi,\xi^{-1}))$, consisting of pure tensors of π' (resp. $\pi(\xi|\cdot|^{i\tau},\xi|\cdot|^{-i\tau})$). For the portion (3.1.7)+(3.1.8), an adelic version of Weyl's law Theorem 1.6.8 is needed. From it we deduce

Lemma 3.1.6. *For a typical term, we have*

$$S_{cusp}(v_1, v'_1, v_2, v'_2) \ll_{\epsilon,F,\pi,\theta,\kappa,h_0} E^2 Q^{-(1/2-\theta)+\epsilon}.$$

Consequently we get

$$(3.1.7) \ll_{\epsilon,F,\pi,\theta,\kappa,h_0} E^2 Q^{-(1/2-\theta)+\epsilon}.$$

Lemma 3.1.7. *For a typical term, we have*

$$S_{Eis}(v_1, v'_1, v_2, v'_2) \ll_{\epsilon,F,\pi,\kappa,h_0} E Q^{(\kappa-1)/2+\epsilon} + E^2 Q^{-1/2+\epsilon}.$$

Consequently we get

$$(3.1.8) \ll_{\epsilon,F,\pi,\kappa,h_0} E Q^{(\kappa-1)/2+\epsilon} + E^2 Q^{-1/2+\epsilon}.$$

Lemmas 3.1.5 to 3.1.7 immediately imply

Lemma 3.1.8. *We have*

$$l^h(n(T)|\sigma'_\chi * \varphi_0|^2) \ll_{\pi,\kappa,\epsilon} E^{\epsilon-2} Q^{(2+\kappa)\epsilon} + E^2 Q^{-(1/2-\theta)+\epsilon} + E Q^{(\kappa-1)/2+\epsilon}.$$

Remark 3.1.9. *A comparison between the eigenvalues appearing here and those appearing in the trace of Δ_∞^{-l} should be taken into account, where $l > 1$ will be specified. We'll see this in detail later.*

Remark 3.1.10. *We should explain what "typical term" means in Lemmas 3.1.6 and 3.1.7. In fact, a full list of the types of positions of v_1, v'_1, v_2, v_2 are*

Case 1: *v_1, v_1', v_2, v_2' are distinct.*
Case 2: *$v_1 = v_2$ or $v_1' = v_2'$, and there are 3 elements in $\{v_1, v_1', v_2, v_2'\}$.*
Case 3: *$v_1 = v_2'$ or $v_1' = v_2$ and there are 3 elements in $\{v_1, v_1', v_2, v_2'\}$.*
Case 4: *$v_1 = v_2$ and $v_1' = v_2'$, and there are 2 elements in $\{v_1, v_1', v_2, v_2'\}$.*
Case 5: *$v_1 = v_2'$ and $v_1' = v_2$ and there are 2 elements in $\{v_1, v_1', v_2, v_2'\}$.*
Case 6: *$v_1 = v_1' = v_2$ or $v_1 = v_1' = v_2'$ or $v_2 = v_2' = v_1$ or $v_2 = v_2' = v_1'$ and there are 2 elements in $\{v_1, v_1', v_2, v_2'\}$.*
Case 7: *$v_1 = v_1' = v_2 = v_2'$.*

***Case 1** is dominant in the sense that there are $\simeq M_E^4$ possibilities for this case but $O(M_E^3)$ for the other cases. Therefore it is considered to be typical. We should consider each case and add together their effects to get the second assertions in Lemmas 3.1.6 and 3.1.7. But it turns out that it is **Case 1** which gives the most significant contribution in any situation that will be considered.*

Now it is clear that Proposition 3.1.2 follows from Lemma 3.1.3, (3.1.5) and Lemma 3.1.8, by solving the equation

$$\min_{\kappa, E} \max(E^{\epsilon-1}, EQ^{-1/4+\theta/2}, Q^{-\kappa/2}, E^{1/2}Q^{(\kappa-1)/4+\epsilon}) = Q^{-\frac{1-2\theta}{8}+\epsilon}$$

An optimal choice is

$$E = Q^{\frac{1-2\theta}{8}}, \kappa = 1/4 + \theta/6.$$

Remark 3.1.11. *If we apply the $n(T)$ translation before the projections in (3.1.7) and (3.1.8), and use a more general result concerning the decay of matrix coefficients, then we find ourselves in the exact setting of [28], where all the technical calculations are folded in the "Ergodic Principle" in Section 2.5.3.*

3.2 Eisenstein Case

Recall that $\pi = \pi(1,1)$ is induced from the trivial character of $B(\mathbb{A})$. Then the corresponding L-function

$$L(1/2, \pi \otimes \chi) = L(1/2, \chi)^2.$$

Therefore, in order to prove Theorem 0.3.4, we only need to give a subconvex bound for $L(1/2, \pi \otimes \chi)$. We choose the local data as in the previous section. Then the local estimation Proposition 3.1.1 is still valid. For this choice, the test function φ_0 is K_f-invariant. We need a proposition similar to Proposition 3.1.2.

Proposition 3.2.1. *Suppose $\pi = \pi(1,1)$ is Eisenstein. For the φ chosen as above, there is an absolute constant $\delta > 0$ such that $\forall \epsilon > 0$*

$$\zeta(1/2, \varphi, \chi) \ll_{\epsilon, \pi} Q^{-\delta+\epsilon}. \tag{3.2.1}$$

We may choose $\delta = \dfrac{1-2\theta}{12}$, *or* $\dfrac{25}{384}$ *using the best known result of [4] i.e.* $\theta = 7/64$.

The truncation on the integral works the same way as in the cuspidal case.

Lemma 3.2.2. *If* $\chi(y) = |y|^{i\tau}$ *for some* $\tau \in \mathbb{R}$, *then*

$$l^{\chi}(\varphi - \varphi_N) = \int_{F^\times\backslash\mathbb{A}^\times} \sigma * h(|y|)(\varphi - \varphi_N)(a(y))\chi(y)d^\times y + O_\epsilon(Q^{-\kappa/2} + Q^{-(1-\kappa)/2+\epsilon}).$$

Otherwise, the estimation is the same as in Lemma 3.1.3

$$l^{\chi}(\varphi - \varphi_N) = \int_{F^\times\backslash\mathbb{A}^\times} \sigma * h(|y|)(\varphi - \varphi_N)(a(y))\chi(y)d^\times y + O(Q^{-\kappa/2}).$$

We are reduced to control

$$l^{\chi,h}(\sigma'_\chi * (\varphi - \varphi_N)) = \int_{F^\times\backslash\mathbb{A}^\times} h(|y|)\sigma'_\chi * (\varphi - \varphi_N)(a(y))\chi(y)d^\times y.$$

We write φ_0 as a sum of $\varphi_{1,\infty}$, $\varphi_{1,1}$ and φ_2. Here

$$\varphi_{1,\infty} + \varphi_{1,1} = \varphi_1 = \Lambda(c,u)(\varphi_0)$$

is the truncated function of φ_0 defined in Section 6.1. $\varphi_{1,1}$ is the one-dimensional portion of φ_1 defined by

$$\varphi_{1,1}(g) = \sum_{\chi' \in \widehat{F^\times\backslash\mathbb{A}^\times}, \chi'^2=1} \frac{\langle \varphi_1, \chi' \circ \det\rangle}{Vol(X(F))} \chi' \circ \det(g).$$

Note that φ_1 is K_f-invariant as φ_0 is, so are $\varphi_{1,\infty}$ and $\varphi_{1,1}$. Consequently, we write

$$l^{\chi,h}(\sigma'_\chi * (\varphi - \varphi_N)) = l^{\chi,h}(\sigma'_\chi * (n(T)\varphi_{1,\infty})) + l^{\chi,h}(\sigma'_\chi * \varphi_{1,1})$$
$$+ l^{\chi,h}(\sigma'_\chi * (n(T)\varphi_2)) - l^{\chi,h}(\sigma'_\chi * \varphi_{0,N}).$$

The treatment of $l^{\chi,h}(\sigma'_\chi * (n(T)\varphi_{1,\infty}))$ is the same as in the cuspidal case discussed in the previous section, except that we should extend the estimation of the constant contribution for $\varphi_{1,\infty}$. This is given in Section 6.3.2, especially by applying Corollary 6.3.2 to $\varphi_1 = \varphi_2 = \varphi_{1,\infty}$. We thus have

$$l^{\chi,h}(\sigma'_\chi * (n(T)\varphi_{1,\infty})) \ll_{F,\epsilon,\kappa} \max\left(EQ^{-(1/4-\theta/2)+\epsilon}, E^{1/2}Q^{(\kappa-1)/4+\epsilon}\right) \times$$
$$\max_{v_1,v_1',v_2,v_2' \in I_E} \|\Delta_\infty^{5/4+2\epsilon}\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_{1,\infty} a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_{1,\infty}\right)\|^{1/2}$$
$$+ Q^\epsilon E^{-1+\epsilon}\|\varphi_{1,\infty}\|^{1/2-\epsilon}\|\Delta_\infty.\varphi_{1,\infty}\|^{1/2+\epsilon}$$
$$+ Q^{(\kappa-1)/2}\|\varphi_{1,\infty}\|^{3/8-\epsilon}\|\Delta_\infty^2.\varphi_{1,\infty}\|^{5/8+\epsilon}.$$

Together with Corollary 6.1.5 (with $r = 1,2$ there), we get

Lemma 3.2.3.

$$l^{\chi,h}(\sigma'_\chi * (n(T)\varphi_{1,\infty})) \ll_{F,\epsilon,\kappa,\varphi_0} c^{1/2+\epsilon} \max\left(EQ^{-(1/4-\theta/2)+\epsilon}, E^{1/2}Q^{(\kappa-1)/4+\epsilon}\right)$$
$$+c^\epsilon Q^\epsilon E^{-1+\epsilon} + c^\epsilon Q^{(\kappa-1)/2}.$$

In order to estimate $l^{\chi,h}(\sigma'_\chi * \varphi_{1,1})$, we first estimate the coefficients $\langle \varphi_1, \chi' \circ \det\rangle$. Note that

$$\langle \varphi_0, \chi' \circ \det\rangle = 0$$

since φ_0 is an Eisenstein series, thus orthogonal to the one-dimensional representations. We deduce

$$|\langle \varphi_1, \chi' \circ \det\rangle| = |\langle \varphi_2, \chi' \circ \det\rangle| \leq \int_{\mathscr{S}} |\varphi_2(g)| dg,$$

where $\mathscr{S}$ is a Siegel domain containing a fundamental domain of $Z(\mathbb{A})G(F)\backslash G(\mathbb{A})$. Recall

$$\varphi_2(g) = \sum_{\gamma \in B(F)\backslash G(F)} \varphi_{0,N}(\gamma g) u(H(\gamma g) - c).$$

For $c > 1$, the number of $\gamma \in B(F)\backslash G(F)$ s.t. $\exists g \in \mathscr{S}, H(\gamma g) > c$ is bounded by a constant depending only on F, thus using (2.1.3) we get

$$\int_{\mathscr{S}} |\varphi_2(g)| dg \ll_F \int_{|y|>c, k\in K} \varphi_{0,N}(a(y)k) \frac{d^\times y}{|y|} dk \ll_\epsilon c^{-1/2+\epsilon} \|\varphi_0\|_{\mathrm{Eis}}^2.$$

Combining with the trivial bound $l^{\chi,\sigma * h}(\chi' \circ \det) \ll_{F,\kappa} \log Q$, we obtain

Lemma 3.2.4.

$$l^{\chi,h}(\sigma'_\chi * \varphi_{1,1}) = l^{\chi,\sigma * h}(\varphi_{1,1}) \ll_{F,\epsilon} c^{-1/2+\epsilon} Q^\epsilon.$$

The implicit constant depends on the norm of φ_0 in the induced model.

Bounding $l^{\chi,h}(\sigma'_\chi * \varphi_{0,N})$ is just an easy consequence of (2.1.3).

Lemma 3.2.5.

$$l^{\chi,h}(\sigma'_\chi * \varphi_{0,N}) = l^{\chi,\sigma * h}(\varphi_{0,N}) \ll_{F,\epsilon} Q^{(\kappa-1)/2+\epsilon}.$$

The implicit constant depends on some Sobolev norm of φ_0 in the induced model.

It remains the term $l^{\chi,h}(\sigma'_\chi * (n(T)\varphi_2)) = l^{\chi,\sigma * h}(n(T)\varphi_2)$. The following idea is again to use the equidistribution property of the torus $\{a(y)n(T)\}$. Note from Corollary 2.3.3 that $\varphi_{0,N}(g) \ll_F H(g)^{1/2} \log H(g) \|\mathscr{C}_{K_\infty}^{3/4+\epsilon}.f_0\|$. By Corollary 2.3.2 and (2.1.3), we have $\varphi_{0,N}^0(g) = 2\gamma_F H(g)^{1/2} + h_F H(g)^{1/2} \log H(g)$, where $\varphi_0^0 = \varphi_{\bar{0}}^0$ is defined in Corollary 2.3.2. Thus for some constant c_F

depending only on F,

$$|\varphi_{0,N}(g)| \ll_F \|\mathscr{C}_{K_\infty}^{3/4+\epsilon}.f_0\| \varphi_{0,N}^0(g), H(g) \geq c_F.$$

If we decompose $\varphi_0^0 = \varphi_{1,\infty}^0 + \varphi_{1,1}^0 + \varphi_2^0$ as what we did for φ_0, then we have

$$|\varphi_2(g)| \ll_F \|\mathscr{C}_{K_\infty}^{3/4+\epsilon}.f_0\| \varphi_2^0(g), \forall c \geq c_F.$$

Consequently, $|l^{\chi,\sigma*h}(n(T)\varphi_2)| \ll_F \|\mathscr{C}_{K_\infty}^{3/4+\epsilon}.f_0\| l^{\sigma*h}(n(T)\varphi_2^0)$. Thus we have a bound

$$|l^{\chi,\sigma*h}(n(T)\varphi_2)| \ll_F \|\mathscr{C}_{K_\infty}^{3/4+\epsilon}.f_0\| \times (|l^{\sigma*h}(n(T)(\varphi_0^0 - \varphi_{0,N}^0)|$$

$$+|l^{\sigma*h}(\varphi_{0,N}^0)| + |l^{\sigma*h}(n(T)\varphi_{1,\infty}^0)| + |l^{\sigma*h}(\varphi_{1,1}^0)|).$$

The discussion in Section 5.4 applies to $l^{\sigma*h}(n(T)(\varphi_0^0 - \varphi_{0,N}^0)$, after discarding the amplification. Thus

$$|l^{\sigma*h}(n(T)(\varphi_0^0 - \varphi_{0,N}^0)| \ll_\epsilon Q^{(\kappa-1)/2+\epsilon}.$$

We have already seen that $\varphi_{0,N}^0(a(y)) = 2\gamma_F|y|^{1/2} + h_F|y|^{1/2}\log H(g)$, thus

$$|l^{\sigma*h}(\varphi_{0,N}^0)| \ll_{F,\epsilon} Q^{(\kappa-1)/2+\epsilon}.$$

The method of Lemma 3.2.4 applies to $l^{\sigma*h}(\varphi_{1,1}^0)$ also gives

$$|l^{\sigma*h}(\varphi_{1,1}^0)| \ll_{F,\epsilon} c^{-1/2}Q^\epsilon.$$

The method of Lemma 3.2.3 applies to $l^{\sigma*h}(n(T)\varphi_{1,\infty}^0)$ gives

$$l^{\sigma*h}(n(T)\varphi_{1,\infty}^0) \ll_{F,\epsilon,\kappa,\varphi_0^0} c^{1/2+\epsilon} \max\left(EQ^{-(1/4-\theta/2)+\epsilon}, E^{1/2}Q^{(\kappa-1)/4+\epsilon}\right)$$

$$+c^\epsilon Q^\epsilon E^{-1+\epsilon} + c^\epsilon Q^{(\kappa-1)/2}.$$

We obtain

Lemma 3.2.6.

$$|l^{\chi,\sigma*h}(n(T)\varphi_2)| \ll_{F,\epsilon,\kappa,\varphi_0} c^{1/2+\epsilon} \max\left(EQ^{-(1/4-\theta/2)+\epsilon}, E^{1/2}Q^{(\kappa-1)/4+\epsilon}\right)$$

$$+c^\epsilon Q^\epsilon E^{-1+\epsilon} + c^\epsilon Q^{(\kappa-1)/2} + Q^{(\kappa-1)/2+\epsilon} + c^{-1/2}Q^\epsilon.$$

Collecting Lemma 3.2.2, 3.2.3, 3.2.4, 3.2.5 and 3.2.6, we are reduced to solving

$$\min_{\kappa,E,c} \max(Q^{-\kappa/2}, Q^{(\kappa-1)/2}, c^{1/2}EQ^{-(1-2\theta)/4}, c^{1/2}E^{1/2}Q^{(\kappa-1)/4}, E^{-1}, c^{-1/2}) = Q^{-\frac{1-2\theta}{12}}.$$

An optimal choice of parameters is given by

$$\kappa = \frac{1+2\theta}{6}, E = Q^{\frac{1-2\theta}{12}}, c = Q^{\frac{1-2\theta}{6}}.$$

Proposition 3.2.1 follows.

4 Choice of φ_0 and Local Estimation

In this chapter we define the vector φ of Proposition 3.1.1. Recall that it is of the shape $\varphi = n(T)\varphi_0$. Here $\varphi_0 \in \pi$ is a pure tensor corresponding to $W_0(g) = \prod_v W_{0,v}(g_v)$ in the Kirillov model of π. Recall also that we only need to specify $W_{0,v}$ for every place $v \in V_F$.

4.1 Archimedean places

We first make the notion "Analytic Conductor" precise. The general definition, for both GL_1 and GL_2 representations, is given in 3.1.8 [28]. In this paper, we're particularly interested in GL_1 case. Using the notations from 3.1.8 [28] and from Chapter XIV § 4 [26], one easily sees that if $F_v = \mathbb{R}$ and $\chi_v(a) = \mathrm{sgn}(a)^m |a|^{i\varphi}$, then $\mu_{\chi_v} = \dfrac{i\varphi + m}{2}$, $m \in \{0,1\}$ thus we may define

$$C(\chi_v) = 2 + |\frac{i\varphi + m}{2}|.$$

If $F_v = \mathbb{C}$ and $\chi_v(a) = (\dfrac{a}{|a|})^m |a|^{i2\varphi}$, then $\mu_{\chi_v} = i\varphi + |m|/2$, we may define

$$C(\chi_v) = (2 + |i\varphi + |m|/2|)^2.$$

Lemma 4.1.1. *Let $\phi \in S(F_v^\times)$ (i.e. ϕ as well as all its derivatives decay faster than any polynomial of $|t^{-1}|$ as $|t| \to +\infty$ and more rapidly than any polynomial of $|t|$ as $|t| \to 0$). Let $C = C(\chi_v)$ be the analytic conductor of χ_v. Set, for $t \in F_v^\times$,*

$$G_\phi(\chi_v, t) = \int_{F_v} \phi(x)\psi_v(tx)\chi_v(x)dx.$$

Then for any $N \in \mathbb{N}, 1/2 \le \alpha < \beta < 1$

$$|G_\phi(\chi_v, t)| \ll_{\phi,N,\alpha,\beta} \min((\frac{1+|t|}{C})^N, (\frac{C}{|t|})^N, C^{1/2-\alpha}|t|^{\alpha-\beta}).$$

This is essentially the Lemma 3.1.14 of [28]. Let's recall the proof: Note that C is comparable with the maximal absolute value among eigenvalues of χ_v for a fixed $F_v^\times$-invariant basis of differential operators of degree $[F_v:\mathbb{R}]$. The first two bounds then follow from two different kinds of integration by parts. For the third bound, apply the local functional equation as in Tate's thesis, we obtain

$$G_\phi(\chi_v, t) = \frac{\int_{F_v} \Phi(x+t)\chi_v^{-1}(x)|x|^\alpha d^\times x}{\gamma(\chi_v, \psi_v, 1-\alpha)}$$

where $\Phi = \widehat{\phi|\cdot|^\alpha} \in S(F_v)$ is the Fourier transform of $\phi(x)|x|^\alpha$. Recall if we fix a small $\epsilon > 0$, and let $\alpha \in [1/2, 1-\epsilon]$, by (3.5) of [28], and the third property after Theorem 3 of §3 [26]

$$|\gamma(\chi_v, \psi_v, 1-\alpha)| \simeq_\epsilon C^{\alpha-1/2}$$

Then after some evident change of variables, one gets

$$|G_\phi(\chi_v, t)| \simeq_\epsilon C^{1/2-\alpha}|t|^\alpha| \int_{F_v} \Phi(tx)|x-1|^{\alpha-1}\chi^{-1}(x-1)dx|$$

But for any $\beta > 0$, $\Phi(x) \ll_{\beta,\phi} |x|^{-\beta}$, thus

$$|G_\phi(\chi_v, t)| \ll_{\epsilon,\beta,\phi} C^{1/2-\alpha}|t|^{\alpha-\beta} \int_{F_v} |x|^{-\beta}|x-1|^{\alpha-1}dx$$

The integral converges if $1/2 \le \alpha < \beta < 1$. Under this condition, we get

$$|G_\phi(\chi_v, t)| \ll_{\alpha,\beta,\phi} C^{1/2-\alpha}|t|^{\alpha-\beta}$$

Corollary 4.1.2. *For any $\epsilon > 0$ there is a C_0 depending only on ϕ and ϵ, such that for $C \ge C_0$ there exists t with $|t| \in [C^{1-\epsilon}, C^{1+\epsilon}]$, s.t. $|G_\phi(\chi, t)| \ge_{\phi,\epsilon} C^{-1/2-\epsilon}$.*

Apply the Plancherel formula for $L^2(F_v)$

$$\int_{F_v} |\phi(x)|^2 dx = \int_{F_v} |G_\phi(\chi_v, t)|^2 dt \ll_{\phi,N} \int_{|t|\le C^{1-\epsilon}} (\frac{1+|t|}{C})^{2N} dt + \int_{|t|\ge C^{1+\epsilon}} (\frac{C}{|t|})^{2N}$$
$$+(C^{1+\epsilon} - C^{1-\epsilon}) \max_{|t|\in[C^{1-\epsilon}, C^{1+\epsilon}]} |G_\phi(\chi_v, t)|^2$$

The result follows by taking $N = 1 + \lceil \frac{1}{2\epsilon} \rceil$ ($N > 1/2 + \frac{1}{2\epsilon}$ suffices) for example.

Choose $W_{0,v} \in S(F_v^\times)$ and $T_v = t$ as in the above corollary, s.t.

$$\zeta(1/2, n(T_v)W_{0,v}, \chi_v, \psi_v) \gg_{\epsilon, W_{0,v}} C(\chi_v)^{-1/2-\epsilon} \tag{4.1.1}$$

Corollary 4.1.3. *For any $0 < \epsilon < 1/2$, and any $\sigma \in \mathbb{R}$ varying in a compact set*

$$|G_\phi(\chi_v|\cdot|_v^\sigma, t)| \ll_{\epsilon,\phi} \min(C^{-1/2+\epsilon}, |t|^{-1/2+\epsilon})$$

We have $|G_\phi(\chi_v,t)| \ll_{\alpha,\beta,N,\phi} \min(C^{1/2-\alpha}|t|^{\alpha-\beta}, |t|^N C^{-N}) \le C^{-\frac{N(\beta-1/2)}{N+\beta-\alpha}}$. Take $\alpha = 1/2$, β approaching 1 and N big enough. Considering $G_\phi(\chi_v|\cdot|_v^\sigma, t) = G_{\phi|\cdot|_v^\sigma}(\chi_v, t)$ gives the result.

Remark 4.1.4. *If $C(\chi_v) < C_0$, note that $G_\phi(\chi_v,t)$ can be extended to an analytic function on t and χ_v not identically 0 for any fixed χ_v. Since $C(\chi_v) \le C_0$ defines a compact region for χ_v, a routine argument gives the existence of a finite set $A \subset \mathbb{R}$ depending on ϕ and C_0 s.t. for any such χ_v, $\exists t \in A$, $|G_\phi(\chi_v,t)| \gg_{\phi,\epsilon} 1$. Thus Corollary 4.1.2 remains true if the condition $|t| \in [C^{1-\epsilon}, C^{1+\epsilon}]$ is replaced by $|t| \in [C/2, 2C]$. We note that $C(\chi_v) > 1$ by definition. We take $T_v = t$ accordingly.*

4.2 Non-Archimedean places

We study the local analog of the generalized Gauss sum as in the previous subsection. For simplicity, we assume that the conductor of ψ_v is $\mathcal{O}_v$. Take the convention $n(\varpi_v^0) = 1$.

Lemma 4.2.1. *Let W transform as ω_v under translations by $a(\mathcal{O}_v^\times)$. Suppose the conductor of $\omega_v\chi_v$ is $1+\varpi_v^r\mathcal{O}_v$ and $\Re(s) = \sigma$. Then if $r > 0, l > 0$*

$$|\zeta(s+1/2, n(\varpi_v^{-l})W, \chi_v, \psi_v)| = q^{-r/2} q^{-\sigma(l-r)} |W(\varpi_v^{l-r})|.$$

If $r > 0, l = 0$,

$$\zeta(s+1/2, n(\varpi_v^{-l})W, \chi_v, \psi_v) = 0.$$

If $r = 0, l > 0$

$$\zeta(s+1/2, n(\varpi_v^{-l})W, \chi_v, \psi_v) = \sum_{k=l}^{\infty} W(\varpi_v^k)\chi_v(\varpi_v)^k q_v^{-sk} - \frac{1}{q_v - 1} W(\varpi_v^{l-1})\chi_v(\varpi_v)^{l-1} q_v^{-s(l-1)}.$$

If $r = 0, l = 0$

$$\zeta(s+1/2, n(\varpi_v^{-l})W, \chi_v, \psi_v) = \sum_k W(\varpi_v^k)\chi_v(\varpi_v)^k q_v^{-sk}.$$

In fact for $r > 0$, $\zeta(1/2, n(\varpi_v^{-l})W, \chi_v) = W(\varpi_v^{l-r})\chi_v(\varpi_v)^{l-r} \int_{\mathcal{O}_v^\times} \psi_v(\varpi_v^{-r}y)\omega_v\chi_v(y)d^\times y$, the integral being a Gauss sum with absolute value $q^{-r/2}$. The following corollary is Lemma 11.7 of [33].

Corollary 4.2.2. *Let r be the conductor of $\omega_v\chi_v$. If π_v is spherical, take $W_{0,v}$ to be the spherical vector. If π_v is ramified, take $W_{0,v}(y) = \omega_v(y)1_{v(y)=0}$. Then if $r > 0$*

$$|\zeta(s, n(\varpi_v^{-r})W_{0,v}, \chi_v, \psi_v)| = q_v^{-r/2}.$$

If $r = 0$,

$$\zeta(s, W_{0,v}, \chi_v, \psi_v) = L(s, \pi_v \otimes \chi_v).$$

As a consequence

$$\prod_{v<\infty}|\frac{L(1/2,\pi_v\otimes\chi_v)}{\zeta(1/2,n(T_v)W_{0,v},\chi_v,\psi_v)}|\leq\prod_{v<\infty,\omega_v\chi_v ramified}\frac{1}{(1-q_v^{-1/2+\theta})^2}C(\omega_v\chi_v)^{1/2}$$

$$\ll_{\epsilon,\pi,F}\prod_{v<\infty}C(\chi_v)^{1/2+\epsilon}. \tag{4.2.1}$$

Note that (3.1.1) is established by (4.1.1) and (4.2.1) once $T=(T_v)_v$ and $\varphi=n(T)\varphi_0$ are chosen, where φ_0 corresponds to $(W_{0,v})_v$.

Proposition 4.2.3. *The function φ_0 corresponding to $\prod_v W_{0,v}$ in the Kirillov model of π verifies $\varphi_0\in R^s$.*

This is an obvious consequence of the discussion in the section 1.8. In fact it is easy to verify $\varphi_0\in R_0^\infty$, then we apply Corollary 1.6.12.

4.3 A Calculation in Unitary Principal Series

First consider a finite place. We are interested in consequences of Lemma 4.2.1 in the case of a unitary principal series representation. We may assume $v(\psi)=0$. For simplicity of notations, we omit the subscript v. Assume that the representation takes the form $\pi=\pi(\xi,\xi^{-1})$ for some unramified unitary character ξ of $F^\times$. For an integer $m\geq 0$, we are interested in vectors of π invariant by $K^0[m]$. Let W_π denote the Whittaker model of π.

Lemma 4.3.1. *If $W\in W_\pi$ is invariant by $K^0[m]$, then we have*

$$|W(a(y))|\ll(v(y)+1)(m+1)q^{m/2}\|W\||y|^{1/2}1_{v(y)\geq 0},$$

the implicit constant being absolute. As a consequence,

$$|l^{|\cdot|^s}(n(\bar{\omega}^{-l})W)|\ll_\epsilon(m+1)q^{m/2}q^{-l(1-\epsilon)}\|W\|,\forall\epsilon>0,\Re(s)=1/2+\epsilon. \tag{4.3.1}$$

Recall that for any $f\in\pi^{K^0[m]}$, there is a sequence of complex numbers $f_n, 0\leq n\leq m$ s.t.

$$f|_{B(\mathcal{O})wN(\mathcal{O})}=f_0, f(\begin{pmatrix}1&0\\x&1\end{pmatrix})=f_n,\forall x\in\mathcal{O}_n.$$

Therefore, if

$$W(a(y))=W_f(a(y))=\xi^{-1}(y)|y|^{1/2}\int_F f(wn(x))\psi(-xy)dx$$

denotes the Whittaker function of f, then we obtain, with $t=\xi(\varpi)$

$$W(a(y))=\xi^{-1}(y)|y|^{1/2}1_{v(y)\geq 0}(f_0-q^{-1}f_{v(y)+1}t^{2(v(y)+1)}+(1-q^{-1})\sum_{n=1}^{v(y)}f_nt^{2n}).$$

If $v(y)\geq m$, we rewrite

$$W(a(y))=\xi^{-1}(y)|y|^{1/2}1_{v(y)\geq 0}(f_0+(1-q^{-1})\sum_{n=1}^{m-1}f_nt^{2n}+$$

$$(\frac{t^{2m}-t^{2(v(y)+1)}}{1-t^2}-q^{-1}\frac{t^{2m}-t^{2(v(y)+2)}}{1-t^2})f_m).$$

If $t=1$ we can analytically continue the above formula. By the discussion in the section 3.1.6 of [28], we have

$$\|W\|^2=(1-q^{-1})^{-1}(|f_0|^2+\sum_{n=1}^{m-1}|f_n|^2q^{-n}(1-q^{-1})+|f_m|^2q^{-m}).$$

We apply Cauchy-Schwarz and get the lemma.

We apply the second case of Lemma 4.2.1 to the above $W=W_f$ and obtain for $\Re(s)=1/2+\epsilon/2$ (4.3.1) by noting

$$|l^{|\cdot|^s}(n(\varpi^{-l})W)|\leq(\sum_{k=l}^{\infty}|W(\varpi^k)|^2)^{1/2}(\sum_{k=l}^{\infty}q^{-k(1-\epsilon)})^{1/2}+|W(\varpi^{l-1})|\frac{q^{-(l-1)/2}}{q-1}.$$

The analog of (4.3.1) at an infinite place is just a consequence of integration by parts. Take the case of a real place for example, if $W\in W_\pi^\infty$ then we know that $W(a(y))$ is of rapid decay as $|y|\to\infty$, controlled by $|y|^{1/2}$ as $|y|\to 0$, as well as $X.W$ for any X in the enveloping algebra of G. Consequently

$$l^{|\cdot|^s}(n(t)W)=-\frac{1}{t}\int_F n(t).U.W(a(y))|y|^{s-2}+(s-1)n(t).W(a(y))|y|^{s-2}dy, \Re(s)=1/2+\epsilon.$$

The right side converges thanks to the upper bounds of $W(a(y)), U.W(a(y))$, where recall $U=\begin{pmatrix}1 & \\ & 0\end{pmatrix}$. We then use the local functional equation to see

$$l^{|\cdot|^{s-1}}(n(t)W)=\gamma(s-1/2,\pi,\psi)^{-1}l^{|\cdot|^{1-s}}(w.n(t).W).$$

The gamma factor $\gamma(s-1/2,\pi,\psi)=\gamma(s-1/2,\xi,\psi)\gamma(s-1/2,\xi^{-1},\psi)$ is of size $\asymp_\epsilon C(\xi)^{1-2\epsilon}|s|^{1-2\epsilon}$, while the integral is bounded, separating the contributions from $|y|\leq 1$ and from $|y|>1$, as $\ll_\epsilon \|w.n(t).W\|+\|T.w.n(t).W\|$, with $T=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ in the Lie algebra of G. We do similar estimation for $n(t).U.W$. Using Theorem 1.7.1, We thus find

$$|l^{|\cdot|^s}(n(t)W)| \ll_\epsilon |t|^{-1}|s|^{-2+\epsilon}C(\xi)^{-1+\epsilon}\|\Delta.W\|, \forall\epsilon>0, \Re(s)=1/2+\epsilon. \tag{4.3.2}$$

Finally, note that if v is a complex place, the proof of Theorem 1.7.1 given here implies that we should replace $\|\Delta.W\|$ by $\|\Delta^8.W\|$ in (4.3.2).

5 Global Estimation: Cuspidal Case

5.1 Truncation

The goal of this section is to establish Lemma 3.1.3.

Fix a function $h_0 \in C^\infty(\mathbb{R}^+)$ supported in $(0,2]$ such that $h_0|_{(0,1]} = 1$ and $0 < h_0 < 1$. Denote by $\mathcal{M}(\cdot)$ the Mellin transform. For any $A > 0$, let $h_{0,A}(t) = h_0(t/A)$. The following relation is immediate:

$$|\mathcal{M}(\sigma * h_{0,Q^{-\kappa-1}})(s)| \leq 4^{|\Re(s)|} Q^{-(\kappa+1)\Re(s)} |\mathcal{M}(h_0)(s)|.$$

For any $t > 0$, choose $y_t \in \mathbb{A}^\times$ such that $|y_t| = t$, and define

$$f(t) = \int_{F^\times \backslash \mathbb{A}^{(1)}} \varphi(a(yy_t))\chi(yy_t) d^\times y,$$

then

$$l^{\chi, \sigma * h_{0,Q^{-\kappa-1}}}(\varphi) = \int_0^{+\infty} \sigma * h_{0,Q^{-\kappa-1}}(t) f(t) d^\times t.$$

Note that $\mathcal{M}(f)(s) = l^{\chi|\cdot|^s}(\varphi)$, Mellin inversion gives,

$$|l^{\chi, \sigma * h_{0,Q^{-\kappa-1}}}(\varphi)| = |\int_{\Re(s) = -1/2-\epsilon} \mathcal{M}(\sigma * h_{0,Q^{-\kappa-1}})(-s) l^{\chi|\cdot|^s}(\varphi) \frac{ds}{2\pi i}|$$

$$\ll Q^{-(\kappa+1)(1/2+\epsilon)} \int_{\Re(s) = -1/2-\epsilon} |\mathcal{M}(h_0)(-s) l^{\chi|\cdot|^s}(\varphi)| ds.$$

According to (1.2.7), one can write

$$l^{\chi|\cdot|^s}(\varphi) = L(\pi \otimes \chi, s+1/2) \prod_{v|\infty} l^{\chi_v|\cdot|_v^s}(n(T_v)W_{0,v}) \prod_{v<\infty} \frac{l^{\chi_v|\cdot|_v^s}(n(T_v)W_{0,v})}{L(\pi_v \otimes \chi_v, s+1/2)}$$

$$= L^{(S)}(\pi \otimes \chi, s+1/2) \prod_{v \in S} l^{\chi_v|\cdot|_v^s}(n(T_v)W_{0,v}),$$

where S is the subset of places v for which $T_v \neq 0$. From Corollary 4.1.3 and Corollary 4.2.2, one sees that for each $v \in S$, $|l^{\chi_v|\cdot|_v^s}(n(T_v)W_{0,v})| \ll_{\epsilon, \varphi_0} C(\chi_v)^{-1/2+\epsilon}$ and the product of the implicit constants tends to 0 as S increases. So

$$\prod_{v\in S} l^{\chi_v|\cdot|_v^s}(n(T_v)W_{0,v}) \ll_{\epsilon,\varphi_0} Q^{-1/2+\epsilon}.$$

By the convexity bound together with bounds towards the Ramanujan-Petersson conjecture, we have

$$L^{(S)}(\pi\otimes\chi, s+1/2) \ll_\epsilon (1+|s|)^2 C(\pi\otimes\chi)^{1/2+\epsilon}, \Re(s) = -1/2-\epsilon.$$

Note that $C(\pi\otimes\chi) \ll C(\pi)C(\chi)^2$, we finally get

$$l^{\chi,\sigma * h_{0,Q^{-\kappa-1}}}(\varphi) \ll_{\epsilon,\varphi_0,h_0} Q^{-\kappa/2+\epsilon}.$$

Similar argument, using Mellin inversion for $\Re s = 1/2+\epsilon$, gives

$$l^{\chi,\sigma*(1-h_{0,Q^{\kappa-1}})}(\varphi) \ll_{\epsilon,\varphi_0,h_0} Q^{-\kappa/2+\epsilon}.$$

Lemma 3.1.3 is proved by taking $h = h_{0,Q^{\kappa-1}} - h_{0,Q^{-\kappa-1}}$.

We will need to exploit the Mellin transform of h further. Since for any $h \in C_c(\mathbb{R}_+)$

$$\mathscr{M}(h)(s) = (-1)^n \frac{\mathscr{M}(h^{(n)})(s+n)}{s(s+1)\cdots(s+n-1)},$$

we have, for $h = h_{0,A}$,

$$\mathscr{M}(h^{(n)})(s) = A^{s-n}\mathscr{M}(h_0^{(n)})(s).$$

For $h = h_{0,Q^{\kappa-1}} - h_{0,Q^{-\kappa-1}}$, we thus have for $n \geq 1$

$$\mathscr{M}(h)(s) = (-1)^n \frac{(Q^{(\kappa-1)s} - Q^{-(\kappa+1)s})\mathscr{M}(h_0^{(n)})(s+n)}{s(s+1)\cdots(s+n-1)}.$$

Note that $h_0^{(n)}$ is supported in $[1,2]$ and

$$|\mathscr{M}(h)(s)| \leq \frac{2\kappa|s|\log Q \max(Q^{(\kappa-1)\Re(s)}, Q^{-(\kappa+1)\Re(s)})\|h_0^{(n)}\|_\infty \int_1^2 t^{\Re(s)+n} d^\times t}{|s(s+1)\cdots(s+n-1)|}$$

$$\ll_{\Re(s)+n} \frac{2\kappa\log Q\|h_0^{(n)}\|_\infty Q^{(\kappa-1)\Re(s)}}{|(s+1)\cdots(s+n-1)|}, \Re(s) \geq 0, \tag{5.1.1}$$

$$\ll_{\Re(s)+n} \frac{2\kappa\log Q\|h_0^{(n)}\|_\infty Q^{(-\kappa-1)\Re(s)}}{|(s+1)\cdots(s+n-1)|}, \Re(s) < 0. \tag{5.1.2}$$

5.2 Estimation of the Constant Contribution

Writing the Fourier expansion

$$\varphi_0(g) = \sum_{\alpha\in F^\times} W_0(a(\alpha)g),$$

we obtain

$$(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0)_N(g) = \sum_{\alpha\in F^\times} W_0(a(\alpha)ga(\frac{\varpi_{v_1}}{\varpi_{v_1'}}))\overline{W_0(a(\alpha)ga(\frac{\varpi_{v_2}}{\varpi_{v_2'}}))}.$$

As a consequence, we get a Rankin-Selberg like equality for $\Re(s)$ big enough,

$$l^{|\cdot|^s}((a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0)_N) = \int_{\mathbb{A}^\times} W_0(a(y)a(\frac{\varpi_{v_1}}{\varpi_{v_1'}}))\overline{W_0(a(y)a(\frac{\varpi_{v_2}}{\varpi_{v_2'}}))}|y|^s d^\times y. \tag{5.2.1}$$

This integral splits into product of local factors

$$\int_{\mathbb{A}^\times} W_0(a(y)a(\frac{\varpi_{v_1}}{\varpi_{v_1'}}))\overline{W_0(a(y)a(\frac{\varpi_{v_2}}{\varpi_{v_2'}}))}|y|^s d^\times y = \prod_{v|\infty}\int_{F_v^\times} |W_{0,v}(a(y))|^2|y|_v^s d^\times y\cdot$$

$$\frac{L(s+1,\pi\times\bar{\pi})}{\zeta_F(2s+2)}\prod_{v<\infty}\frac{\zeta_v(2s+2)\int_{F_v^\times} W_{0,v}(a(y)a(u_v))\overline{W_{0,v}(a(y)a(u_v'))}|y|_v^s d^\times y}{L(s+1,\pi_v\times\bar{\pi}_v)}.$$

Here, u_v, u_v' are suitably chosen according to $\{v_1, v_1', v_2, v_2'\}$. For almost all v, the local term equals 1. This identity admits meromorphic continuation to $\mathbb{C}$ and is holomorphic for $\Re(s) > 0$. By the convergence of $L(s,\pi\times\bar{\pi})$, we have

$$\frac{L(s+1,\pi\times\bar{\pi})}{\zeta_F(2s+2)} \ll_\epsilon 1, \text{ for } \Re(s) = \epsilon > 0.$$

If v is a ramified place for π, we can always say that the corresponding local factor is bounded by some constant depending only on $\Re(s), \pi$. So we may only consider unramified places of π. On such a place, $W_{0,v}$ is spherical and is the new vector (c.f. (1.5.2)). If $\alpha_{1,v}, \alpha_{2,v}$ are the Satake parameters ($|\alpha_{1,v}\alpha_{2,v}| = 1$), then

$$W_{0,v}(a(\varpi_v^m)) = q_v^{-m/2}\frac{\alpha_{1,v}^{m+1}-\alpha_{2,v}^{m+1}}{\alpha_{1,v}-\alpha_{2,v}}, m\geq 0,$$

$$W_{0,v}(a(\varpi_v^m)) = 0, m < 0.$$

Hence the corresponding local term is explicitly computable. We do the calculation of one case and leave the others to the reader. Let $u_v = \varpi_v, u_v' = 1$ (i.e. $v = v_1$ and $v \neq v_1', v_2, v_2'$), then

$$\frac{\zeta_v(2s+2)\int_{F_v^\times} W_{0,v}(a(y)a(u_v))\overline{W_{0,v}(a(y)a(u_v'))}|y|_v^s d^\times y}{L(s+1,\pi_v\times\bar{\pi}_v)} = \frac{(\mathrm{tr}_v - n_v\overline{\mathrm{tr}_v}q_v^{-s-1})q_v^{-1/2}}{1-q_v^{-2s-2}}$$

where $\mathrm{tr}_v = \alpha_{1,v} + \alpha_{2,v}, n_v = \alpha_{1,v}\alpha_{2,v}$. Let $\max(|\alpha_{1,v}|, |\alpha_{2,v}|) = q_v^{\theta_v}$, then $|\mathrm{tr}_v| \ll q_v^{\theta_v}$ we get, for $\epsilon > 0$ small,

$$|\frac{(\mathrm{tr}_v - n_v\overline{\mathrm{tr}_v} q_v^{-s-1})q_v^{-1/2}}{1-q_v^{-2s-2}}| \ll q_v^{-1/2}|\mathrm{tr}_v|, \Re(s) = \epsilon.$$

Similarly, we deal with all the other cases and get

$$\prod_{v<\infty} \frac{\zeta_v(2s+2)\int_{F_v^\times} W_{0,v}(a(y)a(u_v))\overline{W_{0,v}(a(y)a(u_v'))}|y|_v^s d^\times y}{L(s+1,\pi_v\times\bar{\pi}_v)} \ll_{\epsilon,\pi} \prod_{v=v_1,v_1',v_2,v_2'} q_v^{-1/2}|\mathrm{tr}_v|.$$

Inserting it to (5.2.1), we obtain

$$l^{|\cdot|^s}((a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0)_N) \ll_{\epsilon,\pi} \prod_{v=v_1,v_1',v_2,v_2'} q_v^{-1/2}|\mathrm{tr}_v|, \Re(s) = \epsilon > 0.$$

Note that

$$S_{cst}(v_1,v_1',v_2,v_2') = \int_{\Re(s)=\epsilon} \mathcal{M}(h)(-s) l^{|\cdot|^s}((a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0)_N)\frac{ds}{2\pi i},$$

which with (5.1.2) gives

$$S_{cst}(v_1,v_1',v_2,v_2') \ll_{F,\pi,\epsilon} \kappa \log Q Q^{(1+\kappa)\epsilon} E^{-2} \prod_{v=v_1,v_1',v_2,v_2'} |\mathrm{tr}_v|.$$

Lemma 5.2.1. *We have Ramanujan conjecture on average, i.e.*

$$\sum_{q_v\in I_E} |\mathrm{tr}_v| \ll_\epsilon M_E E^\epsilon$$

In fact, by the theory of Rankin-Selberg, $L(s,\pi\times\bar{\pi})$ is meromorphic and only has possible simple poles at $s = 0,1$. This implies $\sum_{\alpha \text{ ideal of } F, N_F(\alpha)\le N} |\lambda_\pi(\alpha)|^2 \ll_\epsilon N^{1+\epsilon}, \forall\epsilon > 0$. Here, $\lambda_\pi(\alpha)$ is the Hecke eigenvalues which coincides with tr_v when α is the prime ideal corresponding with v. Using (3.1.9), we obtain Lemma 3.1.5 from Lemma 5.2.1.

5.3 Estimation of the Cuspidal Constribution

The goal of this section is to establish Lemma 3.1.6. Recall that we are reduced to estimate

$$S_{cusp}(v_1,v_1',v_2,v_2') = \sum_{\pi' \text{cuspidal}} l^h(n(T)P_{\pi'}(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0)).$$

The projector $P_{\pi'}$ is realized by the choice of a basis of π', denoted by $\mathcal{B}(\pi'; v_1, v_1', v_2, v_2')$. It is determined by the choices of local basis of π_v', denoted by $\mathcal{B}_v(\pi'; v_1, v_1', v_2, v_2')$. When there is no confusion, we may write them shortly as $\mathscr{B}$ resp. $\mathscr{B}_v$. There are related with each other by

$$\mathscr{B} = \otimes'_v \mathscr{B}_v, e \leftrightarrow (W_{e,v})_v.$$

Here, $W_{e,v}$ is the component at v of e in the Kirillov model. We may also write it as e_v if there is no confusion. According to Remark 1.6.4, we only need to choose $\mathscr{B}_v$ for $v < \infty$.

Definition 5.3.1. *Denote, for any subgroup $H \subset G(F_v)$ and $g \in G(F_v)$, $H^g = gHg^{-1}$. Then the Harish-Chandra's function $\Xi_v^{g_0}$ associated to the Borel subgroup $B(F_v)^{g_0}$ is given by, with notations in Section 1.9*

$$\Xi_v^{g_0}(g) = \Xi_v(g_0^{-1} g g_0).$$

Definition 5.3.2. *Suppose $v(\pi') = m$. For any integer n, recall that the space of $K_v^0[n]$-invariant vectors of π'_v is of dimension* $\max(n-m+1, 0)$. *A* ***standard basis*** *of level n consists of, for each integer l s.t. $m \le l \le n$, a vector invariant by $K_v^0[l]$ and orthogonal to all the vectors invariant by $K_v^0[l-1]$, and vectors orthogonal to the space of $K_v^0[n]$-invariant vectors. A* ***nice basis*** *of level n w.r.t. $g \in G_v$ consists of the g translates of the vectors of a standard basis of level n. Define* ***the maximal compact subgroup*** *K_v^* of G_v* ***associated with the above nice basis*** *to be*

$$K_v^* = K_v^g.$$

If $\mathscr{B}_v$ is a standard or nice basis of level n, we write $\mathscr{B}_v^*$ to be the elements in $\mathscr{B}_v$ invariant by $K_v^0[n]$ or its corresponding translate. We also call the basis as in Remark 1.6.4 standard. At an infinite place, we define $\mathscr{B}_v^* = \mathscr{B}_v$. We write

$$\mathscr{B}^* = \otimes'_v \mathscr{B}_v^*$$

If $\mathscr{B}_v$ is of level n, then

$$\sum_{e_v \in \mathscr{B}_v^*} \dim(K_v e_v) \ll q_v^n \tag{5.3.1}$$

We choose $\mathscr{B}_v$ and K_v^* explicitly as follows:

Case 1: v is different from v_1, v'_1, v_2, v'_2 resp. $v = v_1 = v_2$ resp. $v = v'_1 = v'_2$. Since $a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0$ is $K_v^0[v(\varphi_0)]$ resp. $K_v^0[v(\varphi_0)]^{a(\varpi_v)}$ resp. $K_v^0[v(\varphi_0)]^{a(\varpi_v^{-1})}$ invariant, we take $\mathscr{B}_v$ to be a standard basis of level $v(\varphi_0)$ resp. a nice basis of level $v(\varphi_0)$ w.r.t. $a(\varpi_v)$ resp. a nice basis of level $v(\varphi_0)$ w.r.t. $a(\varpi_v^{-1})$.

Case 2: $v = v_1$ or v_2 resp. $v = v'_1$ or v'_2. Since $a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0$ is $K_v^0[v(\varphi_0)+1]^{a(\varpi_v)}$ resp. $K_v^0[v(\varphi_0)+1]$ invariant, we take $\mathscr{B}_v$ to be a nice basis of level $v(\varphi_0)+1$ w.r.t. $a(\varpi_v)$ resp. a standard basis of level $v(\varphi_0)+1$.

Case 3: $v = v_1 = v'_2$ or $v = v_2 = v'_1$. Since $a(\frac{\varpi_{v_1}}{\varpi_{v'_1}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v'_2}})\overline{\varphi}_0$ is $K_v^0[v(\varphi_0)+2]^{a(\varpi_v)}$ invariant, we take $\mathscr{B}_v$ to be a nice basis of level $v(\varphi_0)+2$ w.r.t. $a(\varpi_v)$.

Then we rewrite

$$S_{cusp}(v_1, v_1', v_2, v_2') = \sum_{\pi'} \sum_{e \in \mathscr{B}^*} \langle a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0, e\rangle l^h(n(T)e). \tag{5.3.2}$$

We have

$$l^h(n(T)e) = \int_{\Re(s)=0} \mathscr{M}(h)(-s) l^{|\cdot|^s}(n(T)e)\frac{ds}{2\pi i}, \tag{5.3.3}$$

and since the vector e is a pure tensor,

$$l^{|\cdot|^s}(n(T)e) = L(s+1/2, \pi') \prod_{v|\infty} l^{|\cdot|^s}(n(T_v)W_{e,v}) \prod_{v<\infty} \frac{l^{|\cdot|^s}(n(T_v)W_{e,v})}{L(s+1/2, \pi_v')}.$$

The convexity bound gives

$$|L(s+1/2, \pi')| \ll_\epsilon C(\pi' \otimes |\cdot|^s)^{1/4+\epsilon} \ll (1+|s|)^{1/2+\epsilon} C(\pi')^{1/4+\epsilon}, \forall \epsilon > 0.$$

Lemma 5.3.3. *Let I_1 be the set of places v s.t. $v \in \{v_1, v_1', v_2, v_2'\}$ and π_v is unramified, π_v' is ramified. Let I_2 be the set of places v s.t. $v \in \{v_1, v_1', v_2, v_2'\}$ and π_v, π_v' are unramified. Then we have* $\forall \epsilon > 0$

$$|l^{|\cdot|^s}(n(T)e)| \leq_{\epsilon,\theta,\varphi_0} (1+|s|)^{1/2+\epsilon} |T|^{-1/2+\theta+\epsilon} \lambda_{e,\infty}^{3/4+\epsilon} \prod_{v \in I_1} q_v^{v(\pi')/4+\epsilon} \prod_{v \in I_2} \dim(K_v^* e_v)^{1/2}.$$

Write

$$M_v(e) = \sup_{s \in i\mathbb{R}} |C(\pi_v')^{1/4+\epsilon} l^{|\cdot|^s}(n(T_v)W_{e,v})|, \forall v|\infty,$$

$$M_v(e) = \sup_{s \in i\mathbb{R}} |C(\pi_v')^{1/4+\epsilon} \frac{l^{|\cdot|^s}(n(T_v)W_{e,v})}{L(s+1/2, \pi_v')}|, \forall v < \infty.$$

To prove Lemma 5.3.3, we estimate the local terms $M_v(e)$ case by case. This is technical and will be given in the following subsections. Lemma 5.3.3 will be a consequence of Corollary 5.3.5, 5.3.8, Lemma 5.3.6, 5.3.9, 5.3.10, as well as Lemma 1.5.1 and the remark following it (with $\|e\|_{X(F)} = 1$). Recall the following bound resulting from (5.1.1)

$$\int_{i\mathbb{R}} |\mathscr{M}(h)(-s)|(1+|s|)^{1/2+\epsilon} \frac{ds}{2\pi i} \ll_\epsilon 2\kappa \log Q \|h_0^{(3)}\|_\infty.$$

From it we obtain

$$(5.3.2) \leq \sum_{\pi'} \sum_e |\langle a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0, e\rangle| \times |l^h(n(T)e)|$$

$$\ll_{F,\epsilon,\theta,\varphi_0,h_0} |T|^{-1/2+\theta+\epsilon} \|P_{\text{cusp}}\left(\Delta_\infty^{5/4+2\epsilon}\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0\right)\right)\| \times$$

$$\sqrt{\sum_{\pi'} \sum_{e_\infty \in \otimes_{v|\infty}\mathscr{B}_v^*} \lambda_{e,\infty}^{-1-\epsilon} \prod_{v<\infty, T_v \neq 0} (v(\varphi_0)+1) \prod_{v<\infty, T_v=0, v(\varphi_0)>0} (v(\varphi_0)+3) S(I_1,I_2)} \tag{5.3.4}$$

with

$$S(I_1,I_2) = \prod_{v\in I_1} \sum_{e_v \in \mathscr{B}_v^*} q_v^{v(\pi')/2+\epsilon} \prod_{v\in I_2} \sum_{e_v\in\mathscr{B}_v^*} \dim(K_v^* e_v).$$

It is easy to see that $\| P_{\mathrm{cusp}}\left(\Delta_\infty^{5/4+2\epsilon}\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0\right)\right)\|$ is bounded above by some Sobolev norm of φ_0. In a typical situation as mentioned in Remark 3.1.10, distinguishing $v(\pi')=0$ and $v(\pi')=1$ and using (5.3.1) we get

$$S(I_1,I_2) \ll \prod_{v\in\{v_1.v_1',v_2,v_2'\}} q_v \ll E^4.$$

Therefore

$$(5.3.4) \ll_{\varphi_0} E^2 (\text{trace of } \Delta_\infty^{-1-\epsilon})^{1/2},$$

which proves the first part of Lemma 3.1.6. The second part follows from similar bounds for $S(I_1,I_2)$ in the other 6 situations mentioned in Remark 3.1.10.

5.3.1 At v such that $T_v \neq 0$

In this case, $\mathscr{B}_v$ is given by the first case of **Case 1**, hence is standard. Note that

$$|l^{|\cdot|^s}(n(T_v)W_{e,v})|^2 = \int_{F_v^\times} \langle n(-T_v)a(y)n(T_v)W_{e,v}, W_{e,v}\rangle |y|^s d^\times y.$$

By Theorem 1.9.1, we get,

$$|l^{|\cdot|^s}(n(T_v)W_{e,v})|^2 \leq A_v(\epsilon)\dim(K_v e_v)\|W_{e,v}\|^2 \int_{F_v^\times} \Xi_v(n(-T_v)a(y)n(T_v))^{1-2\theta-\epsilon} d^\times y. \tag{5.3.5}$$

Lemma 5.3.4. *For any $\epsilon>0$, we have*

$$|l^{|\cdot|^s}(n(T_v)W_{e,v})| \ll_{\epsilon,\theta} |T_v|_v^{-1/2+\theta+\epsilon} \dim(K_v e_v)^{1/2}\|W_{e,v}\|.$$

Corollary 5.3.5. *There exist a constant $C(\theta,\epsilon)$ depending only on θ and ϵ s.t.*

If $v|\infty$, then we have

$$M_v(e) \leq C(\theta,\epsilon)\lambda_{e,v}^{3/4+\epsilon}|T_v|_v^{-1/2+\theta+\epsilon}\|W_{e,v}\|.$$

If $v<\infty$, then

$$M_v(e) \leq C(\theta,\epsilon)|T_v|_v^{-1/2+\theta+\epsilon} q_v^{3v(\varphi_0)/4}\|W_{e,v}\|.$$

Note that $\dim(K_v e_v), C(\pi'_v) \ll \lambda_{e,v}$ if $v|\infty$, and e_v is $K_v^0[v(\varphi_0)]$ invariant by the choice of $\mathscr{B}_v^*$, $v(\pi') \leq v(\varphi_0)$ if $v < \infty$, we deduce the corollary from the lemma by noting that

$$[K_v : K_v^0[v(\varphi_0)]] \ll q_v^{v(\varphi_0)}.$$

Let us now prove Lemma 5.3.4 place by place.

At a Real Place : $F_v = \mathbb{R}$

Recall the (bi-K_v-invariant, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$-invariant) Harish-Chandra's function as in [15] 5.2.2 is given by:

$$\Xi_v(\begin{pmatrix} e^{r/2} & 0 \\ 0 & e^{-r/2} \end{pmatrix}) = \mathfrak{P}_{-1/2}(\cosh r), r > 0.$$

For some absolute constants $\alpha, \beta > 0$, we have

$$\mathfrak{P}_{-1/2}(\cosh r) \leq e^{-r/2}(\alpha + \beta r).$$

We make a change of variable $t = \dfrac{y + y^{-1}}{2}$ and get

$$\int_{\mathbb{R}^\times} \Xi_v(n(-T_v)a(y)n(T_v))^{1-2\theta} d^\times y$$

$$\leq 2(1+T_v^2)^{-\frac{1-2\theta}{2}}(1+\log(1+T_v^2))^{1-2\theta} \int_1^\infty (t-1)^{-1/2+\theta}(\alpha' + \beta \log t)^{1-2\theta}$$

$$+ t^{-1/2+\theta}(\alpha' + \beta \log(t+1))^{1-2\theta} \frac{dt}{\sqrt{t^2-1}}$$

$$\ll_\theta (1+T_v^2)^{-\frac{1-2\theta}{2}}(1+\log(1+T_v^2))^{1-2\theta}.$$

We get the lemma at v using (5.3.5).

At a Complex Place : $F_v = \mathbb{C}$

The Harish-Chandra's function as in [15] 5.2.1 is given by:

$$\Xi_v(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) = \frac{2\log t}{t - t^{-1}}, t > 0.$$

When we evaluate it at $n(-T_v)a(y)n(T_v)$, the corresponding t satisfies

$$t^2 + t^{-2} = |y| + |y|^{-1} + \frac{|T_v|^2|y-1|^2}{|y|}.$$

This expression being invariant by the change of variable $y \mapsto y^{-1}$, we get, with the change of variable $r = \dfrac{|y| + |y|^{-1}}{2}$

$$\int_{\mathbb{C}^\times} \Xi_v(n(-T_v)a(y)n(T_v))^{1-2\theta} d^\times y = 2\int_{|y|>1} (\frac{2\log t}{t - t^{-1}})^{1-2\theta} d^\times y$$

$$\leq 2(2(1+|T_v|^2))^{-\frac{1-2\theta}{2}} (\log 2(1+|T_v|^2))^{1-2\theta} \cdot 2\pi \int_1^\infty (\frac{1+\frac{\log(r+1)}{\log 2}}{\sqrt{r-1}})^{1-2\theta} \frac{dr}{\sqrt{r^2-1}}$$

$$\ll_\theta (1+T_v^2)^{-\frac{1-2\theta}{2}} (1+\log(1+T_v^2))^{1-2\theta}.$$

We get the lemma at v using (5.3.5).

At a Non Archimedean Place

The values of the Harish-Chandra function associated with the standard Borel subgroup can be inferred from the Macdonald formula, Theorem 4.6.6 of [2], by letting $\alpha_1 \to 1, \alpha_2 = 1$.

$$\Xi_v(n) = \Xi_v(\begin{pmatrix} \varpi_v^n & 0 \\ 0 & 1 \end{pmatrix}) = q_v^{-n/2} + nq_v^{-n/2} \frac{1-q_v^{-1}}{1+q_v^{-1}}, n \geq 0$$

Apply (42) of [15] to the torus $\mathbb{T} = n(-T_v)Z_v A_v n(T_v)$, the local integral can be calculated and bounded as, with $d = \max(0, -v(T_v))$

$$q_v^{d_v/2} \int_{F_v^\times} \Xi_v(n(-T_v)a(y)n(T_v))^{1-2\theta} d^\times y$$

$$= 2\sum_{n>2d} \Xi_v(n)^{1-2\theta} + \sum_{n=1}^{d-1} \frac{q_v^{d-n} - q_v^{d-n-1}}{q_v^d - q_v^{d-1}} \Xi_v(2(d-n))^{1-2\theta}$$

$$+ \frac{1}{q_v^d - q_v^{d-1}} \Xi_v(0)^{1-2\theta} + \frac{q_v^d - 2q_v^{d-1}}{q_v^d - q_v^{d-1}} \Xi_v(2d)^{1-2\theta}$$

$$\ll C(\theta) \max(1, |T_v|)^{-(1-2\theta)} (1 + \max(1, \log|T_v|))^{2-2\theta}.$$

We get the lemma at v using (5.3.5) and conclude the lemma. We record the following estimation: for some constant $C'(\theta)$ depending only on θ,

$$q_v^{d_v/2} \int_{F_v^\times} \Xi_v(a(y))^{1-2\theta} d^\times y \leq 2\sum_{n>0} (n+1) q_v^{-n(1/2-\theta)} + 1 \leq C'(\theta) \tag{5.3.6}$$

5.3.2 At v such that $T_v = 0$, π_v ramified

The number of such places is finite and depends only on π. $\mathscr{B}_v$ is given by **Cases 1,2,3**. Applying (5.3.5) (with $T_v = 0$) and (5.3.6) we get similarly to section 5.3.1

Lemma 5.3.6. *For $\forall \epsilon > 0$, there is a constant $C(\theta, \epsilon)$ s.t.*

$$M_v(e) \leq C(\theta, \epsilon) q_v^{3v(\varphi_0)/4 + 3/2 + \epsilon} \| W_{e,v} \|.$$

This follows from the bound $\dim(K_v^* e_v) \le q_v^{v(\varphi_0)+2}$: a suitable translate of e_v is at most $K_v^0[v(\varphi_0)+2]$ invariant, and $v(\pi') \le v(\varphi_0)+2$.

5.3.3 At v such that $T_v = 0$, π_v unramified, π'_v ramified

In that case $v \in \{v_1, v'_1, v_2, v'_2\}$. So the number of possible places is at most 4 and $v(\pi') \le 2$. By the theory of new vectors and conductor, we know that $y \mapsto W_{e,v}(a(y))$ is supported in $\{y \in F_v^\times : v(y) = m\}$ for some integer $m \ge 0$ (c.f. [10]). Therefore, by Cauchy-Schwarz (since $s \in i\mathbb{R}$) we deduce

Lemma 5.3.7. *We have*

$$|\frac{l^{|\cdot|^s}(W_{e,v})}{L(s+1/2,\pi'_v)}| = |l^{|\cdot|^s}(W_{e,v})| \le q_v^{-d_v/2} \| W_{e,v} \|.$$

Corollary 5.3.8. *For any $\epsilon > 0$, there is $C(\epsilon)$ depending only on ϵ s.t.*

$$M_v(e) \le C(\epsilon) q_v^{v(\pi')/4+\epsilon} q_v^{-d_v/2} \| W_{e,v} \|.$$

5.3.4 At v such that $T_v = 0$, π_v unramified, π'_v unramified

If $v \notin \{v_1, v'_1, v_2, v'_2\}$ then e_v is spherical and we have

Lemma 5.3.9. *For $v \notin \{v_1, v'_1, v_2, v'_2\}$, we have*

$$M_v(e) = \frac{\| W_{e,v} \|}{\sqrt{L(1/2, \pi'_v \times \bar{\pi}'_v)}}.$$

Note that almost all v are in this category.

If $v \in \{v_1, v'_1, v_2, v'_2\}$, then we apply (5.3.5) and (5.3.6) to get

Lemma 5.3.10. *For $v \in \{v_1, v'_1, v_2, v'_2\}$, there is a constant $C(\theta, \epsilon)$ depending only on θ and ϵ,*

$$M_v(e) \le C(\theta, \epsilon) \dim(K_v^* e_v)^{1/2} \| W_{e,v} \|$$

5.4 Estimation of the Eisenstein Contribution

The goal of this section is to establish Lemma 3.1.7. First we rewrite

$$S_{Eis}(v_1, v_1', v_2, v_2') = \sum_{\xi \in \widehat{F^\times \backslash \mathbb{A}^{(1)}}} \sum_{\Phi \in \mathscr{B}(\pi(\xi,\xi^{-1}))} \int_{-\infty}^{\infty} < a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_0 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_0, E(\Phi, i\tau) > \cdot$$

$$l^h(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau)))d\tau.$$

The treatment of $l^h(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau)))$ is similar to that of $l^h(n(T)e)$ in the previous section, except that we can take $\theta = 0$. One starts with

$$l^h(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau))) = \int_{\Re(s)\gg 1} \mathscr{M}(h)(-s) l^{|\cdot|^s}(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau)))\frac{ds}{2\pi i}$$

with

$$l^{|\cdot|^s}(n(T)(E(\Phi, i\tau) - E_N(\Phi, i\tau))) = \frac{\Lambda(s + i\tau + 1/2, \xi)\Lambda(s - i\tau + 1/2, \xi^{-1})}{\Lambda(1 + 2i\tau, \xi^2)} \tag{5.4.1}$$

$$\cdot \prod_v \frac{L(1 + 2i\tau, \xi_v^2) l^{|\cdot|^s}(n(T_v)W_{\Phi_{i\tau},v})}{L(s + i\tau + 1/2, \xi_v)L(s - i\tau + 1/2, \xi_v^{-1})}$$

where $\Lambda(\cdot, \xi)$ is the completed (GL_1) L-function. (5.4.1) has an analytic continuation and admits simple poles at $s = 1/2 \pm i\tau$ only when $\xi = 1$ is the trivial character and $\tau \neq 0$.

If $\xi = 1$, we shift the integral into $\Re s = 1/2 + \epsilon$. The local factors for which $T_v \neq 0$ are bounded by using (4.3.1), (4.3.2). For those for which $v \in \{v_1, v_1', v_2, v_2'\}$, $T_v = 0$, we use instead

$$|l^{|\cdot|^s}(W_{\Phi_{i\tau},v})| \leq \|W_{\Phi_{i\tau},v}\| \int_{\mathrm{supp} W_{\Phi_{i\tau},v}} |y|^{1+2\epsilon} d^\times y.$$

Together with (5.1.2), we deduce that they are of size $O_\epsilon(Q^{(\kappa-1)/2+\epsilon}E)$.

We then shift the contour to $\Re s = 0$ if $\xi \neq 1$, and the estimation is just as in the cuspidal case: In order to bound the contribution on the line $\Re(s) = 0$, we use the bound

$$|l^{|\cdot|^s}(n(T_v)W_{\Phi_{i\tau},v})|^2 \leq \dim(K_v W_{\Phi_{i\tau},v})\|W_{\Phi_{i\tau},v}\|^2 \int_{F_v^\times} \Xi_v(n(-T_v)a(y)n(T_v))d^\times y.$$

Since Ξ_v is a matrix coefficient, one always has $\Xi_v \leq 1$, so $\Xi_v \leq \Xi_v^{1-\epsilon}$ for any $\epsilon > 0$. We get

$$|l^{|\cdot|^s}(n(T_v)W_{\Phi_{i\tau},v})| \ll_\epsilon (1 + |T_v|)^{-1/2+\epsilon}(\dim(K_v W_{\Phi_{i\tau},v}))^{1/2}\|W_{\Phi_{i\tau},v}\|.$$

Similarly to the previous section, using the convexity bounds for GL_1 L-functions, the contribution on the line $\Re(s) = 0$ is bounded by $O_{\epsilon,F,\pi,\kappa,h_0}(E^2 Q^{-1/2+\epsilon})$. This completes the proof of Lemma 3.1.7.

6 Global Estimation: Eisenstein Case

6.1 The Truncation Process

We use a truncation process slightly more general than the one in the section 5 of [20]. We'll assume the knowledge of that paper and use the results of that section implicitly in the sequel. Recall the height function $H(g)$ defined by

$$g \in \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} K \mapsto H(g) = |\frac{a}{b}|.$$

Definition 6.1.1. *A function $f : B(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is an admissible truncator if its support is contained in $\{g : H(g) \geq c\}$ for some $c > 0$ and transforms under $Z(\mathbb{A})$ as a Hecke character ω. This character is called the central character of f.*

Definition 6.1.2. *Given an admissible truncator f, the associated truncation operator $\Lambda(f)$ is defined by*

$$\Lambda(f)(\varphi)(g) = \varphi(g) - \sum_{\gamma \in B(F)\backslash G(F)} f(\gamma g),$$

where $\varphi : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ has the same central character as f. The sum over γ is finite.

If φ is continuous, and $\varphi - f$ is rapidly decaying in any Siegel domain, then $\Lambda(f)(\varphi)$ is rapidly decaying in any Siegel domain, thus lies in $L^2(G(F)\backslash G(\mathbb{A}), \omega)$. Let $u \in C^\infty(\mathbb{R})$ be real valued and satisfy

$$\operatorname{supp} u \subset [1, \infty), u|_{[2,\infty)} = 1, 0 \leq u \leq 1.$$

If φ is a unitary Eisenstein series constructed from a smooth vector in the induced model, then $f(g) = \varphi_N(g) u(H(g)/c)$ is an admissible truncator for any $c > 0$, and $\Lambda(f)(\varphi)$ is rapidly decaying in any Siegel domain. Similarly, if $\varphi = \prod_{i=1}^{r} \varphi^i$ is a finite product of Eisenstein series constructed from smooth vectors in the induced model, then $f(g) = u(H(g)/c) \prod_{i=1}^{r} \varphi_N^i(g)$ is an admissible truncator. We make the convention

$$\Lambda(c,u)(\varphi) = \Lambda(f)(\varphi)$$

for the above defined f when there is no confusion.

Consider $X = \prod_{i=1}^{d} E_i$, where $\forall E_i \exists v|\infty$ s.t. E_i is in some fixed basis of $\mathfrak{g}_v$. We call d the degree of X. For any $\sigma \in \{0,1\}^d$, we write $X^\sigma = \prod_{i=1}^{d} E_i^{\sigma(i)}$, where $E_i^0 = 1$ is the identity in the universal enveloping algebra. Suppose $c > 1$.

If $H(g) > 2c$, then

$$X.\Lambda(c,u)(\varphi)(g) = X.\varphi(g) - X.(\prod_{i=1}^{r} \varphi_N^i)(g) = \sum_{\sigma_i \in \{0,1\}^d, \sum_i \sigma_i = (1,\cdots,1)} \left(\prod_{i=1}^{r} X^{\sigma_i}.\varphi^i(g) - \prod_{i=1}^{r} X^{\sigma_i}.\varphi_N^i(g) \right).$$

But it is easy to see

$$\prod_{i=1}^{r} X^{\sigma_i}.\varphi^i - \prod_{i=1}^{r} X^{\sigma_i}.\varphi_N^i = \sum_{j=1}^{r} (X^{\sigma_j}.\varphi^j - X^{\sigma_j}.\varphi_N^j) \prod_{i<j} X^{\sigma_i}.\varphi_N^i \prod_{i>j} X^{\sigma_i}.\varphi^i.$$

Since $X^{\sigma_i}.\varphi^i - X^{\sigma_i}.\varphi_N^i$ is rapidly decaying, $X.\Lambda(c,u)(\varphi)$ is, too. Corollary 2.3.3 gives

$$X.\Lambda(c,u)(\varphi)(g) \ll_{F,N} \prod_i S_l^{\mathrm{Ind}}(\varphi^i) H(g)^{-N}.$$

Here l depends on the degree of $F/\mathbb{Q}$, $d = \deg X$ and N.

If $H(g) \leq 2c$, then

$$X.\Lambda(c,u)(\varphi)(g) = X.\varphi(g) - \sum_{\sigma_i \in \{0,1\}^d, \sum_i \sigma_i = (1,\cdots,1)} X^{\sigma_0}.(u(H(g)/c)) \prod_{i=1}^{r} X^{\sigma_i}.\varphi_N^i(g)$$

Lemma 6.1.3. *For any X as above, we have*

$$X.H(g) \ll_d H(g),$$

where d is the degree of X.

Corollary 6.1.4. *We write $|\sigma_0| = \sum_{j=1}^{d} \sigma_0(j)$, then*

$$X^{\sigma_0}.(u(H(g)/c)) \ll_{u,|\sigma_0|} 1.$$

The implicit constant is some Sobolev norm of u of order depending on $|\sigma_0|$.

In fact, H is the spherical function in $\mathrm{Ind}_B^G(1,1)$, thus if $g = znak$ is the Cartan decomposition with k_∞ being the infinite part of k,

$$X.H(g) = H(g)X.H(k_\infty).$$

Then it is routine to check $X.H(k_\infty) \ll_d 1$ by the compactness of K_∞ and the boundedness of its action on $\mathfrak{g}_\infty$. The corollary follows easily by the chain rule of derivative. We conclude

$$X.\Lambda(c,u)(\varphi)(g) = X.\varphi(g) \ll_{F,d,r} \prod_i S_l^{\mathrm{Ind}}(\varphi^i)\left(H(g)^{1/2}\log H(g)\right)^r, \forall H(g) \geq c_0.$$

for some l depending on the degree of $F/\mathbb{Q}$ and $d = \deg X$ by Corollary 2.3.3.

Proposition 6.1.5. *The L^2 norm of $X.\Lambda(c,u)(\varphi)$ satisfies*

$$\|X.\Lambda(c,u)(\varphi)\| \ll_{\epsilon,F,d,r} \prod_i S_l^{Ind}(\varphi^i)c^{(r-1)/2+\epsilon}, \forall \epsilon > 0.$$

Where l depends on F, $d = \deg X$, r.

6.2 Truncation on the Integral

In view of Proposition 2.1.1, we only need to consider the poles of

$$\mathscr{M}(\sigma * h_{0,Q^{-\kappa-1}})(-s)l^{\chi|\cdot|^s}(\varphi - \varphi_N)$$

at $s = -1/2 - i\tau$ and $1/2 - i\tau$. Here recall that we are assuming $\chi(y) = |y|^{i\tau}$. The residues are equal to

$$\mathscr{M}(\sigma * h_{0,Q^{-\kappa-1}})(1/2 + i\tau)\varphi_N(1)\text{resp.}\mathscr{M}(\sigma * h_{0,Q^{-\kappa-1}})(-1/2 + i\tau)\varphi_N(w),$$

which, in view of $\varphi_N(g) = \varphi_{0,N}(gn(T))$, (5.1.1) and (5.1.2), are of size

$$O_\epsilon\left(\varphi_{0,N}(1)Q^{(\kappa-1)/2+\epsilon}\right)\text{resp.}O_\epsilon\left(\varphi_{0,N}(wn(T))Q^{(\kappa+1)/2+\epsilon}\right).$$

The first one is already absorbed in $O_\epsilon\left(Q^{(\kappa-1)/2+\epsilon}\right)$. Recall the equation (2.1.3), we deduce that

$$\varphi_{0,N}(wn(T)) = O_\epsilon\left(|T|^{-1+\epsilon}\right) = O_\epsilon\left(Q^{-1+\epsilon}\right).$$

Proposition 3.2.2 follows.

6.3 Estimation of the Constant Contribution (Continued)

We extend our result in Section 5.2 to any function φ_0 which lies not necessarily in one irreducible cuspidal representation.

Lemma 6.3.1. *Let π_1, π_2 be two generic automorphic representation of R_ω, which are unram-*

ified at every finite place. Given $\varphi_1 \in \pi_1^\infty, \varphi_2 \in \pi_2^\infty$ invariant by K_f, if at least one of π_1, π_2 is cuspidal, then

$$l^h\left(\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_1 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_2\right)_N\right) \ll_{F,\epsilon} \kappa \log Q Q^{(\kappa+1)\epsilon}$$
$$\cdot \|\varphi_1\|^{1-\epsilon}\|\varphi_2\|^{1-\epsilon}\|\Delta_\infty^4.\varphi_1\|^\epsilon\|\Delta_\infty^4.\varphi_2\|^\epsilon$$
$$\cdot E^{-2}\prod_{v=v_1,v_1',v_2,v_2'}\max(|\mathrm{tr}_{1,v}|,|\mathrm{tr}_{2,v}|).$$

If both π_1, π_2 are (unitary) Eisenstein, then

$$l^h\left(\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_1 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_2\right)_N\right) \ll_{F,\epsilon} \kappa \log Q Q^{(\kappa+1)\epsilon}E^{-2+\epsilon}$$
$$\cdot \|\varphi_1\|_{\mathrm{Eis}}^{1-\epsilon}\|\varphi_2\|_{\mathrm{Eis}}^{1-\epsilon}\|\Delta_\infty^4.\varphi_1\|_{\mathrm{Eis}}^\epsilon\|\Delta_\infty^4.\varphi_2\|_{\mathrm{Eis}}^\epsilon$$
$$+Q^{1-\kappa}\|\varphi_1\|_{\mathrm{Eis}}^{1/4-\epsilon}\|\varphi_2\|_{\mathrm{Eis}}^{1/4-\epsilon}\|\Delta_\infty.\varphi_1\|_{\mathrm{Eis}}^{3/4+\epsilon}\|\Delta_\infty.\varphi_2\|_{\mathrm{Eis}}^{3/4+\epsilon}.$$

First we consider the case that at least one of π_1, π_2 is cuspidal. If W_1, W_2 are the functions in the Whittaker model of π_1, π_2 corresponding to φ_1, φ_2, then we get, with similar notations as in Section 5.2,

$$l^{|\cdot|^s}\left(\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_1 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_2\right)_N\right) = \prod_{v|\infty}\int_{F_v^\times} W_{1,v}(a(y))\overline{W_{2,v}(a(y))}|y|_v^s d^\times y \cdot$$

$$\frac{L(s+1,\pi_1\times\bar{\pi}_2)}{\zeta_F(2s+2)}\prod_{v<\infty}\frac{\zeta_v(2s+2)\int_{F_v^\times} W_{1,v}(a(y)a(u_v))\overline{W_{2,v}(a(y)a(u_v'))}|y|_v^s d^\times y}{L(s+1,\pi_{1,v}\times\bar{\pi}_{2,v})}. \tag{6.3.1}$$

On the line $\Re(s) = \epsilon > 0$, similar argument shows, for example for $u_v = \varpi_v, u_v' = 1$

$$\frac{\zeta_v(2s+2)\int_{F_v^\times} W_{0,v}(a(y)a(u_v))\overline{W_{0,v}(a(y)a(u_v'))}|y|_v^s d^\times y}{L(s+1,\pi_v\times\bar{\pi}_v)} = \frac{(\mathrm{tr}_{1,v}-n_{1,v}\overline{\mathrm{tr}_{2,v}}q_v^{-s-1})q_v^{-1/2}}{1-q_v^{-2s-2}}$$

$$\ll_\epsilon \prod_{v=v_1,v_1',v_2,v_2'} q_v^{-1/2}\max(|\mathrm{tr}_{1,v}|,|\mathrm{tr}_{2,v}|), \tag{6.3.2}$$

where $\mathrm{tr}_{1,v}$ (resp. $\mathrm{tr}_{2,v}$) is the Hecke eigenvalue of φ_1 (resp.φ_2) at v. At infinite places, we use Cauchy-Schwarz to see

$$\prod_{v|\infty}\int_{F_v^\times} W_{1,v}(a(y))\overline{W_{2,v}(a(y))}|y|_v^s d^\times y \le \prod_{v|\infty}\sqrt{\int_{F_v^\times}|W_{1,v}(a(y))|^2|y|_v^\epsilon d^\times y\int_{F_v^\times}|W_{2,v}(a(y))|^2|y|_v^\epsilon d^\times y}. \tag{6.3.3}$$

Using Hölder's inequality and Proposition 1.5.4, we see

$$\prod_{v|\infty}\int_{F_v^\times}|W_{1,v}(a(y))|^2|y|_v^{\epsilon}d^\times y \le \prod_{v|\infty}\left(\int_{F_v^\times}|W_{1,v}(a(y))|^2 d^\times y\right)^{1-\epsilon/2}\left(\int_{F_v^\times}|W_{1,v}(a(y))|^2|y|^2 d^\times y\right)^{\epsilon/2}$$

$$\ll_F \frac{\zeta_F(2)}{(\mathrm{disc}F)^{1/2}L^*(1,\pi_1\times\bar{\pi}_1)}\|\varphi_1\|^{2-\epsilon}\cdot\left(\sum_{\sigma_v\in\{1,2\}}\|\prod_{v\mathrm{real}}T_v^1\prod_{v\mathrm{complex}}T_v^{\sigma_v}.\varphi_1\|^2\right)^{\epsilon/2}, \tag{6.3.4}$$

where $T_v^1=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $T_v^2=\begin{pmatrix}0&i\\0&0\end{pmatrix}$ are elements in the corresponding Lie algebras. As the proof of Theorem 1.7.1 shows, we have a bound for a single term

$$\|\prod_{v\mathrm{real}}T_v^1\prod_{v\mathrm{complex}}T_v^{\sigma_v}.\varphi_1\|\ll_F\|\Delta_\infty^4.\varphi_1\|. \tag{6.3.5}$$

Putting (6.3.2), (6.3.3), (6.3.4) and (6.3.5) into (6.3.1) we get, for $\Re(s)=\epsilon>0$

$$l^{|\cdot|^s}\left(\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_1 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_2\right)_N\right)\ll_{F,\epsilon}C(\pi_1)^\epsilon C(\pi_2)^\epsilon\|\varphi_1\|^{1-\epsilon/2}\|\varphi_2\|^{1-\epsilon/2}\|\Delta_\infty^4.\varphi_1\|^{\epsilon/2}\|\Delta_\infty^4.\varphi_2\|^{\epsilon/2}$$

$$\cdot\prod_{v=v_1,v_1',v_2,v_2'}q_v^{-1/2}\max(|\mathrm{tr}_{1,v}|,|\mathrm{tr}_{2,v}|). \tag{6.3.6}$$

With (5.1.2) and $C(\pi_1)^4\ll_F\dfrac{\|\Delta_\infty^4.\varphi_1\|}{\|\varphi_1\|}$, we obtain the first part of Lemma 6.3.1 from (6.3.6).

We turn to the case when both π_1 and π_2 are Eisenstein. We assume $\pi_1=\pi(\chi_1,\omega\chi_1^{-1}),\pi_2=\pi(\chi_2,\omega\chi_2^{-1})$ with χ_1,χ_2 two Hecke characters. We have an obvious equality

$$l^h\left(\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_1 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_2\right)_N\right)=l^h\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_{1,N}a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_{2,N}\right)+$$

$$l^h\left(\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_1 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_2\right)_N-a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_{1,N}a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_{2,N}\right).$$

The second term at the right side of the above equation can be treated in the same way as in the case where at least one of π_1,π_2 is cuspidal, except that we should use Proposition 1.5.6 instead of Proposition 1.5.4. For the first term, suppose φ_i is constructed from f_i $(i=1,2)$ in the induced model by the formula (1.5.1). Then we have

$$\varphi_{i,N}=f_i+M(f_i), i=1,2,$$

where $M(\cdot)$ is the intertwining operator. Hence, we are reduced to bound 4 terms like

$$\chi_1(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\overline{\chi_2(\frac{\varpi_{v_2}}{\varpi_{v_2'}})}|\frac{\varpi_{v_1}}{\varpi_{v_1'}}|^{1/2}|\frac{\varpi_{v_2}}{\varpi_{v_2'}}|^{1/2}\int_{F^\times\backslash\mathbb{A}^\times}\chi_1(y)\overline{\chi_2(y)}|y|h(|y|)d^\times y\cdot f_1(1)\overline{f_2(1)}.$$

It is non zero only if $\chi_1\overline{\chi_2}$ is trivial on $F^\times\backslash\mathbb{A}^{(1)}$, in which case the integral is bounded by a constant multiplied by

$$\int_{F^\times\backslash\mathbb{A}^\times}|y|h(|y|)d^\times y = O_F(Q^{\kappa-1}).$$

The restriction of f_i to K_v, $v|\infty$ lies in $\mathrm{Ind}_{T_v}^{K_v}(\chi_i,\omega\chi_i^{-1})$, where $T_v = B_v\cap K_v$ is a sub-torus of K_v. Let $f = f_{i,v}$, and decompose it into K_v-isotypic part as

$$f = \sum_{\sigma\in\hat{K}} a_\sigma(f)f_\sigma$$

with

$$f_\sigma(k) = \sqrt{d_\sigma}\langle k.v, v_0\rangle, a_\sigma(f) = \langle f, f_\sigma\rangle.$$

Here $v, v_0 \in V_\sigma$ are unitary and satisfy

$$t.v_0 = (\chi_i,\omega\chi_i^{-1})(t^{-1})v_0.$$

We get a Sobolev inequality

$$|f_\sigma(k)| \le \sum_{\sigma\in\hat{K}}|a_\sigma(f)\sqrt{d_\sigma}| \ll_{v,\epsilon} \|\mathscr{C}_{K_v}^{\sigma_v+\epsilon}.f\| \le \|f\|^{1-\sigma_v-\epsilon}\|\mathscr{C}_{K_v}.f\|^{\sigma_v+\epsilon}, \forall\epsilon>0$$

where $\sigma_v = 1/2$ if v is real and $\sigma_v = 3/4$ if v is complex. We have used results from Section 2.2, especially (2.2.5), and Hölder's inequality in the last inequalities. We conclude by noticing $\|\mathscr{C}_{K_v}.f\| \ll_v \|\Delta_v.f\|$ that

$$l^h\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_{1,N}a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_{2,N}\right) \ll_{F,\epsilon} \|\varphi_1\|_{\mathrm{Eis}}^{1/4-\epsilon}\|\varphi_2\|_{\mathrm{Eis}}^{1/4-\epsilon}\|\Delta_\infty.\varphi_1\|_{\mathrm{Eis}}^{3/4+\epsilon}\|\Delta_\infty.\varphi_2\|_{\mathrm{Eis}}^{3/4+\epsilon}Q^{\kappa-1}.$$

Corollary 6.3.2. *Let $\varphi_1,\varphi_2 \in R_\omega^\infty$ be K_f-invariant, without one dimensional part in the spectral decomposition in the sense of Theorem 1.6.1. Then*

$$\frac{1}{M_E^4}\sum_{v_1,v_1',v_2,v_2'\in I_E}\chi(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\chi^{-1}(\frac{\varpi_{v_2}}{\varpi_{v_2'}})l^h\left(\left(a(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\varphi_1 a(\frac{\varpi_{v_2}}{\varpi_{v_2'}})\overline{\varphi}_2\right)_N\right) \ll_{F,\epsilon}$$

$$\kappa\log Q Q^{(\kappa+1)\epsilon}E^{-2+\epsilon}\cdot\|\varphi_1\|^{1/2-\epsilon}\|\varphi_2\|^{1/2-\epsilon}\|\Delta_\infty.\varphi_1\|^{1/2+\epsilon}\|\Delta_\infty.\varphi_2\|^{1/2+\epsilon}$$
$$+Q^{\kappa-1}\cdot\|\varphi_1\|^{3/8-\epsilon}\|\varphi_2\|^{3/8-\epsilon}\|\Delta_\infty^2.\varphi_1\|^{5/8+\epsilon}\|\Delta_\infty^2.\varphi_2\|^{5/8+\epsilon}.$$

In fact, since the decomposition in Theorem 1.6.1 converges uniformly and absolutely on any compact, the operator $\frac{1}{M_E^4}\sum_{v_1,v_1',v_2,v_2'\in I_E}\chi(\frac{\varpi_{v_1}}{\varpi_{v_1'}})\chi^{-1}(\frac{\varpi_{v_2}}{\varpi_{v_2'}})l^h$ commutes with the spectral decomposition of φ_1,φ_2. We then apply the above lemma to each term after the spectral decomposition.

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  — Mathematisches Forschungsinstitut, Oberwolfach, Deutschland, August 28 - September 3, 2011
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  — Ein Gedi, Israel, January 3 - 7, 2011
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