Simple relaying strategies for half-duplex networks

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Abstract—**We consider the diamond network where a source communicates with the destination through N non-interfering half-duplex relays. Using simple outer bounds on the capacity of the network, we show that simple relaying strategies having exactly two states and avoiding broadcast and multiple access communication can still achieve a significant constant fraction of the capacity of the 2 relay network, independent of the SNR values. The results are extended to the case of 3 relays for the special class of antisymmetric networks. We also study the structure of (approximately) optimal relaying strategies for such networks. Simulations show that optimal schedules have at most $N+1$ states, which we conjecture to be true in general. We prove the conjecture for $N=2$ and in special cases for $N=3$.**

I. Introduction

Calculating the capacity of wireless relay networks is a hard problem; calculating the capacity when the relays are half-duplex is even harder. Indeed, in half duplex relay networks, an additional dimension of optimization comes into play: scheduling the relay states, i.e., whether each relay transmits (T) or listens (L) at any given time instance [6]. For example, for the N-relay diamond network in Fig. 1, there exist 2^N possible combinations of L, T states, and any capacity achieving strategy would need to optimize for how long each of these occurs.

Our position in this paper is that, at least for small diamond networks, there might be no need for such an exponential size optimization. We base this claim on two observations.

First, following the network simplification approach of [5], we show that even very simple strategies that use only two states and employ point-to-point connections (no broadcasting and no multiple access) can (approximately) achieve a significant multiplicative fraction of the capacity of the whole network. This factor is independent of the strength of the links in the 2 and 3 relay diamond networks. The approximations are derived using the simple bounds to the capacity of half-duplex relaying schemes developed in [7].

Second, the approximately (in the sense of [7]) optimal schedule has at most $N+1$ active states, instead of the possible 2^N. That is, for 2 relays, although 4 states are possible, at most 3 are employed (this directly follows from the work in [2]) and for 3 relays, only 4 out of the 8 possible states are employed. This observation is based on experimental results and we prove it for a few special cases.

Our aim in this paper is to study the effect of schedule complexity on capacity. We will work with approximate bounds on capacities and not worry about how the schedules are implemented. In the rest of the paper, Section II provides the framework of our work, i.e., the network model, the bounds in [7], and a Linear Programming (LP) problem formulation; Section III establishes bounds on the performance of simple strategies; Section IV presents our conjecture regarding the linear number of active states in the (approximately) optimal schedule and Section V concludes the paper.

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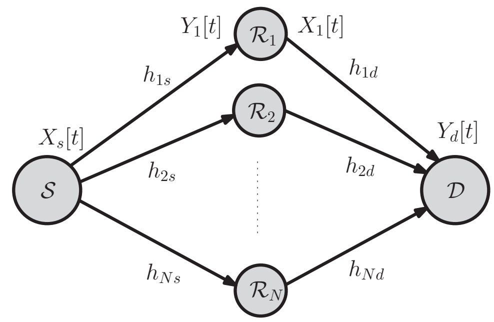

Fig. 1. The Gaussian N-relay half-duplex diamond network.

II. Network Model and Problem Formulation

A. Network Model

We consider the Gaussian N-relay diamond network where a source $\mathcal{S}$ transmits information to a destination $\mathcal{D}$ with the help of half-duplex relays. At any given time t, each relay $\mathcal{R}_i$ can either listen (L) or transmit (T), but not both; we denote with $M_i[t] \in \{L, T\}$ its state. For consistency, we denote with $M_s[t]$ and $M_d[t]$ the states of the source and the destination, respectively.

Let $X_s[t]$ be the signal transmitted by $\mathcal{S}$ at time t, $X_i[t]$ be the signal transmitted by relay $\mathcal{R}_i$, $Y_d[t]$ and $Y_i[t]$ the signals received by $\mathcal{D}$ and $\mathcal{R}_i$, respectively. Then

$$\begin{aligned}
Y_i[t] &= h_{is}X_s[t] + Z_i[t] \text{ when } M_j[t] = L \\
&= 0 \text{ when } M_j[t] = T \\
Y_d[t] &= \sum_{i=1}^{N} h_{id}X_i[t] + Z[t] \text{ when } M_d[t] = L \\
&= 0 \text{ when } M_d[t] = T
\end{aligned}$$

where h_{is}, h_{id} are the complex channel coefficients between $\mathcal{S}$ and $\mathcal{R}_i$ and $\mathcal{R}_i$ and $\mathcal{D}$, respectively. $Z_i[t]$ and $Z[t]$ are independent and identically distributed white Gaussian random processes of power spectral density $N_0/2$ Watts/Hz.

The power constraints for the source and all the relays are fixed to P. We can then calculate the individual link capacities from $\mathcal{S}$ to $\mathcal{R}_i$ as $R_{is} = \log(1 + |h_{is}|^2 P)$ and from $\mathcal{R}_i$ to $\mathcal{D}$ as $R_{id} = \log(1 + |h_{id}|^2 P)$. $[N]$ represents the set $\{1, \cdots, N\}$ and the relays are ordered such that $R_{is} \geq R_{js}$ for $i < j$.

B. Simple Bounds on Capacity

For half-duplex relay networks, the capacity depends not only on the strength of the channel coefficients, but crucially also on the L-T scheduling. Let $m \in M = \{L,T\}^N$ denote a particular state of the relays and let $L(m)$ and $T(m)$ be the set of relays in listening and transmitting state in m, respectively. In a particular schedule, let $p(m)$ denote the fraction of time the relays are in state m. From [7], the cut-set upper bound to the N-relay half-duplex diamond network, denoted by C_{cs}^N, can be bounded as follows.

Theorem 2.1: [7]

$$C_{LP}^N \leq C_{cs}^N \leq C_{LP}^N + G(N) \text{ where} \tag{1}$$

$$C_{LP}^N = \max_{\substack{p(m)\\ m\in M}} \min_{\Lambda \subseteq [N]} \sum_{m\in M} p(m) \left(\max_{i\in \bar{\Lambda}\cap L(m)} R_{is} + \max_{i\in \Lambda\cap T(m)} R_{id} \right) \tag{2}$$

and $G(N) = N + 3\log N - 2.75$.

Unless otherwise stated, the term "constant" will mean a quantity that depends only on the number of relays and is independent of the channel SNRs. Let C_{hd}^N be the capacity of the N-relay half-duplex diamond network. From [1] we get

$$C_{cs}^N - G'(N) \leq C_{hd}^N \leq C_{cs}^N \tag{3}$$

for some constant $G'(N)$. Combining (2) and (3) we get:

Lemma 2.2: For a N relay half-duplex diamond network, there exist constants $G(N)$ and $G'(N)$ such that

$$C_{LP}^N - G'(N) \leq C_{hd}^N \leq C_{LP}^N + G(N) \tag{4}$$

C. Simple Strategies

We define *simple strategies* to be relaying strategies that use exactly two states and avoid broadcast at the source and multiple access at the destination. We can have:

1-relay simple strategy: A single relay $\mathcal{R}_i$ is used to convey information by operating in the L and T states. The capacity of the *1-relay simple strategy* (C_{s1}) is obtained by using the best relay, i.e., maximizing ($C_{s1,i}$), the capacity of the one hop network $\mathcal{S}$-$\mathcal{R}_i$-$\mathcal{D}$:

$$C_{s1} = \max_{i\in[N]} C_{s1,i} \tag{5}$$

2-relay simple strategy: We select a pair of relays and operate them in a complementary fashion, alternating between the two states $\{L,T\}$ and $\{T,L\}$. If we select $\mathcal{R}_i$ and $\mathcal{R}_j$, essentially we use the two one hop networks - $\mathcal{S}$-$\mathcal{R}_i$-$\mathcal{D}$ and $\mathcal{S}$-$\mathcal{R}_j$-$\mathcal{D}$ that have together capacity $C_{s2,ij}$ (also see MDF in [2]). Thus the capacity of the *2-relay simple strategy* C_{s2} is

$$C_{s2} = \max_{i,j\in[N], i<j} C_{s2,ij} \tag{6}$$

$C_{s1,i}$ and $C_{s2,ij}$ can also be characterized by a simpler quantity by extending Lemma 2.2. For example,

$$C_{s2,ij,LP} - G'(2) \leq C_{s2,ij} \leq C_{s2,ij,LP} + G(2) \tag{7}$$

where $C_{s2,ij,LP}$ is the optimal value obtained by solving (2) for the 2-relay network consisting of $\mathcal{R}_i, \mathcal{R}_j$ and only the $\{L,T\}$ and $\{T,L\}$ states having non-zero time fractions.

Now, suppose we have a relaying strategy on a subnetwork with k relays that has capacity C' for which the corresponding value of (2) is C'_{LP}. Then the following lemma holds.

Lemma 2.3: If $C'_{LP} \geq \alpha C_{LP}^N$, then $C' \geq \alpha C_{hd}^N - (\alpha G(N) + G'(k))$, where $\alpha \in \mathbb{R}^+$.

Proof: Applying Theorem 2.2 to the subnetwork, we have $C' \geq C'_{LP} - G'(k) \geq \alpha C_{LP}^N - G'(k)$. Using Theorem 2.2 again, we get the desired result. ∎

Using the fact that C_{LP}^N characterizes the capacity of a relaying strategy within constant factors, we will derive universal bounds on the ratio of the capacity of the strategy to the capacity of the network.

D. Linear Programming Formulation

The calculation of C_{LP}^N in (2) involves a linear program which can be rewritten in the following form.

$$\textbf{LP1}: \text{ Maximize } C$$

$$\sum_{i=1}^{2^N} p_i \left(\max_{j\in\bar{\Lambda}\cap L(m_i)} R_{js} + \max_{j\in\bar{\Lambda}\cap T(m_i)} R_{jd} \right) \geq C; \forall \Lambda \subseteq [N]$$

$$\sum_{i=1}^{2^N} p_i = 1;\ \forall i, p_i \geq 0, C \geq 0$$

The 2^N variables of type $p(m)$ have been numbered as p_i with m_i being the corresponding state. Note that the set of allowed $p(m)$ values depends on the kind of relaying strategy we are using. **LP1** can be visualized in a matrix form as follows. (All vectors are column vectors)

$$\text{Maximize } \mathbf{c}^T[\mathbf{p}\, C] \tag{\textbf{LP1}}$$
$$\mathbf{A}[\mathbf{p}\, C] \geq \mathbf{b};\ [\mathbf{p}\, C] \geq \mathbf{0}$$

where the objective function vector $\mathbf{c}^T$ is of size $1\times(2^N+1)$, with all zero entries except the last one which is +1. $\mathbf{A}$ is a $(2^N+1)\times(2^N+1)$ matrix with

$$\begin{aligned} A_{k,i} &= \max_{j\in\Lambda(k)\cap L(m_i)} R_{js} + \max_{j\in\overline{\Lambda(k)}\cap T(m_i)} R_{jd} \\ &\quad \text{for } 1 \leq k \leq 2^N; 1 \leq i \leq 2^N \\ &= -1 \text{ for } 1 \leq k \leq 2^N; i = 2^N+1 \\ &= -1 \text{ for } k = 2^N+1; 1 \leq i \leq 2^N \\ &= 0 \text{ for } k = 2^N+1; i = 2^N+1 \end{aligned}$$

where $\Lambda(k)$ is the k-th subset of $[N]$. $\mathbf{b}$ is a $(2^N+1)\times 1$ vector with all zero entries except the last one which is -1. The variable vector $\mathbf{p}$ consists of 2^N variables p_i and the capacity variable C. It will also be useful for us to define the dual program of **LP1**. Using the symmetry of the program, it is not difficult to see that the dual of **LP1**, denoted by **DLP1**, is a minimization problem defined as follows.

$$\text{Minimize } \mathbf{c}^T[\mathbf{p}_d\, C_d] \tag{\textbf{DLP1}}$$
$$\mathbf{A}[\mathbf{p}_d\, C_d] \leq \mathbf{b};\ [\mathbf{p}_d\, C_d] \geq \mathbf{0}$$

The definitions of $\mathbf{A}, \mathbf{b}, \mathbf{c}$ are the same as above and $[\mathbf{p}_d\, C_d]$ is the corresponding variable vector in the dual program.

III. Performance of Simple Strategies

A. Capacity of Simple Strategies

The capacity for a specific relaying strategy can also be bounded using Lemma 2.2 once we solve the corresponding form of **LP1**. In the case of the 1-relay simple strategy, let

p_1, p_2 be the fraction of time $\mathcal{R}_i$ is in the L and T state, respectively. Then, the corresponding linear program is

$$\begin{aligned} &\text{Maximize } C \\ &R_{is}p_1 \geq C;\ R_{id}p_2 \geq C \\ &p_1 + p_2 = 1;\ p_1, p_2, C \geq 0 \end{aligned}$$

Therefore, $C_{s1,i,LP} = R_{is}R_{id}/(R_{is} + R_{id})$. For the 2-relay simple strategy, let us consider $\mathcal{R}_i$ and $\mathcal{R}_j$, $i < j$. Let p_1 be the fraction of time $\mathcal{R}_i$ is in L and $\mathcal{R}_j$ is in T and p_2 be the fraction of time $\mathcal{R}_i$ is in T and $\mathcal{R}_j$ is in L, respectively. Then $C_{s2,ij,LP}$ can be calculated as

$$\begin{aligned} &\text{Maximize } C \\ &R_{is}p_1 + R_{js}p_2 \geq C;\ (R_{id} + R_{js})p_2 \geq C \\ &(R_{is} + R_{jd})p_1 \geq C;\ R_{jd}p_1 + R_{id}p_2 \geq C \\ &p_1 + p_2 = 1;\ p_1, p_2, C \geq 0 \end{aligned}$$

Since $R_{is} \geq R_{js}$, this can be solved to obtain

$$\begin{aligned} C_{s2,ij,LP} &= \frac{R_{is}(R_{js} + R_{id})}{R_{is} + R_{id}} \text{ if } R_{is}R_{js} < R_{id}R_{jd} \\ &= \frac{R_{id}(R_{is} + R_{jd})}{R_{is} + R_{id}} \text{ if } R_{is}R_{js} \geq R_{id}R_{jd}, R_{jd} < R_{id} \\ &= \frac{R_{jd}(R_{js} + R_{id})}{R_{js} + R_{jd}} \text{ if } R_{is}R_{js} \geq R_{id}R_{jd}, R_{jd} \geq R_{id} \end{aligned}$$

In [5], it was shown that for full-duplex N-relay diamond networks, we can always find a k-relay subnetwork that approximately achieves $\frac{k}{k+1}$ of the full-duplex network capacity within an additive constant factor; for half-duplex, this implies the following lemma.

Lemma 3.1: For a N-relay half-duplex diamond network, there exist a k relay subnetwork that approximately achieves $\frac{k}{2(k+1)}$ of the capacity of the whole network within constant additive factors.

Therefore, a 1-relay subnetwork can approximately achieve 1/4 and a 2 relay subnetwork 1/3 of the network's capacity for any N. Network simplification [5] for half-duplex relays involves both using fewer relays and fewer number of states in the schedule. Therefore, what we show below can be thought of as improved simplification bounds for $N = 2$ and $N = 3$, using a restricted set of simple strategies.

B. 2 Relay Networks

As shown in [2], the linear program for C^2_{LP} can be solved exactly to obtain closed form expressions. Using them, we can prove the following result.

Lemma 3.2: For a 2-relay half-duplex diamond network,

$$C_{s1} \geq \frac{1}{2}C^2_{hd} - c_1,\ C_{s2} \geq \frac{8}{9}C^2_{hd} - c_2$$

for some constants c_1, c_2.

Proof: We show that $C_{s1,LP} \geq \frac{1}{2}C^2_{LP}$ and $C_{s2,LP} \geq \frac{8}{9}C^2_{LP}$, whence the result follows from Lemma 2.3. For the detailed proof see [3]. ■

The multiplicative factors are essentially the best we can obtain for C_{s1} and C_{s2}.

Lemma 3.3: There exist 2-relay half-duplex diamond networks where

$$C_{s1} = \frac{1}{2}C^2_{LP},\ C_{s2} \approx \frac{8}{9}C^2_{LP}$$

Proof: For the first claim, consider the network where $R_{1s} = a, R_{2s} = b, R_{1d} = b, R_{2d} = a$ for some $a, b \in \mathbb{R}^+$, $(a > b)$. In this case, $C_{s1}/C^2_{LP} = \frac{ab/(a+b)}{2ab/(a+b)} = 1/2$. For the second claim, consider the network with $R_{1s} = 2a, R_{2s} = a, R_{1d} = a, R_{2d} = ka$ for some $k > 2$. Then, plugging in the expressions for capacities, we have

$$\frac{C_{s2}}{C^2_{LP}} = \frac{4(2+2k)}{3(2+3k)} \rightarrow \frac{8}{9} \text{ as } k \rightarrow \infty$$

■

To summarize, we have shown that for the 2-relay diamond network, we can universally achieve approximately 50% of the capacity using the *1-relay simple strategy* and 88% by using the *2-relay simple strategy*, independent of the channel SNRs.

C. 3 Relay Antisymmetric Networks

For the case of $N = 3$ relays, it is difficult to obtain closed form expressions for C^3_{LP} involving the six capacities $(R_{1s}, R_{2s}, R_{3s}, R_{1d}, R_{2d}, R_{3d})$. We distinguish the relay networks according to the order of the relative values of these capacities. Assuming that $R_{1s} \geq R_{2s} \geq R_{3s}$, the R_{id} values can occur in six possible permutations. Although bounds can be obtained for each of the cases separately, we present here the results for the special case of *antisymmetric networks* where $R_{1s} \geq R_{2s} \geq R_{3s}$ and $R_{1d} \leq R_{2d} \leq R_{3d}$.

Lemma 3.4: For the anti-symmetric 3-relay half-duplex diamond network

$$C_{s1} \geq \frac{1}{3}C^3_{hd} - c_3,\ C_{s2} \geq \frac{1}{2}C^3_{hd} - c_4$$

for some constants c_3, c_4.

Proof: To prove the result we show that $C_{s1,LP} \geq \frac{1}{3}C^3_{LP}$ and $C_{s2,LP} \geq \frac{1}{2}C^3_{LP}$ whence the result follows from Lemma 2.3. For brevity, we assume $R_{1s} = a, R_{2s} = b, R_{3s} = c, R_{1d} = d, R_{2d} = e, R_{3d} = f$. Let

$$\begin{aligned} &x = \max\{d, e\},\ y = \max\{e, f\}\ z = \max\{d, f\} \\ &t = \max\{d, e, f\} \end{aligned}$$

Hence, the **LP1** matrix for the 3-relay network is

$$\begin{pmatrix} a & a & a & b & a & b & c & 0 & -1 \\ a & a+f & a & b & a+f & b+f & 0 & f & -1 \\ a & a & a+e & c & a+e & 0 & c+e & e & -1 \\ b & b & c & b+d & 0 & b+d & c+d & d & -1 \\ a & a+f & a+e & 0 & a+y & f & e & y & -1 \\ b & b+f & 0 & b+d & f & b+z & d & z & -1 \\ c & 0 & c+e & c+d & e & d & c+x & x & -1 \\ 0 & f & e & d & y & z & x & t & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}$$

For the anti-symmetric network, $a \geq b \geq c$ and $d \leq e \leq f$. Hence $x = e$ and $y, z, t = f$. We will construct three upper bounds to the optimum value by picking three different dual feasible solutions. They are (written as $[p_1, \cdots, p_8, C]$)

$$\bar{\alpha}_d = [\frac{d}{d+a-b}, 0, 0, \frac{a-b}{d+a-b}, 0, 0, 0, 0, \frac{ad+ab-b^2}{d+a-b}]$$

$$\bar{\gamma}_d = [0, 0, 0, 0, 0, 0, \frac{f-e}{c+f-e}, \frac{c}{c+f-e}, \frac{fc+fe-e^2}{c+f-e}]$$

The third one $\bar{\beta}_d$ is defined as follows. When $e \neq d$ or $b \neq c$,

$$\bar{\beta}_d = [0,0,0,\frac{e-d}{e-d+b-c},0,0,\frac{b-c}{e-d+b-c},0, \frac{(b+d)(e-d)+(c+d)(b-c)}{e-d+b-c}]$$

and when $e=d, b=c$, we define

$$\bar{\beta}_d = [0,0,0,\frac{1}{2},0,0,\frac{1}{2},0,b+d]$$

We define $\alpha_0 = \frac{ad+ab-b^2}{d+a-b}$, $\gamma_0 = \frac{fc+fe-e^2}{c+f-e}$ and $\beta_0 = \frac{(b+d)(e-d)+(c+d)(b-c)}{e-d+b-c}$ or $b+d$ depending on the parameter values. It can be verified that these three solutions are dual feasible and hence by weak duality [4] their objective values are upper bounds to C_{LP}^3. Hence, $\alpha_0,\beta_0,\gamma_0 \geq C_{LP}^3$, which implies $\min\{\alpha_0,\beta_0,\gamma_0\} \geq C_{LP}^3$.

We claim that the following holds,

$$\frac{\frac{ad}{a+d}}{\alpha_0} + 2\frac{\frac{be}{b+e}}{\beta_0} + \frac{\frac{cf}{c+f}}{\gamma_0} \geq \frac{4}{3}$$

This can be shown by expanding the terms and using the fact that $a \geq b \geq c$ and $d \leq e \leq f$. Therefore

$$\frac{4C_{s1,LP}}{\min\{\alpha_0,\beta_0,\gamma_0\}} \geq \frac{\frac{ad}{a+d}}{\beta_0} + 2\frac{\frac{be}{b+e}}{\gamma_0} + \frac{\frac{cf}{c+f}}{\alpha_0} \geq \frac{4}{3}$$

which implies that $C_{s1,LP} \geq \frac{1}{3}C_{LP}^3$. Now for the second claim, let us consider the pairs of relays $(\mathcal{R}_1,\mathcal{R}_2)$ and $(\mathcal{R}_2,\mathcal{R}_3)$. If $C' = C_{s2,12,LP} + C_{s2,23,LP}$, using the expressions above for the 2-relay simply strategy, we have

$$\begin{aligned} C' &= \frac{a(b+d)}{a+d} + \frac{b(e+c)}{b+e} \text{ if } \frac{e}{b} \geq \frac{a}{d} \geq \frac{c}{f} \\ &= \frac{e(b+d)}{b+e} + \frac{b(e+c)}{b+e} \text{ if } \frac{a}{d} \geq \frac{e}{b} \geq \frac{c}{f} \\ &= \frac{e(b+d)}{b+e} + \frac{f(e+c)}{f+c} \text{ if } \frac{a}{d} \geq \frac{c}{f} \geq \frac{e}{b} \end{aligned}$$

Since $C_{LP}^3 \leq \min\{\alpha_0,\beta_0,\gamma_0\}$, we have the following

If $\left(\frac{e}{b} \geq \frac{a}{d} \geq \frac{c}{f}\right)$

$$\frac{C'}{C_{LP}^3} \geq \frac{C_{s2,12,LP}+C_{s2,23,LP}}{\alpha_0} = \frac{n_1(a,b,c,d,e,f)}{d_1(a,b,c,d,e,f)} \geq 1$$

If $\left(\frac{a}{d} \geq \frac{e}{b} \geq \frac{c}{f}\right)$

$$\frac{C'}{C_{LP}^3} \geq \frac{C_{s2,12,LP}}{\alpha_0} + \frac{C_{s2,23,LP}}{\gamma_0} = \frac{n_2(a,b,c,d,e,f)}{d_2(a,b,c,d,e,f)} \geq 1$$

If $\left(\frac{a}{d} \geq \frac{c}{f} \geq \frac{e}{b}\right)$

$$\frac{C'}{C_{LP}^3} \geq \frac{C_{s2,12,LP}+C_{s2,23,LP}}{\gamma_0} = \frac{n_3(a,b,c,d,e,f)}{d_3(a,b,c,d,e,f)} \geq 1$$

where n_1,n_2,n_3,d_1,d_2,d_3 are polynomials in (a,b,c,d,e,f) and the last inequalities in each of the three cases can be easily proved by substituting, expanding the terms and using the fact that $a \geq b \geq c$ and $d \leq e \leq f$. Therefore $C_{s2,12,LP} + C_{s2,23,LP} \geq C_{LP}^3$. Picking the maximum of the two pairs, we get

$$C_{s2,LP} \geq \max\{C_{s2,12,LP}, C_{s2,23,LP}\} \geq \frac{1}{2}C_{LP}^3$$

■

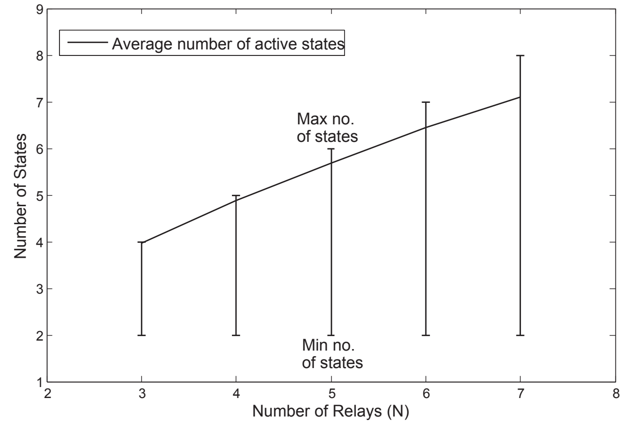

Fig. 2. Average, minimum and maximum number of active states for C_{LP}^N

The best lower bound multiplicative ratios we have been able to establish are the following.

Lemma 3.5: There exist 3-relay half-duplex diamond networks where

$$C_{s1,LP} \approx 0.4C_{LP}^3,\ C_{s2,LP} \approx 0.625C_{LP}^3$$

Proof: Consider the network $a = kr, b = 3r, c = 3r, d = 2r, e = 5r, f = 5r$ for some $k > 30, r > 0$. For this case, $C_{LP}^3 = \frac{(5k-9)r}{k-1}$, $C_{s1,LP} = \frac{2kr}{k+2}$, $C_{s2,LP} = \frac{25r}{8}$. Therefore, as $k \to \infty$,

$$\frac{C_{s1,LP}}{C_{LP}^3} \to \frac{2}{5} = 0.4,\ \frac{C_{s2,LP}}{C_{LP}^3} \to \frac{5}{8} = 0.625$$

■

IV. The Complexity of Optimal Schedules

In general, the optimal schedule in **LP1** corresponding to C_{LP}^N can have 2^N active states; we here present our conjecture that in fact, there always exists an optimal schedule with a linear number of active states. If true, this offers a significant reduction (from exponential to linear) to the number of states needed for optimal operation, making it more feasible to implement such schedules in practice.

Conjecture: For a N relay half-duplex diamond network, there exists a schedule that optimizes the value of C_{LP}^N and has at most $N+1$ active states.

We support this conjecture in two ways:

Experimental results: Fig. 2 shows numerical evaluation results for **LP1**. We plot the average number of active states in the optimal schedule as a function of the number of relays N. The average is taken over several random instances of the networks, where the SNRs of the source to relay and relay to destination channels are chosen independently and uniformly at random from the interval $[1, 1000]$. The maximum and the minimum number of active states observed for each N is also shown. In each of the cases, the maximum number of non-zero states in the optimal schedule of a N relay network turned out to be at most $N+1$.

Proof for special cases: For the case of $N = 2$ relays, the claim follows easily by directly evaluating the optimal schedule [2] and checking that there are at most three states, instead of four. We have not been able to come up with a general proof for $N > 2$. In what follows, we prove the conjecture for a special case of $N = 3$.

Again, for brevity, we will assume $R_{1s} = a$, $R_{2s} = b$, $R_{3s} = c$, $R_{1d} = d$, $R_{2d} = e$, $R_{3d} = f$. The special case we will consider is when the point to point capacities of all the relay to destination links dominates those of the source to relay links or vice-versa.

Lemma 4.1: Consider a 3-relay half-duplex diamond network where $\min\{d, e, f\} \geq \max\{a, b, c\}$ or $\min\{a, b, c\} \geq \max\{d, e, f\}$. Then the optimal solution for **LP1** has exactly 4 non-zero states.

Proof: Consider the case when $\min\{d, e, f\} \geq \max\{a, b, c\}$. The matrix corresponding to **LP1** is the same as the one mentioned in the proof of Lemma 3.4. Consider the sub-matrix $\mathbf{S}$ formed using rows I_1, I_4, I_7, I_8, I_9 and columns J_1, J_2, J_3, J_4, J_9 and the corresponding form of **LP1** with equality.

$$\mathbf{S}[p_1\, p_2\, p_3\, p_4\, C] = [0\,0\,0\,0\ -1]$$

This can be solved to get the following result.

$$\{p_1, p_2, p_3, p_4\} = \{\frac{\Delta_1}{(a-b+d)(b-c+e)(c+f)}, \frac{c}{c+f}, \frac{bc+(b-c)f}{(b-c+e)(c+f)}, \frac{e(a-b)(c+f)+(b-c)(ac+f(a-c))}{(a-b+d)(b-c+e)(c+f)}\}$$

and

$$C = \frac{(a((c+d)(e-d)+b(c+e))+d(b(c+e)+c(e-d)))fe}{(a+d)(b+e-d)(c+f-e)} - \frac{e\,(ad(e-d)+be(a+d))}{(a+d)(b+e-d)(c+f-e)} = a(p_1+p_2+p_3)+bp_4$$

where

$$\Delta_1 = b^2c - c^2f + def + bc(e+f-d) + a(c(c+f-e) - b(2c+f))$$

and I_4 is the all one 4×1 column vector. Since $a \geq b \geq c$, it is easy to see that $p_2, p_3, p_4 \geq 0$. For p_1, we need to show that $\Delta_1 \geq 0$ for our case. Since $\min\{d, e, f\} \geq \max\{a, b, c\} = a$, we have $f = a + l_1, e = a + l_2, d = a + l_3$, for some $l_1, l_2, l_3 \geq 0$. Therefore,

$$\Delta_1 = (a^2 - bc)(a-b) + l_1(a(a-b) + c(b-c) + ac) + l_2(a(a-c)+bc) + l_3(a^2-bc) + a(l_1l_2 + l_2l_3 + l_3l_1) + l_1l_2l_3$$

Since $a \geq b \geq c$, $\Delta_1 \geq 0$ and $C \geq 0$. If we define $\mathbf{p} = \{p_1, p_2, p_3, p_4, 0, 0, 0, 0\}$ and C is the same as above, then

$$I_1[\mathbf{p}\, C] = I_4[\mathbf{p}\, C] = I_7[\mathbf{p}\, C] = I_8[\mathbf{p}\, C] = \mathbf{0}$$

It can be easily verified that this implies

$$I_2[\mathbf{p}\, C], I_3[\mathbf{p}\, C], I_5[\mathbf{p}\, C], I_6[\mathbf{p}\, C] \geq 0$$

In other words $[\mathbf{p}\, C]$ is a feasible solution for **LP1**. We will now consider the dual program and solve for the submatrix of the dual consisting of columns J_1, J_2, J_3, J_4, J_9 and rows I_1, I_4, I_7, I_8, I_9, which is the transpose of $\mathbf{S}$ considered above. Note that the dual variables in the **DP1** correspond to the rows in **LP1**. The corresponding form of **DLP1** with equality is as follows.

$$\mathbf{S}[p_1^d\ p_4^d\ p_7^d\ p_8^d\ C] = [0\,0\,0\,0\ -1]$$

On solving, we get

$$\{p_1^d, p_4^d, p_7^d, p_8^d\} = \{\frac{d}{a-b+d}, \frac{(a-b)e}{(a-b+d)(b-c+e)}, \frac{(a-b)(b-c)f}{(a-b+d)(b-c+e)(c+f)}, \frac{(a-b)(b-c)c}{(a-b+d)(b-c+e)(c+f)}\}$$

and where

$$C^d = ap_1^d + bp_4^d + cp_7^d = C$$

It follows from anti-symmetry that $p_1^d, p_4^d, p_7^d, p_8^d \geq 0$. If we define $\mathbf{p}^d = \{p_1^d, 0, 0, p_4^d, 0, 0, p_7^d, p_8^d\}$, then

$$J_1^T[\mathbf{p}^d\, C^d] = J_2^T[\mathbf{p}^d\, C^d] = J_3^T[\mathbf{p}^d\, C^d] = J_4^T[\mathbf{p}^d\, C^d] = 0$$

It can be easily verified that this implies

$$J_5^T[\mathbf{p}^d\, C^d], J_6^T[\mathbf{p}^d\, C^d], J_7^T[\mathbf{p}^d\, C^d], J_8^T[\mathbf{p}^d\, C^d] \leq 0$$

In other words, $[\mathbf{p}^d\, C^d]$ is feasible for **DLP1**. Thus, the objective value of $C = C^d$ corresponds to both a dual feasible and primal feasible solution, which means it is the optimum value of **LP1**. Since the optimal schedule given by $[\mathbf{p}\, C]$ has just 4 non-zero states and there are 3 relays, the conjecture is valid for this case. The case when $\min\{a, b, c\} \geq \max\{d, e, f\}$ can be proved in a similar manner by reordering the relays so that the relay to destination link capacities are in sorted order. ■

V. Conclusion

In this paper, we have considered simple relaying strategies for half-duplex diamond networks that have exactly two states and avoid broadcast and multiple access. We show that these strategies approximately achieve a significant fraction of the capacity of the whole network with $N = 2, 3$ relays. It would be interesting to develop techniques for proving such bounds for larger values of N and for other networks. The definition of simple strategies can also be generalized to include other practical and easily implementable modes of operation. Finally, the linear complexity conjecture of optimal schedules in half-duplex diamond networks poses an intriguing open problem.

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