Convex computation of the region of attraction for polynomial control systems

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Region of Attraction (ROA)

$$\dot{x} = f(x,u)\,, \quad x(t) \in X\,, \quad u(t) \in U \quad \text{for all } t \in [0,T]$$

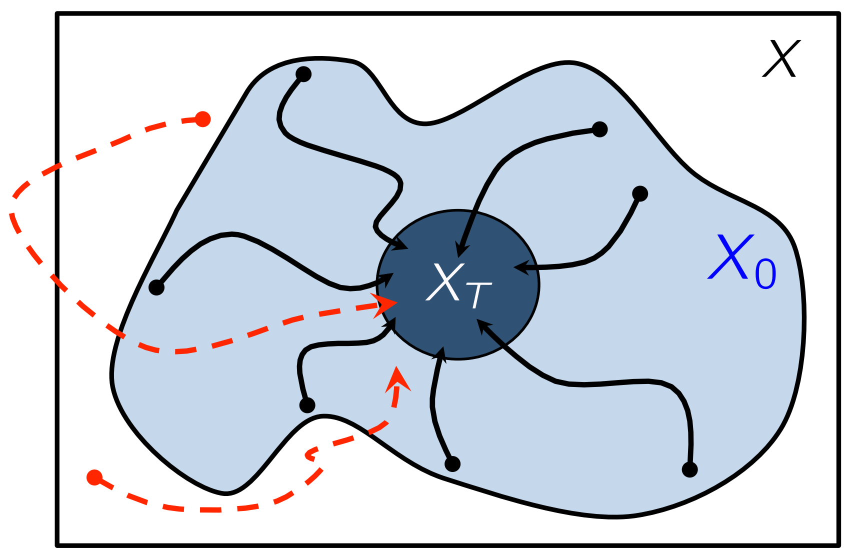
$$x(T) \in X_T$$

Polynomial

Compact Semialgebraic

Region of attraction (ROA) a.k.a. Backward reachable set

The set of all initial states that can be admissibly steered to the target set at a given time



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Polynomial

Compact Semialgebraic

Everything extends to rational or trigonometric setting

Region of Attraction (ROA)

Fundamentally **difficult to determine**

Long history - typically tackled using non-convex BMIs or gridding

Our contribution

Convex formulation

Infinite dimensional LP formulation for ROA computation
Converging hierarchy of SDP relaxations providing **outer approximations**

Readily modeled using freely available tools (Gloptipoly, Yalmip, etc.)

No initialization data required!

How is it done?

Approach

Approach

Study how **ensembles** of initial conditions evolve, not single trajectories

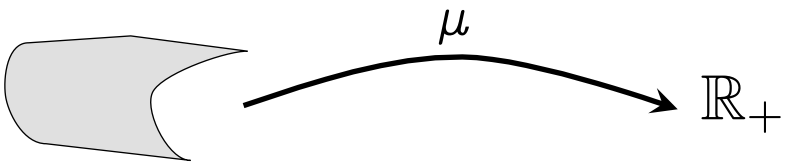
Common approach for stochastic or chaotic systems

How to model these ensembles? **Using measures.**

Measures

Measures

Mappings from sets to nonnegative real numbers



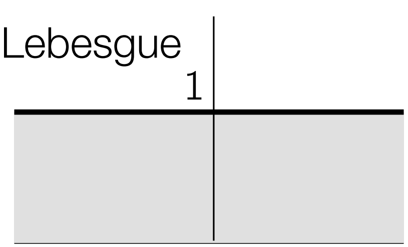
Integration

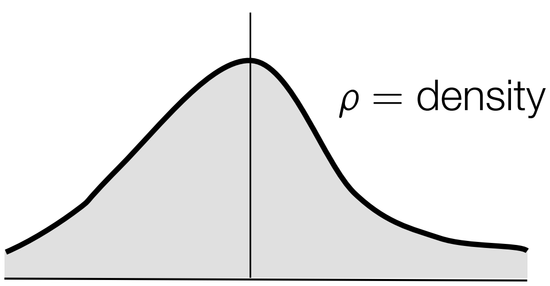
$$\int_A g(x)\, d\mu(x)$$

Intuition: integration w.r.t. to a weighting function or **density** $\rho(x)$

$$\int_A g(x)\, d\mu(x) = \int_A g(x)\, \rho(x) dx$$

density of Lebesgue





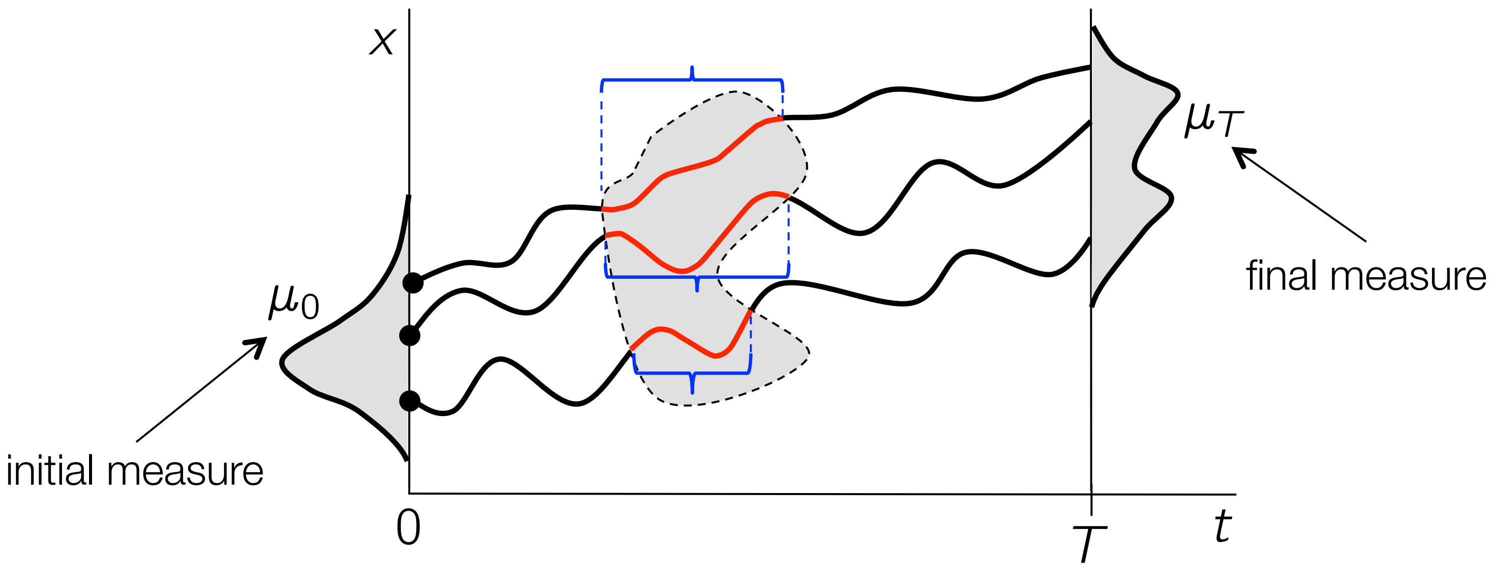
$\rho =$ density of μ

Measures in control

Initial measure μ_0 – distribution of the state at time 0

Final measure μ_T – distribution of the state at time T

Occupation measure μ – average time spent by $(t, x(t), u(t))$ in subsets of $[0, T] \times X \times U$



Liouville's equation

Linear equation linking the measures μ_0, μ and μ_T

$$\frac{\partial v}{\partial t} + \nabla_x v \cdot f$$

$$\int_X v(T,x)\,d\mu_T(x) - \int_X v(0,x)\,d\mu_0(x) = \int_{[0,T]\times X\times U} \mathcal{L}v(t,x,u)\,d\mu(t,x,u)$$

for all **test functions** $v \in C^1([0,T]\times X)$

Key fact

Liouville's equation

$\Updownarrow^*$

System dynamics $\dot{x} = f(x,u)$

Optimization over system trajectories	$\overset{*}{\Leftrightarrow}$	Optimization over measures satisfying Liouville's equation

Characterization of ROA using measures

Dynamics

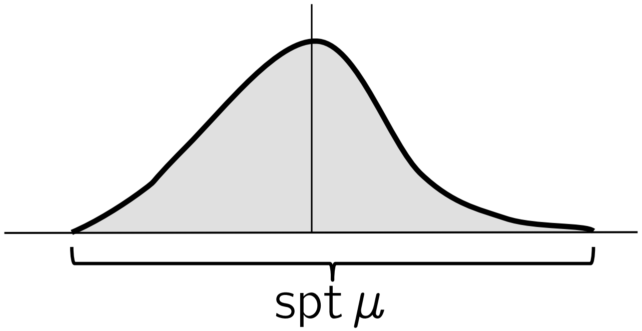
Dynamics $\overset{*}{\Longleftrightarrow}$ Liouville's equation

Constraints

Constraints $\Leftrightarrow$ Support constraints

Support of a measure

Smallest closed set whose complement has zero measure



$$x(0) \in X \longleftrightarrow \operatorname{spt} \mu_0 \subset X$$

$$(t, x(t), u(t)) \in [0, T] \times X \times U \longleftrightarrow \operatorname{spt} \mu \subset [0, T] \times X \times U$$

$$x(T) \in X_T \longleftrightarrow \operatorname{spt} \mu_T \subset X_T$$

Characterization of ROA using measures

Need to

Maximize the support of μ_0 subject to the Liouvillel's equation and the support constraints

Non-convex

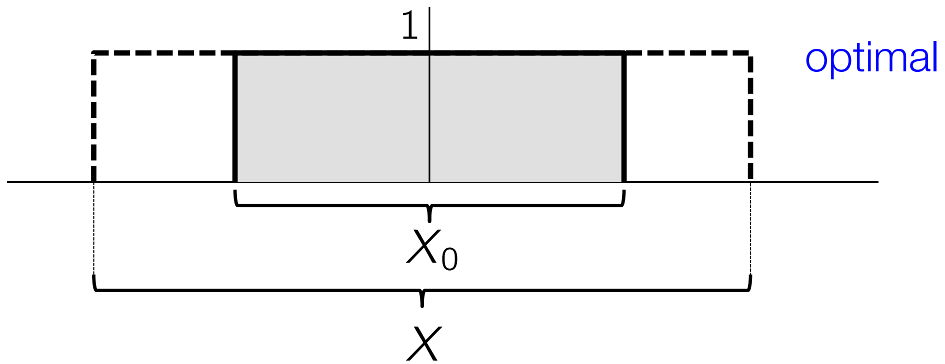
Convex characterization of ROA

Key idea

Maximize the **mass** of μ_0 subject to the constraint $\mu_0 \leq \lambda$

Lebesgue measure

Optimal solution is the restriction of λ to the ROA X_0



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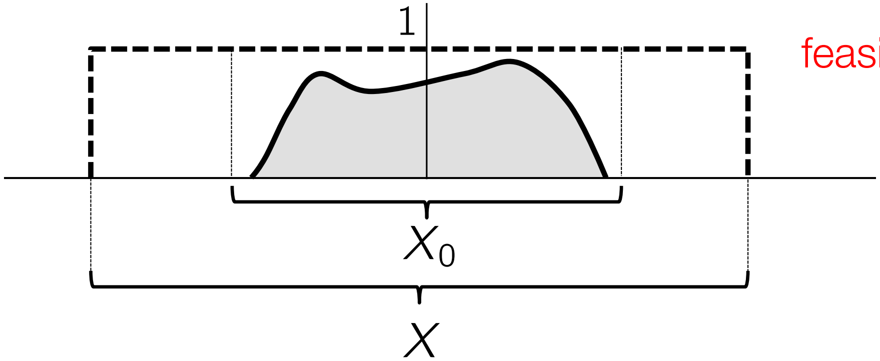

feasible but not optimal

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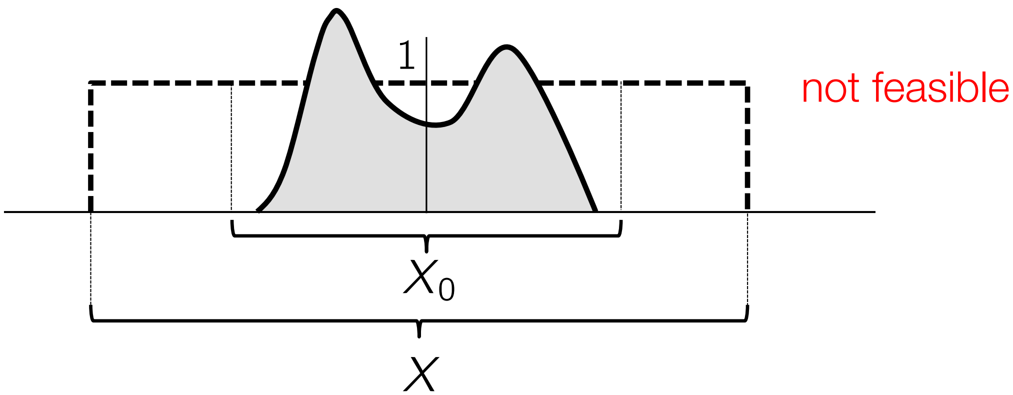

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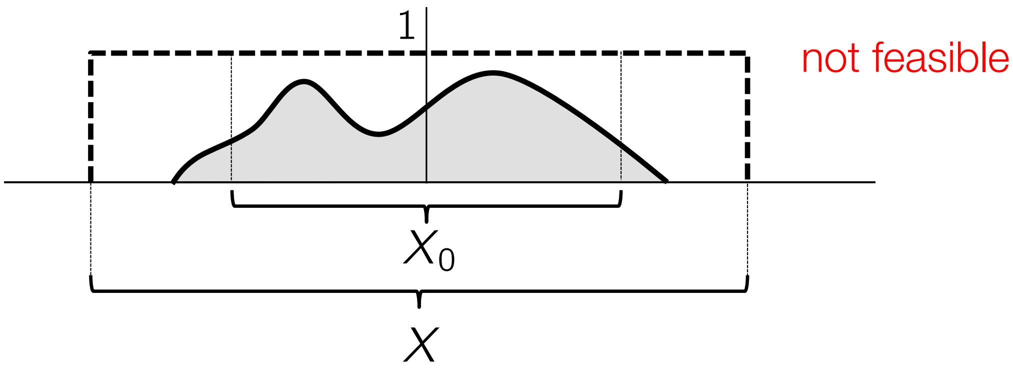

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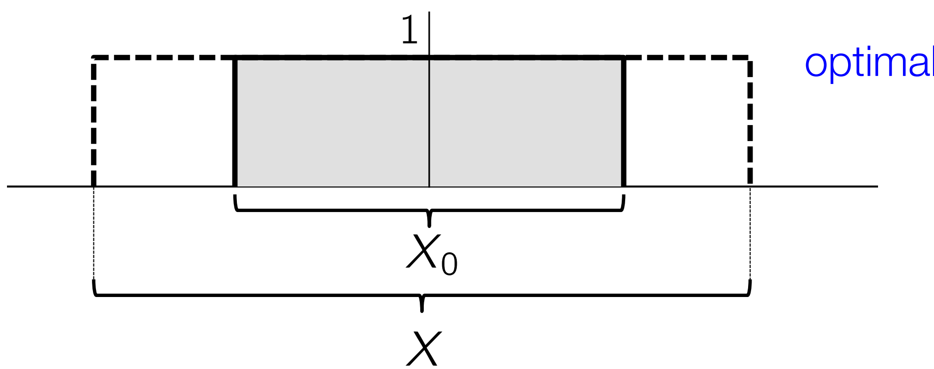


Primal LP

The ROA is characterized by the optimization problem

mass of μ_0

Primal LP

$$
\begin{array}{ll}
\sup & \mu_0(X) \\
\text{s.t.} & \int_{X_T} v \, d\mu_T - \int_X v \, d\mu_0 = \int_{[0,T]\times X\times U} \mathcal{L}v \, d\mu \quad \forall v \in C^1 \\
& \mu_0 \le \lambda \\
& \text{spt}\, \mu \subset [0,T] \times X \times U, \;\; \text{spt}\, \mu_0 \subset X, \;\; \text{spt}\, \mu_T \subset X_T
\end{array}
$$

Linear constraint

Conic constraints

Infinite dimensional **linear program** in the cone of nonnegative measures

Dual LP on continuous functions

Dual LP

$$
\begin{array}{lll}
\inf & \displaystyle\int_X w(x)\,dx & \\
\text{s.t.} & \mathcal{L}v(t,x,u) \le 0, & \forall\,(t,x,u) \in [0,T]\times X \times U \\
 & v(T,x) \ge 0, & \forall\, x \in X_T \\
 & w(x) \ge v(0,x) + 1, & \forall\, x \in X \\
 & w(x) \ge 0, & \forall\, x \in X,
\end{array}
$$

Decrease along trajectories

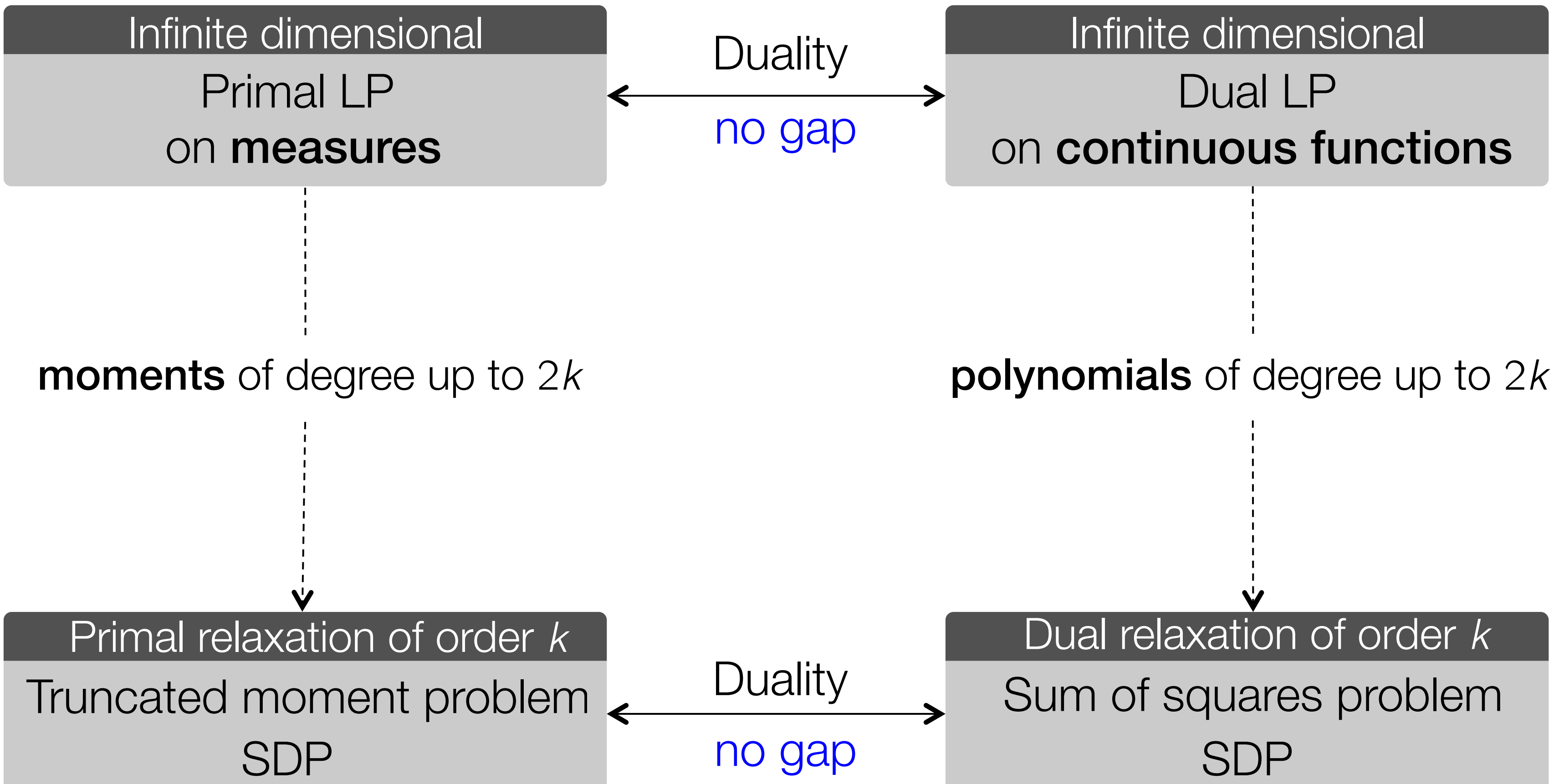
where the infimum is over $v \in C^1([0,T]\times X)$ and $w \in C(X)$

Key observation

$w \ge I_{X_0}$ and $\{x \mid w(x) \ge 1\} \supset X_0$ for any feasible w

Finite-dimensional relaxations

Big picture



Infinite dimensional
Primal LP
on **measures**
Duality
no gap
Infinite dimensional
Dual LP
on **continuous functions**
moments of degree up to $2k$
polynomials of degree up to $2k$
Primal relaxation of order k
Truncated moment problem
SDP
Duality
no gap
Dual relaxation of order k
Sum of squares problem
SDP

Convergence results

Convergence of primal and dual relaxations

Optimal values of the primal and dual SDPs converge to the optimal value of the two infinite dimensional LPs, which is equal to the **volume** of the ROA X_0

- Let $w_k(x)$ be the optimal solution to the dual SDP relaxation of order k

Functional convergence

$w_k \searrow I_{X_0}$ in L_1 and $\min_{i \leq k} w_i \searrow I_{X_0}$ almost uniformly as $k \to \infty$

- Define $X_{0k} := \{x \mid w_k(x) \geq 1\}$

Set-wise convergence

$X_{0k} \supset X_0$ and $\mathrm{volume}(X_{0k} \setminus X_0) \to 0$ as $k \to \infty$

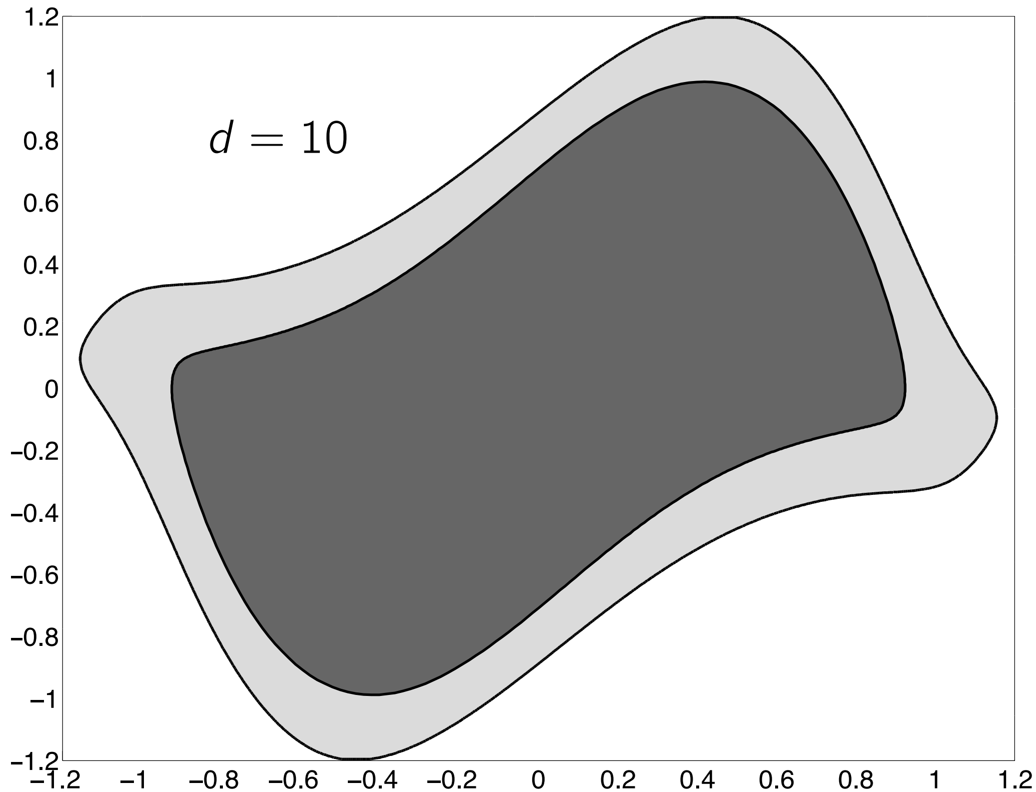
Numerical examples

Backward Van der Pol oscillator

$\dot{x}_1 = -2x_2$
$\dot{x}_2 = 0.8x_1 + 10(x_1^2 - 0.21)x_2$
$X = [-1.2, -1.2]^2$
$X_T = \{x \mid \|x\|_2 \leq 0.01\},\ T = 100$

Stable equilibrium at the origin with a bounded region of attraction

outer approximations of X_0



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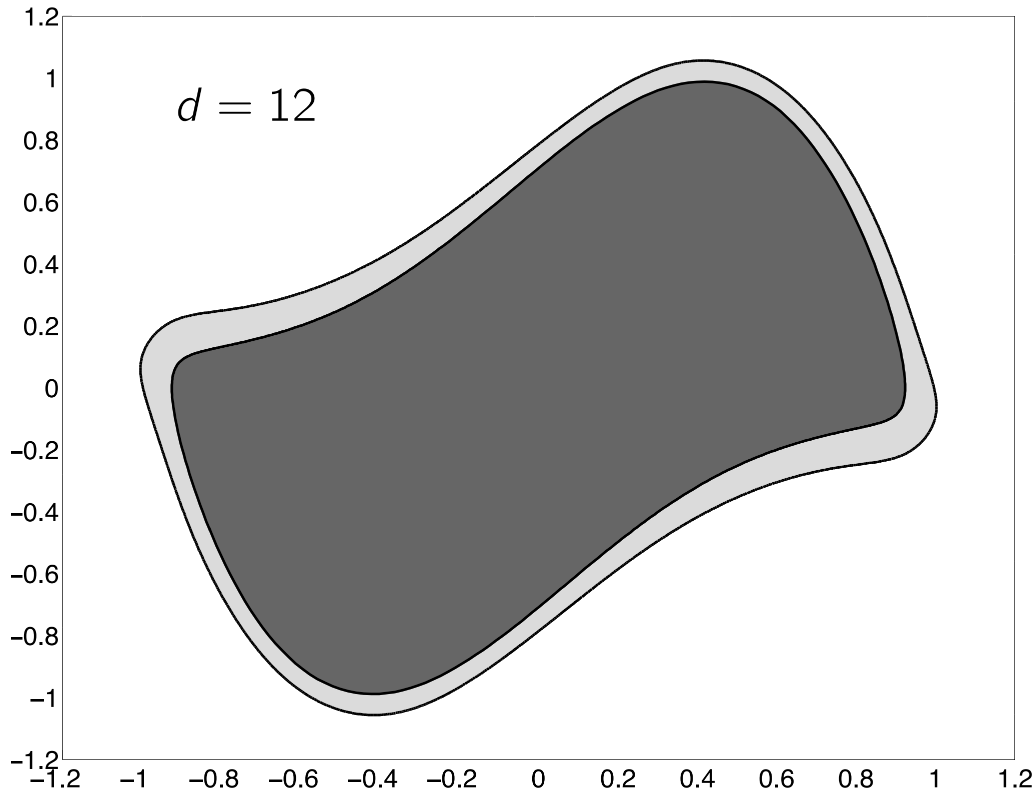

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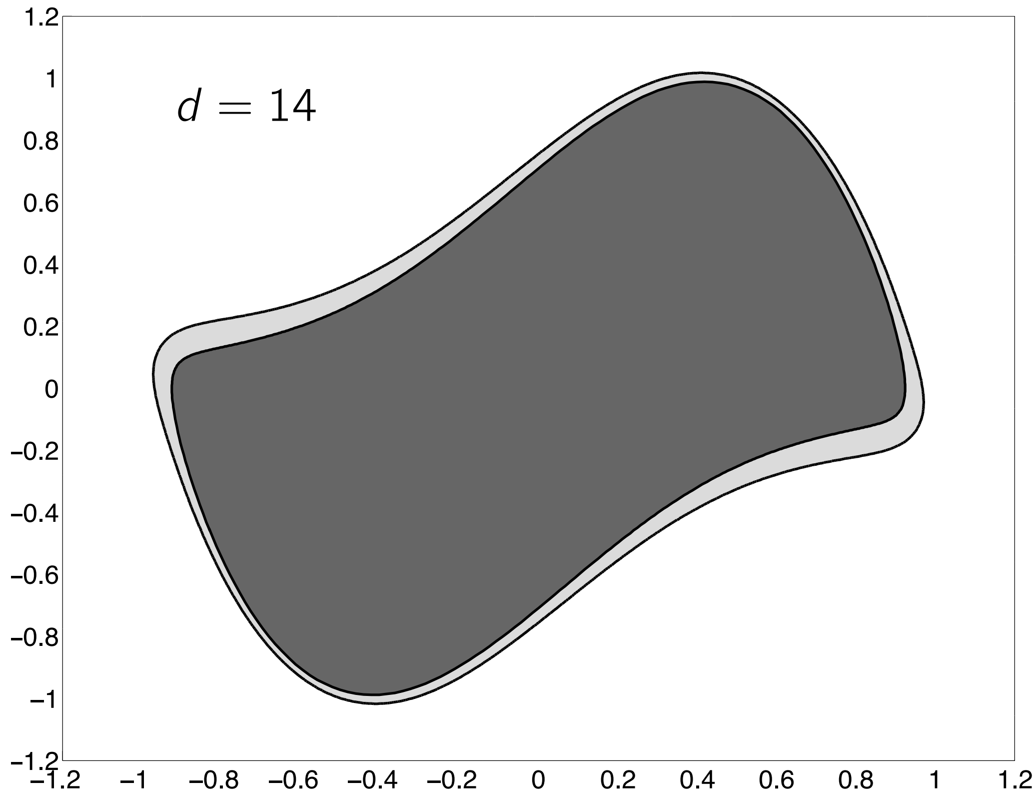

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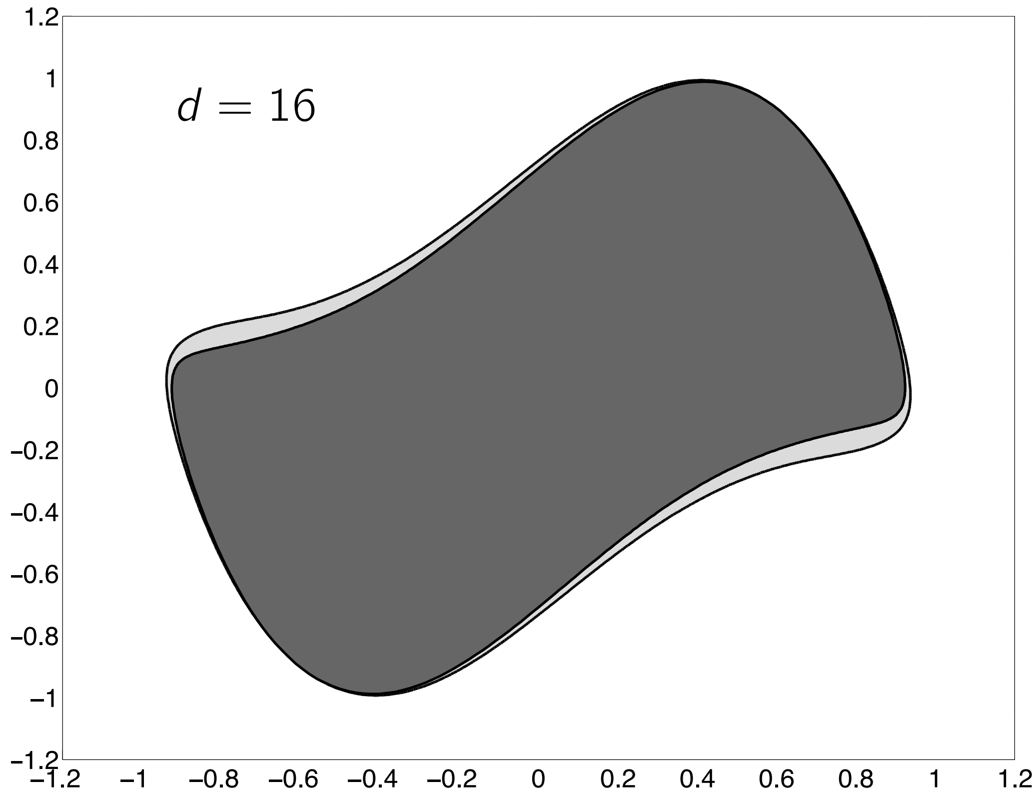

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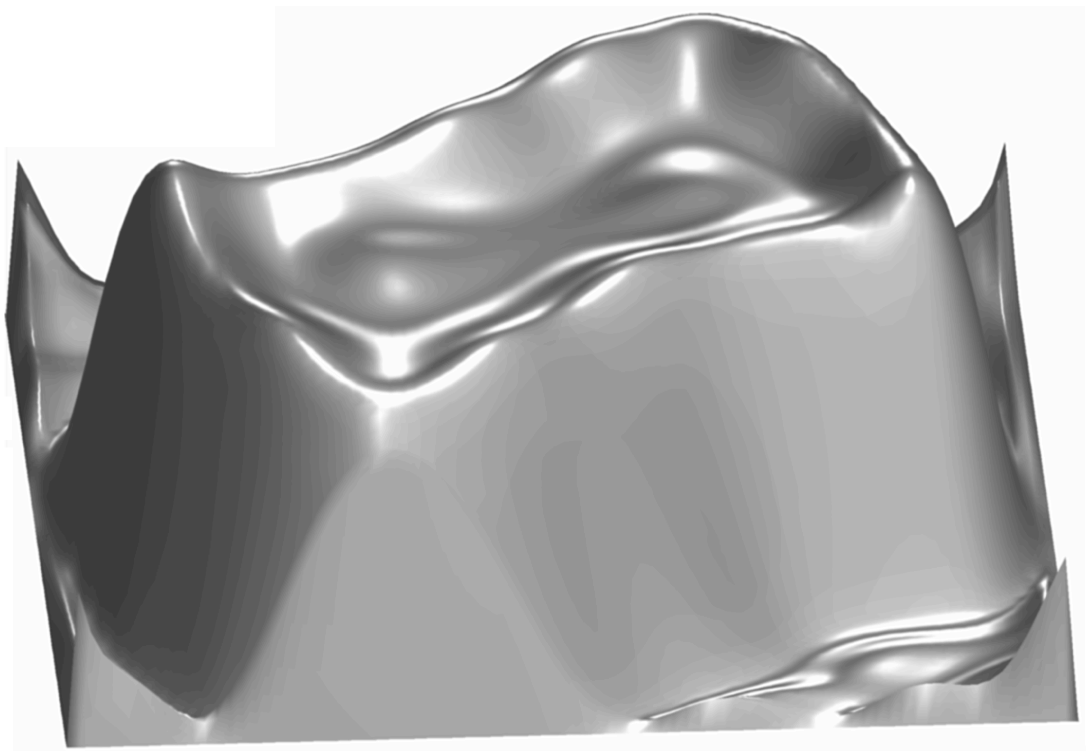
degree 18 approximation to I_{X_0}



Numerical examples

Brockett integrator

$$\dot{x}_1 = u_1$$
$$\dot{x}_2 = u_2$$
$$\dot{x}_3 = u_1 x_2 - u_2 x_1$$
$$X = \{x \mid ||x||_\infty \leq 1\}$$
$$U = \{u \mid ||u||_2 \leq 1\}$$
$$X_T = \{0\}, \quad T = 1$$

ROA known semi-analytically

$d = 6$

Numerical examples

Brockett integrator

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$$X_T = \{0\}, \quad T = 1$$

ROA known semi-analytically

$d = 10$

More examples

Extended version: `http://homepages.laas.fr/henrion/Papers/roa.pdf`

Examples from robotics + **control law extraction**:

[A. Majumdar, et al. Convex Optimization of Nonlinear Feedback Controllers via Occupation Measures, 2013]

Computational issues

Need to solve **large** SDPs

$\rightarrow$ Interior-point methods - **Mosek**, Sedumi, SDPA

Conditioning has secondary effect

``Medium” scale only

$\rightarrow$ First-order methods - **DSA-BD**, SDPNAL

Conditioning is very important

Large scale

Monomial basis $\rightarrow$ bad conditioning

Chebyshev basis $\rightarrow$ better conditioning

Conclusion

- **Convex** characterization of the ROA
- SDP relaxations $\rightarrow$ **converging** outer approximations
  - Additional properties (e.g. convexity) can be easily enforced
  - Covers a broad class of systems
- Easy modeling using Gloptipoly, Yalmip, SOSTOOLS, etc.

Extremely simple to use!

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Question time

Thank you