CENTER FOR FISCAL POLICY

On the Benefits of a Monetary Union: Does it Pay to Be Bigger?

by

Chiara Forlati

July 2012

Center for Fiscal Policy Working Paper Series

Working Paper 03-2013

ON THE BENEFITS OF A MONETARY UNION: DOES IT PAY TO BE BIGGER?*

Chiara Forlati[†]

This Draft: July 2012

Abstract

This paper revisits the question of the appropriate domain of a currency area using a New-Keynesian open economy model in which the world is split in two areas, each framed as a continuum of small open regions. We show that the adoption of a common currency like the euro can be beneficial for the members of the monetary union, since the spill-over effects within and across areas generated by the inflationary policies of the small open economies are likely to outweigh the costs of not tailoring monetary policy to country-specific shocks. We also show that while enlargement of the monetary union to another group of small open economies can bring about welfare gains for all countries involved, monetary integration of two large economies, such as the euro area and the U.S., will not. These findings can rationalize the process of the creation and enlargement of the eurozone.

Keywords: New Keynesian Open Economy Macroeconomics, Optimal Monetary Policy, Currency Area, Terms-of-Trade Externality.

JEL Classification: E52, E61, F41.

1 Introduction

What is the appropriate domain of a currency area? It might seem at first that the question is purely academic since it hardly appears within the realm of political feasibility that national currencies would ever be abandoned in favor of any other arrangement.(...) Certain parts of the world are undergoing processes of economic integration and disintegration, new experiments are being made and a conception of what constitutes an optimum currency area can clarify the meaning of these experiments.[Mundell (1961)]

Today Robert Mundell's words appear prophetic. The debate on the creation and enlargement of the Economic and Monetary Union of the European Union (EMU) and more recently the euro crisis have brought to the fore the question which countries should form or join a currency area. The costs of losing monetary autonomy are well known: when countries share the same currency, monetary policy cannot properly stabilize country-specific shocks. By contrast, the sources of welfare benefits that can rationalize the existence of a currency area have been less clearly identified,[1] casting serious doubts on the desirability of sharing a common currency like the euro.

This paper revisits the issue of the appropriate domain of a currency area, and specifically of the euro area, within a multi-country New-Keynesian open economy framework in which the objectives of the policy makers are fully micro-founded – i.e., derived directly from the welfare of the representative household.[2] To our knowledge, we are the first to study within this class of models to what extent the process of formation and enlargement of a monetary union like the eurozone entails beneficial effects for its citizens by comparing the welfare gains of the adoption of a common currency or the extension of the currency area with the costs of renouncing country-specific stabilization policies. According to our main results, there can be welfare gains from sharing a common currency as long as the currency area is formed by a group of small open economies. Similarly, it can be desirable to enlarge the currency area to another group of small open economies. Conversely, integrating the monetary union with another big country cannot bring about sizable welfare benefits. Put differently, according to our findings, while the adoption of the euro and the process of eurozone enlargement including the Eastern European countries is likely to entail welfare benefits for all the countries involved, there is no reason to try to implement a monetary union between the eurozone and the U.S.

In the model, the source of welfare gains from adopting a common currency is the internalization of a standard terms of trade externality according to which open economy policy makers try to manipulate the terms of trade in order to outsource labor effort at other countries' expense. This externality has been extensively studied in the open macro literature,[3] which however, tends to underestimate the ensued welfare losses, by usually considering a two country setup.[4] Differently, we use a multi-country

---

[1]As emphasized by the so-called Delors report (1989), there are microeconomic benefits from adopting a common currency like, for instance, savings in transaction costs. Alesina and Barro (2002) incorporate these kinds of costs into a model and show that the desirability of currrency unions increases as the world number of countries rises and the average country size falls. Yet, in their analysis the objectives of the policy makers are *ad hoc.*

[2]See Rotemberg and Woodford (1997) and Benigno and Woodford (2005).

[3]See e.g. Corsetti and Pesenti (2001), Corsetti and Pesenti (2005), Pappa (2004), Benigno and Benigno (2003), Benigno and Benigno (2006), Benigno and Benigno (2008) and De Paoli (2009a).

[4]An exception in this respect is represented by Epifani and Gancia (2009). Within a multi-country open economy framework, they show that the incentive of fiscal open economy authorities to manipulate the terms of trade can explain the relationship between trade openness and the size of governments. However,

small open economy model. This modeling choice seems more appropriate to study the welfare gains of the creation of the EMU – including initially eleven countries – and its enlargement. In our setting, the welfare gains of a currency area formed by small open economies are generally larger than those of a monetary union formed by big economies. Intuitively, when economies are small, policy makers take as given what happens in the rest of the world, disregarding completely how their independent policies jointly affect the global economy and the efficient use of world's resources. By contrast, policy makers of big economies internalize to a large extent the impact of their decisions on the world aggregate outcomes.

The framework of our analysis is a dynamic stochastic general equilibrium open economy model in which the world is split into two areas, called H and F. In each area, there is a continuum of small open regions. Each region produces a bundle of differentiated goods. Preferences exhibit home bias for goods produced within both the region and the area. The trade elasticity is allowed to be different from one to nest both the cases in which home and foreign bundles are substitutes and complements. Results are always shown for different values of this elasticity, since this elasticity plays a crucial role in determining the strength and the direction of the terms of trade externality.[5] Financial markets are complete, while labor is immobile across regions. There is no capital. Prices are staggered, implying a cost for the adoption of a common currency due to the impossibility to properly stabilize asymmetric shocks.

In this setup, we consider three different policy regimes (called A, B and C). Under regime A, in area H exchange rates are flexible and each small open economy has its own autonomous central bank; by contrast, in area F all regions share a common currency and monetary policy is delegated to a single authority (e.g. FED). Under regime B there is a single currency in each area and monetary policy is under the control of two independent central banks (e.g. ECB and FED). Finally, under regime C there is a common central bank for the world economy. Moreover, in all regimes monetary policies are chosen under commitment and are optimal from the *timeless* perspective.[6]

Both under regimes A and B optimal policies are biased by the desire of the monetary authorities to affect the terms of trade in their favor.[7] This incentive stems from a free riding problem. Through the manipulation of their terms of trade, open economy policy makers try to externalize labor effort at other countries' expense. The direction of this externality – i.e., whether this incentive leads to the attempt to worsen or to improve the domestic terms of trade – depends critically on the value of the trade elasticity. If the trade elasticity is sufficiently low, domestic and foreign goods are complements in the utility. Then, open economy authorities seek to worsen their terms of trade and increase the home demand for *both* domestic and foreign produced goods. In this way, domestic households can raise their consumption without reducing leisure of the same amount – as it would happen in a closed economy – by virtue of the foreign labor effort. Vice versa, if the trade elasticity is sufficiently large domestic and foreign goods are substitutes in the utility. In this case, policy makers try to improve their terms of trade to render foreign goods cheaper and to decrease the demand for domestically produced goods. The reduction of the labor effort more than compensates in the utility for the fall in consumption which is partially dampened by the rise in the

---

differently from us, they focus on the external effects of uncoordinated fiscal policies.

[5]For a discussion see Tille (2001), Obstfeld and Rogoff (2002), Benigno and Benigno (2003) and Pappa (2004).

[6]See Woodford (2003), Benigno and Woodford (2005) and Benigno and Woodford (2012).

[7]Notice that other policy instruments to affect terms of trade, such as tariffs, cannot be used in the WTO.

domestic demand for the goods produced abroad – again this effect would be absent in a closed economy.

These incentives are common to the policy makers of both large and small open economies. However, the difference in size of these economies shapes their optimal monetary conduct. In the limiting case in which the economy is small, the economic performance of the small country is irrelevant for the behavior of the aggregate economy. As a consequence, from the small open economy's point of view, strategically manipulating the terms of trade exclusively has effects on domestic output, while leaving the rest of the world unaffected. In equilibrium, however, the opposite will be true: since the group of small open economies is large, aggregate distortions are substantial. For instance, small open economy authorities do not realize that if they jointly try to improve their terms of trade by reducing the demand for domestic goods, the demand for goods of the foreign area increases, potentially amplifying the equilibrium effects on their terms of trade. For this reason, they are more prone to adopt highly inflationary policies that generate strong negative externalities. In contrast, when the economy is big, even if they do not internalize the effects of their policies on other countries' welfare, policy makers take into account the impact of their decisions on the world economy equilibrium. So they disagree on how much to produce and consume individually and they try to manipulate their terms of trade by affecting domestic and foreign outputs in opposite directions to allow domestic households to enjoy *relatively* more leisure or consumption. Nevertheless, they take into account the feedback effects of their policies stemming from the other area and they recognize the importance of using efficiently the resources available in the world economy.

The differences in the conduct of monetary policies explain the differences in outcomes across policy regimes. Under regime B, policy makers of areas H and F are exactly symmetric. Both of them try to manipulate the terms of trade between areas. Hence, being under regime C instead of regime B eliminates this externality. However, independently of the value of the trade elasticity, this welfare benefit is always outweighed by the costs due to the impossibility of properly stabilizing area-specific shocks in a monetary union. This result suggests that adopting a common currency for two large economies like the U.S. and the eurozone is not desirable. Conversely, under regime A, while the common central bank in area F seeks to manipulate the terms of trade between areas, monetary policy makers of the small open economies try to influence those between their region and the areas H and F. By so doing, they neither internalize the spill-over effects *within* nor *across* areas. As a consequence, for values of the trade elasticity above 1.8, there are welfare benefits – which can be substantial[8] – for the households living in areas H and F not only from being under regime B instead of A, but also from being under regime C instead of A. These findings help to explain the process of the EMU formation and enlargement.

This paper is organized as follows. Section 2 describes the basic setup, section 3 determines the equilibrium, section 4 characterizes the Pareto efficient allocation, section 5 describes the welfare approximations, section 6 discusses the dynamic simulations and section 7 reports the results of the welfare evaluation.

---

[8]Especially if compared with those found in previous studies on monetary policy coordination (e.g. Obstfeld and Rogoff (2002)) and in the business cycle literature (e.g. Lucas (1987) and Lucas (2003)).

2 The basic framework

The world consists of a continuum of small open regions indexed by $i \in [0, 1]$. The regions are split in two areas – H and F – of equal size. In area H, there is a continuum of regions indexed by $i \in [0, \frac{1}{2})$, which are independent countries. Area F consists of regions indexed by $i \in [\frac{1}{2}, 1]$, which belong to a monetary union. In each region i, households supply a continuum of imperfectly substitutable labor services, which are immobile both across regions and areas. Each region produces a continuum of imperfectly substitutable goods using all the labor services available in the domestic economy. There is no capital. Moreover, there are complete financial markets and the law of one price holds.

2.1 Preferences

Agents are infinitely lived and maximize the expected value of the discounted sum of the per-period utility. Preferences of a generic household s of region i are defined over a private consumption bundle, C_t^i, and labor service s, $N_t^i(s)$:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{N_t^i(s)^{\varphi+1}}{\varphi+1} \right] \quad 0 < \beta < 1 \tag{1}$$

where β is the intertemporal preferences discount factor. Agents consume all the goods produced in the world economy, but preferences exhibit home bias. The private consumption index is a CES aggregation of the following type:

$$C_t^i \equiv \left[\alpha_s^{\frac{1}{\eta}} C_{i,t}^i{}^{\frac{\eta-1}{\eta}} + (\alpha_b - \alpha_s)^{\frac{1}{\eta}} C_{H,t}^i{}^{\frac{\eta-1}{\eta}} + (1-\alpha_b)^{\frac{1}{\eta}} C_{F,t}^i{}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \tag{2}$$

for all $i \in \left[0, \frac{1}{2}\right)$ $\eta > 0$, $0 < \alpha_s < \alpha_b$ and $\frac{1}{2} < \alpha_b < 1$. α_s and α_b are the degrees of home bias for the goods produced within region i and the area to which region i belongs, respectively. Hence, if $\alpha_s = 1$, the region becomes closed, whereas if it goes to zero, the CES aggregation in (2) turns into that of a two-country model with home bias and countries of equal size. Conversely, when $\alpha_b = 1$, the area is closed. In this case, the small open regions are like the small open economy laid out in Galí and Monacelli (2005) and De Paoli (2009a) with the difference that the external world is framed as a continuum of small open economies – as in Galí and Monacelli (2009) – and not as a closed country.

The parameter η denotes the elasticity of substitution between $C_{H,t}^i$, $C_{F,t}^i$ and $C_{i,t}^i$, which are defined as:

$$C_{H,t}^i \equiv \left[2^{\frac{1}{\eta}} \int_0^{\frac{1}{2}} C_{j,t}^i{}^{\frac{\eta-1}{\eta}} dj \right]^{\frac{\eta}{\eta-1}} \quad C_{F,t}^i \equiv \left[2^{\frac{1}{\eta}} \int_{\frac{1}{2}}^1 C_{j,t}^i{}^{\frac{\eta-1}{\eta}} dj \right]^{\frac{\eta}{\eta-1}} \tag{3}$$

$$C_{j,t}^i \equiv \left(\int_0^1 c_t^i(h^j)^{\frac{\varepsilon-1}{\varepsilon}} dh^j \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad j \in [0, \frac{1}{2}) \quad C_{j,t}^i \equiv \left(\int_0^1 c_t^i(f^j)^{\frac{\varepsilon-1}{\varepsilon}} df^j \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad j \in [\frac{1}{2}, 1] \tag{4}$$

with ε being the elasticity of substitution among goods produced in the same region. The definition of the private consumption index (2) enables us to determine a consistent definition for the consumers' price index of region i, given by:

$$P_{C^i,t} \equiv \left[\alpha_s P_{i,t}^{1-\eta} + (\alpha_b - \alpha_s) P_{H,t}^{1-\eta} + (1-\alpha_b) P_{F,t}^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{5}$$

5

for all $i \in [0, \frac{1}{2})$, where all prices are denominated in the currency of the home country. The variables $P_{i,t}$, $P_{H,t}$ and $P_{F,t}$ are producers' price indexes that are defined consistently with the other consumption indexes (3) and (4). The law of one price is assumed to hold in all single-good markets. However, given the home bias in preferences, in general purchasing power parity does not hold for indexes $P_{C^i,t}$. Symmetric definitions apply to the region of area F.

2.2 Consumption demand, portfolio choices and labor supply

The consumption and price index definitions allow to solve the consumer problem in two stages. In the first stage, agents decide how much real net income to allocate to goods produced at home and abroad, leading to the following demand functions:

$$C_{i,t}^i = \alpha_s \left(\frac{P_{i,t}}{P_{C^i,t}} \right)^{-\eta} C_t^i \quad C_{H,t}^i = (\alpha_b - \alpha_s) \left(\frac{P_{H,t}}{P_{C^i,t}} \right)^{-\eta} C_t^i \quad C_{F,t}^i = (1 - \alpha_b) \left(\frac{P_{F,t}}{P_{C^i,t}} \right)^{-\eta} C_t^i \tag{6}$$

for all $i \in [0, \frac{1}{2})$ and $j \in [0, 1]$. Similarly, we can use (3) and (4) to retrieve consistent demand function for $C_{j,t}^i$, $c_t^i(h^j)$ and $c_t^i(f^j)$. In the second stage, agents maximize (1) with respect to C_t^i, D_{t+1}^i and $N_t^i(s)$ subject to the following sequence of budget constraints:

$$E_t\{Q_{t,t+1}^i D_{t+1}^i\} = D_t^i + W_{i,t}(s) N_t^i(s) - P_{C^i,t} C_t^i + T_t^i \tag{7}$$

$$N_t^i(s) = \left(\frac{W_{i,t}(s)}{W_{i,t}} \right)^{-v_t^i} N_t^i \tag{8}$$

where $W_{i,t} \equiv \left[\int_0^1 W_{i,t}(s)^{1-v_t^i} ds \right]^{\frac{1}{1-v_t^i}}$ is the aggregate wage index of individual labor services. Condition (7) is the budget constraint, which states that nominal saving, net of lump sum transfers, has to equalize the nominal value of a state contingent portfolio. Here $W_{i,t}(s)$ stands for the per hour nominal wage, $Q_{t,t+1}^i$ denotes the stochastic discount factor and D_{t+1}^i is the payoff of a one-period-maturity portfolio of firm shares. Constraint (8) is a consequence of a CES aggregation of labor inputs, which will be specified below and states that the labor market is monopolistically competitive. Indeed, each agent offers a different kind of labor service. Note that v_t^i, the elasticity of demand of labor is region-specific and time-varying as in Clarida, Galí and Gertler (2002). Domestic and international markets are assumed to be complete.

The optimality conditions of the household's problem imply:

$$(1 + \mu_t^i) N_t^i(s)^{\varphi} C_t^{i\sigma} = \frac{W_{i,t}}{P_{C^i,t}} \tag{9}$$

$$\beta \left(\frac{C_{t+1}^i}{C_t^i} \right)^{-\sigma} \left(\frac{P_{C^i,t}}{P_{C^i,t+1}} \right) = Q_{t,t+1}^i \tag{10}$$

which hold in all states of nature and in all periods and where $\mu_t^i \equiv \frac{1}{v_t^i - 1}$. According to (9), workers set the real wage as mark-up over the marginal rate of substitution between consumption and leisure, while the value of the intertemporal marginal rate of substitution of consumption equals the stochastic discount factor expressed in terms of the currency of region i. Notice that since wages are perfectly flexible, $N_t^i(s) = N_t^i$ and $W_{i,t}(s) = W_{i,t}$ for all s and t. Moreover the wage mark-up, μ_t^i, is shocked exogenously, entailing exogenous movements in the wedge between the real wage and the marginal rate of substitution between consumption and leisure.

2.3 Firms, technology and price setting

In each region i there is a continuum of firms. Each of them produces a single differentiated good with a constant return to scale technology of the type:

$$y_t(h^i) = A_t^i N_t(h^i) \tag{11}$$

with $N_t(h^i) = \left[\int_0^1 N_t(s,h^i)^{\frac{v_t^i-1}{v_t^i}} ds \right]^{\frac{v_t^i}{v_t^i-1}}$ being labor input bundle, composed of a continuum of imperfectly substitutable labor services.[9] Moreover, A_t^i is the region-specific technology shock. Given (11) and the fact that $N_t(s,h^i) = N_t(h^i)$ for all h^i and s, the aggregate relationship between output and labor can be written as:

$$N_t^i = \frac{Y_t^i}{A_t^i} Z_t^i \tag{12}$$

where $Y_t^i \equiv \left[\int_0^1 y_t(h^i)^{\frac{\varepsilon-1}{\varepsilon}} dh^i \right]^{\frac{\varepsilon}{\varepsilon-1}}$, $Z_t^i \equiv \int_0^1 \frac{y_t(h^i)}{Y_t^i} dh^i$ and $N_t^i \equiv \int_0^1 N_t(h^i) dh^i$. Using the demand functions one can show that $Z_t^i = \int_0^1 \left(\frac{p_t(h^i)}{P_{i,t}} \right)^{-\varepsilon} dh^i$; thus Z_t^i can be interpreted as an index of the relative price dispersion or output dispersion across firms. We assume that goods prices adjust according to a staggered mechanism *à la* Calvo. Therefore, in each period a given firm can re-optimize its price only with probability $1-\theta$. As a result, the fraction of firms that set a new price is fixed and the aggregate producer price index of the intermediate goods evolves accordingly to:

$$P_{i,t}^{(1-\varepsilon)} = \theta P_{i,t-1}^{(1-\varepsilon)} + (1-\theta)\tilde{p}_t(h^i)^{(1-\varepsilon)} \tag{13}$$

with $\tilde{p}_t(h^i)$ being the optimal price. Firms maximize the discounted expected sum of the future profits that would be collected if the optimal price could not be changed.

$$\sum_{s=0}^{\infty} (\theta)^s E_t \left\{ Q_{t,t+s}^i y_{t+s}(h^i) \left[\tilde{p}_t(h^i) - MC_{i,t+s}^n \right] \right\} \tag{14}$$

where $y_t(h^i) = \left(\frac{p_t(h^i)}{P_{i,t}} \right)^{-\varepsilon} Y_t^i$ and $MC_{i,t}^n = \frac{(1-\tau^i)W_{i,t}}{A_t^i}$ is the nominal marginal cost with τ^i denoting a constant labor subsidy. Taking into account (10) and that $MC_{i,t} \equiv \frac{MC_{i,t}^n}{P_{i,t}}$, the optimality condition of the firm problem can be written as:

$$\sum_{s=0}^{\infty} (\beta\theta)^s E_t \left\{ C_{t+s}^{i}{}^{-\sigma} \left(\frac{\tilde{p}_t(h^i)}{P_{i,t+s}} \right)^{-\varepsilon} Y_{t+s}^i \frac{P_{i,t}}{P_{C^i,t+s}} \left[\frac{\tilde{p}_t(h^i)}{P_{i,t}} - \frac{\varepsilon}{\varepsilon-1} \frac{P_{i,t+s}}{P_{i,t}} MC_{i,t+s} \right] \right\} = 0 \tag{15}$$

Condition (15) states implicitly that firms reset their prices as a mark-up over a weighted average of the current and expected marginal costs, where the weight of the expected marginal cost at some date $t+s$ depends on the probability that the price is still effective at that date.

---

[9]By assumption, therefore, every household works in all firms.

3 Equilibrium

3.1 International risk sharing

The assumption of complete markets implies:

$$\frac{C_t^{i\,-\sigma}}{P_{C^i,t}} = \frac{C_t^{j\,-\sigma}}{\mathcal{E}_{ij,t}P_{C^j,t}} \tag{16}$$

for all t, $i \in [0,\frac{1}{2})$ and $j \in (\frac{1}{2},1]$, where $\mathcal{E}_{ij,t}$ denotes the nominal exchange rate of region j currency relative to region i currency. According to (16), the values of marginal utilities of consumption are equal across regions. However, given the home bias in consumption, even if the law of one price holds, purchasing power parity does not. As a consequence, consumption can be different both across regions and areas.

By appropriately integrating this equation we obtain:

$$\frac{C_t^{i\,-\sigma}}{P_{C^i,t}} = \frac{C_{H,t}^{*\,-\sigma}}{\mathcal{E}_{iH,t}P_{H,t}^*} \quad i \in [0,\frac{1}{2}) \quad \frac{C_t^{i\,-\sigma}}{P_{C^i,t}} = \frac{C_{F,t}^{*\,-\sigma}}{\mathcal{E}_{iF,t}P_{F,t}^*} \quad i \in (\frac{1}{2},1] \quad \frac{C_{H,t}^{*\,-\sigma}}{P_{H,t}^*} = \frac{C_{F,t}^{*\,-\sigma}}{\mathcal{E}_{HF,t}P_{F,t}^*} \tag{17}$$

for all i, where $C_{H,t}^* \equiv \left[2\int_0^{\frac{1}{2}} C_t^{j\,-\sigma(1-\eta)}dj\right]^{\frac{-1}{\sigma(1-\eta)}}$ and $P_{H,t}^* \equiv \left[2\int_0^{\frac{1}{2}}\left(\mathcal{E}_{Hj,t}P_{C^j,t}\right)^{(1-\eta)}dj\right]^{\frac{1}{(1-\eta)}}$.
Symmetric definitions apply to $C_{F,t}^*$ and $P_{F,t}^*$.

Here $\mathcal{E}_{Hi,t}$ stands for the nominal exchange rate of region i currency to a common unit of account of area H. Regarding conditions (17), notice the following. Within area F, there is always a common currency, independently of the policy regime. Thus, $\mathcal{E}_{Fi,t} = 1$ for all $i \in [\frac{1}{2},1]$. Conversely within area H, $\mathcal{E}_{Hi,t} = 1$ for all $i \in [0,\frac{1}{2})$ only under regimes B and C, when there is a common currency and the exchange rates are fixed. Finally, in general, $\mathcal{E}_{HF,t}$ is floating under both regimes A and B while is fixed to 1 under regime C.

As shown in Appendix A, it follows from (16) and (17) that:

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\gamma_s + (\gamma_b - \gamma_s)\left(\frac{C_{H,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)} + (1-\gamma_b)\left(\frac{C_{F,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)}\right]^{\frac{1}{1-\eta}} \tag{18}$$

for $i \in [0,\frac{1}{2})$ and where $\gamma_s \equiv \frac{1}{\alpha_s}$ and $\gamma_b \equiv \frac{\alpha_b}{2\alpha_b-1}$. A corresponding condition can be retrieved for area F. At the same time, the price index (5) can be log-linearized as:

$$\hat{p}_{i,t} - \hat{p}_{c,t}^i = -(\alpha_b - \alpha_s)\hat{s}_{iH,t} - (1-\alpha_b)\hat{s}_{iF,t} \quad i \in [0,\frac{1}{2}) \tag{19}$$

where $\hat{s}_{iH,t} \equiv e_{iH,t} + \hat{p}_{H,t} - \hat{p}_{i,t}$ and $\hat{s}_{iF,t} \equiv e_{iF,t} + \hat{p}_{F,t} - \hat{p}_{i,t}$ denote the terms of trade between the small open region i and areas H and F respectively[10] and where $\hat{c}_{H,t} \equiv 2\int_0^{\frac{1}{2}}\hat{c}_t^j dj$ and $\hat{c}_{F,t} \equiv 2\int_{\frac{1}{2}}^1\hat{c}_t^j dj$.[11]

---

[10]...namely the average price of the goods produced in areas H or F relative to the average price of the goods produced in the small open economy i. With a notational abuse $\hat{p}_{F,t}$ indicates the log-deviation of the average price in area F expressed in terms of the common currency of that area. Similar interpretation applies to $\hat{p}_{H,t}$.

[11]We will use this as a general notation. For a given variable X_t^j, $\hat{x}_t^j \equiv \log X_t^j - \log X^j$ is the log-deviation of X_t^j from the steady state, while $\hat{x}_{H,t} \equiv 2\int_0^{\frac{1}{2}}\hat{x}_t^j dj$ and $\hat{x}_{F,t} \equiv 2\int_{\frac{1}{2}}^1\hat{x}_t^j dj$ are the average area log-deviation from the steady state.

By combining (5) with (19) and using (17) we obtain:

$$\hat{s}_{iH,t} = -\sigma\gamma_s(\hat{c}_{H,t} - \hat{c}_t^i) \qquad \hat{s}_{iF,t} = -\sigma\gamma_s(\hat{c}_{H,t} - \hat{c}_t^i) - \sigma(2\gamma_b - 1)(\hat{c}_{F,t} - \hat{c}_{H,t}) \quad (20)$$

for all $i \in [0, \frac{1}{2})$. Moreover, by integrating the log-linear approximation of (18) and (19), it is easy to show that:

$$\hat{s}_{HF,t} = -\sigma(2\gamma_b - 1)(\hat{c}_{F,t} - \hat{c}_{H,t}) \qquad (21)$$

where $\hat{s}_{HF,t} \equiv \hat{e}_{HF,t} + \hat{p}_{F,t} - \hat{p}_{H,t}$ is the terms of trade between area F and area H. According to (21), in equilibrium a rise in the terms of trade at home reduces its relative consumption ratio since, as previously assumed, $\alpha_b > \frac{1}{2}$ and thus $\gamma_b > \frac{1}{2}$. A terms-of-trade worsening (i.e., an increase of $\hat{s}_{HF,t}$) induces home consumers to substitute goods produced in area F with goods produced in area H and increases their overall consumption because given the home bias in consumption they prefer relatively more the bundle produced in their own area. Notice that the impact of a terms-of-trade deterioration on consumption differentials depends critically on households' risk aversion (or the inverse of the intertemporal elasticity of substitution of consumption) σ. The higher σ is, the lower is the difference in average consumption across areas associated with a given movement in the terms of trade. More risk adverse households are more willing to share risk across different states of the world (or more willing to smooth consumption across periods). Similarly, the lower the degree of home bias α_b is, the more movements in the terms of trade translate into consumption differentials, implying that more open areas are more sensitive to changes in the terms of trade. Finally, by taking differences of (20) and (21), it follows that:

$$\Delta e_{iH,t} + \pi_{H,t} - \pi_{i,t} = -\sigma\gamma_s(\Delta\hat{c}_{H,t} - \Delta\hat{c}_t^i) \qquad i \in [0, \frac{1}{2}) \qquad (22)$$

$$\pi_{F,t} - \pi_{i,t} = -\sigma\gamma_s(\Delta\hat{c}_{F,t} - \Delta\hat{c}_t^i) \qquad i \in [\frac{1}{2}, 1] \qquad (23)$$

$$\Delta e_{HF,t} + \pi_{F,t} - \pi_{H,t} = -\sigma(2\gamma_b - 1)(\Delta\hat{c}_{F,t} - \Delta\hat{c}_{H,t}) \qquad (24)$$

Equation (23) – and under regimes B and C, when $e_{iH,t} = 1$ and $e_{HF,t} = 1$, also equations (22) and (24) – can be interpreted as a constraint imposed by the adoption of a common currency according to which, in response to asymmetric shocks, the terms of trade cannot adjust instantaneously because of the sluggish price adjustment and the fixed exchange rates. Differently, under regime A in area H, when there is monetary autonomy, the fluctuations of the nominal exchange rate assure that condition (22) is always satisfied.

3.2 IS curves

Condition (10) implies that:

$$\frac{1}{1 + r_t^i} = \beta E_t \left\{ \left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\sigma} \Pi_{C^i,t+1}^{-1} \right\} \qquad (25)$$

for all i, where $\frac{1}{1+r_{i,t}} = E_t\{Q_{t,t+1}^i\}$. When markets are complete, the expected value of the intertemporal marginal rate of substitution of private consumption, namely the price of a riskless portfolio, equalizes the price of the riskless bond, being $r_{i,t}$ the nominal interest rate. Notice that under regime A, r_t^i can be different across the regions in area H, since national central banks are independent in their policy decisions, while

$r_t^i = r_{F,t}$ for all $i \in [\frac{1}{2}, 1]$ with $r_{F,t}$ being the nominal interest of area F set by the common central bank of the monetary union. Conversely, under regime B, $r_t^i = r_{H,t}$ for all $i \in [0, \frac{1}{2})$, $r_t^i = r_{F,t}$ for all $i \in [\frac{1}{2}, 1]$. Finally, under regime C $r_t^i = r_{W,t}$ for all i.

By using (17), we can log-linearize (25) and rewrite it as:

$$r_{F,t} - \rho - E_t\{\pi_{F,t+1}\} = \sigma E_t\{\Delta\hat{c}_{F,t+1} + (1-\gamma_b)(\Delta\hat{c}_{H,t+1} - \Delta\hat{c}_{F,t+1})\} \qquad (26)$$

$$r_t^i - \rho - E_t\{\pi_{i,t+1}\} = \sigma E_t\{\Delta\hat{c}_{t+1}^i + (\gamma_b - \gamma_s)(\Delta\hat{c}_{H,t+1} - \Delta\hat{c}_{t+1}^i) \\ + (1-\gamma_b)(\Delta\hat{c}_{F,t+1} - \Delta\hat{c}_{t+1}^i)\} \qquad (27)$$

$$r_{H,t} - \rho - E_t\{\pi_{H,t+1}\} = \sigma E_t\{\Delta\hat{c}_{H,t+1} + (1-\gamma_b)(\Delta\hat{c}_{F,t+1} - \Delta\hat{c}_{H,t+1})\} \qquad (28)$$

where $\rho \equiv -log(\beta)$. Conditions (26), (27) and (28) are the so called IS-curves. Condition (26) holds under regimes A and B, condition (27) under regime A and condition (28) under regime B.

3.3 Aggregate demand

In each region i of area H the demand for a specific good, $y_t(h^i)$, is determined by the demand of home and foreign consumers, namely:

$$y_t(h^i) = c_t^i(h^i) + \int_0^{\frac{1}{2}} c_t^j(h^i)dj + \int_{\frac{1}{2}}^1 c_t^j(h^i)dj \qquad (29)$$

for all $i \in [0, \frac{1}{2})$ and where $c_t^j(h^i)$ is the foreign demand for the good h^i. Given (6), from condition (29) we can find the aggregate demand for region i:

$$Y_t^i = \alpha_s \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta} C_t + 2(\alpha_b - \alpha_s)\int_0^{\frac{1}{2}} \left(\frac{P_{i,t}}{P_{C^j,t}}\right)^{-\eta} C_t^j dj + 2(1-\alpha_b)\int_{\frac{1}{2}}^1 \left(\frac{P_{i,t}}{P_{C^j,t}}\right)^{-\eta} C_t^j dj \qquad (30)$$

with $Y_t^i \equiv \left[\int_0^1 y_t(h^i)^{\frac{\varepsilon-1}{\varepsilon}} dh^i\right]^{\frac{\varepsilon}{\varepsilon-1}}$. Because of (16), (30) can be written as:

$$Y_t^i = \left(\frac{P_{i,t}}{P_{C^i,t}}\right)^{-\eta}\left[\alpha_s C_t^i + (\alpha_b - \alpha_s)C_t^{i\sigma\eta}\mathcal{C}_{H,t} + (1-\alpha_b)C_t^{i\sigma\eta}\mathcal{C}_{F,t}\right] \qquad (31)$$

with $\mathcal{C}_{H,t} \equiv 2\int_0^{\frac{1}{2}} C_t^{j1-\sigma\eta}dj$ and $\mathcal{C}_{F,t} \equiv 2\int_{\frac{1}{2}}^1 C_t^{j1-\sigma\eta}dj$ for all $i \in [0, \frac{1}{2})$. A symmetric condition can be stated for all $i \in [\frac{1}{2}, 1]$. It is easy to show that the log-linear approximation to (31) corresponds to:

$$\hat{y}_t^i = \delta_s\hat{c}_t^i + (\delta_b - \delta_s)\hat{c}_{H,t} + (1-\delta_b)\hat{c}_{F,t} \quad i \in [0, \frac{1}{2}) \qquad (32)$$

where $\delta_s \equiv \gamma_s\eta\sigma + \alpha_s(1-\eta\sigma)$ and $\delta_b \equiv \gamma_b\eta\sigma + \alpha_b(1-\eta\sigma)$. Notice that δ_s can be interpreted as the elasticity of region i output to region i consumption. By using (20) and (21), we can rewrite (32) as:

$$\hat{y}_t^i = \hat{c}_t^i + \left[\frac{1-\delta_b}{\sigma(2\gamma_b - 1)} - \frac{1-\delta_s}{\sigma\gamma_s}\right]\hat{s}_{iH,t} - \frac{1-\delta_b}{\sigma(2\gamma_b - 1)}\hat{s}_{iF,t} \qquad (33)$$

Thus, the fluctuations in the aggregate demand for region i produced goods depend crucially on the terms-of-trade movements. Aggregating (33), we obtain:

$$\hat{y}_{H,t} = \hat{c}_{H,t} - \frac{1-\delta_b}{\sigma(2\gamma_b - 1)}\hat{s}_{HF,t} \qquad (34)$$

10

According to (34), when $\delta_b = 1$ area H aggregate consumption and output perfectly co-move one to one as if the area were closed. Conversely, if $\delta_b > 1$ ($\delta_b < 1$), in response to a terms of trade improvement (worsening) between area H and F output in area H falls (rises). An intuition of this result is provided in Section 5.1.

3.4 Aggregate supply

Following Benigno and Woodford (2005), we can rewrite the firms' optimality conditions, (13) and (15), recursively as follows:

$$\frac{1 - \theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta} = \left(\frac{F_t^i}{K_t^i}\right)^{\varepsilon-1} \tag{35}$$

$$Z_t^i = \theta Z_{t-1}^i \Pi_{i,t}^{\varepsilon} + (1-\theta)\left(\frac{1 - \theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{36}$$

where:

$$K_t^i = C_t^{i-\sigma} Y_{i,t}^i \frac{P_{i,t}}{P_{C^i,t}} \frac{\varepsilon}{\varepsilon-1} MC_{i,t} + \beta\theta E_t\left\{\Pi_{i,t+1}^{\varepsilon} K_{t+1}^i\right\} \tag{37}$$

$$F_t^i = C_t^{i-\sigma} Y_t^i \frac{P_{i,t}}{P_{C^i,t}} + \beta\theta E_t\left\{\Pi_{i,t+1}^{\varepsilon-1} F_{t+1}^i\right\} \tag{38}$$

with $\Pi_{i,t} \equiv \frac{P_{i,t}}{P_{i,t-1}}$ and $\frac{K_t^i}{F_t^i} = \frac{\tilde{p}_t(h^i)}{P_{i,t}}$. By the log-linear approximation of (35), (37) and (38):

$$\pi_{i,t} = \lambda\widehat{mc}_{i,t} + \beta E_t\{\pi_{i,t+1}\} \tag{39}$$

with $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$ and where $\widehat{mc}_{i,t} = (\hat{w}_t^i - \hat{p}_{c,t}^i) - (\hat{p}_{i,t} - \hat{p}_{c,t}^i) - \hat{a}_t^i$ for all t and i. Condition (39) is the New Keynesian Phillips Curve, which results from the Calvo mechanism. As usual, current domestic inflation depends on the expectation on future domestic inflation and the current real marginal cost of producing goods. In equilibrium this cost is determined by the real wage, which – according to (9) – is a mark-up over the marginal rate of substitution between consumption and leisure, the product price index relative to the consumption price index (18) and labor productivity. By substituting the log-linear approximation of (9) and condition (19) into $\widehat{mc}_{i,t}$ we obtain:

$$\widehat{mc}_{i,t} = \varphi\hat{y}_t^i + \sigma\hat{c}_t^i + (\alpha_b - \alpha_s)\hat{s}_{iH,t} + (1-\alpha_b)\hat{s}_{iF,t} - (1+\varphi)\hat{a}_t^i + \hat{\mu}_t^i$$

$$= (\varphi+\sigma)\hat{y}_t^i + \left[\frac{\gamma_s - \delta_s}{\gamma_s} - \frac{\gamma_b - \delta_b}{2\gamma_b - 1}\right]\hat{s}_{iH,t} + \frac{\gamma_b - \delta_b}{2\gamma_b - 1}\hat{s}_{iF,t} - (1+\varphi)\hat{a}_t^i + \hat{\mu}_t^i \tag{40}$$

for all $i \in [0, \frac{1}{2})$ where the last equality can be recovered by using (33). According to the second equation in (40) – and as discussed in more detail below in Sections 4, 5.1 and 5.3 – in general firms' marginal cost is influenced by terms-of-trade fluctuations. However, when $\eta\sigma = 1$, the firms' marginal cost of small open regions is completely insulated from terms-of-trade movements and becomes *isomorphic* to that of a closed economy. In fact, under this parametric configuration $\delta_s - \gamma_s = (\gamma_s - \alpha_s)(\eta\sigma - 1) = 0$ and $\delta_b - \gamma_b = (\gamma_b - \alpha_b)(\eta\sigma - 1) = 0$.

The rational-expectations equilibrium can be determined using (9), (17), (18), (25), (31), (35), (36), (37), (38) and their foreign counterparts. To close the model, it remains to determine the optimal monetary policies under regimes A, B and C.

4 The Pareto-efficient allocation

In Appendix B we retrieve the conditions that characterize the Pareto-efficient allocation. As common in the New-Keynesian literature, in this model in the absence of mark-up shocks, the efficient allocation can be supported as a market equilibrium under the following conditions: First, the *steady-state* distortion due to monopolistic competition is corrected in all regions by the use of an appropriate labor subsidy; second, the *dynamic* distortion caused by price and output dispersion is eliminated by completely stabilizing firms' marginal costs in all markets. While a formal derivation of this result is beyond the scope of this paper, we will show how the second condition applies to the average fluctuations in the areas. This will prove useful for the analysis of the optimal policies to which we turn below. By combining the two conditions in (73) with their symmetric counterparts for the foreign area, we obtain:

$$\hat{y}_{H,t}^e = \frac{(\varphi+1)}{\sigma+\varphi}\hat{a}_{H,t} + \frac{(\delta_b - \gamma_b)\sigma(\varphi+1)}{(\sigma+\varphi)((2\gamma_b-1)\sigma + (2\delta_b-1)\varphi)}(\hat{a}_{H,t} - \hat{a}_{F,t}) \qquad (41)$$

where the suffix e stands for efficient. Condition (41) expresses the percentage deviations of the efficient level of output in area H in terms of area H productivity and the productivity differentials between area H and F. At the same time, we can define

$$\widehat{mc}_{H,t}^e \equiv \varphi\hat{y}_{H,t} + \sigma\hat{c}_{H,t} + (1-\alpha_b)\hat{s}_{HF,t} - (1+\varphi)\hat{a}_{H,t} \qquad (42)$$

as firms' marginal cost in the absence of the mark-up shocks. A symmetric definition applies to $\widehat{mc}_{F,t}^e$, the efficient firms' marginal cost in area F. Then, stabilizing such costs, i.e.,

$$\widehat{mc}_{H,t}^e = 0$$
$$\widehat{mc}_{F,t}^e = 0 \qquad (43)$$

ensures jointly with the equilibrium conditions (21), (34) and its foreign counterpart that the average output fluctuations mimic those of the efficient allocation which are determined by equation (41). Intuitively, when a technology shock occurs and the firms' marginal costs are held constant, the dynamic distortion due to price stickiness is corrected and the average area fluctuations in the marginal rates of substitution and transformation between consumption and labor are equal.

Regarding condition (41) recall that $\delta_b - \gamma_b = (\gamma_b - \alpha_b)(\eta\sigma - 1)$. As a result, the sign of the coefficient of productivity shock differentials depends critically on whether the trade elasticity η is greater or smaller than the intertemporal elasticity of substitution $1/\sigma$ – i.e., on whether domestic and foreign goods are substitutes in the utility so that an increase in the demand for foreign goods at home reduces the marginal utility from consuming domestic goods. Therefore, the impact of a foreign productivity shock on domestic output is determined by relative strength of two effects. The first effect is an *intertemporal* effect: in response to an inprovement in the foreign productivity, the foreign interest rate falls and – since $\sigma \geq 1$ – foreign consumers increase their supply of state-state contingent assets in order to smooth consumption across periods. As a consequence, domestic households can borrow more. This effect is stronger, the higher is σ. The second effect is an *intratemporal* effect: as productivity improves in the foreign economy, domestic terms of trade amiliorate. Then, domestic and foreign households shift the composition of their consumption bundle, reducing (raising) their demand for domestic goods as long as η, the trade elasticity, is greater (smaller) than one. Given these two effects, when $\eta\sigma > 1$ (substitutes in the utility), domestic households

increase their borrowing and their leisure in response to a rise in foreign productivity, while the improvement in the domestic terms of trade possibly contracts the demand for domestically produced goods via the *expenditure switching* effect. Overall output in area H falls. Vice versa, when $\eta\sigma < 1$ (complements in the utility), a positive productivity shock in the foreign area raises domestic output. In this case, the demand for domestically produced goods increases in response to the rise in the demand for foreign goods despite the terms-of-trade improvement. Finally, if $\eta\sigma = 1$, output in area H is insulated from foreign productivity shocks and the associated shifts in the terms of trade and behaves as if the economy were closed.[12] In this special case, as made clear by condition (40), firms' marginal costs are independent of terms-of-trade fluctuations since the intertemporal and the intratemporal effects exactly offset each other and there is no incentive for domestic households to change their consumption and labor decisions in response to a technological improvement in the foreign economy.

5   Welfare

As anticipated in the introduction, the main objective of this paper is to compare welfare costs and benefits of a monetary union in a fully micro-founded New-Keynesian model under three policy regimes. Under regime A, there is a common currency in area F, while countries in area H retain their own central banks; under regime B, there are two monetary unions, one in area F and the other one in area H. Under regime C, there is a single authority that sets the interest rate for the world economy as a whole.

In order to solve the optimal monetary policy problems, we make the following assumptions. Independently of the policy regime all monetary authorities (the central banks of the monetary unions and of the small open economies) are benevolent, take as given other policy makers' choices and can commit credibly to *past* and future promises. In other words, policies are optimal from the timeless perspective. As in Benigno and Benigno (2006), policy makers' strategies of the policy game under regimes A and B are specified in terms of the entire path of inflation,[13] defined as a time-varying function of the shocks hitting the economy.[14] So when maximizing domestic utility at time 0, these authorities commit credibly to implement the desired state-contingent path of inflation, taking as given the state-contingent path of inflation chosen by foreign policy makers.[15]

Under these hypotheses, we can use the linear-quadratic approach pioneered by Benigno and Woodford (2005) and Benigno and Woodford (2012) to determine the optimal monetary policies.[16] We implement the linear quadratic approach as follows.[17] First, we formulate the non-linear optimal policy problems (Ramsey problems). Second, we recover the zero-inflation deterministic steady state of these problems. Third, we employ the second-order approximation of the structural equations to retrieve a

---

[12]These results are well known in the open-macro literature. See for a discussion Corsetti, Dedola and Leduc (2010).

[13]The producer price inflation.

[14]Namely, we study the open-loop Nash equilibrium.

[15]Notice that given this assumption, under regime A, small open economy central banks take as given *all* the aggregate variables. Intuitively, if the economy is *infinitesimally* small, its performance cannot influence aggregate variables.

[16]Actually, it would be difficult to compute these policies using standard numerical methods, since the economy is framed as a continuum of small open regions.

[17]For more specific details on the non-linear optimal policy problems, on the zero-inflation steady state and on the quadratic approximation, see Appendices C and D.

purely quadratic approximation to the objectives of both the small open economy and the monetary union authorities expressed as function of output, inflation and terms of trade in deviations from their *welfare-relevant* targets. As emphasized below, identifying these targets proves particularly useful to disentangle what drives policy makers' incentives. Fourth, we determine the optimal monetary policies by maximizing these quadratic approximations subject to the log-linear approximation of the structural constraints. Finally, we quantify welfare differences for the households of areas H and F across policy regimes and we identify which regime is preferable depending on the deep parameters of the model.

5.1 The steady-state distortion

We assume that τ, the employment subsidy, is equal across countries and regimes. As shown in Appendix C, under this assumption, there exists a symmetric deterministic steady state at which zero inflation is a Nash equilibrium policy for all policy makers in areas H and F under both regimes A and B. Similarly, it can be shown that at the deterministic steady state zero inflation is an optimal policy for the world monetary policy maker under regime C.[18]

At the steady state the following condition holds:

$$Y = C = (1 - \tilde{\tau})^{-\frac{1}{\sigma + \varphi}} \tag{44}$$

where $\tilde{\tau} \equiv 1 - (1 - \tau)(1 + \mu)\frac{\varepsilon}{\varepsilon - 1}$. As made clear by (44), $\tilde{\tau}$ determines the steady-state wedge between the marginal rates of substitution and transformation between consumption and leisure. Indeed, according to (44) $(1 - \tilde{\tau})MRS = (1 - \tilde{\tau})C^\sigma Y^\varphi = 1 = MRT$.

Given their different objectives, the policy makers of the small open economies and of the monetary unions have different incentives and different ideas about what is the efficient steady-state level of domestic output.

Consider first the case of a cooperative policy maker. In order to determine the optimal level of domestic output from the world authority point of view we maximize:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\int_0^1 \left(\frac{C_t^{i\,1-\sigma}}{1 - \sigma} - \frac{1}{\varphi + 1} \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi + 1} \right) di \right] \tag{45}$$

with respect to C_t^i and Y_t^i for all $i \in [0, 1]$, subject to (31) and its foreign counterpart and taking into account that $P_{i,t}/P_{C^i,t}$ are determined according to (18) and its foreign analogue. Not surprisingly, at the deterministic steady state the cooperative policy maker finds it optimal to choose $\tilde{\tau} = 1 - (1 - \tau))(1 + \mu)\frac{\varepsilon}{\varepsilon - 1} = 0$ so that $Y_w = 1$. With this policy, she manages to implement the Pareto optimum, offsetting the monopolistic distortions in both the labor and the goods markets and closing completely the wedge between marginal rates of substitution and transformation. Then we can define:

$$\Phi_w \equiv \tilde{\tau} \tag{46}$$

as a parameter that governs the steady-state distortion from the world economy policy maker viewpoint.

Consider now the case of the small open economy i. The efficient steady state from the small open policy maker perspective can be retrieved by maximizing:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{i\,1-\sigma}}{1 - \sigma} - \frac{1}{\varphi + 1} \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi + 1} \right]$$

---

[18]See Benigno and Benigno (2006) for a formal proof of this result.

f with respect to C_t^i and Y_t^i, subject to (31) and where $P_{i,t}/P_{C^i,t}$ are determined consistently with (18), while – differently from the case of cooperation – $C_{H,t}^*$, $C_{F,t}^*$, $\mathcal{C}_{H,t}$ and $\mathcal{C}_{F,t}$ are taken as given. According to the first order conditions, at the symmetric deterministic steady state:

$$Y_s = \delta_s^{\frac{-1}{\sigma+\varphi}} \tag{47}$$

As above, $\delta_s \equiv \gamma_s \eta\sigma + \alpha_s(1 - \eta\sigma)$. According to (44) and (47), the optimal labor subsidy that decentralizes – at the steady state – the efficient allocation from the small open economy viewpoint as a market equilibrium is given by $\tilde{\tau}_s = 1 - \delta_s$. Then, we can define:

$$\Phi_s \equiv \tilde{\tau} - \tilde{\tau}_s \tag{48}$$

as a parameter that measures the steady-state distortion from the small open economy perspective.

As made clear by (47), the small open economy policy makers do not aim to reach the Pareto-efficient steady state at which the monopolistic distortions are exactly eliminated unless $\delta_s = 1$. As long as $\delta_s > 1$ ($\delta_s < 1$), they would rather prefer a lower (higher) level of steady-state production. Equations (32) and (33) provide an intuition for this result. Consider first the case in which domestic and foreign goods are substitutes in the utility (i.e., $\eta\sigma > 1$ and $\delta_s > 1$)[19] and the economy experiences a terms-of-trade improvement. As the terms of trade ameliorates, consumers at home borrow more (*intertemporal* effect), increase their leisure and possibly reduce the demand for domestic goods (*intratemporal* effect). Domestic consumption falls relative to foreign consumption. However, its fall is dampened by the rise in borrowing and in the demand for goods produced abroad. Hence, domestic consumption decreases less than output. As a consequence, small open economy authorities have an incentive to try to improve the terms of trade: the rise in leisure due to the reduction in the labor supply more than compensates households for the fall in consumption generated by the terms-of-trade improvement. In other words, small open economies want to reduce domestic production in order to improve the terms of trade and to externalize labor effort at other countries' expense. If $\frac{\alpha_s}{1+\alpha_s} < \eta\sigma < 1$, δ_s is still greater than one. Intuitively, even if home and foreign goods are complements in the utility, consumers find it optimal to borrow more. In this way, they can dampen the reduction of the overall expenditure for consumption reducing at the same time their labor supply. Still, home consumption falls less than output and small open economy policy makers will try to improve their terms of trade. However, when $\eta\sigma < \frac{\alpha_s}{1+\alpha_s}$, $\delta_s < 1$ and home and foreign goods are strong complements, this incentive turns around. Small country authorities seek to worsen their terms of trade. In this case, the demand for *both* home and foreign goods increase, as the terms of trade deteriorate. Then, a rise in one unit in domestic production allows to raise home consumption *by more* than a unit thanks to the increase in foreign labor supply. As a result, the increase in consumption more than outweighs the reduction in leisure due to the terms-of-trade worsening in the utility. Finally, if $\delta_s = 1$, $\Phi_s = \Phi_w$ and the steady-state level of output efficient from the small open economy viewpoint coincides with the Pareto-efficient level i.e., $Y_s = Y_w$.

In the case of the policy maker of the monetary union, the desired level of steady-state output can be determined by maximizing:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[2 \int_0^{\frac{1}{2}} \left(\frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1} \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi+1} \right) di \right] \tag{49}$$

---

[19]Notice that $\delta_s > 1$ as long as $\eta\sigma > 1$.

15

with respect to C_t^i and Y_t^i for all $i \in [0, 1]$ and subject to:

$$\frac{P_{i,t}}{P_{C^i,t}} = \frac{(1 - \tilde{\tau})}{A_t^{i\varphi+1}} \frac{Y_t^{i\varphi}}{C_t^{i-\sigma}} \tag{50}$$

for all $i \in [\frac{1}{2}, 1]$, constraint (31) and its foreign counterpart.[20] Again $P_{i,t}/P_{C^i,t}$ is determined according to (18) and its foreign counterpart. From the first-order conditions of this problem it follows that at the symmetric deterministic steady state:

$$Y_b = \left[1 - \frac{(1 - \delta_b)(\sigma + \varphi)}{(\delta_b\varphi + \gamma_b\sigma)}\right]^{\frac{-1}{\sigma+\varphi}} \tag{51}$$

This allocation can be achieved as market equilibrium by choosing a labor subsidy of $\tilde{\tau}_b = \frac{(1-\delta_b)(\sigma+\varphi)}{(\delta_b\varphi+\gamma_b\sigma)}$ where $\delta_b \equiv \gamma_b\eta\sigma + \alpha_b(1 - \eta\sigma)$. At the same time, we can define:

$$\Phi_b \equiv \tilde{\tau} - \tilde{\tau}_b \tag{52}$$

as the analogue of the parameters Φ_w and Φ_s for the case of the big economy. According to (51), even in the case of the big economy, policy makers seek to manipulate the terms of trade to their own advantage. But differently from the case of the small open economy, the key parameter that governs this incentive is $1 - \delta_b$. If $\delta_b > 1$, policy makers of the big economies would like to under-subsidize labor, to improve the terms of trade and reduce domestic production with respect to what would be Pareto efficient. Vice versa, if $\delta_b < 1$ and home and foreign goods are complements, the policy authorities of the area try to over-subsidize domestic production to worsen the terms of trade. In either case, they seek to externalize labor effort at other countries' expense. Again when $\delta_b = 1$ there is no incentive to manipulate the terms of trade. Thus, in this case domestic output is distorted from the big economy viewpoint only when the steady state is not Pareto efficient, i.e., $Y_b \neq Y_w$.

Finally, Figure 1 plots the desired level of domestic output[21] for the three cases considered above as a function of η, – the trade elasticity – and using the baseline parametrization: i.e., $\alpha_s = 0.6$, $\alpha_b = 0.85$ $\sigma = 2$ and $\varphi = 3$.[22] The violet line with triangles shows the level of steady-state output under the baseline parametrization.

For most of the values of η policy makers of both small and big economies want to under-subsidize labor to improve their terms of trade. However, this incentive is significantly stronger for the small open economy policy makers. For instance, when $\eta = 2$, Y_b and Y_s are respectively 11.5% and 27.13% lower than Y_w, the Pareto efficient level of output. The reasons for this outcome are threefold. First of all, bigger countries are less open. As a consequence, the incentive of their policy makers to improve the terms of trade is weaker. Secondly, big economy authorities realize that they hold monopoly power only on the terms of trade between areas and they internalize the external effects produced within the monetary union. Finally, they take into account the impact of their policies on the foreign economy. For example, they are aware that when $\delta_b > 1$ a terms-of-trade improvement can raise foreign production in response to the increase in the demand for foreign produced goods due to the expenditure-switching

---

[20]Condition (50) implicitly states that the policy maker of area H takes as given the strategy $\tilde{\tau}$ chosen by a symmetric policy maker in area F. Notice that in the case of the two areas and differently from the case of the small open economy, in general, a steady state that is efficient from the viewpoint of both the policy makers of the areas F and H and which is akin to the Pareto efficient steady state in a closed economy does not exist.

[21]We could have alternatively shown the level of the desired subsidies.

[22]For a detailed discussion of the baseline calibration see section 6.

effect. So they recognize that a lower labor tax rate (lower than the one set by the small open economy policy makers, who take as given what happens in the foreign economies) allows to reach the same terms-of-trade improvement. All these motives contribute to the weakening of the desire to influence the terms of trade.

Summing up, the difference in size between small and big countries affects the incentives of their policy makers to generate externalities at the steady state. Specifically, under the baseline parametrization, for almost all the values of the trade elasticity, the desired steady-state level of domestic output is closer to Pareto efficiency in the case of the monetary union than in the case of small open economies. As it will become evident in the next sections, the fact that open economy policy makers have different "perceptions" of the steady-state distortions is key to explain the differences in their optimal policies over the business cycle. Indeed as we will clarify below, if the steady state is distorted from the policy maker's perspective (i.e., if steady state output does not coincide with the desired level) then, monetary policy makers seek to manipulate household and firm behavior over the business cycle in order to drive what we call the per period output and the per period terms of trade (i.e., the expected level of output and of the terms of trade) towards their desired levels.

5.2 The benchmark: the case of cooperation

In this section we consider the benchmark case of cooperation. In this scenario there is a common authority that maximizes world welfare.[23] In Appendix D.4, we show that the objective of the cooperative policy maker can be approximated in a purely quadratic way as follows:

$$
\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0 \Big[\frac{1}{2}\varpi_{1,w}\int_0^1 (\hat{y}_t^i)^2 di + \varpi_{2,w}\int_0^1 \hat{c}_t^i \hat{y}_t^i di + \frac{1}{2}\varpi_{3,w}\int_0^1 (\hat{c}_t^i)^2 di
$$

$$
+ \frac{1}{2}\varpi_{4,w}\int_0^1 (\pi_{i,t})^2 di + \frac{1}{2}\varpi_{5,w}\left(\hat{y}_{H,t}\hat{c}_{H,t} + \hat{y}_{F,t}\hat{c}_{F,t}\right) + \frac{1}{4}\varpi_{6,w}\left(\hat{c}_{H,t}^2 + \hat{c}_{F,t}^2\right)
$$

$$
+ \frac{1}{2}\varpi_{7,w}\left(\hat{y}_{F,t}\hat{c}_{H,t} + \hat{y}_{H,t}\hat{c}_{F,t}\right) + \frac{1}{2}\varpi_{8,w}\hat{c}_{H,t}\hat{c}_{F,t} - \varpi_{9,w}\int_0^1 \hat{y}_t^i \hat{a}_t^i di
$$

$$
- \varpi_{10,w}\int_0^1 \hat{y}_t^i \hat{\mu}_t^i di \Big] + t.i.p. \tag{53}
$$

where $\varpi_{1,w}$, $\varpi_{2,w}$, $\varpi_{3,w}$, $\varpi_{4,w}$, $\varpi_{5,w}$ $\varpi_{6,w}$, $\varpi_{7,w}$, $\varpi_{8,w}$, $\varpi_{9,w}$ and $\varpi_{10,w}$ are defined consistently with (128) and depend on the structural parameters of the model and where t.i.p. stands for "terms independent of policies". The second-order welfare approximation in (53) expresses the utility losses as a function of inflation, consumption and output of the single regions and of the areas H and F. We want to rewrite (53) in deviations from what the literature calls *policy targets*. In order to do so, we minimize (53) subject to (32) and its foreign counterpart for all $i \in [0,1]$.[24] The first-order conditions of this minimization problem allow to retrieve the target of the cooperative policy maker in our model. The policy target can then be interpreted as the allocation that a policy maker would choose, taking into account the steady-state distortion due

---

[23]This approximation to the welfare will then be used to retrieve the objective of the central bank of the world monetary union in regime C.

[24]Note that this is exactly the same as substituting these constrains into (53), as done by Benigno and Woodford (2005).

to monopolistic competition – or a distortive labor subsidy –, but abstracting from the dynamic distortion generated by the firm price dispersion. For this reason, as shown below, the analytical expression of these targets helps to disentangle what drives policy maker incentives.

In the case of cooperation, the target of the cooperative authority satisfies the following conditions:[25]

$$[1 - \zeta_w(\varphi + 1)]\,\widehat{mc}_{H,t}^{e,w} = \zeta_w(\varphi + 1)\hat{\mu}_{H,t}$$

$$[1 - \zeta_w(\varphi + 1)]\,\widehat{mc}_{F,t}^{e,w} = \zeta_w(\varphi + 1)\hat{\mu}_{F,t} \tag{55}$$

where we used the convention that $\hat{x}_t^w$ is the target of the world policy maker for the variable $\hat{x}_t$, while $\widehat{mc}_{H,t}^{e,w}$ and $\widehat{mc}_{F,t}^{e,w}$ are defined consistently with (42). From (55), we draw the following conclusions about the goals of the cooperative authority:[26]

1. *The trade-off.* The cooperative monetary policy maker faces a trade-off between: a) stabilizing completely the firms' marginal costs at their efficient level (i.e., $\widehat{mc}_{H,t}^{e,w} = 0$ and $\widehat{mc}_{F,t}^{e,w} = 0$), b) allowing $\widehat{mc}_{H,t}^{e,w}$ and $\widehat{mc}_{F,t}^{e,w}$ to fluctuate in response to mark-up shocks.

2. *The weights of the trade-off and the steady-state distortion.* The coefficients $1 - \zeta_w(\varphi + 1)$ and $\zeta_w(\varphi + 1)$ correspond to the relative weights attached to the trade-off and depend critically on the Φ_w, the parameter that governs the steady-state distortion. In fact, since $\zeta_w = \frac{\Phi_w}{(\sigma + \varphi)}$, $\zeta_w = 0$ if and only if $\Phi_w = 0$. As a result, if the steady state is Pareto efficient (i.e., $\Phi_w = 0$) completely stabilizing firms' marginal costs at their efficient level is the target of the monetary policy maker. In this case, we go back to the standard finding of closed economy literature[27] stating that the central bank aims to close the gap between the marginal rates of substitution and transformation between consumption and labor and to reach the first-best allocation.

3. *The cases for price stability.* Independently of whether the steady state is distorted or not, if shocks are only to technology, strict-inflation targeting is optimal. Intuitively, even when $\Phi_w \neq 0$, the monetary policy maker does not try to correct – not even partially – the steady-state distortions because the welfare gains derived from the improvement of per period output towards efficiency are exactly offset by the costs associated with firms' output dispersion. Therefore, she abstracts from the steady-state distortion and seeks to replicate the fluctuations of the first-best allocation. In other words, under technological shocks the flexible-price allocation is constrained efficient.

4. *Mark-up shocks.* Under mark-up shocks the monetary authority is willing to allow $\widehat{mc}_{H,t}^{e,w}$ and $\widehat{mc}_{F,t}^{e,w}$ to fluctuate over the cycle to the extent to which the steady state is distorted. In particular, if the per period output is inefficiently low – if i.e.,

---

[25]See condition (132) in Appendix D.4 for the full set of equations that determine the target of the cooperative policy maker. For the sake of simplicity, we limit the discussion of the target of the cooperative authority to the average-area variables. However, it can be shown that similar conditions hold for single regions:

$$[1 - \zeta_w(\varphi + 1)]\,\widehat{mc}_{i,t}^{e,w} = \zeta_w(\varphi + 1)\hat{\mu}_t^i \tag{54}$$

[26]Most of these conclusions are the same as those of Benigno and Benigno (2006).

[27]See among others Galí (2008) and Woodford (2003). On the open economy case, see Benigno and Benigno (2006).

$\Phi_w < 0$ –, the monetary policy maker wants output to negatively co-move with mark-up shocks. As a result, she *under-stabilizes* output – i.e., she stabilizes it less than what she would do if the steady state were efficient. Put differently, the cooperative authority changes her inflation output stabilization trade-off and stabilizes inflation more than output. The stronger this incentive, the higher Φ_w is, the latter being the parameter that governs the average wedge between the marginal rates of substitution and transformation at the steady state.

At first, the third result is quite puzzling: mark-up shocks generate inefficient fluctuations in consumption and output. Intuitively, we could expect that the central bank then aims to completely stabilize output and consumption (as in fact it is willing to do so when the steady state is efficient). Instead, it wants output and consumption to co-move with these shocks. The underlying reason of this behavior can be understood by looking at condition (100) in its closed economy counterpart. When prices are flexible and there are no shocks to technology then:

$$E\left\{\frac{W_t}{P_t}\right\} = E\left\{Y_t^{\varphi+\sigma}\right\} E\left\{(1+\mu_t)\right\} + Cov\left\{Y_t^{\varphi+\sigma}(1+\mu_t)\right\} \tag{56}$$

According to (56), the lower the covariance between mark-up shocks and output, the lower is the per period output for a given level of per period real wage. Then, condition (56) can explain the incentive of cooperative authorities (as well as of the closed economy policy maker) to affect the risk of output fluctuations. These authorities can try to manipulate strategically the effects of uncertainty on households behavior. Intuitively, policies can provide better – or worse – insurance against the risk of output fluctuations due to mark-up shocks. For instance, when policies are less expansionary in response to mark-up shocks, output fluctuates more. Then, households, who are risk averse, would like to save more and raise their wealth stored to face bad states of the world.[28] For this reason, given the per period real wage an increase in the risk of output fluctuations in response to mark-up shocks – i.e., a fall in the covariance between mark-up shock and output – induces households to raise their average labor effort. As a consequence, if per period output is inefficiently low, policies that increase the negative co-movements between output and mark-up shocks have beneficial welfare effects: they shift the average labor supply curve downward, allowing for an efficient increase in the expected level of production. By using (20), (21) and (131), we can rewrite (53) in terms of deviations from the target:

$$\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\Big[\varpi_{11,w}\int_0^{\frac{1}{2}}(\tilde{s}_{iH,t}^w)^2 di + \varpi_{11,w}\int_{\frac{1}{2}}^1(\tilde{s}_{iF,t}^w)^2 di + \varpi_{12,w}(\tilde{y}_{H,t}^w)^2$$

$$+ \varpi_{12,w}(\tilde{y}_{F,t}^w)^2 + \varpi_{13,w}(\tilde{s}_{HF,t}^w)^2 + \frac{1}{2}\varpi_{4,w}\int_0^1(\pi_{i,t})^2 di\Big] + t.i.p. \tag{57}$$

where $\varpi_{11,w}$, $\varpi_{12,w}$ and $\varpi_{13,w}$ are defined in (135) and where we use the convention by which $\tilde{x}_t^w \equiv \hat{x}_t - \hat{x}_t^w$. Consistently with the open macro literature,[29] the welfare loss in (57) is expressed as a function of regional inflations, and gaps of area-specific output and of terms of trade between areas and between regions and areas. Inflation

---

[28]Condition (56) can be interpreted alternatively as follows. As the covariance between mark-up shocks rises, labor and labor income become lower in those states of world in which consumption is low. In other words, labor becomes a worse hedge against bad states of the world. Hence, the expected level of the price of labor (i.e., of the real wage) decreases.

[29]See for instance Benigno and Benigno (2006) and Corsetti et al. (2010).

and terms of trade account for the inefficient dispersion of output across firms and across regions and areas respectively. Finally, we use (57) to recover the objective of the monetary policy maker of the world currency area under regime C, to formulate her optimal policy problem and to retrieve the associated first-order conditions (see conditions (136), (138) and (139) respectively).

5.3   The case of the small open economy

As shown in Appendix D.2, the objective of the small open economy policy maker of country i in area H can be approximated up to the second order as:

$$\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\Big[\frac{1}{2}\varpi_{1,s}(\hat{y}_t^i)^2 + \varpi_{2,s}\hat{c}_t^i\hat{y}_t^i + \frac{1}{2}\varpi_{3,s}(\hat{c}_t^i)^2 + \frac{1}{2}\varpi_{4,s}(\pi_{i,t})^2 - \varpi_{5,s}\hat{y}_t^i\hat{c}_{H,t}$$

$$ - \varpi_{6,s}\hat{y}_t^i\hat{c}_{F,t} - \varpi_{7,s}\hat{c}_t^i\hat{c}_{H,t} - \varpi_{8,s}\hat{c}_t^i\hat{c}_{F,t} - \varpi_{9,s}\hat{y}_t^i\hat{a}_t^i - \varpi_{10,s}\hat{y}_t^i\hat{\mu}_t^i\Big] + t.o.c. \qquad (58)$$

for all $i \in [0,\frac{1}{2})$ and where $\varpi_{1,s}$, $\varpi_{2,s}$, $\varpi_{3,s}$, $\varpi_{4,s}$, $\varpi_{5,s}$, $\varpi_{6,s}$, $\varpi_{7,s}$, $\varpi_{8,s}$, $\varpi_{9,s}$ and $\varpi_{10,s}$ are functions of the underlying parameters of the model as shown in (88). In addition, *t.o.c.* stands for "terms out of the control" of the policy makers and include – beside the terms independent of monetary policy – the aggregate variables of both areas H and F. As for the case of cooperation, we want to write the second-order approximation in (58) in deviations from the policy target of the small open policy makers. To this end, we minimize (58) subject to (32). According to the first-order conditions of this problem, the policy target of the small open country authorities satisfies the following condition:[30]

$$[1 - \zeta_s(\varphi+1)]\widehat{mc}_{i,t}^{e,s} = \zeta_s(\varphi+1)\hat{\mu}_t^i + \kappa_H^s\hat{s}_{iH,t}^s + \kappa_F^s\hat{s}_{iF,t}^s \qquad (59)$$

where $\hat{x}_t^s$ indicates the target of the small open economy authority for the variable $\hat{x}_t^i$, $\widehat{mc}_{i,t}^{e,s}$ is the efficient level of firms' marginal cost in country i, defined as the small open economy counterpart of (42) and the parameters of the model κ_H^s and κ_F^s are specified in (94). Comparing (59) with their analogue for the cooperative case, namely (54), allows to stress the following findings:

1. *The trade-off.* The small open economy policy maker faces a trade-off between: a) stabilizing firms' marginal cost at its efficient level, b) affecting the covariance between output and mark-up shocks to drive the per period terms of trade towards its efficient level, c) manipulating the terms of trade that are relevant from small open economy's viewpoint over the cycle. The relative strength of these three incentives depends critically on the coefficients $[1 - \zeta_s(\varphi+1)]$, $\zeta_s(\varphi+1)$ and both κ_H^s and κ_F^s respectively.

2. *The weights of the trade-off and the steady-state distortion.* As was discussed for the cooperative case, ζ_s is determined by Φ_s, the parameter that determines the steady-state distortion from the perspective of the policy makers of the small regions. In fact $\zeta_s = \frac{\Phi_s}{\delta_s\varphi + \gamma_s\sigma}$. Then, $\zeta_s = 0$ if and only if $\Phi_s = 0$, i.e., the steady state is efficient from the viewpoint of the small open economy. However, differently from the cooperative case, even when $\Phi_s = 0$ stabilizing marginal cost fluctuations at their efficient level is not the target of regional authorities: they still trade off between this incentive and the desire to influence terms of trade

---

[30]See condition (93) in Appendix D.2 for the full set of equations that determine the target of the small open economy.

volatility to their own advantage. As a result, in general, strict inflation targeting cannot be optimal even when shocks are only to technology and the steady state is not distorted from the small open economy's perspective.

3. *The case for price stability.* Under the parametric restriction $\eta\sigma = 1$, it follows that $\kappa_H^s = \kappa_F^s = 0$ and the target of the small open economy policy makers in the presence of technological shocks turns out to replicate the flexible-price allocation. This finding is independent of whether the steady-state distortion has been eliminated by an appropriate labor subsidy (i.e., independent of whether $\Phi_s = 0$) and is consistent with the conclusions of the previous literature (see Benigno and Benigno (2003), Galí and Monacelli (2005) and De Paoli (2009a)). Intuitively, when $\eta\sigma = 1$, domestic output fluctuations are independent of the terms of trade since the *intertemporal* and the *intratemporal* effects exactly compensate and the terms of trade movements do not induce domestic households to change their consumption and their labor decisions. Then, the policy makers of the small open economies anticipate that they cannot affect aggregate households' decisions by using the terms of trade over the cycle strategically.

4. *Mark-up shocks.* As in the case of cooperation, the target of the small open economy authorities reacts to domestic mark-up shocks if and only if the steady state is inefficient from the small open economy perspective (i.e., $\Phi_s \neq 0$). In particular, suppose that the small open region policy maker has an incentive to improve their per period terms of trade (i.e., $\Phi_s > 0$). Then, she can affect the covariance between output and mark-up shocks. The mechanism works exactly as in the case of the cooperative policy maker. By stabilizing output more than inflation[31] in response to mark-up shocks, small country authorities can induce domestic households to lower their per period labor effort. As a result, per period domestic output – which is perceived as too low – can fall, improving the terms of trade.

5. *The terms-of-trade volatility.* When $\eta\sigma \neq 1$, it follows that $\kappa_H^s \neq 0$ and $\kappa_F^s \neq 0$. In other words, if $\eta\sigma \neq 1$, independently of whether the steady state is efficient from the small open economy viewpoint, policy makers of the small open countries are influenced by the desire of manipulating the terms of trade over the cycle. Notice that since κ_H^s and κ_F^s depend critically on Φ_s, policy makers of the small open economies use the manipulation of the terms-of-trade volatility also to drive the expected levels of the terms of trade and output towards the desired levels.

According to Point (5), the reason that explains the incentive of the small region authorities to manipulate terms-of-trade volatility is twofold. First, by affecting the volatility of their terms of trade, small open country authorities can try to externalize part of the cost of business cycle fluctuations at expenses of other countries. To give a specific example let us assume that area H is closed (i.e., $\alpha_b = 1$ and $\kappa_F^s = 0$). In this case, it can be shown that, as long as $\Phi_s = 0$ and $\eta\sigma > 1$, it follows that $\kappa_H^s < 0$.[32] Hence, under this parametric configuration, small open economy policy makers seek to *under-stabilize* firms' marginal costs in response to productivity shocks (see condition (59)). For instance, following an improvement in domestic technology they worsen their terms of trade less than needed to keep firms' marginal costs constant – as the cooperative policy maker would do. Dampening the terms-of-trade movements reduces

---

[31]as long as $\frac{\delta_s\varphi + \gamma_s\sigma}{\varphi + 1} > \delta_s - (1 - \tilde{\tau})$.

[32]The case $\alpha_b < 1$ is more complicated. Still whether $\eta\sigma$ is greater or smaller than 1 is crucial in determining the sign of κ_H^s and κ_F^s.

the shifts of state-contingent assets and the possible expenditure-switching effects of the households' demand from foreign to domestic goods. In this way, the volatilities of consumption and output and their associated welfare costs can decrease.[33]

Second, small open economy policy makers attempt to manipulate the terms-of-trade volatility to drive per period domestic output and per period terms of trade toward their desired level. Indeed, as made clear by (31) and (39), terms-of-trade movements influence both the aggregate demand and the aggregate supply curves. Here, for the sake of simplicity, we discuss only the potential effects of terms-of-trade movements on the average labor demand curve when prices are flexible. In this simple case, since $\frac{P_{i,t}}{P_{C^i,t}} = \frac{(1-\tau)}{(A_t^i)} \frac{\varepsilon}{(\varepsilon-1)} \frac{W_t^i}{P_{C^i,t}}$, it follows that:

$$E\left\{\frac{W_t^i}{P_{C^i,t}}\right\} = \frac{1}{(1-\tau)} \frac{\varepsilon-1}{\varepsilon} E\left\{\frac{P_{i,t}}{P_{C^i,t}}\right\} E\left\{A_t^i\right\} + \frac{1}{(1-\tau)} \frac{\varepsilon-1}{\varepsilon} Cov\left\{\frac{P_{i,t}}{P_{C^i,t}} A_t^i\right\} \quad (60)$$

To interpret (60), recall that $\frac{P_{i,t}}{P_{C^i,t}}$ is a decreasing function of the terms of trade. According to (60) the higher is the covariance between the terms of trade and productivity shocks, the more the per period terms of trade improve for a given real wage. As a result, to improve their terms of trade, small country authorities may seek to raise this covariance. In this way, labor demand is shifted downward, per period output falls and the per period terms of trade can improve.

By using (20) and (92) we can rewrite (58) in terms of output gap, terms of trade gap and inflation as:

$$\frac{1}{1-\tilde{\tau}} \sum_{t=0}^{\infty} \beta^t E_0 \left[\frac{1}{2}\varpi_{1,s}(\tilde{y}_t^s)^2 + \varpi_{11,s}(\tilde{s}_{iH,t}^s)^2 + \varpi_{12,s}(\tilde{s}_{iF,t}^s)^2 + \frac{1}{2}\varpi_{4,s}(\pi_{i,t})^2 \right] + t.o.c. \quad (61)$$

where $\varpi_{11,s}$ and $\varpi_{12,s}$ are define in (99) and $\tilde{x}_t^s$ stands for the deviation of $\hat{x}_t$ from its target – i.e., $\tilde{x}_t^s \equiv \hat{x}_t - \hat{x}_t^s$. Finally, the welfare losses in (61) allows us to formulate the optimal monetary policy problem of the small open economy policy maker and retrieve the corresponding first-order conditions (see (100) and (101) respectively).

5.4 The case of the big economy

As shown in Appendix D.3, if there is a monetary union in area H, the objective of the monetary policy maker can be approximated in a purely quadratic way as:

$$\frac{1}{(1-\tilde{\tau})} \sum_{t=0}^{\infty} \beta^t E_0 \Big[\frac{1}{2}\varpi_{1,b}\hat{y}_{H,t}^2 + \varpi_{2,b}\hat{c}_{H,t}\hat{y}_{H,t} + \frac{1}{2}\varpi_{3,b}\hat{c}_{H,t}^2 + \frac{1}{2}\varpi_{4,b}\pi_{H,t}^2 + \frac{1}{2}\varpi_{5,b}\hat{y}_{F,t}^2$$

$$+ \varpi_{6,b}\hat{c}_{F,t}\hat{y}_{F,t} + \frac{1}{2}\varpi_{7,b}\hat{c}_{F,t}^2 + \varpi_{8,b}\hat{y}_{H,t}\hat{c}_{F,t} + \varpi_{9,b}\hat{y}_{F,t}\hat{c}_{H,t} + \varpi_{10,b}\hat{c}_{H,t}\hat{c}_{F,t} - \varpi_{11,b}\hat{y}_{H,t}\hat{a}_{H,t}$$

$$- \varpi_{12,b}\hat{y}_{H,t}\hat{\mu}_{H,t} - \varpi_{13,b}\hat{y}_{F,t}\hat{a}_{F,t} - \varpi_{14,b}\hat{y}_{F,t}\hat{\mu}_{F,t} \Big] + t.o.c. \quad (62)$$

---

[33]In contrast, if $\eta\sigma < 1$, then $\kappa_H^s > 0$ and small open economy policy makers have an incentive to *over-stabilize* firms' marginal costs. When, for instance, domestic technology improves, a stronger deterioration of the terms of trade – still stronger than needed to keep firms' marginal costs constant – prevents domestic and foreign households from increasing their supply of state-contingent assets and their demand for foreign goods. As a result, since home and foreign goods are complements in the utility, the demand for domestic goods increases less, reducing output and consumption volatilities. In the case of $\eta\sigma < 1$, it is then by amplifying terms of trade responses to productivity shocks that small open countries try to cut the costs of consumption and labor fluctuations.

where $\varpi_{1,b}$, $\varpi_{2,b}$, $\varpi_{3,b}$, $\varpi_{4,b}$, $\varpi_{5,b}$, $\varpi_{6,b}$, $\varpi_{7,b}$, $\varpi_{8,b}$, $\varpi_{9,b}$, $\varpi_{10,b}$, $\varpi_{11,b}$, $\varpi_{12,b}$, $\varpi_{13,b}$ and $\varpi_{14,b}$ are listed in condition (112) and depend on the fundamental parameters of the model. In addition *t.o.c.*, "the terms out of control" of the policy maker, include the state-contingent path of $\pi_{F,t}$ decided by the policy maker of the monetary union in area F and the differentials between country-specific and average-union variables.[34] Similarly to the cooperative and the small open economy cases, the target of the monetary union's central bank can be determined by minimizing (62) subject to constraints (113).

To grasp some insights about the incentives driving the policy maker of the monetary union, we use condition (21) to retrieve from (116) the following expression:

$$[1 - \zeta_b(\varphi + 1)]\,\widehat{mc}^{e,b}_{H,t} = \zeta_b(\varphi+1)\hat{\mu}_{H,t} + \kappa^b_H \hat{s}^b_{HF,t}$$

$$-(\zeta_w - \zeta_b)(\varphi+1)\widehat{mc}^{e,b}_{F,t} = (\zeta_w - \zeta_b)(\varphi+1)\hat{\mu}_{F,t} + \kappa^b_F \hat{s}^b_{HF,t} \qquad (63)$$

where $\hat{x}^b_t$ stands for the target of the policy maker of the monetary union in area H for the variable $\hat{x}_t$, $\widehat{mc}^{e,b}_{H,t}$ and $\widehat{mc}^{e,b}_{F,t}$ are defined in (42), while κ^b_H and κ^b_F are listed in condition (118).[35]

By contrasting condition (63) with its counterparts – (55) and (59) –, we reach the following conclusions:

1. *The trade-off.* According to (63), big economy policy makers face a trade-off between: a) stabilizing the fluctuations of average-area efficient marginal costs, b) manipulating the covariances between output and mark-up shocks of both areas, c) influencing the volatility of the terms of trade between areas. The relative strength of these incentives depends critically on the coefficients $[1 - \zeta_b(\varphi + 1)]$, $\zeta_b(\varphi + 1)$, κ^b_H, $-(\zeta_w - \zeta_b)(\varphi + 1)$, $(\zeta_w - \zeta_b)(\varphi + 1)$ and κ^b_F.

2. *The weights of the trade-off and the steady-state distortion.* In the case of the big economy – differently from the cooperative case – the monetary authority of area H attaches different weights to the stabilization of domestic and foreign efficient marginal costs.[36] This is because the per period terms of trade at the symmetric steady state are inefficient from the area H policy maker's perspective. An improvement in the per period terms of trade (or a terms-of-trade worsening if η is sufficiently low) is beneficial for domestic households since it generates per period output differentials and allows to externalize labor effort. But despite this desire and differently from the small open economy policy makers – who care only about the performance of the domestic economy –, the central bank of the area takes into account how its decisions affect the demand and supply of foreign produced goods and the related feedback effects on its own economy. As a result, the policy maker of the big economy tries to induce domestic households to work *relatively* less (and possibly or alternatively to consume *relatively* more) than foreign households, aiming at the same time, to allocate the world resources efficiently. All these incentives are reflected in the coefficients of $\widehat{mc}^{e,b}_{H,t}$ and $\widehat{mc}^{e,b}_{F,t}$. The weight of domestic marginal cost is $1 - \zeta_b(\varphi + 1)$ where ζ_b is determined by Φ_b. Indeed, $\zeta_b = \frac{\Phi_b(\gamma_b\sigma + \delta_b\varphi)}{(\sigma + \varphi)((2\gamma_b - 1)\sigma + (2\delta_b - 1)\varphi)}$. It follows that $\zeta_b = 0$ if and only if

---

[34]Indeed, without loss of generality, we assume that by choosing the average-union inflation, the common central bank cares only about the average performance of the currency area. However, these terms have to be taken into account for the welfare evaluation.

[35]The full set of conditions that jointly with (63) determine the target of the big economy is spelled out in (117).

[36]Similar reasoning applies to the impact on the target of domestic and foreign mark-up shocks.

domestic output is efficient from the big economy's viewpoint (i.e., if and only if $\Phi_b = 0$). Then, as long as $\delta_b > 1$ and $\Phi_b \neq 0$, the authority of the monetary union tries to push domestic per period output toward a level that is inefficiently low from the cooperative viewpoint. In contrast, the weight of the foreign marginal cost is $-(\zeta_w - \zeta_b)(\varphi + 1)$ and corresponds – as long as $\delta_b > 1$ – to the intention of driving foreign per period output towards a level inefficiently *high*.[37] In other words, the policy maker in area H seeks to push domestic and foreign per period output in opposite directions to generate – as long as $\delta_b > 1$ – an improvement of the cross-area terms of trade. However, the average weight corresponds to that of the cooperative authorities, i.e., $1 - \zeta_b(\varphi + 1) - (\zeta_w - \zeta_b)(\varphi + 1) = 1 - \zeta_w(\varphi + 1)$. Despite the conflicts with other policy makers, the monetary authority of the large economy wishes to make efficient use of the resources available in the world economy.

3. *The case for price stability.* When $\eta\sigma = 1$, it follows that $\gamma_b = \delta_b$. As a consequence, $\kappa_H = 0$ and $\kappa_F = 0$ and $\widehat{mc}^{e,b}_{H,t} = 0$ and $\widehat{mc}^{e,b}_{F,t} = 0$. Then, under productivity shocks, implementing the flexible-price allocation becomes the target of the common central bank and strict inflation targeting the optimal policy. This result is consistent with other findings in the open macro literature – for instance Benigno and Benigno (2003) and Benigno and Benigno (2006). The intuition is exactly the same as explained for the case of the small open economies. Open economy policy makers internalize that if $\eta\sigma = 1$ there is no reason to manipulate the terms of trade because they cannot influence domestic output.

4. *Mark-up shocks.* Only when $\Phi_b \neq 0$ or $-(\zeta_w - \zeta_b)(\varphi + 1) \neq 0$, the common central has the additional incentive to try to manipulate strategically the covariance between outputs and mark-up shocks: by shifting the average labor supply curves she tries to push domestic and foreign per period outputs towards their efficient levels.

5. *The terms-of-trade volatility.* When $\eta\sigma \neq 1$, $\kappa^b_H \neq 0$ and $\kappa^b_F \neq 0$ and similarly to small open economy authority, the monetary authority of the area seeks to affect strategically the terms of trade volatility both to drive domestic and foreign output towards the desired levels and to try to decrease the welfare costs of consumption and leisure volatilities.

Big countries recognize the importance of an efficient use of the resources available in the world economy. Nevertheless, they disagree on how much to produce and consume individually. This free riding problem generates a conflict that translates into the desire of manipulating the terms of trade between areas. Countries wish to out-source production to allow domestic households to enjoy *relatively* more leisure or consumption. This mechanism can even clarify why the size of the economy can shape optimal policy decisions. When the economy is big, policy makers behave strategically: they realize that they can influence the terms of trade by affecting domestic and foreign outputs in opposite directions. At the same time, they internalize the effects that their policies produce on the foreign economy – even if they disregard the effects on foreign household welfare. They take into account, for instance, that when $\eta\sigma > 1$ domestic terms of trade improve, not only the demand of domestic households but even that of foreign consumers switches from domestic towards foreign goods, amplifying the effects on output differentials. Differently, in the limiting case of a small economy, the only

---

[37]Indeed $-(\zeta_w - \zeta_b)(\varphi + 1) = 0$ corresponds to $\tilde{\tau} = -\tilde{\tau}_b$. Only when $-(\zeta_w - \zeta_b)(\varphi + 1) = 0$ does the target become independent of $\widehat{mc}^{e,b}_{F,t}$ and $\widehat{\mu}_{F,t}$: in this case foreign steady-state output is at the desired level from the area H policy maker's perspective.

way monetary policy can manipulate the per period terms of trade in its favor is by affecting per period domestic production. In fact, the economic performance of a small open country is irrelevant for the behavior of the aggregate economy. However, the joint policies of small open regions in one area can produce large business cycle effects on both the domestic and the foreign area.

By using (21) and (116), the welfare approximation in (62) can be rewritten in terms of the welfare relevant gaps as:

$$\frac{1}{(1-\tilde{\tau})}\sum_{t=0}^{\infty}\beta^t E_0 \left[\varpi_{15,b}(\tilde{y}_{H,t}^b)^2 + \varpi_{16,b}(\tilde{s}_{HF,t}^b)^2 + \varpi_{17,b}(\tilde{y}_{F,t}^b)^2 + \varpi_{4,b}\pi_{H,t}^2 \right] + t.o.c. \quad (64)$$

where $\varpi_{15,b}$, $\varpi_{16,b}$ and $\varpi_{17,b}$ are defined in condition (120). Finally, we use the welfare loss in (64) to formulate the optimal monetary policy problem of the policy maker of the large open economy and to determine the associated first-order conditions (see (122) and (123) respectively). Then, symmetric conditions can be stated for the foreign area to determine the Nash equilibrium policies under regimes A and B.

6   Optimal policies

The purely-quadratic approximations enable us to solve the optimal policy problems of both the small open economy and the monetary union policymakers and to simulate the impact responses under the optimal policies of regimes A, B and C. Before examining the results of the dynamic simulations, we discuss the baseline parametrization.

6.1   Parametrization

The baseline parametrization of the model shown in Table 1 is chosen having the eurozone as a benchmark. The discount factor is set to 0.99 in order to match a steady-state annual interest rate of 4 percent. The intertemporal elasticity of substitution is set equal to $\frac{1}{2}$, as in Pappa (2004) and Corsetti et al. (2010), while the labor supply elasticity is $\frac{1}{3}$ and the import share over GDP of the small open region is 40 percent (i.e., $\alpha_s = 0.6$) consistently with Galí and Monacelli (2005) and De Paoli (2009a). Conversely, the degree of home bias of the area is set equal to 0.85 in line with Coenen, Lombardo, Smets and Straub (2010) and Pappa (2004). The elasticity of substitution across labor types and across varieties produced within the same region are set equal to 6 and correspond to a 20 percent steady-state mark-up in the goods and in the labor markets respectively. Furthermore, we assume that governments do not subsidize labor – i.e., $\tau = 0$. As a result, $\tilde{\tau}$ is equal to -0.44 and the steady-state wedge between the marginal rates of substitution and transformation is equal to 1.44, close to 1.38, the calibration chosen by Benigno and Benigno (2006). In addition, θ the probability of resetting the price is 0.75, implying an average price duration of 3 quarters. Moreover, we parameterize the autocorrelation and the standard deviation of productivity shocks, $\hat{a}_t$ as in Coenen et al. (2010). Following De Paoli (2009a) and Benigno and Benigno (2006) and in line with the findings of Adolfson, Laséesen, Lindé and Villani (2007) we assume $AR(1)$ mark-up shocks, $\hat{\mu}_t$, with an autocorrelation coefficient, ρ_μ, equal to 0.9. Then, we set the standard deviation of mark-up shocks consistently with Coenen et al. (2010). Finally, the correlation of productivity shocks across regions is set according to Galí and Monacelli (2005), while the correlation of mark-up shocks across regions is

half of that of productivity.[38] A more detailed description of the shock parametrization is provided in Appendix E and in Section 7 we check the robustness of our findings to a change in the calibration of various parameters.

There is no clear consensus in the literature on the parametrization of the trade elasticity. If we just focus on the more recent studies, Broda and Weinstein (2006) and Imbs and Méjean (2011), for instance, suggest that a reasonable parametrization of the trade elasticity should lie within 4 and 6 and above 6 respectively. Conversely using Bayesian techniques, Rabanal and Tuesta (2010) estimate this elasticity to lie below 1, consistently with the results of Corsetti, Dedola and Leduc (2008). Then, in line with De-Paoli (2009b) and Rabitsch (2012) we allow the trade elasticity to vary from 0 to 3.9[39] and within this range we study the impact responses of the model under optimal policies and the welfare gains across the different regimes.

6.2   Dynamic responses

Figure 2 depicts the impact responses to a one percent increase in domestic and foreign mark-ups as a function of the trade elasticity under optimal policies in regimes A, B and C. Specifically, the plots show how the inflation output stabilization trade-off generated by a global mark-up shock is modified by a rise in η (i.e., by a change in the desired levels of per period output and in the corresponding perceptions of the steady-state distortion).

Regime C (light blue line with triangles). In this case, the monetary union of the world economy behaves like a closed economy and, not surprisingly, impact responses are symmetric and independent of η. Given the increase in their marginal costs, both domestic and foreign firms raise prices and reduce output supply. Workers cut consumption and increase leisure. Then, the central bank of the monetary union faces a trade-off between output and inflation stabilization and since under the baseline parametrization per period output is inefficiently low, it wishes to stabilize inflation more than output – compared to the case in which the steady state is Pareto efficient. As emphasized above, amplifying the negative response of output reduces the covariance between output and mark-up shocks, pushing households to increase their per period labor effort. As a consequence, monetary policy is slightly expansionary and while inflation increases up to 0.017 percent per quarter, output falls by 0.15 percent per quarter.

Regime B (starred blue line). Under regime B, when there are two currency unions, impact responses are symmetric across areas but not independent of the value of the trade elasticity. For low values of η, the central banks of the monetary unions aim to worsen the per period terms of trade. In this way per period consumption in the domestic economy increases due to the additional labor effort that foreigners provide to produce abroad. In order to deteriorate the per period terms of trade, monetary policy makers attempt to *over-stabilize* foreign output – relatively more than the domestic one – compared to what would be Pareto efficient. Put differently, monetary policy makers try to generate a negative covariance between the terms of trade and global mark-up shocks to induce domestic workers to raise their labor supply more than foreign households. As a result, for low values of η the monetary policy is more contractionary than

---

[38]Notice that this corresponds to a conservative configuration of this parameter. We check the sensitivity of our results to a change in $\varsigma_{s,b}$ in Tables 14 – 16.

[39]This range ensures that under the baseline calibration second-order conditions for the optimal policy problems are satisfied in all regimes and purely random policies cannot be welfare improving. For a discussion on this issue, see Benigno and Benigno (2006) and Benigno and Woodford (2005).

under regime C. Obviously, in equilibrium none of the policy makers in areas H and F reach their goal. Given symmetry, home and foreign outputs perfectly co-move in such a way that the terms of trade are always equal to one. As the trade elasticity becomes larger, and domestic and foreign goods become substitutes in the utility, the incentives of the monetary authority turn around. Big economy policy makers seek to *under-stabilize* foreign output in order to push the terms of trade to positively co-vary with global mark-up shocks. In this way the per period terms of trade can improve and domestic households can work less than foreign workers. Welfare in the home economy can rise. Indeed, as long as the per period terms of trade ameliorate, per period consumption falls but not as much as the increase in leisure would require in a closed economy; the risk sharing in consumption allows domestic households to benefit from the increase in the per period foreign production. Then, given this incentive, when η is large enough, monetary policy is more expansionary in response to a global mark-up shock than under regime C.

Regime A (red line with squares). Under regime A, small open economies in area H still retain monetary policy independence, whereas there is a monetary union in area F. In this case, impact responses are neither symmetric nor independent of the value of η. Since the mark-up shock is global, all the cross-area differences are driven by the differences between monetary policies of the small open economies and of the currency union. Similarly to what happens under regime B, the monetary policy of the small country can be more or less expansionary than under regime C, depending on the values of η. The rationale of this behavior is again explained by the desire of manipulating the terms of trade. When η is low enough – in particular when domestic and foreign goods are complements – small open economy policy makers try to worsen the per period terms of trade and to raise per period consumption to take advantage of the increase in the foreign country labor supply. For this reason, they *under-stabilize* domestic output. Vice versa, for η large enough the incentive works in the opposite direction. In this case small open economy authorities aim to improve the per period terms the terms of trade to reduce household per period labor effort by over-stabilizing domestic output.

However, as η increases and domestic and foreign goods become better substitutes in the utility, the incentive of the small country policy makers to improve per period terms of trade becomes stronger and stronger.[40] Then, their monetary policy stance becomes more and more expansionary – compared to those of the monetary authorities under regime B and C – and exacerbates the ensued negative externalities both within and across areas substantially. Their economies being small, the regional central banks take as given what happens in the world economy. Thus, they disregard (as opposed to the monetary authority of a currency area) how their joint policy decisions affect the efficient allocation of resources of the world economy and the average performances of both areas H and F. They are not aware that the welfare costs of their optimal policies are potentially large. Inflation within area H rises considerably more than under regimes B and C. At the same time, the benefits associated with the over-stabilizing monetary policies cannot compensate the increase in the costs of inflation. Indeed, on the one hand, under global shocks terms of trade do not improve *within* the area in equilibrium – implying that there are no welfare benefits from domestic output over-stabilization. On the other hand, even if small open economy authorities do not realize that they monopoly power over the terms of trade *between* areas, these terms

---

[40]This incentive is the stronger the higher the trade elasticity, because the higher this elasticity, the more households are willing to substitute the domestic goods with the foreign ones.

of trade deteriorate – even more than required by the target of the policy maker of the monetary union. Yet, the behavior of the central bank in area F suggests that the welfare gains associated with this term-of-trade worsening cannot outweigh the welfare costs of the increase in inflation. In fact, it is the central bank of area F that allows for a deterioration of cross-area terms of trade by cutting the nominal interest rate relatively more than under regime B, but less than the policy makers of area H. It realizes what the welfare costs of the increase in inflation for the area as a whole are and has a clear trade-off between opposing the expansionary policies of the small open countries and properly stabilizing domestic price dynamics.[41] While for low values of η the balance tips in favor of inflation stabilization and inflation responses in area F are similar across regimes, as the externalities become more severe, inflation rises in the foreign area to dampen the fall in output.

There is a crucial question that is still left unanswered. What is the appropriate domain of a monetary union from the perspective of area H (and F) households? When are the consumers of areas H and F better off? Under regimes A, B or C? These questions are addressed in the next section.

7 Welfare evaluation

The analysis of the previous sections reveals that in our setting there are potential welfare benefits from the adoption of a common currency. Moreover, it has been made clear what the sources of these benefits are: more efficient use of the resources of the economy, the internalization of the spill-over effects generated within area H and the gains from controlling the cross-area terms of trade.

7.1 Welfare gains of a monetary union

The household welfare based criteria derived in (57) and (111) allows us to quantify the welfare gains from being in a currency union expressed as a fraction of the corresponding steady-state consumption. Figure 3 shows the differences in welfare under the baseline calibration between being under regimes A or B (first two plots), B or C (third plot) and A or C (fourth plot) for both domestic and foreign households as function of the trade elasticity. There are three noteworthy conclusions we can draw from this figure:

1 As the first plot suggests, if there is a group of small open economies that decide to be in a monetary union, welfare gains may be large. When η is greater than 1.8, the net benefits for the domestic households are positive and can reach the considerable size of 27.8 percent of steady-state consumption as η approaches the value of 3.9.[42] At the same time, there can be welfare losses if η is sufficiently small. However, these losses are on average lower than the potential gains especially if we exclude the smallest values of η, which are not very plausible. The interpretation of this finding is quite clear. There is a trade-off for small open countries between maintaining their monetary autonomy and being in a currency area. The cost of a monetary union is due to the impossibility of properly stabilizing region-specific shocks. The benefits are associated with the internalization of the negative externalities produced by the *beggar-thy-neighbor* policies of the

---

[41]Notice that if there were a common central bank in the area H, it would face the same trade-off changing its policy accordingly.

[42]We have not plotted in the figure the values of the welfare gains for η equal to 3.9 because altering its scale renders the plot unreadable.

small open economies, which disregard the effects that their joint decisions generate within and outside the area. Nevertheless, as the analysis of the impact responses has demonstrated, these externalities are more severe for high values of η, i.e., when home and foreign goods become more substitutable. As η increases, the incentive of the small open economy policy makers to manipulate their per period terms of trade and to increase inflation volatility becomes stronger and hence the welfare benefits of a currency area are more substantial.

2 According to the third plot of Figure 3, under the baseline calibration there are no welfare gains from being in a monetary union between the two areas. In this case welfare losses fall in the range of -0.0096 and -0.0012 of the steady-state consumption. Intuitively, if in this case the costs of being in a currency area are smaller, benefits are smaller too because the monetary authorities of the big economies are more concerned about inflation stabilization and less prone to – compared to the small open economy central banks – adopting policies that generate negative externalities for the other area. In fact, monetary authorities of a currency area take into account how their decisions affect not only the domestic but also the foreign economy performance. They realize the importance of an efficient allocation of resources of the world economy as a whole, whereas small open economy policy makers do not.

3 According to the fourth plot, if η is large enough, households of area F – and even those in area H – benefit substantially from being in a monetary union with the small open economies of area H. Again, the rationale of this result is explained by the fact that joining the monetary union can discipline the highly inflationary behavior of the small open economy policy makers. Put differently, as is evident from the second plot, small country policies are welfare detrimental even for the foreign area. Therefore, according to this result there is rationale for the households of the area F to be in a monetary union that includes the small open countries.

We can now use the previous findings to try to characterize the appropriate domain of a currency area such as the European monetary union. According to our results for sufficiently large values of η, there are welfare gains from the adoption of the euro for the consumers living in the eurozone. From the perspective of these consumers it might be a good idea to try to foster the process of enlargement of the euro to other European countries. Conversely, it seems that being in a monetary union with a big country like the U.S. cannot bring about sizable welfare benefits for the domestic and foreign households.

7.2 Sensitivity analysis

In this section we analyze how these benefits vary according to the deep parameters of the model.

The intertemporal elasticity of substitution. Both the intratemporal elasticity of substitution between home and foreign bundles, η, and the intertemporal elasticity of substitution of consumption, σ^{-1}, are crucial in order to determine the size of the welfare gains (or losses) of abandoning monetary autonomy. The higher $\eta\sigma$ is, the more domestic and foreign goods are substitutes in the utility, the larger the expenditure-switching effect of the demand from domestic towards foreign goods (intratemporal effect) and the stronger the incentive to smooth consumption across periods (intertemporal effect). Then, the stronger the rise in foreign output due to a terms-of-trade

improvement and the more domestic production falls allowing domestic households to enjoy more leisure. In other words, these parameters determine the size of the negative externality entailed by small open economy optimal policies and hence the benefits of policy coordination that arise from being in a monetary union. Tables 2, 3 and 4 show how welfare benefits increase as η and σ increase. η varies between 0 and 3.1, while σ varies between 1 and 3.[43] Within this range, the gains from being under regime B instead of A for the consumers living in area H reach a maximum of 21 percent of steady-state consumption. However, for sufficiently low levels of η the adoption of a common currency brings about welfare losses up to 0.12 percent of steady-state consumption.

The degree of home bias. The welfare benefits of a monetary union are due to the internalization of the external effects produced both within and across areas. In order to investigate which of these channels contributes more to explain these gains, we study to what extent they depend on the degree of home bias of the area, α_b. Tables 5, 6 and 7 report the welfare benefits with η and α_b varying between 0 and 3.9 and between 0.6001 and 1 respectively. When α_b is equal to 1, the area is closed and the only source of benefits is the internalization of the external effects within the area. When α_b approaches to 0.6, there is very little trade within the areas. In this case, there are almost no externalities within the area due to the small open economy optimal policies and the welfare gains come from the reduction – or their complete elimination, if regime C is considered – of the spill-over effects produced on the foreign area. When regimes A and B are compared (Table 5), the case α_b approaching 0.6 is especially interesting because it makes it clear to what extent there are welfare benefits from being in a monetary union even when there are no gains due to the internalization of the spill-over effects within the area. In other words, when α_b tends to 0.6, if welfare in area H improves across regimes A and B, it is only because policy makers of the big economies – differently from those of the small open economies – take into account the equilibrium feedback effects stemming from the foreign area. According to our simulation, the benefits can still be substantial even when α_b approaches 0.6.

Finally, in Tables 8-22, we check the sensitivity of our findings to a change in other parameters: $\varsigma_{s,\mu}$ and $\varsigma_{b,\mu}$, the correlations of mark-up shocks across regions and areas, τ, the employment subsidy, ρ_μ, the autocorrelation of mark-up shocks and σ_μ, the standard deviation of mark-up shocks. Results are quite robust to these changes.

8   Conclusion

This paper provides a possible rationale for the process of the creation and enlargement of the eurozone. It shows that as long as the monetary union is formed by a group of small open economies, there can be sizable welfare gains from sharing the same currency and from extending the currency area to another group of small open economies. Our findings stem from the incentive of small open countries to generate potentially strong negative externalities, which lead to large inflation volatility both in the foreign area, which has a single monetary authority, and in the domestic area, in which the small open countries still retain monetary autonomy. In this case, delegating the monetary policy to a common central bank that sets the interest rate for the whole area acts as a disciplining device against the potentially inflationary policies of the small open

---

[43]Again, we choose this range to ensure that the second-order conditions for the optimal policy problems are satisfied in all regimes and purely random policies cannot be welfare improving.

economies.

There are at least two extensions of the model that could be considered for future research. First, it would be interesting to explore how results change once we allow for incomplete risk sharing, i.e., when there is either financial autarchy or areas and/or regions trade only a riskless bond.[44] Under incomplete financial markets, we might expect that the negative spill-overs due to open economy policies are stronger when home and foreign goods are complements in the utility than when they are substitutes (as suggested by De-Paoli (2009b)), reversing our findings under perfect risk sharing. To see why this could be the case, recall that under complete markets consumption is highly correlated across regions. For this reason, the more goods are substitutes, the more households are indifferent between the consumption of domestic and foreign goods and the stronger is the incentive of domestic policy makers to externalize labor effort because they can reduce home production without suffering a large fall in home consumption. The flip side of this result is that when goods are complements, the negative spill-over effects are weak and then losing monetary autonomy is welfare detrimental for the households living in the currency area. Conversely, under incomplete financial markets, domestic production is more correlated with domestic consumption – under financial autarky, for instance, consumption expenditure is completely financed by domestic production. Then, when domestic and foreign goods are highly substitutable, the incentive to manipulate the terms of trade might turn around. It is plausible that as the terms of trade improve, the expenditure-switching effect causes consumption to fall more than output, lowering household utility. On the other hand, if the trade elasticity is low enough, a terms-of-trade improvement might generate welfare benefits. In fact, the complementarity between home and foreign goods dampens the response of consumption to a terms-of-trade improvement because of the increase in the demand for foreign goods. Hence, domestic utility could rise given the possible contraction in domestic production. In summary, we can expect that relaxing the assumption of complete financial markets may change the set of parametric configurations under which there are gains from being in a monetary union. However, we also expect that the main conclusion of our analysis survives under imperfect risk sharing i.e., that even under these circumstances the incentive to generate negative externalities – and then the case for a monetary union – is stronger for the authorities of a group of small open economies, since these authorities disregard the joint effects of their policies on foreign economies.

A second interesting extension could be to allow for sticky wages.[45] Introducing sticky wages would render mark-up shocks endogenous, possibly strengthening the negative externalities of open economy policies under productivity shocks. Indeed, given the impossibility to fully adjust the nominal wages, the fluctuations in the labor supply due to fluctuations in the terms of trade – through their effects on the household purchasing power – might be larger. In other words, labor supply could become more sensitive to changes in the terms of trade. Consequently, with sticky wages, the strategic use of the terms-of-trade volatility might become more effective in influencing the hedging properties of labor and in driving the per period output and the per period terms of trade towards their desired levels. Thus the incentive of the policymakers to produce negative spill-overs might become stronger even if it needs to be weighted

---

[44]For a contribution that studies models with incomplete financial markets, see e.g. Corsetti et al. (2008). They show how imperfect risk sharing combined with low trade elasticities helps to reconcile international business cycle models with the "Backus Smith" puzzle – i.e., the empirical evidence that there is a low and negative correlation between the real exchange rate and the relative consumption.

[45]For a small open economy model with sticky wages see e.g. Campolmi (2012).

up against the effects on wage and price inflations implied by the manipulation of the term-of-trade volatility.

References

Adolfson, Malin, Stefan Laséesen, Jesper Lindé, and Mattias Villani, "Bayesian Estimation of an Open Economy DSGE Model with Incomplete Pass-through," *Journal of International Economics*, 2007, *72* (2), 481–511.

Alesina, Alberto and Robert J. Barro, "Currency Unions," *Quarterly Journal of Economics*, 2002, *17* (2), 409–36.

Benigno, Gianluca and Pierpaolo Benigno, "Price Stability in Open Economies," *Review of Economic Studies*, 2003, *70* (4), 743–64.

_ **and** _ , "Designing Targeting Rules for International Monetary Policy Cooperation," *Journal of Monetary Economics*, 2006, *53* (3), 473–506.

_ **and** _ , "Implementing International Monetary Cooperation through Inflation Targeting," *Macroeconomic Dynamics*, 2008, *12* (1), 45–59.

Benigno, Pierpaolo and Michael Woodford, "Inflation Stabilization and Welfare: The Case of a Distorted Steady State," *Journal of the European Economics Association*, 2005, *3* (6), 1185–236.

_ **and** _ , "Linear-quadratic Approximation of Optimal Policy Problems," *Journal of Economic Theory*, 2012, *147* (1), 1–42.

Broda, Christian and David E. Weinstein, "Globalization and the Gains from Variety," *Quarterly Journal of Economics*, 2006, *121* (2), 541–85.

Campolmi, Alessia, "Which Inflation to Target? A Small Open Economy with Sticky Wages," *Macroeconomic Dynamics*, 2012. Forthcoming.

Clarida, Richard, Jordi Galí, and Mark Gertler, "A Simple Framework for International Monetary Policy Analysis," *Journal of Monetary Economics*, 2002, *49* (5), 879–904.

Coenen, Günter, Giovanni Lombardo, Frank Smets, and Roland Straub, "International Transmission and Monetary Policy Cooperation," in Jordi Galí and Mark J. Gertler, eds., *International Dimensions of Monetary Policy*, University of Chicago Press, 2010.

Corsetti, Giancarlo and Paolo Pesenti, "Welfare and Macroeconomic Interdependence," *Quarterly Journal of Economics*, 2001, *16* (2), 421–46.

_ **and** _ , "International Dimensions of Optimal Monetary Policy," *Journal of Monetary Economics*, 2005, *52* (2), 282–305.

_ **, Luca Dedola, and Sylvan Leduc**, "International Risk Sharing and the Transmission of Productivity Shocks," *Review of Economic Studies*, 2008, *75* (2), 443–73.

_ , _ , and _ , "Optimal Monetary Policy in Open Economies," in Ben Friedman and Michael Woodford, eds., *The Handbook of Monetary Economics*, Vol. III, Elsevier, 2010.

De Paoli, Bianca, "Monetary Policy and Welfare in a Small Open Economy," *Journal of International Economics*, 2009, *77* (1), 11–22.

_ , "Monetary Policy in a Small Open Economy: the Role of the Asset Market Structure," *Journal of Money, Credit and Banking*, 2009, *41* (7), 1301–30.

Epifani, Paolo and Gino Gancia, "Openness, Government Size and the Terms of Trade," *Review of Economic Studies*, 2009, *76* (2), 629–68.

Galí, Jordi, *Monetary Policy, Inflation, and the Business Cycle*, Princeton University Press, 2008.

_ **and Tommaso Monacelli**, "Monetary Policy and Exchange Rate Volatility in a Small Open Economy," *Review of Economic Studies*, 2005, *72* (3), 707–734.

_ **and** _ , "Optimal Monetary and Fiscal Policy in a Currency Union," *Journal of International Economics*, 2009, *76* (1), 116–132.

Imbs, Jean and Isabelle Méjean, "Elasticity Optimism," 2011. Mimeo.

Lucas, Robert E. Jr., *Models of Business Cycles*, Oxford, UK: Basil Blackwell, 1987.

_ , "Macroeconomic Priorities," *American Economic Review*, 2003, *93* (1), 1–14.

Mundell, Robert A., "A Theory of Optimum Currency Areas," *American Economic Review*, 1961, *51* (4), 657–75.

Obstfeld, Maurice and Kenneth Rogoff, "Global Implications of Self-Oriented National Monetary Policy Rules," *Quarterly Journal of Economics*, 2002, *117* (2), 503–536.

Pappa, Evi, "Do the ECB and the Fed Really Need to Cooperate? Optimal Monetary Policy in a Two-Country World," *Journal of Monetary Economics*, 2004, *51* (4), 753–779.

Rabanal, Pau and Vincente Tuesta, "Euro-Dollar Real Exchange Rate Dynamics in an Estimated Two-Country Model: An Assessment," *Journal of Economic Dynamics and Control*, 2010, *34* (4), 780–97.

Rabitsch, Katrin, "The Role of Financial Market Structure and the Trade Elasticity for Monetary Policy in Open Economies,," *Journal of Money, Credit and Banking*, 2012, *44* (4), 603–29.

Rotemberg, Julio J. and Michael Woodford, "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy," *NBER Macroeconomic Annual*, 1997.

Tille, Cedric, "The Role of Consumption Substitutability in the International Transmission of Shocks," *Journal of International Economics*, 2001, *53* (2), 421–444.

Woodford, Michael, *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press, 2003.
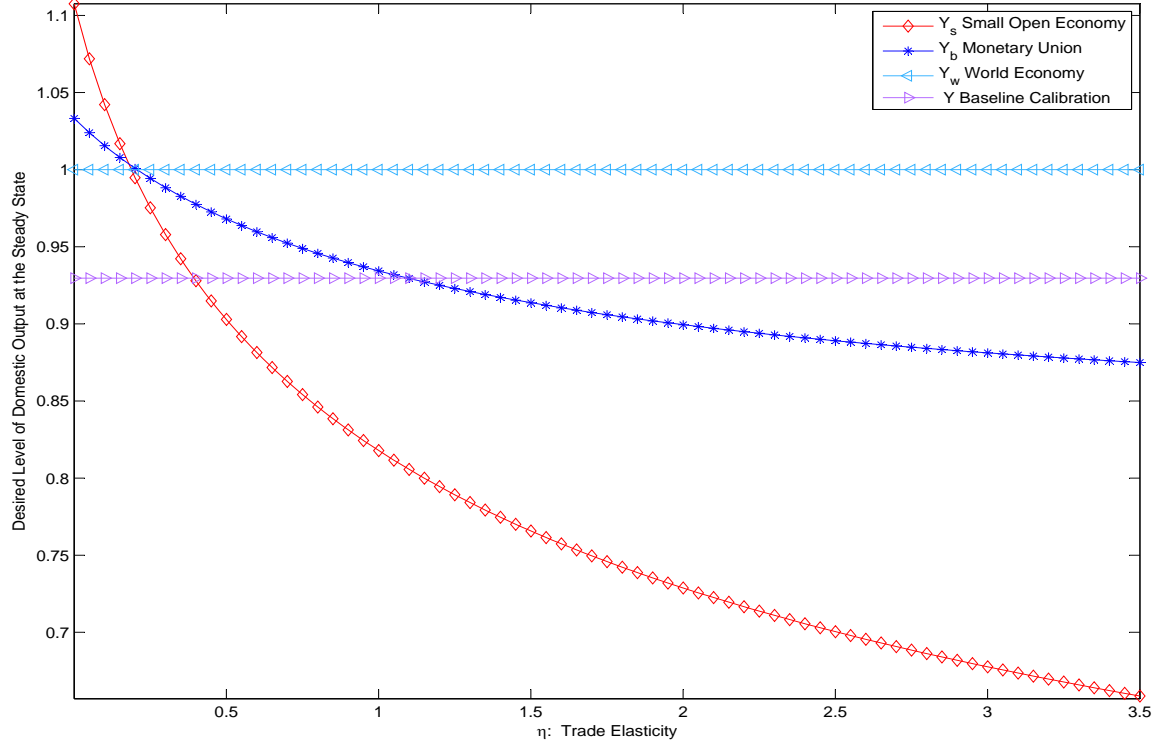
Figure 1: Desired level of steady state domestic output as function of the trade elasticity η.

Parameter	Value	Description
β	0.99	Preferences discount factor
σ^{-1}	1/2	Intertemporal elasticity of substitution of consumption
φ^{-1}	1/3	Frish elasticity of labor supply
α_s	0.6	Degree of home bias in the regions
α_b	0.85	Degree of home bias in the areas
ε	6	Elasticity of substitution across varieties
ν	6	Elasticity of substitution across labor types
θ	0.75	Probability of resetting prices
ρ_a	0.9	Autocorrelation of technological shocks
ρ_μ	0.9	Autocorrelation of mark-up shocks
τ	0	Steady-state labor subsidy
σ_a	0.0055	Standard deviation of the technological shocks
σ_μ	0.0262	Standard deviation of mark-up shock
$\varsigma_{a,b}$	0.258	Correlation of productivity shocks across areas
$\varsigma_{a,s}$	0.3	Correlation of productivity shocks across regions
$\varsigma_{\mu,b}$	0.129	Correlation of mark-up shocks across areas
$\varsigma_{\mu,s}$	0.15	Correlation of mark-up shocks across regions

Table 1: Baseline Parametrization.

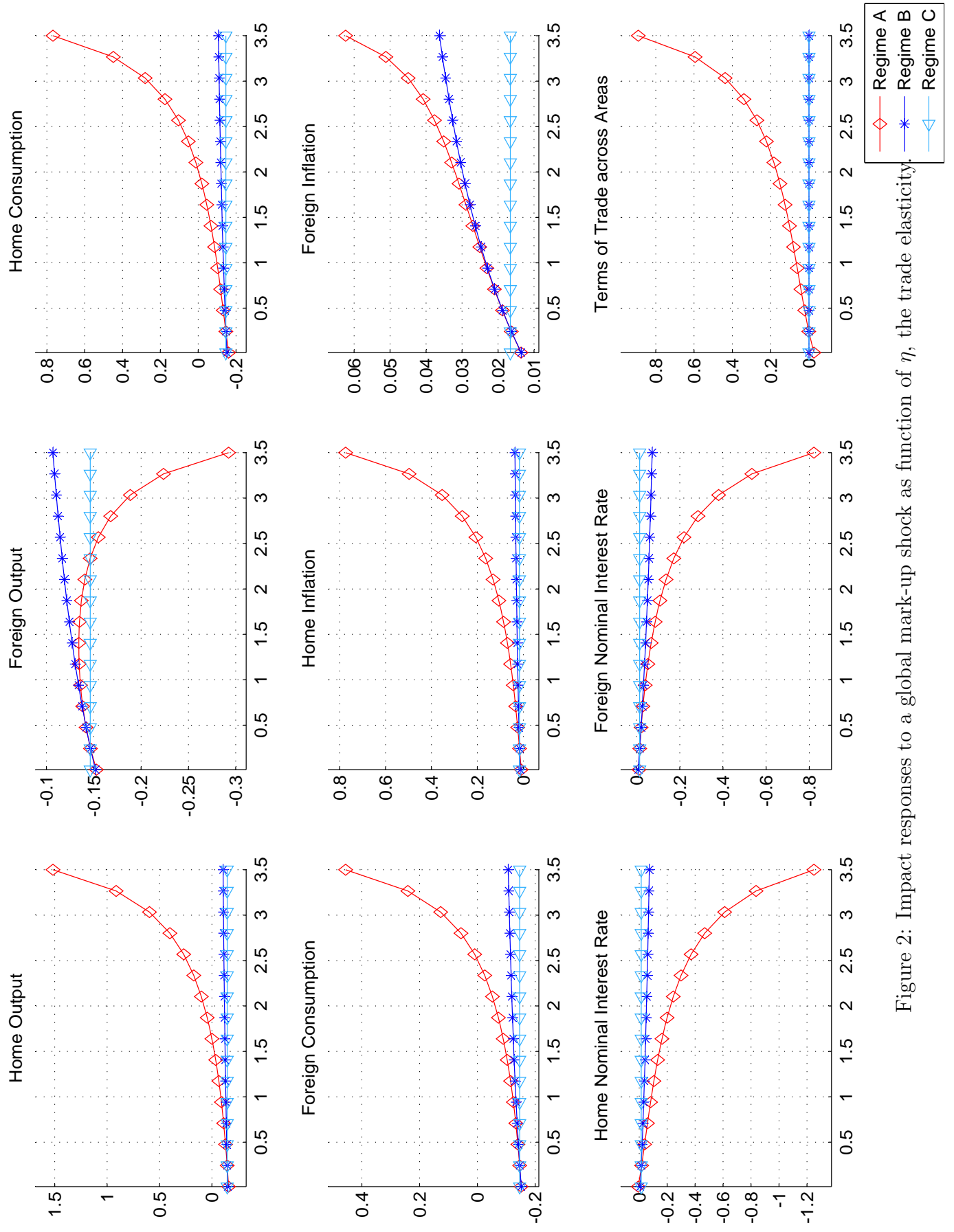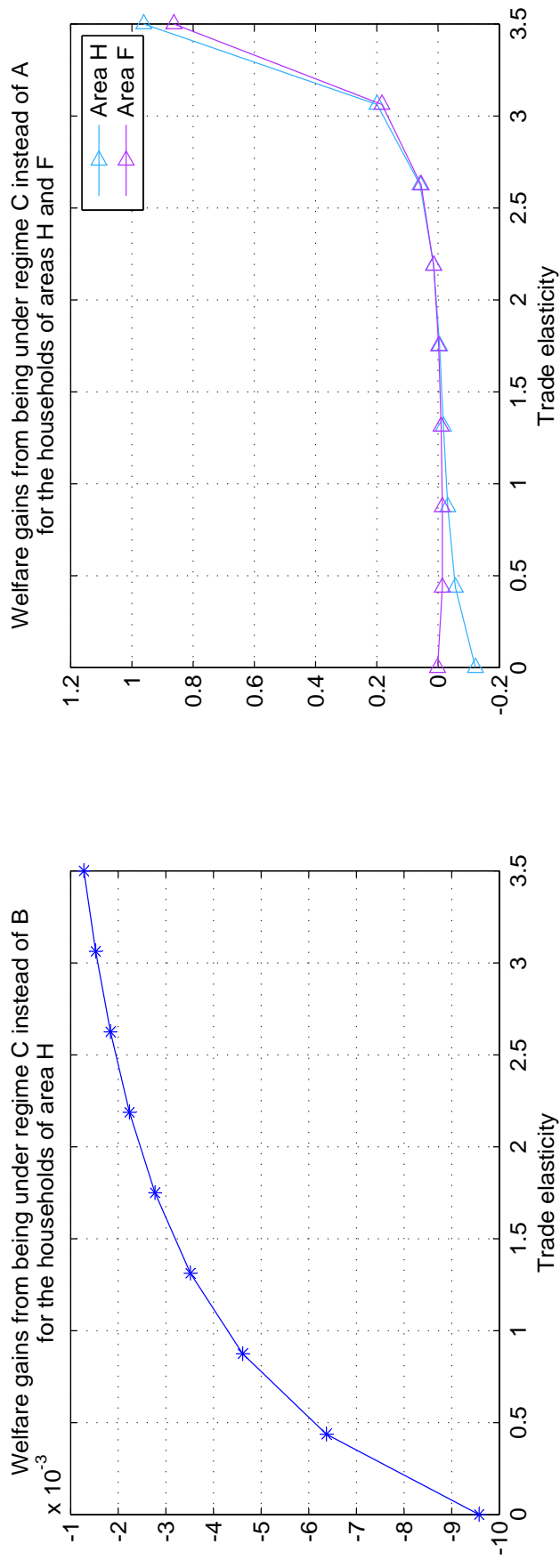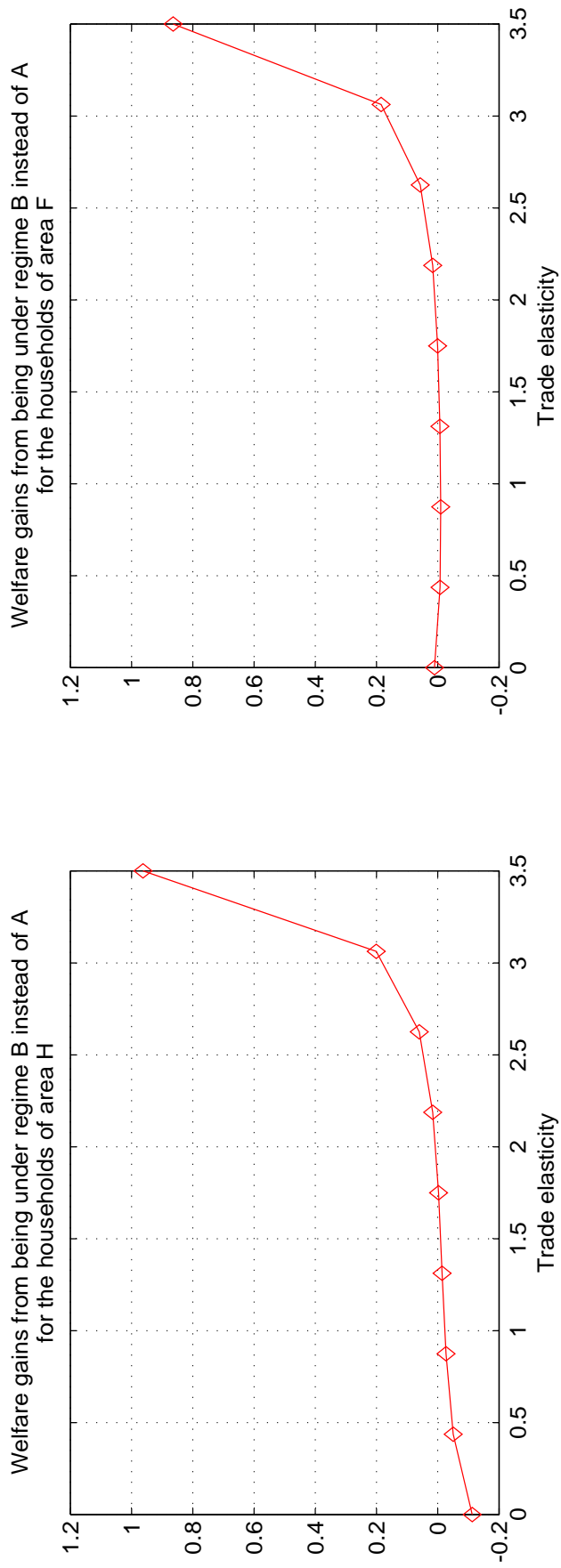Figure 2: Impact responses to a global mark-up shock as function of η, the trade elasticity.

Figure 3: Welfare gains of a monetary union.

Tables

The relative risk aversion σ

η \ σ	1.0	1.29	1.57	1.86	2.14	2.43	2.71	3.0
0.0001	-0.0868	-0.0973	-0.105	-0.11	-0.114	-0.117	-0.12	-0.122
0.388	-0.0487	-0.0516	-0.0532	-0.054	-0.0545	-0.0546	-0.0546	-0.0545
0.775	-0.0306	-0.0313	-0.0313	-0.0311	-0.0307	-0.0303	-0.0298	-0.0294
1.16	-0.0204	-0.02	-0.0193	-0.0185	-0.0176	-0.0167	-0.0157	-0.0148
1.55	-0.0136	-0.0124	-0.0109	-0.00902	-0.00696	-0.00468	-0.0022	0.0005
1.94	-0.00839	-0.00584	-0.00246	0.0018	0.00706	0.0134	0.0211	0.0303
2.33	-0.00337	0.00169	0.00907	0.0195	0.0341	0.0543	0.0822	0.121
2.71	0.00245	0.0125	0.0292	0.0566	0.102	0.18	0.32	0.592
3.1	0.0104	0.0306	0.0705	0.152	0.336	0.825	2.65	21.0

Table 2: Welfare gains for households living in area H from being under regime B instead of A expressed as a percentage of steady-state consumption.

η \ σ	1.0	1.29	1.57	1.86	2.14	2.43	2.71	3.0
0.0001	0.0062	0.00486	0.00344	0.00208	0.0008	-0.0003	-0.00132	-0.00222
0.388	-0.00564	-0.00844	-0.0107	-0.0126	-0.0142	-0.0154	-0.0165	-0.0174
0.775	-0.00876	-0.0111	-0.0128	-0.0141	-0.015	-0.0158	-0.0163	-0.0168
1.16	-0.00898	-0.0105	-0.0114	-0.0118	-0.012	-0.012	-0.0118	-0.0115
1.55	-0.00785	-0.00825	-0.00796	-0.00713	-0.00589	-0.00429	-0.00239	-0.0002
1.94	-0.00574	-0.00453	-0.00218	0.00125	0.00578	0.0115	0.0184	0.0268
2.33	-0.00254	0.00139	0.00782	0.0173	0.0306	0.0492	0.0749	0.11
2.71	0.0022	0.0111	0.0264	0.0517	0.0935	0.164	0.291	0.534
3.1	0.00936	0.028	0.0647	0.139	0.305	0.741	2.35	18.4

Table 3: Welfare gains for households living in area F from being under regime C instead of A expressed as a percentage of steady-state consumption.

η \ σ	1.0	1.29	1.57	1.86	2.14	2.43	2.71	3.0
0.0001	-0.00659	-0.00771	-0.00859	-0.00928	-0.00984	-0.0103	-0.0107	-0.011
0.388	-0.00499	-0.00565	-0.00613	-0.00649	-0.00677	-0.00698	-0.00715	-0.00728
0.775	-0.00394	-0.00436	-0.00465	-0.00486	-0.00501	-0.00512	-0.0052	-0.00527
1.16	-0.0032	-0.00348	-0.00367	-0.00379	-0.00388	-0.00394	-0.00398	-0.00401
1.55	-0.00265	-0.00285	-0.00297	-0.00305	-0.0031	-0.00313	-0.00316	-0.00317
1.94	-0.00224	-0.00237	-0.00246	-0.00251	-0.00253	-0.00255	-0.00256	-0.00256
2.33	-0.00191	-0.00201	-0.00206	-0.00209	-0.0021	-0.00211	-0.00211	-0.00211
2.71	-0.00164	-0.00171	-0.00175	-0.00176	-0.00177	-0.00177	-0.00176	-0.00176
3.1	-0.00143	-0.00147	-0.0015	-0.0015	-0.0015	-0.0015	-0.00149	-0.00148

Table 4: Welfare gains for households living in area H (or F) from being under regime C instead of B expressed as a percentage of steady-state consumption.

The degree of home bias in the area α_b

η \ α_b	0.6	0.657	0.714	0.771	0.829	0.886	0.943	1.0
0.0001	-0.11	-0.115	-0.117	-0.116	-0.114	-0.11	-0.106	-0.101
0.488	-0.0308	-0.035	-0.0386	-0.042	-0.0453	-0.0485	-0.0516	-0.0547
0.975	-0.0154	-0.017	-0.0187	-0.0205	-0.0227	-0.0253	-0.0285	-0.0328
1.46	-0.00753	-0.00798	-0.0085	-0.00913	-0.00994	-0.011	-0.0126	-0.0151
1.95	0.00197	0.00242	0.00292	0.00356	0.0044	0.00562	0.00756	0.0112
2.44	0.0241	0.0258	0.028	0.0307	0.0346	0.0403	0.0502	0.0714
2.93	0.0983	0.104	0.11	0.118	0.13	0.148	0.181	0.262
3.41	0.512	0.532	0.556	0.589	0.636	0.714	0.861	1.26
3.9	28.1	25.5	25.1	25.6	26.9	29.5	35.3	55.9

Table 5: Welfare gains for households living in area H from being under regime B instead of A expressed as a percentage of steady-state consumption.

η \ α_b	0.6	0.657	0.714	0.771	0.829	0.886	0.943	1.0
0.001	-0.00864	-0.00176	0.00194	0.003	0.00212	-0.0008	-0.00286	-0.00606
0.488	-0.03	-0.0259	-0.0223	-0.0188	-0.0156	-0.0123	-0.00917	-0.00606
0.975	-0.0208	-0.0193	-0.0178	-0.0162	-0.0143	-0.0122	-0.0095	-0.00606
1.46	-0.00979	-0.00944	-0.00908	-0.0087	-0.00825	-0.00772	-0.00704	-0.00606
1.95	0.00766	0.00712	0.00646	0.0056	0.0044	0.00261	-0.0003	-0.00606
2.44	0.0464	0.0445	0.0421	0.0391	0.0349	0.0286	0.0176	-0.00606
2.93	0.164	0.158	0.151	0.142	0.13	0.111	0.0768	-0.00606
3.41	0.767	0.739	0.709	0.671	0.62	0.54	0.392	-0.00606
3.9	38.1	32.5	29.7	27.5	25.3	22.2	16.6	-0.00606

Table 6: Welfare gains for households living in area F from being under regime C instead of A expressed as a percentage of steady-state consumption.

η \ α_b	0.6	0.657	0.714	0.771	0.829	0.886	0.943	1.0
0.0001	-0.0171	-0.0156	-0.0138	-0.012	-0.0102	-0.00861	-0.00723	-0.00606
0.488	-0.00614	-0.00615	-0.00615	-0.00614	-0.00612	-0.00611	-0.00609	-0.00606
0.975	-0.00329	-0.00342	-0.0036	-0.00384	-0.00417	-0.00461	-0.00521	-0.00606
1.46	-0.00202	-0.00216	-0.00236	-0.00264	-0.00304	-0.00362	-0.00452	-0.00606
1.95	-0.00132	-0.00146	-0.00164	-0.00191	-0.00231	-0.00292	-0.00397	-0.00606
2.44	-0.0009	-0.00101	-0.00118	-0.00143	-0.0018	-0.00241	-0.00351	-0.00606
2.93	-0.0006	-0.0007	-0.0009	-0.00109	-0.00144	-0.00201	-0.00314	-0.00606
3.41	-0.0004	-0.0005	-0.0006	-0.0008	-0.00116	-0.0017	-0.00281	-0.00606
3.9	-0.0002	-0.0003	-0.0004	-0.0006	-0.0009	-0.00145	-0.00254	-0.00606

Table 7: Welfare gains for households living in area H (or F) from being under regime C instead of B expressed as a percentage of steady-state consumption.

The labor subsidy τ

τ \ η	-0.39	-0.291	-0.191	-0.0919	0.00749	0.107	0.206	0.306
0.0001	-0.107	-0.108	-0.109	-0.111	-0.113	-0.115	-0.117	-0.121
0.388	-0.0534	-0.0536	-0.0538	-0.054	-0.0543	-0.0547	-0.0552	-0.0558
0.775	-0.0316	-0.0315	-0.0313	-0.0311	-0.0309	-0.0306	-0.0303	-0.0299
1.16	-0.02	-0.0197	-0.0192	-0.0187	-0.018	-0.0172	-0.0161	-0.0145
1.55	-0.0122	-0.0115	-0.0106	-0.00944	-0.00788	-0.00571	-0.00253	0.00246
1.94	-0.00532	-0.00385	-0.0019	0.0008	0.00464	0.0105	0.0201	0.0375
2.33	0.00293	0.00606	0.0105	0.017	0.0272	0.0444	0.0771	0.149
2.71	0.0154	0.0223	0.0328	0.0497	0.0794	0.138	0.28	0.765
3.1	0.0376	0.054	0.0812	0.131	0.236	0.519	1.75	41.4

Table 8: Welfare gains for households living in area H from being under regime B instead of A expressed as a percentage of steady-state consumption. Here, as in the following tables, τ, the labor subsidy, varies between -0.39, a value which corresponds a 39 percent labor tax and 0.306, the value of τ for which the monopolistic distortion in the labor and in the good markets are completely eliminated.

τ \ η	-0.39	-0.291	-0.191	-0.0919	0.00749	0.107	0.206	0.306
0.0001	-0.001	-0.0005	0.00005	0.0007	0.0015	0.00248	0.00369	0.00525
0.388	-0.0136	-0.0136	-0.0135	-0.0135	-0.0134	-0.0134	-0.0133	-0.0132
0.775	-0.0145	-0.0145	-0.0145	-0.0146	-0.0146	-0.0146	-0.0146	-0.0146
1.16	-0.0126	-0.0125	-0.0124	-0.0122	-0.0119	-0.0115	-0.0109	-0.00989
1.55	-0.00952	-0.00907	-0.00847	-0.00764	-0.00645	-0.00471	-0.002	0.00245
1.94	-0.00501	-0.0038	-0.00214	0.0002	0.00369	0.00907	0.018	0.0341
2.33	0.0019	0.00474	0.00879	0.0148	0.0243	0.0403	0.0703	0.136
2.71	0.0134	0.0198	0.0296	0.0453	0.0726	0.126	0.253	0.678
3.1	0.0342	0.0494	0.0746	0.12	0.215	0.466	1.54	35.3

Table 9: Welfare gains for households living in area F from being under regime C instead of A expressed as a percentage of steady-state consumption.

τ \ η	-0.39	-0.291	-0.191	-0.0919	0.00749	0.107	0.206	0.306
0.0001	-0.00931	-0.00936	-0.00942	-0.0095	-0.00958	-0.00969	-0.00982	-0.01
0.388	-0.00651	-0.00654	-0.00657	-0.0066	-0.00664	-0.0067	-0.00676	-0.00685
0.775	-0.00488	-0.0049	-0.00491	-0.00492	-0.00494	-0.00496	-0.00499	-0.00504
1.16	-0.00383	-0.00383	-0.00384	-0.00384	-0.00384	-0.00385	-0.00385	-0.00386
1.55	-0.0031	-0.0031	-0.00309	-0.00309	-0.00308	-0.00307	-0.00306	-0.00304
1.94	-0.00257	-0.00256	-0.00255	-0.00254	-0.00252	-0.0025	-0.00248	-0.00244
2.33	-0.00217	-0.00215	-0.00214	-0.00212	-0.0021	-0.00207	-0.00203	-0.00198
2.71	-0.00185	-0.00184	-0.00182	-0.00179	-0.00177	-0.00173	-0.00168	-0.00161
3.1	-0.0016	-0.00158	-0.00156	-0.00153	-0.0015	-0.00146	-0.0014	-0.00131

Table 10: Welfare gains for households living in area H (or F) from being under regime C instead of B expressed as a percentage of steady-state consumption.

The mark-up shock correlation across regions $\varsigma_{s,\mu}$

$\varsigma_{s,\mu}$ η	0	0.143	0.286	0.429	0.571	0.714	0.857	1.0
0.0001	-0.125	-0.113	-0.101	-0.0899	-0.0784	-0.0668	-0.0553	-0.0437
0.388	-0.0511	-0.0467	-0.0423	-0.0378	-0.0334	-0.029	-0.0245	-0.0201
0.775	-0.0258	-0.0237	-0.0216	-0.0194	-0.0173	-0.0151	-0.013	-0.0108
1.16	-0.0115	-0.0104	-0.00918	-0.00801	-0.00683	-0.00565	-0.00447	-0.0033
1.55	0.00404	0.00476	0.00548	0.00621	0.00693	0.00765	0.00837	0.0091
1.94	0.0359	0.0364	0.0369	0.0374	0.0379	0.0384	0.0389	0.0394
2.33	0.135	0.136	0.136	0.136	0.137	0.137	0.138	0.138
2.71	0.659	0.66	0.662	0.664	0.666	0.667	0.669	0.671
3.1	27.0	27.7	28.3	29.0	29.7	30.4	31.0	31.7

Table 11: Welfare gains for households living in area H from being under regime B instead of A expressed as a percentage of steady-state consumption.

$\varsigma_{s,\mu}$ η	0	0.143	0.286	0.429	0.571	0.714	0.857	1.0
0.0001	0.00669	0.00169	-0.0033	-0.0083	-0.0133	-0.0183	-0.0233	-0.02834
0.488	-0.0128	-0.0143	-0.0157	-0.0171	-0.0186	-0.02	-0.0214	-0.0228
0.975	-0.0131	-0.0135	-0.014	-0.0144	-0.0149	-0.0153	-0.0157	-0.0162
1.46	-0.00799	-0.00807	-0.00815	-0.00823	-0.00831	-0.00839	-0.00847	-0.00855
1.95	0.00378	0.00381	0.00385	0.00388	0.00391	0.00395	0.00398	0.00402
2.44	0.0329	0.0329	0.0329	0.0329	0.0329	0.0329	0.0328	0.0328
2.92	0.124	0.124	0.124	0.124	0.123	0.123	0.123	0.123
3.41	0.595	0.595	0.595	0.594	0.594	0.594	0.594	0.594
3.9	23.7	24.2	24.8	25.3	25.9	26.5	27.0	27.6

Table 12: Welfare gains for households living in area F from being under regime C instead of A expressed as a percentage of steady-state consumption.

$\varsigma_{s,\mu}$ η	0	0.143	0.286	0.429	0.571	0.714	0.857	1.0
0.0001	-0.00554	-0.00939	-0.0132	-0.0171	-0.0209	-0.0248	-0.0286	-0.0324
0.488	-0.00371	-0.00601	-0.00831	-0.0106	-0.0129	-0.0152	-0.0175	-0.0198
0.975	-0.00273	-0.00424	-0.00576	-0.00728	-0.00879	-0.0103	-0.0118	-0.0133
1.46	-0.00213	-0.00318	-0.00423	-0.00528	-0.00633	-0.00738	-0.00843	-0.00948
1.95	-0.00173	-0.00247	-0.00321	-0.00396	-0.0047	-0.00545	-0.00619	-0.00694
2.44	-0.00144	-0.00197	-0.0025	-0.00303	-0.00356	-0.00409	-0.00462	-0.00515
2.92	-0.00123	-0.0016	-0.00197	-0.00234	-0.00271	-0.00308	-0.00345	-0.00382
3.41	-0.00106	-0.00131	-0.00156	-0.00181	-0.00206	-0.00231	-0.00256	-0.00281
3.9	-0.0009	-0.00109	-0.00124	-0.00139	-0.00155	-0.0017	-0.00186	-0.00201

Table 13: Welfare gains for households living in area H (or F) from being under regime C instead of B expressed as a percentage of steady-state consumption.

The mark-up shock correlation across areas $\varsigma_{b,\mu}$

$\varsigma_{b,\mu}$ η	0	0.143	0.286	0.429	0.571	0.714	0.857	1.0
0.0001	-0.112	-0.112	-0.112	-0.112	-0.112	-0.112	-0.112	-0.112
0.488	-0.0465	-0.0465	-0.0465	-0.0465	-0.0465	-0.0465	-0.0465	-0.0465
0.975	-0.0236	-0.0236	-0.0236	-0.0236	-0.0236	-0.0236	-0.0236	-0.0236
1.46	-0.0103	-0.0103	-0.0103	-0.0103	-0.0103	-0.0103	-0.0103	-0.0103
1.95	0.0048	0.0048	0.0048	0.00479	0.00479	0.00479	0.00478	0.00478
2.44	0.0364	0.0364	0.0364	0.0364	0.0364	0.0364	0.0364	0.0364
2.93	0.136	0.136	0.136	0.136	0.135	0.135	0.135	0.135
3.41	0.661	0.661	0.66	0.66	0.66	0.66	0.66	0.66
3.9	27.7	27.7	27.7	27.7	27.7	27.7	27.7	27.7

Table 14: Welfare gains for households living in area H from being under regime B instead of A expressed as a percentage of steady-state consumption.

$\varsigma_{b,\mu}$ η	0	0.143	0.286	0.429	0.571	0.714	0.857	1.0
0.0001	0.0008	0.0015	0.00217	0.00283	0.00349	0.00415	0.00482	0.00548
0.488	-0.0147	-0.0143	-0.0139	-0.0135	-0.0131	-0.0127	-0.0123	-0.0119
0.975	-0.0138	-0.0135	-0.0133	-0.013	-0.0127	-0.0125	-0.0122	-0.0119
1.46	-0.00824	-0.00805	-0.00786	-0.00766	-0.00747	-0.00727	-0.00708	-0.00688
1.95	0.00368	0.00383	0.00399	0.00414	0.0043	0.00445	0.00461	0.00476
2.44	0.0328	0.0329	0.033	0.0332	0.0333	0.0335	0.0336	0.0337
2.93	0.124	0.124	0.124	0.124	0.125	0.125	0.125	0.125
3.41	0.595	0.595	0.595	0.595	0.595	0.595	0.596	0.596
3.9	24.2	24.2	24.2	24.2	24.2	24.2	24.2	24.2

Table 15: Welfare gains for households living in area F from being under regime C instead of A expressed as a percentage of steady-state consumption.

$\varsigma_{b,\mu}$ η	0	0.143	0.286	0.429	0.571	0.714	0.857	1.0
0.0001	-0.0102	-0.00951	-0.00885	-0.00819	-0.00752	-0.00686	-0.0062	-0.00554
0.488	-0.00648	-0.00608	-0.00568	-0.00529	-0.00489	-0.00449	-0.0041	-0.0037
0.975	-0.00456	-0.00429	-0.00403	-0.00376	-0.0035	-0.00323	-0.00297	-0.00271
1.46	-0.0034	-0.00321	-0.00302	-0.00283	-0.00264	-0.00246	-0.00227	-0.00208
1.95	-0.00263	-0.00249	-0.00235	-0.00221	-0.00207	-0.00193	-0.00179	-0.00164
2.44	-0.00209	-0.00198	-0.00187	-0.00176	-0.00166	-0.00155	-0.00144	-0.00133
2.93	-0.00169	-0.00161	-0.00152	-0.00143	-0.00134	-0.00126	-0.00117	-0.00108
3.41	-0.00139	-0.00132	-0.00124	-0.00117	-0.0011	-0.00103	-0.001	-0.0009
3.9	-0.00115	-0.00109	-0.00103	-0.001	-0.0009	-0.0008	-0.0008	-0.0007

Table 16: Welfare gains for households living in area H (or F) from being under regime C instead of B expressed as a percentage of steady-state consumption.

The autocorrelation of the mark-up shocks ρ_μ

η \ ρ_μ	0	0.141	0.283	0.424	0.566	0.707	0.849	0.99
0.0001	-0.0219	-0.022	-0.0224	-0.0234	-0.0257	-0.0312	-0.0457	-0.0907
0.488	-0.0101	-0.0102	-0.0103	-0.0108	-0.0119	-0.0142	-0.0196	-0.0337
0.975	-0.00601	-0.00602	-0.00608	-0.00628	-0.00674	-0.00781	-0.0103	-0.0161
1.46	-0.00407	-0.00402	-0.0039	-0.00377	-0.0037	-0.00385	-0.00459	-0.0068
1.95	-0.00298	-0.00279	-0.00226	-0.00146	-0.0004	0.0007	0.00201	0.00289
2.44	-0.00229	-0.00172	-0.0001	0.00246	0.00596	0.0104	0.0159	0.0225
2.93	-0.00183	0.0001	0.00522	0.0136	0.0252	0.0404	0.0596	0.0838
3.41	-0.00149	0.00681	0.0305	0.069	0.123	0.196	0.29	0.411
3.9	-0.0004	0.219	0.895	2.11	4.05	7.02	11.6	18.8

Table 17: Welfare gains for households living in area H from being under regime B instead of A expressed as a percentage of steady-state consumption.

η \ ρ_μ	0	0.141	0.283	0.424	0.566	0.707	0.849	0.99
0.0001	-0.0007	-0.0007	-0.0007	-0.0006	-0.0005	-0.0003	0.0003	0.00205
0.488	-0.00342	-0.00344	-0.00351	-0.00366	-0.00399	-0.00467	-0.00619	-0.00997
0.975	-0.00338	-0.00339	-0.00343	-0.00356	-0.00385	-0.0045	-0.00591	-0.00914
1.46	-0.00299	-0.00295	-0.00285	-0.00275	-0.00271	-0.00288	-0.00355	-0.00541
1.95	-0.0026	-0.00242	-0.00194	-0.00121	-0.0003	0.0007	0.00165	0.00212
2.44	-0.00227	-0.00172	-0.0002	0.0022	0.00545	0.00953	0.0145	0.0201
2.93	-0.00198	-0.0003	0.00458	0.0123	0.0231	0.037	0.0546	0.0765
3.41	-0.00174	0.00578	0.0272	0.062	0.111	0.176	0.261	0.37
3.9	-0.0008	0.192	0.785	1.85	3.55	6.15	10.2	16.5

Table 18: Welfare gains for households living in area F from being under regime C instead of A expressed as a percentage of steady-state consumption.

η \ ρ_μ	0	0.141	0.283	0.424	0.566	0.707	0.849	0.99
0.0001	-0.00277	-0.00278	-0.0028	-0.00286	-0.003	-0.00334	-0.00419	-0.0067
0.488	-0.00185	-0.00186	-0.00188	-0.00192	-0.00202	-0.00224	-0.00273	-0.00401
0.975	-0.00136	-0.00137	-0.00138	-0.00142	-0.00149	-0.00164	-0.00196	-0.00271
1.46	-0.00106	-0.00107	-0.00108	-0.0011	-0.00116	-0.00126	-0.00148	-0.00196
1.95	-0.0008	-0.0008	-0.0008	-0.0009	-0.0009	-0.00101	-0.00116	-0.00148
2.44	-0.0007	-0.0007	-0.0007	-0.0007	-0.0008	-0.0008	-0.0009	-0.00116
2.93	-0.0006	-0.0006	-0.0006	-0.0006	-0.0006	-0.0007	-0.0007	-0.0009
3.41	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0006	-0.0006	-0.0007
3.9	-0.0004	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005	-0.0005

Table 19: Welfare gains for households living in area H (or F) from being under regime C instead of B expressed as a percentage of steady-state consumption.

The standard deviation of the mark-up shocks σ_μ

η \ σ_μ	0	0.00471	0.00943	0.0141	0.0189	0.0236	0.0283	0.033
0.0001	-0.0438	-0.046	-0.0527	-0.0638	-0.0794	-0.0994	-0.124	-0.153
0.488	-0.0202	-0.0211	-0.0236	-0.0279	-0.0338	-0.0415	-0.0509	-0.0619
0.975	-0.012	-0.0124	-0.0135	-0.0154	-0.018	-0.0214	-0.0255	-0.0304
1.46	-0.00813	-0.0082	-0.00842	-0.00877	-0.00926	-0.00989	-0.0107	-0.0116
1.95	-0.00595	-0.0056	-0.00456	-0.00282	-0.0004	0.00276	0.0066	0.0111
2.44	-0.00459	-0.00326	0.0007	0.00738	0.0167	0.0287	0.0433	0.0606
2.93	-0.00366	0.0008	0.0144	0.037	0.0686	0.109	0.159	0.218
3.41	-0.00298	0.0185	0.0831	0.191	0.341	0.535	0.772	1.05
3.9	-0.0008	0.897	3.59	8.08	14.4	22.4	32.3	44.0

Table 20: Welfare gains for households living in area H from being under regime B instead of A expressed as a percentage of steady-state consumption.

η \ σ_μ	0	0.00471	0.00943	0.0141	0.0189	0.0236	0.0283	0.033
0.0001	-0.00147	-0.00137	-0.00109	-0.0006	0.0004	0.0009	0.00192	0.00315
0.488	-0.00685	-0.00709	-0.00782	-0.00904	-0.0107	-0.0129	-0.0156	-0.0187
0.975	-0.00676	-0.00698	-0.00764	-0.00874	-0.0103	-0.0123	-0.0147	-0.0176
1.46	-0.00598	-0.00605	-0.00625	-0.00659	-0.00706	-0.00767	-0.00842	-0.0093
1.95	-0.0052	-0.00491	-0.00403	-0.00257	-0.0005	0.00211	0.00532	0.00912
2.44	-0.00453	-0.00332	0.0003	0.00639	0.0149	0.0258	0.0392	0.0549
2.93	-0.00397	0.0009	0.0126	0.0334	0.0625	0.0998	0.146	0.199
3.41	-0.00348	0.0159	0.0741	0.171	0.307	0.482	0.695	0.947
3.9	-0.00161	0.785	3.14	7.07	12.6	19.7	28.3	38.5

Table 21: Welfare gains for households living in area F from being under regime C instead of A expressed as a percentage of steady-state consumption.

η \ σ_μ	0	0.00471	0.00943	0.0141	0.0189	0.0236	0.0283	0.033
0.0001	-0.00554	-0.00567	-0.00606	-0.00672	-0.00763	-0.00881	-0.0103	-0.012
0.488	-0.0037	-0.00378	-0.00402	-0.00441	-0.00496	-0.00566	-0.00652	-0.00754
0.975	-0.00273	-0.00278	-0.00293	-0.00319	-0.00355	-0.00402	-0.00458	-0.00526
1.46	-0.00213	-0.00216	-0.00227	-0.00245	-0.0027	-0.00302	-0.00341	-0.00388
1.95	-0.00172	-0.00175	-0.00183	-0.00195	-0.00213	-0.00236	-0.00264	-0.00297
2.44	-0.00144	-0.00146	-0.00151	-0.0016	-0.00173	-0.00189	-0.00209	-0.00232
2.93	-0.00123	-0.00124	-0.00128	-0.00134	-0.00143	-0.00154	-0.00168	-0.00184
3.41	-0.00106	-0.00107	-0.0011	-0.00114	-0.0012	-0.00127	-0.00137	-0.00148
3.9	-0.0009	-0009	-0.001	-0.001	-0.00102	-0.00106	-0.00112	-0.00119

Table 22: Welfare gains for households living in area H (or F) from being under regime C instead of B expressed as a percentage of steady-state consumption.

Appendix

A Retrieving condition (18)

Given the definitions of $P_{H,t}^*$ and $P_{F,t}^*$ it is easy to show that:

$$\mathcal{E}_{iH,t}P_{H,t}^* = [\alpha_b P_{H,t}{}^{1-\eta} + (1-\alpha_b)P_{F,t}{}^{1-\eta}]^{\frac{1}{1-\eta}} \qquad \mathcal{E}_{iF,t}P_{F,t}^* = [\alpha_b P_{F,t}{}^{1-\eta} + (1-\alpha_b)P_{H,t}{}^{1-\eta}]^{\frac{1}{1-\eta}} \tag{65}$$

By (65):

$$\frac{\mathcal{E}_{iH,t}P_{H,t}^*}{P_{H,t}} = \left[\alpha_b + (1-\alpha_b)\left(\frac{P_{F,t}}{P_{H,t}}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \qquad \frac{\mathcal{E}_{iF,t}P_{F,t}^*}{P_{F,t}} = \left[\alpha_b + (1-\alpha_b)\left(\frac{P_{H,t}}{P_{F,t}}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \tag{66}$$

which jointly with (17) leads to:

$$\left(\frac{C_{F,t}^*}{C_{H,t}^*}\right) = \left[\frac{\alpha_b \left(\frac{P_{F,t}}{P_{H,t}}\right)^{1-\eta} + (1-\alpha_b)}{(1-\alpha_b)\left(\frac{P_{F,t}}{P_{H,t}}\right)^{1-\eta} + \alpha_b}\right]^{-\frac{1}{\sigma(1-\eta)}} \tag{67}$$

Moreover thanks to (5):

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\frac{1}{\alpha_s} - \frac{\alpha_b - \alpha_s}{\alpha_s}\left(\frac{P_{H,t}}{P_{C^i,t}}\right)^{1-\eta} - \frac{(1-\alpha_b)}{\alpha_s}\left(\frac{P_{F,t}}{P_{C^i,t}}\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \qquad i \in [0,\tfrac{1}{2}) \tag{68}$$

from which using (17) it follows that:

$$\frac{P_{i,t}}{P_{C^i,t}} = \left[\frac{1}{\alpha_s} - \frac{\alpha_b - \alpha_s}{\alpha_s}\left(\frac{P_{H,t}}{\mathcal{E}_{i,H}P_{H,t}^*}\right)^{(1-\eta)}\left(\frac{C_{H,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)} - \frac{(1-\alpha_b)}{\alpha_s}\left(\frac{P_{F,t}}{\mathcal{E}_{i,F}P_{F,t}^*}\right)^{(1-\eta)}\left(\frac{C_{F,t}^*}{C_t^i}\right)^{-\sigma(1-\eta)}\right]^{\frac{1}{1-\eta}} \tag{69}$$

Finally, by using (66) and (67) we can rewrite (69) as (18).

B The Pareto-efficient allocation

In this appendix we solve the social planner problem and find the conditions that characterize the Pareto-efficient allocation. The social planner maximizes the world welfare with respect to Y_t^i, N_t^i and $C_{j,t}^i$ for all i and j:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \int_0^1 \frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1}\left(N_t^i\right)^{\varphi+1} di \right\} \tag{70}$$

where C_t^i is defined consistently with (2) and (3).

subject to resource and the technological constraints:

$$Y_t^i = C_{i,t}^i + \int_0^{\frac{1}{2}} C_{i,t}^j dj + \int_{\frac{1}{2}}^1 C_{i,t}^j dj \qquad i \in [0,1]$$
$$Y_t^i = A_t^i N_t^i \qquad i \in [0,1] \tag{71}$$

Notice that as in Galí and Monacelli (2009) the previous constraints already incorporate the optimal condition whereby consumption and production of each variety is identical within each region i. According to the first-order conditions:

$$C_t^{i-\sigma} = \left[\alpha_s \left(\frac{Y_t^{i\varphi+1}}{A_t^{i\varphi}} \right)^{1-\eta} + 2(\alpha_b - \alpha_s) \int_0^{\frac{1}{2}} \left(\frac{Y_t^{j\varphi+1}}{A_t^{j\varphi}} \right)^{1-\eta} dj + 2(1-\alpha_b) \int_{\frac{1}{2}}^1 \left(\frac{Y_t^{j\varphi+1}}{A_t^{j\varphi}} \right)^{1-\eta} dj \right]^{\frac{1}{1-\eta}}$$

$$\frac{Y_t^{i1+\varphi\eta}}{A_t^{i(\varphi+1)\eta}} = \left[\alpha_s C_t^{i1-\sigma\eta} + 2(\alpha_b - \alpha_s) \int_0^{\frac{1}{2}} C_t^{j1-\sigma\eta} dj + 2(1-\alpha_b) \int_{\frac{1}{2}}^1 C_t^{j1-\sigma\eta} dj \right] \tag{72}$$

for all $i \in [0, \frac{1}{2})$. Symmetric conditions apply to the each region i with $i \in [\frac{1}{2}, 1]$. The first condition in (72) states that at the margin the utility embedded into an additional unit of the composite consumption bundle C_t^i should be equal to a CES aggregation of the additional disutilities entailed by its production. Conversely, the second condition in (72) rewrites the resource constraint in terms of aggregate consumption of each region j by using the technological constraints and the other optimality conditions.

By taking the log-linear approximation to (72) and integrating them over $i \in [0, \frac{1}{2})$ we find that:

$$-\sigma \hat{c}_{H,t}^e = \alpha_b(\varphi \hat{y}_{H,t}^e - (\varphi+1)\hat{a}_{H,t}) + (1-\alpha_b)(\varphi \hat{y}_{F,t}^e - (\varphi+1)\hat{a}_{F,t})$$

$$(1+\varphi\eta)\hat{y}_{H,t}^e - (1+\varphi)\eta \hat{a}_{H,t} = (1-\eta\sigma)(\alpha_b \hat{c}_{H,t}^e + (1-\alpha_b)\hat{c}_{F,t}^e) \tag{73}$$

where the suffix e stands for efficient.

C Zero-inflation deterministic steady state

In this appendix we argue that, given appropriate initial conditions, zero inflation is a Nash equilibrium policy at the deterministic steady state under both regimes A and B.

Under the regime A, the *timelessly* optimal policy problem of a monetary authority of country i in the area H can be formulated as the maximization of the following Lagragian:

$$
\begin{aligned}
L_s = E_0 \sum_{t=0}^\infty \beta^t & \left\{ \frac{C_t^{i1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1} \left(\frac{Y_t^i Z_t^i}{A_t^i} \right)^{\varphi+1} \right. \\
& + \zeta_{1,t}^{s,i} \left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}} \right)^{-\eta} \left(\alpha_s C_t^i + (\alpha_b - \alpha_s) C_t^{i\sigma\eta} \mathcal{C}_{H,t} + (1-\alpha_b) C_t^{i\sigma\eta} \mathcal{C}_{F,t} \right) \right] \\
& + \zeta_{2,t}^{s,i} \left[K_t^i - \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi+1} Z_t^{i\varphi}(1+\mu_t^i)(1-\tau^i) \frac{\varepsilon}{\varepsilon-1} \right] - \zeta_{2,t-1}^{s,i} \theta \Pi_{i,t}^\varepsilon K_t^i \\
& + \zeta_{3,t}^{s,i} \left[F_t^i - Y_t^i C_t^{i-\sigma} \frac{P_t^i}{P_{C^i,t}} \right] - \zeta_{3,t-1}^{s,i} \theta \Pi_{i,t}^{(\varepsilon-1)} F_t^i \\
& + \zeta_{4,t}^{s,i} \left[F_t^i - K_t^i \left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{\varepsilon-1}} \right] \\
& \left. + \zeta_{5,t}^{s,i} \left[Z_t^i - \theta Z_{t-1}^i \Pi_{i,t}^\varepsilon - (1-\theta) \left(\frac{1-\theta\Pi_{i,t}^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} \right] \right\}
\end{aligned}
$$

where $P_{i,t}/P_{C^i,t}$ are determined consistently with (18), while $C^*_{H,t}$, $C^*_{F,t}$, $\mathcal{C}_{H,t}$ and $\mathcal{C}_{F,t}$ are taken as given.[46] Assume that $\mu_t^j = \mu$, $A_t^j = 1$, $\tau^j = \tau$, $Z_t^j = \Pi_{j,t} = 1$ and $Z_{-1}^j = 1$ for all t and $j \neq i$ with $j \in [0, 1]$. In addition, assume that $Z_{-1}^i = 1$ for all $i \in [0, 1]$. Recalling that $\tilde{\tau} = 1 - (1 - \tau)(1 + \mu)\frac{\varepsilon}{\varepsilon-1}$ it can be shown that at the deterministic steady state zero-inflation is an optimal policy, i.e., being a solution of the first-order conditions of the Lagrangian stated in (74), zero-inflation is best response to the zero-inflation policies of the other policymakers in area F and H. Indeed, if $Z_t^i = \Pi_{i,t} = 1$ at all t, then according to the first-order conditions of (74) with respect to C_t^i, Y_t^i, Z_t^i, K_t^i, F_t^i and $\Pi_{i,t}$ at the symmetric deterministic steady state:

$$C^{-\sigma} = \zeta_1^s \delta_s - \zeta_3^s \sigma \gamma_s Y C^{-\sigma-1}$$

$$Y^\varphi = \zeta_1^s - \zeta_2^s(\varphi + 1)Y^\varphi(1 - \tilde{\tau}) - \zeta_3^s C^{-\sigma}$$

$$Y^{\varphi+1} = -\zeta_2^s \varphi Y^{\varphi+1}(1 - \tilde{\tau}) + \zeta_5^s(1 - \beta\theta)$$

$$\zeta_2^s(1 - \theta) = \zeta_4^s$$

$$\zeta_3^s(1 - \theta) = -\zeta_4^s$$

$$\zeta_2^s \theta \varepsilon K = -\zeta_3^s \theta(\varepsilon - 1)F + \zeta_4^s \frac{\theta}{1 - \theta}K \tag{74}$$

with $\gamma_s = \frac{1}{\alpha_s}$ and $\delta_s = \alpha_s(1 - \sigma\eta) + \gamma_s\eta\sigma$. Then:

$$Y = (1 - \tilde{\tau})^{-\frac{1}{\sigma+\varphi}}$$

$$C = Y$$

$$F = K = \frac{YC^{-\sigma}}{1 - \beta\theta} = \frac{Y^{\varphi+1}(1 - \tilde{\tau})}{1 - \beta\theta}$$

$$\Pi = \Pi_H = \Pi_F = Z = 1$$

$$\zeta_1^s = Y^\varphi(1 - \varphi\zeta_s) \qquad \zeta_2^s = -\frac{\zeta_s}{(1 - \tilde{\tau})} \qquad \zeta_3^s = \frac{\zeta_s}{(1 - \tilde{\tau})}$$

$$\zeta_4^s = -\frac{(1 - \theta)\zeta_s}{(1 - \tilde{\tau})} \qquad \zeta_5^s = \frac{Y^{\varphi+1}(1 - \varphi\zeta_s)}{(1 - \beta\theta)}$$

where $\zeta_s \equiv \frac{\delta_s - (1 - \tilde{\tau})}{\gamma_s\sigma + \delta_s\varphi}$ is a steady-state symmetric solution of the optimal policy problem just stated. Consider now the monetary union in area F.[47] Suppose that for all $i \in [0, \frac{1}{2})$, $\Pi_t^i = Z_t^i = 1$ at all times. Hence, $F_t^i = F$, $K_t^i = K$ and $\frac{F_t^i}{K_t^i} = 1$ for all i and t. We want to argue that even in this case, given other policy makers' strategies, $\Pi_t^i = 1$ for all t and $i \in [\frac{1}{2}, 1]$ is the optimal best response of the central bank in area F. If for all $i \in [\frac{1}{2}, 1]$, $\Pi_t^i = 1$ at all times, the optimal policy problem of the monetary

---

[46]Notice that this Lagrangian incorporates additional constraints for time 0 that render the policy timelessly optimal.

[47]We follow closely Benigno and Benigno (2006).

authority in area F can be written as maximizing:

$$L_b = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \int_{\frac{1}{2}}^{1} \left[\frac{C_t^{i1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1} \left(\frac{Y_t^i Z_t^i}{A_t^i} \right)^{\varphi+1} \right] \right.$$

$$+ \zeta_{1,t}^{b,i} \left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}} \right)^{-\eta} \left(\alpha_s C_t^i + (\alpha_b - \alpha_s) C_t^{i\sigma\eta} \mathcal{C}_{F,t} + (1-\alpha_b) C_t^{i\sigma\eta} \mathcal{C}_{H,t} \right) \right]$$

$$+ \zeta_{2,t}^{b,i} \left[K_t^i - \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi+1} Z_t^{i\varphi} (1+\mu_t^i)(1-\tau^i) \frac{\varepsilon}{\varepsilon-1} \right] - \zeta_{2,t-1}^{b,i} \theta \Pi_{i,t}^{\varepsilon} K_t^i$$

$$+ \zeta_{3,t}^{b,i} \left[F_t^i - Y_t^i C_t^{i-\sigma} \frac{P_t^i}{P_{C^i,t}} \right] - \zeta_{3,t-1}^{b,i} \theta \Pi_{i,t}^{(\varepsilon-1)} F_t^i$$

$$+ \zeta_{4,t}^{b,i} \left[F_t^i - K_t^i \left(\frac{1-\theta \Pi_{i,t}^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{\varepsilon-1}} \right]$$

$$+ \zeta_{5,t}^{b,i} \left[Z_t^i - \theta Z_{t-1}^i \Pi_{i,t}^{\varepsilon} - (1-\theta) \left(\frac{1-\theta \Pi_{i,t}^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} \right]$$

$$+ \zeta_{6,t}^{b,i} \left[\left(\frac{C_{F,t}^*}{C_{F,t-1}^*} \right)^{-\sigma} \frac{P_{F,t}}{P_{F,t}^*} \frac{P_{F,t-1}^*}{P_{F,t-1}} \Pi_{F,t}^{-1} - \left(\frac{C_t^i}{C_{t-1}^i} \right)^{-\sigma} \frac{P_{i,t}}{P_{C^i,t}} \frac{P_{C^i,t-1}}{P_{i,t-1}} \Pi_{i,t}^{-1} \right] di$$

$$+ \int_0^{\frac{1}{2}} \zeta_{7,t}^{b,i} \left[Y_t^i - \left(\frac{P_{i,t}}{P_{C^i,t}} \right)^{-\eta} \left(\alpha_s C_t^i + (\alpha_b - \alpha_s) C_t^{i\sigma\eta} \mathcal{C}_{H,t} + (1-\alpha_b) C_t^{i\sigma\eta} \mathcal{C}_{F,t} \right) \right]$$

$$+ \zeta_{8,t}^{b,i} \left[(1+\mu_t^i)(1-\tau) \frac{\varepsilon}{\varepsilon-1} \left(\frac{Y_t^i}{A_t^i} \right)^{\varphi+1} - \frac{P_{i,t}}{P_{C^i,t}} Y_t^i C_t^{i-\sigma} \right] di \right\} \tag{75}$$

where $\mathcal{C}_{H,t} \equiv 2 \int_0^{\frac{1}{2}} C_t^{i1-\sigma\eta} di$ and $\mathcal{C}_{F,t} \equiv 2 \int_{\frac{1}{2}}^{1} C_t^{i1-\sigma\eta} di$ and $P_{i,t}/P_{C^i,t}$ and $P_{F,t}^*/P_{F,t}$ are determined consistently with (18), the foreign counterpart of (18), (66) and (67). Assume that $\mu_t^i = \mu$, $A_t^i = 1$, $\tau^i = \tau$ and $Z_{-1}^i = 1$ for all $i \in [0,1]$ and t. Moreover, suppose that $Z_{-1}^i = 1$ for all $i \in [0,1]$. Then, it can be shown that zero-inflation is a optimal policy, because such a policy is consistent with the first-order conditions of (75). Indeed, if $Z_t^i = \Pi_{i,t} = 1$ for all t and $i \in [\frac{1}{2}, 1]$, the first-order conditions with respect to C_t^i, Y_t^i for all i and Z_t^i, K_t^i, F_t^i and $\Pi_{i,t}$ all $i \in [\frac{1}{2}, 1]$ at the symmetric deterministic steady state can written as:

$$C^{-\sigma} = \zeta_1^b \delta_b + \zeta_7^b (1-\delta_b) - \zeta_3^b \sigma \gamma_b Y C^{-\sigma-1} - \zeta_8^b \sigma (1-\gamma_b) Y C^{-\sigma-1}$$

$$Y^{\varphi} = \zeta_1^b - \zeta_2^b (\varphi+1) Y^{\varphi} (1-\tilde{\tau}) - \zeta_3^b C^{-\sigma}$$

$$0 = \zeta_1^b (1-\delta_b) + \zeta_7^b \delta_b - \zeta_3^b \sigma (1-\gamma_b) Y C^{-\sigma-1} - \zeta_8^b \sigma \gamma_b Y C^{-\sigma-1}$$

$$0 = \zeta_7^b + \zeta_8^b \left[(\varphi+1) Y^{\varphi} (1-\tilde{\tau}) - C^{-\sigma} \right]$$

$$Y^{\varphi+1} = -\zeta_2^b \varphi Y^{\varphi+1} (1-\tilde{\tau}) + \zeta_5^b (1-\beta\theta)$$

$$\zeta_2^b (1-\theta) = \zeta_4^b$$

$$\zeta_3^b (1-\theta) = -\zeta_4^b$$

$$\zeta_2^b \theta \varepsilon K = -\zeta_3^b \theta (\varepsilon-1) F + \zeta_4^b \frac{\theta}{1-\theta} K \tag{76}$$

where $\gamma_b = \frac{\alpha_b}{2\alpha_b - 1}$ and $\delta_b \equiv (1 - \sigma\eta)\alpha_b + \eta\sigma\gamma_b$. As a consequence:

$$Y = (1 - \tilde{\tau})^{-\frac{1}{\sigma+\varphi}}$$

$$C = Y$$

$$F = K = \frac{YC^{-\sigma}}{1 - \beta\theta} = \frac{Y^{\varphi+1}(1 - \tilde{\tau})}{1 - \beta\theta}$$

$$\Pi = \Pi_H = \Pi_F = Z = 1$$

$$\zeta_1^b = Y^\varphi \left(1 - \varphi\zeta_b\right) \qquad \zeta_2^b = -\frac{\zeta_b}{(1 - \tilde{\tau})} \qquad \zeta_3^b = \frac{\zeta_b}{(1 - \tilde{\tau})} \qquad \zeta_4^b = -\frac{(1 - \theta)\zeta_b}{(1 - \tilde{\tau})}$$

$$\zeta_5^b = \frac{Y^{\varphi+1}\left(1 - \varphi\zeta_b\right)}{1 - \beta\theta} \qquad \zeta_7^b = -Y^\varphi \varphi \left(\zeta_w - \zeta_b\right) \qquad \zeta_8^b = \frac{(\zeta_w - \zeta_b)}{(1 - \tilde{\tau})} \qquad (77)$$

where $\zeta_b \equiv \frac{1}{2}\frac{\tilde{\tau}}{\sigma+\varphi} - \frac{\delta_b - 1 + (1/2)\tilde{\tau}}{(1 - 2\gamma_b)\sigma + (1 - 2\delta_b)\varphi}$ and $\zeta_w \equiv \frac{\tilde{\tau}}{\sigma+\varphi}$ Hence, being the best response for both monetary union and the small open economy policy makers, zero inflation is a Nash equilibrium solution under regime A.

Consider now the case of regime B and suppose that the central bank of area H set $\Pi_{H,t} = 1$ for all t. In this case, given the symmetry of area H small open economies, it has to be that $\Pi_{i,t} = 1$ for all t and $i \in [0, \frac{1}{2})$. Hence, under regime B the optimal policy problem of the policy maker of the area F is identical to the one stated in (75) for regime A and zero inflation is a best response of policymaker in area F to a zero-inflation policy of the policy maker in the area H. However, under regime B, the optimal policy problem of the policy maker in area H is symmetric to the one of area F. Thus, we can conclude that zero inflation is a Nash equilibrium policy at the deterministic steady state even under regime B.

D   The purely quadratic approximations to the welfare and the optimal policies under regimes A, B and C

In order to recover the optimal policies, we need to approximate up to the second order single country representative agent utility given by (1).

D.1   The second-order approximation to the utility

Recall that according to (1), the period t utility is $U_t \equiv \frac{C_t^{i\,1-\sigma}}{1-\sigma} - \frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1}$. First, we approximate the utility derived from private consumption for generic region i as:

$$\frac{C_t^{i\,1-\sigma}}{1-\sigma} \simeq \frac{C^{1-\sigma}}{1-\sigma} + C^{1-\sigma}\left(\hat{c}_t^i + \frac{1}{2}(\hat{c}_t^i)^2\right) - \frac{\sigma}{2}C^{1-\sigma}(\hat{c}_t^i)^2 + t.i.p. \qquad (78)$$

where $\hat{c}_t^i$ stands for the log-deviations of private consumption from the deterministic steady state and $t.i.p.$ stand for "terms independent of policy". Then, we approximate labor disutility. Since $Z_t^i = \int_0^1 \left(\frac{p_t(h^i)}{P_{i,t}}\right)^{-\varepsilon} dh^i$, as showed by Galí and Monacelli (2005), it follows that:

$$\hat{z}_t^i \simeq \frac{\varepsilon}{2}Var_{h^i}(p_t(h^i)) \qquad (79)$$

In words, the approximation of Z_t^i around the symmetric steady state is purely quadratic. Moreover, following Woodford (2001, NBER WP8071) it is possible to show that $\sum_{t=0}^{\infty} \beta^t Var_{h^i}(p_t(h^i)) = \frac{1}{\lambda}\sum_{t=0}^{\infty} \beta^t \pi_{i,t}^2$ with $\lambda \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta}$. Therefore, labor disutility can be approximated up to the second order as:

$$\frac{1}{\varphi+1}\left(\frac{Y_t^i Z_t^i}{A_t^i}\right)^{\varphi+1} \simeq \frac{1}{\varphi+1}Y^{\varphi+1} + Y^{\varphi+1}\left(\hat{y}_t^i + \frac{1}{2}(\hat{y}_t^i)^2\right) + Y^{\varphi+1}\frac{\varepsilon}{2\lambda}(\pi_{i,t})^2 + \frac{\varphi}{2}Y^{\varphi+1}(\hat{y}_t^i)^2$$

$$- (\varphi+1)Y^{\varphi+1}\hat{y}_t^i a_t^i + t.i.p. \tag{80}$$

By combining (78) and (80) and recalling that at the steady state $C^{-\sigma} = (1-\tilde{\tau})Y^{\varphi}$, we can express the second-order approximation of U_t as fraction of steady state consumption in the following way:

$$\frac{U_t - U}{U_C C} \simeq \frac{1}{1-\tilde{\tau}}\left[(1-\tilde{\tau})\hat{c}_t^i - (1-\tilde{\tau})\frac{(\sigma-1)}{2}(\hat{c}_t^i)^2 - \hat{y}_t^i - \frac{\varphi+1}{2}(\hat{y}_t^i)^2 - \frac{\varepsilon}{2\lambda}(\pi_{i,t})^2\right] \tag{81}$$

Therefore, the second-order approximation to the welfare of the region i representative agent is given by:

$$\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[(1-\tilde{\tau})\hat{c}_t^i - (1-\tilde{\tau})\frac{(\sigma-1)}{2}(\hat{c}_t^i)^2 - \hat{y}_t^i - \frac{\varphi+1}{2}(\hat{y}_t^i)^2 - \frac{\varepsilon}{2\lambda}(\pi_{i,t})^2\right] + t.i.p. \tag{82}$$

D.2 The case of the small open economy

The expression in (82) can be rewritten in matrix form as:

$$\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\hat{s}_t^{i'}w_s - \frac{1}{2}\hat{s}_t^{i'}W_{s,s}\hat{s}_t^i + \hat{s}_t^{i'}W_{s,e}\hat{e}_t^i\right] + t.i.p. \tag{83}$$

where:

$$\hat{s}_t^{i'} \equiv \left[\hat{y}_t^i,\ \hat{c}_t^i,\ \pi_{i,t}\right] \qquad w_s' \equiv [-1,\ (1-\tilde{\tau}),\ 0] \qquad \hat{e}_t^{i'} \equiv \left[\hat{c}_{H,t},\ \hat{c}_{F,t},\ \hat{a}_t^i,\ \hat{\mu}_t^i\right]$$

$$W_{s,s} \equiv \begin{bmatrix} (\varphi+1) & 0 & 0 \\ 0 & (1-\tilde{\tau})(\sigma-1) & 0 \\ 0 & 0 & \frac{\varepsilon}{\lambda} \end{bmatrix} \qquad W_{s,e} \equiv \begin{bmatrix} 0 & 0 & (\varphi+1) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and with $i \in \left[0, \frac{1}{2}\right)$, $\hat{c}_{H,t} \equiv 2\int_0^{\frac{1}{2}}\hat{c}_t^j dj$ and $\hat{c}_{F,t} \equiv 2\int_{\frac{1}{2}}^1 \hat{c}_t^j dj$. In order to recover a purely quadratic approximation to the welfare of the small open economy, we have to use the second-order approximations both to the aggregate demand and the Phillips curves.

The second-order approximation to the demand curve can be written as:

$$0 \simeq \left[\hat{s}_t^{i'}g_s - \hat{e}_t^{i'}g_e + \frac{1}{2}\hat{s}_t^{i'}G_{s,s}\hat{s}_t^i - \hat{s}_t^{i'}G_{s,e}e_t^i\right] + t.o.c. \tag{84}$$

where:

$$g_s' \equiv [-1,\ \delta_s,\ 0] \qquad g_e' \equiv [-(\delta_b - \delta_s),\ -(1-\delta_b),\ 0,\ 0]$$

$$G_{s,s} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & \delta_s + \omega_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s,e} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega_1 + \omega_2 & -\omega_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $\delta_s \equiv \alpha_s(1-\eta\sigma) + \eta\sigma\gamma_s$, $\delta_b \equiv \alpha_b(1-\eta\sigma) + \gamma_b\eta\sigma$, $\gamma_s \equiv \frac{1}{\alpha_s}$ and $\gamma_b \equiv \frac{\alpha_b}{2\alpha_b-1}$, while:

$$\omega_1 \equiv \frac{(1-\alpha_s)\eta\sigma(\sigma - (1-\alpha_s)\alpha_s(1-\eta\sigma))}{\alpha_s^2}$$

$$\omega_2 \equiv \frac{(1-\alpha_b)\eta\sigma\left(\sigma + \left(\alpha_s^2 + (1-2\alpha_b)\right)(1-\eta\sigma)\right)}{\alpha_s(1-2\alpha_b)}$$

Moreover $t.o.c.$ stands for terms out of control of the policy maker which include average area variables besides the terms independent of policy.

As in Benigno and Woodford (2005), the second-order approximation to (35) can be combined with (37) and (38) to obtain the second-order approximation to the Phillips curve:

$$V_0 = \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\left[\hat{s}_t^{i'}v_s - \hat{e}_t^{i'}v_e + \frac{1}{2}\hat{s}_t^{i'}V_{s,s}\hat{s}_t^i - \hat{s}_t^{i'}V_{s,e}\hat{e}_t^i\right] + t.o.c. \quad (85)$$

where

$$v_s' \equiv [\varphi,\ \sigma\gamma_s,\ 0] \qquad v_e' \equiv [\sigma(\gamma_s - \gamma_b),\ -\sigma(1-\gamma_b),\ -(\varphi+1),\ 1]$$

$$V_{s,s} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & -\eta\sigma^2(\gamma_s-1)\gamma_s - \sigma^2\gamma_s & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix}$$

$$V_{s,e} \equiv \begin{bmatrix} -\sigma(\gamma_b-\gamma_s) & -\sigma(1-\gamma_b) & (\varphi+1)^2 & -(\varphi+1) \\ \eta\sigma^2\gamma_s(\gamma_b-\gamma_s) & \eta\sigma\gamma_s^2(1-\gamma_b) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Given (84) and (85), it is possible to rewrite (83) in a purely quadratic way. Indeed, it is easy to show that:

$$w_s = (1-\varphi\zeta_s)g_s - \zeta_s v_s$$

where $\zeta_s = \frac{\delta_s - (1-\tilde{\tau})}{\delta_s\varphi+\gamma_s\sigma}$.[48] Then, we can write the second-order approximation to region i welfare as:

$$-\frac{1}{(1-\tilde{\tau})}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}_t^{i'}\Omega_{s,s}\hat{s}_t^i - \hat{s}_t^{i'}\Omega_{s,e}\hat{e}_t^i\right] + t.o.c. \quad (86)$$

where:

$$\Omega_{s,s} \equiv W_{s,s} + (1-\varphi\zeta_s)G_{s,s} - \zeta_s V_{s,s} \qquad\qquad \Omega_{s,e} \equiv W_{s,e} + (1-\varphi\zeta_s)G_{s,e} - \zeta_s V_{s,e}$$

and $\Omega_{s,s}$ and $\Omega_{s,e}'$ are respectively equal to:

$$\begin{bmatrix} (1-\zeta_s(\varphi+1))\varphi & -\zeta_s\gamma_s\sigma & 0 \\ -\zeta_s\gamma_s\sigma & (1-\tilde{\tau})(\sigma-1) - \zeta_s\sigma^2\eta\gamma_s(1-\gamma_s) + \zeta_s\sigma^2\gamma_s + (1-\zeta_s\varphi)(\delta_s+\omega_1) & 0 \\ 0 & 0 & \frac{(1-\zeta_s(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}$$

$$\begin{bmatrix} \zeta_s\sigma(\gamma_b-\gamma_s) & -\zeta_s\eta\sigma^2\gamma_s(\gamma_b-\gamma_s) + (1-\zeta_s\varphi)(\omega_1+\omega_2) & 0 \\ \zeta_s\sigma(1-\gamma_b) & -\zeta_s\eta\sigma^2\gamma_s(1-\gamma_b) - (1-\zeta_s\varphi)\omega_2 & 0 \\ (1-\zeta_s(\varphi+1))(\varphi+1) & 0 & 0 \\ \zeta_s(\varphi+1) & 0 & 0 \end{bmatrix}$$

---

[48]Notice that ζ_s determines also the lagrange multipliers previously recovered for the optimal policy problem of the small economy policy maker. See Benigno and Woodford (2005).

Alternatively, (86) can be written as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\Big[\frac{1}{2}\varpi_{1,s}(\hat{y}_t^i)^2 + \varpi_{2,s}\hat{c}_t^i\hat{y}_t^i + \frac{1}{2}\varpi_{3,s}(\hat{c}_t^i)^2 + \frac{1}{2}\varpi_{4,s}(\pi_{i,t})^2 - \varpi_{5,s}\hat{y}_t^i\hat{c}_{H,t}$$

$$-\varpi_{6,s}\hat{y}_t^i\hat{c}_{F,t} - \varpi_{7,s}\hat{c}_t^i\hat{c}_{H,t} - \varpi_{8,s}\hat{c}_t^i\hat{c}_{F,t} - \varpi_{9,s}\hat{y}_t^i\hat{a}_t^i - \varpi_{10,s}\hat{y}_t^i\hat{\mu}_t^i\Big] + t.o.c. \qquad (87)$$

where:

$$\varpi_{1,s} \equiv [1 - \zeta_s(\varphi+1)]\,\varphi$$

$$\varpi_{2,s} \equiv -\zeta_s\gamma_s\sigma$$

$$\varpi_{3,s} \equiv (1-\tilde{\tau})(\sigma-1) - \zeta_s\sigma^2\eta\gamma_s(1-\gamma_s) + \zeta_s\sigma^2\gamma_s + (1-\zeta_s\varphi)(\delta_s+\omega_1)$$

$$\varpi_{4,s} \equiv [1 - \zeta_s(\varphi+1)]\frac{\varepsilon}{\lambda}$$

$$\varpi_{5,s} \equiv \zeta_s\sigma(\gamma_b - \gamma_s)$$

$$\varpi_{6,s} \equiv \zeta_s\sigma(1-\gamma_b)$$

$$\varpi_{7,s} \equiv -\zeta_s\eta\sigma^2\gamma_s(\gamma_b-\gamma_s) + (1-\zeta_s\varphi)(\omega_1+\omega_2)$$

$$\varpi_{8,s} \equiv -\zeta_s\eta\sigma^2\gamma_s(1-\gamma_b) - (1-\zeta_s\varphi)\,\omega_2$$

$$\varpi_{9,s} \equiv [1 - \zeta_s(\varphi+1)](\varphi+1)$$

$$\varpi_{10,s} \equiv \zeta_s(\varphi+1) \qquad (88)$$

The next step is to express the approximations in (86) in terms of deviations from the policy target of the small open economy policy maker. This target can be determined by maximizing (86) subject to good market clearing written as in (32), i.e.,:

$$\hat{y}_t^i = \delta_s\hat{c}_t^i + (\delta_b - \delta_s)\hat{c}_{H,t} + (1-\delta_b)\hat{c}_{F,t} \qquad i \in [0, \tfrac{1}{2}) \qquad (89)$$

The Lagrangian associated to this problem can be written as:

$$L^s = \sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}_t^{s'}\Omega_{s,s}\hat{s}_t^s - \hat{s}_t^{s'}\Omega_{s,e}\hat{e}_t^i + \phi_t^s\left(\hat{s}_t^{s'}g_s - \hat{e}_t^{i'}g_e\right)\right] \qquad (90)$$

where $\hat{s}_t^s$ indicates the target for $\hat{s}_t^i$. The first-order conditions of L^s with respect to $\hat{s}_t^{s'}$ and ϕ_t^s can be read as:

$$\Omega_{s,s}\hat{s}_t^s - \Omega_{s,e}\hat{e}_t^i = -\phi_t^s g_s$$

$$g_s\hat{s}_t^s - g_e\hat{e}_t^i = 0 \qquad (91)$$

Alternatively, condition (91) can be read as:

$$\varpi_{1,s}\hat{y}_t^s + \varpi_{2,s}\hat{c}_t^s - \varpi_{5,s}\hat{c}_{H,t} - \varpi_{6,s}\hat{c}_{F,t} - \varpi_{9,s}\hat{a}_t^i - \varpi_{10,s}\hat{\mu}_t^i = \phi_t^s$$

$$\varpi_{3,s}\hat{c}_t^s + \varpi_{2,s}\hat{y}_t^s - \varpi_{7,s}\hat{c}_{H,t} - \varpi_{8,s}\hat{c}_{F,t} = -\delta_s\phi_t^s$$

$$\varpi_{4,s}\pi_{i,t} = 0$$

$$\hat{y}_t^s = \delta_s\hat{c}_t^s + (\delta_b - \delta_s)\hat{c}_{H,t} + (1-\delta_b)\hat{c}_{F,t} \qquad (92)$$

for all $i \in [0, \tfrac{1}{2})$ and where ϕ_t^s is the lagrange multiplier of (89). Notice that from the perspective of the small open monetary authority $\hat{c}_{H,t}$ and $\hat{c}_{F,t}$ are taken as exogenous. Then, it can be shown that by using equations (20),[49] conditions in (92) can be

---

[49]In fact, given the system of equations in (92) – which allows us to determine the target for $\hat{c}_t^i$ $\hat{y}_t^i$ and π_t^i –, we can recover the target for $\hat{s}_{iH,t}$ and $\hat{s}_{iF,t}$, using conditions (20).

rewritten as:

$$[1 - \zeta_s(\varphi + 1)]\widehat{mc}_{i,t}^{e,s} = \zeta_s(\varphi + 1)\hat{\mu}_t^i + \kappa_H^s \hat{s}_{iH,t}^s + \kappa_F^s \hat{s}_{iF,t}^s$$

$$\hat{y}_t^s = \hat{c}_t^s + \left[\frac{1 - \delta_b}{\sigma(2\gamma_b - 1)} - \frac{1 - \delta_s}{\sigma\gamma_s}\right]\hat{s}_{iH,t}^s - \frac{1 - \delta_b}{\sigma(2\gamma_b - 1)}\hat{s}_{iF,t}^s$$

$$\hat{s}_{iF,t}^s = \hat{s}_{iH,t}^s - \sigma(2\gamma_b - 1)(\hat{c}_{F,t} - \hat{c}_{H,t})$$

$$\hat{s}_{iH,t}^s = \kappa_a^s \hat{a}_t^i + \kappa_\mu^s \hat{\mu}_t^i + \kappa_{cH}^s \hat{c}_{H,t} + \kappa_{cF}^s \hat{c}_{F,t} \tag{93}$$

where $\widehat{mc}_t^{e,s} \equiv \varphi\hat{y}_t^s + \sigma\hat{c}_t^s - (1 + \varphi)\hat{a}_t^i + (\alpha_b - \alpha_s)\hat{s}_{iH,t}^s + (1 - \alpha_b)\hat{s}_{iF,t}^s$, while κ_H^s, κ_F^s, κ_a^s, κ_μ^s, κ_{cH}^s and κ_{cF}^s are defined as:

$$\kappa_H^s \equiv -\frac{a_1^s}{\sigma\gamma_s\delta_s} + \frac{a_2^s}{\sigma(2\gamma_b - 1)\delta_s} + \left(\frac{\gamma_b}{2\gamma_b - 1} - \frac{1}{\gamma_s}\right)[1 - \zeta_s(\varphi + 1)]$$

$$\kappa_F^s \equiv -\frac{a_2^s}{\sigma(2\gamma_b - 1)\delta_s} - \frac{1 - \gamma_b}{2\gamma_b - 1}[1 - \zeta_s(\varphi + 1)]$$

$$\kappa_a^s \equiv \frac{1}{a_3^s}[1 - \zeta_s(\varphi + 1)](1 + \varphi)$$

$$\kappa_\mu^s \equiv \frac{1}{a_3^s}\zeta_s(1 + \varphi)$$

$$\kappa_{cH}^s \equiv -\frac{1}{a_3^s}\left[\frac{a_2^s}{\delta_s} + (1 - \zeta_s(1 + \varphi))(\sigma + \varphi\delta_b)\right]$$

$$\kappa_{cF}^s \equiv \frac{1}{a_3^s}\left[\frac{a_2^s}{\delta_s} - (1 - \zeta_s(1 + \varphi))(1 - \delta_b)\varphi\right] \tag{94}$$

with:

$$a_1^s \equiv -\zeta_s\sigma^2\eta\gamma_s(1 - \gamma_s) + (1 - \zeta_s\varphi)\omega_1 + \zeta_s\sigma\delta_s(1 - \gamma_s) + \zeta_s\sigma\gamma_s(1 - \delta_s)$$

$$a_2^s \equiv -\zeta_s\sigma^2\eta\gamma_s(1 - \gamma_b) - (1 - \zeta_s\varphi)\omega_2 + \zeta_s\sigma\delta_s(1 - \gamma_b) + \zeta_s\sigma\gamma_s(1 - \delta_b)$$

$$a_3^s \equiv [1 - \zeta_s(1 + \varphi)]\left[(\varphi + \sigma)\frac{\delta_s}{\gamma_s\sigma} + \frac{(1 - \delta_s)}{\gamma_s}\right] + \frac{a_1^s}{\gamma_s\delta_s\sigma} \tag{95}$$

The first condition in (93) expresses the target for the fluctuations of the efficient firms' marginal cost as a function of mark-up shocks and two terms of trade that are relevant for small open economy viewpoint; the second and the third conditions rewrite the market clearing equation (89) and condition (20) using conditions (20) and (21). Finally, the last condition expresses $\hat{s}_{iH,t}^s$ in terms of the exogenous shocks. By using (20) and (92), we can rewrite (58) in terms of gaps as:[50]

$$-\frac{1}{(1 - \tilde{\tau})}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\varpi_{1,s}(\tilde{y}_t^s)^2 + \frac{1}{2}\varpi_{11,s}(\tilde{s}_{iH,t}^s)^2 + \frac{1}{2}\varpi_{12,s}(\tilde{s}_{iF,t}^s)^2 + \frac{1}{2}\varpi_{4,s}(\pi_{i,t})^2\right] + t.o.c. \tag{96}$$

---

[50]To recover condition (87) first we rewrote the approximation in (58) in deviations from the target using (92). Then, we use the conditions in (20) in deviations from the target to express the welfare approximation as a function solely of $\tilde{y}_t^s$, $\tilde{s}_{iH,t}^s$, $\tilde{s}_{iF,t}^s$ and $\pi_{i,t}$.

where

$$\varpi_{11,s} \equiv \frac{1}{\sigma\gamma_s} \left[2\varpi_{2,s} \left(\frac{\delta_s}{\sigma\gamma_s} + \frac{1-\delta_b}{\sigma(2\gamma_b - 1)} \right) + \frac{1}{\sigma^2\gamma_s^2} \varpi_{3,s} \right]$$

$$\varpi_{12,s} \equiv -2 \frac{\varpi_{2,s}}{\sigma\gamma_s} \frac{1-\delta_b}{\sigma(2\gamma_b - 1)} \tag{97}$$

The *timelessly* optimal monetary policy can be retrieved by maximizing (96) with respect to $\tilde{y}_t^s$, $\tilde{s}_{iH,t}^s$, $\tilde{s}_{iF,t}^s$ and $\pi_{i,t}$ subject to the following sequence of constraints:

$$\tilde{y}_t^s = \left[\frac{\delta_s}{\sigma\gamma_s} + \frac{1-\delta_b}{\sigma(2\gamma_b - 1)} \right] \tilde{s}_{iH,t}^s - \frac{1-\delta_b}{\sigma(2\gamma_b - 1)} \tilde{s}_{iF,t}^s$$

$$\pi_{i,t} = \lambda \left[(\varphi + \sigma)\tilde{y}_t^s + \left(\frac{\gamma_s - \delta_s}{\gamma_s} - \frac{\gamma_b - \delta_b}{2\gamma_b - 1} \right) \tilde{s}_{iH,t}^s + \frac{\gamma_b - \delta_b}{2\gamma_b - 1} \tilde{s}_{iF,t}^s \right] + \beta E_t\{\pi_{i,t+1}\} + \upsilon_{i,t}^s \tag{98}$$

for all t where:

$$\upsilon_{i,t}^s = \frac{\lambda}{1 - \zeta_s(\varphi + 1)} \left[\left(1 + \kappa_\mu^s(\kappa_H^s + \kappa_F^s) \right) \hat{\mu}_t^i + \left(\kappa_a^s(\kappa_H^s + \kappa_F^s) \right) \hat{a}_t^i \right.$$

$$\left. + \left(\kappa_{cH}^s(\kappa_H^s + \kappa_F^s) + \kappa_F^s\sigma(2\gamma_b - 1) \right) \hat{c}_{H,t} + \left(\kappa_{cF}^s(\kappa_H^s + \kappa_F^s) - \kappa_F^s\sigma(2\gamma_b - 1) \right) \hat{c}_{F,t} \right] \tag{99}$$

and the constraint on $\pi_{i,0}$ implied by the timeless perspective. The constraints in (98) are recovered from (89) and (39), using again (20). The Lagrangian of the optimal monetary policy problem of the small open economy can then be written as:

$$\mathcal{L}^s = \sum_{t=0}^{\infty} \beta^t E_0 \left\{ \left[\frac{1}{2}\varpi_{1,s}(\tilde{y}_t^s)^2 + \frac{1}{2}\varpi_{11,s}(\tilde{s}_{iH,t}^s)^2 + \frac{1}{2}\varpi_{12,s}(\tilde{s}_{iF,t}^s)^2 + \frac{1}{2}\varpi_{4,s}(\pi_{i,t})^2 \right] \right.$$

$$+ \psi_{1,t}^s \left[\pi_{i,t} - \lambda \left((\varphi + \sigma)\tilde{y}_t^s + \left(\frac{\gamma_s - \delta_s}{\gamma_s} - \frac{\gamma_b - \delta_b}{2\gamma_b - 1} \right) \tilde{s}_{iH,t}^s + \frac{\gamma_b - \delta_b}{2\gamma_b - 1} \tilde{s}_{iF,t}^s \right) - \upsilon_{i,t}^s \right] - \psi_{1,t-1}^s\pi_{i,t}$$

$$\left. + \psi_{2,t}^s \left[\tilde{y}_t^s - \left(\frac{\delta_s}{\sigma\gamma_s} + \frac{1-\delta_b}{\sigma(2\gamma_b - 1)} \right) \tilde{s}_{iH,t}^s + \frac{1-\delta_b}{\sigma(2\gamma_b - 1)} \tilde{s}_{iF,t}^s \right] \right\} \tag{100}$$

Minimizing $\mathcal{L}^s$ with respect to $\tilde{y}_t^s$, $\tilde{s}_{iH,t}^s$, $\tilde{s}_{iF,t}^s$ and $\pi_{i,t}$ lead to the following first-order conditions:

$$\varpi_{1,s}\tilde{y}_t^s = \psi_{1,t}^s\lambda(\varphi + \sigma) - \psi_{2,t}^s$$

$$\varpi_{11,s}\tilde{s}_{iH,t}^s = \psi_{1,t}^s\lambda \left(\frac{\gamma_s - \delta_s}{\gamma_s} - \frac{\gamma_b - \delta_b}{2\gamma_b - 1} \right) + \psi_{2,t}^s \left(\frac{\delta_s}{\sigma\gamma_s} + \frac{1-\delta_b}{\sigma(2\gamma_b - 1)} \right)$$

$$\varpi_{12,s}\tilde{s}_{iF,t}^s = \psi_{1,t}^s\lambda\frac{\gamma_b - \delta_b}{2\gamma_b - 1} - \psi_{2,t}^s\frac{1-\delta_b}{\sigma(2\gamma_b - 1)}$$

$$\varpi_{4,s}\pi_t^i = -\left(\psi_{1,t}^s - \psi_{1,t-1}^s \right) \tag{101}$$

D.3 The case of the monetary union

By (82), the second-order approximation to the average welfare of the household living in the area H can be read as:

$$\frac{1}{(1-\tilde{\tau})}\sum_{t=0}^{\infty}\beta^t 2 \int_0^{\frac{1}{2}} E_0 \left[\hat{s}_t^{i'}w_s - \frac{1}{2}\hat{s}_t^{i'}W_{s,s}\hat{s}_t^i + \hat{s}_t^{i'}W_{s,u}\hat{u}_t^i \right] di + t.i.p. \tag{102}$$

53

$$\hat{s}_t^{i'} \equiv \begin{bmatrix} \hat{y}_t^i, & \hat{c}_t^i, & \pi_{i,t} \end{bmatrix} \qquad w_s' \equiv \begin{bmatrix} -1, & (1-\tilde{\tau}), & 0 \end{bmatrix} \qquad \hat{u}_t^{i'} \equiv \begin{bmatrix} a_t^i, & \mu_t^i \end{bmatrix}$$

$$W_{s,s} \equiv \begin{bmatrix} (\varphi+1) & 0 & 0 \\ 0 & (1-\tilde{\tau})(\sigma-1) & 0 \\ 0 & 0 & \frac{\varepsilon}{\lambda} \end{bmatrix} \qquad W_{s,u} \equiv \begin{bmatrix} (\varphi+1) & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

As for the case of the small open economy, we can retrieve a purely quadratic approximation to the welfare of the households living in area H thanks to the second-order approximations to the demand and supply curves.

The second-order approximation to the demand curve of a generic region i in the area H can be read as:

$$0 \simeq \hat{s}_t^{i'} g_s + \int_0^{\frac{1}{2}} \hat{s}_t^i di' g_{S_H} + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' g_{S_F} + \frac{1}{2} \hat{s}_t^{i'} G_{s,s} \hat{s}_t^i di + \frac{1}{2} \int_0^{\frac{1}{2}} \hat{s}_t^{i'} G_{s_H,s_H} \hat{s}_t^i di$$

$$+ \frac{1}{2} \int_{\frac{1}{2}}^1 \hat{s}_t^{i'} G_{s_F,s_F} \hat{s}_t^i di + \frac{1}{2} \int_0^{\frac{1}{2}} \hat{s}_t^i di' G_{S_H,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \frac{1}{2} \int_{\frac{1}{2}}^1 \hat{s}_t^i di' G_{S_F,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di$$

$$+ \hat{s}_t^{i'} G_{s,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \hat{s}_t^{i'} G_{s,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' G_{S_H,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di + s.o.t.i.p. \quad (103)$$

where:

$$g_s' \equiv \begin{bmatrix} -1, & \delta_s, & 0 \end{bmatrix} \qquad g_{S_H}' \equiv \begin{bmatrix} 0, & 2(\delta_b - \delta_s), & 0 \end{bmatrix} \qquad g_{S_F}' \equiv \begin{bmatrix} 0, & 2(1-\delta_b), & 0, \end{bmatrix}$$

$$G_{s,s} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & \delta_s + \omega_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(1-\delta_b) + 2\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_s^2) + 2(\delta_b - \delta_s) - 2(\omega_1 + \omega_3) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4\left(\eta\sigma^2(1-\gamma_s^2) - \eta\sigma^2\gamma_b(1-\gamma_b) + 2\omega_1 + 2\omega_2 + \omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\left(\eta\sigma^2\gamma_b(1-\gamma_b) + \omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{s,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\left(\omega_1 + \omega_2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G_{s,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\omega_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad G_{S_H,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4\left(\eta\sigma^2\gamma_b(1-\gamma_b) - \omega_2\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and with:

$$\omega_3 \equiv \frac{(1-\alpha_b)\eta\sigma(\sigma + 2(1-\alpha_b)(1-\eta\sigma))}{1-2\alpha_b}$$

By integrating (103) over $i \in [0, \frac{1}{2})$, we obtain:

$$0 \simeq \int_0^{\frac{1}{2}} \hat{s}_t^i di' h_{S_H} + \int_{\frac{1}{2}}^1 \hat{s}_t^i di' h_{S_F} + \frac{1}{2} \int_0^{\frac{1}{2}} \hat{s}_t^{i'} H_{s_H,s_H} \hat{s}_t^i di + \frac{1}{2} \int_{\frac{1}{2}}^1 \hat{s}_t^{i'} H_{s_F,s_F} \hat{s}_t^i di$$

$$+ \frac{1}{2} \int_0^{\frac{1}{2}} \hat{s}_t^i di' H_{S_H,S_H} \int_0^{\frac{1}{2}} \hat{s}_t^i di + \frac{1}{2} \int_{\frac{1}{2}}^1 \hat{s}_t^i di' H_{S_F,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' H_{S_H,S_F} \int_{\frac{1}{2}}^1 \hat{s}_t^i di$$

$$+ s.o.t.i.p.$$

with

$$h'_{S_H} \equiv [-1,\ \delta_b,\ 0] \qquad h'_{S_F} \equiv [0,\ (1-\delta_b),\ 0,]$$

$$H_{s_H,s_H} \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\eta\sigma^2(1-\gamma_s^2)+\delta_b-\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1-\delta_b)+\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\eta\sigma^2(1-\gamma_s^2)-2\eta\sigma^2\gamma_b(1-\gamma_b)+2\omega_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\left(\eta\sigma^2(1-\gamma_b)\gamma_b+2\omega_3\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad H_{S_H,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\eta\sigma^2\gamma_b(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A symmetric approximation can be retrieved for the good market-clearing condition of the regions in area F, namely:

$$0 \simeq \int_{\frac{1}{2}}^{1} \hat{s}_t^i di' f_{S_F} + \int_{0}^{\frac{1}{2}} \hat{s}_t^i di' f_{S_H} + \frac{1}{2}\int_{\frac{1}{2}}^{1} \hat{s}_t^{i'} F_{s_F,s_F} \hat{s}_t^i di + \frac{1}{2}\int_{0}^{\frac{1}{2}} \hat{s}_t^{i'} F_{s_H,s_H} \hat{s}_t^i di$$

$$+\frac{1}{2}\int_{\frac{1}{2}}^{1} \hat{s}_t^i di' F_{S_F,S_F} \int_{\frac{1}{2}}^{1} \hat{s}_t^i di + \frac{1}{2}\int_{0}^{\frac{1}{2}} \hat{s}_t^i di' F_{S_H,S_H} \int_{0}^{\frac{1}{2}} \hat{s}_t^i di + \int_{\frac{1}{2}}^{1} \hat{s}_t^i di' F_{S_F,S_H} \int_{0}^{\frac{1}{2}} \hat{s}_t^i di$$

$$+s.o.t.i.p.$$

where $f_{S_F} = h_{S_H}$, $f_{S_H} = h_{S_F}$, $F_{s_F,s_F} = H_{s_H,s_H}$, $F_{s_H,s_H} = H_{s_F,s_F}$, $F_{S_F,S_F} = H_{S_H,S_H}$, $F_{S_H,S_H} = H_{S_F,S_F}$ and $F_{S_F,S_H} = H_{S_H,S_F}$. Conversely, the second-order approximation to the Phillips curve for the area F can be recovered:

$$\begin{aligned} V_0 &= \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\hat{s}_t^{i'}v_s + \int_{\frac{1}{2}}^{1}\hat{s}_t^i di' v_{S_F} + \int_{0}^{\frac{1}{2}}\hat{s}_t^i di' v_{S_H} - \hat{u}_t^{i'}v_u + \frac{1}{2}\hat{s}_t^{i'}V_{s,s}\hat{s}_t^i \\ &\quad +\hat{s}_t^{i'}V_{s,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \hat{s}_t^{i'}V_{s,S_H}\int_{0}^{\frac{1}{2}}\hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_{t'}^{i}V_{S_F,s_F}\hat{s}_t^i di + \frac{1}{2}\int_{0}^{\frac{1}{2}}\hat{s}_t^{i'}V_{s_H,s_H}\hat{s}_t^i di \\ &\quad +\frac{1}{2}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di' V_{S_F,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di + \frac{1}{2}\int_{0}^{\frac{1}{2}}\hat{s}_t^i di' V_{S_H,S_H}\int_{0}^{\frac{1}{2}}\hat{s}_t^i di + \int_{0}^{\frac{1}{2}}\hat{s}_t^i di' V_{S_H,S_F}\int_{\frac{1}{2}}^{1}\hat{s}_t^i di \\ &\quad -\hat{s}_t^{i'}V_{s,u}\hat{u}_t^i\Big] + s.o.t.i.p. \end{aligned} \tag{104}$$

where:

$$v_s' \equiv [\varphi,\ \sigma\gamma_s,\ 0] \quad v_{S_F}' \equiv [0,\ 2\sigma(\gamma_b-\gamma_s),\ 0] \quad v_{S_H}' \equiv [0,\ 2\sigma(1-\gamma_b),\ 0] \quad v_u' \equiv [\,(\varphi+1),\ -1]$$

$$V_{s,s} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & \eta\sigma^2(1-\gamma_s)\gamma_s-\sigma^2\gamma_s & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix} \qquad V_{s,S_F} \equiv \begin{bmatrix} 0 & 2\sigma(1-\gamma_b) & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_b)\gamma_s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s,S_H} \equiv \begin{bmatrix} 0 & 2\sigma(\gamma_b-\gamma_s) & 0 \\ 0 & 2\eta\sigma^2\gamma_s(\gamma_s-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad V_{s_F,s_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(\eta-1)\sigma^2(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2(\eta-1)\sigma^2(\gamma_b-\gamma_s) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad V_{S_F,S_F} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(1-\gamma_b)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(\gamma_b-\gamma_s)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad V_{S_F,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4\eta\sigma^2(1-\gamma_b)(\gamma_b-\gamma_s) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V_{s,u} \equiv \begin{bmatrix} (\varphi+1)^2 & -(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By integrating (104) over $i \in [\frac{1}{2}, 1]$, we find that:

$$
\begin{aligned}
\frac{1}{2}V_0 &= \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\int_{\frac{1}{2}}^1 \hat{s}_t^i di' r_{S_F} + \int_0^{\frac{1}{2}} \hat{s}_t^i di' r_{S_H} - \int_{\frac{1}{2}}^1 \hat{u}_t^i di' r_u + \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^{i'} R_{s_F,s_F}\hat{s}_t^i di \\
&+ \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^{i'} R_{s_H,s_H}\hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^i di' R_{S_F,S_F}\int_{\frac{1}{2}}^1 \hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^i di' R_{S_H,S_H}\int_0^{\frac{1}{2}} \hat{s}_t^i di \\
&+ \int_0^{\frac{1}{2}} \hat{s}_t^i di' R_{S_F,S_H}\int_{\frac{1}{2}}^1 \hat{s}_t^i di - \int_{\frac{1}{2}}^1 \hat{s}_t^{i'} R_{s_F,u}\hat{u}_t^i di\Big] + s.o.t.i.p. \quad (105)
\end{aligned}
$$

where:

$$r'_{S_F} \equiv [\varphi, \ \sigma\gamma_b, \ 0] \qquad r'_{S_H} \equiv [0, \ \sigma(1-\gamma_b), \ 0] \qquad r'_u \equiv [(\varphi+1), \ -1]$$

$$R_{s_F,s_F} \equiv \begin{bmatrix} \varphi(\varphi+2) & \sigma\gamma_s & 0 \\ \sigma\gamma_s & -\eta\gamma_s^2\sigma^2 + \eta\gamma_b\sigma^2 - \gamma_b\sigma^2 & 0 \\ 0 & 0 & \frac{\varepsilon(\varphi+1)}{\lambda} \end{bmatrix} \quad R_{s_H,s_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & (\eta-1)\sigma^2(1-\gamma_b) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{S_F,S_F} \equiv \begin{bmatrix} 0 & 2\sigma(\gamma_b-\gamma_s) & 0 \\ 2\sigma(\gamma_b-\gamma_s) & 2\eta\sigma^2(\gamma_s^2-\gamma_b^2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_{S_H,S_H} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_b)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{S_F,S_H} \equiv \begin{bmatrix} 0 & 2\sigma(1-\gamma_b) & 0 \\ 0 & -2\eta\sigma^2(1-\gamma_b)\gamma_b & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_{s_F,u} \equiv \begin{bmatrix} (\varphi+1)^2 & -(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Again, a symmetric condition can be retrieved for the regions of the area H, namely:

$$
\begin{aligned}
\frac{1}{2}V_0 &= \frac{1-\theta}{\theta}(1-\beta\theta)\sum_{t=0}^{\infty}\beta^t E_0\Big[\int_{\frac{1}{2}}^1 \hat{s}_t^i di' k_{S_H} + \int_0^{\frac{1}{2}} \hat{s}_t^i di' k_{S_F} - \int_{\frac{1}{2}}^1 \hat{u}_t^i di' k_u + \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^{i'} K_{s_H,s_H}\hat{s}_t^i di \\
&+ \frac{1}{2}\int_{\frac{1}{2}}^1 \hat{s}_t^i di' K_{S_H,S_H}\int_{\frac{1}{2}}^1 \hat{s}_t^i di + \frac{1}{2}\int_0^{\frac{1}{2}} \hat{s}_t^i di' K_{S_F,S_F}\int_0^{\frac{1}{2}} \hat{s}_t^i di + \int_0^{\frac{1}{2}} \hat{s}_t^i di' K_{S_H,S_F}\int_{\frac{1}{2}}^1 \hat{s}_t^i di \\
&- \int_{\frac{1}{2}}^1 \hat{s}_t^{i'} K_{s_H,u}\hat{u}_t^i di\Big] + s.o.t.i.p.
\end{aligned}
$$

with $k_{S_H} = r_{S_F}$, $k_{S_F} = r_{S_H}$, $k_u = r_u$, $K_{s_H,s_H} = R_{s_F,s_F}$, $K_{S_H,S_H} = R_{S_F,S_F}$, $K_{S_F,S_F} = R_{S_H,S_H}$, $K_{S_H,S_F} = R_{S_F,S_H}$ and $K_{s_H,u} = R_{s_F,u}$. Then, it can be shown that:

$$w_s = (1-\varphi\zeta_b)h_{S_H} - (\zeta_w-\zeta_b)\varphi f_{S_H} - \zeta_b k_{S_H} - (\zeta_w-\zeta_b)r_{S_H}$$

$$0 = (1-\varphi\zeta_b)h_{S_F} - (\zeta_w-\zeta_b)\varphi f_{S_F} - \zeta_b k_{S_F} - (\zeta_w-\zeta_b)r_{S_F}$$

56

with $\zeta_b = \frac{1}{2}\frac{\tilde{\tau}}{\sigma+\varphi} - \frac{\delta_b - 1 + (1/2)\tilde{\tau}}{(1-2\gamma_b)\sigma + (1-2\delta_b)\varphi}$ and $\zeta_w = \frac{\tilde{\tau}}{\sigma+\varphi}$. Hence, we can write the second-order approximation to the average welfare of area H as:

$$
-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega_{s_H,s_H}\hat{s}_t^i di + \int_{\frac{1}{2}}^1\hat{s}_t^{i'}\Omega_{s_F,s_F}\hat{s}_t^i di + 2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di\right.
$$

$$
\left. +2\int_{\frac{1}{2}}^1\hat{s}_t^i di'\Omega_{S_F,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di + 4\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di - 2\int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega_{s_H,u}\hat{u}_t^i di\right.
$$

$$
\left. -2\int_{\frac{1}{2}}^1\hat{s}_t^{i'}\Omega_{s_F,u}\hat{u}_t^i di\right] + t.o.c. \tag{106}
$$

where:

$$
\Omega_{s_H,s_H} \equiv W_{s,s} + (1-\varphi\zeta_b)H_{s_H,s_H} - (\zeta_w - \zeta_b)\varphi F_{s_H,s_H} - \zeta_b K_{s_H,s_H} - (\zeta_w - \zeta_b)R_{s_H,s_H}
$$

$$
\Omega_{s_F,s_F} \equiv (1-\zeta_b\varphi)H_{s_F,s_F} - (\zeta_w - \zeta_b)\varphi F_{s_F,s_F} - \zeta_b K_{s_F,s_F} - (\zeta_w - \zeta_b)R_{s_F,s_F}
$$

$$
\Omega_{S_H,S_H} \equiv \tfrac{1}{2}(1-\zeta_b\varphi)H_{S_H,S_H} - \tfrac{1}{2}(\zeta_w - \zeta_b)\varphi F_{S_H,S_H} - \tfrac{1}{2}\zeta_b K_{S_H,S_H} - \tfrac{1}{2}(\zeta_w - \zeta_b)R_{S_H,S_H}
$$

$$
\Omega_{S_F,S_F} \equiv \tfrac{1}{2}(1-\zeta_b\varphi)H_{S_F,S_F} - \tfrac{1}{2}(\zeta_w - \zeta_b)\varphi F_{S_F,S_F} - \tfrac{1}{2}\zeta_b K_{S_F,S_F} - \tfrac{1}{2}(\zeta_w - \zeta_b)R_{S_F,S_F}
$$

$$
\Omega_{S_H,S_F} \equiv \tfrac{1}{2}(1-\zeta_b\varphi)H_{S_H,S_F} - \tfrac{1}{2}(\zeta_w - \zeta_b)\varphi F'_{S_F,S_H} - \tfrac{1}{2}\zeta_b K_{S_H,S_F} - \tfrac{1}{2}(\zeta_w - \zeta_b)R'_{S_F,S_H}
$$

$$
\Omega_{s_H,u} \equiv W_{s,u} - \zeta_b K_{s_H,u} \qquad \Omega_{s_F,u} \equiv -(\zeta_w - \zeta_b)R_{s_F,u} \tag{107}
$$

and Ω_{s_H,s_H}, Ω_{s_F,s_F}, Ω_{S_H,S_H}, Ω_{S_F,S_F}, Ω_{S_H,S_F}, $\Omega_{s_H,u}$ and $\Omega_{s_F,u}$ are respectively equal to:

$$
\begin{bmatrix} (1-\zeta_b(\varphi+1))\varphi & -\zeta_b\sigma\gamma_s & 0 \\ -\zeta_b\sigma\gamma_s & \omega_{sHsH} & 0 \\ 0 & 0 & \frac{(1-\zeta_b(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}
$$

$$
\begin{bmatrix} -(\zeta_w-\zeta_b)(\varphi+1)\varphi & -(\zeta_w-\zeta_b)\sigma\gamma_s & 0 \\ -(\zeta_w-\zeta_b)\sigma\gamma_s & \omega_{sFsF} & 0 \\ 0 & 0 & -\frac{((\zeta_w-\zeta_b)(\varphi+1))\varepsilon}{\lambda} \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & -\zeta_b\sigma(\gamma_b-\gamma_s) & 0 \\ -\zeta_b\sigma(\gamma_b-\gamma_s) & \omega_{SHSH} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & -(\zeta_w-\zeta_b)\sigma(\gamma_b-\gamma_s) & 0 \\ -(\zeta_w-\zeta_b)\sigma(\gamma_b-\gamma_s) & \omega_{SFSF} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & -\zeta_b\sigma(1-\gamma_b) & 0 \\ -(\zeta_w-\zeta_b)\sigma(1-\gamma_b) & \omega_{SHSF} & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\begin{bmatrix} (1-\zeta_b(\varphi+1))(\varphi+1) & \zeta_b(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -(\zeta_w-\zeta_b)(\varphi+1)^2 & (\zeta_w-\zeta_b)(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

with:

$$
\begin{aligned}
\omega_{sHsH} \equiv\ & (\sigma-1)(1-\tilde{\tau}) \\
& + (1-\zeta_b\varphi)\left(-\eta\sigma^2(1-\gamma_s^2) + \delta_b - \omega_3\right) \\
& - (\zeta_w-\zeta_b)\varphi(1-\delta_b+\omega_3) \\
& - \zeta_b\left(-\eta\sigma^2\gamma_s^2 + \eta\sigma^2\gamma_b - \sigma^2\gamma_b\right) \\
& - (\zeta_w-\zeta_b)\left(\eta\sigma^2(1-\gamma_b) - \sigma^2(1-\gamma_b)\right)
\end{aligned}
$$

$$\omega_{sFsF} \equiv (1 - \zeta_b\varphi)(1 - \delta_b + \omega_3)$$
$$- (\zeta_w - \zeta_b)\varphi(-\eta\sigma^2\left(1 - \gamma_s^2\right) + \delta_b - \omega_3)$$
$$- \zeta_b\left(\eta\sigma^2\left(1 - \gamma_b\right) - \sigma^2\left(1 - \gamma_b\right)\right)$$
$$+ (\zeta_w - \zeta_b)\left(\eta\sigma^2\gamma_s^2 - \eta\sigma^2\gamma_b + \sigma^2\gamma_b\right)$$

$$\omega_{SHSH} \equiv (1 - \zeta_b\varphi)(\eta\sigma^2\left(1 - \gamma_s^2 - \gamma_b\left(1 - \gamma_b\right)\right) + \omega_3)$$
$$+ (\zeta_w - \zeta_b)\varphi(\eta\sigma^2\gamma_b\left(1 - \gamma_b\right) + \omega_3)$$
$$+ \zeta_b\eta\sigma^2\left(\gamma_b^2 - \gamma_s^2\right)$$
$$+ (\zeta_w - \zeta_b)\eta\sigma^2\left(1 - \gamma_b\right)^2$$

$$\omega_{SFSF} \equiv -(1 - \zeta_b\varphi)(\eta\sigma^2\gamma_b\left(1 - \gamma_b\right) + \omega_3)$$
$$- (\zeta_w - \zeta_b)\varphi(\eta\sigma^2\left(1 - \gamma_s^2 - \gamma_b\left(1 - \gamma_b\right)\right) + \omega_3)$$
$$+ \zeta_b\eta\sigma^2\left(1 - \gamma_b\right)^2$$
$$+ (\zeta_w - \zeta_b)\eta\sigma^2\left(\gamma_b^2 - \gamma_s^2\right)$$

$$\omega_{SHSF} \equiv (1 - \zeta_b\varphi)\eta\sigma^2\gamma_b\left(1 - \gamma_b\right)$$
$$- (\zeta_w - \zeta_b)\varphi\eta\sigma^2\gamma_b\left(1 - \gamma_b\right)$$
$$+ \zeta_b\eta\sigma^2\left(1 - \gamma_b\right)\gamma_b$$
$$+ (\zeta_w - \zeta_b)\eta\sigma^2\left(1 - \gamma_b\right)\gamma_b$$

Now, we rewrite the welfare approximation in (106) as:

$$-\frac{2}{1 - \tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\int_0^{\frac{1}{2}}\left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right)'\Omega_{s_H,s_H}\left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right)di\right.$$

$$+ \frac{1}{2}\int_{\frac{1}{2}}^1\left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right)'\Omega_{s_F,s_F}\left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right)di$$

$$- \int_0^{\frac{1}{2}}\left(\hat{s}_t^i - 2\int_0^{\frac{1}{2}} \hat{s}_t^i di\right)'\Omega_{s_H,u}\left(\hat{u}_t^i - 2\int_0^{\frac{1}{2}} \hat{u}_t^i di\right)di$$

$$- \int_{\frac{1}{2}}^1\left(\hat{s}_t^i - 2\int_{\frac{1}{2}}^1 \hat{s}_t^i di\right)'\Omega_{s_F,u}\left(\hat{u}_t^i - 2\int_{\frac{1}{2}}^1 \hat{u}_t^i di\right)di$$

$$+ \int_0^{\frac{1}{2}}\hat{s}_t^i di'(\Omega_{s_H,s_H} + \Omega_{S_H,S_H})\int_0^{\frac{1}{2}}\hat{s}_t^i di$$

$$+ \int_{\frac{1}{2}}^1\hat{s}_t^i di'(\Omega_{s_F,s_F} + \Omega_{S_F,S_F})\int_{\frac{1}{2}}^1\hat{s}_t^i di$$

$$+ 2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di$$

$$- 2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{s_H,u}\int_0^{\frac{1}{2}}\hat{u}_t^i di$$

$$\left.- 2\int_{\frac{1}{2}}^1\hat{s}_t^i di'\Omega_{s_F,u}\int_{\frac{1}{2}}^1\hat{u}_t^i di\right] + t.o.c. \qquad (108)$$

Notice that the components expressed as the difference between specific-country and average-union variables can be considered terms out of control of the policy maker (even if they should be taken into account in the welfare evaluation). Indeed, movements in the common nominal interest rate can just influence the average union economic performance. Thus, (108) can be read as:

$$-\frac{2}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\int_0^{\frac{1}{2}}\hat{s}_t^i di'(\Omega_{s_H,s_H}+\Omega_{S_H,S_H})\int_0^{\frac{1}{2}}\hat{s}_t^i di+\int_{\frac{1}{2}}^1\hat{s}_t^i di'(\Omega_{s_F,s_F}+\Omega_{S_F,S_F})\int_{\frac{1}{2}}^1\hat{s}_t^i di\right.$$

$$\left.+2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di-2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{s_H,u}\int_0^{\frac{1}{2}}\hat{u}_t^i di-2\int_{\frac{1}{2}}^1\hat{s}_t^i di'\Omega_{s_F,u}\int_{\frac{1}{2}}^1\hat{u}_t^i di\right]$$

$$+ t.o.c. \tag{109}$$

The last step consists in rewriting (109) in terms of gaps with respect to the target of the policymaker of the monetary union. In order to do so, first consider that by (109) it follows that:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\hat{s}_{H,t}'(\Omega_{s_H,s_H}+\Omega_{S_H,S_H})\hat{s}_{H,t}+\frac{1}{2}\hat{s}_{F,t}'(\Omega_{s_F,s_F}+\Omega_{S_F,S_F})\hat{s}_{F,t}\right.$$

$$\left.+\hat{s}_{H,t}'\Omega_{S_H,S_F}\hat{s}_{F,t}-\hat{s}_{H,t}'\Omega_{s_H,u}\hat{u}_{H,t}-\hat{s}_{F,t}'\Omega_{s_F,u}\hat{u}_{F,t}\right]+t.o.c. \tag{110}$$

Notice that the welfare approximation in (110) can be rewritten as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\varpi_{1,b}\hat{y}_{H,t}^2+\varpi_{2,b}\hat{c}_{H,t}\hat{y}_{H,t}+\frac{1}{2}\varpi_{3,b}\hat{c}_{H,t}^2+\frac{1}{2}\varpi_{4,b}\pi_{H,t}^2+\frac{1}{2}\varpi_{5,b}\hat{y}_{F,t}^2\right.$$

$$+\varpi_{6,b}\hat{c}_{F,t}\hat{y}_{F,t}+\frac{1}{2}\varpi_{7,b}\hat{c}_{F,t}^2+\varpi_{8,b}\hat{y}_{H,t}\hat{c}_{F,t}+\varpi_{9,b}\hat{y}_{F,t}\hat{c}_{H,t}+\varpi_{10,b}\hat{c}_{H,t}\hat{c}_{F,t}$$

$$\left.-\varpi_{11,b}\hat{y}_{H,t}\hat{a}_{H,t}-\varpi_{12,b}\hat{y}_{H,t}\hat{\mu}_{H,t}-\varpi_{13,b}\hat{y}_{F,t}\hat{a}_{F,t}-\varpi_{14,b}\hat{y}_{F,t}\hat{\mu}_{F,t}\right]+t.o.c. \tag{111}$$

where:

$$\varpi_{1,b}\equiv[1-\zeta_b(\varphi+1)]\varphi$$

$$\varpi_{2,b}\equiv-\zeta_b\gamma_b\sigma$$

$$\varpi_{3,b}\equiv(\sigma-1)(1-\tilde{\tau})+(1-\zeta_b\varphi)\delta_b-(\zeta_w-\zeta_b)\varphi(1-\delta_b)-\eta\sigma^2\gamma_b(1-\gamma_b)(1-\zeta_w\varphi+\zeta_w)$$

$$+\sigma^2[\gamma_b\zeta_b+(\zeta_w-\zeta_b)(1-\gamma_b)]$$

$$\varpi_{4,b}\equiv[1-\zeta_b(\varphi+1)]\frac{\varepsilon}{\lambda}$$

$$\varpi_{5,b}\equiv-(\zeta_w-\zeta_b)(\varphi+1)\varphi$$

$$\varpi_{6,b}\equiv-(\zeta_w-\zeta_b)\gamma_b\sigma$$

$$\varpi_{7,b}\equiv(1-\zeta_b\varphi)(1-\delta_b)-(\zeta_w-\zeta_b)\varphi\delta_b-\eta\sigma^2\gamma_b(1-\gamma_b)(1-\zeta_w\varphi+\zeta_w)$$

$$+\sigma^2[\zeta_b(1-\gamma_b)+(\zeta_w-\zeta_b)\gamma_b]$$

$$\varpi_{8,b}\equiv-\zeta_b(1-\gamma_b)\sigma$$

$$\varpi_{9,b}\equiv-(\zeta_w-\zeta_b)(1-\gamma_b)\sigma$$

$$\varpi_{10,b}\equiv\eta\sigma^2\gamma_b(1-\gamma_b)(1-\zeta_w\varphi+\zeta_w)$$

$$\varpi_{11,b}\equiv[1-\zeta_b(\varphi+1)](\varphi+1)$$

$$\varpi_{12,b}\equiv\zeta_b(\varphi+1)$$

$$\varpi_{13,b}\equiv-(\zeta_w-\zeta_b)(\varphi+1)^2$$

$$\varpi_{14,b}\equiv(\zeta_w-\zeta_b)(\varphi+1) \tag{112}$$

59

To determine the target of the monetary authority in area H we maximize (108) subject to:

$$\hat{y}_{H,t} = \delta_b \hat{c}_{H,t} + (1 - \delta_b)\hat{c}_{F,t}$$

$$\hat{y}_{F,t} = \delta_b \hat{c}_{F,t} + (1 - \delta_b)\hat{c}_{H,t} \tag{113}$$

The Lagrangian associated to this problem can be written:

$$L^b = \sum_{t=0}^{\infty} \beta^t E_0 \left[\frac{1}{2} \hat{s}_{H,t}^{b\prime}(\Omega_{s_H,s_H} + \Omega_{S_H,S_H})\hat{s}_{H,t}^b + \frac{1}{2}\hat{s}_{F,t}^{b\prime}(\Omega_{s_F,s_F} + \Omega_{S_F,S_F})\hat{s}_{F,t}^b \right.$$

$$+ \hat{s}_{H,t}^{b\prime}\Omega_{S_H,S_F}\hat{s}_{F,t}^b - \hat{s}_{H,t}^{b\prime}\Omega_{s_H,u}\hat{u}_{H,t} - \hat{s}_{F,t}^{b\prime}\Omega_{s_F,u}\hat{u}_{F,t}$$

$$+ \phi_{H,t}^b \left(\hat{s}_{H,t}^{b\prime}h_{S_H} + \hat{s}_{F,t}^{b\prime}h_{S_F} \right)$$

$$\left. + \phi_{F,t}^b \left(\hat{s}_{F,t}^{b\prime}f_{S_F} + \hat{s}_{H,t}^{b\prime}f_{S_H} \right) \right] \tag{114}$$

By the first-order conditions of L^b with respect to $\hat{s}_{H,t}^{b\prime}$, $\hat{s}_{F,t}^{b\prime}$, $\phi_{H,t}^b$ and $\phi_{F,t}^b$ we obtain:

$$(\Omega_{s_H,s_H} + \Omega_{S_H,S_H})\hat{s}_{H,t}^b + \Omega_{S_H,S_F}\hat{s}_{F,t}^b - \Omega_{s_H,u}\hat{u}_{H,t} = -\phi_{H,t}^b h_{S_H} - \phi_{F,t}^b f_{S_H}$$

$$(\Omega_{s_F,s_F} + \Omega_{S_F,S_F})\hat{s}_{F,t}^b + \Omega'_{S_H,S_F}\hat{s}_{H,t}^b - \Omega_{s_F,u}\hat{u}_{F,t} = -\phi_{F,t}^b f_{S_F} - \phi_{H,t}^b h_{S_F}$$

$$h_{S_H}\hat{s}_{H,t}^b + h_{S_F}\hat{s}_{F,t}^b = 0$$

$$f_{S_F}\hat{s}_{F,t}^b + f_{S_H}\hat{s}_{H,t}^b = 0 \tag{115}$$

Alternatively, we can rewrite (115) as:

$$\varpi_{1,b}\hat{y}_{H,t}^b + \varpi_{2,b}\hat{c}_{H,t}^b + \varpi_{8,b}\hat{c}_{F,t}^b - \varpi_{11,b}\hat{a}_{H,t} - \varpi_{12,b}\hat{\mu}_{H,t} = \phi_{H,t}^b$$

$$\varpi_{5,b}\hat{y}_{F,t}^b + \varpi_{6,b}\hat{c}_{F,t}^b + \varpi_{9,b}\hat{c}_{H,t}^b - \varpi_{13,b}\hat{a}_{F,t} - \varpi_{14,b}\hat{\mu}_{F,t} = \phi_{F,t}^b$$

$$\varpi_{2,b}\hat{y}_{H,t}^b + \varpi_{3,b}\hat{c}_{H,t}^b + \varpi_{9,b}\hat{y}_{F,t}^b + \varpi_{10,b}\hat{c}_{F,t}^b = -(\delta_b\phi_{H,t}^b + (1-\delta_b)\phi_{F,t}^b)$$

$$\varpi_{6,b}\hat{y}_{F,t}^b + \varpi_{7,b}\hat{c}_{F,t}^b + \varpi_{8,b}\hat{y}_{H,t}^b + \varpi_{10,b}\hat{c}_{H,t}^b = -(\delta_b\phi_{F,t}^b + (1-\delta_b)\phi_{H,t}^b)$$

$$\varpi_{4,b}\pi_{H,t} = 0$$

$$\hat{y}_{H,t}^b = \delta_b\hat{c}_{H,t}^b + (1-\delta_b)\hat{c}_{F,t}^b$$

$$\hat{y}_{F,t}^b = \delta_b\hat{c}_{F,t}^b + (1-\delta_b)\hat{c}_{H,t}^b \tag{116}$$

Given (21), it can be shown that according to (116):[51]

$$[1 - \zeta_b(\varphi + 1)]\widehat{mc}_{H,t}^{e,b} = \zeta_b(\varphi + 1)\hat{\mu}_{H,t} + \kappa_H^b\hat{s}_{HF,t}^b$$

$$- (\zeta_w - \zeta_b)(\varphi + 1)\widehat{mc}_{F,t}^{e,b} = (\zeta_w - \zeta_b)(\varphi + 1)\hat{\mu}_{F,t} + \kappa_F^b\hat{s}_{HF,t}^b$$

$$\hat{y}_{H,t}^b = \hat{c}_{H,t}^b - \frac{(1 - \delta_b)}{\sigma(2\gamma_b - 1)}\hat{s}_{HF,t}^b$$

$$\hat{y}_{F,t}^b = \hat{c}_{F,t}^b + \frac{(1 - \delta_b)}{\sigma(2\gamma_b - 1)}\hat{s}_{HF,t}^b$$

$$\hat{s}_{HF,t}^b = \kappa_a^b(\hat{a}_{F,t} - \hat{a}_{H,t}) + \kappa_\mu^b(\hat{\mu}_{F,t} - \hat{\mu}_{H,t}) + \kappa_{\mu H}^b\hat{\mu}_{H,t} \tag{117}$$

---

[51]This result is not straightforward. A formal proof is available under request.

where $\widehat{mc}_{H,t}^{e,b}$ and $\widehat{mc}_{F,t}^{e,b}$ are defined consistently with (42) and

$$\kappa_H^b \equiv \frac{(\gamma_b - \delta_b)\left(1 - (2\zeta_b - \zeta_w)(\varphi + 1)\right)}{2(2\gamma_b - 1)}$$

$$\kappa_F^b \equiv \frac{(\gamma_b - \delta_b)\left(1 - (2\zeta_b - \zeta_w)(\varphi + 1)\right)}{2(2\gamma_b - 1)}$$

$$\kappa_a^b \equiv \frac{4(\zeta_w - \zeta_b)(\varphi + 1)^2\sigma(2\gamma_b - 1)}{a^b}\left[1 - \zeta_b(\varphi + 1)\right]$$

$$\kappa_\mu^b \equiv -\frac{4(\zeta_w - \zeta_b)(\varphi + 1)\sigma(2\gamma_b - 1)}{a^b}\left[1 - \zeta_b(\varphi + 1)\right]$$

$$\kappa_{\mu H}^b \equiv -\frac{4(\zeta_w - \zeta_b)(\varphi + 1)\sigma(2\gamma_b - 1)}{a^b} \tag{118}$$

where:

$$a^b \equiv (\sigma(2\delta_b - 1) + \varphi(2\gamma_b - 1))\left(1 - \zeta_w(\varphi + 1)\right)^2 - (\sigma + \varphi)(2\delta_b - 1)\left(1 - (2\zeta_w - \zeta_w)(\varphi + 1)\right)^2$$

The first two conditions in (117) determine implicitly the target for the fluctuations in the efficient marginal cost in areas H and F, whereas the other conditions correspond to the good-market clearings and the condition that expresses the target for terms-of-trade fluctuations as a function of the exogenous shocks.

By conditions (21) and (116), we can rewrite the objective of the policy maker of the monetary union in area H in deviations from the policy targets as:[52]

$$-\frac{1}{(1 - \tilde{\tau})}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\varpi_{15,b}(\tilde{y}_{H,t}^b)^2 + \frac{1}{2}\varpi_{16,b}(\tilde{s}_{HF,t}^b)^2 + \frac{1}{2}\varpi_{17,b}(\tilde{y}_{F,t}^b)^2 + \frac{1}{2}\varpi_{4,b}\pi_{H,t}^2\right] + t.o.c. \tag{119}$$

where:

$$\varpi_{15,b} \equiv \frac{1}{(2\delta_b - 1)}\left[\delta_b\left(\varpi_{2,b} - \varpi_{6,b}\right) + \frac{1}{2}\left(\varpi_{3,b} - \varpi_{7,b}\right) - (1 - \delta_b)\left(\varpi_{8,b} - \varpi_{9,b}\right)\right]$$

$$+ \varpi_{1,b} + \varpi_{2,w} + \frac{1}{2}\varpi_{3,w} + \varpi_{5,w} + \frac{1}{2}\varpi_{6,w} + \varpi_{7,w} + \varpi_{10,b}$$

$$\varpi_{16,b} \equiv \frac{1}{\sigma^2(2\gamma_b - 1)^2}\left[(1 - \delta_b)(2\delta_b - 1)(\varpi_{2,w} + \varpi_{5,w}) + (1 - \delta_b)\delta_b(\varpi_{3,w} + \varpi_{6,w})\right.$$

$$\left.- (1 - \delta_b)(2\delta_b - 1)\varpi_{7,w} - (1 - \delta_b)^2\varpi_{10,b}\right]$$

$$\varpi_{17,b} \equiv -\frac{1}{(2\delta_b - 1)}\left[\delta_b\left(\varpi_{2,b} - \varpi_{6,b}\right) + \frac{1}{2}\left(\varpi_{3,b} - \varpi_{7,b}\right) - (1 - \delta_b)\left(\varpi_{8,b} - \varpi_{9,b}\right)\right]$$

$$+ \varpi_{5,b} + \varpi_{2,w} + \frac{1}{2}\varpi_{3,w} + \varpi_{5,w} + \frac{1}{2}\varpi_{6,w} + \varpi_{7,w} + \varpi_{10,b} \tag{120}$$

with:

$$\varpi_{2,w} \equiv -\zeta_w\gamma_s\sigma$$

$$\varpi_{3,w} \equiv (1 - \tilde{\tau})(\sigma - 1) + (\zeta_w\sigma^2 + (1 - \zeta_w\varphi)) - (1 - \gamma_s^2)\eta\sigma^2(\zeta_w + (1 - \zeta_w\varphi))$$

$$\varpi_{5,w} \equiv -\zeta_w\sigma(\gamma_b - \gamma_s)$$

$$\varpi_{6,w} \equiv \left(1 - \gamma_s^2 - 2(1 - \gamma_b)\gamma_b\right)\eta\sigma^2\left(\zeta_w + (1 - \varphi\zeta_w)\right)$$

$$\varpi_{7,w} \equiv -\sigma(1 - \gamma_b)\zeta_w$$

At the same time, we can rewrite the constraints to the optimal policy problem as:

$$\tilde{y}^b_{H,t} = \tilde{y}^b_{F,t} + \frac{(2\delta_b - 1)}{\sigma(2\gamma_b - 1)}\tilde{s}^b_{HF,t}$$

$$\pi_{H,t} = \lambda\left[(\varphi + \sigma)\tilde{y}^b_{H,t} + \frac{(\gamma_b - \delta_b)}{(2\gamma_b - 1)}\tilde{s}^b_{HF,t}\right] + \beta E_t\{\pi_{H,t+1}\} + \upsilon^b_{H,t}$$

$$\pi_{F,t} = \lambda\left[(\varphi + \sigma)\tilde{y}^b_{F,t} - \frac{(\gamma_b - \delta_b)}{(2\gamma_b - 1)}\tilde{s}^b_{HF,t}\right] + \beta E_t\{\pi_{F,t+1}\} + \upsilon^b_{F,t}$$

for all t where:

$$\upsilon^b_{H,t} \equiv \frac{\lambda}{1 - \zeta_b(\varphi + 1)}\left[\left(1 + \kappa^b_H\kappa^b_{\mu H}\right)\hat{\mu}_{H,t} + \kappa^b_H\kappa^b_\mu(\hat{\mu}_{F,t} - \hat{\mu}_{H,t}) + \kappa^b_H\kappa^b_a(\hat{a}_{F,t} - \hat{a}_{H,t})\right]$$

$$\upsilon^b_{F,t} \equiv -\frac{\lambda}{(\zeta_w - \zeta_b)(\varphi + 1)}\left[\left(1 + \kappa^b_F\kappa^b_{\mu H}\right)\hat{\mu}_{H,t} + \kappa^b_F\kappa^b_\mu(\hat{\mu}_{F,t} - \hat{\mu}_{H,t}) + \kappa^b_F\kappa^b_a(\hat{a}_{F,t} - \hat{a}_{H,t})\right]$$

(121)

Then, the optimal policy problem of the big economy policy maker can be solved by minimizing the following Lagrangian:

$$\mathcal{L}^b = \sum_{t=0}^{\infty}\beta^t E_0\left\{\left[\frac{1}{2}\varpi_{15,b}(\tilde{y}^b_{H,t})^2 + \frac{1}{2}\varpi_{16,b}(\tilde{s}^b_{HF,t})^2 + \frac{1}{2}\varpi_{17,b}(\tilde{y}^b_{F,t})^2 + \frac{1}{2}\varpi_{4,b}\pi^2_{H,t}\right]\right.$$

$$+ \psi^b_{1,t}\left[\pi_{H,t} - \lambda\left((\varphi + \sigma)\tilde{y}^b_{H,t} + \frac{\gamma_b - \delta_b}{2\gamma_b - 1}\tilde{s}^b_{HF,t}\right) - \upsilon^b_{H,t}\right] - \psi^b_{1,t-1}\pi_{H,t}$$

$$+ \psi^b_{2,t}\left[\pi_{F,t} - \lambda\left((\varphi + \sigma)\tilde{y}^b_{F,t} - \frac{\gamma_b - \delta_b}{2\gamma_b - 1}\tilde{s}^b_{HF,t}\right) - \upsilon^b_{F,t}\right] - \psi^b_{2,t-1}\pi_{F,t}$$

$$\left. + \psi^b_{3,t}\left[\tilde{y}^b_{H,t} - \tilde{y}^b_{F,t} - \frac{2\delta_b - 1}{\sigma(2\gamma_b - 1)}\tilde{s}^b_{HF,t}\right]\right\}$$

(122)

with respect to $\tilde{y}^b_{H,t}$, $\tilde{s}^b_{HF,t}$, $\tilde{y}^b_{F,t}$ and $\pi_{H,t}$. The corresponding first-order conditions are:

$$\varpi_{15,b}\tilde{y}^b_{H,t} = \psi^b_{1,t}\lambda(\varphi + \sigma) - \psi^b_{3,t}$$

$$\varpi_{16,b}\tilde{s}^b_{FH,t} = \psi^b_{1,t}\lambda\frac{\gamma_b - \delta_b}{2\gamma_b - 1} - \psi^b_{2,t}\lambda\frac{\gamma_b - \delta_b}{2\gamma_b - 1} + \psi^b_{3,t}\frac{2\delta_b - 1}{\sigma(2\gamma_b - 1)}$$

$$\varpi_{17,b}\tilde{y}^b_{F,t} = \psi^b_{2,t}\lambda(\varphi + \sigma) + \psi^b_{3,t}$$

$$\varpi_{4,b}\pi_{H,t} = -\left(\psi^s_{1,t} - \psi^s_{1,t-1}\right)$$

(123)

The solution to this problem allows us to determine the best response of area H policy maker under regime B, given the state-contingent path of $\pi_{F,t}$. Notice that once the average-union variables are determined, the region-specific variables can be recovered directly from the equilibrium conditions, namely (22), (23), (32), its foreign counterpart, (39) and its foreign akin. At the same time, a symmetric problem can be solved for the policy maker of the monetary union of area F to find the optimal best response both under regime A and B. Indeed, under this formulation, the optimal policy problem of the authority in area F is independent of whether in area H small open economies are under monetary autonomy or share the same currency.

D.4 The case of cooperation

If there is one single policy maker for the entire economy, then the average welfare of the world economy can be approximated to the second order as:

$$\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left\{\int_0^{\frac{1}{2}}\left[\hat{s}_t^{i'}w_s-\frac{1}{2}\hat{s}_t^{i'}W_{s,s}\hat{s}_t^i+\hat{s}_t^{i'}W_{s,u}\hat{u}_t^i\right]di\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left[\hat{s}_t^{i'}w_s-\frac{1}{2}\hat{s}_t^{i'}W_{s,s}\hat{s}_t^i+\hat{s}_t^{i'}W_{s,u}\hat{u}_t^i\right]di\right\}+t.i.p. \tag{124}$$

where w_s, $W_{s,s}$ and $W_{s,u}$ were defined in the Appendix D.2. It is easy to show that

$$w_s=(1-\varphi\zeta_w)h_{S_H}+(1-\varphi\zeta_w)\varphi f_{S_H}-\zeta_w k_{S_H}-\zeta_w r_{S_H}$$
$$w_s=(1-\varphi\zeta_w)h_{S_F}+(1-\varphi\zeta_w)\varphi f_{S_F}-\zeta_w k_{S_F}-\zeta_w r_{S_F} \tag{125}$$

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega_{s_H,s_H}^w\hat{s}_t^i di+\frac{1}{2}\int_{\frac{1}{2}}^1\hat{s}_t^{i'}\Omega_{s_F,s_F}^w\hat{s}_t^i di+\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_H}^w\int_0^{\frac{1}{2}}\hat{s}_t^i di\right.$$
$$+\int_{\frac{1}{2}}^1\hat{s}_t^i di'\Omega_{S_F,S_F}^w\int_{\frac{1}{2}}^1\hat{s}_t^i di+2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega_{S_H,S_F}^w\int_{\frac{1}{2}}^1\hat{s}_t^i di-\int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega_{s_H,u}^w\hat{u}_t^i di$$
$$\left.-\int_{\frac{1}{2}}^1\hat{s}_t^{i'}\Omega_{s_F,u}^w\hat{u}_t^i di\right]+t.i.p. \tag{126}$$

where:

$$\Omega_{s_H,s_H}^w\equiv W_{s,s}+(1-\varphi\zeta_w)H_{s_H,s_H}+(1-\varphi\zeta_w)F_{s_H,s_H}-\zeta_w K_{s_H,s_H}-\zeta_w R_{s_H,s_H}$$
$$\Omega_{s_F,s_F}^w\equiv W_{s,s}+(1-\varphi\zeta_w)H_{s_F,s_F}+(1-\varphi\zeta_w)F_{s_F,s_F}-\zeta_w R_{s_F,s_F}-\zeta_w K_{s_F,s_F}$$
$$\Omega_{S_H,S_H}^w\equiv\tfrac{1}{2}(1-\varphi\zeta_w)H_{S_H,S_H}+\tfrac{1}{2}(1-\varphi\zeta_w)F_{S_H,S_H}-\tfrac{1}{2}\zeta_w K_{S_H,S_H}-\tfrac{1}{2}\zeta_w R_{S_H,S_H}$$
$$\Omega_{S_F,S_F}^w\equiv\tfrac{1}{2}(1-\varphi\zeta_w)H_{S_F,S_F}+\tfrac{1}{2}(1-\varphi\zeta_w)F_{S_F,S_F}-\tfrac{1}{2}\zeta_w K_{S_F,S_F}-\tfrac{1}{2}\zeta_w R_{S_F,S_F}$$
$$\Omega_{S_H,S_F}^w\equiv\tfrac{1}{2}(1-\varphi\zeta_w)H_{S_H,S_F}+\tfrac{1}{2}(1-\varphi\zeta_w)F_{S_F,S_H}'-\tfrac{1}{2}\zeta_w K_{S_H,S_F}-\tfrac{1}{2}\zeta_w R_{S_F,S_H}'$$
$$\Omega_{s_H,u}^w\equiv W_{s,u}-\zeta_w K_{s_H,u}\qquad\Omega_{s_F,u}^w\equiv W_{s,u}-\zeta_w R_{s_F,u}$$

with $\zeta_w=\frac{\tilde{\tau}}{\sigma+\varphi}$ and Ω_{s_H,s_H}^w, Ω_{S_H,S_H}^w, Ω_{S_H,S_F}^w and $\Omega_{s_H,u}^w$ respectively equal to:

$$\begin{bmatrix} (1-\zeta_w(\varphi+1))\varphi & -\sigma\gamma_s\zeta_w & 0 \\ -\sigma\gamma_s\zeta_w & (1-\tilde{\tau})(\sigma-1)+(\zeta_w\sigma^2+(1-\zeta_w\varphi))-(1-\gamma_s^2)\eta\sigma^2(\zeta_w+(1-\zeta_w\varphi)) & 0 \\ 0 & 0 & \frac{\varepsilon(1-\zeta_w(\varphi+1))}{\lambda} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\sigma\left(\gamma_b-\gamma_s\right)\zeta_w & 0 \\ -\sigma\left(\gamma_b-\gamma_s\right)\zeta_w & \left(1-\gamma_s^2-2\left(1-\gamma_b\right)\gamma_b\right)\eta\sigma^2\left(\zeta_w+(1-\varphi\zeta_w)\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\sigma\left(1-\gamma_b\right)\zeta_w & 0 \\ -\sigma\left(1-\gamma_b\right)\zeta_w & 2\left(1-\gamma_b\right)\gamma_b\eta\sigma^2\left(\zeta_w+(1-\varphi\zeta_w)\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} (1-\zeta_w(\varphi+1))(\varphi+1) & \zeta_w(\varphi+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

At the same time, $\Omega^w_{s_F,s_F} = \Omega^w_{s_H,s_H}$, $\Omega^w_{S_F,S_F} = \Omega^w_{S_H,S_H}$ and $\Omega^w_{s_F,u} = \Omega^w_{s_H,u}$. Alternatively, (126) can be written as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\varpi_{1,w}\int_0^1 (\hat{y}_t^i)^2 di + \varpi_{2,w}\int_0^1 \hat{c}_t^i\hat{y}_t^i di + \frac{1}{2}\varpi_{3,w}\int_0^1 (\hat{c}_t^i)^2 di + \frac{1}{2}\varpi_{4,w}\int_0^1 (\pi_{i,t})^2 di\right.$$

$$+ 2\varpi_{5,w}\left(\int_0^{\frac{1}{2}}\hat{y}_t^i di\int_0^{\frac{1}{2}}\hat{c}_t^i di + \int_{\frac{1}{2}}^1\hat{y}_t^i di\int_{\frac{1}{2}}^1\hat{c}_t^i di\right) + \varpi_{6,w}\left(\int_0^{\frac{1}{2}}\hat{c}_t^i di\int_0^{\frac{1}{2}}\hat{c}_t^i di + \int_{\frac{1}{2}}^1\hat{c}_t^i di\int_{\frac{1}{2}}^1\hat{c}_t^i di\right)$$

$$+ 2\varpi_{7,w}\left(\int_{\frac{1}{2}}^1\hat{y}_t^i di\int_0^{\frac{1}{2}}\hat{c}_t^i di + \int_0^{\frac{1}{2}}\hat{y}_t^i di\int_{\frac{1}{2}}^1\hat{c}_t^i di\right) + 2\varpi_{8,w}\int_{\frac{1}{2}}^1\hat{c}_t^i di\int_0^{\frac{1}{2}}\hat{c}_t^i di - \varpi_{9,w}\int_0^1\hat{y}_t^i\hat{a}_t^i di$$

$$\left. - \varpi_{10,w}\int_0^1\hat{y}_t^i\hat{\mu}_t^i di\right] + t.i.p. \tag{127}$$

where:

$$\varpi_{1,w} \equiv [1 - \zeta_w(\varphi+1)]\varphi$$
$$\varpi_{2,w} \equiv -\zeta_w\gamma_s\sigma$$
$$\varpi_{3,w} \equiv (1-\tilde{\tau})(\sigma-1) + (\zeta_w\sigma^2 + (1-\zeta_w\varphi)) - (1-\gamma_s^2)\eta\sigma^2(\zeta_w + (1-\zeta_w\varphi))$$
$$\varpi_{4,w} \equiv [1 - \zeta_w(\varphi+1)]\frac{\varepsilon}{\lambda}$$
$$\varpi_{5,w} \equiv -\zeta_w\sigma(\gamma_b-\gamma_s)$$
$$\varpi_{6,w} \equiv \left(1-\gamma_s^2 - 2(1-\gamma_b)\gamma_b\right)\eta\sigma^2(\zeta_w + (1-\varphi\zeta_w))$$
$$\varpi_{7,w} \equiv -\sigma(1-\gamma_b)\zeta_w$$
$$\varpi_{8,w} \equiv 2(1-\gamma_b)\gamma_b\eta\sigma^2(\zeta_w + (1-\varphi\zeta_w))$$
$$\varpi_{9,w} \equiv [1 - \zeta_w(\varphi+1)](\varphi+1)$$
$$\varpi_{10,w} \equiv \zeta_w(\varphi+1) \tag{128}$$

The target of the cooperative policy maker can be determined as maximizing (126) subject to (32) and its foreign akin. The Lagrangian associated to this problem can be written as:

$$L^w = \sum_{t=0}^{\infty}\beta^t E_0\left[\frac{1}{2}\int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega^w_{s_H,s_H}\hat{s}_t^i di + \frac{1}{2}\int_{\frac{1}{2}}^1\hat{s}_t^{i'}\Omega^w_{s_F,s_F}\hat{s}_t^i di + \int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega^w_{S_H,S_H}\int_0^{\frac{1}{2}}\hat{s}_t^i di\right.$$

$$+ \int_{\frac{1}{2}}^1\hat{s}_t^i di'\Omega^w_{S_F,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di + 2\int_0^{\frac{1}{2}}\hat{s}_t^i di'\Omega^w_{S_H,S_F}\int_{\frac{1}{2}}^1\hat{s}_t^i di$$

$$- \int_0^{\frac{1}{2}}\hat{s}_t^{i'}\Omega^w_{s_H,u}\hat{u}_t^i di - \int_{\frac{1}{2}}^1\hat{s}_t^{i'}\Omega^w_{s_F,u}\hat{u}_t^i di$$

$$+ \int_0^{\frac{1}{2}}\phi_{i,t}^w\left(\hat{s}_t^{i'}g_s + \int_0^{\frac{1}{2}}\hat{s}_t^i di'g_{S_H} + \int_{\frac{1}{2}}^1\hat{s}_t^i di'g_{S_F}\right)di$$

$$\left. + \int_{\frac{1}{2}}^1\phi_{i,t}^w\left(\hat{s}_t^{i'}g_s + \int_{\frac{1}{2}}^1\hat{s}_t^i di'g_{S_H} + \int_0^{\frac{1}{2}}\hat{s}_t^i di'g_{S_F}\right)di\right] \tag{129}$$

By integrating the first-order conditions of L^w with respect to $\hat{s}_t^{i'}$ and $\phi_{i,t}^w$, we obtain:

$$(\Omega_{s_H,s_H}^w + \Omega_{S_H,S_H}^w)\hat{s}_{H,t} + \Omega_{S_H,S_F}^w \hat{s}_{F,t} - \Omega_{s_H,u}^w \hat{u}_{H,t} = -\phi_{H,t}^w(g_s + \frac{1}{2}g_{S_H}) - \phi_{F,t}^w \frac{1}{2}g_{S_F}$$

$$(\Omega_{s_F,s_F}^w + \Omega_{S_F,S_F}^w)\hat{s}_{F,t} + \Omega_{S_H,S_F}^{w'} \hat{s}_{H,t} - \Omega_{s_F,u}^w \hat{u}_{F,t} = -\phi_{F,t}^w(g_s + \frac{1}{2}g_{S_H}) - \phi_{H,t}^w \frac{1}{2}g_{S_F}$$

$$(g_s + \frac{1}{2}g_{S_H})\hat{s}_{H,t} + \frac{1}{2}g_{S_F}\hat{s}_{F,t} = 0$$

$$(g_s + \frac{1}{2}g_{S_H})\hat{s}_{F,t} + \frac{1}{2}g_{S_F}\hat{s}_{H,t} = 0 \tag{130}$$

Alternatively:

$$\varpi_{1,w}\hat{y}_{H,t}^w + (\varpi_{2,w} + \varpi_{5,w})\hat{c}_{H,t}^w + \varpi_{7,w}\hat{c}_{F,t}^w - \varpi_{9,w}\hat{a}_{H,t} - \varpi_{10,w}\hat{\mu}_{H,t} = \phi_{H,t}^w$$

$$\varpi_{1,w}\hat{y}_{F,t}^w + (\varpi_{2,w} + \varpi_{5,w})\hat{c}_{F,t}^w + \varpi_{7,w}\hat{c}_{H,t}^w - \varpi_{9,w}\hat{a}_{F,t} - \varpi_{10,w}\hat{\mu}_{F,t} = \phi_{F,t}^w$$

$$(\varpi_{2,w} + \varpi_{5,w})\hat{y}_{H,t}^w + (\varpi_{3,w} + \varpi_{6,w})\hat{c}_{H,t}^w + \varpi_{7,w}\hat{y}_{F,t}^w + \varpi_{8,w}\hat{c}_{F,t}^w = -(\delta_b\phi_{H,t}^w + (1 - \delta_b)\phi_{F,t}^w)$$

$$(\varpi_{2,w} + \varpi_{5,w})\hat{y}_{F,t}^w + (\varpi_{3,w} + \varpi_{6,w})\hat{c}_{F,t}^w + \varpi_{7,w}\hat{y}_{H,t}^w + \varpi_{8,w}\hat{c}_{H,t}^w = -(\delta_b\phi_{F,t}^w + (1 - \delta_b)\phi_{H,t}^w)$$

$$\varpi_{4,w}\pi_{H,t} = 0$$

$$\varpi_{4,w}\pi_{F,t} = 0$$

$$\hat{y}_{H,t}^w = \delta_b\hat{c}_{H,t}^w + (1 - \delta_b)\hat{c}_{F,t}^w$$

$$\hat{y}_{F,t}^w = \delta_b\hat{c}_{F,t}^w + (1 - \delta_b)\hat{c}_{H,t}^w \tag{131}$$

It can be shown that the above conditions implies that:

$$[1 - \zeta_w(\varphi + 1)]\,\widehat{mc}_{H,t}^{e,w} = \zeta_w(\varphi + 1)\hat{\mu}_{H,t}$$

$$[1 - \zeta_w(\varphi + 1)]\,\widehat{mc}_{F,t}^{e,w} = \zeta_w(\varphi + 1)\hat{\mu}_{F,t}$$

$$\hat{y}_{H,t}^w = \hat{c}_{H,t}^w - \frac{(1 - \delta_b)}{\sigma(2\gamma_b - 1)}\hat{s}_{HF,t}^w$$

$$\hat{y}_{F,t}^w = \hat{c}_{F,t}^w + \frac{(1 - \delta_b)}{\sigma(2\gamma_b - 1)}\hat{s}_{HF,t}^w$$

$$\hat{s}_{HF,t}^w = \kappa_a^w(\hat{a}_{F,t} - \hat{a}_{H,t}) + \kappa_\mu^w(\hat{\mu}_{F,t} - \hat{\mu}_{H,t}) \tag{132}$$

where $\widehat{mc}_{H,t}^{e,w}$ and $\widehat{mc}_{F,t}^{e,w}$ are defined as in (42) and:

$$\kappa_a^w \equiv \frac{(2\delta_b - 1)(\varphi + 1)}{(2\delta_b - 1)\varphi + (2\gamma_b - 1)\sigma}$$

$$\kappa_\mu^w \equiv \frac{(2\delta_b - 1)\zeta_w(\varphi + 1)}{(1 - \zeta_w(\varphi + 1))\left[(2\delta_b - 1)\varphi + (2\gamma_b - 1)\sigma\right]} \tag{133}$$

The first two conditions in (132) determine the target of the cooperative policy maker for the fluctuations of the efficient marginal cost. The other conditions are the market clearing conditions and the condition that expresses the target of the terms of trade fluctuations as a function of the underlying shocks of the model. By using (20), (21)

and the first order conditions of L^w, we can rewrite (127) in terms of deviations from the policy target as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^t\frac{1}{2}E_0\left[\varpi_{11,w}(\int_0^{\frac{1}{2}}(\tilde{s}_{iH,t}^w)^2di+\int_{\frac{1}{2}}^1(\tilde{s}_{iF,t}^w)^2di)+\varpi_{12,w}((\tilde{y}_{H,t}^w)^2+(\tilde{y}_{F,t}^w)^2)\right.$$

$$\left.+\varpi_{13,w}(\tilde{s}_{HF,t}^w)^2+\varpi_{4,w}\int_0^1(\pi_{i,t})^2di\right]+t.i.p. \tag{134}$$

where:

$$\varpi_{11,w}\equiv\frac{1}{\sigma^2\gamma_s^2}\left[\delta_s^2\varpi_{1,w}+2\delta_s\varpi_{2,w}+\varpi_{3,w}\right]$$

$$\varpi_{12,w}\equiv\frac{1}{2}\varpi_{1,w}+\varpi_{2,w}+\frac{1}{2}\varpi_{3,w}+\varpi_{5,w}+\frac{1}{2}\varpi_{6,w}+\varpi_{7,w}+\frac{1}{2}\varpi_{8,w}$$

$$\varpi_{13,w}\equiv\frac{1}{\sigma^2(2\gamma_b-1)^2}\left[(2\delta_b-1)(1-\delta_b)(\varpi_{2,w}+\varpi_{3,w}+\varpi_{5,w})+\delta_b((1-\delta_b)\varpi_{6,w}\right.$$

$$\left.-(2\delta_b-1))\varpi_{7,w}-\frac{1}{2}\left((1-\delta_b)^2+\delta_b^2\right)\varpi_{8,w}\right] \tag{135}$$

Given (134), the objective of policy maker of world monetary union can be written as:

$$-\frac{1}{1-\tilde{\tau}}\sum_{t=0}^{\infty}\beta^tE_0\left[\frac{1}{2}\varpi_{12,w}((\tilde{y}_{H,t}^w)^2+(\tilde{y}_{F,t}^w)^2)+\frac{1}{2}\varpi_{13,w}(\tilde{s}_{HF,t}^w)^2+\frac{1}{2}\varpi_{4,w}(\pi_{H,t}^2+\pi_{F,t}^2)\right]$$

$$+t.o.c. \tag{136}$$

Similarly to what done in the case of the single area currency union, we consider the difference between region-specific and average-area variables as terms out o control of the policy maker of the monetary union. This assumption does not affect the equilibrium optimal policy. Then, the *timelessly* optimal monetary policy of the world monetary union can be retrieved by maximizing (136) subject to the following sequence of constraints:

$$\tilde{y}_{H,t}^w=\tilde{y}_{F,t}^w+\frac{1-\delta_b}{\sigma(2\gamma_b-1)}\tilde{s}_{HF,t}^w$$

$$\pi_{H,t}=\lambda\left[(\varphi+\sigma)\tilde{y}_{H,t}^w+\frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}_{HF,t}^w\right]+\beta E_t\{\pi_{H,t+1}\}+v_{H,t}^w$$

$$\pi_{F,t}=\lambda\left[(\varphi+\sigma)\tilde{y}_{F,t}^w-\frac{\gamma_b-\delta_b}{2\gamma_b-1}\tilde{s}_{HF,t}^w\right]+\beta E_t\{\pi_{F,t+1}\}+v_{F,t}^w$$

$$\pi_{F,t}-\pi_{H,t}=\Delta\tilde{s}_{HF,t}^w+v_{HF,t}^w \tag{137}$$

for all t and the constraints on $\pi_{H,0}$ and $\pi_{F,0}$ that render the policy timelessly optimal. In addition,

$$v_{H,t}^w\equiv\frac{\lambda\mu_t^H}{1-\zeta_w(\varphi+1)}$$

$$v_{F,t}^w\equiv\frac{\lambda\mu_t^F}{1-\zeta_w(\varphi+1)}$$

$$v_{HF,t}^w\equiv\kappa_a^w(\Delta\hat{a}_{F,t}-\Delta\hat{a}_{H,t})+\kappa_\mu^w(\Delta\hat{\mu}_{F,t}-\Delta\hat{\mu}_{H,t})$$

The associated Lagrangian can be written as:

$$\mathcal{L}^w = \sum_{t=0}^{\infty} \beta^t E_0 \Bigg\{ \left[\frac{1}{2}\varpi_{12,w}((\tilde{y}_{H,t}^w)^2 + (\tilde{y}_{F,t}^w)^2) + \frac{1}{2}\varpi_{13,w}(\tilde{s}_{HF,t}^w)^2 + \frac{1}{2}\varpi_{4,w}(\pi_{H,t}^2 + \pi_{F,t}^2) \right]$$

$$+ \psi_{1,t}^w \left[\pi_{H,t} - \lambda\left((\varphi+\sigma)\tilde{y}_{H,t}^w + \frac{\gamma_b - \delta_b}{2\gamma_b - 1}\tilde{s}_{HF,t}^w \right) - \upsilon_{H,t}^w \right] - \psi_{1,t-1}^w \pi_{H,t}$$

$$+ \psi_{2,t}^w \left[\pi_{F,t} - \lambda\left((\varphi+\sigma)\tilde{y}_{F,t}^w - \frac{\gamma_b - \delta_b}{2\gamma_b - 1}\tilde{s}_{HF,t}^w \right) - \upsilon_{F,t}^w \right] - \psi_{2,t-1}^w \pi_{F,t}$$

$$+ \psi_{3,t}^w \left[\tilde{y}_{H,t}^w - \tilde{y}_{F,t}^w - \frac{2\delta_b - 1}{\sigma(2\gamma_b - 1)}\tilde{s}_{HF,t}^w \right]$$

$$+ \psi_{4,t}^w \left[\pi_{F,t} - \pi_{H,t} - \Delta \tilde{s}_{HF,t}^w - \upsilon_{HF,t}^w \right] \Bigg\} \tag{138}$$

The first-order conditions of $\mathcal{L}^w$ with respect to $\tilde{y}_{H,t}^w$, $\tilde{s}_{HF,t}^w$, $\tilde{y}_{F,t}^w$, $\pi_{H,t}$ and $\pi_{F,t}$ are:

$$\varpi_{12,w}\tilde{y}_t^w = \psi_{1,t}^w \lambda(\varphi+\sigma) - \psi_{3,t}^w$$

$$\varpi_{13,w}\tilde{s}_{FH,t}^b = \psi_{1,t}^w \lambda \frac{\gamma_b - \delta_b}{2\gamma_b - 1} - \psi_{2,t}^w \lambda \frac{\gamma_b - \delta_b}{2\gamma_b - 1} + \psi_{3,t}^w \frac{2\delta_b - 1}{\sigma(2\gamma_b - 1)} + \psi_{4,t}^w - \beta E_t\left\{ \psi_{4,t+1}^w \right\}$$

$$\varpi_{12,w}\tilde{y}_{F,t}^b = \psi_{2,t}^w \lambda(\varphi+\sigma) + \psi_{3,t}^w$$

$$\varpi_{4,w}\pi_{H,t} = -\left(\psi_{1,t}^w - \psi_{1,t-1}^w \right) + \psi_{4,t}^w$$

$$\varpi_{4,w}\pi_{F,t} = -\left(\psi_{2,t}^w - \psi_{2,t-1}^w \right) - \psi_{4,t}^w \tag{139}$$

E Parametrization of shocks

As anticipated in Section 6.1 we assume that:

$$\hat{a}_{t+1}^i = \rho_a \hat{a}_t^i + \varepsilon_{a,t}^i$$
$$\hat{\mu}_{t+1}^i = \rho_\mu \hat{\mu}_t^i + \varepsilon_{\mu,t}^i \tag{140}$$

where $\varepsilon_{a,t}^i$ and $\varepsilon_{\mu,t}^i$ are white noise innovations with zero mean and standard deviation equal to σ_a and σ_μ respectively. Moreover the innovations to productivity and mark-up shocks can be decomposed into purely idiosyncratic and common components, i.e.,:

$$\varepsilon_{a,t}^i = \eta_{a,t} + \eta_{a,t}^i$$
$$\varepsilon_{\mu,t}^i = \eta_{\mu,t} + \eta_{\mu,t}^i$$

with $\eta_{a,t}^i \equiv \varepsilon_{a,t}^i - \eta_{a,t}$ and $\eta_{\mu,t}^i \equiv \varepsilon_{\mu,t}^i - \eta_{\mu,t}$ being the idiosyncratic component where we impose that:

$$Cov\left\{ \eta_{a,t}^i, \eta_{a,t}^j \right\} = \begin{cases} \sigma_{a,\eta^i}^2 & i = j \\ 0 & i \neq j \end{cases}$$

and

$$Cov\left\{ \eta_{\mu,t}^i, \eta_{\mu,t}^j \right\} = \begin{cases} \sigma_{\mu,\eta^i}^2 & i = j \\ 0 & i \neq j \end{cases}$$

Notice that as a consequence:

$$Cov\left\{ \varepsilon_{a,t}^i, \varepsilon_{a,t}^j \right\} = \begin{cases} \sigma_{a,\eta}^2 + \sigma_{a,\eta^i}^2 & i = j \\ \sigma_{a,\eta}^2 & i \neq j \end{cases}$$

and

$$Cov\left\{\varepsilon_{\mu,t}^{i}, \varepsilon_{\mu,t}^{j}\right\} = \begin{cases} \sigma_{\mu,\eta}^{2} + \sigma_{\mu,\eta^{i}}^{2} & i = j \\ \sigma_{\mu,\eta}^{2} & i \neq j \end{cases}$$