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Note

Application of the spatial averaging theorem to radiative heat transfer in two-phase media

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ABSTRACT

The spatial averaging theorem is applied to rigorously derive continuum-scale equations of radiative transfer in two-phase media consisting of arbitrary-type phases in the limit of geometrical optics. The derivations are based on the equations of radiative transfer and the corresponding boundary conditions applied at the discrete-scale to each phase, and on the discrete-scale radiative properties of each phase and the interface between the phases. The derivations confirm that radiative transfer in two-phase media consisting of arbitrary-type phases in the range of geometrical optics can be modeled by a set of two continuum-scale equations of radiative transfer describing the variation of the average intensities associated with each phase. Finally, a Monte Carlo based methodology for the determination of average radiative properties is discussed in the light of previous pertinent studies.

1. Introduction

Radiative heat transfer in multi-phase media is commonly encountered in chemical processing, combustion, nuclear and civil engineering, atmospheric sciences, and solar technology. Of particular interest are stochastic two-phase media to which laws of geometrical optics are applicable [1]. They find applications as reacting packed beds, porous heat exchangers, radiant absorbers and burners, and insulations. Radiative heat transfer in such media can often be predicted based on the phenomenological theory of radiative energy transfer with the knowledge of appropriate radiative properties [2–6]. Examples of previous pertinent studies include the determination of radiative properties of porous two-phase media consisting of an optically-thin phase and an opaque phase [7–9] and of an optically-thin and a semi-transparent phase [10–14]. Equations of radiative transfer (RTEs) were derived from an RTE applied at the discrete scale in a multi-phase porous medium having an optically-thin phase and an opaque phase consisting of large opaque particles [15], formulated for a medium having an optically-thin phase and a semi-transparent phase [10], derived for heterogeneous two-phase media with irregular phase boundaries [16], and derived directly from statistical electromagnetics for a medium containing arbitrarily shaped and oriented particles in an optically-thin host medium [17]. In [8,10,14] the radiative characteristics were determined based on the exact morphology of the porous media, obtained from computer tomography. In the studies dealing with porous media in the range of geometrical optics, either the continuum-scale RTEs were derived for specific cases or the physical meaning of the radiative intensity appearing in these equations was not clearly explained, making the physical interpretation of the continuum-scale radiative properties difficult.[1]

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[1] Variables and properties associated with the length scale of single porous structures (particles, pores, struts, etc.) are referred to as discrete-scale. Variables and properties associated with the length scale

Nomenclature

A	surface area, m^2
f_V	volume fraction
I	average intensity, $\mathrm{W\,m^{-3}\,sr^{-1}}$
L	discrete-scale intensity, $\mathrm{W\,m^{-3}\,sr^{-1}}$
n	refractive index
$\hat{\mathbf{n}}$	normal unit vector
$\mathbf{r}$	global position vector, m
s	thickness, m
$\hat{\mathbf{s}}$	direction unit vector
T	temperature, K
V	averaging volume, m^3
$\mathbf{x}$	global position vector of the center of an averaging volume, m

Greek symbols

ε	emissivity
κ	absorption coefficient, m^{-1}
ρ	reflectivity
σ	Stefan-Boltzman constant, $\mathrm{W\,m^{-2}\,K^{-4}}$
σ_{s}	scattering coefficient, m^{-1}
τ	transmissivity
Φ	phase function
ψ	scalar quantity
ψ'	difference between a scalar ψ and its superficial volume average
$\psi_i'\vert^i$	difference between a scalar ψ and its intrinsic volume average
Ω	solid angle, sr

Subscripts

b	blackbody
c	collision
d	discrete-scale
i, j	phase indices
in	incoming
int	interface
k	ray index
r	reflection
w	boundary of the two-phase medium

Superscripts

“	bi-directional
‘⌢	directional-hemispherical
i, j	indices of the averaging phases

In this note, the spatial averaging theorem (SAT), derived from the Gauss–Ostrogradsky theorem and previously applied to mass transfer as well as conduction and convection heat transfer in porous media, is applied to the radiative heat transfer in a two-phase medium consisting of either two semi-transparent or one semi-transparent and one opaque homogeneous phases in the limit of geometrical optics [18–20]. RTEs and the corresponding boundary conditions are rigorously derived for volume-averaged radiative intensities by utilizing RTEs and the corresponding boundary conditions applied at the discrete scale to each phase, and the discrete-scale radiative properties of each phase. Unambiguous definitions of the continuum-scale radiative properties are postulated when deriving the continuum-scale RTEs. A Monte Carlo (MC) based methodology for the determination of continuum-scale radiative properties is discussed in the light of the studies presented in [7,8,10,14].

2. Basic definitions

Consider a two-phase medium shown in Fig. 1. An averaging volume V is enclosed by the dashed line. Phases i and j are characterized by their partial volumes V_i and V_j within V, and the corresponding volume fractions $f_{V,i}$ and $f_{V,j}$, respectively. In the following text, all equations written for phase i apply analogously to phase j unless stated otherwise. A given discrete-scale scalar quantity ψ_i associated with phase i can be expressed as

$$\psi_i = \langle\psi_i\rangle + \psi_i', \tag{1}$$

$$\psi_i = \langle\psi_i\rangle^i + \psi_i'|^i, \tag{2}$$

where the superficial and the intrinsic averages, $\langle\psi_i\rangle$ and $\langle\psi_i\rangle^i$ respectively, are calculated as

$$\langle\psi_i\rangle = \frac{1}{V}\int_V \psi_i\,\mathrm{d}V, \quad V = \sum_i V_i, \tag{3}$$

$$\langle\psi_i\rangle^i = \frac{1}{V_i}\int_{V_i} \psi_i\,\mathrm{d}V, \quad V_i = f_{V,i}V. \tag{4}$$

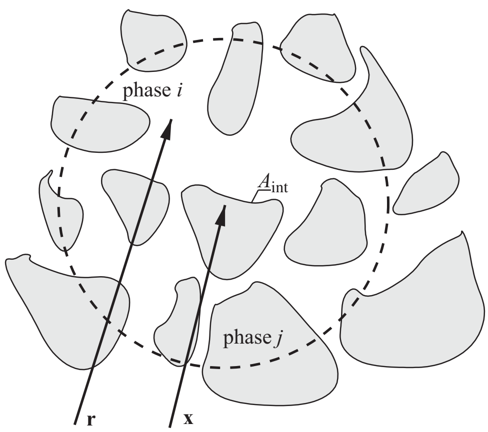

Fig. 1. Averaging volume for the medium consisting of phases i and j.

(*footnote continued*)
of a sufficiently-large volume of the two-phase medium that can be treated as a continuum, are referred to as continuum-scale.

A scalar quantity associated with a particular phase is required to vanish outside this phase. Thus,

$$\langle \psi_i \rangle = f_{V,i} \langle \psi_i \rangle^i. \tag{5}$$

The volume average of the discrete-scale gradient of ψ_i is related to the continuum-scale gradient of $\langle \psi_i \rangle$ as given by the spatial averaging theorem (SAT) [18–20],

$$\langle \nabla_{\mathbf{r}} \psi_i \rangle = \nabla_{\mathbf{x}} \langle \psi_i \rangle + \frac{1}{V} \int_{A_{\text{int}}} \psi_i \hat{\mathbf{n}}_i \, \mathrm{d}A_{\text{int}}, \tag{6}$$

where $\mathbf{r}$ and $\mathbf{x}$ are a global position vector of a given point within V_i and a global position vector of the center of the averaging volume V, respectively [18]. $A_{\text{int}} = \delta V_i = \delta V_j$ and $\hat{\mathbf{n}}_i$ are the surface area of the i–j interface contained within V and the unit normal vector at the interface pointing out of phase i, respectively. V is assumed to be (i) large enough to include all typical morphological structures of the real medium and (ii) small enough as compared to the overall size of the two-phase medium.

In this paper, the discrete-scale, superficial average and intrinsic average intensities associated with phase i are denoted by $L_i(\mathbf{r}, \hat{\mathbf{s}})$, $I_i(\mathbf{x}, \hat{\mathbf{s}})$, and $I_i^i(\mathbf{x}, \hat{\mathbf{s}})$, respectively.

3. Derivation of continuum-scale RTEs

The subsequent derivations are subject to the following assumptions: (i) the two-phase medium is isotropic at the continuum scale, (ii) both phases are internally homogeneous and their discrete-scale spectral radiative properties are independent of position, (iii) the refractive index is constant in each phase, (iv) each phase is isothermal in the averaging volume V, (v) each phase is at rest as compared to the speed of light, (vi) each phase is non-polarizing and the state of polarization can be neglected, (vii) the local thermodynamic equilibrium is valid in each phase, (viii) geometrical optics is valid in each phase, (ix) diffraction is negligible, and (x) radiative transfer in each phase is quasi-steady.

3.1. Medium consisting of two semi-transparent phases

In a medium consisting of the two semi-transparent phases, the radiative intensities exist in each phase. The discrete-scale intensity variations are governed by discrete-scale RTEs applied to each phase. For phase i, the discrete-scale RTE reads [2–4]:

$$\hat{\mathbf{s}} \cdot \nabla_{\mathbf{r}} L_i(\mathbf{r}, \hat{\mathbf{s}}) = -[\kappa_{\text{d},i} + \sigma_{\text{s,d},i}] L_i(\mathbf{r}, \hat{\mathbf{s}}) + n_i^2 \kappa_{\text{d},i} L_{\text{b},i}(\mathbf{r}) + \frac{\sigma_{\text{s,d},i}}{4\pi} \int_{4\pi} L_i(\mathbf{r}, \hat{\mathbf{s}}_{\text{in}}) \Phi_{\text{d},i}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) \, \mathrm{d}\Omega_{\text{in}}, \tag{7}$$

where the spectral subscript has been omitted for brevity. The corresponding boundary condition at $A_{\text{int}} : \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i < 0$ reads [2–4] (see Fig. 2)

$$L_i(\mathbf{r}_{\text{int}}, \hat{\mathbf{s}}) = \int_{\Omega_{\text{in}}:\hat{\mathbf{s}}_{\text{in}} \cdot \hat{\mathbf{n}}_i > 0} \rho''_{ij}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) L_i(\mathbf{r}_{\text{int}}, \hat{\mathbf{s}}_{\text{in}}) \hat{\mathbf{s}}_{\text{in}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}\Omega_{\text{in}} - \int_{\Omega_{\text{in}}:\hat{\mathbf{s}}_{\text{in}} \cdot \hat{\mathbf{n}}_i < 0} \tau''_{ji}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) L_j(\mathbf{r}_{\text{int}}, \hat{\mathbf{s}}_{\text{in}}) \hat{\mathbf{s}}_{\text{in}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}\Omega_{\text{in}}, \tag{8a}$$

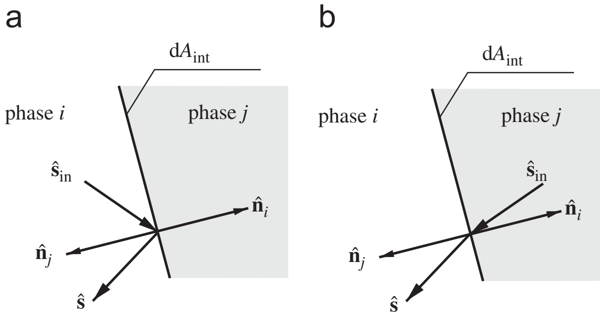

Fig. 2. Interface phenomena in a medium consisting of two semi-transparent phases: (a) reflection and (b) transmission.

where

$$-\int_{\Omega:\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i < 0} \rho''_{ij}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}\Omega + \int_{\Omega:\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i > 0} \tau''_{ij}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}\Omega = \rho'^{\frown}_{ij}(\hat{\mathbf{s}}_{\text{in}}) + \tau'^{\frown}_{ij}(\hat{\mathbf{s}}_{\text{in}}) = 1. \tag{8b}$$

Analogous equations are obtained for phase j by interchanging the subscripts i and j. Eq. (7) is volume-averaged by applying Eq. (3) to each term. Thus, for constant discrete-scale radiative properties

$$\langle \hat{\mathbf{s}} \cdot \nabla_{\mathbf{r}} L_i(\mathbf{r}, \hat{\mathbf{s}}) \rangle = -[\kappa_{\text{d},i} + \sigma_{\text{s,d},i}] I_i(\mathbf{x}, \hat{\mathbf{s}}) + n_i^2 \kappa_{\text{d},i} I_{\text{b},i}(\mathbf{x}) + \frac{\sigma_{\text{s,d},i}}{4\pi} \left\langle \int_{4\pi} L_i(\mathbf{r}, \hat{\mathbf{s}}_{\text{in}}) \Phi_{\text{d},i}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) \, \mathrm{d}\Omega_{\text{in}} \right\rangle. \tag{9}$$

The first term on the LHS of Eq. (9) is developed by applying SAT. The order of integration with respect to Ω_{in} and V in the incoming scattering term, the third term on the RHS of Eq. (9), can be changed because Ω_{in} and V are independent variables. Since $\hat{\mathbf{s}}$ is only a parameter, it follows that

$$\hat{\mathbf{s}} \cdot \nabla_{\mathbf{x}} I_i(\mathbf{x}, \hat{\mathbf{s}}) = -[\kappa_{\text{d},i} + \sigma_{\text{s,d},i}] I_i(\mathbf{x}, \hat{\mathbf{s}}) + n_i^2 \kappa_{\text{d},i} I_{\text{b},i}(\mathbf{x}) + \frac{\sigma_{\text{s,d},i}}{4\pi} \int_{4\pi} I_i(\mathbf{x}, \hat{\mathbf{s}}_{\text{in}}) \Phi_{\text{d},i}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) \, \mathrm{d}\Omega_{\text{in}} - \frac{1}{V} \int_{A_{\text{int}}} L_i(\mathbf{r}_{\text{int}}, \hat{\mathbf{s}}) \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}A_{\text{int}}. \tag{10}$$

The last term on the RHS of Eq. (10) gives the net radiative heat transfer rate per unit volume and solid angle around the direction $\hat{\mathbf{s}}$ resulting from the radiative intensities entering and leaving phase i at $A_{\text{int}} : \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i < 0$ and $A_{\text{int}} : \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i > 0$, respectively. The surface integral in Eq. (10) is developed by using the boundary condition, Eq. (8a),

$$\begin{aligned}
\int_{A_{\text{int}}} & L_i(\mathbf{r}_{\text{int}}, \hat{\mathbf{s}}) \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}A_{\text{int}} \\
&= \int_{A_{\text{int}}:\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i < 0} \int_{\Omega_{\text{in}}:\hat{\mathbf{s}}_{\text{in}} \cdot \hat{\mathbf{n}}_i > 0} \rho''_{ij}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) L_i(\mathbf{r}_{\text{int}}, \hat{\mathbf{s}}_{\text{in}}) \hat{\mathbf{s}}_{\text{in}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}\Omega_{\text{in}} \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}A_{\text{int}} \\
&\quad - \int_{A_{\text{int}}:\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i < 0} \int_{\Omega_{\text{in}}:\hat{\mathbf{s}}_{\text{in}} \cdot \hat{\mathbf{n}}_i < 0} \tau''_{ji}(\hat{\mathbf{s}}_{\text{in}}, \hat{\mathbf{s}}) L_j(\mathbf{r}_{\text{int}}, \hat{\mathbf{s}}_{\text{in}}) \hat{\mathbf{s}}_{\text{in}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}\Omega_{\text{in}} \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}A_{\text{int}} \\
&\quad + \int_{A_{\text{int}}:\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i > 0} L_i(\mathbf{r}_{\text{int}}, \hat{\mathbf{s}}) \hat{\mathbf{s}} \cdot \hat{\mathbf{n}}_i \, \mathrm{d}A_{\text{int}}.
\end{aligned} \tag{11}$$

The first two terms on the RHS of Eq. (11) give the augmentation (with negative sign) of radiative heat transfer rate per unit solid angle around direction $\hat{\mathbf{s}}$ in phase i resulting from intensities reflected from phase i to

i and transmitted from phase j to i, respectively. The third term on the RHS of Eq. (11) gives the attenuation (with the negative sign) of radiative heat transfer rate per unit solid angle around direction $\hat{\mathbf{s}}$. This term is further developed by using Eq. (8b). The following continuum-scale scattering coefficients and scattering phase functions associated with the superficial average intensity $I_i(\mathbf{x},\hat{\mathbf{s}}_{\rm in})$ are postulated:

$$\sigma_{\rm s,int,\mathit{ii}} \equiv \frac{\int_{A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0} \rho_{ij}^{\prime\frown}(\hat{\mathbf{s}})L_i(\mathbf{r}_{\rm int},\hat{\mathbf{s}})\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,{\rm d}A_{\rm int}}{I_i(\mathbf{x},\hat{\mathbf{s}})V}, \tag{12}$$

$$\sigma_{\rm s,int,\mathit{ij}} \equiv \frac{\int_{A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0} \tau_{ij}^{\prime\frown}(\hat{\mathbf{s}})L_i(\mathbf{r}_{\rm int},\hat{\mathbf{s}})\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,{\rm d}A_{\rm int}}{I_i(\mathbf{x},\hat{\mathbf{s}})V}, \tag{13}$$

$$\Phi_{{\rm int},ii}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}}) \equiv \frac{\int_{A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i<0} \rho_{ij}''(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})L_i(\mathbf{r}_{\rm int},\hat{\mathbf{s}}_{\rm in})\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,{\rm d}A_{\rm int}}{(-4\pi)^{-1}\sigma_{{\rm s,int},ii}I_i(\mathbf{x},\hat{\mathbf{s}}_{\rm in})V}, \quad \hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i>0, \tag{14}$$

$$\Phi_{{\rm int},ji}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}}) \equiv \frac{\int_{A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i<0} \tau_{ji}''(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})L_j(\mathbf{r}_{\rm int},\hat{\mathbf{s}}_{\rm in})\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i\;\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,{\rm d}A_{\rm int}}{(4\pi)^{-1}\sigma_{{\rm s,int},ji}I_j(\mathbf{x},\hat{\mathbf{s}}_{\rm in})V}, \quad \hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i<0. \tag{15}$$

Eq. (10) is rewritten by using Eqs. (11)–(15) as

$$\begin{aligned}\hat{\mathbf{s}}\cdot\nabla_{\mathbf{x}}I_i(\mathbf{x},\hat{\mathbf{s}}) = &-[\kappa_{{\rm d},i}+\sigma_{{\rm s,d},i}+\sigma_{{\rm s,int},ii}+\sigma_{{\rm s,int},ij}]I_i(\mathbf{x},\hat{\mathbf{s}})+n_i^2\kappa_{{\rm d},i}I_{{\rm b},i}(\mathbf{x})\\ &+\frac{1}{4\pi}\int_{4\pi}I_i(\mathbf{x},\hat{\mathbf{s}}_{\rm in})[\sigma_{{\rm s,d},i}\Phi_{{\rm d},i}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})\\ &+\sigma_{{\rm s,int},ii}\Phi_{{\rm int},ii}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})]\,{\rm d}\Omega_{\rm in}\\ &+\frac{\sigma_{{\rm s,int},ji}}{4\pi}\int_{4\pi}I_j(\mathbf{x},\hat{\mathbf{s}}_{\rm in})\Phi_{{\rm int},ji}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})\,{\rm d}\Omega_{\rm in}.\end{aligned} \tag{16}$$

Then, defining

$$\kappa_i \equiv \kappa_{{\rm d},i},\quad \sigma_{{\rm s},ii}\equiv\sigma_{{\rm s,d},i}+\sigma_{{\rm s,int},ii},\quad \sigma_{{\rm s},ij}\equiv\sigma_{{\rm s,int},ij},\quad \sigma_{{\rm s},i}\equiv\sigma_{{\rm s},ii}+\sigma_{{\rm s},ij},\quad \beta_i=\kappa_i+\sigma_{{\rm s},i}, \tag{17a}$$

$$\Phi_{ii}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}}) \equiv \frac{\sigma_{{\rm s,d},i}\Phi_{{\rm d},i}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})+\sigma_{{\rm s,int},ii}\Phi_{{\rm int},ii}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})}{\sigma_{{\rm s},ii}},\quad \Phi_{ji}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})\equiv\Phi_{{\rm int},ji}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}}), \tag{17b}$$

and introducing the intrinsic intensity by virtue of Eq. (5), it follows from Eq. (16)

$$\begin{aligned}f_{V,i}\hat{\mathbf{s}}\cdot\nabla_{\mathbf{x}}I_i^i(\mathbf{x},\hat{\mathbf{s}}) = &-f_{V,i}\beta_iI_i^i(\mathbf{x},\hat{\mathbf{s}})+f_{V,i}n_i^2\kappa_iI_{{\rm b},i}^i(\mathbf{x})\\ &+f_{V,i}\frac{\sigma_{{\rm s},ii}}{4\pi}\int_{4\pi}I_i^i(\mathbf{x},\hat{\mathbf{s}}_{\rm in})\Phi_{ii}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})\,{\rm d}\Omega_{\rm in}\\ &+f_{V,j}\frac{\sigma_{{\rm s},ji}}{4\pi}\int_{4\pi}I_j^j(\mathbf{x},\hat{\mathbf{s}}_{\rm in})\Phi_{ji}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})\,{\rm d}\Omega_{\rm in}.\end{aligned} \tag{18}$$

Eq. (18) is identical with Eq. (1) in [10], given without any derivation, and is consistent with Eqs. (8) and (9) in [16]. The boundary condition for Eq. (18), formulated at $A_{\rm w}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i<0$, reads:

- for a semi-transparent boundary

$$\begin{aligned}I^i(\mathbf{x}_{\rm w},\hat{\mathbf{s}}) = &\int_{\Omega_{\rm in}:\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i>0}\rho_{i,{\rm w}}''(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})I^i(\mathbf{x}_{\rm w},\hat{\mathbf{s}}_{\rm in})\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i\,{\rm d}\Omega_{\rm in}\\ &-f_{V,i}\int_{\Omega_{\rm in}:\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i<0}\tau_{{\rm w},i}''(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})I_{\rm w}(\mathbf{x}_{\rm w},\hat{\mathbf{s}}_{\rm in})\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i\,{\rm d}\Omega_{\rm in},\end{aligned} \tag{19a}$$

- for an opaque boundary

$$\begin{aligned}I^i(\mathbf{x}_{\rm w},\hat{\mathbf{s}}) = &\int_{\Omega_{\rm in}:\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i>0}\rho_{i,{\rm w}}''(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})I^i(\mathbf{x}_{\rm w},\hat{\mathbf{s}}_{\rm in})\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i\,{\rm d}\Omega_{\rm in}\\ &+f_{V,i}n_i^2\varepsilon_{{\rm w},i}'(\hat{\mathbf{s}})I_{\rm b,w}(\mathbf{x}_{\rm w}).\end{aligned} \tag{19b}$$

3.2. Medium consisting of one semi-transparent phase and one opaque phase

For a semi-transparent phase i and an opaque phase j, the discrete-scale radiative intensity exists only within phase i, and the RTE is written only for this phase,

$$\begin{aligned}\hat{\mathbf{s}}\cdot\nabla_{\mathbf{r}}L_i(\mathbf{r},\hat{\mathbf{s}}) = &-[\kappa_{{\rm d},i}+\sigma_{{\rm s,d},i}]L_i(\mathbf{r},\hat{\mathbf{s}})+n_i^2\kappa_{{\rm d},i}L_{{\rm b},i}(\mathbf{r})\\ &+\frac{\sigma_{{\rm s,d},i}}{4\pi}\int_{4\pi}L_i(\mathbf{r},\hat{\mathbf{s}}_{\rm in})\Phi_{{\rm d},i}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})\,{\rm d}\Omega_{\rm in}.\end{aligned} \tag{20}$$

The corresponding boundary condition at $A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i<0$ reads:

$$\begin{aligned}L_i(\mathbf{r}_{\rm int},\hat{\mathbf{s}}) = &\int_{\Omega_{\rm in}:\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i>0}\rho_{ij}''(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})L_i(\mathbf{r}_{\rm int},\hat{\mathbf{s}}_{\rm in})\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i\,{\rm d}\Omega_{\rm in}\\ &+n_i^2\varepsilon_{ji}'(\hat{\mathbf{s}})L_{{\rm b},j}(\mathbf{r}_{\rm int}).\end{aligned} \tag{21}$$

Performing the volume-averaging on Eq. (20) and subsequently applying SAT, one obtains an equation identical to Eq. (10), to which the boundary condition (21) is then applied. Consequently, the following continuum-scale scattering and absorption coefficients and the scattering phase function associated with the superficial average intensities are postulated:

$$\sigma_{\rm s,int} \equiv \frac{\int_{A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0} \rho_{ij}^{\prime\frown}(\hat{\mathbf{s}})L_i(\mathbf{r}_{\rm int},\hat{\mathbf{s}})\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,{\rm d}A_{\rm int}}{I_i(\mathbf{x},\hat{\mathbf{s}})V}, \tag{22}$$

$$\kappa_{\rm int} \equiv \frac{\int_{A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0} \alpha_{ij}'(\hat{\mathbf{s}})L_i(\mathbf{r}_{\rm int},\hat{\mathbf{s}})\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,{\rm d}A_{\rm int}}{I_i(\mathbf{x},\hat{\mathbf{s}})V}, \tag{23}$$

$$\kappa_{\rm int} \equiv -\frac{\int_{A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i<0} \varepsilon_{ji}'(\hat{\mathbf{s}})L_{{\rm b},j}(\mathbf{r}_{\rm int})\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,{\rm d}A_{\rm int}}{I_{{\rm b},j}(\mathbf{x})V}, \tag{24}$$

$$\Phi_{\rm int}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}}) \equiv \frac{\int_{A_{\rm int}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i<0} \rho_{ij}''(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})L_i(\mathbf{r}_{\rm int},\hat{\mathbf{s}}_{\rm in})\hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,{\rm d}A_{\rm int}}{(-4\pi)^{-1}\sigma_{\rm s,int}I_i(\mathbf{x},\hat{\mathbf{s}}_{\rm in})V}, \quad \hat{\mathbf{s}}_{\rm in}\cdot\hat{\mathbf{n}}_i>0. \tag{25}$$

Since, according to Kirchhoff's law, $\alpha_{ij}'(\hat{\mathbf{s}})=\varepsilon_{ji}'(\hat{\mathbf{s}})$, the definitions of the absorption coefficient given by Eqs. (23) and (24) satisfy the condition of local thermodynamic equilibrium for a small isothermal volume of a non-scattering medium irradiated from a black source, $L_i(\mathbf{r},\hat{\mathbf{s}})=L_{{\rm b}j}(\mathbf{r},\hat{\mathbf{s}})$ [2,3]. Furthermore, defining

$$\kappa_i\equiv\kappa_{{\rm d},i}+\kappa_{\rm int},\quad \sigma_{{\rm s},i}=\sigma_{{\rm s,d},i}+\sigma_{\rm s,int},\quad \beta_i=\kappa_i+\sigma_{{\rm s},i}, \tag{26a}$$

$$\Phi_i(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})=\frac{\sigma_{{\rm s,d},i}\Phi_{{\rm d},i}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})+\sigma_{\rm s,int}\Phi_{\rm int}(\hat{\mathbf{s}}_{\rm in},\hat{\mathbf{s}})}{\sigma_{{\rm s},i}}, \tag{26b}$$

leads to

$$\hat{\mathbf{s}}\cdot\nabla_{\mathbf{x}}I_i^i(\mathbf{x},\hat{\mathbf{s}}) = -\beta_i I_i^i(\mathbf{x},\hat{\mathbf{s}}) + n_i^2\kappa_{\mathrm{d},i}I_{\mathrm{b},i}^i(\mathbf{x}) + n_i^2\kappa_{\mathrm{int}}I_{\mathrm{b},j}^i(\mathbf{x}) + \frac{\sigma_{\mathrm{s},i}}{4\pi}\int_{4\pi} I_i^i(\mathbf{x},\hat{\mathbf{s}})\Phi_i(\hat{\mathbf{s}}_{\mathrm{in}},\hat{\mathbf{s}})\,\mathrm{d}\Omega_{\mathrm{in}}. \tag{27}$$

Eq. (27) reduces to that underlying the analysis in [7,8] for optically-thin phase i, i.e. $\kappa_{\mathrm{d}} = \sigma_{\mathrm{s,d}} = 0$. It is also consistent with the results of the derivation presented in [16]. The boundary condition for Eq. (27) is the same as for Eq. (18) and is given by Eq. (19).

4. Continuum-scale radiative properties

The radiative coefficients and the scattering phase functions appearing in Eqs. (18) and (27) can be determined based on their definitions, Eqs. (12)–(15) and (22)–(25). Here, an example is given how to obtain $\sigma_{\mathrm{s,int}}$ and $\Phi_{\mathrm{int}}(\hat{\mathbf{s}}_{\mathrm{in}},\hat{\mathbf{s}})$ for a medium consisting of a semi-transparent phase i and an opaque phase j but the other radiative properties can be determined analogously. Consider an averaging volume $V = As$ in a form of a slab, which is uniformly illuminated with collimated radiation at a single slab boundary A within the semi-transparent phase i. The thickness s of the volume is small enough so that the contribution to the intensity into direction $\hat{\mathbf{s}}$ by multiple internal scattering and interface reflections, and the shadowing effects can be neglected. The boundary area A is large enough so that all typical morphological interface elements are present in V. It follows from Eqs. (22) and (25):

$$\sigma_{\mathrm{s,int}} = \frac{\int_{A_{\mathrm{int}}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0}\rho_{ij}'^{\frown}(\hat{\mathbf{s}})\exp[-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})s^*]\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,\mathrm{d}A_{\mathrm{int}}}{\int_{V_i}\exp[-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})s^*]\,\mathrm{d}V}, \tag{28}$$

$$\Phi_{\mathrm{int}}(\hat{\mathbf{s}},\hat{\mathbf{s}}_{\mathrm{r}}) \equiv \frac{\int_{A_{\mathrm{int}}:\hat{\mathbf{s}}_{\mathrm{r}}\cdot\hat{\mathbf{n}}_i<0}\rho_{ij}''(\hat{\mathbf{s}},\hat{\mathbf{s}}_{\mathrm{r}})\exp[-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})s^*]\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,\hat{\mathbf{s}}_{\mathrm{r}}\cdot\hat{\mathbf{n}}_i\mathrm{d}A_{\mathrm{int}}}{(-4\pi)^{-1}\int_{A_{\mathrm{int}}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0}\rho_{ij}'^{\frown}(\hat{\mathbf{s}})\exp[-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})s^*]\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,\mathrm{d}A_{\mathrm{int}}},\quad \hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0, \tag{29}$$

where s^* is the distance from the illuminated boundary to the nearest interface element $\mathrm{d}A_{\mathrm{int}}$ measured along the actual path into the direction $\hat{\mathbf{s}}$. $\hat{\mathbf{s}}_{\mathrm{in}}$ and $\hat{\mathbf{s}}$ in Eq. (25) have been replaced by $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}_{\mathrm{r}}$ in Eq. (29), respectively. In the particular case of optically-thin phase i:

$$\sigma_{\mathrm{s,int}} = \frac{\int_{A_{\mathrm{int}}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0}\rho_{ij}'^{\frown}(\hat{\mathbf{s}})\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\mathrm{d}A_{\mathrm{int}}}{V_i}, \tag{30}$$

$$\Phi_{\mathrm{int}}(\hat{\mathbf{s}},\hat{\mathbf{s}}_{\mathrm{r}}) = \frac{\int_{A_{\mathrm{int}}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i<0}\int\rho_{ij}''(\hat{\mathbf{s}},\hat{\mathbf{s}}_{\mathrm{r}})\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\hat{\mathbf{s}}_{\mathrm{r}}\cdot\hat{\mathbf{n}}_i\,\mathrm{d}A_{\mathrm{int}}}{(-4\pi)^{-1}\int_{A_{\mathrm{int}}:\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0}\rho_{ij}'^{\frown}(\hat{\mathbf{s}})\hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i\,\mathrm{d}A_{\mathrm{int}}},\quad \hat{\mathbf{s}}\cdot\hat{\mathbf{n}}_i>0. \tag{31}$$

Eqs. (30) and (31) are consistent with the results of derivation of radiative properties for uniformly-irradiated large opaque spheres, given in standard textbooks [2,3]. Eqs. (28) and (29) can be readily solved by the Monte Carlo method for known exact morphology and radiative properties of each phase and the interface. A large number N_i of stochastic rays, which represent the discrete-scale intensity, are launched within phase i at the illuminated boundary. $N_{\mathrm{c},i}$ rays will collide with the interface. Since for each ray the associated fraction of A is the same and equal to $1/N$, it follows that

$$\sigma_{\mathrm{s,int}} \approx \frac{-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})\sum_{k=1}^{N_{\mathrm{c},i}}\{\rho_{ij}'^{\frown}(\hat{\mathbf{s}})\exp[-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})s_k]\}}{\sum_{k=1}^{N_i}\{\exp[-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})s_k]\}}, \tag{32}$$

$$\Phi_{\mathrm{int}}(\hat{\mathbf{s}},\hat{\mathbf{s}}_{\mathrm{r}}) \equiv \frac{\sum_{k=1}^{N_{\mathrm{c},i}}\{\rho_{ij}''(\hat{\mathbf{s}},\hat{\mathbf{s}}_{\mathrm{r}})\exp[-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})s_k]\hat{\mathbf{s}}_{\mathrm{r}}\cdot\hat{\mathbf{n}}_i\}}{(4\pi)^{-1}\sum_{k=1}^{N_{\mathrm{c},i}}\{\rho_{ij}'^{\frown}(\hat{\mathbf{s}})\exp[-(\kappa_{\mathrm{d},i}+\sigma_{\mathrm{s,d},i})s_k]\}}. \tag{33}$$

In practice, the size of the base A is limited by the available size of a medium sample, which can be insufficient to include all typical morphological interface elements. In this case, the thin slab V can be replaced by a statistically equivalent volume consisting of a large number N of randomly located and oriented sub-volumes, each of volume As/N. The number of the stochastic rays, the thickness s and the boundary area A should be chosen by considering the convergence criteria applied to the radiative properties.

The approach presented in this paper is based on the differential interpretation of the radiative properties associated with the average intensities, as given by Eqs. (12)–(15), and (22)–(25), i.e. effective attenuation cross-sections are considered. It is applicable to two-phase media, in which both phases have high connectivity, or in which one phase has low connectivity, e.g. it consists of closed cells or semi-transparent large particles. The methodology presented in [7,8,10,14] is based on the integral interpretation of the radiative properties, i.e. mean penetration paths are considered. MC is directly used there to determine the mean penetration path, which in turn is used to calculate the extinction coefficient for the volume-averaged RTE. This is equivalent to the assumption $L_i = I_i^i$. However, no limitations are imposed on the maximum possible path length of a stochastic ray, which leads to discontinuity in the distribution of discrete-scale intensity for a low-connectivity phase, to which in turn the exponential function should be fitted. In this case, the radiative properties may still be determined when fitting is performed in the limit of sufficiently short path lengths. However, this may lead to high uncertainties in the radiative coefficients and the approach presented in this paper may be preferred.

Finally, the intrinsic average intensities obtained from Eq. (18) or (27) are used to calculate the intrinsic average divergence of radiative flux as [2,3]

$$\langle\nabla\cdot\mathbf{q}_{\mathrm{r},i}(\mathbf{r})\rangle^i = \kappa_{\mathrm{d},i}\left[4\pi n_i^2 I_{\mathrm{b},i}^i(\mathbf{x}) - \int_0^{4\pi} I_i^i(\mathbf{x},\hat{\mathbf{s}})\,\mathrm{d}\Omega\right]. \tag{34}$$

Since Eqs. (18) and (27) are subject to the assumptions of Sections 2 and 3, Eq. (34) can be used only to solve a volume-averaged energy equation for a given phase to obtain its volume-averaged temperature [19,20]. A further development will be pursued to develop continuum-scale equations for media with spatially varying properties due to their temperature dependency, space variation of the internal properties and the morphological characteristics.

5. Summary and conclusions

The spatial averaging theorem has been applied to radiative heat transfer in a two-phase medium consisting of either two semi-transparent or one semi-transparent and one opaque homogeneous phases. The continuum-scale equations of radiative transfer and the corresponding boundary conditions were rigorously derived based on the RTEs and the corresponding boundary conditions applied at the discrete scale, and by utilizing discrete-scale radiative properties of each phase and the interface between the phases. The continuum-scale RTEs are consistent with those derived independently in [16] by using a different approach. This confirms that radiative transfer in heterogeneous two-phase media consisting of arbitrary-type phases in the range of geometrical optics can be modeled by a set of two continuum-scale equations of radiative transfer describing the variation of the average intensities associated with each phase. A Monte Carlo methodology for the determination of the continuum-scale radiative properties was demonstrated for a medium consisting of an optically-thin phase and an opaque phase. The approach presented in this paper is based on the differential interpretation of the continuum-scale radiative properties. It can be applied to two-phase media, in which the dispersed phase has arbitrary connectivity. This makes it an alternative to the integral approach presented in the previous pertinent studies. Further work will focus on detailed numerical examination of the proposed Monte Carlo methodology and on combined continuum-scale conduction-radiation heat transfer analysis in two-phase media consisting of arbitrary-type phases in the range of geometrical optics.

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