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ON EQUALITIES INVOLVING INTEGRALS OF THE LOGARITHM OF THE RIEMANN ζ-FUNCTION AND EQUIVALENT TO THE RIEMANN HYPOTHESIS

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By using the generalized Littlewood theorem about a contour integral involving the logarithm of an analytic function, we show how an infinite number of integral equalities involving integrals of the logarithm of the Riemann ζ-function and equivalent to the Riemann hypothesis can be established and present some of them as an example. It is shown that all earlier known equalities of this type, viz., the Wang equality, Volchkov equality, Balazard–Saias–Yor equality, and an equality established by one of the authors, are certain special cases of our general approach.

1. Introduction

Among numerous statements shown to be equivalent to the Riemann hypothesis (RH; see, e.g., [1] for details and the general discussion of the Riemann ζ-function and Riemann hypothesis), there are a few stating that certain integrals involving the logarithm of the Riemann ζ-function must have a certain well-defined value. Actually, we know four of them established by Wang [2], Volchkov [3], Balazard, Saias, and Yor [4], and one of us [5]. As it stands today, these equalities look as isolated and not connected with each other, and the methods used to obtain them were quite different. In the present paper based on our hitherto unpublished notes posted in ArXiv [6], we apply the generalized Littlewood theorem on the contour integrals of the logarithm of an analytic function and introduce a general recipe of getting infinitely many integral equalities of the above-mentioned type. This recipe is illustrated by presenting a number of equalities of this sort. In particular, all earlier known integral equalities equivalent to the Riemann hypothesis are obtained as certain special cases of our general approach.

2. Generalized Littlewood Theorem

The Littlewood theorem on the contour integrals of the logarithm of an analytic function [7] is well known and widely used in the theory of functions in general and in the Riemann-function research in particular; see, e.g., [1]. The following generalization of the theorem is quite straightforward and was actually used by Wang [2].

Throughout the paper, we use the notation $z = x + iy$ and/or $z = \sigma + it$ on the equal footing. Consider two functions $f(z)$ and $g(z)$. Let $F(z) = \ln(f(z))$. We introduce and analyze the integral

$$\int\limits_C F(z)g(z)dz$$

along a rectangular contour C formed by a broken line composed of four straight segments connecting the vertices with the coordinates $X_1 + iY_1$, $X_1 + iY_2$, $X_2 + iY_2$, and $X_2 + iY_1$. Suppose that $X_1 < X_2$, $Y_1 < Y_2$, that

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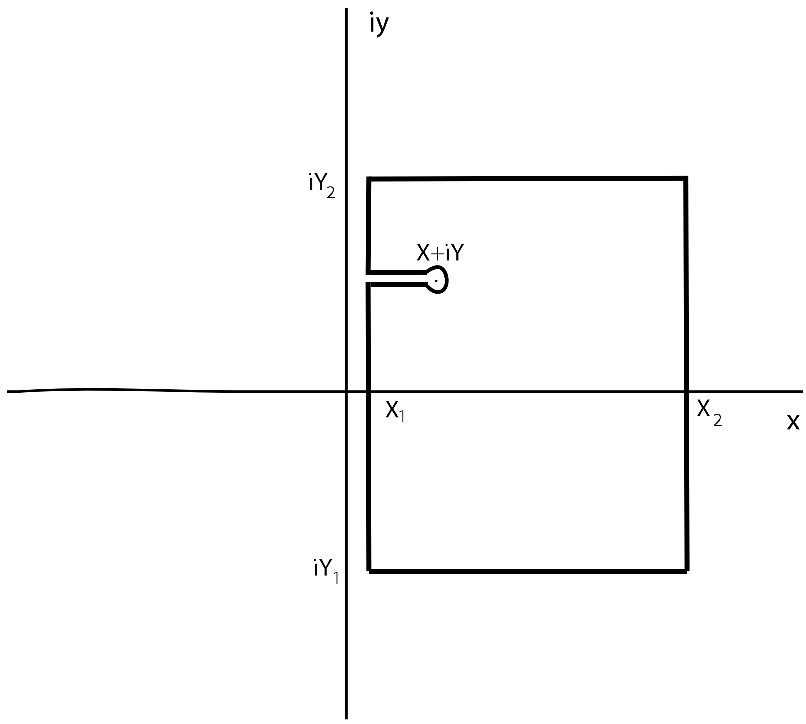

Fig. 1. Contour C used in the proof of the generalized Littlewood theorem

$f(z)$ is analytic and nonzero on C and meromorphic inside it, and that $g(z)$ is analytic on C and meromorphic inside it. First, we assume that the poles and zeros of the functions $F(z)$, $g(z)$ do not coincide and that the function $f(z)$ has no poles but has only one simple zero located at the point $X+iY$ in the interior of the contour. Further, we make a cut along the segment of the straight line X_1+iY, $X+iY$ and consider a new contour C', namely, the initial contour C plus the cut indenting the point $X+iY$ (see Fig. 1). The Cauchy residue theorem can be applied to the contour C' which means that the value of

$$\int_{C'} F(z)g(z)dz \qquad \text{is} \qquad 2\pi i \sum_{\rho} F(\rho)\,\mathrm{res}(g(\rho)),$$

where the sum is over all simple poles ρ of the function $g(z)$ lying inside the contour.

An appropriate choice of the branches of the logarithm function assures that the difference between two branches of the logarithm function appearing when the integration path indents the point $X+iY$ is $2\pi i$ (see the exact definition in what follows), which means that the value of the integral along the initial contour C is

$$2\pi i \left(\sum_{\rho} F(\rho)\,\mathrm{res}(g(\rho)) - \int_{X_1+iY}^{X+iY} g(z)dz \right).$$

Recall that, in this case, the integral is taken along the straight segment parallel to the real axis and, hence, for this integral, $\mathrm{Im}(z) = \mathrm{const} = Y$ and $dz = dx$. Similarly, a single pole of the function $f(z)$ occurring at the (other) point $X+iY$ contributes

$$2\pi i \int_{X_1+iY}^{X+iY} g(z)dz$$

to the value of the integral along the contour. Thus, by finding the sum of all terms arising from different zeros and poles of the function $f(z)$, we get the following generalization of the Littlewood theorem (its modification for the case of presence of the poles/zeros of the function $f(z)$ of order $m > 1$ is evident):

Theorem 1. *Let C be a rectangle bounded by the lines $x = X_1$, $x = X_2$, $y = Y_1$, and $y = Y_2$, where $X_1 < X_2$ and $Y_1 < Y_2$, and let $f(z)$ be a function analytic and nonzero on C and meromorphic inside this contour. Also let $g(z)$ be analytic on C and meromorphic inside this contour and let the logarithm*

$$F(z) = \ln(f(z))$$

be defined as follows: we start from a particular determination on $x = X_2$ and obtain the values at other points by continuous variations along $y = \text{const}$ from $\ln(X_2 + iy)$. If, however, this path crosses a zero or a pole of $f(z)$, then we take $F(z)$ to be $F(z \pm i0)$ as we approach the path from above or from below. Assume that the poles and zeros of the functions $f(z)$, $g(z)$ do not coincide. Then

$$\int_C F(z)g(z)dz = 2\pi i \left(\sum_{\rho_g} \operatorname{res}(g(\rho_g)F(\rho_g)) - \sum_{\rho_f^0} \int_{X_1+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz + \sum_{\rho_f^{\text{pol}}} \int_{X_1+iY_\rho^{\text{pol}}}^{X_\rho^{\text{pol}}+iY_\rho^{\text{pol}}} g(z)dz \right)$$

where the sum is over all ρ_g that are simple poles of the function $g(z)$ lying inside C, all $\rho_f^0 = X_\rho^0 + iY_\rho^0$ that are zeros of the function $f(z)$ counted taking into account their multiplicities (i.e., the corresponding term is multiplied by m for any zero of order m) and lying inside C, and all $\rho_f^{\text{pol}} = X_\rho^{\text{pol}} + iY_\rho^{\text{pol}}$ that are poles of the function $f(z)$ counted taking into account their multiplicities and lying inside C. In order that this assertion be true, all relevant integrals on the right-hand side of the equality must exist.

Clearly, the exact nature of the contour is actually irrelevant for the theorem, as it is for the Littlewood theorem. As for the last remark concerning the existence of integrals taken along the segments $\left[X_1 + iY_\rho,\ X_\rho + iY_\rho\right]$ parallel to the real axis (their inexistence might happen, e.g., if a pole of the $g(z)$ function occurs on the segment), the corresponding integration path can often be modified in such a way that it becomes possible to take these integrals. The case of coincidence of poles and zeros of the functions $f(z)$, $g(z)$ often does not pose real problems and can easily be analyzed. We consider several cases of this sort in what follows.

It is also clear that the theorem remains true if there are poles or zeros of $f(z)$ on the left border of the contour $x = X_1$: they contribute nothing to the value of the integral along the contour but the integral over the line $x = X_1$ should be understood as the principal value. The presence of poles/zeroes of the function $f(z)$ on the right border of the contour $x = X_2$ can be treated similarly.

3. Application of the Generalized Littlewood Theorem in Deducing Equalities Equivalent to the Riemann Hypothesis

In what follows, all integrals involving the logarithm of the Riemann ζ-function are understood in the sense of the generalized Littlewood theorem.

3.1. Power Functions. The most "natural" starting point to apply the generalized Littlewood theorem with an aim to deduce an equality equivalent to the Riemann hypothesis is to take in the conditions of the above-mentioned theorem

$$f(z)=\zeta(z),\quad g(z)=\frac{1}{z^2},\quad X_1=b,\quad X_2=b+T,\quad Y_1=-T,\quad \text{and}\quad Y_2=T$$

with real $b\geq 1/2,\ b\neq 1,$ as $T\to\infty$ and consider the contour integral

$$\int_C \ln(f(z))g(z)dz.$$

The known asymptotic properties of the Riemann function for large values of the argument guarantee that the value of the contour integral taken along the "external" three straight lines containing at least one of the points $b+T+iT$ or $b+T-iT$ tends to zero as $T\to\infty$ (this was shown for a similar $O(1/z^2)$ function $g(z)$ by Wang [2] and his considerations can be repeated without changes for any function $O(1/z^{\alpha})$, $\operatorname{Re}(\alpha)>1$) and, thus, we conclude that the value of the contour integral tends to

$$-i\int_{-\infty}^{\infty}\frac{\ln(\zeta(b+it))}{(b+it)^2}dt;$$

the sign "$-$" is explained here by the necessity to describe the contour in the counterclockwise direction; the integral is taken along the line $z=b+it$, where $dz=idt$ and

$$g(z)=\frac{1}{(b+it)^2}.$$

Suppose that the RH is true and there are no zeros inside the contour. Then, by the Littlewood theorem, the value of the integral is equal to the contribution of the simple pole of the Riemann function observed for $z=1$:

$$-i\int_{-\infty}^{\infty}\frac{\ln(\zeta(b+it))}{(b+it)^2}dt=2\pi i\int_b^1\frac{dx}{x^2}=-2\pi i(1-1/b)$$

and, thus,

$$\int_{-\infty}^{\infty}\frac{\ln(\zeta(b+it))}{(b+it)^2}dt=2\pi(1-1/b)$$

for $b<1$ and

$$\int_{-\infty}^{\infty}\frac{\ln(\zeta(b+it))}{(b+it)^2}dt=0$$

for $b > 1$. If there is a zero $\rho = \sigma_k + it_k$ with the real part $\sigma_k > b$, then it makes the contribution

$$-2\pi i \int\limits_{b+it}^{\sigma+it} \frac{dx}{x^2}.$$

Hence, the joint contribution of two complex conjugate zeroes is equal to

$$4\pi i \left(\frac{1}{\sigma^2 + t^2} - \frac{1}{b^2 + t^2} \right),$$

and we arrive at the equality

$$\frac{1}{2\pi i} \int\limits_{b-i\infty}^{b+i\infty} \frac{\ln(\zeta(z))}{z^2} dz = 1 - 1/b - 2 \sum_{\rho:\sigma_k > b, t_k > 0} \left(\frac{1}{\sigma_k^2 + t_k^2} - \frac{1}{b^2 + t_k^2} \right).$$

All terms under the sign of the sum are definitely negative and, thus, we have established our first equality equivalent to the Riemann hypothesis:

Theorem 2. *The equality*

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{\ln(\zeta(b + it))}{(b + it)^2} dt = 1 - \frac{1}{b}, \tag{1}$$

where $1 > b \geq 1/2$, *is true for some b if and only if there are no zeros of the Riemann function with* $\sigma > b$. *For* $b = 1/2$, *this equality is equivalent to the Riemann hypothesis.*

The case $b = 1/2$, i.e.,

$$\int\limits_{-\infty}^{\infty} \frac{\ln(\zeta(1/2 + it))}{(1/2 + it)^2} dt = -2\pi,$$

is analyzed numerically.

The same theorem takes a more elegant form if we apply the generalized Littlewood theorem to the function $f(z) = \zeta(z)(z - 1)$, thus killing the contribution of the simple pole:

Theorem 2a. *The equality*

$$\frac{1}{2\pi} \int\limits_{b-i\infty}^{b+i\infty} \frac{\ln(\zeta(z)(z - 1))}{z^2} dz = 0,$$

where $1 > b \geq 1/2$, *is true for some* b *if and only if there are no zeros of the Riemann function with* $\sigma > b$. *For* $b = 1/2$, *this equality is equivalent to the Riemann hypothesis.*

Remark 1. The use of inverse square function $g(z) = 1/z^2$ is not necessary under the conditions of Theorems 2 and 2a and a very broad set of power functions $g(z) = 1/z^\alpha$, where α is real and greater than 1, although not excessively large, can be used instead of this function. From the results of numerical calculations performed by Wedeniwski, as cited by Ramaré and Saouter [8], it follows that there are no "incorrect" Riemann zeros with $\sigma > 1/2$ if $t < t_{\min} = 3.3 \cdot 10^9$. (We can also mention more recent calculations by Gourdon and Demichel, where it is reported that the first 10^{13} Riemann zeros are located on the critical line [9] but we fail to get an exact value of T from their paper.) This means that the argument of the "incorrect" zeros lies in very narrow intervals: $(\pi/2, \ \cong \pi/2 - 1/t_{\min})$ for positive t and $(-\pi/2, \ \cong -\pi/2 + 1/t_{\min})$ for negative t. Hence, the argument of the integrand in the contribution of this Riemann zero to the values of the contour integral

$$\int\limits_{b+it_k}^{\sigma_k+it_k} \frac{1}{z^\alpha} dz$$

lies inside the intervals $(\pm\alpha\pi/2, \ \cong \ \pm\alpha\pi/2 \mp \alpha/t_{\min})$ and, therefore, the same sign of the real part of the sum of two paired contributions of complex conjugate zeros is certain, except the cases where α takes the form $\alpha = 2n + 1 + \delta$, where n is integer and δ is positive and very small, or α is very large. Thus, for α close to 1, it suffices to take α "somewhat larger" than

$$1 + \frac{2}{\pi t_{\min}} \cong 1 + 2 \cdot 10^{-10}$$

to guarantee the same signs of all "incorrect" contributions of Riemann zeros. Similarly, α is bounded from above:

$$\alpha < \frac{\pi t_{\min}}{2} \cong 5.18 \cdot 10^9.$$

All necessary details can be easily and accurately studied but we do not do this here and restrict ourselves to several examples of the equalities equivalent to the Riemann hypothesis:

$$\int\limits_{1/2-i\infty}^{1/2+i\infty} \frac{\ln(\zeta(z)(z-1))}{z^{1+3\cdot 10^{-10}}} dz = 0,$$

$$\int\limits_{1/2-i\infty}^{1/2+i\infty} \frac{\ln(\zeta(z)(z-1))}{z^3} dz = 0,$$

$$\int\limits_{1/2-i\infty}^{1/2+i\infty} \frac{\ln(\zeta(z)(z-1))}{z^{5\cdot 10^9}} dz = 0,$$

etc.

Despite their very simple form, these equalities have a drawback connected with the fact that they are "non-symmetric" with respect to the real and imaginary parts and cannot be represented in the form containing integrals only of the argument or of the logarithm of the module. This drawback can be removed by taking "more symmetric" functions $g(z)$. Thus,

$$g(z) = \frac{1}{a^2 - (z-b)^2}$$

seems to be the most natural first example. Here, a and b are arbitrary real positive numbers, $1 > b \geq 1/2$, $a + b \neq 1$, and the contour integral

$$\int_C \ln(f(z))g(z)dz$$

is reduced to the form

$$I = -i \int_{-\infty}^{\infty} \frac{\ln(\zeta(b+it))}{a^2+t^2} dt.$$

Thus, inside the contour, we now have a simple pole at $z = b + a$, which gives the contribution

$$-\frac{i\pi}{a} (\ln(\zeta(a+b)))$$

to the value of the integral. The contribution of a zero of the Riemann function $\rho = \sigma_k + it_k$ with real part $\sigma_k > b$ is given by the elementary integral

$$\int_0^{\sigma_k - b} \frac{dp}{a^2 - (p+it_k)^2},$$

whose real part is equal to

$$\int_0^{\sigma_k - b} \frac{a^2 - p^2 + t_k^2}{(a^2 - p^2 + t_k^2)^2 + 4p^2 t_k^2} dp.$$

It is well known that possible values of t_k for the zeros of the Riemann function with $\sigma_k > 1/2$ (if they exist) are very large, while $0 < p < 1/2$. Thus, the expression $a^2 - p^2 + t_k^2$ is always positive and, therefore, all contributions of these Riemann zeros are also positive. The evaluation of the contribution of the simple pole at $z = 1$ is trivial, and we get the following theorem:

Theorem 3. *The equality*

$$\frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\ln|\zeta(b+it)|}{a^2+t^2} dt = \ln\left|\frac{\zeta(a+b)(a+b-1)}{a-b+1}\right|, \tag{2}$$

where a and b are arbitrary real positive numbers and $1 > b \geq 1/2$, $a + b \neq 1$, *is true for some b if and only if there are no zeros of the Riemann function with* $\sigma > b$. *For* $b = 1/2$, *this equality is equivalent to the Riemann hypothesis.*

Note that, in this case, the integral is taken over the logarithm of the module. The corresponding equality for the integrals involving the argument is trivially satisfied due to the parity of the involved functions. As above, if we consider $f(z) = \zeta(z)(z-1)$ then the integral can be represented in the form

$$\frac{a}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\zeta(b+it)(b-1+it)|}{a^2+t^2}dt = \ln|\zeta(a+b)(a+b-1)|.$$

This same hint can always be applied and, hence, we do not mention this fact in what follows. In this stage, it is also reasonable to note that we have performed the numerical analysis not only of equality (2) with $a = 1$ and $b = 1/2$ but also of a more general equality

$$\frac{a}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\zeta(b+it)|}{a^2+t^2}dt = \ln\left|\frac{\zeta(a+b)(a+b-1)}{a-b+1}\right| + \sum_{\sigma_k > b} \ln\left|\frac{a+\sigma_k-b+it_k}{a-\sigma_k+b-it_k}\right|$$

with $a = 1$ and $b = 1/4$, where the first 649 zeros of the Riemann function were taken into account [6].

The case $a+b = 1$ does not pose any problems as the evident limit of the equalities presented above. Setting $a = 1-b+\delta$ and taking the limit as $\delta \to 0$, we find

$$\frac{1-b}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\zeta(b+it)|}{(1-b)^2+t^2}dt = \ln|\zeta(1+\delta)\,\delta| - \ln|2-2b|.$$

By using the well-known Laurent expansion of the Riemann zeta-function in the vicinity of $z = 1$, i.e.,

$$\zeta(1+\delta) = \frac{1}{\delta} + \ldots,$$

we readily conclude that

$$\frac{1-b}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\zeta(b+it)|}{(1-b)^2+t^2}dt = -\ln|2-2b|$$

and, thus, arrive at the following theorem:

Theorem 3a. *The equality*

$$\frac{1-b}{\pi}\int_{-\infty}^{\infty} \frac{\ln|\zeta(b+it)|}{(1-b)^2+t^2}dt = -\ln(2-2b), \tag{3}$$

where b is an arbitrary positive number such that $1 > b \geq 1/2$, holds for some b if and only if there are no zeros of the Riemann function with $\sigma > b$. For $b = 1/2$, this equality, i.e.,

$$\int_{-\infty}^{\infty} \frac{\ln|\zeta(1/2+it)|}{1/4+t^2}dt = 0,$$

is equivalent to the Riemann hypothesis.

The last equality in Theorem 3a is, clearly, the Balazard–Saias–Yor equality [4].

Similarly, by considering "impaired" functions $g(z)$ (the most straightforward example of these functions is given by a function

$$g(z) = -i\frac{z-b}{(c^2-(z-b)^2)(d^2-(z-b)^2)},$$

which takes the form

$$g(t) = \frac{t}{(c^2+t^2)(d^2+t^2)}$$

along the line $z = b+it$), we can get integral equalities with integrals over the argument. In this way, we establish the following theorem:

Theorem 4. *The equality*

$$\int_0^{\infty} \frac{t\arg(\zeta(b+it))}{(c^2+t^2)(d^2+t^2)}dt = \frac{\pi}{2(d^2-c^2)}\ln\left|\frac{\zeta(b+d)}{\zeta(b+c)}\right| + \frac{\pi}{2(d^2-c^2)}\ln\left|\frac{(d^2-(1-b)^2)c^2}{(c^2-(1-b)^2)d^2}\right|, \tag{4}$$

where b, c, and d are real positive numbers such that $c\cdot d \le 3.2\cdot 10^{19}$, $c \ne d$, $b+c \ne 1$, $b+d \ne 1$, *and* $1 > b \ge 1/2$, *holds for some b if and only if there are no zeros of the Riemann function with* $\sigma > b$. *For* $b = 1/2$, *this equality is equivalent to the Riemann hypothesis.*

The details of the proof are straightforward and, thus, omitted. Actually, they can be found in our ArXiv submissions [6]. The appearance of the condition $c\cdot d \le 3.2\cdot 10^{19}$ is explained by the fact that the contribution of the zero of the Riemann function $\rho = \sigma_k + it_k$ with $\sigma_k > b$ is given by the integral

$$I_\rho = -2\pi\int_0^{\sigma_k-b} \frac{q+it_k}{(c^2-q^2+t_k^2-2iqt_k)(d^2-q^2+t_k^2-2iqt_k)}dq.$$

Hence, the real part of this integral is the integral of the expression

$$-q\frac{-3t_k^4-2t_k^2(q^2+c^2+d^2)-q^2(c^2+d^2)+c^2d^2+q^4}{((c^2-q^2+t_k^2)^2+4q^2t_k^2)((d^2-q^2+t_k^2)^2+4q^2t_k^2)}.$$

As follows from our choice of b, $0 \le q < 1/2$ (all Riemann zeros lie in the critical strip), while $t < t_{\min} = 3.3\cdot 10^9$; see Remark 1. Hence, if the product cd is not very large, then the sign of this expression is definitely positive: It suffices that $c\cdot d \le 3t_{\min}^2$, whence we get the condition of the theorem.

As an example, we can take $b = 1/2$, $c = 3/2$, and $d = 7/2$ and then, in view of the equalities $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, we obtain the following rather elegant criterion equivalent to the RH:

$$\int_0^\infty \frac{t \arg(\zeta(1/2 + it))}{(9/4 + t^2)(49/4 + t^2)} dt = \frac{\pi}{20} \ln(18\pi^2/245).$$

As some other conditions, this condition was checked numerically [6].

Further, the case $b + c = 1$ or $b + d = 1$ does not create any problems because the corresponding limit is quite transparent. Proceeding as above, we get the following theorem:

Theorem 4a. *The equality*

$$\int_0^\infty \frac{t \arg(\zeta(b + it))}{(c^2 + t^2)((1-b)^2 + t^2)} dt = \frac{\pi}{2(c^2 - (1-b)^2)} \ln \left| \frac{\zeta(b+c)(c^2 - (1-b)^2)(1-b)}{2c^2} \right|, \tag{5}$$

where b and c are real positive numbers such that $c(1-b) \leq 3.2 \cdot 10^{19}$, $b + c \neq 1$, and $1 > b \geq 1/2$, is true for some b if and only if there are no zeros of the Riemann function with $\sigma > b$. For $b = 1/2$, this equality is equivalent to the Riemann hypothesis.

As an illustration, we can take, e.g., $b = 1/2$ and $c = 3/2$. Thus, in view of $\zeta(2) = \pi^2/6$, we obtain another fairly elegant equality equivalent to the RH (it was also tested numerically):

$$\int_0^\infty \frac{t \arg(\zeta(1/2 + it))}{(9/4 + t^2)(1/4 + t^2)} dt = \frac{\pi}{4} \ln\left(\frac{\pi^2}{27}\right).$$

The case of the integrals

$$\int_0^\infty \frac{t \arg(\zeta(b + it))}{(a^2 + t^2)^2} dt$$

is studied similarly by introducing a function

$$g(z) = -i \frac{z - b}{(a^2 - (z-b)^2)^2}$$

and analyzing the contour integral

$$\int_C \ln(\zeta(z)) g(z) dz$$

taken around the same contour C. In this case, in the interior of the contour, we have a pole of the second order at the point $z = a + b$ which, according to the residue theorem, contributes

$$2\pi i \frac{d}{dz} \left(-i \frac{z-b}{(a + z - b)^2} \ln(\zeta(z)) \right) \Big|_{z=b+a} = \frac{\pi}{2a} \frac{\zeta'(a+b)}{\zeta(a+b)}$$

to the value of the contour integral. Proceeding as earlier, we arrive at the following theorem:

Theorem 5. *The equality*

$$\int_0^\infty \frac{t \arg(\zeta(b+it))}{(a^2+t^2)^2} dt = \frac{\pi}{4a} \frac{\zeta'(a+b)}{\zeta(a+b)} + \frac{\pi}{2} \left(\frac{1}{a^2-(1-b)^2} - \frac{1}{a^2} \right), \tag{6}$$

where a and b are real and positive, $a+b \neq 1$, $a \leq 5.1 \cdot 10^9$, *and* $1 > b \geq 1/2$, *holds for some b if and only if there are no zeros of the Riemann function with* $\sigma > b$. *For* $b = 1/2$, *this equality is equivalent to the Riemann hypothesis.*

Again, the case $a+b=1$ is the evident limit of the expressions presented above for $a+b=1+\delta$ as $\delta \to 0$ (see [6], for further details):

Theorem 5a. *The equality*

$$\int_0^\infty \frac{t \arg(\zeta(b+it))}{((1-b)^2+t^2)^2} dt = \frac{\pi}{4(1-b)} \left(\gamma - \frac{3}{2(1-b)} \right), \tag{7}$$

where b *is real,* $1/2 \leq b < 1$, *and* γ *is the Euler-Mascheroni constant, is true for some* b *if and only if there are no zeros of the Riemann function with* $\sigma > b$. *For* $b = 1/2$, *this equality is equivalent to the Riemann hypothesis.*

For $b = 1/2$, Theorem 5a gives

$$\frac{1}{\pi} \int_0^\infty \frac{2t \arg(\zeta(1/2+it))}{(1/4+t^2)^2} dt = \gamma - 3. \tag{8}$$

This is nothing else than the Volchkov criterion [3].

Remark 2. Volchkov's paper contains this relation more or less explicitly but actually he gave another form of his criterion. Indeed, he deduced (8) and then considered the function

$$S_1(x) = \int_0^x \arg(\zeta(1/2+it))dt.$$

In view of $S_1(x) = O(\ln x)$ [1, p. 214], as a result of the integration by parts, he obtained

$$\int_0^\infty \arg(\zeta(1/2+ix))g(x)dx = -\int_0^\infty S_1(x)\frac{d}{dx}g(x)dx.$$

Then Volchkov introduced the relation

$$S_1(x) = \int_{1/2}^\infty (\ln|\zeta(\sigma+ix)| - \ln|\zeta(\sigma)|)d\sigma$$

and, in view of the fact that

$$\int_{1/2}^{\infty} \ln|\zeta(\sigma)|d\sigma = \text{const},$$

which contributes nothing to the integral

$$\int_{0}^{\infty} S(x)g_1(x)dx,$$

obtained

$$\frac{32}{\pi}\int_{0}^{\infty} \frac{1-12t^2}{(1+4t^2)^3}dt \int_{1/2}^{\infty} \ln|\zeta(\sigma+it)|d\sigma = 3-\gamma.$$

Volchkov's criterion was first published just in this form.

Remark 3. The equalities established above and involving integrals containing the function $\arg(\zeta(b+it))$ as a factor under the integral sign may be interpreted as certain sums over zeros of the Riemann function. Thus, for $b = 1/2$, we have a factor $\arg(\zeta(1/2+it))$ and this factor is exactly the "nontrivial" part of the expression for the number of zeros of the Riemann function lying in the critical strip and having the imaginary part $0 < t < x$:

$$N(x) = 1 - \frac{x\ln\pi}{2\pi} + \frac{1}{\pi}\mathrm{Im}\Big(\ln\Gamma(1/4) + \frac{ix}{2}\Big) + \frac{1}{\pi}\arg(\zeta(1/2+ix)); \tag{9}$$

see, e.g., [1, p. 212] for details. The integral

$$\int_{0}^{\infty} G(x)dN(x)$$

(clearly, if this integral exists) is equal to the sum

$$\sum_{\rho, t_k>0} G(t_k)$$

taken over all Riemann zeros. In practice, this calculation is of interest if the function $G(x)$ is such that it is possible to use the integration by parts:

$$\int_{0}^{\infty} G(x)dN(x) = G(x)N(x)|_0^{\infty} - \int_{0}^{\infty} g(x)N(x)dx,$$

where, evidently, $g(x) = dG(x)/dx$. Volchkov established his criterion in exactly the same way.

Clearly, an infinite number of other integral equalities equivalent to the Riemann hypothesis can be obtained in the same way. We also note that, by considering the "semicontour" C with $X_1 = b$, $X_2 = b+T$, $Y_1 = 0$, and $Y_2 = T$, we can get a number of interesting equalities establishing the relationships between the integrals of the logarithm of the Riemann function taken along the real axis and along the line $(b,\ b + i\infty)$. In some cases, the analysis of the "square-root type" functions, such as

$$g(z) = \frac{1}{(a^2 - (z-b)^2)^{3/2}},$$

might also be of interest; for details, see our contributions [6].

3.2. Exponential Functions. The case of exponential functions $g(z)$, i.e.,

$$g(z) = \frac{i}{\cos(a(z-b))}, \qquad g(z) = \frac{-i(z-b)}{\cos^2(a(z-b))}, \qquad \text{etc.}$$

can be studied similarly. Here, we restrict ourselves to the first function mentioned above. For the analysis of the second function, see [6].

We take the same contour as above. For $z - b = x + iy$, where x, y are real, we conclude that $|\cos(a(z-b))|^{-1} = O(e^{-a|y|})$ for large y provided that $\arg(z-b) \neq 0$ and $\arg(z-b) \neq \pi$. This asymptotic, together with the known asymptotic of $\ln(\zeta(z))$, guaranties the disappearance of the integral taken along the external lines of the contour (i.e., along the lines other than its left border): the problems might appear only for real positive $z - b$ but, in this case, we have $\ln(\zeta(z)) \cong 2^{-z}$ and, hence, it suffices to avoid the values of X for which $g(z)$ has poles as $X \to \infty$.

In the interior of the contour, we have simple poles at the points $z = b + \pi/(2a) + \pi n/a$, where n is either an integer number or a zero and, if $b < 1$, also a simple pole of the Riemann function at $z = 1$. If $b < 1$, then we can also have a number of zeros of the Riemann function in the interior of the contour. Moreover, we definitely have infinitely many zeros for $b < 1/2$. The contribution of the poles of $g(z)$ to the value of the contour integral is, by the residue theorem, equal to

$$2\pi i \left(i \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{a} \ln(\zeta(b + \pi/(2a) + \pi n/a) \right).$$

To find the corresponding contribution of the pole of the Riemann function, we use the generalized Littlewood theorem: For $b < 1$, this contribution is equal to

$$2\pi i \int_b^1 \frac{i}{\cos(a(z-b))} dz = -2\pi \int_0^{1-b} \frac{1}{\cos(ax)} dx = \frac{\pi}{a} \ln \frac{1 - \sin(a(1-b))}{1 + \sin(a(1-b))}.$$

We take $a(1-b) < \pi/2$ to avoid the problems with this integral. Similarly, for $b < \sigma_k$, the zero of the Riemann function $\rho = \sigma_k + it_k$ of order l_k makes the following contribution to the value of the contour integral:

$$-2\pi i \int_{b+it_k}^{\sigma_k + it_k} \frac{i l_k}{\cos(a(z-b))} dz = 2\pi \int_0^{\sigma_k - b} \frac{l_k}{\cos(ax)\cosh(at_k) - i \sin(ax)\sinh(at_k)} dx.$$

Pairing the complex conjugate zeros, we see that, due to the symmetry of the distribution of zeros of the Riemann function, the imaginary part of their contributions vanishes and, thus, collecting everything together, we get the following equality: For $-2 \le b < 1$,

$$\int_0^\infty \frac{\ln|\zeta(b+it)|}{\cosh(at)}dt = \pi \sum_{\rho, t_k > 0} 2l_k \int_0^{\sigma_k - b} \frac{\cos(ax)\cosh(at_k)}{\cos^2(ax)\cosh^2(at_k) + \sin^2(ax)\sinh^2(at_k)}dx$$

$$+ \frac{\pi}{2a}\ln\frac{1-\sin(a(1-b))}{1+\sin(a(1-b))} + \pi\sum_{n=0}^\infty \frac{(-1)^n}{a}\ln(\zeta(b+\pi/(2a)+\pi n/a)). \tag{10}$$

As usual, the sum in this relation is taken over all zeros $\rho = \sigma_k + it_k$ with $b < \sigma_k$ taking into account their multiplicities. In deducing (10), we paired the contributions of the complex conjugate zeros $\rho = \sigma_k \pm it_k$. Hence, $t_k > 0$.

Thus, it is easy to see that if we take $a(1-b) < \pi/2$, then the integrand in

$$\int_0^{\sigma_k - b} \frac{\cos(ax)\cosh(at_k)}{\cos^2(ax)\cosh^2(at_k) + \sin^2(ax)\sinh^2(at_k)}dx$$

is always positive (the value of $\sigma_k - b$ cannot exceed $1-b$). Thus, we have proved the following theorem:

Theorem 6. *The equality*

$$\frac{a}{\pi}\int_0^\infty \frac{\ln|\zeta(b+it)|}{\cosh(at)}dt = \frac{1}{2}\ln\frac{1-\sin(a(1-b))}{1+\sin(a(1-b))} + \sum_{n=0}^\infty (-1)^n \ln(\zeta(b+\pi/(2a)+\pi n/a)), \tag{11}$$

where b and a are real positive numbers such that $1 > b \ge 1/2$ and $a(1-b) < \pi/2$ holds for some b if and only if there are no zeros of the Riemann function with $\sigma > b$. For $b = 1/2$, this equality is equivalent to the Riemann hypothesis.

It is of interest to take $b = 1/2$ and consider the limit as $a \to \pi$. There are no zeros of the Riemann function with $\operatorname{Re} s = 1$ and, hence, the positivity of the contributions of zeros of the Riemann function that do not lie on the critical line is still definite and it remains to consider the limit

$$\lim_{a\to\pi}\left(\frac{1}{2}\ln\frac{1-\sin(a(1-b))}{1+\sin(a(1-b))} + \ln\left(\zeta\left(1/2+\frac{\pi}{2a}\right)\right)\right).$$

This can be done without problems, and we arrive at the following equality equivalent to the Riemann hypothesis:

$$\int_0^\infty \frac{\ln|\zeta(1/2+it)|}{\cosh(\pi t)}dt = \ln\frac{\pi}{2} + \sum_{n=2}^\infty (-1)^{n+1}\ln(\zeta(n)), \tag{12}$$

where the logarithms of the Riemann function corresponding to even n can be expressed via the Bernoulli numbers

$$B_{2m}, \quad m = 1, 2, 3, \ldots : \ \zeta(2m) = (-1)^{m+1}(2\pi)^{2m}\frac{B_{2m}}{2(2m)!},$$

and the following remarkable property of the Riemann ζ-function:

$$\prod_{n=2}^{\infty} \zeta(n) = C = 2.29485\ldots,$$

i.e.,

$$\sum_{n=2}^{\infty} \ln(\zeta(n)) = \ln C = 0.8306\ldots$$

can also be used for calculations; see [10, p. 16]. Here, C is the residue of the pole at $s = 1$ for the Dirichlet series whose coefficients $g(n)$ are the numbers of nonisomorphic Abelian group of order n. Equation (12) was also tested numerically.

Conclusions

In the present paper, we established numerous criteria involving the integrals of the logarithm of the Riemann ζ-function and equivalent to the Riemann hypothesis. Our results include all criteria of this kind known earlier [3, 6, 9]. They are obtained as special cases of the proposed general approach. An infinite number of other criteria of the form: "the integral

$$\int_{b-i\infty}^{b+i\infty} g(z)\ln(\zeta(z))dz = f(b)$$

is equivalent to the RH" can be constructed following the lines of the present paper, i.e., by selecting an appropriate function $g(z)$ and computing the values of the contour integral with the help of the generalized Littlewood theorem.

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