Geom Dedicata (2012) 161:85–107
DOI 10.1007/s10711-012-9695-3

ORIGINAL PAPER

On the second successive minimum of the Jacobian of a Riemann surface

Bjoern Muetzel

Received: 5 January 2010 / Accepted: 9 January 2012 / Published online: 29 February 2012

Abstract To a compact Riemann surface of genus g can be assigned a *principally polarized abelian variety* (PPAV) of dimension g, the *Jacobian* of the Riemann surface. The Schottky problem is to discern the Jacobians among the PPAVs. Buser and Sarnak showed that the square of the first successive minimum, the squared norm of the shortest non-zero vector in the lattice of a Jacobian of a Riemann surface of genus g is bounded from above by $\log(4g)$, whereas it can be of order g for the lattice of a PPAV of dimension g. We show that in the case of a hyperelliptic surface this geometric invariant is bounded from above by a constant and that for any surface of genus g the square of the second successive minimum is equally of order $\log(g)$. We obtain improved bounds for the kth successive minimum of the Jacobian, if the surface contains small simple closed geodesics.

Keywords Compact Riemann surfaces · Hyperelliptic surfaces · Jacobian · Schottky problem

Mathematics Subject Classification (2000) 14H40 · 14H42 · 30F15 · 30F45

1 Introduction

A *principally polarized abelian variety (PPAV)* of dimension g may be defined as a pair (A, H), where $A = \mathbb{C}^g/L$ is a complex torus of dimension g, the quotient of $\mathbb{C}^g$ modulo a lattice L. Furthermore, H, the *polarization*, is a positive definite hermitian form, whose imaginary part ImH is integral on the lattice points of the lattice L. It is *principal*, if it satisfies certain conditions (see [3], Section 4.1.).

The author has been supported by the Swiss National Science Foundation (grant nr. 200021-125324).

B. Muetzel (✉)
Department of Mathematics, Ecole Polytechnique Fédérale de Lausanne Station 8,
1015 Lausanne, Switzerland
e-mail: bjorn.mutzel@epfl.ch

Certain PPAV arise from compact Riemann surfaces in the following way. Let $(\alpha_i)_{i=1\ldots 2g}$ be a *canonical homology basis* of closed cycles on the surface S. This basis is assumed to be given in the following way. Each α_i is a simple closed curve and the curves are paired, such that for each α_i there exists exactly one $\alpha_{\tau(i)} \in \left(\alpha_j\right)_{j=1\ldots 2g}$ that intersects α_i in exactly one point and there are no other intersection points. In the vector space of harmonic forms on S let $(u_k)_{k=1\ldots 2g}$ be a *dual basis for* $(\alpha_i)_{i=1\ldots 2g}$ defined by

$$\int_{\alpha_i} u_k = \delta_{ik}.$$

The *period matrix* P_S of a compact Riemann surface (R.S.) S of genus g is the Gram matrix

$$P_S = \left(\langle u_i, u_j\rangle\right)_{i,j=1\ldots 2g} = \left(\int_S u_i \wedge {}^*u_j\right)_{i,j=1\ldots 2g}$$

This period matrix P_S defines a complex torus, the *Jacobian* or *Jacobian variety* $J(S)$ of the Riemann surface S. By Riemann's period relations the Jacobian is a PPAV.

The moduli or parameter space $\mathscr{A}_g$ for PPAVs of dimension g has dimension $\frac{1}{2}g(g+1)$, whereas the moduli space of compact Riemann surfaces of genus g, $\mathscr{M}_g$, has dimension $3g-3$. The assignment of the Jacobian $J(S)$ to the R.S. S provides a mapping $t : \mathscr{M}_g \rightarrow \mathscr{A}_g$. By Torelli's theorem, this mapping is injective. In general, it is not known, if a given PPAV is the image of a Jacobian under t. The Schottky problem is to describe the sublocus $t(\mathscr{M}_g)$ in $\mathscr{A}_g$.

The closure of the sublocus of Jacobian varieties $\overline{t(\mathscr{M}_g)}$ in the parameter space $\mathscr{A}_g$ is only equal to $\mathscr{A}_g$ for $g = 2$ and 3. For $g \geq 4$ it is a proper closed subset. Several analytic approaches have been used to further characterize $t(\mathscr{M}_g)$. Notably van Gemen proved in van Geemen [15] that $t(\mathscr{M}_g)$ is an irreducible component of the locus $\mathscr{S}_g$ defined by the Schottky–Jung polynomials. However, an exact description of the locus $t(\mathscr{M}_g)$, given in terms of polynomials of theta constants that vanish on $t(\mathscr{M}_g)$, but not on $\mathscr{A}_g$, is only known for $g = 4$ [13]. Shiota [14] showed that an indecomposable PPAV is the Jacobian of a Riemann surface, if the corresponding theta function fulfills the K-P differential equation. However, a solution to this equation can not as yet be determined explicitly.

In Buser and Sarnak [5], it was shown that the Jacobians can be characterized among the PPAVs with the help of a geometric invariant of the lattice of the PPAVs, the first successive minimum or shortest non-zero lattice vector, whose square is also called the *minimal period length* of the PPAV. Here the *kth successive minimum of a complex lattice L* of dimension g is defined by

$$m_k(L) = \min\left\{r \in \mathbb{R}^+ \,|\, \exists \{l_1, \ldots, l_k\} \subset L,\ \text{lin. independent over } \mathbb{R},\ \|l_i\| \leqslant r \forall i\right\}$$

The *kth successive minimum of a PPAV* $(A = \mathbb{C}^g/L, H)$, $m_k(A, H)$, is defined as the kth successive minimum of its lattice L. Here the norm $\|\cdot\| = \|\cdot\|_H$ is the norm induced by the hermitian form H. If $(l_i)_{i\in\{1\ldots 2g\}}$ is a lattice basis of L, then the corresponding Gram matrix

$$P_H = \left(\langle l_i, l_j\rangle_H\right)_{i,j=1\ldots 2g}$$

has determinant 1, due to the fact that the PAV is principal. Therefore we can apply the general upper bounds on the successive minima from Minkowski's theorems (see [8]) to the case of a PPAV (A, H) of dimension g, whereas the lower bound for Hermite's constant over the PPAVs

$$\delta_{2g} = \max_{(A,H)\in\mathscr{A}_g} m_1(A, H)^2.$$

was proven in Buser and Sarnak [5]:

$$\frac{g}{\pi e} \approx \frac{1}{\pi}\sqrt[g]{2g!} \leqslant \delta_{2g} \leqslant \frac{4}{\pi}\sqrt[g]{g!} \approx \frac{4g}{\pi e}$$

The approximation applies to large g. By Minkowski's second theorem we have

$$\prod_{k=1}^{2g} m_k(A, H)^2 \leqslant \left(\frac{4}{\pi}\right)^g g!^2$$

Buser and Sarnak showed in [5] that the shortest non-zero lattice vector of the Jacobian of a compact Riemann surface of genus g can be maximally of order $\log(g)$:

Theorem 1.1 *If* $\eta_{2g} = \max_{(A,H)\in t(\mathscr{M}_g)} m_1(A, H)^2$*, then*

$$c \log g \leq \eta_{2g} \leq \frac{3}{\pi}\log(4g-2),$$

where c is a constant.

To extend this theorem, we are going to prove the following:

Theorem 1.2 *Let S be a compact R.S. of genus g and let $J(S)$ be its Jacobian. Then*

$$m_1(J(S))^2 \leq \log(4g-2) \quad \textit{and} \quad m_2(J(S))^2 \leq 3.1\log(8g-7).$$

For the second successive minimum of a PPAV (A, H) of dimension g we obtain by Minkowski's second theorem:

$$m_2(A, H)^2 \leqslant \frac{1}{\sqrt[g]{m_1(A, H)}}\left(\frac{4\sqrt[g]{g!}}{\pi}\right)^{2g/(2g-1)} \approx \frac{4g}{\sqrt[g]{m_1(A, H)}\pi e},$$

where the approximation applies for large g. Furthermore there exist examples of PPAVs where $m_2(A, H)^2$ is of order g. This follows from the fact that PPAVs, whose shortest lattice vector is maximal, have a basis of minimal non-zero vectors (see [2], Theorem 7.6). In this case all $m_k(A, H)^2$ are of order g. In contrast, we have for the Jacobian of a Riemann surface, $J(S)$ that $m_1(J(S))^2$ and $m_2(J(S))^2$ are both of order $\log(g)$, independent of the length of the shortest non-zero lattice vector.

If a R.S. contains a certain number of mutually disjoint small simple closed geodesics, we obtain the following corollary of Theorem 1.2:

Corollary 1.3 *Let S be a compact R.S. of genus g that contains n disjoint simple closed geodesics $\left(\eta_j\right)_{j=1,\dots,n}$ of length smaller than t. If we cut open S along these geodesics, then the decomposition contains m R.S. S_i of signature (g_i, n_i), with $g_i > 0$. There exist m linear independent vectors $(l_i)_{i=1,\dots,m}$ in the lattice of the Jacobian $J(S)$, such that*

$$\|l_i\|_H{}^2 \leq \frac{(n_i+1)\max\{4\log(4g_i+2n_i-3), t\}}{\pi - 2\arcsin(M)} \quad \textit{for} \quad i \in \{1,\dots,m\},$$

where $M = \min\left\{\frac{\sinh(\frac{t}{2})}{\sqrt{\sinh(\frac{t}{2})^2+1}}, \frac{1}{2}\right\}$.

As the vectors l_i are linearly independent, the corollary implies improved bounds for a certain number of $m_k(J(S))$. This corollary is related to a Theorem of Fay. In Fay [6, chap. III] a sequence of Riemann surfaces S_t is constructed, where t denotes the length of a separating simple closed geodesic η. η divides S_t into two surfaces S_i of signature $(g_i, 1), i \in \{1, 2\}$. If $t \to 0$ then the period matrix for a suitable canonical homology basis converges to a block matrix, where each block is in $M_{2g_i}(\mathbb{R})$.

If η is any separating geodesic that separates a R.S. S into two surfaces S_i of signature $(g_i, 1), i \in \{1, 2\}$.is small enough. Applying Lemma 3.4, we obtain a slightly better bound than in the corollary:

$$m_i(J(S))^2 \leq \log(8g_i - 2) \quad \text{for} \quad i \in \{1, 2\}.$$

The corollary shows that indeed the first two successive minima of the Jacobian of the surfaces S_i can only be of order $\log(g_i)$ and gives explicit bounds depending on the length of t.

It shows that $m_1(J(S))^2$ and $m_2(J(S))^2$ of a R.S. with a sufficiently small separating scg, is at most of the order of the first successive minimum of a R.S. of genus g_1 and g_2, respectively.

Using the same methods as in Buser and Sarnak [5], we furthermore show that

Theorem 1.4 *If S is a hyperelliptic R.S. of genus g and $J(S)$ its Jacobian, then*

$$m_1(J(S))^2 \leq \frac{3\log(3 + 2\sqrt{3} + 2\sqrt{5 + 3\sqrt{3}})}{\pi} = 2.4382\ldots$$

It is worth mentioning that this result follows from a simple refinement of the proof that the systole of hyperelliptic surfaces is bounded from above by a constant, which was shown in Bavard [1] and Jenni [9].

2 Relating the length of lattice vectors of the Jacobian to geometric data of the surface

In Buser and Sarnak [5] an upper bound for the norm of a certain lattice vector of a Jacobian of a surface S is obtained by linking the norm of the vector to the length of a non-separating simple closed geodesic on S and the width of its collar, a topological tube around this geodesic. This approach can be further expanded.

For simplification the following expressions will be abbreviated. A simple closed geodesic will be denoted *scg* and a non-separating simple closed geodesic *nsscg* and a separating simple closed geodesic *sscg*. By abuse of notation we will denote the length of a geodesic arc by the same letter as the arc itself, if it is clear from the context.

Let S be a compact R.S. and $(\alpha_i)_{i=1\ldots 2g}$ a canonical homology basis on S. The collar of a scg γ, $C(\gamma)$, is defined by

$$C(\gamma) = \{x \in S \mid \text{dist}(x, \gamma) < w\}.$$

Here w is the supremum of all ω, such that the geodesic arcs of length ω emanating perpendicularly from γ are pairwise disjoint. For a given α_i, let $\alpha_{\tau(i)}$ be the unique scg in the canonical homology basis that intersects α_i. As in Buser and Sarnak [5, p. 36] test forms u_i' may be defined on the collar of an α_i that satisfy

$$\int_{\alpha_j} u_i' = \begin{cases} 1 & \text{if } j = \tau(i) \\ 0 & \text{if } j \neq \tau(i). \end{cases} \tag{1}$$

Among all differential forms on S that satisfy Eq. 1, the corresponding harmonic form $u_{\tau(i)}$ in the dual basis for the homology basis $(\alpha_i)_{i=1\ldots 2g}$ is minimizing with respect to the scalar product $\langle\cdot,\cdot\rangle = \int_S \cdot \wedge {}^*\cdot$. Therefore

$$\int_S u_{\tau(i)} \wedge {}^*u_{\tau(i)} \leq \int_S u_i' \wedge {}^*u_i' \quad \text{for all} \quad i \in \{1\ldots 2g\}. \tag{2}$$

If (A, H), where $A = \mathbb{C}^g/L$ is the Jacobian $J(S)$ of the surface S, then

$$\left(\langle l_i, l_j\rangle_H\right)_{i,j=1\ldots 2g} = P_H = P_S = \left(\int_S u_i \wedge {}^*u_j\right)_{i,j=1\ldots 2g}$$

by Riemann's period relations and therefore we have for all i

$$\langle l_i, l_i\rangle_H = \int_S u_i \wedge {}^*u_i.$$

The $(l_i)_{i=1\ldots 2g}$ are linear independent vectors of the lattice L and if we can obtain an upper bound for the test forms $\left(u_i'\right)_{i=1\ldots k}$, we obtain an upper bound on $m_k(A, H)^2$ by Eq. 2,

$$m_k(A, H)^2 \leq \max_{i\in\{1,\ldots,k\}} \int_S u_i' \wedge {}^*u_i'. \tag{3}$$

The capacity of $C(\alpha_i)$ (see [5, p. 36]), $\text{cap}(C(\alpha_i))$ provides this upper bound for the squared norm of an u_i'. The bound is given by

$$\left\langle u_i', u_i'\right\rangle \leq \text{cap}(C(\alpha_i)) = \frac{l(\alpha_i)}{\pi - 2\cdot\arcsin\left(\frac{1}{\cosh(w_i)}\right)}, \tag{4}$$

where w_i denotes the width of the collar $C(\alpha_i)$. $\text{cap}(C(\alpha_i))$ is a strictly increasing function with respect to w_i. The following values W and W' for the width of a collar occur frequently in our proof:

$$W = \text{arccosh}(2) = 1.3169\ldots \quad \text{and} \quad W' = \text{arctanh}(2/3) = 0.8047\ldots$$

If $w_i = W$, we have that

$$\text{cap}(C(\alpha_i)) = \frac{3l(\alpha_i)}{2\pi} \leq 0.5l(\alpha_i).$$

If $w_i = W'$, we obtain that

$$\text{cap}(C(\alpha_i)) = \frac{l(\alpha_i)}{\pi - 2\arcsin\left(\frac{\sqrt{5}}{3}\right)} \leq 0.7l(\alpha_i).$$

The upper bound in Theorem 1.1 follows from the fact that a canonical homology basis $(\alpha_i)_{i=1\ldots 2g}$ can always be constructed, such that α_1 is the shortest nsscg on a Riemann surface S. It was shown in Buser and Sarnak [5] that the length of the shortest nsscg α_1 is smaller than $2\log(4g-2)$ for any R.S. of genus g and that its collar width w_1 is bounded from below

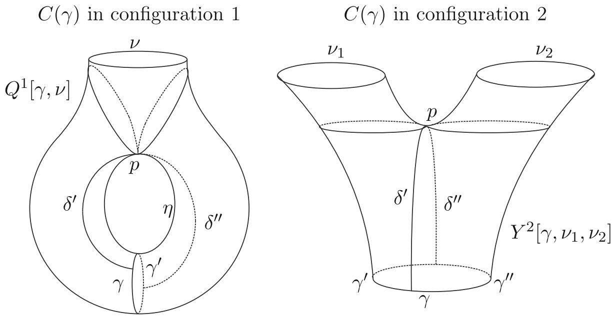


Fig. 1 $C(\gamma)$ in configuration 1 and 2

by W'. It follows from the above equations that $m_1(J(S))^2$ is bounded from above. A more refined analysis shows that $m_1(J(S))^2 \leq \frac{3}{\pi}\log(4g-2)$.

We obtain Theorem 1.2 by showing that there exist two short nsscg, α_1 and α_2, whose collar widths are bounded from below and which can be incorporated together into the canonical homology basis. In principle this approach would provide further bounds for the consecutive $m_k(J(S))$, but finding bounds for both collar width and length of the nsscgs has already proven to be very technical for $k = 2$.

Theorem 1.4 follows from the fact that the non-separating systole of a hyperelliptic surface is bounded from above by a constant, independent of the genus.

3 General upper bounds for the length of short scgs on a Riemann surface

To prove the main theorems we will have to consider on many occasions the configuration in which the closure of the collar of a scg self-intersects. The closure of the collar of a scg γ, $\overline{C(\gamma)}$ self-intersects in a single point p. There exist two geodesic arcs of length w emanating from γ and perpendicular to γ having the endpoint p in common. These two arcs, δ' and δ'', form a smooth geodesic arc δ. Two cases are possible—either δ arrives at γ on opposite sites of γ or it arrives on the same side (see Fig. 1).

Definition 3.1 The closure of the collar of a scg γ, $\overline{C(\gamma)}$ self-intersects in a point p. We say that $C(\gamma)$ *is in configuration 1* if the shortest geodesic arcs δ' and δ'' emanating from the intersection point p and meeting γ perpendicularly arrive at γ on opposite sides. We say that $C(\gamma)$ *is in configuration 2*, if they arrive on the same side of γ.

For both configurations we have a corresponding Y-piece, a topological three-holed sphere, whose interior is isometrically embedded in S. If $C(\gamma)$ is in configuration 1, we cut open S along γ. We call S' the surface obtained in this way from S. Let γ^1 and γ^2 the two scg in S' corresponding to γ in S. Let ν be the shortest scg in the free homotopy class of $\gamma^1\delta\gamma^2\delta^{-1}$. Then γ^1,γ^2 and ν bound a three-holed sphere Y^1, whose interior lies in S'. As this decomposition occurs frequently, we will refer to it as $Y^1[\gamma,\nu]$, *the Y-piece for* γ *from configuration 1*. If we close $Y^1[\gamma,\nu]$ again at γ, we obtain a R.S. of signature $(1, 1)$, $Q^1[\gamma,\nu] \subset S$ (see Fig. 1). Note that in this case we obtain

$$\nu < 2\gamma + 2\delta = 2\gamma + 4w, \tag{5}$$

as ν is in the free homotopy class of $\gamma^1\delta\gamma^2\delta^{-1}$. However, we can also calculate the exact value of ν by decomposing $Y^1[\gamma,\nu]$ into two isometric hexagons, H_1 and H_2. Here we cut open $Y^1[\gamma, \nu]$ along the shortest geodesic arcs connecting the boundary curves. In $H_1 \delta$ is the shortest geodesic arc connecting $\frac{\gamma^1}{2}$ and $\frac{\gamma^2}{2}$ and $\frac{\nu}{2}$ is the side opposite of δ. From the geometry of right-angled hexagons (see [4, p. 454]) we obtain

$$\cosh\left(\frac{\nu}{2}\right) = \sinh\left(\frac{\gamma}{2}\right)^2 \cosh(\delta) - \cosh\left(\frac{\gamma}{2}\right)^2$$

As $\cosh(x)^2 = \sinh(x)^2 + 1$ this is equal to

$$\nu = 2 \operatorname{arccosh}\left(\sinh\left(\frac{\gamma}{2}\right)^2 (\cosh(2w) - 1) - 1\right). \tag{6}$$

We note furthermore that there exists a geodesic arc γ' in γ connecting the two endpoints of δ on γ, whose length is restricted by $\gamma' \leq \frac{\gamma}{2}$. The shortest scg η in the free homotopy class of $\gamma'\delta$ has length smaller than

$$\eta < \frac{\gamma}{2} + \delta = \frac{\gamma}{2} + 2w. \tag{7}$$

If $C(\gamma)$ is in configuration 2, δ divides γ in two parts, γ' and γ''. Let ν_1 and ν_2 be the scg in the free homotopy class of $\gamma'\delta$ and $\gamma''\delta$. The three scg γ, ν_1 and ν_2 then bound a Y-piece, we will refer to it as $Y^2[\gamma, \nu_1, \nu_2]$, *the Y-piece for γ from configuration 2* (see Fig. 1). Note that $\nu_1 < \gamma' + \delta$ and $\nu_2 < \gamma'' + \delta$. Let WLOG $\gamma' \leq \gamma''$. As $\gamma = \gamma' \cup \gamma''$, we have

$$\nu_1 < \gamma' + \delta \leq \frac{\gamma}{2} + \delta = \frac{\gamma}{2} + 2w \quad \text{and} \quad \nu_2 < \gamma'' + \delta < \gamma + 2w. \tag{8}$$

For small values of γ, we obtain a better bound for ν by decomposing $Y^2[\gamma, \nu_1, \nu_2]$ into two isometric hexagons, H_1 and H_2, by cutting it open along the shortest geodesic arcs connecting the boundary curves. Here $\frac{\delta}{2}$ is the unique geodesic arc in H_1 perpendicular to $\frac{\gamma}{2}$ and the geodesic arc between $\frac{\nu_1}{2}$ and $\frac{\nu_2}{2}$ and with endpoints on both arcs. $\frac{\delta}{2}$ divides H_1 into two pentagons, P_1 and P_2. Let P_1 be the pentagon that contains $\frac{\gamma'}{2}$ as a boundary arc. From the geometry of right-angled pentagons (see [4, p. 454]), we get

$$\cosh\left(\frac{\nu_1}{2}\right) = \sinh\left(\frac{\gamma'}{2}\right) \sinh\left(\frac{\delta}{2}\right) \quad \text{and} \quad \nu_1 \leq 2 \operatorname{arccosh}\left(\sinh\left(\frac{\gamma}{4}\right) \sinh(w)\right), \tag{9}$$

as sinh and arccosh are strictly increasing functions on $\mathbb{R}^+$.

With the help of this decomposition, the following lemma is proven in Buser and Sarnak [5, pp. 40–42]:

Lemma 3.2 *Let S be a compact R.S. of genus g and γ a scg in S. Let $C(\gamma)$ be the collar of γ of width w. If $C(\gamma)$ is in configuration 1, let δ be the geodesic arc emanating from both sides of γ and perpendicular to γ. δ divides γ into two arcs. Let γ' be the shorter of the two. Let furthermore η be the scg in the free homotopy class of $\gamma'\delta$. If $\eta \geq \gamma$, then*

$$w \geq \max\left\{\operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{\gamma}{2}\right)}\right), \operatorname{arccosh}\left(\frac{\cosh\left(\frac{\gamma}{2}\right)}{\cosh\left(\frac{\gamma}{4}\right)}\right)\right\} \geq W'$$

If $C(\gamma)$ is in configuration 2, let $Y^2[\gamma, \nu_1, \nu_2]$ be the Y-piece for γ from configuration 2. If either ν_1 or ν_2 is bigger than γ, then

$$w \geq \operatorname{arccosh}(2) = W$$

The lower bound for the width of the geodesic γ depends on the constant K, where

$$K = 3.326.$$

As a consequence of this lemma we have that if $C(\gamma)$ is in configuration 1 and $\gamma < 3.326 = K$ then its width w is bigger than $\operatorname{arctanh}(2/3) = W'$. If $C(\gamma)$ is in configuration 1 and $\gamma > K$, then its width w is bigger than $\operatorname{arccosh}(2) = W$. If $C(\gamma)$ is in configuration 2 its width w is always bigger than W.

For the proof of Theorem 1.2 we also need the following lemma from Buser and Sarnak [5, p. 38]:

Lemma 3.3 *Let F be a compact Riemann surface of signature $(h, 1)$, such that $1 \leq h$ and assume that the boundary η of F has length $\eta < 2\log(8h-2)$ Then F contains a nsscg α of length smaller than $2\log(8h-2)$ in its interior.*

A consequence of this lemma is that every compact R.S. of genus g contains a nsscg of length smaller than $2\log(4g-2)$ in its interior (see [5, p. 38]). With the help of this lemma, we prove the following:

Lemma 3.4 *Let F be a compact Riemann surface of signature $(h, 1)$ and assume that the boundary η of F has length η. Then F contains a nsscg α of length smaller than $L = \max\left\{\frac{\eta}{2} + \log(8h-2), 2\log(8h-2)\right\}$ in its interior.*

Proof The collar of η, $C(\eta)$ of width w in F is in configuration 2. Let $Y^2[\eta, \nu_1, \nu_2]$ be the Y-piece for η from configuration 2 and $\nu_1 \leq \nu_2$. We have by 8

$$\nu_1 < \frac{\eta}{2} + 2w$$

We now show that either F contains a nsscg α of length $2\log(8h-2)$ in its interior or that $2w < \log(8h-2)$, from which follows that $\nu_1 < L$. If ν_1 is non-separating, then we are done. If not, we cut open F along ν_1 into two parts. The part F^1 that does not contain η has signature $(h^1, 1)$, where $h^1 \leq h-1$ and its boundary is $\nu_1 \leq L$. In this case we argue as before and divide F^1 again into two parts. As long as the shorter scg in the Y-piece for the boundary geodesic from configuration 2 is separating, we can successively cut off pieces F^k from F. Let $(h^k, 1)$ be the signature of F^k, where $h^k \leq h-k$. Repeating the argument for ν_1 we obtain that the boundary geodesic of a F^k has length smaller than L. This procedure ends at least, when F^k is a Q-piece, a Riemann surface of signature (1,1). Then the decomposition of F^k yields a nsscg α of length smaller than L in the interior of $F^k \subset F$.

To conclude the proof, we have to show that $2w < \log(8h-2)$. Consider the surface $F' = F + F/\eta$, which is obtained by attaching the mirror image of F along the boundary η. It has genus $2h$. As a consequence of Lemma 3.3 there exists a nsscg α of length smaller than $2\log(8h-2)$ in the interior of of F'. Note that $\alpha \neq \eta$, as η is separating in F'. If $\alpha \cap \eta = \emptyset$, then α is contained in F and we are done. If $\alpha \cap \eta \neq \emptyset$, then it has to traverse the collar of η, $C(\eta)$ in F' at least twice and therefore $2\log(8h-2) > \alpha > 4w$, from which follows that $2w < \log(8h-2)$. □

With the help of the previous lemmata we establish an upper bound for the second shortest scg on a compact R.S. in the following lemma. Other methods were applied in Buser [4, p. 123] to obtain such an upper bound, however the one obtained here is lower.

Lemma 3.5 *Let S be a compact Riemann surface of genus $g \geq 2$ and let γ_1 be the systole of S and γ_2 be the second shortest scg on S. Then $\gamma_1 \leq 2\log(4g-2)$ and $\gamma_2 \leq 3\log(8g-7)$.*

Proof By an area argument (see [4, p. 124]) the length of the shortest scg, γ_1 of a compact Riemann surface S of genus g is bounded from above by $2\log(4g-2)$. If γ_1 is separating, we cut open S along γ_1 into two parts S^1 and S^2. Let WLOG S^1 be the part, such that S^1 is of signature $(h, 1)$, such that $h \leq \frac{g}{2}$. By Lemma 3.3 there exists a nsscg α of length smaller or equal to $2\log(4g-2)$ in the interior of S^1. In this case we have $\gamma_2 \leq 2\log(4g-2)$.

If γ_1 is non-separating, we have to take another approach. The collar of γ_1, $C(\gamma_1)$ intersects in the point p_1 and has width w_1. We furthermore know that the interior of $C(\gamma_1)$ is isometrically embedded into S and therefore its area can not exceed the area of S. Therefore

$$2\gamma_1 \cdot \sinh(w_1) = \text{area}\, C(\gamma_1) < \text{area}\, S = 4\pi(g-1).$$

Hence

$$w_1 \leq \operatorname{arcsinh}\left(\frac{2\pi(g-1)}{\gamma_1}\right).$$

If $\frac{\pi}{2} \leq \gamma_1 \leq 2\log(4g-2)$, we obtain an upper bound for w_1, using that $\operatorname{arcsinh}(x) \leq \log(2x+1)$.

$$w_1 \leq \log(8(g-1)+1) < \log(8g-7).$$

In this case, we can conclude that there is a scg $\gamma_2 \neq \gamma_1$ in S of length smaller than

$$\gamma_2 < \frac{\gamma_1}{2} + 2w_1 < 3\log(8g-7).$$

To see this, we apply either Eq. 7 or Eq. 8, depending on, whether the collar of γ_1 is in configuration 1 or 2, respectively.

If $\gamma_1 < \frac{\pi}{2}$, we have to consider again both possible configurations. If γ_1 is in configuration 1, we obtain by Eq. 6, using the decomposition of the Y-piece from configuration 1, $Y^1[\gamma_1, \nu]$ into hexagons and as $w_1 \leq \operatorname{arcsinh}\left(\frac{2\pi(g-1)}{\gamma_1}\right)$ that

$$\begin{aligned}\nu &\leq 2\operatorname{arccosh}\left(\left(\sinh\left(\frac{\gamma_1}{2}\right)^2\left(\cosh\left(2\operatorname{arcsinh}\left(\frac{2\pi(g-1)}{\gamma_1}\right)\right)-1\right)\right)-1\right)\\ &\leq 4\log(8g-7).\end{aligned}$$

Here the upper bound of $4\log(8g-7)$ was determined using MAPLE. If we cut open S along ν, we obtain two pieces, one corresponding to $Y^1[\gamma_1, \nu]$ and a second piece S' of signature $(g-1, 1)$. Applying Lemma 3.4 to S', we conclude that there exists a scg γ_2 in $S' \subset S$, whose length is bounded from above by $\frac{\nu}{2} + \log(8(g-1)-2) \leq 3\log(8g-7)$.

If γ_1 is in configuration 2, we obtain from the decomposition of the Y-piece from configuration 2, $Y^2[\gamma_1, \nu_1, \nu_2]$ into pentagons Eq. 9 and as $w_1 \leq \operatorname{arcsinh}\left(\frac{2\pi(g-1)}{\gamma_1}\right)$ that

$$\nu_1 \leq 2\operatorname{arccosh}\left(\sinh\left(\frac{\gamma}{4}\right)\frac{2\pi(g-1)}{\gamma_1}\right) \leq 3\log(8g-7).$$

Again the upper bound of $3\log(8g-7)$ was determined using MAPLE. □

A useful result for Riemann surfaces with boundary was obtained in Gendulphe [7]:

Lemma 3.6 *Let S be a Riemann surface of signature (g, n). Let γ_1 be the systole of S and $l(\partial S)$ be the length of the boundary of S. Then $\gamma_1 \leq 4\log(4g+2n+3) + l(\partial S)$.*

For a Q-piece, a R.S. of signature $(1, 1)$ we have the following inequalities for a short canonical homology basis (α_1, α_2) by Parlier [10, pp. 59–62]:

Lemma 3.7 *Let Q be a Riemann surface of signature $(1, 1)$ and γ be the boundary geodesic of Q. There exists a canonical homology basis (α_1, α_2), $\alpha_1 \leq \alpha_2$ of Q satisfying the following inequalities:*

$$\cosh\left(\frac{\alpha_1}{2}\right) \leq \cosh\left(\frac{\gamma}{6}\right) + \frac{1}{2} \quad \textit{and}$$

$$\cosh\left(\frac{\alpha_2}{2}\right) \leq \sqrt{\frac{\cosh^2\left(\frac{\gamma}{4}\right) + \cosh^2\left(\frac{\alpha_1}{2}\right) - 1}{2(\cosh\left(\frac{\alpha_1}{2}\right) - 1)}}.$$

The result is stated differently in Parlier [10]. In Parlier [10] α_1 is the shortest scg in the interior of Q and α_2 the shortest scg in Q that intersects α_1. But due to this construction, both α_1 and α_2 are non-separating. Furthermore α_2 intersects α_1 only once due to its minimality. Hence α_1 and α_2 have the required properties for a canonical homology basis.

Another lemma needed for the proof of the main theorem concerns comparison surfaces and can be found in Parlier [11, p. 234]:

Lemma 3.8 *Let S be a Riemann surface of signature (g, n) with $n > 0$. Let $\beta_1, \ldots, \beta_n$ be the boundary geodesics of S. For $(\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{R}^+)^n$ with at least one $\varepsilon_i > 0$, there exists a comparison surface S_c with boundary geodesics of length $\beta_1 + \varepsilon_1, \ldots, \beta_n + \varepsilon_n$ such that for each simple closed geodesic γ_c in the interior of the comparison surface S^c, there exists a geodesic γ in the interior of S, such that $\gamma < \gamma^c$.*

We finally state a consequence of the collar lemma stated in Buser [4, p. 106]:

Lemma 3.9 *Let S be a Riemann surface of genus g with $g \geq 2$. Let γ be a simple closed geodesic in S. If η is another scg that does not intersect γ, then*

$$\operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\gamma}{2})}\right) < \operatorname{dist}(\eta, \gamma)$$

and if w is the width of $C(\gamma)$, the collar of γ, then $w > \operatorname{arcsinh}(\frac{1}{\sinh(\frac{\gamma}{2})})$.

4 Main theorems

Proof of Theorem 1.2 It is well known that two nsscg α_1 and α_2 can be incorporated together into a canonical homology basis, if $\alpha_1 \cup \alpha_2$ does not separate S into two parts and if α_1 and α_2 have either exactly one or no intersection point. To prove Theorem 1.2 we have to show that there exist two short nsscg, α_1 and α_2, with these properties and whose collar width is bounded from below. Then we can obtain Theorem 1.2 from Eq. 4. The proof of Theorem 1.2 depends on whether the shortest scg γ_1 in S, the systole, is separating or non-separating. We will distinguish several cases. These cases are depicted in Fig. 2.

Case 1 *The systole γ_1 of S is a separating scg*

By Lemma 3.5, γ_1 has length smaller than $2\log(4g - 2)$. We cut open S along γ_1, which yields two R.S., S_1 and S_2 of signature $(h_1, 1)$ and $(h_2, 1)$, respectively, such that $h_1 \leq h_2$. In both surfaces the half-collar of γ_1 is in configuration 2. By Lemma 3.2 the width of a half-collar of γ_1 is bigger than W. We now show that there exist two short nsscg, α_1 in S_1

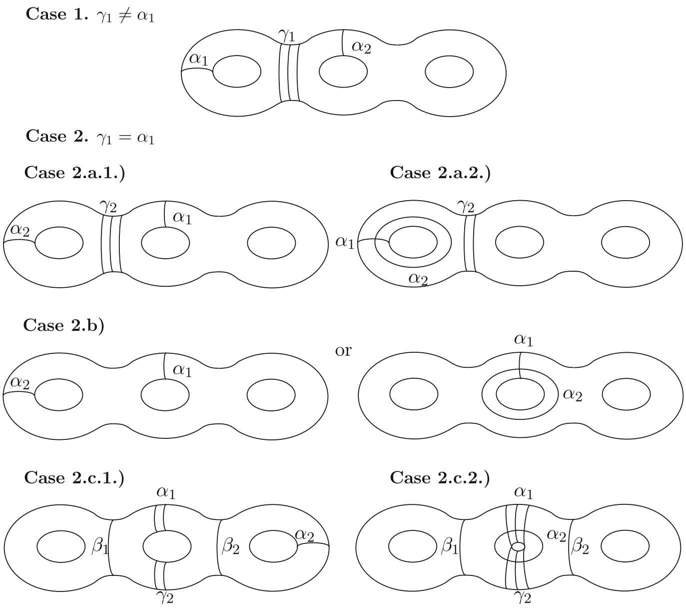


Fig. 2 Relative positions of α_1 and α_2 in the different cases of the proof of Theorem 1.2

and α_2 in S_2, whose collars in S have width of at least W. As $\alpha_1 \cup \alpha_2$ cannot divide S into two parts and as α_1 and α_2 do not intersect, they can be both together incorporated into a canonical homology basis of S. By Lemma 3.3 and Lemma 3.4, we have that

$$\alpha_1 < 2\log(4g-2) \quad \text{and} \quad \alpha_2 < 2\log(8(g-1)-2) = 2\log(8g-10).$$

We now show that each α_i, $i \in \{1,2\}$ has a collar, whose width w_i is bounded from below. Namely, if $\alpha_i < K$ then $w_i > W'$ and if $\alpha_i > K$, then $w_i > W$.

Consider WLOG the collar of α_1 in S_1. Its closure self-intersects in a point $p \in S_1$ or a geodesic arc δ of length smaller than w_1, emanating perpendicularly from α_1 meets the boundary of S_1 first.

In the first case, we apply Lemma 3.2. We obtain that if $\alpha_1 < K$ then $w_1 > W'$ and if $\alpha_1 > K$, then $w_2 > W$. In the second case, we show that $\overline{C(\alpha_1)}$ can not self-intersect in $C(\gamma_1) \cap S_2$. Therefore it self-intersects a point $p \in S_2 \backslash C(\gamma_1)$. In this case every geodesic arc δ' emanating perpendicularly from α_1 with endpoint p has to traverse $C(\gamma_1) \cap S_2$ and hence has length bigger than W in S.

To prove that $\overline{C(\alpha_1)} \subset S$ can not self-intersect in $S_1 \cup C(\gamma_1)$, we lift $C(\gamma_1)$ into the universal covering. Here γ_1 lifts to γ_1' and $\delta \cap C(\gamma_1)$ to δ^* (see Fig. 3). The lift δ^* is a geodesic. Let s_1 and s_2 be the first intersection points of δ^* and γ_1', the lift of γ_1 on opposite sides of p', the lift of p. There exists an unique geodesic arc connecting s_1 and s_2, which is an arc in γ_1', as γ_1' is a geodesic. But s_1 and s_2 also lie on δ^*, which implies that δ^* is contained in γ_1', a contradiction.

Summary of Case 1 If the systole γ_1 of S is a separating scg, then we can always find two short nsscg $\alpha_1 < 2\log(4g-2)$ and $\alpha_2 < 2\log(8g-10)$ for a homology basis of S. Let w_1 and w_2 be the collar width of α_1 and α_2, respectively. If $\alpha_i < K$ then $w_i > W'$ and if

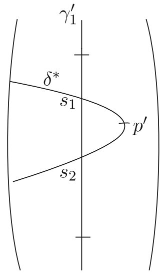


Fig. 3 Lift of $C(\gamma_1)$ in the universal covering

$\alpha_i > K$, then $w_i > W$, for $i \in \{1, 2\}$. It follows from Eqs. 3 and 4 and the subsequent remark that $m_1(J(S))^2$ and $m_2(J(S))^2$ satisfy the inequalities from Theorem 1.2.

Case 2 *The systole γ_1 of S is a non-separating scg*

In this case we can find a homology basis of S, such that $\gamma_1 = \alpha_1$. As α_1 is the shortest nsscg, it follows from Eqs. 3 and 4 that $m_1(J(S))^2$ satisfies the inequalities from Theorem 1.2.

To find a second short scg that does not separate S together with α_1, we have to consider several subcases. Let γ_2 be the second shortest scg on S. By Lemma 3.5 its length smaller than $3\log(8g-7)$. We will have to examine different cases, depending on whether γ_2 is separating, non-separating and non-separating with α_1 or non-separating but separating together with α_1.

Case 2.a *γ_2 is separating*

Note that γ_1 and γ_2 can not intersect. It is easy to see that otherwise we could find a scg in S that is smaller than γ_2. We separate S into two parts, S_1 and S_2 along γ_2. Let S_1 be the part, which contains α_1 and S_2 be the remaining part of signature $(h_2, 1)$, such that $h_2 \leq g - 1$. In this case γ_2 is smaller than $2\log(8g - 10)$, due to the minimality of γ_2. Otherwise we would arrive again at a contradiction, if we apply Lemma 3.4 to S_2. The collar of γ_2 is in configuration 2. Let $Y^2[\gamma_2, \nu_1, \nu_2]$ be the Y-piece for γ_2 from configuration 2. We have to distinguish two cases for the choice of α_2, where the choice depends on $Y^2[\gamma_2, \nu_1, \nu_2]$.

Case 2.a.1 $Y^2[\gamma_2, \nu_1, \nu_2] \neq Y^2[\gamma_2, \alpha_1, \alpha_1]$

In this case let α_2 be the shortest nsscg in S_2. As $\gamma_2 < 2\log(8g - 10)$ it follows from Lemma 3.4 that $\alpha_2 < 2\log(8g-10)$. α_1 and α_2 can be incorporated together into a canonical homology basis. As α_1 does not occur twice in the boundary curves of $Y^2[\gamma_2, \nu_1, \nu_2]$ and as γ_2 is the second shortest scg in S, we conclude by Lemma 3.2 that the collar of γ_2 has width $w' \geq W$. We now determine a lower bound for the width of $C(\alpha_2)$, w_2. α_2 is the shortest nsscg in the interior of S_2. Hence we can argue as in the case of the collar of α_2 in Case 1 to obtain a lower bound for the width of $C(\alpha_2)$, w_2. If $\alpha_2 < K$ then $w_2 > W'$ and if $\alpha_2 > K$, then $w_2 > W$.

Case 2.a.2 $Y^2[\gamma_2, \nu_1, \nu_2] = Y^2[\gamma_2, \alpha_1, \alpha_1]$

If $\nu_1 = \nu_2 = \alpha_1$, then the interior of $Y^2[\gamma_2, \nu_1, \nu_2]$ is embedded in the Q-piece $Q_1 = S_1$, a R.S. of signature $(1, 1)$. This case can not occur, if $2.1 \leq \alpha_1 = \gamma_1 \leq \gamma_2$, because otherwise there would exist a scg $\alpha_2{}' \neq \alpha_1$ in Q_1 that is smaller than γ_2 by Lemma 3.7.

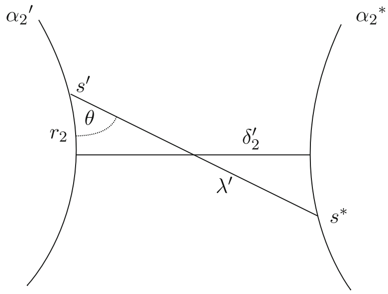


Fig. 4 Two lifts of α_2 in the universal covering

In this case let β be the shortest nsscg in S_2. As $\gamma_2 < 2\log(8g-10)$ it follows from Lemma 3.4 that $\beta < 2\log(8g-10)$. Let α_2 be the shortest nsscg in S that does not intersect α_1. We have $\alpha_2 \leq \beta < 2\log(8g-10)$.

α_2 has a collar $C(\alpha_2)$, whose width w_2 is bounded from below. To see this, we cut open S along α_1 to obtain S'. Consider the collar of α_2 in S'. Its closure self-intersects in a point $p \in S'$ or a geodesic arc δ of length smaller than w_2, emanating perpendicularly from α_2 meets the boundary of S' first.

By Lemma 3.9 $\mathrm{dist}(\alpha_1, \alpha_2) > \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\alpha_1}{2})}\right) > \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{2.1}{2})}\right)$, as $\alpha_1 \leq 2.1$. It follows from the same arguments as in Case 1 that w_2 has the lower bound

$$w_2 > \min\left\{\operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{2.1}{2})}\right), W'\right\} > 0.73.$$

Summary of Case 2.a We can always find two short nsscg α_1 and α_2 for a homology basis of S,whose lengths satisfy the same upper bounds as in as in Case 1 and whose collar width is bounded from below, such that $m_1(J(S))^2$ and $m_2(J(S))^2$ satisfy the inequalities from Theorem 1.2.

Case 2.b γ_2 *is non-separating and non-separating with* $\gamma_1 = \alpha_1$

In this case we have to distinguish two cases, $\alpha_2 = \gamma_2$ and $\alpha_2 \neq \gamma_2$.

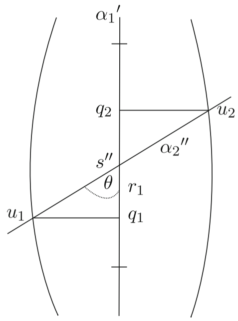
Case 2.b.1 $\alpha_2 = \gamma_2$

We have $\alpha_2 = \gamma_2 < 3\log(8g-7)$. Note that α_2 can not intersect α_1 more than once, as otherwise there would exist a scg, which is shorter than α_2. We now determine a lower bound for the width of the collar of α_2, $C(\alpha_2)$.

If $C(\alpha_2)$ is in configuration 2, let $Y^2[\alpha_2, \nu_1, \nu_2]$ be the Y-piece for α_2 from configuration 2. If ν_1 and ν_2 are both smaller than α_2, then both must be α_1. If $Y^2[\alpha_2, \nu_1, \nu_2]$=$Y^2[\alpha_2, \alpha_1, \alpha_1]$, then $Y^2[\alpha_2, \alpha_1, \alpha_1]$ is embedded in S as a Q-piece with boundary α_2 and α_2 would be separating, a contradiction. Hence we conclude by Lemma 3.2 that the width of $C(\alpha_2)$ is bigger than W.

If $C(\alpha_2)$ is in configuration 1, $\overline{C(\alpha_2)}$ self-intersects in a single point p. There exist two geodesic arcs of length w_2 emanating from α_2 and perpendicular to α_2 having the endpoint p in common. These two arcs form a smooth geodesic arc δ_2. We lift α_2 and δ_2 in the universal covering. Here α_2 lifts to $\alpha_2{}'$ and $\alpha_2{}^*$ and δ_2 to δ_2'. In the covering there exist two points, $s' \in \alpha_2{}'$ and $s^* \in \alpha_2{}^*$, on opposite sites of δ_2' and at the same distance $r_2 \leq \frac{\alpha_2}{4}$ from δ_2', such

Fig. 5 Lift of $C(\alpha_1)$ in the universal covering



that s' and s^* are mapped to the same point $s \in \alpha_2$ by the covering map. By drawing the geodesic λ' from s' to s^*, we obtain two isometric right-angled geodesic triangles (see Fig. 4).

We have to consider two subcases, $\lambda' \neq \alpha_1$ and $\lambda' = \alpha_1$ and $\alpha_1 > 1.1$. In the case $\lambda' = \alpha_1 \leq 1.1$, we will switch to Case 2.b.2

$\lambda' \neq \alpha_1$

If $\lambda' \neq \alpha_1$, then we can again argue as in Case 1. We obtain that if $\alpha_2 < K$ then $w_2 > W'$ and if $\alpha_2 > K$, then $w_2 > W$.

$\lambda' = \alpha_1$ and $\alpha_1 > 1.1$

To intersect α_1, α_2 has to traverse the collar of α_1. We can use this fact to derive a lower bound for the width of the collar of α_2, $C(\alpha_2)$.

Lift $C(\alpha_1)$ in the universal covering and let $C'(\alpha_1)$ be the lift of $C(\alpha_1)$ (see Fig. 5). α_2 traverses the collar $C(\alpha_1)$ of width w_1. It lifts to $\alpha_2{}''$ and in the lift it enters $C'(\alpha_1)$ at a point u_1 and leaves at a point u_2. Consider the geodesic arcs emanating from u_1 and u_2 respectively and meeting the lift of α_1, $\alpha_1{}'$ perpendicularly. Their length is w_1. Let q_1 and q_2 be the endpoints of these geodesic arcs on $\alpha_1{}'$. $\alpha_2{}''$ intersects $\alpha_1{}'$ in the midpoint s'' of the geodesic arc between q_1 and q_2 under angle θ. Here s'' is a lift of $s \in Q_1$. Set $r_1 = \mathrm{dist}(q_1, s'') = \mathrm{dist}(q_2, s'')$. Then r_1 is smaller or equal to $\frac{\alpha_1}{4}$, as α_2 is the third shortest scg in Q_1 and otherwise there exists another point u_2', such that u_2 and u_2' map to the same point on Q_1 under the universal covering map, such that $\mathrm{dist}(u_1, u_2') < \mathrm{dist}(u_1, u_2)$, a contradiction to the fact that α_2 is minimal. Consider the right-angled triangle with vertices u_1, q_1 and s''. From the geometry of hyperbolic triangles (see [4, p. 454]) we have for θ:

$$\sin(\theta) = \frac{\sinh(w_1)}{\sinh(\mathrm{dist}(u_1, s''))} \quad \text{and} \quad \cosh(\mathrm{dist}(u_1, s'')) = \cosh(w_1) \cdot \cosh(r_1)$$

from which follows, as $\sinh^2(x) = \cosh^2(x) - 1$ that

$$\sin(\theta) = \frac{\sinh(w_1)}{\sqrt{\cosh^2(r_1) \cdot \cosh^2(w_1) - 1}}.$$

The point s'' corresponds to s' in the other lift of α_2 (see Fig. 4) and the angle θ to the interior angle of the right-angled geodesic triangle at the vertex s'. From the geometry of this triangle we get:

$$\sin(\theta) = \frac{\sinh(w_2)}{\sinh\left(\frac{\alpha_1}{2}\right)}$$

and therefore, as $\sinh(w_2)$ is decreasing with increasing $r_1 \leq \frac{\alpha_1}{4}$

$$\sinh(w_2) \geq \frac{\sinh(w_1) \cdot \sinh\left(\frac{\alpha_1}{2}\right)}{\sqrt{\cosh^2\left(\frac{\alpha_1}{4}\right) \cdot \cosh^2(w_1) - 1}}. \tag{10}$$

Note that the left hand side in 10 is increasing with increasing w_1 and increasing α_1. As the width of $C(\alpha_1)$, w_1 is bigger than W', we get a lower bound for w_2, if we set $w_1 = W'$. In this case we obtain from Eq. 10

$$w_2 \geq w_2{}^Q = \operatorname{arcsinh}\left(\frac{\frac{2\sqrt{5}}{5} \cdot \sinh\left(\frac{\alpha_1}{2}\right)}{\sqrt{\frac{9}{5}\cosh^2\left(\frac{\alpha_1}{4}\right) - 1}}\right). \tag{11}$$

In this case we obtain, with $\alpha_1 > 1.1$:

$$m_2(J(S))^2 < \frac{3\log(8g-7)}{\pi - 2 \cdot \arcsin\left(\frac{1}{\cosh(w_2{}^Q)}\right)} \leq 3.1\log(8g-7).$$

Summary of Case 2.b.1 We can always find two short nsscg $\alpha_1 = \gamma_1 < 2\log(4g-2)$ and $\alpha_2 = \gamma_2 < 3\log(8g-7)$ for a homology basis of S. Their collar width is bounded from below, such that $m_1(J(S))^2$ and $m_2(J(S))^2$ satisfy the inequalities from Theorem 1.2.

Case 2.b.2 $\alpha_2 \neq \gamma_2$

This case treats the remaining case for $C(\gamma_2)$ in configuration 1, and the geodesic λ' (see Fig. 4) is α_1, with $\alpha_1 \leq 1.1$.

In this case we cut open S along α_1 to obtain the surface S' of signature $(g-1, 2)$. Let $\alpha_1{}'$ and $\alpha_1{}''$ be the boundary. In this case we let α_2 be the shortest nsscg in S' that does not intersect α_1. We first show the following claim.

Claim 4.1 *The shortest nsscg $\alpha_2 \subset S'$ has length smaller than* $2\log(24g-23) + 2.2$.

Proof We first show that there exists a scg of length smaller than $2\log(24g-23) + 2.2$ in the interior of S'. It is sufficient to proof this statement for the case $\alpha_1 = 1.1$. It follows from Lemma 3.8 that this is also true for $\alpha_1 < 1.1$. If there exists a scg of length smaller than $2\log(24g-23) + 2.2$ in S' and this geodesic is non-separating, we are done. If it is separating, we apply Lemma 3.4 and conclude that there exists a nsscg in S' that is smaller than $\log(24g-23) + 1.1 + \log(8g-10) < 2\log(24g-23) + 2.2$, which proves the claim.

Let $\alpha_1{}' = 1.1$. The closure of the half-collar $\overline{C(\alpha_1{}')} \subset S'$ self-intersects in a point in S' or a geodesic arc emanating perpendicularly from $\alpha_1{}'$, of length smaller than w' meets $\alpha_1{}''$ perpendicularly in a point p_1. We examine two cases, which depend on how $\overline{C(\alpha_1{}')}$ intersects itself.

(i) *The closure of the half-collar of $\alpha_1{}'$ intersects $\alpha_1{}''$ in p_1 before self-intersecting in S'*

A geodesic arc $\sigma \subset S'$ meets $\alpha_1{}'$ and $\alpha_1{}''$ perpendicularly on both endpoints where p_1 is the endpoint on $\alpha_1{}''$. The distance set $Z_r(\alpha_1{}') \subset S'$ is defined by

$$Z_r(\alpha_1{}') = \{x \in S' \mid \operatorname{dist}(x, \alpha_1{}') < r\}.$$

As long as r is small enough, such that $Z_r(\gamma) \subset C(\alpha_1{}')$, we have from hyperbolic geometry that $\operatorname{area}(Z_r(\alpha_1{}')) = \alpha_1{}' \cdot \sinh(r)$. Consider $Z_\sigma(\alpha_1{}')$. It is embedded into S' and therefore its area can not exceed the area of $S' = S$, which is smaller than $4\pi(g-1)$. Therefore

$$\alpha_1{}' \cdot \sinh(\sigma) = \text{area } Z_\sigma(\alpha_1{}') < \text{area}(S) = 4\pi(g-1)$$

As $\alpha_1{}' = 1.1$ and as $\text{arcsinh}(x) \leq \log(2x+1)$, we obtain an upper bound for σ.

$$\sinh(\sigma) \leq \frac{4\pi(g-1)}{1.1} \Rightarrow \sigma \leq \log(24g-23)$$

Hence we conclude that the shortest scg $\beta_1 \subset S'$ in the free homotopy class of $\alpha_1{}'\,\sigma\,\alpha_1{}''\,\sigma^{-1}$ has length smaller than $2\log(24g-23)+2.2$.

(ii) *The closure of the collar of* α_1 *self-intersects in* $p_1 \in S'$

A geodesic arc σ passes through p_1 and meets α_1 perpendicularly on both endpoints. Let $Y^2[\alpha_1, \nu_1, \nu_2]$ be the Y-piece for α_1 from configuration 2.

As $\alpha_1 = 1.1$, we conclude by the same area argument as in case *i* that $\sigma < 2\log(24g-23)$. From Eq. 8 it follows that both ν_1 and ν_2 are smaller than

$$\alpha_1 + \sigma \leq 2\log(24g-23) + 1.1.$$

At least one of them is not $\alpha_1{}''$. Hence there exists a scg of length smaller than $2\log(24g-23)+2.2$ in S'. Hence we have proven the claim. □

Let w_2 be the width of the collar of α_2. In this case we conclude as in Case 2.a.2 that $\text{dist}(\alpha_1, \alpha_2) > \text{arcsinh}\left(\frac{1}{\sinh(\frac{\alpha_1}{2})}\right) > \text{arcsinh}\left(\frac{1}{\sinh(\frac{1.1}{2})}\right)$, as $\alpha_1 \leq 1.1$. It follows from the same arguments as in Case 2.a.2 that w_2 has the lower bound

$$w_2 > \min\left\{\text{arcsinh}\left(\frac{1}{\sinh\left(\frac{1.1}{2}\right)}\right), W'\right\} > W'.$$

Summary of Case 2.b.2 We have that $\alpha_2 < 2\log(24g-23)+2.2$ and $w_2 > W'$. We obtain that

$$m_2(J(S))^2 < \frac{2\log(24g-23)+2.2}{\pi - 2\arcsin\left(\frac{1}{\cosh(W')}\right)} \leq 3.1\log(8g-7)$$

Case 2.c γ_2 *is non-separating, but separating with* $\gamma_1 = \alpha_1$

By Lemma 3.5 we know that the length of γ_2 is bounded by $3\log(8g-7)$. It is easy to see that γ_2 can not intersect α_1. It can not intersect α_1 more than once, due to the minimality of the two geodesics and it can not intersect α_1 once due to the fact that it is separating with α_1. As γ_2 is separating with α_1, we conclude by Lemma 3.2 that its collar width is bounded from below. If $\gamma_2 < K$ then the width of its collar is bigger than W' and if $\gamma_2 > K$, then the width of its collar is bigger than W.

We cut open S along γ_2 and α_1. The two geodesics divide S into S_1 and S_2. We first show, the following claim:

Claim 4.2 *The shortest nsscg* $\alpha_2{}^i \subset S_i$ *has length smaller than* $4.5\log(8g-7)$ *for* $i \in \{1, 2\}$.

The proof is similar to the proof of Claim 4.1.

Proof Consider WLOG S_1. We proof the claim for the cases $\alpha_1 < \pi$ and $\alpha_1 \geq \pi$:

(a) $\alpha_1 \geq \pi$

$\overline{C(\alpha_1)}$ self-intersects in a point in S_1 or a geodesic arc emanating perpendicularly from α_1, of length smaller than w_1 meets γ_2 perpendicularly in a point p_1. We examine two cases, which depend on how $\overline{C(\alpha_1)}$ intersects itself.

(i) *The closure of the collar of* α_1 *intersects* γ_2 *in* p_1 *before self-intersecting in* S_1

A geodesic arc $\sigma \subset S_1$ meets α_1 and γ_2 perpendicularly on both endpoints where p_1 is the endpoint on γ_2. We now define for a scg γ in S and an $r > 0$, the distance set of distance r of γ, $Z_r(\gamma)$ by

$$Z_r(\gamma) = \{x \in S \mid \text{dist}(x, \gamma) < r\}.$$

As long as r is small enough, such that $Z_r(\gamma) \subset C(\gamma)$, we have from hyperbolic geometry that $\text{area}(Z_r(\gamma)) = 2\gamma \cdot \sinh(r)$. Consider $Z_\sigma(\alpha_1) \cap S_1$. It is embedded into S_1 and therefore its area can not exceed the area of S_1, which is smaller than $4\pi((g-2)-1)$. Therefore

$$\alpha_1 \cdot \sinh(\sigma) = \text{area } Z_\sigma(\alpha_1) \cap S_1 < \text{area } S_1 = 4\pi(g-3).$$

As $\pi \leq \alpha_1$ and as $\text{arcsinh}(x) \leq \log(2x+1)$, we obtain an upper bound for σ.

$$\sinh(\sigma) \leq \frac{4\pi(g-3)}{\pi} \Rightarrow \sigma \leq \log(8(g-3)+1) < \log(8g-7)$$

Hence we conclude that the shortest scg β_1 in the free homotopy class of $\alpha_1 \sigma \gamma_2 \sigma^{-1}$ has length smaller than $7\log(8g-7)$. It is a separating scg. Applying Lemma 3.4 we conclude that there exists a nsscg of length smaller than $4.5\log(8g-7)$ in S_1. Note that, using the hexagon decomposition (see [5, p. 454]) of the Y-piece with boundary geodesics β_1, α_1 and γ_2, we can obtain the exact value of the length of β_1, which will be useful later for small values of α_1. It is

$$\cosh\left(\frac{\beta_1}{2}\right) = \sinh\left(\frac{\alpha_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right)\cosh(\sigma) - \cosh\left(\frac{\alpha_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right). \tag{12}$$

(ii) *The closure of the collar of* α_1 *self-intersects in* $p_1 \in S_1$

A geodesic arc σ passes through p_1 and meets α_1 perpendicularly on both endpoints. Let $Y^2[\alpha_1, \nu_1, \nu_2]$ be the Y-piece for α_1 from configuration 2. If $\alpha_1 \geq \pi$, we conclude by the same area argument as in case *i* that $\sigma < \log(8g-7)$. From Eq. 8 it follows that both ν_1 and ν_2 are smaller than

$$\alpha_1 + \sigma \leq 3\log(8g-7).$$

At least one of them is not γ_2. Therefore, if this scg is non-separating, we are done. If it is separating, we cut off the part of S_1 that contains α_1 and conclude by Lemma 3.4 that this part contains a nsscg of length smaller than $3\log(8g-7) < 4.5\log(8g-7)$. This settles the claim in the case $\alpha_1 \geq \pi$.

(b) $\alpha_1 < \pi$

If $\alpha_1 < \pi$, we use the fact that there exists a comparison surface S_1^c for S_1, as described in Lemma 3.8, such that one boundary geodesic has length π and the other has length γ_2 and conclude that it contains a scg of length smaller than $4.5\log(8g-7)$ in its interior. Therefore there exists a scg of length smaller than $4.5\log(8g-7)$ in S_1, by Lemma 3.8. If this geodesic is separating, we apply again Lemma 3.4 and conclude that there exists a nsscg in S_1 that is smaller than $4.5\log(8g-7)$. □

In total, we obtain that the shortest nsscg ${\alpha_2}^1$ in S_1 and the shortest nsssg ${\alpha_2}^2$ in S_2 are both smaller than $4.5\log(8g-7)$. Both can be incorporated with α_1 into a canonical homology basis. Consider the sets $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$, with $W' = \operatorname{arctanh}(2/3)$. We now choose a nsscg α_2 that is non-separating with α_1. The choice depends on how $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ intersect. We distinguish two cases.

Case 2.c.1 *$Z_{W'}(\alpha_1) \cap Z_{W'}(\gamma_2) \cap S_1 = \emptyset$ or $Z_{W'}(\alpha_1) \cap Z_{W'}(\gamma_2) \cap S_2 = \emptyset$*

If $Z_{W'}(\alpha_1) \cap Z_{W'}(\gamma_2) \cap S_1 = \emptyset$, then we choose $\alpha_2 = {\alpha_2}^2 \subset S_2$ and if $Z_{W'}(\alpha_1) \cap Z_{W'}(\gamma_2) \cap S_2 = \emptyset$ we choose $\alpha_2 = {\alpha_2}^1 \subset S_1$. Consider WLOG the first case. We show that the collar of ${\alpha_2}^2$ has width bigger than W'. If the closure of the collar of ${\alpha_2}^2$ self-intersects in S_1, it has to traverse either $S_1 \cap Z_{W'}(\alpha_1)$ or $S_1 \cap Z_{W'}(\gamma_2)$ and hence its width is bigger than W'. This follows from the same arguments as in Case 1. If $\overline{C({\alpha_2}^2)}$ self-intersects in S_2, we conclude by Lemma 3.2 that ${\alpha_2}^2$ has a collar of with bigger than W'.

Summary of Case 2.c.1 α_2 is the shortest nsscg in either S_1 or S_2, Its length is restricted by $\alpha_2 < 4.5\log(8g-7)$, the width of its collar is bigger than W'. It follows from Eqs. 3 and 4 that

$$m_2(J(S))^2 < 3.1\log(8g-7).$$

Case 2.c.2 *$Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ intersect both in S_1 and S_2*

If $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ intersect both in S_1 and S_2 we have to argue in a different way. We choose another small nsscg to be α_2. We may assume that $\alpha_1 \geq 1.5$ and $\gamma_2 \geq 2.1$. Otherwise it would follow from Eq. 12, with $\sigma = 2W'$ that $\beta_1 < \gamma_2$, a contradiction. We now choose α_2. Let δ' and δ'' be the shortest geodesic arcs in S_1 and S_2, respectively, connecting α_1 and γ_2. Their length is bounded from above by $2W'$ as $Z_{W'}(\alpha_1)$ and $Z_{W'}(\gamma_2)$ intersect. The endpoints of δ' and δ'' divide each α_1 and γ_2 into two geodesic arcs. Let ${\alpha_1}^*$ and γ_2^* be the shorter of these arcs. We define α_2 to be the shortest scg in the free homotopy class of $\delta'\,{\alpha_1}^*\,\delta''\gamma_2^*$. It intersects α_1 only once. The length of α_2 is restricted by

$$\alpha_2 < \frac{\alpha_1}{2} + \frac{\gamma_2}{2} + 4W' < 2.5\log(8g-7) + 4W'.$$

Let $Y' \subset S_1$ be the Y-piece with boundaries α_1, γ_2 and the shortest scg β_1 in the free homotopy class of $\alpha_1\,\delta'\gamma_2\delta'^{-1}$. Let $Y'' \subset S_2$ be the Y-piece constructed in the same way in S_2, having as third boundary β_2. The union $Y' \cup Y''$ in S is embedded as a Riemann surface F of signature (1,2) (see Fig. 6).

However, if the length of γ_2 is small, the upper bound for α_2 given above is not sufficient to establish an appropriate lower bound for the collar of α_2. Therefore we will establish a better upper bound for the length of α_2.

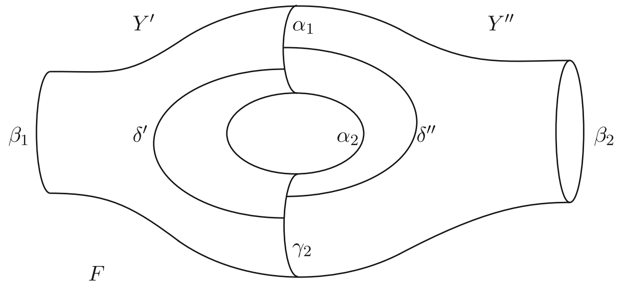


Fig. 6 The Riemann surface F of signature (1,2)

Lift α_1 to $\alpha_1{}'$ and equally δ' and δ'' into the universal covering (see Fig. 7). By abuse of notation we will denote the lift of these two arcs by the same letter. To δ' and δ'' attach the adjacent lifts of γ_2, γ_2' and γ_2'' on opposite sides of $\alpha_1{}'$. Let q' be the endpoint of δ' on $\alpha_1{}'$ and q'' be the endpoint of δ'' on $\alpha_1{}'$, such that $\mathrm{dist}(q', q'') \leq \frac{\alpha_1}{2}$. Let furthermore be s' be the endpoint of δ' on γ_2' and s'' be the endpoint of δ'' on γ_2''. Let s^* be the point on γ_2'' that maps to the same point on γ_2 under the covering map as s', such that $\mathrm{dist}(s'', s^*) \leq \frac{\gamma_2}{2}$. Let equally be s^{**} be the point on γ_2' that maps to the same point on γ_2 under the covering map as s'', such that $\mathrm{dist}(s^{**}, s') = \mathrm{dist}(s'', s^*)$. Let η' be the geodesic arc connecting the midpoint of s' and s^{**} on γ_2' and midpoint of s'' and s^* on γ_2''.

The image of η' under the covering map, η forms a closed geodesic arc on S. As η is in the same free homotopy class as α_2, its length provides an upper bound for the length of α_2. In Fig. 7, the points s^{**} and s^* lie on opposite sides of δ' and δ''. We will derive an upper bound for this case. In any other case the length of η' is either shorter or the situation is a mirror image of the depicted one. It is clear that η' is maximal, if $\mathrm{dist}(s^{**}, s') = \mathrm{dist}(s'', s^*) = \frac{\gamma_2}{2}$ and $\mathrm{dist}(q', q'') = \frac{\alpha_1}{2} = \frac{\gamma_2}{2}$. Therefore it is sufficient to discuss this case.

In this case we obtain from the geometry of hyperbolic triangles:

$$\cosh\left(\frac{\alpha_2}{4}\right) < \cosh\left(\frac{\eta'}{4}\right) = \cosh\left(\frac{\gamma_2}{4}\right) \cdot \cosh\left(\frac{\delta'}{2}\right) \leq \cosh\left(\frac{\gamma_2}{4}\right) \cosh(W'). \quad (13)$$

as $\delta' \leq 2W'$. We now determine a lower bound for the width of the collar of α_2. We have to distinguish several subcases:

Case 2.c.2.a *The collar of α_2 is in configuration 1*

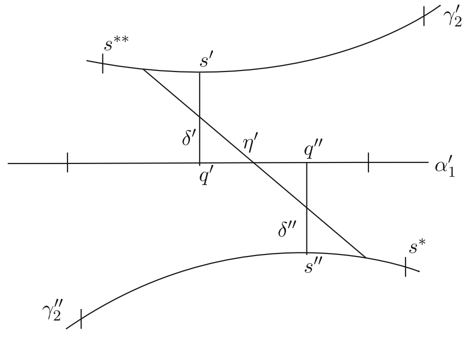


Fig. 7 Lifts of α_2 and γ_2 in the universal covering

$\overline{C(\alpha_2)}$ has width w_2. It self-intersects in a point p. Lift α_2 into the hyperbolic plane as described Case 2.b.1 (see Fig. 4) We have to discuss two cases, $\lambda' = \alpha_1$ and $\lambda' \neq \alpha_1$ (λ' is shown in Fig. 4).

$\lambda' = \alpha_1$

This case was discussed in Case 2.b.1. We may assume that $\alpha_1 \geq 1.5$ from Eq. 12. Hence we can apply Eq. 10 with $\alpha_1 = 1.5$ and $w_1 = W'$ and obtain

$$w_2 > 0.66$$

$\lambda' \neq \alpha_1$

If $\lambda' \geq \alpha_2$ we can apply Lemma 3.2 and conclude that $w_2 \geq W'$. If $\lambda' < \alpha_2$, we conclude from Eq. 13 and as γ_2 is the second shortest scg in S that

$$\gamma_2 \leq \lambda' < \alpha_2 < 4 \operatorname{arccosh}\left(\cosh\left(\frac{\gamma_2}{4}\right)\cosh(W')\right). \tag{14}$$

From the geometry of right-angled hyperbolic triangles (see [4, p. 454]), we obtain from Fig. 4 that

$$\cosh\left(\frac{\alpha_2}{4}\right)\cosh(w_2) \geq \cosh(r_2)\cosh(w_2) = \cosh\left(\frac{\lambda'}{2}\right).$$

Using the upper bound for α_2 and the lower bound for λ' from Eq. 14 in this inequality we obtain:

$$\cosh(w_2) \geq \frac{\cosh\left(\frac{\gamma_2}{2}\right)}{\cosh\left(\frac{\gamma_2}{4}\right)\cosh(W')}. \tag{15}$$

Case 2.c.2.b *The collar of α_2 is in configuration 2*

Let $Y^2[\alpha_2, \nu_1, \nu_2]$ be the Y-piece for α_2 in configuration 2. $\overline{C(\alpha_2)}$ self-intersects in the point p_2, such that $\operatorname{dist}(p_2, \alpha_2) = w_2 = \frac{\delta_2}{2}$. The geodesic arc δ_2 emanating perpendicularly from α_2 passes through this point and its endpoints divide α_2 into two parts, α_2' and α_2''. The common perpendiculars of the boundary geodesics of $Y^2[\alpha_2, \nu_1, \nu_2]$ separate the Y-piece into two isometric hexagons and δ_2 decomposes these hexagons into pentagons. By the pentagon formula (see [4, p. 454]) we have

$$\sinh\left(\frac{\delta_2}{2}\right)\sinh\left(\frac{\alpha_2'}{2}\right) = \cosh\left(\frac{\nu_1}{2}\right) \quad \text{and} \quad \sinh\left(\frac{\delta_2}{2}\right)\sinh\left(\frac{\alpha_2''}{2}\right) = \cosh\left(\frac{\nu_2}{2}\right).$$

None of these boundary geodesics can be α_1, as α_1 intersects α_2. If either ν_1 or ν_2 is bigger, than α_2, we obtain from Lemma 3.2 that $w_2 > W$. If not, then both must be bigger than γ_2. Additionally $\frac{\alpha_2'}{2} + \frac{\alpha_2''}{2} = \frac{\alpha_2}{2}$. Therefore either $\frac{\alpha_2'}{2}$ or $\frac{\alpha_2''}{2}$ is bigger than $\frac{\alpha_2'}{4}$. Let WLOG α_2' be the bigger one. We obtain from Eq. 13:

$$\sinh(w_2)\sinh\left(\operatorname{arccosh}\left(\cosh\left(\frac{\gamma_2}{4}\right)\cosh(W')\right)\right) \geq \sinh\left(\frac{\delta_2}{2}\right)\sinh\left(\frac{\alpha_2'}{2}\right) = \cosh\left(\frac{\nu_1}{2}\right) \geq \cosh\left(\frac{\gamma_2}{2}\right).$$

or equally, as $\sinh(x) = \sqrt{\cosh^2(x) - 1}$ for $x \geq 0$:

$$\sinh(w_2) \geq \frac{\cosh\left(\frac{\gamma_2}{2}\right)}{\sqrt{\cosh^2\left(\frac{\gamma_2}{4}\right)\cosh^2(W') - 1}}.$$

It follows from this equation that in Case 2.c.2.b

$$w_2 > 0.96.$$

Summary of Case 2.c.2 Let α_2 to be the shortest scg in the free homotopy class of $\delta' \alpha_1{}^* \delta'' \gamma_2^*$ (see Fig. 6). Its length is restricted by

$$\alpha_2 < 4 \operatorname{arccosh}\left(\cosh\left(\frac{\gamma_2}{4}\right)\cosh(W')\right).$$

From the discussion of the subcases Case 2.c.2.a we conclude that the width of the collar w_2 is bounded from below by

$$w_2 \geq \min\left\{0.66, \operatorname{arccosh}\left(\frac{\cosh\left(\frac{\gamma_2}{2}\right)}{\cosh\left(\frac{\gamma_2}{4}\right)\cosh(W')}\right)\right\}.$$

As a consequence of Eq. 12,we have that $2.1 \leq \gamma_2$. With the help of this lower bound it follows from the above equation that w_2 is bounded from below. As α_2 is bounded from above, it follows from Eqs. 3 and 4 that $m_2(J(S))^2$ is bounded from above. A refined analysis shows that

$$m_2(J(S))^2 < 3.1\log(8g-7).$$

This proves that Theorem 1.2 is valid. □

Proof of Corollary 1.3 The proof is very similar to the discussion of Case 1 of Theorem 1.2.

Let $\eta_i \leq t$ be one of the simple closed geodesics that divide S. By Lemma 3.9 the width of a half-collar of η_i is bigger than $\operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{t}{2})}\right)$ on both sides of η_i. It follows also from the collar theorem that any other scg in S has a distance greater than $\operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{t}{2})}\right)$ from η_i. Let S_i be a surface of genus (g_i, n_i), $g_i > 0$ from the decomposition of S. Let WLOG $\eta_1, \ldots, \eta_{n_i}$ be its boundary geodesics. We first prove that the shortest nsscg, α_i in S_i is smaller than $(n_i+1)\max\{4\log(4g_i+2n_i+3), t\}$. Then we show that it has a collar in S whose width is bounded from below. By Lemma 3.6, there exists a scg γ_i^1 in S_i of length $\gamma_i^1 \leq 4\log(4g_i+2n_i+3)+n_i t$.

We have that

$$4\log(4g_i+2n_i+3)+n_i t \leq (n_i+1)\max\{4\log(4g_i+2n_i+3), t\}.$$

Hence, if γ_i^1 is non separating, we are done. If γ_i^1 is separating, we cut open S_i along γ_i^1. S_i decomposes into two surfaces, such that one of these two, S_i^2 has signature (g_i', n_i'), with $g_i' > 0$ and $n_i' \leq n_i - 1$. The length of its boundary is smaller than $4\log(4g_i+2n_i+3)+(n_i-1)t$. We can again apply Lemma 3.6 to this surface to obtain an upper bound for the length of a scg in S_i. Repeating this process iteratively we obtain that there exists a nsscg in S_i, whose length is smaller than $(n_i+1)\max\{4\log(4g_i+2n_i+3), t\}$.

Each $\alpha_i, i \in \{1, \ldots, m\}$ has a collar, whose width w_i is bounded from below. Namely, if $\alpha_i < K$ then $w_i > \min\left\{\operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{t}{2})}\right), W'\right\}$ and if $\alpha_i > K$, then $w_i > \min\left\{\operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{t}{2})}\right), W\right\}$. This follows from the same line of argumentation as in Case 1 of Theorem 1.2. The $(\alpha_i)_{i=1,\ldots,m}$ can be together incorporated into a canonical homology basis of S. From the bounds on the length and the width of the collars of the geodesics follows the bound on the norm of the lattice vectors of the Jacobian of S. In total we obtain Corollary 1.3. □

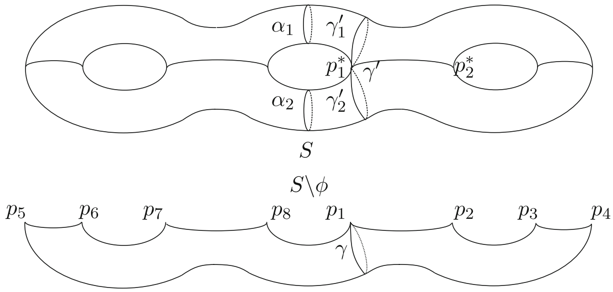


Fig. 8 A hyperelliptic surface S and the quotient surface $S\backslash\phi$

Proof of Theorem 1.4 For the proof of Theorem 1.4 we first give a suitable definition of a hyperelliptic surface.

Definition 4.3 Let S be a compact Riemann surface of genus g. An *involution* is an isometry $\phi : S \rightarrow S, \phi \neq id$, such that $\phi^2 = id$. The surface S is *hyperelliptic*, if it has an involution that has exactly $2g + 2$ fixed points. These fixed points are called the *Weierstrass points (WPs).*

It is well known that the above definition is equivalent to the usual one. Let S be a hyperelliptic surface of genus g with involution ϕ. We will show that the shortest nsscg α_1 of S is bounded by a constant, independent of the genus. It was shown in Buser and Sarnak [5] that the width of the collar of the shortest nsscg on a R.S. S is bounded from below. It then follows from Eqs. 4 and 3 that Theorem 1.4 holds.

Consider the quotient surface $S\backslash\phi$. This surface is a topological sphere with $2g+2$ cones of angle π, whose vertices $\{p_i\}_{i=1\ldots 2g+2}$ are the images of the WPs $\{p_i^*\}_{i=1\ldots 2g+2}$ under the projection (see Fig. 8).

Let $B_r(p_i)$ be a disk of radius r around a vertex of a cone. As long as these disks are small enough, they are embedded in $S\backslash\phi$. In this case the area of a disk of radius r around a vertex of a cone p_i, $B_r(p_i)$ is half the area of a disk of radius r in the hyperbolic plane, $\text{area}(B_r(p_i)) = \pi(\cosh(r) - 1)$. Now expand all disks around the cone points until either a disk self-intersects or two different disks intersect for the first time at radius R. In this limit case we still obtain:

$$(2g+2)\pi(\cosh(R) - 1) = \text{area}\left(\bigcup_{i=1}^{2g+2} B_R(p_i)\right) < \text{area}(S\backslash\phi) = 2\pi(g-1).$$

As $\frac{g-1}{g+1} < 1$ we conclude that $R < \text{arccosh}(2)$.

When the radii of the disks reach R and two different disks intersect the geodesic arc that forms lifts to a simple closed geodesic in S. When a disk self-intersects at radius R, the geodesic arc that forms lifts to a Fig. 8 geodesic in S. This Fig. 8 geodesic consists of two loops. The scg in the free homotopy class of such a loop is smaller than the loop itself. Hence there exists a scg of length smaller than $4R$ in S. It follows, for the systole γ_1 in S that $\gamma_1 < 4\,\text{arccosh}(2) = 5.2678\ldots$

By a refinement of this area estimate Bavard obtains a better upper bound in Bavard [1], which is

$$\gamma_1 < 4\,\mathrm{arccosh}\left(\left(2\sin\left(\frac{\pi(g+1)}{12g}\right)\right)^{-1}\right) < 2\log\left(3+2\sqrt{3}+2\sqrt{5+3\sqrt{3}}\right) = 5.1067\ldots \tag{16}$$

We now show that this upper bound is equally valid for the shortest non-separating scg in S. Consider the case, where two different disks intersect at radius R. In this case a geodesic arc of length smaller than $2R$ connects WLOG p_1 and p_2. It is easy to see that it lifts to a scg α_1 of length $4R$ in the double covering S (see Fig. 8). α_1 passes the two WPs p_1^* and p_2^*. It is a well-known fact that such a geodesic is non-separating. Consider now the case, where WLOG $B_R(p_1)$ self-intersects. The geodesic arc γ that passes p_1 and the intersection point, lifts to a figure 8 geodesic γ' in S (see Fig. 8). Let γ_1' and γ_2' be the two different lifts of γ in S with intersection point p_1^*. Let α_1 and α_2 be the scg in the free homotopy class of γ_1' and γ_2', respectively. The length of both is bounded from above by $2R$. It is easy to convince oneself that the situation depicted in Fig. 8. is the correct one and that both α_1 or α_2 are non-separating.

In any case there exists a nsscg α_1 in S, whose length is smaller than the constant from Eq. 16. Hence we obtain an upper bound for $m_1(J(S))^2$:

$$m_1(J(S))^2 < \frac{3\log(3+2\sqrt{3}+2\sqrt{5+3\sqrt{3}})}{\pi} \leq 2.4382\ldots$$

This proves Theorem 1.4. □

References

1. Bavard, C.: La systole des surfaces hyperelliptiques, vol. 71. Prepublication de l'ENS Lyon (1992)
2. Bergé, A.-M., Martinet, J.: Densité dans des familles de réseaux. Application aux réseaux isoduaux. L'Enseignement Mathématique **41**, 335–365 (1995)
3. Birkenhake, C., Lange, H.: Complex Abelian Varieties, Grundlehren der mathematischen Wissenschaften, vol. 302. Springer, Berlin (2004)
4. Buser, P.: Geometry and Spectra of Compact Riemann Surfaces, Progress in Mathematics, vol. 106. Birkhäuser Verlag, Boston (1992)
5. Buser, P., Sarnak, P.: On the period matrix of a Riemann surface of large genus (with an appendix by Conway, J. H. and Sloane, N. J. A.). Inventiones Mathematicae **117**(1), 27–56 (1994)
6. Fay, J.D.: Theta Functions on Riemann Surfaces, Lecture Notes in Mathematics, vol. 352. Springer, Berlin (1973)
7. Gendulphe, M.: Découpages et inégalités systoliques pour les surfaces hyperboliques à bord. Geometriae Dedicata **142**, 23–35 (2009)
8. Gruber, P.M., Lekkerkerker, C.G.: Geometry of Numbers, North Holland Mathematical Library. Elsevier, Amsterdam (1987)
9. Jenni, F.: The 1st eigenvalue of Laplace operators in selected examples of Riemannian surfaces. Commentarii Mathematici Helvetici **59**(2), 193–203 (1984)
10. Parlier, H.: On the geometry of simple closed geodesics. PhD thesis, Ecole Polytechnique Fédérale de Lausanne (2004)
11. Parlier, H.: Lengths of geodesics on Riemann surfaces with boundary. Ann. Acad. Sci. Fenn. Math. **30**(2), 227–236 (2005)
12. Schmutz, P.: Riemann surfaces with shortest geodesic of maximal length. Geometric and Functional Analysis **3**(6), 564–631 (1993)
13. Schottky, F.: Zur Theorie der Abelschen Functionen von vier Variablen. Journal für Die Reine Und Angewandte Mathematik **102**, 304–352 (1888)
14. Shiota, T.: Characterization of Jacobian varieties in terms of soliton-equations. Inventiones Mathematicae **83**(2), 333–382 (1986)
15. van Geemen, B.: Siegel modular forms vanishing on the moduli space of curves. Inventiones Mathematicae **78**(2), 329–349 (1984)