

ON TYPE I BLOW UP FORMATION FOR THE CRITICAL NLW

JOACHIM KRIEGER, WILLIE WONG

ABSTRACT. We introduce a suitable concept of weak evolution in the context of the radial quintic focussing semilinear wave equation on \mathbb{R}^{3+1} , that is adapted to continuation past type II singularities. We show that the weak extension leads to type I singularity formation for initial data corresponding to: (i) the Kenig-Merle blow-up solutions with initial energy below the ground state and (ii) the Krieger-Nakanishi-Schlag blow-up solutions sitting initially near and “above” the ground state static solution.

1. INTRODUCTION

We consider the critical focussing nonlinear wave equation on \mathbb{R}^{3+1} , given by

$$\square u := -u_{tt} + \Delta u = -u^5, \tag{1.1}$$

which has a (possibly negative) conserved energy

$$E(u) := \int_{\mathbb{R}^3} \left(\frac{1}{2}(u_t^2 + |\nabla_x u|^2) - \frac{u^6}{6} \right) dx.$$

We restrict to radial solutions of the form $u(t, x) = v(t, |x|)$. It is well-known and easy to show that this model admits finite time blow up solutions with finite initial free energy

$$E_{\text{free}}(u)(t) := \int_{\mathbb{R}^3} \left[\frac{1}{2}(u_t^2 + |\nabla_x u|^2) \right] dx.$$

One can start with the explicit ODE-type solutions

$$u(t, x) = \frac{\left(\frac{3}{4}\right)^{\frac{1}{4}}}{(T-t)^{\frac{1}{2}}}$$

for any $T \in \mathbb{R}_+$. By truncation to a backward (or forward) light cone and invocation of Huygens’ principle, one can modify these to solutions for which the initial free energy is finite. Indeed, one may consider data $u[0] = (u(0, \cdot), u_t(0, \cdot))$ with

$$u(0, \cdot) = \chi_{|x| < 3T} \frac{\left(\frac{3}{4}\right)^{\frac{1}{4}}}{T^{\frac{1}{2}}}, \quad u_t(0, \cdot) = \chi_{|x| < 3T} \frac{\left(\frac{3}{64}\right)^{\frac{1}{4}}}{T^{\frac{3}{2}}}$$

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where the cut-off function $\chi_{|x|<3T}|_{|x|\leq 2T} = 1$ and smoothly truncates to the region $|x| < 3T$. Observe that these solutions satisfy

$$\limsup_{t \nearrow T} \int_{|x|<T} \left[\frac{1}{2}(u_t^2(t) + |\nabla_x u(t)|^2) \right] dx = +\infty,$$

and thus cannot be continued past time T (though a singularity may form at some earlier time, depending on the choice of cut-off $\chi_{|x|<3T}$).

Motivated by these ODE type blow-ups, we say a blow-up solution u with maximum forward time of existence T is of *type I* if

$$\limsup_{t \nearrow T} E_{\text{free}}(u)(t) = +\infty$$

and *type II* otherwise, that is

$$\limsup_{t \nearrow T} E_{\text{free}}(u)(t) < +\infty.$$

Recent works by Duyckaerts-Kenig-Merle [3–6] have provided a complete classification near the blow-up time of type II solutions for (1.1), while existence of solutions of this type was established in [12] and [1]. Here, we would like to discuss the formation of type I blow up. To the best of the authors' knowledge, previously demonstrated type I blow up mechanisms all derive in principle from u_t having a sign pointwise. In addition to the explicit ODE solutions (and perturbations thereof as in [2]), Duyckaerts-Kenig-Merle showed in [5] that monotonicity in time of a radial solution close to the blow up time implies type I blow up, which they then used to show that the W^+ solution of [7] as well as solutions given by initial data $u[0] = (cW, 0)$ with $c > 1$ all evolve into type I blow ups.

In a recent work [10], the study of all possible dynamics which result as *perturbations of the static solution* $W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-\frac{1}{2}}$ was begun. Note that these static solutions are a special feature of the energy critical case. Also, crucially for the analysis of [10], the perturbations are close to W with respect to a norm strictly stronger than the energy. It was then shown in [10] that there exists a co-dimension one Lipschitz manifold Σ passing through W such that within a sufficiently close neighbourhood to W , data 'above' Σ result in finite time blow up while data 'below' Σ scatter to zero, all in forward time. Further, data precisely located on Σ lead to solutions in forward time scattering toward a re-scaling of W .

In this note, we would like to study the finite-time blow up solutions corresponding to data slightly above Σ . Conjecturally, a generic set within these solutions ought to correspond to type I blow up solutions. At this time we cannot show this. Instead, our goal here is to introduce a suitable concept of *canonical weak solution* and show that such solutions will result eventually, in finite time, in a type I blow up scenario. This will be seen to directly result from a combination of the recent breakthrough characterization of type II blow up solutions by Duyckaerts-Kenig-Merle [6] with the techniques developed in [9]. Along the way, we will also show that the canonical weak extensions of the blow-up solutions exhibited by Kenig-Merle [8], whose initial energy is below that of the ground state, terminates in finite time with exploding free energy.

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2. A CANONICAL CONCEPT OF WEAK EVOLUTION

Let $u(t, x)$ be a (radial) Shatah-Struwe energy class solution (see e.g. [14], [8]) of (1.1), existing on an interval $I = [0, T)$, $T > 0$. Also, assume that I is a maximal such interval. If $T < \infty$, then the solution either has a type I singularity at T , or else a type II singularity. Assume the latter situation. According to the seminal work [6], the solution admits a decomposition (writing $W_\lambda(x) = \lambda^{\frac{1}{2}} W(\lambda x)$ for the \dot{H}^1 invariant scaling)

$$u(t, \cdot) = \sum_{i=1}^N \kappa_i W_{\lambda_i(t)}(\cdot) + u_1(t, \cdot) + o_{\dot{H}^1}(1), \quad \kappa_i \in \{\pm 1\},$$

$$u_t(t, \cdot) = u_{1,t}(t, \cdot) + o_{L^2}(1)$$

as $t \nearrow T$, with¹

$$(T - t)^{-1} \ll \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_N(t)$$

and $u_1(t, \cdot)$ is an energy class solution of (1.1) in a neighborhood around time $t = T$. One easily verifies that this $u_1(t, \cdot)$ is indeed uniquely determined by u and T . It is then natural, assuming that there is a type II singularity at time $t = T$, to continue the evolution past time T by imposing data

$$u[T] = (u(T, \cdot), u_t(T, \cdot)) := (u_1(T, \cdot), u_{1,t}(T, \cdot))$$

and then using the Shatah-Struwe evolution of $u[T]$ starting from time $T =: T_1$. Then there exists $T_2 \in (T_1, +\infty]$, such that if $T_2 < \infty$, there is either a type I or type II singularity at T_2 , and then in the latter case again the Duyckaerts-Kenig-Merle profile decomposition applies at time $t = T_2$, allowing us to write

$$u(t, \cdot) = \sum_{i=1}^{N_2} \kappa_i^{(2)} W_{\lambda_i^{(2)}(t)}(\cdot) + u_2(t, \cdot) + o_{\dot{H}^1}(1), \quad t \nearrow T_2,$$

where u_2 is now a solution of (1.1) in a neighborhood containing $t = T_2$. In this way, we obtain a sequence of times

$$T_1 < T_2 < T_3 < \dots, \quad T_i \in (0, +\infty]$$

with the following possibilities: (i) the sequence is finite, and the last $T_{\text{terminal}} := T_N = +\infty$, with all previous T_i being type II blow up times; (ii) the sequence is finite, and the last $T_{\text{terminal}} := T_N < \infty$ being the first type I blow up time in the evolution; (iii) the sequence is infinite and we define $T_{\text{terminal}} := \lim_{i \rightarrow \infty} T_i \in (0, \infty]$. Note that that except in case (ii) we have no *a priori* knowledge as to whether the solution blows up at T_{terminal} .

We now define the *canonical evolution* of the data $u[0]$ on $[0, T_{\text{terminal}})$ to be the function $\tilde{u}(t, \cdot)$ given by $u(t, \cdot)$ on $[0, T_1)$, by $u_1(t, \cdot)$ on $[T_1, T_2)$ etc.

¹The notation $a(t) \ll b(t)$ here means $\lim_{t \nearrow T} \frac{b(t)}{a(t)} = +\infty$.

On the other hand, we define $u(t, x) \in L^\infty([0, T_*], \dot{H}^1) \cap W^{1,\infty}([0, T_*], L^2)$ to be a *weak solution* of (1.1), provided for every $\phi \in C_0^\infty((-\infty, T_*) \times \mathbb{R}^3)$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} u_t(0, \cdot) \phi(0, \cdot) dx + \int_0^{T_*} \int_{\mathbb{R}^3} (u_t \phi_t - \nabla_x u \cdot \nabla_x \phi) dx dt \\ = - \int_0^{T_*} \int_{\mathbb{R}^3} u^5 \phi dx dt. \end{aligned} \quad (2.1)$$

Note that our concept of canonical weak evolution is in fact more regular than $L^\infty([0, T_*], \dot{H}^1) \cap W^{1,\infty}([0, T_*], L^2)$, since

$$\tilde{u}|_{[T_{i-1}, T_i]} \in C^0([T_{i-1}, T_i], \dot{H}^1) \cap C^1([T_{i-1}, T_i], L^2).$$

In particular, for the canonical evolution \tilde{u} is right-continuous at time 0, that is $\lim_{t \searrow 0} \tilde{u}(t, \cdot) = u(0, \cdot)$ with respect to \dot{H}^1 . Then

Lemma 2.1. *Let $\tilde{u}(t, \cdot)$ be the canonical evolution of $u[0] \in \dot{H}^1 \times L^2$, defined on $[0, T_{\text{terminal}}]$. Then \tilde{u} is a weak solution of (1.1) in the above sense with $T_* = T_{\text{terminal}}$.*

Proof. Let $\phi \in C_0^\infty((-\infty, T_{\text{terminal}}) \times \mathbb{R}^3)$. Then recalling the construction of \tilde{u} , there exist finitely many $T_i, i = 1, 2, \dots, k, T_0 := 0$, with $T_i \in \pi_t(\text{supp}(\phi))$ with $\pi_t : \mathbb{R}^{3+1} \rightarrow \mathbb{R}$ the projection onto the time coordinate. Then we have $\tilde{u}|_{[T_i, T_{i+1}]} = u_i, u_0 = u$ being the evolution of the data $u[0]$. Now for each i , pick a function $\chi \in C_0^\infty([T_i, T_{i+1}])$ with $\chi(T_i) = 1$; then integrating by parts the Shatah-Struwe energy class solution u_i we have

$$\begin{aligned} \int_{\mathbb{R}^3} u_{i,t}(T_i, \cdot) \phi(T_i, \cdot) dx + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} (u_{i,t}(\chi \phi)_t - \nabla_x u_i \cdot \nabla_x(\chi \phi)) dx dt \\ = - \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} u_i^5 \chi \phi dx dt \end{aligned} \quad (2.2)$$

Pick a sequence $\chi^{(k)} \in C_0^\infty([T_i, T_{i+1}])$ with $\chi^{(k)} \rightarrow \chi|_{[T_i, T_{i+1}]}$ pointwise and locally uniformly and such that

$$\lim_{k \rightarrow \infty} \int_{T_i}^{T_{i+1}} (\chi^{(k)})'(t) f(t) dt = -f(T_{i+1})$$

for $f \in C^0([T_i, T_{i+1}])$. Since we can write (as $t \nearrow T_{i+1}$ and where $\kappa_k^{(i)} \in \{\pm 1\}$)

$$u_i(t, \cdot) \xrightarrow{\dot{H}^1} \sum_{k=1}^{N_i} \kappa_k^{(i)} W_{\lambda_k^{(i)}(t)}(\cdot) + u_{i+1}(t, \cdot), \quad u_{i,t} \xrightarrow{L^2} u_{i+1,t}(T_{i+1}, \cdot),$$

we infer

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} (u_{i,t}(\chi^{(k)} \phi)_t - \nabla_x u_i \cdot \nabla_x(\chi^{(k)} \phi)) dx dt \\ = \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} (u_{i,t} \phi_t - \nabla_x u_i \cdot \nabla_x(\phi)) dx dt - \int_{\mathbb{R}^3} u_{i+1,t}(T_{i+1}, \cdot) \phi(T_{i+1}, \cdot) dx \end{aligned}$$

and so we obtain

$$\begin{aligned} & \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} (u_{i,t} \phi_t - \nabla_x u_i \cdot \nabla_x(\phi)) \, dx \, dt + \int_{T_i}^{T_{i+1}} \int_{\mathbb{R}^3} u_i^5 \phi \, dx \, dt \\ &= \int_{\mathbb{R}^3} u_{i+1,t}(T_{i+1}, \cdot) \phi(T_{i+1}, \cdot) \, dx - \int_{\mathbb{R}^3} u_{i,t}(T_i, \cdot) \phi(T_i, \cdot) \, dx. \end{aligned} \quad (2.3)$$

Summation of the relations (2.3) over $i = 1, 2, \dots, k$, we find the relation

$$\begin{aligned} & \int_{\mathbb{R}^3} \tilde{u}_t(0, \cdot) \phi(0, \cdot) \, dx + \int_0^{T_{\text{terminal}}} \int_{\mathbb{R}^3} (\tilde{u}_t \phi_t - \nabla_x \tilde{u} \cdot \nabla_x \phi) \, dx \, dt \\ &= - \int_0^{T_{\text{terminal}}} \int_{\mathbb{R}^3} \tilde{u}^5 \phi \, dx \, dt, \end{aligned} \quad (2.4)$$

proving the lemma. \square

An important consequence of the profile decomposition of Duyckaerts-Kenig-Merle [6] is that, per the asymptotic separation of profiles and energy conservation of the regular evolution, the energy of our canonical weak evolution is *strictly decreasing*. In fact, we have that

$$E(u_i) = N_i \cdot E(W) + E(u_{i+1}), \quad N_i \geq 1 \quad (2.5)$$

since the soliton energy is scale invariant. Evaluating

$$E(W) = \frac{1}{3} \int_{\mathbb{R}^3} |\nabla W|^2 \, dx > 0$$

we see explicitly the energy jump as solitons get bubbled off. This also implies that our canonical weak evolution concept is time *irreversible*.

3. FORMATION OF TYPE I SINGULARITIES: GENERAL CASE

Before considering the blow-up solutions of [10], we first prove some general lemmas about the eventual formation of type I singularities for our canonical weak evolution.

Lemma 3.1. *If a canonical weak solution \tilde{u} satisfies $T_{\text{terminal}} < +\infty$, then it satisfies the “type I” condition*

$$\limsup_{t \nearrow T_{\text{terminal}}} E_{\text{free}}(\tilde{u})(t) = +\infty. \quad (3.1)$$

Proof. As discussed in Section 2, the only possibility when $T_{\text{terminal}} < +\infty$ is either $T_{\text{terminal}} = T_N < \infty$ ending in a type I blow-up, or there exists an infinite sequence $T_i \nearrow T_{\text{terminal}}$ of type II blow-up points. In the first case (3.1) follows by definition. In the second case we appeal to the energy evolution (2.5) which implies that $\lim_{i \rightarrow \infty} E(u_i) = -\infty$. Then from Sobolev’s embedding we obtain, as claimed,

$$\lim_{i \rightarrow \infty} \|\nabla_{t,x} u_i(T_i, \cdot)\|_{L_x^2} = +\infty.$$

\square

Next, we show one of the main advantages of our canonical weak solution construction: it preserves the virial type functional used in [9]. Define, for some large large $\tau > 0$ to be fixed later, and a cutoff $\chi \in C_0^\infty([0, \infty))$ with $\chi|_{[0,1]} = 1$, the functions

$$w(t, x) = \chi\left(\frac{|x|}{t + \tau}\right), \quad y(t) := \langle w(t, \cdot) \tilde{u}(t, \cdot), \tilde{u}(t, \cdot) \rangle \quad (3.2)$$

Here the spatial L^2 pairing $\langle \cdot, \cdot \rangle$ is well-defined as long as $\tilde{u} \in \dot{H}^1$, due to Sobolev's embedding and the cutoff. This is definitely the case on each interval $I_j := (T_{i-1}, T_i)$. In fact, from the computations in [9, Section 5], we have that $y(t), \dot{y}(t), \ddot{y}(t)$ are continuous functions on each open interval I_j . Now suppose that \tilde{u} is a canonical weak solution maximally defined on $[0, T_{\text{terminal}})$.

Lemma 3.2. *The functions $y(t), \dot{y}(t)$, and $\ddot{y}(t)$ extend continuously to $(0, T_{\text{terminal}})$.*

Proof. It suffices to check that the three functions are continuous at each time T_i . Recall first the representation for \tilde{u} from its definition

$$\begin{aligned} \tilde{u}(t, \cdot) - \left[\sum_{k=1}^{N_i} \kappa_k^{(i)} W_{\lambda_k^{(i)}(t)}(\cdot) + u_{i+1}(t, \cdot) \right] &\xrightarrow[t \nearrow T_{i+1}]{\dot{H}^1} 0, \\ \tilde{u}_t(t, \cdot) - u_{i+1,t}(T_{i+1}, \cdot) &\xrightarrow[t \nearrow T_{i+1}]{L^2} 0. \end{aligned} \quad (3.3)$$

Now observe that with a finite radius cutoff

$$\int_{|x| < R} W_\lambda^2 dx = 4\pi \int_0^R \frac{\lambda r^2}{1 + \lambda^2 r^2/3} dr \leq \frac{12\pi R}{\lambda}.$$

This implies that for each $\lambda_k^{(i)}$ we have

$$\sqrt{w} W_{\lambda_k^{(i)}(t)} \xrightarrow[t \nearrow T_{i+1}]{L^2} 0$$

and hence

$$\lim_{t \nearrow T_{i+1}} \langle w \tilde{u}, \tilde{u} \rangle = \langle w u_{i+1}, u_{i+1} \rangle|_{t=T_{i+1}}$$

showing the continuity of $y(t)$.

For the derivatives, we follow the computations in [9]. In particular, observe that

$$\dot{y}(t) = \langle \dot{w} \tilde{u} + 2w \dot{\tilde{u}}, \tilde{u} \rangle$$

provided $t \in [T_i, T_{i+1})$. Now, the same argument as above shows that using the uniformly bounded support of \dot{w} and w near T_{i+1} ,

$$\lim_{t \nearrow T_{i+1}} \langle \dot{w} \tilde{u}, \tilde{u} \rangle = \langle \dot{w} u_{i+1}, u_{i+1} \rangle|_{t=T_{i+1}}.$$

Together with the L^2 convergence of $w \tilde{u} \rightarrow u_{i+1}(T_{i+1})$ and $\dot{\tilde{u}} \rightarrow u_{i+1,t}(T_{i+1})$ as $t \nearrow T_{i+1}$ we get

$$\lim_{t \nearrow T_{i+1}} \dot{y}(t) = \langle \dot{w} u_{i+1}(T_{i+1}, \cdot) + 2w u_{i+1,t}(T_{i+1}, \cdot), u_{i+1}(T_{i+1}, \cdot) \rangle.$$

Since $\tilde{u}|_{[T_{i+1}, T_{i+2})} = u_{i+1}$, the continuity of $\dot{y}(t)$ across $t = T_{i+1}$ is evident. Next, consider $\ddot{y}(t)$, which according to [9] is given by the expression

$$\ddot{y}(t) = \langle 2w, \dot{\tilde{u}}^2 - |\nabla \tilde{u}|^2 + \tilde{u}^6 \rangle + \langle \dot{w}\tilde{u}, \tilde{u} \rangle + \langle 4\dot{w}\tilde{u}, \dot{\tilde{u}} \rangle - 2\langle \tilde{u}\nabla w, \nabla \tilde{u} \rangle, \quad t \in I_{i+1}. \quad (3.4)$$

The continuity of the middle two terms at times T_{i+1} is obtained exactly as shown previously. For the last term, in addition to the representation formulae (3.3) above we use also the fact that the derivative ∇w has compact spatial support uniformly (near $t = T_{i+1}$) away from the origin and so kills the contributions from $\nabla W_{\lambda_k^{(i)}}(t)$.

We examine the remaining term. The convergence in L^2 of $\dot{\tilde{u}}$ to \dot{u}_{i+1} as $t \nearrow T_{i+1}$ implies

$$\lim_{t \nearrow T_{i+1}} \langle 2w(t, \cdot), \dot{\tilde{u}}^2(t, \cdot) \rangle = \langle 2w, u_{i+1,t}^2(T_{i+1}, \cdot) \rangle.$$

The expressions $\langle 2w, |\nabla \tilde{u}|^2 \rangle$ and $\langle 2w, \tilde{u}^6 \rangle$ must be taken together as they *do not individually extend continuously across* T_{i+1} . Their difference, however, does. Indeed, it is straightforward to check that

$$\lim_{t \nearrow T_{i+1}} \langle 2w, |\nabla h_i|^2 - h_i^6 \rangle = 0, \quad h_i(t, \cdot) := \sum_{k=1}^{N_i} \kappa_k^{(i)} W_{\lambda_k^{(i)}}(t)$$

where we exploit of course the fact that W is the ground state, i.e. $\Delta W + W^5 = 0$, as well as the fact that the solitons separate in scale, i.e. $\lambda_{k-1}^{(i)} \ll \lambda_k^{(i)}$. It follows that

$$\lim_{t \nearrow T_{i+1}} \langle 2w, -|\nabla \tilde{u}|^2 + \tilde{u}^6 \rangle = \langle 2w, -|\nabla u_{i+1}|^2 + u_{i+1}^6 \rangle|_{t=T_{i+1}}.$$

The fact that $\ddot{y}(t)$ extends continuously across T_{i+1} follows easily. \square

We conclude this section with the following result, obtained as a modification of the classical blow-up theorem of Levine [13].

Lemma 3.3. *Let \tilde{u} be a maximally extended canonical weak solution, and suppose that at some positive time its energy $E(\tilde{u}) < 0$. Then $T_{\text{terminal}} < +\infty$ for \tilde{u} .*

Proof. Following [9], we observe that, due to the cut-off function w in (3.2), we can write

$$\ddot{y}(t) = 2(4\|\dot{\tilde{u}}\|_{L^2}^2 + 4\|\nabla \tilde{u}\|_{L^2}^2 - 6E(\tilde{u})) + O(E_{\text{ext}}) \quad (3.5)$$

where

$$E_{\text{ext}}(t) := \int_{|x| > t+\tau} (|\dot{u}|^2 + |\nabla u|^2) dx \lesssim E_{\text{ext}}(0)$$

via a continuity argument and Huygens' principle, and the observation that the bubbling off of solitons happen "at the origin". By picking the initial cutoff $\tau > 0$ sufficiently large, we can force $E_{\text{ext}}(0)$ as small as we want (as long as $E_{\text{free}}(0)$ is finite). Suppose now (as given by the hypothesis of our lemma) that for some T_i that $E(\tilde{u})|_{(T_i, T_{\text{terminal}})} < -2\varepsilon_* < 0$, where we used the monotonicity of energy. Then a suitably large choice of τ would guarantee that

$$\ddot{y}(t) \geq 8\|\dot{\tilde{u}}\|_{L^2}^2 + \varepsilon_* \quad (3.6)$$

holds on $(T_i, T_{\text{terminal}})$.

Now assume, for contradiction, that $T_{\text{terminal}} = +\infty$. Note that (3.6) establishes a lower bound on \ddot{y} in (T_i, ∞) , which implies that after some large finite time $\dot{y} > 0$ and $y > y_0$ is bounded below. Hence if one remarks just as in [9] that by Cauchy-Schwartz

$$|\dot{y}(t)| = 2\langle w\dot{\tilde{u}}, \tilde{u} \rangle + O(E_{\text{ext}}) \leq 2\sqrt{y}\|\dot{\tilde{u}}\|_{L^2}^{1/2} + O(E_{\text{ext}})$$

we have that at all sufficiently late times past T_i we can upgrade (3.6) to

$$\ddot{y}(t) \geq \frac{3\dot{y}^2}{2y} + \frac{1}{2}\varepsilon_* . \quad (3.7)$$

From this inequality, however, we can apply the exact same argument as in [9]: (3.7) implies that $\frac{d^2}{dt^2}y^{-\frac{1}{2}} < 0$ at all $t > T_i$ sufficiently large; that $\dot{y}, y > 0$ implies that $\frac{d}{dt}y^{-\frac{1}{2}} < 0$ at all $t > T_i$ sufficiently large. Together the concavity implies $y(t)$ must blow up in finite time, ruling out the possibility $T_{\text{terminal}} = +\infty$ and proving our lemma. \square

Theorem 3.4. *The maximally extended canonical weak solution for the Kenig-Merle type [8] blow-up initial data with $E(u) < E(W)$ and $\|\nabla_x u\|_{L^2} > \|\nabla W\|_{L^2}$ terminates in finite T_{terminal} with the type I condition (3.1) satisfied.*

Proof. If T_1 ends in a type I blow-up, we are done. If not, by the profile decomposition and (2.5) we have that $E(\tilde{u})(T_1) = E(u)(0) - N_1 \cdot E(W) < 0$. The theorem then follows from Lemmas 3.1 and 3.3. \square

4. FORMATION OF TYPE I SINGULARITIES: ABOVE THRESHOLD SOLUTIONS

Theorem 3.4 above settles the problem for initial data with energy below that of the ground state, in view of the dichotomy proven in [8]. We now turn our attention to whether there exist *generic sets* (not necessarily in the energy topology) of solutions which satisfy $T_{\text{terminal}} < +\infty$ with initial energy above that of the ground state. We note that by appropriately time-translating the type II blow-up solutions constructed in [12], we obtain one that satisfies $T_1 > 0$ and $T_2 = +\infty$. In order to rule out these type of behaviour, we move to a stronger topology²: here we review the results of [10].

First we recall that linearising (1.1) around the solution W leads us to consider the linearised operator $-\Delta - 5W^4$. On radial functions, this linearised operator has a unique negative eigenvalue $-k_d^2$ with eigenfunction g_d satisfying $g_d > 0$; this contributes to the linear instability of the ground state W . In [11], it was shown, for initial data supported in a fixed ball with the topology $H_{\text{rad}}^3(\mathbb{R}^3) \times H_{\text{rad}}^2(\mathbb{R}^3)$, that there exists a Lipschitz manifold Σ in a small neighbourhood of the ground state W , which contains the soliton curve \mathcal{S} (i. e. rescalings of W), such that initial data given on Σ exists globally and scatters to \mathcal{S} . Moreover, this Lipschitz manifold Σ is transverse to $(g_d, 0)$: indeed, Σ is written as a Lipschitz graph over the subspace orthogonal to g_d of the tangent space at W . Therefore an ε_d -neighbourhood of W can

²This is analogous to [11], where conditional stability of the ground state W is shown for a stronger topology than energy. The same question is open in energy topology. We refer the readers to [10] for a summary.

be divided into the portion ‘above’ Σ (i. e. those that can be written as $\sigma + \delta(g_d, 0)$ for $\sigma \in \Sigma$ and $0 < \delta < \varepsilon$) and those ‘below’ Σ (with a minus sign instead). In [10], it was shown that this division provides a dichotomy: those data sitting above Σ blows up in finite time, while those data sitting below Σ has global existence in forward time and scatters to zero in energy space.

Our main theorem concerns the blow-up solutions sitting above Σ :

Theorem 4.1. *Let $u(t, \cdot)$ be one of the blow up solutions with initial data of the form $\sigma + \delta(g_d, 0)$ with $\delta > 0$ as described above. Then the canonical weak extension of this solution will satisfy $T_{\text{terminal}} < +\infty$. Furthermore, the canonical weak solutions satisfy (3.1).*

Proof. By Lemma 3.1 it suffices to rule out the case $T_{\text{terminal}} = +\infty$. Now, if $T_2 < T_{\text{terminal}}$ or $T_1 < T_{\text{terminal}}$ with $N_i > 1$, by the energy jump condition (2.5) we have that the conditions of Lemma 3.3 is satisfied, since our initial energy is close to that of a single soliton, and thus $T_{\text{terminal}} < +\infty$.

It remains to rule out the case where $T_1 < T_2 = T_{\text{terminal}} = +\infty$, where *exactly one* soliton has bubbled off at T_1 . For this, we will appeal to the one pass theorem of [9], which states roughly that, for initial data close to the soliton curve \mathcal{S} , once the solution leaves a small neighbourhood of \mathcal{S} it can never return. More precisely, we can write

$$u(t, \cdot) \approx \kappa W_{\lambda(t)} + u_1(t, \cdot), \quad \kappa \in \{\pm 1\}, \quad t \in [0, T_1].$$

For $t \in [T_1, T_2)$, the energy satisfies

$$E(\tilde{u})(t) = \int_{\mathbb{R}^3} \left(\frac{1}{2}(\tilde{u}_t)^2 + |\nabla \tilde{u}|^2 - \frac{1}{6}\tilde{u}^6 \right) dx < \varepsilon,$$

and by using Sobolev’s inequality we get that for some constant $C_* > 0$

$$\frac{1}{2} \|\nabla \tilde{u}\|_{L^2}^2 - C_* \|\nabla \tilde{u}\|_{L^2}^6 < \varepsilon$$

which implies that if the constant ε (which we recall measures the distance from the soliton curve of our initial data) is chosen sufficiently small, by continuity we must have that throughout $t \in [T_1, T_2)$, either

$$\|\nabla \tilde{u}\|_{L^2} \lesssim \sqrt{\varepsilon} \tag{4.1a}$$

or

$$\|\nabla \tilde{u}\|_{L^2} \gtrsim 1. \tag{4.1b}$$

We rule out the case (4.1a): it would necessarily require a bound

$$\|\tilde{u}_t\|_{L^2} \lesssim \sqrt{\varepsilon}$$

which implies that

$$\|\nabla_{t,x} u_1(T_1, \cdot)\|_{L_x^2} \lesssim \sqrt{\varepsilon}.$$

This requires that there exists \tilde{t} less than but arbitrarily close to T_1 such that the inequality

$$\text{dist}_{\dot{H}^1 \times L^2}(u[\tilde{t}], \mathcal{S} \cup -\mathcal{S}) \lesssim \sqrt{\varepsilon} \tag{4.2a}$$

holds. But from Proposition 1.2 and the proof of Theorem 1.1 in [10] we see that, assuming $\varepsilon > 0$ is sufficiently small, for some $t \in [0, T_1)$ we must have

$$\text{dist}_{\dot{H}^1 \times L^2}(u[t], \mathcal{S} \cup -\mathcal{S}) \gg \sqrt{\varepsilon} \quad (4.2b)$$

due to the exponential growth of the unstable mode. The two equations (4.2a) and (4.2b) are contradictory in view of Theorem 4.1 in [9].

It follows that the alternative (4.1b) must hold. Looking at (3.5) again we see that if τ is chosen sufficiently large, and if ε is sufficiently small, the expression (3.6) would also apply in $[T_1, T_2)$ for a suitable ε_* . We can then conclude exactly as in the proof of Lemma 3.3. \square

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JOACHIM KRIEGER

Bâtiment des Mathématiques, EPFL
Station 8, CH-1015 Lausanne, Switzerland
joachim.krieger@epfl.ch

WILLIE WONG

Bâtiment des Mathématiques, EPFL
Station 8, CH-1015 Lausanne, Switzerland

willie.wong@epfl.ch