

The functor of p -permutation modules for abelian groups

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Abstract

Let k be a field of characteristic p , where p is a prime number, let $\text{pp}_k(G)$ be the Grothendieck group of p -permutation kG -modules, where G is a finite group, and let $\mathbb{C}\text{pp}_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \text{pp}_k(G)$. In this article, we find all the composition factors of the biset functor $\mathbb{C}\text{pp}_k$ restricted to the category of abelian groups.

Keywords: biset functor, p -permutation modules, composition factors, abelian groups

1. Introduction

Let k be an algebraically closed field of characteristic p , where p is a prime number or 0.

Let G be a finite group and let $\text{pp}_k(G)$ be the Grothendieck group of p -permutation kG -modules. If we tensor everything with \mathbb{C} and if G varies, $\mathbb{C}\text{pp}_k$ becomes a \mathbb{C} -linear biset functor (Definition 31).

Recall that the simple biset functors $S_{H,V}$ are parametrized by pairs (H, V) , where H is a finite group and V a simple $\mathbb{C}\text{Out}(H)$ -module.

We want to describe the composition factors of the biset functor $\mathbb{C}\text{pp}_k$ restricted to the category of abelian groups. This is similar in spirit to the work [1] dealing with the subfunctor of permutation modules.

In order to work with abelian group, we will define the tensor product of biset functors on groups of coprime order. Let $\mathcal{C}_{p \times p'}$ be the category whose objects are finite groups of the form $P \times Q$ where P is a p -group and Q is a p' -group (the morphism are defined using bisets).

In this article, we will prove the following theorem:

Theorem 1. *The composition factors of $\mathbb{C}\text{pp}_k$ on $\mathcal{C}_{p \times p'}$ are the simple factors $S_{C_p \times C_p \times C_m, \mathbb{C}_\xi}$ and S_{C_m, \mathbb{C}_ξ} , where (m, ξ) runs over the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$. Their multiplicity as composition factor is 1.*

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Restricting to abelian groups, we obtain the following corollary:

Corollary 2. *If G is a finite abelian group and V is a simple $\mathbb{C}\text{Out}(G)$ -module then $S_{G,V}$ is a composition factor of $\mathbb{C}\text{pp}_k$ if and only if there exists a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$ such that $G \cong C_m$ or $G \cong C_p \times C_p \times C_m$, where V is the 1-dimensional module \mathbb{C}_ξ . Moreover, the multiplicity of $S_{G,V}$ as composition factor is 1.*

This article begins with some background results on biset functors and composition factors. Then we define the tensor product of biset functors on group of coprime order and show that the tensor product of simple functors is also a simple functor. Using this and the composition factors of the Burnside functor $\mathbb{C}B$ and of the functor $\mathbb{C}R_k$ of ordinary representation obtained by Serge Bouc, [2] chapter 5 and 7, we will find the composition factors of the functor of p -permutation modules $\mathbb{C}\text{pp}_k$. So we will have the proof of Theorem 1. Then we will restrict the category $\mathcal{C}_{p \times p'}$ to abelian groups to obtain Corollary 2.

All groups are supposed finite, all vector spaces are finite dimensional and all modules are finitely generated left modules. Let G and H be finite groups. All G -sets and all (H, G) -bisets are finite. We denote by $[U]$ the isomorphism class of U (where U can be a group, a vector space, a module, a G -set, an (H, G) -biset, ...).

2. Background on biset functors

2.1. The category of biset functors

We begin with some recall on biset functors. See [2] for more details.

Definition 3. Let G and H be finite groups. Then $B(H, G)$ is the Grothendieck group of the isomorphism classes of finite (H, G) -bisets (for the disjoint union).

Notation 4 ([2], Definition 2.4.9 and Notation 2.4.10). Let G be a group. We denote by Id_G the (G, G) -biset G where the two actions are defined by left and right multiplication in G . We also denote by Id_G the image of Id_G in $B(G, G)$.

The product \times_H can be extended to a map

$$\times_H : B(G, H) \times B(H, K) \rightarrow B(G, K)$$

such that $[V] \times_H [U] = [V \times_H U]$ for every (G, H) -biset V and every (H, K) -biset U .

Definition 5 ([2], Definition 3.1.1).

We define the biset category $\underline{\text{GrB}}$ of finite groups as follows:

- The objects of $\underline{\text{GrB}}$ are all finite groups;
- If G and H are finite groups, then

$$\text{Hom}_{\underline{\text{GrB}}}(G, H) = B(H, G);$$

- If G, H and K are finite groups, then the composition $v \circ u$ is equal to $v \times_H u$, for all morphisms $u \in B(H, G)$ and for all morphisms $v \in B(K, H)$;
- For any finite group G , the identity morphism of G in $\underline{\text{GrB}}$ is equal to Id_G .

Let R be a commutative ring with identity element.

Definition 6 ([2], Definition 3.1.6).

We define the category $R\underline{\text{GrB}}$ as follows:

- The objects of $R\underline{\text{GrB}}$ are all finite groups;
- If G and H are finite groups, then

$$\text{Hom}_{R\underline{\text{GrB}}}(G, H) = R \otimes_{\mathbb{Z}} B(H, G);$$

- The composition of morphisms in $R\underline{\text{GrB}}$ is the R -linear extension of the composition in $\underline{\text{GrB}}$;
- For any finite group G , the identity morphism of G in $R\underline{\text{GrB}}$ is equal to $\text{Id}_R \otimes_{\mathbb{Z}} \text{Id}_G$.

Definition 7 ([2], Definition 3.2.2).

Let \mathcal{D} be a preadditive subcategory of $\underline{\text{GrB}}$. A *biset functor defined on \mathcal{D}* with values in $R\text{-mod}$ is a R -linear functor from $R\mathcal{D}$ to the category $R\text{-mod}$ of all finite generated R -modules.

Biset functors over $R\mathcal{D}$, with values in $R\text{-mod}$, are the objects of a category, denoted by $\mathcal{F}_{\mathcal{D}, R}$, where morphisms are natural transformations of functors, and composition of morphisms is composition of natural transformations.

Proposition 8 ([2], Proposition 3.2.8).

Let \mathcal{D} be a preadditive subcategory of $\underline{\text{GrB}}$.

1. The category $\mathcal{F}_{\mathcal{D}, R}$ is a R -linear abelian category: If $f : F \rightarrow F'$ is a morphism of biset functors, then for every object G of \mathcal{D}

$$(\text{Ker}(f))(G) = \text{Ker}(f(G)), \quad (\text{Coker}(f))(G) = \text{Coker}(f(G)).$$

2. A sequence $0 \rightarrow F \xrightarrow{f} F' \xrightarrow{f'} F'' \rightarrow 0$ is an exact sequence of $\mathcal{F}_{\mathcal{D}, R}$ if and only if for any object G of \mathcal{D} , the sequence

$$0 \rightarrow F(G) \xrightarrow{f^{(G)}} F'(G) \xrightarrow{f'^{(G)}} F''(G) \rightarrow 0$$

is an exact sequence of R -modules.

Definition 9 ([2], Definition 4.1.7).

A class \mathcal{D} of finite groups is said to be *closed under taking subquotients* if any group isomorphic to a subquotient of an element of \mathcal{D} is in \mathcal{D} .

A subcategory \mathcal{D} of $\underline{\text{GrB}}$ is called *replete* if it is a full subcategory whose class of objects is closed under taking subquotients.

A subcategory \mathcal{C} of $R\underline{\text{GrB}}$ is called *replete* if there exists a replete subcategory \mathcal{D} of $\underline{\text{GrB}}$ such that $\mathcal{C} = R\mathcal{D}$.

2.2. Simple functors and composition factors

Let R be a commutative ring with identity element and \mathcal{D} be a replete subcategory of $\underline{\text{GrB}}$ containing group isomorphism. We set $\mathcal{C} = R\mathcal{D}$

Now we will give some recall on simple functors of the category $\mathcal{F}_{\mathcal{D},R}$. For more details, see [3], [2] and [4].

Definition 10. A functor is said *simple* if he's non-zero and its only subfunctors are itself and the zero functor.

Remark 11. Proposition 4.2.2 of [2] implies that the simple functors doesn't depends of the chosen category \mathcal{D} .

Definition 12 ([2], Example 3.3.5). Let G be an object of \mathcal{C} and V be a simple $R\text{Out}(G)$ -module. Then V is also a $\text{End}_{\mathcal{C}}(G)$ -module and we define the biset functor $L_{G,V}$ by:

- For all object H of \mathcal{C} , we set

$$L_{G,V}(H) = \text{Hom}_{\mathcal{C}}(G, H) \otimes_{\text{End}_{\mathcal{C}}(G)} V = RB(H, G) \otimes_{RB(G, G)} V.$$

- For all morphism $\varphi : H \rightarrow H'$ in \mathcal{C} , $L_{G,V}(\varphi) : L_{G,V}(H) \rightarrow L_{G,V}(H')$ is defined by:

$$\begin{aligned} L_{G,V}(\varphi) : L_{G,V}(H) &\rightarrow L_{G,V}(H') \\ f \otimes v &\mapsto (\varphi \circ f) \otimes v. \end{aligned}$$

We can notice that $L_{G,V}(G) \cong V$.

Proposition 13 ([2], Corollary 4.2.4). Let G be an object of \mathcal{C} and V be a simple $\text{End}_{\mathcal{C}}(G)$ -module. Then the functor $L_{G,V}$ has a unique proper maximal subfunctor $J_{G,V}$, and the quotient $S_{G,V} = L_{G,V}/J_{G,V}$ is a simple object of $\mathcal{F}_{\mathcal{D},R}$, such that $S_{G,V}(G) \cong V$.

Remark 14 ([2], Remark 4.2.6). Let H be an object of \mathcal{C} . Then $J_{G,V}(H)$ is equal to the set of finite sums $\sum_{i=1}^n \varphi_i \otimes v_i$ in $L_{G,V}(H)$, where $\varphi_i \in \text{Hom}_{\mathcal{C}}(G, H)$ and $v_i \in V$, such that $\sum_{i=1}^n (\psi \circ \varphi_i) \cdot v_i = 0$ for any $\psi \in \text{Hom}_{\mathcal{C}}(H, G)$, where $(\psi \circ \varphi_i) \cdot v_i$ denote the image of the element v_i of V under the action of the endomorphism $\psi \circ \varphi_i$ of G .

The simple objects of the category $\mathcal{F}_{\mathcal{D},R}$ are labelled by pairs (G, V) , where G is a finite group and V a simple $R\text{Out}(G)$ -module. We denote by $S_{G,V}$ the simple functor associated to (G, V) . If $F \in \mathcal{F}_{\mathcal{D},R}$ is a simple functor, then $F \cong S_{G,V}$ where G is the smallest group (unique up to isomorphism) such that $F(G) \neq \{0\}$ and $V = F(G)$. We can define a notion of isomorphism on those pairs such that two simple functors are isomorphic if and only if the corresponding pairs are isomorphic ([2], Theorem 4.3.10).

Definition 15. Let F be a biset functor on \mathcal{C} . A simple functor S is a *composition factor* of F if there exists subfunctors $F' \subseteq F'' \subseteq F$ such that $F''/F' \cong S$.

Definition 16. Let G be an object of \mathcal{C} . We define the category $\mathcal{C} \downarrow_G$ as the full subcategory of \mathcal{C} whose objects are subquotients of G .

Definition 17 ([5], page 20). Let G be a fixed object of \mathcal{C} and F be a biset functor on \mathcal{C} . The functor F has a *composition series over G* if there is a series of subfunctors

$$0 = T_0 \subseteq B_1 \subset T_1 \subseteq \dots \subseteq B_m \subset T_m \subseteq B_{m+1} = F$$

such that

- T_i/B_i is a simple functor, whose restriction to $\mathcal{C} \downarrow_G$ is non-zero for all $i = 1, \dots, m$.
- $\text{Res}_{\mathcal{C} \downarrow_G}^{\mathcal{C}}(B_{i+1}/T_i) = 0$ for all $i = 0, \dots, m$.

If such a composition series exists, we will call the set of simple functors T_i/B_i together with their multiplicities the *composition factors of F over G* .

Remark 18. The composition factors of F over G corresponds exactly to the composition factors of F which are non-zero on G . In particular, this allows us to have a notion of multiplicity for composition factors (thanks to the following Proposition and Theorem).

Proposition 19 ([5], Proposition 3.1). *Let G be a fixed object of \mathcal{C} and F a biset functor on \mathcal{C} . If F has a composition series over G , then any other composition series over G of F has the same length and the composition factors on G (taken with multiplicities) are the same.*

Theorem 20. *We suppose that R is a field. Then every biset functor on \mathcal{C} has a composition sequence on G , for all group G in \mathcal{C} .*

PROOF. This is a consequence of Theorem 3.3, page 22 of [5].

3. Tensor product of biset functors on groups of coprime order

Let k be a algebraically closed field. Let π be a set of prime numbers. Let \mathcal{P} be the full subcategory of $k\text{-GrB}$ whose objects are all finite π -groups and let \mathcal{Q} be the full subcategory of $k\text{-GrB}$ whose objects are all finite π' -groups. Moreover, let $\mathcal{C}_{\mathcal{P}, \mathcal{Q}}$ be the full subcategory of $k\text{-GrB}$ whose objects are all the groups of the form $P \times Q$ where P is a object of \mathcal{P} and Q is an object of \mathcal{Q} . Those three category are replete.

Let $F_{\mathcal{P}}$ be a biset functor on the category \mathcal{P} and $F_{\mathcal{Q}}$ a biset functor on the category \mathcal{Q} . The aim is to construct a biset functor F on the category $\mathcal{C}_{\mathcal{P}, \mathcal{Q}}$ such that the restriction of F on \mathcal{P} and \mathcal{Q} correspond to $F_{\mathcal{P}}$ and $F_{\mathcal{Q}}$ respectively.

We start by defining F on the objects of $\mathcal{C}_{\mathcal{P}, \mathcal{Q}}$: Let $G = P \times Q$, where P is an object of \mathcal{P} and Q is an object of \mathcal{Q} . We set

$$F(G) = F_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q).$$

As $F_{\mathcal{P}}(P)$ and $F_{\mathcal{Q}}(Q)$ are finite dimensional k -vector spaces, $F(G)$ is also a finite dimensional k -vector space.

Now we have to define $F(\mu)$ when μ is an element of $kB(H, G)$ (where P and P' are objects of \mathcal{P} and Q and Q' are objects of \mathcal{Q} such that $G = P \times Q$ and $H = P' \times Q'$). In order to do this, we need the following proposition:

Proposition 21. *Let G, G', H and H' be finite groups. If U is a (G, G') -biset and V is a (H, H') -biset, then $U \times V$ is a $(G \times H, G' \times H')$ -biset for the structure given by:*

$$(g, h) \cdot (u, v) \cdot (g', h') = (gxg', hyh'),$$

for all $g \in G$, for all $g' \in G'$, for all $h \in H$, for all $h' \in H'$, for all $u \in U$ and for all $v \in V$. The correspondence $(U, V) \mapsto U \times V$ induces a bilinear map from $B(G, G') \times B(H, H')$ to $B(G \times H, G' \times H')$, hence a linear map

$$\varepsilon : B(G, G') \otimes_{\mathbb{Z}} B(H, H') \longrightarrow B(G \times H, G' \times H'),$$

which is an injective \mathbb{Z} -module homomorphism preserving identity elements. If $G \times G'$ and $H \times H'$ have coprime order, this map is an isomorphism.

PROOF. The fact that the correspondence induces a bilinear map comes from Lemma 8.1.2, page 135 of [2]. It is a generalization of Proposition 2.5.14 b), pages 38-39 of [2]. The proof is analogous.

Remark 22. The previous proposition remains true if we replace \mathbb{Z} by k . We obtain a homomorphism of k -vector spaces

$$\varepsilon : kB(G, G') \otimes_k kB(H, H') \longrightarrow kB(G \times H, G' \times H')$$

which becomes an isomorphism if $G \times G'$ et $H \times H'$ have coprime order. If moreover $G = G'$ and $H = H'$, it becomes an isomorphism of k -algebras.

Let $G = P \times Q$ and $H = P' \times Q'$, where P and P' are objects of \mathcal{P} and Q and Q' are objects of \mathcal{Q} . Let u be an element of $\mathbb{C}B(P \times Q, P' \times Q')$. Then we define $F(u) : F(G) \longrightarrow F(H)$ as the map

$$\sum_{i=1}^n \lambda_i F_{\mathcal{P}}(u_{\mathcal{P},i}) \otimes_k F_{\mathcal{Q}}(u_{\mathcal{Q},i}) : F_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \longrightarrow F_{\mathcal{P}}(P') \otimes_k F_{\mathcal{Q}}(Q'),$$

where $\varepsilon^{-1}(u) = \sum_{i=1}^n \lambda_i u_{\mathcal{P},i} \otimes_k u_{\mathcal{Q},i}$, with $u_{\mathcal{P},i} \in kB(P, P')$ and $u_{\mathcal{Q},i} \in kB(Q, Q')$, for all $i = 1, \dots, n$.

Proposition 23. *Defined in this way, F is a biset functor.*

PROOF. Since ε preserves identity elements, it is easy to check that $F(\text{Id}_G) = \text{Id}_{F(G)}$ for all object G of $\mathcal{C}_{\mathcal{P}, \mathcal{Q}}$.

It remains to check that $F(u_1 \circ u_2) = F(u_1) \circ F(u_2)$ for all $u_1 \in kB(P'' \times Q'', P' \times Q')$ and for all $u_2 \in kB(P' \times Q', P \times Q)$. By k -linearity, it is enough to verify this in the case of transitive bisets. Let P, P' and P'' be objects of \mathcal{P} , Q, Q' and Q'' be objects of \mathcal{Q} , U be a transitive $(P'' \times Q'', P' \times Q')$ -biset and V be a transitive $(P' \times Q', P \times Q)$ -biset. Then, by Proposition 21, there exists a (P'', P') -biset $U_{\mathcal{P}}$, a (P', P) -biset $V_{\mathcal{P}}$, a (Q'', Q') -biset $U_{\mathcal{Q}}$ and a (Q', Q) -biset $V_{\mathcal{Q}}$ such that:

$$U = U_{\mathcal{P}} \times U_{\mathcal{Q}} \text{ et } V = V_{\mathcal{P}} \times V_{\mathcal{Q}}.$$

Then we have:

$$\begin{aligned} F([U]) \circ F([V]) &= (F_{\mathcal{P}}([U_{\mathcal{P}}]) \otimes_k F_{\mathcal{Q}}([U_{\mathcal{Q}}])) \circ (F_{\mathcal{P}}([V_{\mathcal{P}}]) \otimes_k F_{\mathcal{Q}}([V_{\mathcal{Q}}])) \\ &= (F_{\mathcal{P}}([U_{\mathcal{P}}]) \circ F_{\mathcal{P}}([V_{\mathcal{P}}])) \otimes_k (F_{\mathcal{Q}}([U_{\mathcal{Q}}]) \circ F_{\mathcal{Q}}([V_{\mathcal{Q}}])) \\ &= F_{\mathcal{P}}([U_{\mathcal{P}} \times_{P'} V_{\mathcal{P}}]) \otimes_k F_{\mathcal{Q}}([U_{\mathcal{Q}} \times_{Q'} V_{\mathcal{Q}}]) \\ &= F([U \times_{P' \times Q'} V]) \end{aligned}$$

because there exists an isomorphism of $(P'' \times Q'', P \times Q)$ -bisets

$$(U_{\mathcal{P}} \times_{P'} V_{\mathcal{P}}) \times (U_{\mathcal{Q}} \times_{Q'} V_{\mathcal{Q}}) \cong (U_{\mathcal{P}} \times U_{\mathcal{Q}}) \times_{P' \times Q'} (V_{\mathcal{P}} \times V_{\mathcal{Q}}).$$

Proposition 24. *Let $0 \longrightarrow F_{\mathcal{P}} \xrightarrow{\mu} F'_{\mathcal{P}} \xrightarrow{\eta} F''_{\mathcal{P}} \longrightarrow 0$ be an exact sequence of biset functors on the category \mathcal{P} and let $F_{\mathcal{Q}}$ be a biset functor on the category \mathcal{Q} . We can define an exact sequence of biset functors on $\mathcal{C}_{\mathcal{P}, \mathcal{Q}}$:*

$$0 \longrightarrow F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}} \xrightarrow{\mu \otimes_k \text{Id}_{F_{\mathcal{Q}}}} F'_{\mathcal{P}} \otimes_k F_{\mathcal{Q}} \xrightarrow{\eta \otimes_k \text{Id}_{F_{\mathcal{Q}}}} F''_{\mathcal{P}} \otimes_k F_{\mathcal{Q}} \longrightarrow 0.$$

PROOF. We begin by defining $\mu \otimes_k \text{Id}_{F_{\mathcal{Q}}}$ and verifying that this is a natural transformation. Let $G = P \times Q$, where P is an object of \mathcal{P} and Q is an object of \mathcal{Q} . We define $(\mu \otimes_k \text{Id}_{F_{\mathcal{Q}}})(G) : (F_{\mathcal{P}} \otimes_k F_{\mathcal{Q}})(G) \longrightarrow (F'_{\mathcal{P}} \otimes_k F_{\mathcal{Q}})(G)$ by

$$(\mu \otimes_k \text{Id}_{F_{\mathcal{Q}}})(P \times Q) = \mu(P) \otimes_k \text{Id}_{F_{\mathcal{Q}}(Q)} : F_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \longrightarrow F'_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q).$$

It is easy to check that it is a natural transformation, by verifying that it is the case for transitive bisets, using the fact that we have k -linear maps. The map $\eta \otimes_k \text{Id}_{F_{\mathcal{Q}}}$ is defined in the same way.

Now we have to verify that we really obtain an exact sequence. By Proposition 8, it is enough to verify that it is an exact sequence when evaluating in a group G of $\mathcal{C}_{\mathcal{P}, \mathcal{Q}}$. Let $G = P \times Q$, where P is an object of \mathcal{P} and Q is an object of \mathcal{Q} . We have to check that the following sequence of \mathbb{C} -vector spaces is exact:

$$0 \longrightarrow F_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \longrightarrow F'_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \longrightarrow F''_{\mathcal{P}}(P) \otimes_k F_{\mathcal{Q}}(Q) \longrightarrow 0.$$

But, by Proposition 8, the following sequence is exact:

$$0 \longrightarrow F_{\mathcal{P}}(P) \xrightarrow{\mu(P)} F'_{\mathcal{P}}(P) \xrightarrow{\eta(P)} F''_{\mathcal{P}}(P) \longrightarrow 0,$$

which implies that the previous sequence is also exact.

3.1. The product of simple functors

Let P be an object of \mathcal{P} , Q be an object of \mathcal{Q} , V be a simple $k \text{Out}(P)$ -module and W be a simple $k \text{Out}(Q)$ -module. We want to show the following theorem:

Theorem 25. *We consider $S_{P,V}$ as a simple functor on \mathcal{P} and $S_{Q,W}$ as a simple functor on \mathcal{Q} . The functor $S_{P,V} \otimes_k S_{Q,W}$ is, on the category $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$, isomorphic to the simple functor $S_{P \times Q, V \otimes_k W}$.*

In order to prove this theorem, we first need the following proposition:

Proposition 26. *We consider $L_{P,V}$ as a functor on \mathcal{P} and $L_{Q,W}$ as a functor on \mathcal{Q} . The functor $L_{P,V} \otimes_k L_{Q,W}$ is, on the category $\mathcal{C}_{\mathcal{P},\mathcal{Q}}$, isomorphic to the functor $L_{P \times Q, V \otimes_k W}$.*

PROOF. Let G be an object of \mathcal{P} and H be an object of \mathcal{Q} . Then:

$$(L_{P,V} \otimes_k L_{Q,W})(G \times H) = (kB(G, P) \otimes_{kB(P,P)} V) \otimes_k (kB(H, Q) \otimes_{kB(Q,Q)} W)$$

and

$$\begin{aligned} (L_{P \times Q, V \otimes_k W})(G \times H) &= kB(G \times H, P \times Q) \otimes_{kB(P \times Q, P \times Q)} (V \otimes_k W) \\ &\cong (kB(G, P) \otimes_k kB(H, Q)) \otimes_{kB(P,P) \otimes_k kB(Q,Q)} (V \otimes_k W). \end{aligned}$$

Those are isomorphic k -vector spaces, by the map $\varphi(G \times H)$ defined by:

$$\begin{aligned} \varphi(G \times H) : (L_{P,V} \otimes_k L_{Q,W})(G \times H) &\longrightarrow L_{P \times Q, V \otimes_k W}(G \times H) \\ (u_1 \otimes_{kB(P,P)} v) \otimes_k (u_2 \otimes_{kB(Q,Q)} w) &\longmapsto \varepsilon(u_1 \otimes_k u_2) \otimes_{kB(P \times Q, P \times Q)} (v \otimes_k w). \end{aligned}$$

This induce a natural transformation $\varphi : L_{P,V} \otimes_k L_{Q,W} \longrightarrow L_{P \times Q, V \otimes_k W}$.

Now we have the following commutative diagram:

$$\begin{array}{ccc} L_{P,V} \otimes_k L_{Q,W} & \xrightarrow{\text{proj} \otimes_k \text{proj}} & S_{P,V} \otimes_k S_{Q,W} \\ \downarrow \varphi & \searrow \psi & \\ L_{P \times Q, V \otimes_k W} & \xrightarrow{\text{proj}} & S_{P \times Q, V \otimes_k W} \end{array}$$

We set $\psi = \text{proj} \circ \varphi : L_{P,V} \otimes_k L_{Q,W} \longrightarrow S_{P \times Q, V \otimes_k W}$.

Lemma 27. *The kernel of $\text{proj} \otimes_k \text{proj}$ is contained in the kernel of ψ .*

PROOF. By Proposition 8, it is enough to prove it for a group $G \times H$, where G is an object of \mathcal{P} and H is an object of \mathcal{Q} . Moreover,

$$\text{Ker}(\text{proj} \otimes_k \text{proj}) = J_{P,V} \otimes_k L_{Q,W} + L_{P,V} \otimes_k J_{Q,W}$$

and Remark 14 gives a description of $J_{P,V}(G)$ and $J_{Q,W}(H)$.

Let $n \in \mathbb{N}$, $u_i \in \text{Hom}_{\mathcal{P}}(G, P)$ and $v_i \in V$ such that $\sum_{i=1}^n (\alpha \times_G u_i) \cdot v_i = 0$ for all $\alpha \in \text{Hom}_{\mathcal{P}}(P, G)$ (that is $\sum_{i=1}^n u_i \otimes v_i \in J_{P,V}(G)$). Let $u \otimes w \in L_{Q,W}$. We have to verify that $\psi(G \times H)((\sum_{i=1}^n u_i \otimes v_i) \otimes_k (u \otimes w)) = 0$, which is equivalent to check that $\varphi(G \times H)((\sum_{i=1}^n u_i \otimes v_i) \otimes_k (u \otimes w)) \in J_{P \times Q, V \otimes_k W}(G \times H)$. But

$$\varphi(G \times H) \left(\left(\sum_{i=1}^n u_i \otimes v_i \right) \otimes_k (u \otimes w) \right) = \sum_{i=1}^n \varepsilon(u_i \otimes_k u) \otimes (v_i \otimes_k w).$$

Let U be a transitive $(P \times Q, G \times H)$ -biset. Then it exists a (P, G) -biset $U_{\mathcal{P}}$ and a (Q, H) -biset $U_{\mathcal{Q}}$ such that $U = U_{\mathcal{P}} \times U_{\mathcal{Q}}$. Then

$$\begin{aligned} \sum_{i=1}^n ([U] \times_{G \times H} \varepsilon(u_i \otimes_k u)) (v_i \otimes_k w) &= \sum_{i=1}^n (\varepsilon([U_{\mathcal{P}}] \times_G u_i) \otimes_k ([U_{\mathcal{Q}}] \times_H u)) (v_i \otimes_k w) \\ &= \sum_{i=1}^n \underbrace{([U_{\mathcal{P}}] \times_G u_i)(v_i) \otimes_k ([U_{\mathcal{Q}}] \times_H u)(w)}_{= 0} \\ &= 0 \end{aligned}$$

By k -linearity, this implies that

$$\varphi(G \times H) \left(\left(\sum_{i=1}^n u_i \otimes v_i \right) \otimes_k (u \otimes w) \right) \in J_{P \times Q, V \otimes_k W}(G \times H).$$

Similarly, we can show that

$$\varphi(G \times H)(L_{P,V}(G) \otimes_k J_{Q,W}(H)) \subseteq J_{P \times Q, V \otimes_k W}(G \times H).$$

As $\text{Ker}(proj \otimes_k proj) \subseteq \text{Ker } \psi$, it exists a natural transformation $\mu : S_{P,V} \otimes_k S_{Q,W} \rightarrow S_{P \times Q, V \otimes_k W}$. We will show that it is an isomorphism. It is enough to show it for a group $G \times H$ of $\mathcal{C}_{\mathcal{P}, \mathcal{Q}}$.

Lemma 28. *The k -vector space $S_{P \times Q, V \otimes_k W}(G \times H)$ is zero if and only if $S_{P,V}(G)$ or $S_{Q,W}(H)$ is zero.*

PROOF. If $S_{P,V}(G)$ or $S_{Q,W}(H)$ is zero, then $S_{P,V}(G) \otimes_k S_{Q,W}(H) = 0$. As $\mu(G \times H)$ is surjective (because φ and $proj$ are surjective and so is ψ), we have that $S_{P \times Q, V \otimes_k W}(G \times H) = 0$.

Reciprocally, we suppose that $S_{P \times Q, V \otimes_k W}(G \times H) = 0$. If $S_{P,V}(G) = 0$, we have finish, so we can suppose that $S_{P,V}(G) \neq 0$, that is $J_{P,V}(G) \subsetneq L_{P,V}(G)$. Therefore, it exists $n \in \mathbb{N}$, $\varphi_i \in kB(G, P)$, $v_i \in V$ for all $1 \leq i \leq n$ and $\rho \in kB(P, G)$ such that $\sum_{i=1}^n (\rho \times_G \varphi_i)(v_i) \neq 0$. We will show that $S_{Q,W}(H) = 0$. It is enough to show that the elements $u \otimes w$, where $u \in kB(H, Q)$ and $w \in W$, are in $J_{Q,W}(H)$, because, this implies that $J_{Q,W}(H) = L_{Q,W}(H)$ and so $S_{Q,W}(H) = 0$. Let $u \in kB(H, Q)$ and $w \in W$. For all $\tilde{u} \in kB(Q, H)$, as

$\sum_{i=1}^n \varepsilon(\varphi_i \otimes_k u) \otimes (v_i \otimes_k w)$ is in $L_{P \times Q, V \otimes_k W}(G \times H) = J_{P \times Q, V \otimes_k W}(G \times H)$:

$$\begin{aligned}
0 &= \sum_{i=1}^n (\varepsilon(\rho \otimes_k \tilde{u}) \varepsilon(\varphi_i \otimes_k u)) (v_i \otimes_k w) \\
&= \sum_{i=1}^n \varepsilon((\rho \times_G \varphi_i) \otimes_k (\tilde{u} \times_H u)) (v_i \otimes_k w) \\
&= \sum_{i=1}^n ((\rho \times_G \varphi_i)(v_i)) \otimes_k ((\tilde{u} \times_H u)(w)) \\
&= \underbrace{\left(\sum_{i=1}^n (\rho \times_G \varphi_i)(v_i) \right)}_{\neq 0} \otimes_k ((\tilde{u} \times_H u)(w))
\end{aligned}$$

Consequently, we must have that $(\tilde{u} \times_H u)(w) = 0$. As \tilde{u} is arbitrary, this implies that $u \otimes_k w \in J_{Q, W}(H)$.

Now we can suppose that $S_{P \times Q, V \otimes_k W}(G \times H)$ and $S_{P, V}(G) \otimes_k S_{Q, W}(H)$ are non-zero because otherwise, they are both zero and we have

$$S_{P, V}(G) \otimes_k S_{Q, W}(H) \cong S_{P \times Q, V \otimes_k W}(G \times H).$$

But then, by Corollaire 4.2.4, page 58 of [2], we know that $S_{P, V}(G)$ is a simple $kB(G, G)$ -module, $S_{Q, W}(H)$ is a simple $kB(H, H)$ -module and $S_{P \times Q, V \otimes_k W}(G \times H)$ is a simple $kB(G \times H, G \times H)$ -module. As k is algebraically closed, the tensor product of two simple modules on finite dimensional k -algebras is a simple module on the tensor product of the two algebras (Proposition 3.56, page 65 of [6] and Schur's lemma, [7], Lemma 27.3, page 181), so $S_{P, V}(G) \otimes_k S_{Q, W}(H)$ is a simple $kB(G, G) \otimes_k kB(H, H)$ -module. But $kB(G, G) \otimes_k kB(H, H) \cong kB(G \times H, G \times H)$ as k -algebras (Remarque 22), so we can consider $S_{P, V}(G) \otimes_k S_{Q, W}(H)$ as a simple $kB(G \times H, G \times H)$ -module. Now we have a non-zero homomorphism of module between two simple $kB(G \times H, G \times H)$ -modules, which implies that it is an isomorphism through Schur's lemma ([7], Lemma 27.3, page 181). Now we have proved the wanted theorem:

Theorem 29. *We consider $S_{P, V}$ as a simple functor on \mathcal{P} and $S_{Q, W}$ as a simple functor on \mathcal{Q} . The functor $S_{P, V} \otimes_k S_{Q, W}$ is, on the category $\mathcal{C}_{\mathcal{P}, \mathcal{Q}}$, isomorphic to the simple functor $S_{P \times Q, V \otimes_k W}$.*

4. The biset functor of p -permutation modules

Let k be an algebraically closed field of characteristic $p \in \mathbb{P}$.

We want to prove the following theorem:

Theorem 30. *If G is a finite abelian group and V is a simple $\mathbb{C} \text{Out}(G)$ -module, then $S_{G, V}$ is a composition factor of $\mathbb{C} \text{pp}_k$ on $\mathbb{C} \text{GrB}$ if and only if there*

exists an integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\star \longrightarrow \mathbb{C}^\star$ such that $G \cong C_m$ or $G \cong C_p \times C_p \times C_m$, where V is the simple $\mathbb{C}\text{Out}(G)$ -module \mathbb{C}_ξ (where $\text{Out}(C_p \times C_p)$ acts trivially in the second case). Moreover, the multiplicity of $S_{G,V}$ as composition factor is equal to 1.

First, we have to define $\mathbb{C}\text{pp}_k$ and two other biset functors we need for the proof:

Definition 31. Let G be a finite group. We define $\text{pp}_k(G)$ as the Grothendieck group of the set of isomorphism classes of p -permutation kG -modules (i.e. direct sums of indecomposable trivial source kG -modules) with respect to direct sums. For every (H, G) -biset U we define:

$$\begin{aligned} \text{pp}_k([U]) : \text{pp}_k(G) &\longrightarrow \text{pp}_k(H) \\ [M] &\longmapsto [kU \otimes_{kG} M] \end{aligned}$$

for every trivial source kG -module M . This extends by \mathbb{C} -linearity to a map $\mathbb{C}\text{pp}_k([U]) : \mathbb{C}\text{pp}_k(G) \longrightarrow \mathbb{C}\text{pp}_k(H)$, where $\mathbb{C}\text{pp}_k(G) = \mathbb{C} \otimes_{\mathbb{Z}} \text{pp}_k(G)$.

Now we can define $\mathbb{C}\text{pp}_k(u)$ for every $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$. Let $u = \sum_{i=1}^n \lambda_i [U_i]$ where $\lambda_i \in \mathbb{C}$ and U_i is a (H, G) -biset, for every $i = 1, \dots, n$. Then $\mathbb{C}\text{pp}_k(u) = \sum_{i=1}^n \lambda_i \mathbb{C}\text{pp}_k([U_i])$. This defines a structure of biset functor $\mathbb{C}\text{pp}_k$.

Definition 32. Let G be a finite group and $B(G)$ be the Grothendieck group of the set of isomorphism classes of finite G -sets (for disjoint union). Then $B(G)$ is a ring (called the Burnside ring of G), where the multiplication is defined by

$$[U] \cdot [V] = [U \times V]$$

for all G -sets U and V (extended to $B(G)$ by bilinearity).

Let G et H be two finite groups. For every (finite) (H, G) -biset U , we can define the following map:

$$\begin{aligned} B([U]) : B(G) &\longrightarrow B(H) \\ [V] &\longmapsto [U \times_G V] \end{aligned}$$

for every (finite) G -set V .

This map can be extended by \mathbb{C} -linearity to a map $\mathbb{C}B([U]) : \mathbb{C}B(G) \rightarrow \mathbb{C}B(H)$, where $\mathbb{C}B(G) = \mathbb{C} \otimes_{\mathbb{Z}} B(G)$. As for $\mathbb{C}\text{pp}_k$, we can now define $\mathbb{C}B(u)$ for every $u \in \mathbb{C} \otimes_{\mathbb{Z}} B(H, G)$. This defines a structure of biset functor $\mathbb{C}B$.

Definition 33 ([2], Notation 5.2.2 and Definition 5.4.6).

If G is a finite group, and N is a normal subgroup of G , denote by $m_{G,N}$ the rational number defined by

$$m_{G,N} = \frac{1}{|G|} \sum_{XN=G} |X| \mu(X, G),$$

where μ is the Möbius function of the poset of subgroups of G .

A finite group G is called a B -group if for any non trivial normal subgroup N of G , the constant $m_{G,N}$ is equal to zero.

Theorem 34 ([2], Remark 5.5.2).

Let \mathcal{D} be a replete subcategory of $\underline{\text{GrB}}$. The composition factors of the Burnside functor $\mathbb{C}B$ on $\mathbb{C}\mathcal{D}$ are exactly the functors $S_{G,\mathbb{C}}$, where G is an object of \mathcal{D} which is a B -group.

Definition 35 ([2], Notation 7.1.1).

Let G be a finite p' -group. Denote by $R_k(G)$ the Grothendieck group of the category of finite dimensional kG -modules (for exact sequences). If H is a other finite p' -group and U is a (H, G) -biset, denote by $R_k([U]) : R_k(G) \rightarrow R_k(H)$ the group homomorphism defined by:

$$R_k([U])([E]) = [kU \otimes_{kG} E],$$

for all kG -module E . By \mathbb{Z} -linearity, we can extend this definition to $R_k(G)$.

As for $\mathbb{C}\text{pp}_k$ and $\mathbb{C}B$, this construction can be extend by \mathbb{C} -linearity to a biset functor $\mathbb{C}R_k$ defined on $\mathbb{C}\mathcal{D}$ with values in $\mathbb{C}\text{-mod}$, where \mathcal{D} is the full subcategory of $\underline{\text{GrB}}$ of all finite p' -groups.

Definition 36 ([2], Definition 7.3.1).

Let $m \in \mathbb{N}^*$. A character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$ is called *primitive* if it cannot be factored through any proper quotient $(\mathbb{Z}/n\mathbb{Z})^*$ of $(\mathbb{Z}/m\mathbb{Z})^*$.

Theorem 37 ([2], Corollary 7.3.5).

The functor $\mathbb{C}R_k$ is a semisimple object of $\mathcal{F}_{\mathcal{D},\mathbb{C}}$ (where \mathcal{D} is the full subcategory of $\underline{\text{GrB}}$ of all finite p' -group). More precisely

$$\mathbb{C}R_k \cong \bigoplus_{(m,\xi)} S_{\mathbb{Z}/m\mathbb{Z},\mathbb{C}_\xi},$$

where (m,ξ) runs through the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$ and \mathbb{C}_ξ is the vector space \mathbb{C} on which the group $\text{Out}(\mathbb{Z}/m\mathbb{Z}) \cong (\mathbb{Z}/m\mathbb{Z})^*$ acts via ξ .

Let \mathcal{C}_p be the full subcategory of $\mathbb{C}\underline{\text{GrB}}$ whose objects are all the finite p -groups and let $\mathcal{C}_{p'}$ be the full subcategory of $\mathbb{C}\underline{\text{GrB}}$ whose objects are all the finite p' -groups. Let moreover $\mathcal{C}_{p \times p'}$ be the full subcategory of $\mathbb{C}\underline{\text{GrB}}$ whose objects are all the groups of the form $P \times Q$ where P is a finite p -group and Q is a finite p' -group.

Theorem 38. The biset functor $\mathbb{C}\text{pp}_k$ on the category $\mathcal{C}_{p \times p'}$ is isomorphic to the functor $\mathbb{C}B \otimes_{\mathbb{C}} \mathbb{C}R_k$, where $\mathbb{C}B$ is considered as a functor on \mathcal{C}_p and $\mathbb{C}R_k$ as a functor on $\mathcal{C}_{p'}$.

PROOF. Let P be a finite p -group and Q be a finite p' -group. We define $\mu(P \times Q)$ by:

$$\begin{aligned} \mu(P \times Q) : \mathbb{C}B(P) \otimes \mathbb{C}R_k(Q) &\longrightarrow \mathbb{C}\text{pp}_k(P \times Q), \\ [X] \otimes_{\mathbb{C}} [V] &\longmapsto [kX \otimes_k V] \end{aligned}$$

for all P -set X and for all kQ -module V . The action of $P \times Q$ on $kX \otimes_k V$ is such that P acts on kX and Q acts on V . The $k(P \times Q)$ -module $kX \otimes_k V$ is a p -permutation module because V is a p -permutation kQ -module. We extend the definition of $\mu(P \times Q)$ to $\mathbb{C}B(P) \otimes \mathbb{C}R_k(Q)$ by \mathbb{C} -linearity. It is a bijective \mathbb{C} -linear map: we can define its inverse by:

$$\begin{aligned} \mu^{-1}(P \times Q) : \mathbb{C} \operatorname{pp}_k(P \times Q) &\longrightarrow \mathbb{C}B(P) \otimes \mathbb{C}R_k(Q), \\ [M] &\longmapsto [P/D] \otimes [V] \end{aligned}$$

for all indecomposable p -permutation $k(P \times Q)$ -module, where D is a vertex of M and V is a simple kQ -module such that $M = \operatorname{Ind}_{D \times Q}^{P \times Q} \operatorname{Inf}_Q^{D \times Q} V$

Proposition 39. *A such D and a such V always exist.*

PROOF. By the classification of the p -permutation indecomposable modules (Theorem 3.2 of [8] or Theorem 27.10 of [9]), there exists a vertex D of M and a projective indecomposable $kN_P(D)/D \times Q$ -module W such that M is a direct factor of

$$\operatorname{Ind}_{N_P(D) \times Q}^{P \times Q} \operatorname{Inf}_{N_P(D)/D \times Q} W.$$

But then, there exists a simple kQ -module such that $W = \operatorname{Ind}_Q^{N_P(D)/D \times Q} V$ and so M is a direct factor of

$$\operatorname{Ind}_{N_P(D) \times Q}^{P \times Q} \operatorname{Inf}_{N_P(D)/D \times Q} \operatorname{Ind}_Q^{N_P(D)/D \times Q} V.$$

But $\operatorname{Ind}_{N_P(D) \times Q}^{P \times Q} \operatorname{Inf}_{N_P(D)/D \times Q} \operatorname{Ind}_Q^{N_P(D)/D \times Q} V$ is indecomposable (Corollary 6.11, [10]) and so equal to M . But then

$$\begin{aligned} M &\cong \operatorname{Ind}_{N_P(D) \times Q}^{P \times Q} \operatorname{Inf}_{N_P(D)/D \times Q} \operatorname{Ind}_Q^{N_P(D)/D \times Q} V \\ &\cong \operatorname{Ind}_{N_P(D) \times Q}^{P \times Q} \operatorname{Ind}_{D \times Q}^{N_P(D)/D \times Q} \operatorname{Inf}_Q^{D \times Q} V \\ &\cong \operatorname{Ind}_{D \times Q}^{P \times Q} \operatorname{Inf}_Q^{D \times Q} V \end{aligned}$$

which concludes the proof.

Lemma 40. *If D is a subgroup of P and V is a kQ -module, then $\operatorname{Ind}_{D \times Q}^{P \times Q} \operatorname{Inf}_Q^{D \times Q} V$ is isomorphic to the $k(P \times Q)$ -module $kP/D \otimes_k V$.*

PROOF. We have

$$\operatorname{Ind}_{D \times Q}^{P \times Q} \operatorname{Inf}_Q^{D \times Q} V \cong k(P \times Q) \otimes_{k(D \times Q)} kQ \otimes_{kQ} V.$$

Now we define the following homomorphism of $k(P \times Q)$ -modules:

$$\begin{aligned} \varphi : kP/D \otimes_k V &\longrightarrow k(P \times Q) \otimes_{k(D \times Q)} kQ \otimes_{kQ} V \\ \sum_{i=1}^n \lambda_i(p_i D) \otimes_k v &\longmapsto \sum_{i=1}^n \lambda_i(p_i, 1) \otimes_k 1 \otimes_k v \end{aligned}$$

for all $\sum_{i=1}^n \lambda_i(p_i D) \in kP/D$ and for all $v \in V$. It is easy to check that this map is well defined and a homomorphism of $k(P \times Q)$ -modules. It remains to check that it's bijective. But

$$\begin{aligned} \dim_k(kP/D \otimes_k V) &= [P : D] \cdot \dim_k V \\ &= [P \times Q : D \times Q] \cdot \dim_k V \\ &= \dim_k(k(P \times Q) \otimes_{k(D \times Q)} kQ \otimes_{kQ} V). \end{aligned}$$

Proposition 41. *The application $\mu^{-1}(P \times Q)$ is the inverse of the application $\mu(P \times Q)$.*

PROOF. Let M be an indecomposable $kP \times Q$ -module of p -permutation. Let D be a vertex of M and V be a simple kQ -module such that $M = \text{Ind}_{D \times Q}^{P \times Q} \text{Inf}_Q^{D \times Q} V$. Then:

$$\begin{aligned} \mu(P \times Q) \mu^{-1}(P \times Q)([M]) &= \mu(P \times Q)([P/D] \otimes [V]) \\ &= [k(P/D) \otimes_k V] \\ &= [\text{Ind}_{D \times Q}^{P \times Q} \text{Inf}_Q^{D \times Q} V] \\ &= [M] \end{aligned}$$

Reciprocally, let D be a subgroup of P and V be a simple kQ -module. Then

$$\mu(P \times Q)([P/D] \otimes_k [V]) = [k(P/D) \otimes_k V]$$

so if we prove that D is a vertex of $k(P/D) \otimes_k V$ then

$$\mu^{-1}(P \times Q) \mu(P \times Q)([P/D] \otimes_k [V]) = [P/D] \otimes_k [V].$$

As $M = k(P/D) \otimes_k V \cong \text{Ind}_{D \times Q}^{P \times Q} \text{Inf}_Q^{D \times Q} V$ is a direct factor of

$$\begin{aligned} \text{Ind}_{D \times Q}^{P \times Q} \text{Inf}_Q^{D \times Q} kQ &\cong \text{Ind}_{D \times Q}^{P \times Q} \text{Inf}_Q^{D \times Q} \text{Ind}_1^Q k \\ &\cong \text{Ind}_{D \times Q}^{P \times Q} \text{Ind}_D^{D \times Q} \text{Inf}_1^D k \\ &\cong \text{Ind}_D^{P \times Q} \text{Inf}_1^D k \\ &\cong \text{Ind}_D^{P \times Q} k, \end{aligned}$$

M is relatively D -projective. So the vertex of M must be a subgroup of D , say \tilde{D} . Then M is a direct factor of $\text{Ind}_{\tilde{D}}^{P \times Q} k$, so $\text{Res}_D^{P \times Q} M$ is a direct factor of $\text{Res}_D^{P \times Q} \text{Ind}_{\tilde{D}}^{P \times Q} k$. But

$$\begin{aligned} \text{Res}_D^{P \times Q} M &= \text{Res}_D^{P \times Q} (k(P/D) \otimes_k V) \\ &\cong \bigoplus_{i=1}^{\dim_k V} k(P/D). \end{aligned}$$

and

$$\begin{aligned} \text{Res}_D^{P \times Q} \text{Ind}_{\tilde{D}}^{P \times Q} k &\cong \sum_{x \in [D \setminus P \times Q / \tilde{D}]} \text{Ind}_{D \cap {}^x \tilde{D}}^D \text{Iso}(\gamma_x) \text{Res}_{D^x \cap \tilde{D}}^{\tilde{D}} k \\ &\cong \sum_{x \in [D \setminus P \times Q / \tilde{D}]} \text{Ind}_x^D k. \end{aligned}$$

So $k(P/D)$ is a direct factor of $\text{Ind}_x^D k$ for some $x \in P \times Q$. This is impossible except if $D = \tilde{D}$ because the kD -module $k(P/D)$ is of vertex D .

It's remain to check that μ is a natural transformation between $\mathbb{C}B \otimes_{\mathbb{C}} \mathbb{C}R_k$ and $\mathbb{C} \text{pp}_k$. Thanks to the \mathbb{C} -linearity of μ , it is enough to verify it for the element of a basis of $\mathbb{C}B(P \times Q, P' \times Q')$, where P and P' are finite p -groups and Q and Q' are finite p' -groups. Let U be a transitive $(P \times Q, P' \times Q')$ -biset. We have to check that the following diagram commuts:

$$\begin{array}{ccc} \mathbb{C}B(P) \otimes_{\mathbb{C}} \mathbb{C}R_k(Q) & \xrightarrow{\mu^{(P \times Q)}} & \mathbb{C} \text{pp}_k(P \times Q) \\ \downarrow (\mathbb{C}B \otimes \mathbb{C}R_k)([U]) & & \downarrow \mathbb{C} \text{pp}_k([U]) \\ \mathbb{C}B(P') \otimes_{\mathbb{C}} \mathbb{C}R_k(Q') & \xrightarrow{\mu^{(P' \times Q')}} & \mathbb{C} \text{pp}_k(P' \times Q') \end{array}$$

There exists a (transitive) (P, P') -biset $U_{\mathcal{P}}$ and a (transitive) (Q, Q') biset $U_{\mathcal{Q}}$ such that $U = U_{\mathcal{P}} \times U_{\mathcal{Q}}$ (Proposition 21). It is enough to verify that the diagram commuts for the elements of a basis of $\mathbb{C}B(P) \otimes_{\mathbb{C}} \mathbb{C}R_k(Q)$. So let X be a P -set and V a kQ -module. Then:

$$\begin{aligned} \mathbb{C} \text{pp}_k([U]) \circ \mu(P \times Q)([X] \otimes_{\mathbb{C}} [V]) &= \mathbb{C} \text{pp}_k([U])([kX \otimes_k V]) \\ &= [kU \otimes_{k(P \times Q)} (kX \otimes_k V)] \\ &= [(kU_{\mathcal{P}} \otimes_k kU_{\mathcal{Q}}) \otimes_{k(P \times Q)} (kX \otimes_k V)] \end{aligned}$$

and

$$\begin{aligned} \mu(P' \times Q') \circ (\mathbb{C}B \otimes_{\mathbb{C}} \mathbb{C}R_k)([U])([X] \otimes_{\mathbb{C}} [V]) &= \\ &= \mu(P' \times Q') \circ (\mathbb{C}B([U_{\mathcal{P}}]) \otimes \mathbb{C}R_k([U_{\mathcal{Q}}]))([X] \otimes_{\mathbb{C}} [V]) \\ &= \mu(P' \times Q')([U_{\mathcal{P}} \times_P X] \otimes_{\mathbb{C}} [kU_{\mathcal{Q}} \otimes_{kQ} V]) \\ &= [k(U_{\mathcal{P}} \times_P X) \otimes_k (kU_{\mathcal{Q}} \otimes_{kQ} V)] \\ &= [(kU_{\mathcal{P}} \otimes_{kP} kX) \otimes_k (kU_{\mathcal{Q}} \otimes_{kQ} V)]. \end{aligned}$$

Given the definition of the action of $P \times Q$ on $kU_{\mathcal{P}} \otimes_k kU_{\mathcal{Q}}$ and $[kX \otimes_k V]$, it is easy to check that there is an isomorphism of $k(P' \times Q')$ -modules

$$(kU_{\mathcal{P}} \otimes_k kU_{\mathcal{Q}}) \otimes_{k(P \times Q)} (kX \otimes_k V) \cong (kU_{\mathcal{P}} \otimes_{kP} kX) \otimes_k (kU_{\mathcal{Q}} \otimes_{kQ} V).$$

But on $\mathcal{C}_{p'}$, $\mathbb{C}R_k$ decomposes as a direct sum of simple functors:

$$\mathbb{C}R_k \cong \bigoplus_{(m, \xi)} S_{\mathbb{Z}/m\mathbb{Z}, \mathbb{C}_{\xi}},$$

where (m, ξ) runs over the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$. Consequently, on $\mathcal{C}_{p \times p'}$, we have the following decomposition:

$$\mathbb{C} \text{pp}_k \cong \bigoplus_{(m, \xi)} \mathbb{C} B \otimes_{\mathbb{C}} S_{C_m, \mathbb{C}_\xi},$$

where (m, ξ) runs over the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$. So, to find the composition factors of $\mathbb{C} \text{pp}_k$ on $\mathcal{C}_{p \times p'}$, it is enough to find the composition factors of $\mathbb{C} B \otimes_{\mathbb{C}} S_{\mathbb{Z}/m\mathbb{Z}, \mathbb{C}_\xi}$ on $\mathcal{C}_{p \times p'}$, for all positive integer m prime to p and for all primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$. Let m be a positive integer prime to p and $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$ be a primitive character. By 5.6.9 “The case of p -groups”, page 94 of [2], we have the following exact sequence, on \mathcal{C}_p :

$$0 \longrightarrow S_{C_p \times C_p, \mathbb{C}} \longrightarrow \mathbb{C} B \longrightarrow S_{1, \mathbb{C}} \longrightarrow 0.$$

But now, applying Proposition 24, we have the following exact sequence (on $\mathcal{C}_{p \times p'}$):

$$0 \longrightarrow S_{C_p \times C_p, \mathbb{C}} \otimes_{\mathbb{C}} S_{C_m, \mathbb{C}_\xi} \longrightarrow \mathbb{C} B \otimes_{\mathbb{C}} S_{C_m, \mathbb{C}_\xi} \longrightarrow S_{1, \mathbb{C}} \otimes_{\mathbb{C}} S_{C_m, \mathbb{C}_\xi} \longrightarrow 0.$$

We can now apply Theorem 29 and we obtain the following exact sequence:

$$0 \longrightarrow S_{C_p \times C_p \times C_m, \mathbb{C}_\xi} \longrightarrow \mathbb{C} B \otimes_{\mathbb{C}} S_{C_m, \mathbb{C}_\xi} \longrightarrow S_{C_m, \mathbb{C}_\xi} \longrightarrow 0$$

on $\mathcal{C}_{p \times p'}$, where the $\mathbb{C} \text{Out}(C_p \times C_p \times C_m)$ -module \mathbb{C}_ξ is defined as follows: the action of $\text{Out}(C_p \times C_p)$ is trivial and $\text{Out}(C_m)$ acts through ξ (as $\text{Out}(C_p \times C_p \times C_m) \cong \text{Out}(C_p \times C_p) \times \text{Out}(C_m)$, we can describe the two actions separately).

Now the composition factors of $\mathbb{C} B \otimes_{\mathbb{C}} S_{\mathbb{Z}/m\mathbb{Z}, \mathbb{C}_\xi}$ on $\mathcal{C}_{p \times p'}$ are $S_{C_p \times C_p \times C_m, \mathbb{C}_\xi}$ and S_{C_m, \mathbb{C}_ξ} . Consequently, we have proved the following Theorem:

Theorem 42. *The composition factors of $\mathbb{C} \text{pp}_k$ on $\mathcal{C}_{p \times p'}$ are the simple factors $S_{C_p \times C_p \times C_m, \mathbb{C}_\xi}$ and S_{C_m, \mathbb{C}_ξ} , where (m, ξ) runs over the set of pairs consisting of a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$. Their multiplicity as composition factor is 1.*

By the description of composition factors in section 2.2, this implies the following Theorem:

Theorem 43. *If G is an object of $\mathcal{C}_{p \times p'}$ (that is the direct product of a finite p -group with a finite p' -group) and V is a simple $\mathbb{C} \text{Out}(G)$ -module, then $S_{G, V}$ is a composition factor of $\mathbb{C} \text{pp}_k$ on $\mathbb{C} \text{GrB}$ if and only if there exists a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^* \rightarrow \mathbb{C}^*$ such that $G \cong C_m$ where $G \cong C_p \times C_p \times C_m$ and V is the module \mathbb{C}_ξ . Moreover, the multiplicity $S_{G, V}$ as composition factor is 1.*

As corollary, we obtain Theorem 30:

Corollary 44. *If G is a finite abelian group and V is a simple $\mathbb{C} \operatorname{Out}(G)$ -module then $S_{G,V}$ is a composition factor of $\mathbb{C} \operatorname{pp}_k$ on $\mathbb{C} \operatorname{GrB}$ if and only if there exists a positive integer m prime to p and a primitive character $\xi : (\mathbb{Z}/m\mathbb{Z})^\star \rightarrow \mathbb{C}^\star$ such that $G \cong C_m$ or $G \cong C_p \times C_p \times C_m$, where V is the module \mathbb{C}_ξ . Moreover, the multiplicity of $S_{G,V}$ as composition factor is 1.*

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