Fixed-order LPV Controller Design for LPV Systems by Convex Optimization [1]

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Abstract: In this paper a new method for fixed-order output-feedback Linear Parameter Varying (LPV) controller design is presented. First, a set of stabilizing LPV controllers is given as an inner convex approximation of the non-convex set of all stabilizing LPV controllers with the same order. This characterization is based on the decoupling of the state and Lyapunov matrices appearing together in the derivative of the Lyapunov function. As a measure of performance the decay-rate related to exponential stability is considered. The efficiency of the proposed method is illustrated on an appropriate simulation example.

1. INTRODUCTION

In recent years the modeling and control of LPV systems has become a very important area of research (e.g. Leith and Leithead [2000]). The motivation is to use of linear systems theory tools on a wide class of nonlinear systems (Shamma and Athans [1991]). Over the years, the theory of LPV systems was successfully used for modeling and control in practical applications, e.g. for wind turbine control (F. D. Bianchi et al. [2004]), turbofan engines (W. Gilbert et al. [2010]), wafer stage (Wassink et al. [2005]) and active braking control (G. Panzani et al. [2012]).

LPV systems are characterized by linear-like models depending on time-varying measured signals which we usually refer to as scheduling parameters. One of the important aspects is the way the scheduling parameters are handled in the design process. In some of the approaches (e.g. Apkarian and Gahinet [1995]) Linear Fractional Transformation (LFT) framework is used to isolate the scheduling parameters, which allows the small-gain theorem to be used for the analysis of system's stability. In this approach conservatism stems from the fact that the scheduling parameters are collected into an uncertainty matrix of a very specific structure, while the small-gain theorem provides stability for a system interconnected with "any" uncertainty from a bounded set. The use of a single quadratic Lyapunov function guarantees stability even for infinitely fast variations of scheduling parameters. In (F. Wu et al. [1996]), however, it is shown that there exist some LPV systems that are not stabilizable using a single quadratic Lyapunov function. So, considering a bound on the variation rate of scheduling parameters, which is realistic in most practical applications, will certainly relax the controller synthesis problem.

Parameter Dependent Lyapunov Functions (PDLF) are used in (F. Wu et al. [1996]) and (Apkarian and Adams [1998]). In (F. Wu et al. [1996]) both state-feedback and full-order output feedback are treated in a relatively general setting. All the stability constraints are polynomial in the scheduling parameters, and their satisfaction over the whole polytope is approximately ensured by gridding in the parameter space. This is justified by the fact that in practice often the number of scheduling parameters does not exceed 3. Also, as a consequence of the way the original stability constraints are transformed into Linear Matrix Inequalities (LMIs), controller matrices will depend on the derivative of the scheduling parameter, which is not measurable in the general case. In (Apkarian and Adams [1998]) similar problems appear, but dependence of the output-feedback controller on the scheduling parameters' derivative can be avoided by fixing part of the structured Lyapunov matrix over the polytope. The conservatism of the approach can be reduced by the use of a scaling matrix. An iterative approach is proposed that leads to a local performance optimum. This approach is improved in (Sato [2011]) by slightly different structuring of Lyapunov matrix and the addition of a scalar parameter. It is proven that this approach will give at worst the same performance as the one in (Apkarian and Adams [1998]), but at the cost of increased computation time related to the line search over the space of the scalar parameter.

In paper (J. C. Geromel et al. [1998]) sufficient stability condition for the continuous-time systems with polytopic uncertainty is derived. Stability is proven through the existence of PDLF. Conditions for the stability of whole polytope are represented by the means of finite number of matrix inequalities, which is enabled by decoupling of state and Lyapunov matrix by the means of slack variable matrices. Obtained stability conditions are similar to conditions derived in this paper, however the idea that leads to conditions is different. These conditions are further developed in the literature, e.g. for the robust output feedback controller design for the continuous-time polytopic systems in (Geromel et al. [2007]).

Practical implementation of LPV controllers is in general a complex task. In the state-feedback case, in addition to measuring/estimating the states, the scheduling parameters should be measured/estimated. For implementation of full-order output-feedback LPV controllers some tedious

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linear algebra has to be applied online, including matrix inversion, which limits the use of such controllers. Also, the order of such controllers is related to the order of the plant, which could be very high. For these reasons, there exists a need for fixed-order output-feedback LPV controller synthesis methods.

In (Henrion et al. [2003]) and (H. Khatibi and A. Karimi [2010]) some methods for stabilization of polytopic uncertain systems by fixed-order robust controllers are proposed. The type of plant models considered in these papers is SISO rational transfer functions with their numerators and denominators belonging to some polytope of polynomials. These approaches are based on the existence of a stable central polynomial, which is used for the convexification of the stability constraints for the whole polytope. These ideas are expanded to LPV systems with a scheduling parameter dependent transfer function representation in (W. Gilbert et al. [2010]) and (Z. Emedi and A. Karimi [2012]). The approach in (W. Gilbert et al. [2010]) is successfully applied to a real turbofan engine control problem. However, the transfer function representation of LPV systems could be very limiting because affine LPV state-space models, coming e.g. from physical modeling, would produce transfer functions which would be polynomial (of the order of the plant) in the scheduling parameters. The other adverse side is the extension of the approach to MIMO systems which could be very involved as all transfer functions would have to be brought to the common denominator, increasing the complexity of the plant models.

In this paper we present a new fixed-order output-feedback LPV controller design method for state-space LPV plant models with affine dependence on the scheduling parameter vector. Some bounds on the scheduling parameters and their variation rates are assumed through the use of affine PDLF. As a performance measure, the decay rate related to exponential stability of the closed-loop system is considered. In Section 2 we introduce the class of LPV plant models and the LPV controller structure, with emphasis on their interconnection. Then, in Section 3, a new convex set of fixed-order LPV controllers, with exponential decay rate as a performance measure is proposed. The effectiveness of the proposed method is illustrated on a simulation example in Section 4, with remarks on some additional features. Finally, the content of the paper is summarized in Section 5.

2. PROBLEM DESCRIPTION

2.1 Plant model

We consider a class of continuous-time LPV systems given by the following model

$$\begin{aligned}\dot{\mathbf{x}}_g(t) &= A_g(\boldsymbol{\theta}(t))\mathbf{x}_g(t) + B_g(\boldsymbol{\theta}(t))\mathbf{u}(t) \\ \mathbf{y}(t) &= C_g\mathbf{x}_g(t)\end{aligned} \tag{1}$$

where $\mathbf{x}_g(t)$ represents the state vector belonging to $\mathbb{R}^n$, $\mathbf{u}(t)$ is the control input vector belonging to $\mathbb{R}^{n_u}$, $\mathbf{y}(t)$ is the plant output vector belonging to $\mathbb{R}^{n_y}$ and $\boldsymbol{\theta}(t)$ represents the vector of scheduling parameters:

$$\boldsymbol{\theta} = [\theta_1, \ldots, \theta_{n_\theta}]^T.$$

The plant model is strictly proper, a characteristic of all physical systems. Proper models can be easily converted to a strictly proper model by considering a high bandwidth filter for the output sensors. For some technical reasons, the scheduling parameter vector appears only in one of the vectors B_g or C_g. Here, the results are given for the case that B_g is a function of the scheduling parameters, but similar results can be developed for the other case, straightforwardly.

One important subclass of LPV systems, related to many practical problems, are LPV systems with affine structure. They are characterized through the affine dependence of state-space matrices on the scheduling parameter vector:

$$A_g(\boldsymbol{\theta}(t)) = A_{g_0} + \sum_{i=1}^{n_\theta} \theta_i(t) A_{g_i}, \tag{2}$$

similarly for $B_g(\boldsymbol{\theta}(t))$. In many applications, the scheduling parameter vector $\boldsymbol{\theta}(t)$ belongs to a hyperrectangle $\Theta \in \mathbb{R}^{n_\theta}$, i.e.

$$\theta_i(t) \in [\underline{\theta}_i, \overline{\theta}_i],\ i = 1, \ldots, n_\theta. \tag{3}$$

The set of vertices of Θ will be denoted by Θ_v.

2.2 Controller structure

Our goal is to parameterize a set of fixed-order LPV dynamic output feedback controllers $K(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t))$ that stabilize the plant $G(\boldsymbol{\theta}(t))$.

The structure of the LPV controller $K(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t))$ is given by

$$\begin{aligned}\dot{\mathbf{x}}_k(t) &= A_k(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t))\mathbf{x}_k(t) + B_k(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t))(\mathbf{r}(t) - \mathbf{y}(t)) \\ \mathbf{u}(t) &= C_k\mathbf{x}_k(t) + D_k(\mathbf{r}(t) - \mathbf{y}(t)),\end{aligned} \tag{4}$$

with $\mathbf{x}_k(t)$ representing the vector of controller states. Similarly to $\boldsymbol{\theta}(t)$, we suppose that $\dot{\boldsymbol{\theta}}(t)$ belongs to a hyperrectangle $\Delta \in \mathbb{R}^{n_\theta}$, i.e.

$$\dot{\theta}_i(t) \in [\underline{\delta}_i, \overline{\delta}_i],\ i = 1, \ldots, n_\theta, \tag{5}$$

and by Δ_v we will denote the set of vertices of Δ.

The dependence of the controller matrices on the scheduling parameter vector $\boldsymbol{\theta}(t)$ and its variation rate $\dot{\boldsymbol{\theta}}(t)$ is affine, as for the plant, with matrix $A_k(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t))$ given by

$$A_k(\boldsymbol{\theta}(t), \dot{\boldsymbol{\theta}}(t)) = A_{k_0} + \sum_{i=1}^{n_\theta} \theta_i(t) A_{k_i} + \sum_{i=1}^{n_\theta} \dot{\theta}_i(t) A_{kd_i}, \tag{6}$$

and accordingly for $B_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}(t))$. In the rest of this paper the dependence of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$ on time is implied.

It is important to emphasize at this stage that in practice $\dot{\boldsymbol{\theta}}$ is not always measurable. This can simply be circumvented by setting to zero the terms of $K(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ related to $\dot{\boldsymbol{\theta}}$.

2.3 Closed-loop system structure

By combining the plant model and the controller we obtain the closed-loop system representation:

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{x}}_g(t) \\ \dot{\mathbf{x}}_k(t)\end{bmatrix} &= \begin{bmatrix} A_g(\boldsymbol{\theta}) - B_g(\boldsymbol{\theta})D_kC_g & B_g(\boldsymbol{\theta})C_k \\ -B_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})C_g & A_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\end{bmatrix}\begin{bmatrix}\mathbf{x}_g(t) \\ \mathbf{x}_k(t)\end{bmatrix} \\ &\quad + \begin{bmatrix} B_g(\boldsymbol{\theta})D_k \\ B_k(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\end{bmatrix}\mathbf{r}(t) \\ \mathbf{y}(t) &= [C_g\ 0]\begin{bmatrix}\mathbf{x}_g(t) \\ \mathbf{x}_k(t)\end{bmatrix}\end{aligned} \tag{7}$$

We can notice that the closed-loop matrices depend affinely on both $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$. To shorten the presentation in the rest of the text we will denote the closed-loop matrices by $A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, $B_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ and C_{cl}, and the closed-loop state vector by $\mathbf{x} = [\mathbf{x}_g^T(t)\ \mathbf{x}_k^T(t)]^T$, so (7) is rewriten as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})(t))\mathbf{x}(t) + B_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})(t))\mathbf{r}(t) \\ \mathbf{y}(t) &= C_{cl}\mathbf{x}(t). \end{aligned} \tag{8}$$

2.4 Stability conditions

Stability of a dynamical system can be examined through the existence of an appropriate Lyapunov function. The standard choice of a Lyapunov function candidate for an LPV system is a quadratic function given as $V(\mathbf{x}) = \mathbf{x}^T P \mathbf{x}$ (with implicit dependence on time). The matrix P should not depend on $\boldsymbol{\theta}$ if the scheduling parameters can vary with infinite variation rate. However, in practice it is reasonable to assume that we can set certain bounds on $\dot{\boldsymbol{\theta}}$, and in this case the use of a unique P over Θ could be too conservative. So, as a Lyapunov function candidate we will consider a Parameter Dependent Lyapunov Function (PDLF) with affine dependence on the scheduling parameter vector:

$$V(\mathbf{x}) = \mathbf{x}^T P(\boldsymbol{\theta})\mathbf{x}, P(\boldsymbol{\theta}) = P_0 + \sum_{i=1}^{n_\theta} \theta_i P_i, \tag{9}$$

where $P(\boldsymbol{\theta}) > 0$ for $\forall \boldsymbol{\theta} \in \Theta$.

The following results are well known in the literature and are listed to improve the presentation. From the fact that $V(\mathbf{x})$ is quadratic in $\mathbf{x}$ and $P(\boldsymbol{\theta})$ is positive definite we can conclude that $V(\mathbf{x})$ is positive for all non-zero state vectors $\mathbf{x}$ and zero only for $\mathbf{x} = 0$. For $V(\mathbf{x})$ to be a Lyapunov function for LPV system (7) its derivative needs to be negative for all non-zero $\mathbf{x}$, which is given by

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \frac{dV}{dt}(\mathbf{x}^T P(\boldsymbol{\theta})\mathbf{x}) \\ &= \dot{\mathbf{x}}^T P(\boldsymbol{\theta})\mathbf{x} + \mathbf{x}^T P(\boldsymbol{\theta})\dot{\mathbf{x}} + \mathbf{x}^T \dot{P}(\boldsymbol{\theta})\mathbf{x}. \end{aligned} \tag{10}$$

This combined with the dynamic equation of the unforced system $\dot{\mathbf{x}} = A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\mathbf{x}$ gives

$$\dot{V}(\mathbf{x}) = \mathbf{x}^T [A_{cl}^T(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})P(\boldsymbol{\theta}) + P(\boldsymbol{\theta})A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \dot{P}(\boldsymbol{\theta})]\mathbf{x}, \tag{11}$$

where

$$\dot{P}(\boldsymbol{\theta}) = \sum_{i=1}^{n_\theta} \dot{\theta}_i P_i = P(\dot{\boldsymbol{\theta}}) - P_0.$$

Looking at the matrix inequality

$$A_{cl}^T(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})P(\boldsymbol{\theta}) + P(\boldsymbol{\theta})A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \dot{P}(\boldsymbol{\theta}) < 0, \quad \forall \boldsymbol{\theta} \in \Theta \wedge \forall \dot{\boldsymbol{\theta}} \in \Delta \tag{12}$$

we can observe that the left hand side of the inequality is polynomial in $(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$. This means that in general the infinite number of inequalities in (12) cannot be straightforwardly replaced by a finite inequality set without loosing either the full guarantee of stability or introducing some conservatism. On the other hand, the controller parameters in A_{cl} are multiplied by P which makes the above inequality bilinear. As it will be presented in the following section, we will replace the given infinite set of bilinear matrix inequalities with a finite set of linear matrix inequalities in which $A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ will be decoupled from $P(\boldsymbol{\theta})$.

3. FIXED-ORDER LPV CONTROLLER DESIGN

Our idea is to present an inner convex approximation of the stability condition (12) for affine LPV state-space plants by decoupling $A_{cl}(\theta, \dot{\theta})$ from $P(\theta)$. To perform this, similarly to (W. Gilbert et al. [2010]) and (Z. Emedi and A. Karimi [2012]), but taking into account that we treat state-space models, we are looking for a "decoupling matrix" whose stability will be related to the stability of the given LPV system.

To proceed, we will in short present some definitions and lemmas which will be useful for the representation of the convex set of fixed-order LPV controllers.

The KYP lemma for continuous-time systems states that the transfer function $H(s) = C(sI - A)^{-1}B + D$ is Strictly Positive Real (SPR) transfer function if and only if there exists a matrix $P = P^T > 0$ such that

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D - D^T \end{bmatrix} < 0. \tag{13}$$

By the Schur complement lemma (Boyd et al. [1994]) the SPRness of the system implies its stability in Lyapunov sense. We will use the expression "Hurwitz stable" for a matrix with all of its eigenvalues in the left-hand side of complex plane. The following lemma relates the SPRness of a transfer function with the SPRness of its inverse.

Lemma 1. These two statements are equivalent:

1) $H(s) = \left[\begin{array}{c|c} A & B \\ \hline C & I \end{array}\right]$ is SPR.

2) $H^{-1}(s) = \left[\begin{array}{c|c} A - BC & B \\ \hline -C & I \end{array}\right]$ is SPR.

Proof. According to the KYP lemma and using the Schur complement lemma, Statement 1 is equivalent to the existence of $P = P^T > 0$ such that

$$A^T P + PA + \frac{1}{2}(PB - C^T)(B^T P - C) < 0. \tag{14}$$

This inequality can be rearranged to

$$(A - BC)^T P + P(A - BC) + \frac{1}{2}(PB + C^T)(B^T P + C) < 0, \tag{15}$$

which is equivalent to Statement 2. □

The following consequence of Lemma 1 will be of great importance for stating the main result of this paper.

Lemma 2. The following two matrix inequalities are equivalent:

$$\begin{bmatrix} M^T P + PM & P - M^T + A^T \\ P - M + A & -2I \end{bmatrix} < 0 \Leftrightarrow \tag{16}$$

$$\begin{bmatrix} A^T P + PA & P - A^T + M^T \\ P - A + M & -2I \end{bmatrix} < 0. \tag{17}$$

Proof. Set $B = I$ and $C = A - M$ for the transfer function H in Lemma 1. Now, writing the KYP lemma conditions for the SPRness of transfer functions H^{-1} and H and taking into account their equivalence, we obtain (16) and (17). For the fact that in the first inequality there is no product of the matrices A and P we will refer to matrix M as a "decoupling matrix". □

The next lemma will be used to represent the convex set of controllers as a finite number of Linear Matrix Inequalities (LMIs).

Lemma 3. Consider a symmetric matrix L which is affine in the parameter vector ϕ, i.e.

$$L(\boldsymbol{\phi}) = L_0 + \sum_{i=1}^{n_\phi} \phi_i L_i, \tag{18}$$

where $\boldsymbol{\phi}$ belongs to the polytope Φ, and the finite set of vertices of Φ is denoted by $\Phi_v = \{\phi_{v_1}, \dots, \phi_{v_q}\}$. Then the infinite set of matrix inequalities

$$L(\boldsymbol{\phi}) < 0, \forall \boldsymbol{\phi} \in \Phi \tag{19}$$

is equivalent to the finite set of matrix inequalities

$$L(\boldsymbol{\phi}) < 0, \forall \boldsymbol{\phi} \in \Phi_v. \tag{20}$$

The proof is easily derived using convex combinations of the vertices.

Finally, we can state the paper's main result.

Theorem 1. Suppose that the LPV plant model is given by (1) and (2) and that the scheduling parameters and their variation rates belong to hyperrectangles Θ and Δ (as in (3) and (5)), with Θ_v and Δ_v denoting vertex sets of Θ and Δ. Then, given a Hurwitz stable matrix M, the controller in (4) and (6) stabilizes the LPV model for any allowable scheduling parameter trajectory if

$$\begin{bmatrix} M^T P(\boldsymbol{\theta}) + P(\boldsymbol{\theta})M + P(\dot{\boldsymbol{\theta}}) - P_0 & (*) \\ P(\boldsymbol{\theta}) - M + A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) & -2I \end{bmatrix} < 0, \tag{21}$$

$$P(\boldsymbol{\theta}) > 0 \quad , \quad \forall \boldsymbol{\theta} \in \Theta_v, \forall \dot{\boldsymbol{\theta}} \in \Delta_v.$$

Symbol $(*)$ substitutes terms which ensure the symmetry of the matrix.

Proof. First we can observe that left-hand side of (21) can be represented as a symmetric matrix expression affine in vector $\phi^T = [\boldsymbol{\theta}^T \, \dot{\boldsymbol{\theta}}^T]$. Noticing that the polytope Φ is given by $\Phi = \Theta \times \Delta$, we can conclude from Lemma 3 that matrix inequality (21) is satisfied for all $\boldsymbol{\theta} \in \Theta$ and $\dot{\boldsymbol{\theta}} \in \Delta$. But now observe Lemma 2 and notice that the addition of a term $\dot{P}(\boldsymbol{\theta})$ to the upper left blocks of both matrices does not spoil the equivalence. Therefore from (21) we obtain the negative definiteness of

$$\begin{bmatrix} A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})^T P(\boldsymbol{\theta}) + P(\boldsymbol{\theta}) A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + P(\dot{\boldsymbol{\theta}}) - P(0) & (*) \\ P(\boldsymbol{\theta}) + M - A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) & -2I \end{bmatrix}.$$

Using the Schur complement lemma we can conclude that the following inequality is implied

$$A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})^T P(\boldsymbol{\theta}) + P(\boldsymbol{\theta}) A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + P(\dot{\boldsymbol{\theta}}) - P_0 < 0$$

for $\forall \boldsymbol{\theta} \in \Theta$ and $\forall \dot{\boldsymbol{\theta}} \in \Delta$, which means that the system is stabilized for bounded scheduling parameter variations. □

Remark 1. As in (21) the controller and Lyapunov matrices appear affinely, it is a set of LMIs as long as the decoupling matrix M is fixed. A proposition for the choice of M will be given in the next subsection.

The set of LMIs in (21) guarantees stability of the closed-loop system for a bounded scheduling parameter variation rate. We can also ensure a good exponential stability decay-rate for the closed-loop system.

Corollary 1. Making the same assumptions as in Theorem 1, the following finite set of matrix inequalities with $\gamma > 0$

$$\begin{bmatrix} M^T P(\boldsymbol{\theta}) + P(\boldsymbol{\theta})M + P(\dot{\boldsymbol{\theta}}) - P_0 + \gamma P(\boldsymbol{\theta}) & (*) \\ P(\boldsymbol{\theta}) - M + A_{cl}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) & -2I \end{bmatrix} < 0, \tag{22}$$

$$P(\boldsymbol{\theta}) > 0 \quad , \quad \forall \boldsymbol{\theta} \in \Theta_v, \forall \dot{\boldsymbol{\theta}} \in \Delta_v$$

describes the convex set of LPV controllers stabilizing the LPV plant and ensuring an exponential stability decay-rate α such that $0 < \alpha < \gamma$ and

$$\|\mathbf{x}(t)\| \le c\|\mathbf{x}(0)\| e^{-2\alpha t} \tag{23}$$

for some $c > 0$. The value of γ can be maximized by the bisection algorithm.

Proof. Based on Theorem 4.10 from (Khalil [2006]).

Remark 2. If in (22) we replace the term $\gamma P(\boldsymbol{\theta})$ by γI, the maximum of γ is a lower bound for the decay-rate α.

3.1 Choice of the decoupling matrix M

The choice of M is crucial in this approach. The role of this matrix is very similar to that of "central polynomial" in Henrion et al. [2003] and W. Gilbert et al. [2010]. For this we propose a method based on the gain-scheduled controllers which are closely related to LPV controllers. As mentioned, gain-scheduling is a control paradigm commonly used in practice. It is based on two main steps: design of linear controllers for some operating points of the system and interpolation of these controllers in the real-time to obtain the controller for a point which is not included in the original operating point set. Now, suppose that our set of operating points corresponds to $\theta \in \Theta_v$. For each of these operating points we can design a controller by any of the classical controller design approaches. However, as the next step we will perform the calculation of M (instead of interpolation). Based on the initial controllers, we can compute closed-loop matrices $A^0_{cl_j}, j = 1, \dots, 2^{n_\theta}$. Then, using Lemma 2 we propose the following semi-definite programming problem:

$$\begin{bmatrix} (A^0_{cl_j})^T P_j + P_j A^0_{cl_j} + \gamma I & (*) \\ P_j + M - A^0_{cl_j} & -2I \end{bmatrix} < 0. \tag{24}$$

If this problem is feasible, considering M and P_j, for $j = 1, \dots, 2^{n_\theta}$ as optimization variables, we choose the M for which the value of γ is maximized using the bisection algorithm.

This means that the presented LPV controller design algorithm comprises of the following steps:

Step 1 : Design of initial controllers, using some classical controller design approach, one per vertex of the scheduling parameter space;

Step 2 : Calculation of the decoupling matrix M from (24), based on these initial controllers;

Step 3 : Design of the LPV controller and Lyapunov function using (21), based on M from Step 2.

4. SIMULATION EXAMPLE

The efficiency of the proposed method is verified on the simulation example taken from (Wu [2001]). The simulation plant represents a simple two-disc system with the following LPV model

$$\dot{\mathbf{x}}_g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \theta_1 - \dfrac{k_0}{m_1} & \dfrac{k_0}{m_1} & \dfrac{b_0}{m_1} & 0 \\ \dfrac{k_0}{m_2} & \theta_2 - \dfrac{k_0}{m_2} & 0 & \dfrac{b_0}{m_2} \end{bmatrix} \mathbf{x}_g + \begin{bmatrix} 0 \\ 0 \\ \dfrac{0.1}{m_1} \\ 0 \end{bmatrix} \mathbf{u} \quad (25)$$

$$\mathbf{y} = [\,0\ 1\ 0\ 0\,]\,\mathbf{x}_g,$$

with constants $k_0 = 200N/m$, $b_0 = 1kg/s$, $m_1 = 1kg$ and $m_2 = 0.5kg$. The scheduling parameters θ_1 and θ_2 belong to the intervals $[0, 9]$ and $[0, 25]$, respectively. For bounds on the scheduling parameter variation rate we choose the reasonable values of $[\underline{\delta}_1, \overline{\delta}_1] = [-30, 30]\dfrac{rad}{s^2}$ and $[\underline{\delta}_2, \overline{\delta}_2] = [-50, 50]\dfrac{rad}{s^2}$. Analysing the plant for frozen values of $\boldsymbol{\theta}$ we can easily observe that for $[\theta_1, \theta_2] = [0, 0]$ there is one pole at 0, and for the other 3 vertices of the hyperrectangle Θ, the open-loop system is unstable. Our aim is to compute a second-order LPV controller that guarantees the exponential stability of the closed-loop system with a good decay rate.

The first step is to design simple initial controllers for 4 vertices of Θ. As a tool, the Frequency-domain Robust Controller Design Toolbox (Karimi [2012]) is used. To show that simple tuning of initial controllers gives good results we choose to design PID controllers, as this is still the first choice of control engineers in practice (Visioli [2006]). The controllers are designed by an open-loop shaping method that guarantees a maximum of 4.5 for the magnitude of the sensitivity function. In this manner we obtain the following controllers:

$$K_1^0(s) = -828.73\frac{(s+0.3278)(s+5.32)}{s(s+50)} \quad \text{for } \boldsymbol{\theta} = [0,\ 0]^T$$

$$K_2^0(s) = -992.27\frac{(s+0.5496)(s+17.19)}{s(s+50)} \quad \text{for } \boldsymbol{\theta} = [0,\ 25]^T$$

$$K_3^0(s) = -919.68\frac{(s+0.7045)(s+15.4)}{s(s+50)} \quad \text{for } \boldsymbol{\theta} = [9,\ 0]^T$$

$$K_4^0(s) = -948.97\frac{(s+0.1187)(s+24.58)}{s(s+50)} \quad \text{for } \boldsymbol{\theta} = [9,\ 25]^T.$$

Obviously, if we could obtain a unique Lyapunov matrix P for 4 closed-loop systems based on these 4 controllers, then any interpolation between these controllers would produce an LPV controller stabilizing the LPV plant for any possible variation of $\boldsymbol{\theta}$. However, examining the system of 4 LMIs in unknown matrix P gives no feasible solution. Therefore, we can conclude that it makes sense to design an LPV controller for given bounds on the scheduling parameter variation rate.

Using initial controllers, the decoupling matrix M is obtained from the 4 LMIs in (24) related to the 4 vertices of Θ.

The next step is to verify the existence of an affine LPV controller $K(\boldsymbol{\theta})$ and an affine Lyapunov matrix $P(\boldsymbol{\theta})$ in (22). All controller state-space matrices, i.e. A_k, B_k, C_k and D_k are chosen to depend affinely on $\boldsymbol{\theta}$. Note that since matrices B_g and C_g are independent of $\boldsymbol{\theta}$, this choice will not change the affineness of A_{cl} with respect to $\boldsymbol{\theta}$. The number of LMIs appearing here is 16, representing the number of vertices of $\Theta \times \Delta$. For solving the described semidefinite programming problem, SDPT3 (Toh et al. [1999]) can be used as a solver, with YALMIP (Löfberg [2004]) as a Matlab environment for describing convex programming problems. The maximum value of γ obtained by the bisection algorithm is 0.6789. The state-space description of the resulting controller K_1 is:

$$A_k(\boldsymbol{\theta}) = \begin{bmatrix} -3.9959 & -0.9679 \\ -1.1207 & -46.2523 \end{bmatrix} + \theta_1 \begin{bmatrix} -0.0994 & 0.2436 \\ -0.2160 & -0.0359 \end{bmatrix} + \theta_2 \begin{bmatrix} 0.1112 & 0.0277 \\ 0.2213 & -0.0066 \end{bmatrix}$$

$$B_k(\boldsymbol{\theta}) = \begin{bmatrix} 10.7943 \\ 177.6351 \end{bmatrix} + \theta_1 \begin{bmatrix} -1.0236 \\ 0.0678 \end{bmatrix} + \theta_2 \begin{bmatrix} -0.1847 \\ -0.0736 \end{bmatrix}$$

$$C_k^T(\boldsymbol{\theta}) = \begin{bmatrix} 12.2934 \\ 109.0334 \end{bmatrix} + \theta_1 \begin{bmatrix} 2.7138 \\ -2.6864 \end{bmatrix} + \theta_2 \begin{bmatrix} -1.0924 \\ -0.2185 \end{bmatrix}$$

$$D_k(\boldsymbol{\theta}) = -639.2795 + 5.5746\theta_1 - 3.5194\theta_2.$$

Remark 3. The results can be improved if we use a different choice of decoupling matrix M based on the LPV controller K_1. This means that matrices $A^0_{cl_j}$ in the algorithm step (A2) are computed based on the LPV controller K_1 (and not on the initial controllers). Obtained decoupling matrix M_1 leads to a new controller K_2 with $\gamma = 1.0769$. If we continue redesigning M followed by the controller in this manner, after 4 more iterations we obtain controller K_6 with $\gamma = 1.8034$. To see how an increased value of γ affect the response of the system, we can perform simulations with a step signal as a reference for the closed-loop system with LPV controllers K_1 and K_6. In both cases the step is applied at $t = 1s$, while θ_1 and θ_2 take exponential trajectories as follows:

$$\theta_i = \begin{cases} \overline{\theta}_i\,, & t < 1 \\ \overline{\theta}_i e^{-(t-1)/\tau_i}\,, & t \geq 1 \end{cases}, i = 1, 2.$$

which corresponds to moving exponentially from $\theta_1 = 9$ and $\theta_2 = 25$ to $\theta_1 = 0$ and $\theta_2 = 0$, respectively. Taking the equation of parameter trajectories into account, we can conclude that the absolute value of the variation rate of θ_i will be maximum for $t = 1s$ where it reaches the value θ_i/τ_i. So, to bring the system to the design limits we choose $\tau_i = \overline{\theta}_i/\overline{\delta}_i$, i.e. $\tau_1 = 0.3s$ and $\tau_2 = 0.5s$. Figure 1 illustrates the step response for K_1 (red) and K_6 (blue). The second one is much less oscillatory (no undershoot and shorter settling time). Note that the open-loop model is unstable and an overshoot of 100% with a second order controller is a reasonable response very close to the limit of achievable performance.

Remark 4. As previously explained, this approach allows us to obtain controllers that depend on both $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$, if variation rate measurements are also available. This provides some additional degrees of freedom that may lead to better decay-rate performance. For this example, however, using the same initial M we can obtain a new controller $K(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ with only a small improvement in $\gamma(=0.7117)$, which does not significantly change the closed-loop performance.

Remark 5. In this approach, some of the poles of the controller can be fixed to a predefined value. For example, to ensure the integral action of the LPV controllers to track the step for frozen values of scheduling parameter, we

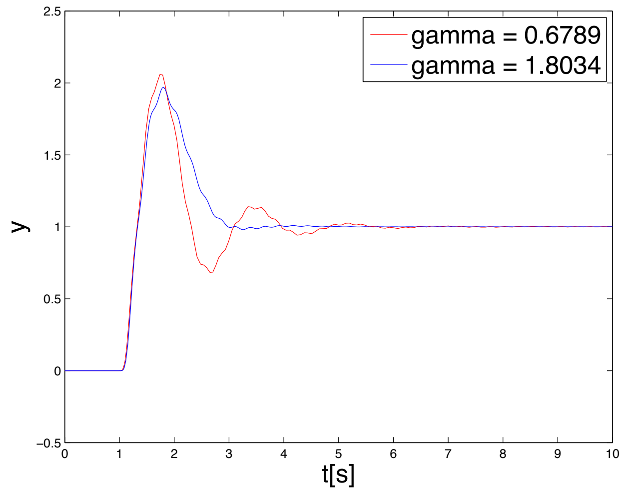

Fig. 1. Step response for two different LPV controllers. Red line: using initial PID controllers; Blue line: after 5 iterations.

could simply constrain one column (or row) of the matrices A_{k_i} to be identically equal to zero.

5. CONCLUSIONS

In this paper a new method for designing fixed-order LPV controllers with state-space representations affine in the scheduling parameters is presented. A set of LMI constraints that guarantee the stability of the LPV system for bounded variation rates of the scheduling parameters is proposed, and a recommendation for the choice of a decoupling matrix is given. An upper bound on the exponential stability decay-rate is also treated. The efficiency of the proposed method is illustrated by means of a simulated example from the literature. It is shown that, by simple application of the proposed method, a fixed-order LPV controller which ensures stability for all values of the scheduling parameters and the given bounds on their variation rates can be computed. The performance of the closed-loop system depends on the choice of a decoupling matrix based on some initial non-LPV controllers. In future work the effect of this choice will be studied and other methods will be investigated. Other types of performance (H_2 and H_∞) will also be considered as well as robustness to uncertainty in the measurement of the scheduling parameter vector.

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