The research program of Stochastic Deformation (with a view toward Geometric Mechanics)

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1 Introduction

The program of Stochastic Deformation was born in 1984-5 as an attempt to understand the paradoxical probabilistic structures involved in quantum mechanics [1]. In the course of this work, it became clear that no mathematically consistent and physically relevant approach was (and perhaps ever will be) available. But also, on the positive side, that a mathematical reinterpretation of Feynman's Path Integral approach to the problem was indeed accessible, providing a general method to produce quantum-like probability measures with qualitative properties quite distinct from what we are used to in statistical physics.

It is our intention to describe here the main features of this method. No familiarity with Feynman path integral approach or even quantum mechanics itself are required. We shall summarize their basic elements allowing the reader to understand why our construction is a rigorous version of his approach.

Quantum Theory provides for sure a kind of deformation of Classical Mechanics. But not a probabilistic one, unfortunately. One way to see our construction is, precisely, as such a quantum-like stochastic deformation. In this respect, it should also be of interest in stochastic approaches to Geometric Mechanics [2].

It may be worth recalling that the project to make sense of Feynman Path Integral method (and not only of a few of his formulas) is still a widely open problem, of a potentially devastating generality for probabilists. Who can doubt, indeed, after all these years and so many applications far beyond what Feynman could imagine in the fifties that a mathematically consistent version of this method should exist ?

This overview of Stochastic Deformation will be organized as follows:

Section 2 will summarize the original ideas of Feynman's reinterpretation of elementary quantum mechanics. For a probabilist, they look like a (very) informal version of Stochastic Analysis, with a twist regarding boundary conditions.This twist will prove to be indeed fundamental for their mathematical interpretation.

Section 3 will provide a probabilistic counterpart of Feynman's approach, i.e. a kind of stochastic boundary value problem whose conditions of existence and uniqueness of solutions will be specified.

In Section 4 the random dynamics of the relevant class of processes, together with its associated symmetries, will be described, in Lagrangian and Hamiltonian form.

The last Section will be devoted to some computational and geometric aspects of our Stochastic Deformation.

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2 Overview of Feynman Path Integral approach to quantum theory

We shall consider the example of a system of a single unit mass (charged) particle under the effect of a (bounded below) scalar potential $V(q)$ and a smooth vector potential $a(q)$. According to elementary quantum mechanics, the Hamiltonian observable of such a system is a densely defined self-adjoint operator (in $L^2(\mathbb{R}^3)$ for simplicity)

$$H = \frac{1}{2}(P - a(Q))^2 + V(Q) \tag{2.1}$$

where the position and momentum observables are defined on appropriate dense domains as

$$\begin{aligned} Q : \mathcal{D}_Q \subset L^2 &\to L^2 \\ \psi(q) &\mapsto q\psi(q) \end{aligned} \tag{2.2}$$

$$\begin{aligned} P : \mathcal{D}_p \subset L^2 &\to L^2 \\ \psi(q) &\mapsto -i\hbar\nabla\psi(q) \end{aligned} \tag{2.3}$$

The initial state ψ of the system evolves in an unitary way, $U_t\psi = \psi_t$, $U_t = e^{-\frac{i}{\hbar}tH}$, so that ψ_t solves

$$i\hbar\frac{d}{dt}\psi_t = H\psi_t \tag{2.4}$$

Any observable, in fact, is a densely defined self-adjoint operator O. The contact with (the language of) probability theory is made by the definition of the "mean value of O in the state ψ_t" at time t:

$$\langle O\rangle_{\psi_t} \equiv \langle\psi_t|O\psi_t\rangle = \langle\psi|O_t\psi\rangle \equiv \langle O_t\rangle_\psi, \quad O_t = U_t^+ O U_t \tag{2.5}$$

where $\langle\cdot|\cdot\rangle$ denotes the L^2 scalar product, antilinear on the left (with associated norm $\|\cdot\|$), and $+$ is the adjoint.

Eq (2.5) is a dual expression of the dynamics of an observable. For H as in (2.1), we obtain

$$\begin{cases} \frac{d}{dt}\langle Q_t\rangle_\psi = \langle P_t - a_t\rangle_\psi \\ \frac{d}{dt}\langle P_t - a\rangle_\psi = \langle (P_t - a_t)\wedge \mathrm{rot} a_t - \nabla V(Q_t)\rangle_\psi \end{cases} \tag{2.6}$$

But of what kind of random variables, really, those expressions are the mean values? We do not know; in fact no probability space has ever been defined in the first place. The only hint at probability theory, in quantum mechanics is Von Neumann's Axiom of "Quantum Static" according to which if one performs a measurement of an observable O for a system in state ψ, the absolute probability to find a result $\leqslant \lambda \in \mathbb{R}$ is

$$\langle E^O(\lambda)\rangle_\psi = \|E^O(\lambda)\psi\|^2 \tag{2.7}$$

where $E^O(\lambda)$ denotes the spectral family of orthogonal projections of O. For instance, when $O = Q$, and any λ in the interval $[a, b]$,

$$\|E^Q([a,b])\psi\|^2 = \int_{[a,b]} |\psi(q)|^2 dq \tag{2.8}$$

called Born interpretation of the wave function ψ. In this sense Eqs (2.5–2.6) are understandable, if not justified probabilistically. Even for the simplest quantum systems (Eq (2.1) with $a = V = O$, i.e the "free case") there is no underlying concept of space-time trajectory. The justification of this prohibition lies in the uncertainty relation:

For ψ in the domains of Q and P,

$$(Q_jP_k - P_kQ_j)\psi = i\hbar\delta_{jk}, \quad 1 \leqslant j,k \leqslant 3 \tag{2.9}$$

is interpreted as the impossibility to localize experimentally the position and the momentum simultaneously, i.e to define a trajectory as in classical Hamiltonian mechanics.

Feynman transformed qualitatively these shaky relations between Quantum Physics and Probability Theory [3].

Let us consider a classical (conservative) system of Lagrangian L, in dynamical evolution on the time interval $I = [s, u]$. If its configuration at time $t \in I$ is denoted by $\omega(t)$, the Action functional of this system is defined by

$$S_L[\omega(\cdot); u - s] = \int_s^u L(\omega(t), \dot\omega(t))dt \tag{2.10}$$

For our system whose quantum Hamiltonian is (2.1),

$$L(\omega, \dot\omega) = \frac{1}{2}|\dot\omega|^2 - V(\omega) + a(\omega)\cdot\dot\omega. \tag{2.11}$$

Feynman starts from two states φ, ψ, say at the final time u of I and rewrites $\langle\varphi_u|\psi_u\rangle \in \mathbb{C}$ in terms of the unitary evolution kernel applied to ψ_s:

$$\int\int \psi_s(x)(e^{-\frac{i}{\hbar}(u-s)H})(x,z)\overline{\varphi}_u(z)\,dx\,dz \tag{2.12}$$

where $\overline{\varphi}$ denotes the complex conjugate.

The key point is that he reinterprets this as the following "Path Integral":

$$\int\int \psi_s(x) e^{\frac{i}{\hbar} S_L[\omega(\cdot);u-s]} \mathcal{D}\omega\, \overline{\varphi}_u(z)\, dx\, dz \tag{2.13}$$

In this symbolic expression, $\Omega_{x,s}^{z,u}$ means the set of continuous paths $\{\omega \in C([s,u],\mathbb{R}^3) | \omega(s) = x, \omega(u) = z\}$ and $\mathcal{D}\omega$ plays the role of a "flat measure" $\prod_{s\leqslant t\leqslant u} d\omega(t)$ on this path space.

Consider any time $s < t < u$. Clearly ψ_s can be regarded as initial condition of

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi, \tag{2.14}$$

and $\overline{\varphi}_u$ as final boundary condition of the adjoint problem

$$-i\hbar \frac{\partial \overline{\varphi}}{\partial t} = H\overline{\varphi} \tag{2.15}$$

Feynman calls "transition amplitude" the (complex) expression (2.12) or (2.13). He will use it as a kind of expectation with weight $\exp \frac{i}{\hbar} S_L$, and denotes it by $\langle \varphi | 1 | \psi \rangle$ or $\langle 1 \rangle_{S_L}$. It should be stressed that, in Feynman's view, (2.13) should simply be regarded as a short notation for a sum over time-discretized paths $\omega(t_j) = x_j$, $t_j = j\frac{(u-s)}{N}$, $1 \leqslant j \leqslant N \in \mathbb{N}$. Along the same line, for "any" functional $F[\omega(\cdot)]$ of such paths, the "expectation" of F is defined by

$$\langle F \rangle_{S_L} = \int\int\int \psi_s(x)\; e^{\frac{i}{\hbar} S_L[\omega(\cdot);u-s]} F[\omega(\cdot)] \overline{\varphi}_u(z)\, \mathcal{D}\omega\, dx\, dz. \tag{2.16}$$

This definition is the starting point of Feynman's "Functional Calculus", whose main result is an Integration by parts formula [3]:

$$\langle \delta\, F[\omega](\delta\omega) \rangle_{S_L} = -\frac{i}{\hbar} \langle F \delta S_L[\omega](\delta\omega) \rangle_{S_L} \tag{2.17}$$

where $\delta G[\omega](\delta\omega)$ denotes the Gateaux derivative of a functional G on ω in the direction $\delta\omega$. Shaky as it is, mathematically, this formula deserves some interest since it is the ancestor of all the integration by parts formulas designed in Stochastic Analysis during the last 25 years !

For $F = 1$, L as in (2.11), Feynman finds the path integral counterpart of (2.6):

$$\langle \ddot{\omega} \rangle_{S_L} = \langle \dot{\omega} \wedge \operatorname{rot} a(\omega) - \nabla V(\omega) \rangle_{S_L} \tag{2.18}$$

where, regarding the meaning of the l.h.s, he observed that "in the few examples with which we had experience, the substitution $\ddot{\omega} = \frac{1}{(\Delta t)^2}(\omega(t+\Delta t) - 2\omega(t) + \omega(t - \Delta t))$ has been adequate".

The most revealing application of the relation (2.17), however, is the one for $F[\omega] = \omega$ and some specific direction $\delta\omega$. Indeed, (2.17) reduces, then, to the (discretized) expression:

$$\left\langle \omega_j(t) \left(\frac{\omega(t) - \omega(t - \Delta t)}{\Delta t} \right)_k \right\rangle_{S_L} - \left\langle \left(\frac{\omega(t + \Delta t) - \omega(t)}{\Delta t} \right)_k \omega_j(t) \right\rangle_{S_L} = i\hbar\delta_{jk}. \tag{2.19}$$

This maybe the most fundamental discovery of Feynman approach, [3] although it is almost never cited, for reasons to be understood afterwards. First, let us observe that (2.19) is a kinematical claim regarding the nature of trajectories. No dynamics is involved here. By construction, $\omega(\cdot) \in \Omega^{z,u}_{x,s}$, but what (2.19) suggests (since Feynman was aware that the desirable $\lim_{\Delta t \downarrow 0}$ is, to say the least, problematic) is that $\dot{\omega}(t_-)$ should be different from $\dot{\omega}(t^+)$, for any $t \in]s, u[$. In other words, the $\omega(\cdot)$ are indeed continuous quantum trajectories, but there are not differentiable.

Now consider $a = 0$ in the Lagrangian (2.11). Then, by definition of the classical momentum, $p = \frac{\partial L}{\partial \dot{\omega}} = \dot{\omega}$ if the configuration $q = \omega$. So Feynman regards (2.19) as the "space-time" version of the uncertainty relation (2.9), providing a deeply different interpretation from the regular one of Quantum Theory in Hilbert space: no reference to any limitation on experimental measurement is involved here.

Notice that, in this perspective and for Lagrangians as before, it would make perfect sense to interpret Quantum Mechanics as a Stochastic Deformation of Classical Mechanics for smooth paths. Clearly, $\hbar$ is the deformation parameter. However, even $S_L[\omega(\cdot)]$ would become singular along such "quantum paths" since $\dot{\omega}$ does not make sense.

Let us come back, for instance, on our question, after Eq (2.6): of what kind of random variables are those equations mean values? Take the above elementary momentum, for instance. By (2.6) (with $a = 0$) it should satisfy, in particular, after quantization,

$$\frac{d}{dt}\langle Q_t \rangle_\psi = \langle P_t \rangle_\psi. \tag{2.20}$$

According to Feynman, however, this corresponds to

$$\frac{d}{dt}\langle \omega(t) \rangle_{S_L} = \text{“}\lim_{\Delta t \downarrow 0}\text{”} \left\langle \frac{\omega(t + \Delta t) - \omega(t)}{\Delta t} \right\rangle_{S_L} \tag{2.21}$$

where $\omega(\cdot)$ denotes some, yet unspecified, random process.

The right hand size can be computed as a difference of 2 functionals like (2.16), using Eqs (2.14–2.15). On the other hand, of course, according to regular quantum theory, and using (2.3), we know that

$$\langle P_t \rangle_\psi = \langle P \rangle_{\psi_t} = \int \overline{\psi}_t(-i\hbar\nabla\psi_t)dq = \int \overline{\psi}_t\psi_t \left(-i\hbar\frac{\nabla\psi_t}{\psi_t} \right) dq. \tag{2.22}$$

In this way Feynman reinterprets each quantum observable as a specific function of his "random process" $\omega(\cdot)$, here $-i\hbar\nabla \log \psi_t$ for P. Unfortunately, R.H.

Cameron proved in 1960 ([4]) that $e^{\frac{i}{\hbar}S_L}\mathcal{D}\omega$ does not make any sense as a countably additive measures on the path space underlying (2.13). This means that there is no such thing as the "expectation" $\langle\ \rangle_{S_L}$ of (2.16), and no $\lim_{\Delta t\downarrow 0}$ in expressions like (2.19) or (2.21).

A complex measure similar to the Wiener one but with a purely complex variance, like the one used informally by Feynman would be in particular of infinite total variation and therefore not appropriated for his quantum purpose.

The challenge we face is therefore to preserve the essential of Feynman's approach but with well defined probability measures on path spaces. In order to tackle this, we will first summarize some of the key qualitative aspects to preserve:

1. Assume that there is a (filtered) probability space (Ω, σ, P) where such a stochastic process, say $X(t)$, $t \in I$, is well defined.

   Then the uncertainty relation (2.19) should mean

$$E\left\{X_j(t) \lim_{\Delta t\downarrow 0} E_t\left[\frac{X(t)-X(t-\Delta t)}{\Delta t}\right]_k\right\} - E\left\{\lim_{\Delta t\downarrow 0} E_t\left[\frac{X(t+\Delta t)-X(t)}{\Delta t}\right]_k X_j(t)\right\} = \hbar\delta_{jk} \quad (2.23)$$

   where E_t should be a conditional expectation since we know that the momentum has to become a function of the process $X(t)$. This means, in particular, that $X(t)$ should be Markovian. Still, the two limit random variables of (2.23) should be distinct, and will be denoted respectively by $D_t^* X$ and $D_t X$. Following R.H. Cameron, we cannot hope to produce complex measures satisfying Eq (2.19). We would content ourselves with real measures.

2. We need to preserve a probabilistic counterpart of the time symmetry in quantum physics (at least for systems whose Lagrangian is time independent) involved in Born interpretation (2.8): Indeed, the relation between ψ and $\overline{\psi}$ can be regarded as a time reversal (cf. (2.14)–(2.15)) therefore their product, in Born interpretation, is unchanged under this symmetry.

3. Although we shall treat here exclusively the class of elementary Lagrangian systems (2.11) considered by Feynman, our construction should rely on general principles compatible with any physical system.

Before describing the program of Stochastic Deformation, a few words about the internal evolution of Stochastic Analysis itself.

After the pioneering works of Wiener and Itô, this field made great progress since 1980 but there is one where it did not; with the notable exception of some approaches of Stochastic Control (we shall come back on this), the field suffered from a chronic lack of dynamical content (in the classical sense of dynamical systems theory). In Itô's original perspective Stochastic differential equations are Stochastic deformation of ordinary differential equation, still the comparison of the history of the two fields is revealing. For (second order) ODE a single, very hard, dynamical problem became the motor of all scientific progress: the N-body problem. Nothing like it was ever considered in Stochastic Analysis. This explains why very basic notions of ODE's theory, like the one of integrability, for instance, are lacking in Stochastic Analysis.

Here is a (dynamical) "paradox" mentioned by Krzysztof Burdzy. Let $\phi : [0,T] \to \mathbb{R}$ such that $\sup_{t\in[0,T]} |\ddot{\phi}(t)| < \infty$. If W_t denotes the Brownian motion, it is known that the probability

$$P\{\phi(t) - \varepsilon < W_t < \phi(t) + \varepsilon, \ \forall t \in [0,T]\} \sim c(\varepsilon) \exp -\frac{1}{2}\int_0^T (\dot{\phi}(\tau))^2 d\tau \equiv F[\phi(\cdot)]$$

The functional $F[\phi]$ is maximized by $\phi(\tau) = 0 \ \ \forall t \in [0,T]$ i.e, in particular, the solution of the second order ODE $\ddot{\phi}(t) = 0$ with $\phi(0) = \dot{\phi}(0) = 0$. But what does it mean, really, in relation with any observation and known properties of Brownian paths ? Notice that, according to Feynman, this is the free case $V = a = 0$ in (2.11) and the dynamical equation of this system should indeed be (cf (2.18)) $\langle \ddot{\omega} \rangle_{S_L} = 0$, whatever meaning can be given to $\langle \cdot \rangle_{S_L}$.

3 Probabilistic counterpart of Feynman's approach

To make sense of (2.23), in the form

$$E\Bigg[X_j(t) \lim_{\Delta t \downarrow 0} E_t \left[\frac{X(t) - X(t-\Delta t)}{\Delta t}\right]_k - \lim_{\Delta t \downarrow 0} E_t \left[\frac{X(t+\Delta t) - X(t)}{\Delta t}\right]_k X_j(t)\Bigg] = \hbar \delta_{jk} \quad (3.1)$$

we need two filtrations, to take into account not only the usual past information on a time interval $I \supset [s,u]$, i.e an increasing one $\mathcal{P}_t, t \in I$, but also a decreasing filtration $\mathcal{F}_t$ taking into account the future. The underlying filtered probability space should, therefore, be of the form $(\Omega, \sigma, \{\mathcal{P}_t\}, \{\mathcal{F}_t\}, P)$ with $t \in [s,u]$.

Feynman's time discretized interpretation of the left hand side of his dynamical Eq (2.18) suggests to limit ourselves to processes X_t such that, for any bounded measurable f and any $s \leqslant s_1 < t < t_1 \leqslant u$,

$$E\left[f(X_t) | \mathcal{P}_{s_1} \cup \mathcal{F}_{t_1}\right] = E\left[f(X_t) | X(s_1), X(t_1)\right]. \quad (3.2)$$

This property is what we call now Local Markov (or two-sided Markov). But it was introduced in 1932 by Sergei Bernstein, who named it "reciprocal" [5]. His motivation was a remark of E. Schrödinger, a year earlier, which seems to be at the origin of all the notion of stochastic reversibility known today to probabilist [6]. But we will refrain to insist here on the (quite tortuous) story of this notion.

To keep track, as suggested by Feynman, of the past and future informations about the system, the traditional Markovian transition probability should be replaced by a more symmetric measure Q, named after Bernstein:

$A \mapsto Q(s,x,t,A,v,z)$, $\forall x,z \in \mathbb{R}$, $s < t < u$ in I, measurable in x,y with $A \in \mathcal{B}(\mathbb{R})$, the Borelian of $\mathbb{R}$ (for simplicity).

For Q there is a 3 points analogue of Chapman-Kolmogorov property, such that, for $X(u) = z$ fixed, Q becomes a forward Markov transition and for $X(s) = x$ fixed Q reduces to a backward Markov property (let us recall, with A.D. Wentzell [7], that these are just two of the 64 ways to express Markov property !).

Of course, without fixing the starting or ending point, $X(\cdot)$ will only be a Bernstein process, satisfying (3.2) and not a Markovian one.

Let us denote by M the joint probability measure on $\mathcal{B} \times \mathcal{B}$ for the pair of initial and final random variables.

Then B. Jamison (1974 [8]) proved the following (cf. [1] or [17], for a version involving $\mathcal{P}_t$ and $\mathcal{F}_t$ and appropriate to the relation with Feynman's approach).

Theorem. *For a given Bernstein transition Q and a given joint measure M,*

a) There is a unique probability measure P_M such that under P_M, $X(t)$, $t \in [s,u]$, satisfies Bernstein property (3.2).

b) $P_M(X(s) \in A_s, X(u) \in A_u) = M(A_s \times A_u)$ Borelians in $\mathcal{B}$, the Borel tribe of $\mathbb{R}^3$.

c) $P_M(X(s) \in A_s, X(t_1) \in A_1, \dots, X(t_n) \in A_n, X(u) \in A_u)$

$$= \int_{A_s \times A_u} dM(x,z) \int_{A_1} Q(s,x,t_1,dx_1,u,z) \int_{A_2} \cdots \int_{A_n} Q(t_{n-1},q_{n-1},t_n,dq_n,u,z)$$

for $s \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n < u$ and $A_i \in \mathcal{B}$, $i = 1, \dots, n$.

The final random variable has been fixed here so that, as said before, Q has the properties of a forward Markovian transition but c) would hold as well with a fixed initial $X(s) = x$. In other words, the construction is perfectly symmetric with respect to the past and future informations, as it should be.

Jamison also proved that only one joint probability measure $M = M_m$ (M_m for Markov) turn $X(t)$ into a Markovian and not only a Bernstein process. Using the same notation as in §1 but, this time, for the (strongly continuous contraction) semigroup generated by the lower bounded operator H, M_m is of the form

$$M_m(A_s \times A_u) = \int_{A_s \times A_u} \eta_s^*(x) \left(e^{-\frac{1}{\hbar}(u-s)H}\right)(x,z)\eta_u(z)\, dx\, dz \tag{3.3}$$

where $\eta_s^*(x)$ and $\eta_u(z)$ are two positive (not necessarily bounded) functions to be determined later. This expression should be compared with Eq (2.12). Now, with

$$Q(s,x,t,dq,u,z) = h(x,u-s,z)^{-1} h(x,t-s,q) h(q,u-t,z) dq, \ s < t < u$$

where the handy notation $(e^{-\frac{1}{\hbar}(u-s)H})(x,z) = h(s,x,u,z)$ has been used, the substitution of (3.3) in the above finite dimensional distributions c) provides, after simplifications, the finite dimensional distributions:

$$\rho_n(dx_1, t_1, dx_2, t_2, \dots, dx_n, t_n), \ s < t_1 < t_2 < \cdots < t_n < u$$

$$= \int_{A_s \times A_u} \eta_s^*(x) h(s, dx, t_1, dx_1) \dots h(t_n, dx_n, u, dz) \eta_u(z)\, dx\, dz \tag{3.4}$$

Now define the following densities of the forward and backward transition probabilities

$$P(s,x,t,dy) = h(s,x,t,y)\frac{\eta_t(y)}{\eta_s(x)} dy \qquad s \leqslant t \tag{3.5}$$

where

$$\eta_s(x) = \int h(s,x,t,y)\eta_t(y)\, dy, \tag{3.6}$$

and

$$P^*(s,dy,t,x) = \frac{\eta_s^*(y)}{\eta_t^*(x)} h^*(t,x,s,y)\, dy \qquad s \leqslant t \tag{3.7}$$

where

$$\eta_t^*(x) = \int \eta_s^*(y) h^*(t,x,s,y)\, dy = \int \eta_s^*(y) h(s,y,t,x)\, dy \tag{3.8}$$

and the classical relation $h^*(t,x,s,y) = h(s,y,t,x)$ between integrals kernels of two adjoint parabolic equations (for H not necessarily symmetric) has been used. Then it is easy to check that (3.4) coincides with the final dimensional distributions of a forward Markovian process of initial probability density $\eta_s^*(x)\eta_s(x)\, dx$ and transition probability density (3.5) or, equivalently, of a backward Markovian with final probability density $\eta_u^*(z)\eta_u(z)\, dz$ and backward transition density of the form (3.7).

As a matter of fact, $\forall\, t \in [s,u]$, it is true that

$$P(X(t) \in A) = \int_A \eta_t^* \eta_t(x)\, dx \tag{3.9}$$

where, as shown by (3.6) and (3.8), η_t^* and η_t are two positive solutions of the two adjoint parabolic PDE, $s \leqslant t \leqslant u$

$$\begin{cases} -\hbar \frac{\partial \eta^*}{\partial t} = H^+ \eta^* \\ \eta^*(s,x) = \eta_s^*(x) \end{cases} \tag{3.10}$$

and

$$\begin{cases} \hbar \frac{\partial \eta}{\partial t} = H\eta \\ \eta(u,x) = \eta_u(x). \end{cases} \tag{3.11}$$

Here, a comment is needed since we said that quantum Hamiltonians, like any observable, are self-adjoint and the first PDE of (3.10) involves the adjoint H^+ of H. Our example (2.1) illustrates this point. When written explicitly, Schrödinger equation (2.4) means

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2}\Delta\psi + \frac{i\hbar}{2}\nabla .a\psi + i\hbar a\nabla\psi + \frac{1}{2}|a|^2\psi + V\psi \tag{3.12}$$

The "Euclidean version" of this corresponds to transform t into it and a into $-iA$ so that the right hand side operator becomes, indeed, a non-symmetric operator

$$H = -\frac{\hbar^2}{2}\Delta + \hbar A\nabla + \frac{\hbar}{2}\nabla .A - \frac{1}{2}|A|^2 + V. \tag{3.13}$$

Notice how close we are, in Eq (3.9), from Born interpretation of the wave function ψ in (2.8). Informally, we have done $t \mapsto it$ and the above transformation of the vector potential a. The product structure of the probability density in (3.9) is fundamental; it expresses manifestly a kind of invariance under time reversal, more general than the one traditionally known by probabilists as "reversibility". For instance, if $P_t(dx)$ denotes this probability at time t, it follows immediately from (3.5) and (3.7) that the following "detailed balance" condition holds

$$P_s(dx)P(s,x,u,z) = P^*(s,dx,u,z)P_u(dz), s \leqslant u \tag{3.14}$$

generalizing Kolmogorov's notion of reversibility [9] to non-stationary situations. In this paper, in fact, Kolmogorov refers to E. Schrödinger (1931,32) [6] who is at the origin of our program of Stochastic Deformation. This has been regrettably forgotten afterwards.

Given (3.5) and (3.7), a simple calculation provides the forward and backward drifts of the underlying diffusion process $X(\cdot)$. Preserving the notations D_tX and D_t^*X introduced after (2.23),

$$D_tX = \lim_{\Delta t \downarrow 0} E_t\left[\frac{X(t+\Delta t) - X(t)}{\Delta t}\right] = \hbar\nabla \log \eta_t(X) - A(X) \tag{3.15}$$

$$D_t^*X = \lim_{\Delta t \downarrow 0} E_t\left[\frac{X(t) - X(t-\Delta t)}{\Delta t}\right] = -\hbar\nabla \log \eta_t^*(X) - A(X) \tag{3.15*}$$

In particular, since $P_t(dx) = \eta_t^*\eta_t(x)\,dx = \rho(x,t)\,dx$

$$D_t^*X = D_tX - \hbar\nabla \log \rho. \tag{3.16}$$

After substitution of (3.16) in the left hand side of (3.1) and an integration by part we obtain an elementary proof of this uncertainty relation (3.1), justifying in this way the presence of two filtrations.

There is a new qualitative aspect in our probabilistic counterpart of Feynman's approach. Not surprisingly it comes from its boundary conditions.

The above construction of the processes, for a bounded below "Hamiltonian" strongly suggests that natural boundary conditions should be two probability (densities) at the initial and final time:

$$P_s(dx) = P_s(x)\,dx \text{ and } P_u(dz) = P_u(z)\,dz \tag{3.17}$$

instead of the boundary conditions of the two adjoint equations (3.10) and (3.11). But let us write the marginals of the joint probability M_m (3.3):

$$\begin{cases} \eta_s^*(x) \int h(s,x,u,z)\eta_u(z)\,dz = P_s(x) \\ \eta_u(z) \int \eta_s^*(x) h(s,x,u,z)\,dx = P_u(z) \end{cases} \tag{3.18}$$

in terms of the single integral kernel h. If P_s and P_u are arbitrarily given, Eq (3.18) is a non linear integral system for (η_s^*, η_u) the two boundary conditions of the underlying adjoint PDE (3.10) and (3.11), $s < t < u$.

Beurling has proved in 1960 the following general result:

Theorem ([10]). *Let the above integral kernel $h(s,x,u,z) = \left(e^{-\frac{1}{\hbar}(u-s)H}\right)(x,z)$ be continuous, positive and defined on any locally compact space. Then the system* (3.18) *has a unique pair* (η_s^*, η_u) *of positive, not necessarily integrable solutions, for any strictly positive probability densities $P_s(x), P_u(z)$.*

The proof of Beurling uses an entropic argument. This approach is quite natural in many respects (cf. [11] for instance) when handling this class of processes. Here, however, our present motivation comes from Mechanics and we shall not elaborate this Statistical Mechanics connection.

The above construction provides the solution of a stochastic boundary value problem, quite distinct from the Cauchy kind of problems originally inspired by Kolmogorov. Remarkably enough the processes solving such boundary value problems are necessarily invariant under time reversal (in a sense to be specified soon) although, as we will see, some of their partial characterizations reintroduce an "arrow of time".

Examples.

1. *Consider a Brownian W_t, $t \in \mathbb{R}^+$, on the real line, with diffusion coefficient $\hbar$ and initial probability density $\chi > 0$. This is the case $A = V = 0$ in* (3.13)*, i.e $H = -\frac{\hbar^2}{2}\Delta$. The traditional interpretation is that, given $\mu(dx) = \chi(x)\,dx$, $P^\mu(W_t \in dx) = \eta^*(x,t)$, where η^* solves*

$$\begin{cases} -\hbar \frac{\partial \eta^*}{\partial t} = H\eta^* \\ \eta^*(x,0) = \chi(x) \end{cases} \tag{3.19}$$

Now, if we wish to look at W_t as a Bernstein reciprocal process $X(t)$, we should start from the same Hamiltonian H, a bounded time interval, say $I = [0,T]$ and the following boundary probability densities in (3.18)*:*

$P_0(x) = \chi(x)$ and $P_T(z) = \eta^*_\chi(x, T)$, where η^*_χ is the (positive) solution of Eq (3.19). *Of course, the kernel h of* (3.18) *is the Gaussian one:*

$$h(0, x, T, z) = (2\pi\hbar T)^{-1/2} \exp -\frac{1}{2\hbar}\frac{|z-x|^2}{T} \tag{3.20}$$

and the solution of Eq (3.18) *(the "Schrödinger system") on* $[0, T]$ *is trivial:*

$$\{\eta^*_0(x) = \chi(x), \eta_T(z) = 1\} \tag{3.21}$$

With those boundary condition, the solutions of the two (heat) Eqs. (3.10) *are clearly,* $\forall\, t \in [0, T]$

$$\eta^*(x, t) = \eta^*_\chi(x, t), \quad \eta(x, t) = 1.$$

According to (3.15) *the forward and backward drifts reduce therefore to*

$$D_t X = 0,\ D^*_t X = -\hbar\nabla \log \eta^*_\chi(X, t)$$

so that, denoting by dX and d_*X, respectively, the Itô differentials under E_t in (3.15), *$X(t)$ solves both SDE*

$$dX(t) = \hbar^{1/2} dW_t,\ d_*X(t) = -\hbar\nabla \log \eta^*_\chi(X(t), t)\, dt + \hbar^{1/2} d_* W^*_t \tag{3.22}$$

where W^*_t denotes a Wiener process with respect to the filtration $\mathcal{F}_t$, $t \in [0, T]$.

With such choice of boundary probability densities (P_0, P_T) it is hard, of course, to see any dynamical time symmetry. But let us switch them both, for the same kernel h as before. Then $\{\hat{\eta}^*_0(x) = 1, \hat{\eta}_T(x) = \chi(x)\}$ are also solutions of the system (3.18). *Indeed, $\hat{\eta}^*(x, t) = 1$ and $\hat{\eta}(x, t) = \eta^*_\chi(x, T - t)$ solve the pair of heat equations* (3.10), (3.11). *The new associated process $\hat{X}(t)$, $t \in [0, T]$ is such that $D_t\hat{X} = \hbar\nabla \log \hat{\eta}(\hat{X}, t)$ and $D^*_t\hat{X} = 1$ and it is as well defined as the above diffusion $X(\cdot)$.*

Notice that it follows easily from the above argument (or directly from the definitions (3.15) *and* (3.15*)*) that*

$$D_t\hat{X}(t) = -D^*_t X(u + s - t), \quad s \leqslant t \leqslant u. \tag{3.23}$$

This rule deforms the classical time reversal of derivatives into a more subtle one, involving necessarily two filtrations.

The full time symmetry of a Bernstein process (or equivalently of its probability measure) appears more clearly when considering processes not of independent increments:

2. *For the same H as in 1. pick the informal limiting case $P_s = \delta_x$ and $P_u = \delta_z$, corresponding to the solution $\eta^*_s(\cdot) = \delta_x$, $\eta_u(\cdot) = \delta_z$ of Eq* (3.18). *So that $\eta^*(q, t) = h(s, x, q, t)$ and $\eta(q, t) = h(q, t, u, z)$ with h as before. Eqs* (3.15) *provide the two drifts*

$$D_t X = \frac{z - X(t)}{u - t}, \quad D^*_t X = \frac{X(t) - x}{t - s} \tag{3.24}$$

$X(t)$ is called the Brownian Bridge between (s,x) and (u,z). Defining $\hat{X}(t)=X(u+s-t)$, $s\leqslant t\leqslant u$, $\hat{X}(t)$ is another bridge traveling backward from $P_u=\delta_z$ to $P_s=\delta_x$.

3. *The above construction is, in fact, independent of the form of the "Hamiltonian" H. For instance, if one considers the (non-symmetric)*

$$H\eta(k)=U(k)\eta-c\nabla\eta-\frac{1}{2}\Delta\eta-\int_{\mathbb{R}^3}(\eta(k+y)-\eta(k)-y\nabla\eta(k)1_{\{|y|\leqslant 1\}})\nu(dy) \quad (3.25)$$

for $U:\mathbb{R}^3\to\mathbb{R}$ continuous, bounded below, c and $k\in\mathbb{R}^3$, $\nu(dy)$ a Lévy measure on $\mathbb{R}^3\backslash\{0\}$, the resulting processes are well defined. They form an interesting class of time reversible Lévy processes ([12], [13]).

The above examples suggest the following general notion of the time reversibility involved here:

Any Bernstein process $X(t)$, $t\in[s,u]$ constructed as before, given a non necessarily symmetric Hamiltonian H with integral kernel h as in Beurling's Theorem, and any given pair of strictly positive probability densities $P_s(x)$ and $P_u(z)$ is invariant under time reversal in the sense that $\hat{X}(t)=X(u+s-t)$, $t\in[s,u]$ is also a well defined process of the same class, evolving backward from P_u to P_s.

We shall conclude this section by a comparison between Bernstein measures and the usual ("Euclidean") approach in Mathematical Physics. Consider the definition (3.3) of the Markovian joint probability measure M_m. For H self-adjoint and η_s^*,η_u real-valued and bounded it can also be regarded as the L^2 scalar product

$$\langle\eta_s^*|e^{-\frac{1}{\hbar}(u-s)H}\eta_u\rangle$$

and expressed in terms of Wiener measure μ_w:

$$\int\eta_s^*(\omega(s))e^{-\frac{1}{\hbar}\int_s^u V(\omega(\tau))d\tau}\eta_u(\omega(u))d\mu_w(\omega). \quad (3.26)$$

This version of Feynman-Kac formula has been known and used since the sixties [57]. A key difference with our construction is that, to produce Bernstein measures, η_s^* and η_u have first to be found as (positive) solutions of the system (3.18), given initial and final probability densities $P_s(dx)$ and $P_u(dz)$. Only then the reversibility of Bernstein measures and therefore their dynamical meeting will show up. Indeed, in Eq (3.26) the Wiener measure does not carry any specific dynamical meaning, in contrast with Bernstein measure, involving two drifts. This is already seen clearly in the Feynman-like Formula (3.1) and, of course, in the equations of motion that we are going to obtain in the next Section.

4 Stochastic Dynamics and Symmetries

There are two approaches to classical dynamics (or the classical calculus of variations) the Lagrangian and the Hamiltonian one. Feynman's method suggests to start from the former one. The classical system with Lagrangian (2.11) will be our guide. We already know that, in relation with the $\mathcal{P}_t$ filtration, the classical $\dot{\omega}$ should become $D_t X$ (apart from its imaginary unit factor) under our stochastic deformation. So, from the transformations used in (3.12), (3.13) the Lagrangian should be proportional to

$$\mathcal{L}(X, D_t X) = \frac{1}{2}|D_t X|^2 + V(X) + A \cdot D_t X + \frac{\hbar}{2}\nabla \cdot A. \tag{4.1}$$

The last term of (4.1) requires some explanation , useful also for later purposes.

In our time-symmetric context, 3 definitions stochastic of integrals are available [14] for the stochastic deformation of a classical expression of the form

$$\int_t^u A(\omega) d\omega(\tau) \tag{4.2}$$

in the Action functional.

With respect to the increasing filtration $\mathcal{P}_\tau$, for A $\mathcal{P}_\tau$-adapted, with $E\int_t^u |A|^2(X(\tau))d\tau < \infty$,

$$\int_t^u A(X)\, dX(\tau) = \underset{\substack{\max|\tau_j - \tau_{j-1}| \to 0 \\ 1 \leqslant j \leqslant N}}{\text{l.i.p}} \sum_{j=1}^N A(X(\tau_{j-1}))(X(\tau_j) - X(\tau_{j-1}))$$

where l.i.p means limit in probability. Then, using (3.15),

$$E\int_t^u A\, dX(\tau) = E\int_t^u A\, D_\tau X\, d\tau. \tag{4.3}$$

With respect to the decreasing filtration $\mathcal{F}_\tau$ and introducing the notation $d_* X(\tau)$ for the backward differential involved in (3.15*),

$$\int_t^u A(X) d_* X(\tau) = \underset{\substack{\max|\tau_j - \tau_{j-1}| \to 0 \\ 1 \leqslant j \leqslant N}}{\text{l.i.p}} \sum_{j=1}^N A(X(\tau_j))(X(\tau_j) - X(\tau_{j-1}))$$

and therefore, under the $\mathcal{F}_\tau$ counterpart of regularity conditions,

$$E\int_t^u A d_* X(\tau) = E\int_t^u A D_\tau^* X\, d\tau. \tag{4.4}$$

The third one is due to Stratonovich: (when, in addition, $E\int_t^u |\nabla A|^2(X(\tau))d\tau < \infty$)

$$\int_t^u A \circ dX(\tau) = \underset{\substack{\max|\tau_j - \tau_{j-1}| \to 0 \\ 1 \leqslant j \leqslant N}}{\text{l.i.p}} \sum_{j=1}^N \frac{1}{2}[A(X(\tau_{j-1})) +$$

$$A(X(\tau_j))](X(\tau_j) - X(\tau_{j-1}))$$

and

$$E\int_t^u A\circ dX(\tau) = E\int_t^u A\cdot\frac{1}{2}(D_\tau X + D_\tau^* X)d\tau. \tag{4.5}$$

It follows from the usual (forward) Itô calculus [15] that

$$A\circ dX(\tau) = A\cdot dX(\tau) + \frac{\hbar}{2}\nabla\cdot A\,d\tau \tag{4.6}$$

Together with (4.2) this justifies the following definition of the Action functional relevant for the dynamics of our system with Lagrangian (4.1):

$$\begin{aligned} J[X] = E_{x,t}&\int_t^u \mathcal{L}(X(\tau), D_\tau X(\tau))\,d\tau \\ &= E_{x,t}\left\{\int_t^u \frac{1}{2}|D_\tau X(\tau)|^2 + V(X(\tau))\,d\tau + \int_t^u A\circ dX(\tau)\right\} \end{aligned} \tag{4.7}$$

where $E_{x,t}$ denotes the conditional expectation given $X(t) = x$, $t < u$.

What about the geometrical meaning of J ? Let us define, for any $\mathcal{L}$, the (forward) Momentum as classically by

$$P = \frac{\partial L}{\partial D_t X}(X, D_t X) \tag{4.8}$$

assuming that Eq (4.8) is solvable in $D_t X = \phi(P, X)$. This is the case for the system (4.1) and

$$D_t X = P - A. \tag{4.9}$$

Then, if $\tilde{\omega}_{PC}$ is the section, on the (X, τ) submanifold, of the Poincaré-Cartan 1-form, the Action can be written as

$$J[X] = E_{t,x}\int_t^u \tilde{\omega}_{PC} = E_{t,x}\int_t^u P\circ dX(\tau) - h(X(\tau), \tau)\,d\tau \tag{4.10}$$

where h is now a scalar field, called the energy function.

For (4.1) it reduces to

$$h(X,\tau) = \frac{1}{2}|D_t X|^2 + \frac{\hbar}{2}\nabla\cdot(D_t X) - V(X) \tag{4.11}$$

We shall come back later on the meaning of h.

Assume now, following Feynman, that we are only given the Lagrangian $\mathcal{L}$ of (4.7). For the sake of generality we shall add a smooth (final) boundary condition: $J[X] = E_{x,t}\left\{\int_t^u \mathcal{L}\,d\tau + S_u(X(u))\right\}$.

When not referring to the $\mathcal{L}$ of (4.1) we shall assume from now on that $\mathcal{L}, S_u$ are continuous and that , for some constants $c, k, |\mathcal{L}| \leqslant c(1+|X|^k) + |DX|^k, |S_u(X)| \leqslant c(1+|X|^k)$.

How can we characterize critical points of J ?

We shall pick, as domain $\mathcal{D}_J$ of J the set of diffusions X absolutely continuous with respect to the Wiener measure $P_W^\hbar$ with diffusion matrix $\hbar I$ (I the 3×3 Identity matrix), and arbitrary Borel measurable drift B of $\mathbb{R}^3\times[s,u]$ into $\mathbb{R}^3$.

Def. *Such a process X is extremal for J if*

$$E_{xt}[\nabla J[X](\delta X)] = E_{xt}\left[\lim_{\varepsilon\to 0}\frac{J[X+\varepsilon\delta X]-J[X]}{\varepsilon}\right] = 0, \tag{4.12}$$

for any variation δX in the Cameron-Martin space preserving the absolute continuity under the shift $X+\varepsilon\delta X$ (cf. [17], [28], [32]).

Then

$$0 = E_{xt}[\nabla J[X](\delta X)] = E_{xt}\int_t^u \left(\frac{\partial\mathcal{L}}{\partial X}\delta X + \frac{\partial\mathcal{L}}{\partial D_\tau X}D_\tau\delta X\right)d\tau + E_{xt}[\nabla S_u(X(u))\delta X(u)].$$

In the second term, notice that we preserve the notation D_τ, this time for the (extended) infinitesimal generator of $X \in \mathcal{D}_J$,

$$D_\tau = \frac{\partial}{\partial\tau} + B\nabla + \frac{\hbar}{2}\Delta. \tag{4.13}$$

When applied, in particular, to X itself, $D_\tau X = B(X,\tau)$, as in (3.15). Under $E_{xt}[\dots]$, D_τ satisfies, by Itô formula, an integration by parts formula. Since $\delta X(\tau)$ is of bounded variation,

$$0 = E_{xt}\int_t^u \left(\frac{\partial\mathcal{L}}{\partial X} - D_\tau\left(\frac{\partial\mathcal{L}}{\partial D_\tau X}\right)\right)\delta X(\tau)\,d\tau + E_{xt}\left[\left(\frac{\partial\mathcal{L}}{\partial D_\tau X} + \nabla S_u\right)(X(u))\delta X(u)\right]$$

so we obtain the

Stochastic Euler-Lagrange Theorem

The critical processes of the Action functional $J[X]$ with boundary condition S_u solve the almost sure Stochastic Euler-Lagrange equation:

$$\text{(SEL)}\qquad \begin{cases} D_\tau\left(\frac{\partial\mathcal{L}}{\partial D_\tau X}\right) - \frac{\partial\mathcal{L}}{\partial X} = 0, & t<\tau<u \\ \frac{\partial\mathcal{L}}{\partial D_\tau X}(X(u), D_\tau X(u)) = -\nabla S_u(X(u)), & X(t) = x. \end{cases} \tag{4.14}$$

For instance, when $\mathcal{L}$ is as in (4.1), (SEL) reduces to

$$\begin{cases} D_\tau D_\tau X(\tau) = \nabla V(X(\tau)) + D_\tau X(\tau)\wedge \operatorname{rot} A + \frac{\hbar}{2}\operatorname{rot}\operatorname{rot} A \\ D_\tau X(u) + A(X(u)) = -\nabla S_u(X(u)). \end{cases} \tag{4.15}$$

One should notice the $\hbar$ dependent term, in addition to the Lorentz force, on the r.h.s of Eq (4.15). It is natural to interpret (SEL) as the ($\mathcal{P}_t$) stochastic deformation of its classical counterpart.

Regarding the Hamiltonian approach of the dynamics, one can define, as classically, using the hypothesis after (4.8),

$$\mathcal{H}(X,P) = P\phi(P,X) - \mathcal{L}(X,\phi(P,X)). \tag{4.16}$$

The deformed version of the Hamiltonian differential equations becomes the almost sure

Stochastic Hamiltonian equations:

$$\begin{cases} D_\tau X = \frac{\partial \mathcal{H}}{\partial P} \\ D_\tau P = -\frac{\partial \mathcal{H}}{\partial X}. \end{cases} \tag{4.17}$$

For instance when $\mathcal{L}$ is the one of (4.1), $\mathcal{H}$ is

$$\mathcal{H}(X,P) = \frac{1}{2}|P - A(X)|^2 - V(X) - \frac{\hbar}{2}\nabla \cdot A \tag{4.18}$$

and the Hamiltonian equations reduce to

$$\text{(SHE)} \quad \begin{cases} D_\tau X = P - A \\ D_\tau P = D_\tau X \nabla . A + D_\tau X \wedge \operatorname{rot} A + \frac{\hbar}{2} \operatorname{rot}\operatorname{rot} A + \frac{\hbar}{2}\Delta A + \nabla V. \end{cases} \tag{4.19}$$

The first equation reduces to another version of (3.15). Its substitution in the second one is consistent with the Euler-Lagrange equation (4.15).

Almost sure equations like SEL (4.14) and SHE (4.17) may seem to be odd but, in fact, they were (rather deeply) hidden behind classical results of Stochastic Control Theory found around the eighties [18, 19, 20] from a very different viewpoint. In this context, it is the classical theory of Characteristics which was stochastically deformed.We shall give here only a hint about this connection, particularly natural in our Path Integral perspective, for the simplest case above where $A = 0$.

Consider, then, the Action functional $J[X]$, where the drift B of the critical process (with fixed diffusion matrix) we are looking for, called the "control" is just supposed to be $\mathcal{P}_t$-measurable and such that $E\int_t^u |B(\tau)|^n d\tau < \infty$, $n \in \mathbb{N}$. Notice that this include, now, non Markovian processes, a natural hypothesis in our Bernsteinian perspective.

Consider any scalar field S in the domain of the infinitesimal generator $A^{B(\tau)} = \frac{\partial}{\partial \tau} + B(\tau)\nabla + \frac{\hbar}{2}\Delta$ of such a process $X(\tau)$, such that

$$E_{xt}\int_t^u |A^{B(\tau)}S(X_\tau,\tau)|\, d\tau < \infty$$

and Dynkin formula holds:

$$E_{xt}\int_t^u A^{B(\tau)}S(X_\tau,\tau)\, d\tau = E_{xt}S(X(u),u) - S(x,t).$$

So, for any $X(\tau)$ in the above class, and $\mathcal{L}(X, D_\tau X) = \frac{1}{2}|D_\tau X|^2 + V(X)$

$$J[X] = E_{xt} \int_t^u \mathcal{L}(X(\tau), B(\tau))\, d\tau + E_{xt} S_u(X(u))$$

Theorem ([18])**.** *Let $S(x,t)$be a classical solution of the deformed Hamilton-Jacobi equation (known as "Hamilton-Jacobi Bellmann", or HJB)*

$$\begin{cases} \frac{\partial S}{\partial t} - \frac{1}{2}|\nabla S|^2 + \frac{\hbar}{2}\Delta S + V = 0 \\ S(x,u) = S_u(x). \end{cases} \tag{4.20}$$

Then $S(x,t) \leqslant J[X]$, $\forall X$ in the above class. Moreover, for

$$B(t) = B(x,t) = -\nabla S(x,t) \tag{4.21}$$

this inequality becomes an equality.

To understand the relation with our construction (for $A = 0$), define

$$\eta(x,t) = e^{-\frac{1}{\hbar}S(x,t)} \tag{4.22}$$

then HJB reduces to Eq (3.11) with a positive final condition,

$$\hbar\frac{\partial \eta}{\partial t} = -\frac{\hbar^2}{2}\Delta\eta + V, \quad \eta(x,u) = e^{-\frac{1}{\hbar}S_u(x)} \tag{4.23}$$

and the critical, indeed minimal, diffusion of $J[X]$ is our Markovian (forward) Bernstein process of drift (3.15) $D_t X = B(X,t) = \hbar\nabla\log\eta(x,t)$.

It is in fact possible to prove in a purely geometric way that the critical points of $J[X]$ are minimal [21].

The hypothesis, in the last Theorem, that S is a classical solution of HJB is, of course, much too restrictive. The construction holds under weaker conditions (cf. [22]). Regarding HJB, the appropriate notion of weak solution is the one of viscosity solution (cf. [18]).

When $\hbar = 0$, the existence of a classical, global solution of Hamilton-Jacobi equation is a condition of Complete Integrability of the system and, if available, the gradient of this equation coincides with the second Hamiltonian equation, $\frac{dp}{dt} = -\frac{\partial\mathcal{H}}{\partial x} = -\nabla V(x)$, when $\mathcal{H}(x,p) = \frac{1}{2}p^2 + V(x)$.

The stochastic deformation of this integrability condition is that, the gradient of (4.20), using $D_t X = \hbar\nabla\log\eta_t$ (i.e (3.15) for $A = 0$) reduces to, almost surely, $D_t D_t X(\tau) = \nabla V(X(\tau))$ namely (4.15) or, equivalently, the second deformed Hamiltonian equation (4.17) in the same special case (cf. [23]).

Coming back to the deformed Hamiltonian $\mathcal{H}$ of (4.18) ones observes that it does not coincides with the energy function h (4.11) of the Poincaré-Cartan 1-form. So, what is the meaning, if any, of $h(X(\tau),\tau)$?

A key observation is that $D_\tau h(X(\tau),\tau) = 0$, for the critical, i.e dynamical, diffusion $X(t)$, with generator

$$D_\tau = \frac{\partial}{\partial\tau} + \hbar\nabla\log\eta\nabla + \frac{\hbar}{2}\Delta. \tag{4.24}$$

In other words, $h(X(\tau),\tau)$ is a $\mathcal{P}_\tau$-martingale, a natural deformation of the classical notion of constant of the motion or first integral. Is it accidental ?

The answer is negative. Before mentioning the Theorem showing that all first integrals of our stochastic dynamical system are indeed martingales, let us stress that Feynman could not find such a result in his informal time discretized account of quantum dynamics.

As it is clear from (3.15) or (4.19), the processes critical for J are entirely built from (positive) solutions of a parabolic equation (3.11). We shall stick to the simple case $A=0$, i.e Eq (4.23).

Consider this equation, written now as

$$\hat{H}\eta = \left(\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2}\Delta - V\right)\eta = 0 \tag{4.25}$$

and a generator of the form

$$N = T(t)\frac{\partial}{\partial t} + Q_i(x,t)\frac{\partial}{\partial x_i} - \frac{1}{\hbar}\phi(x,t) \tag{4.26}$$

where the summation convention is used, for $1 \leqslant i \leqslant 3$ and the coefficient T, Q, ϕ are analytic in x and t. Then N is infinitesimal generator of a symmetry Lie Algebra $\mathcal{A}_s$ of Eq (4.25) if

$$\hat{H}\eta = 0 \Rightarrow \hat{H}N\eta = 0. \tag{4.27}$$

Lie proved long ago that this is the case if (T, Q, ϕ) solve the following "determining equations":

$$\begin{cases} \frac{dT}{dt} = 2\frac{\partial Q_i}{\partial x_i} & \frac{\partial Q_i}{\partial x_j} + \frac{\partial Q_j}{\partial x_i} = 0, \quad i = 1,2,3, \quad j \neq i \\ \frac{\partial Q_i}{\partial t} = \frac{\partial \varphi}{\partial x^i} & \\ \frac{\partial \phi}{\partial t} + \frac{\hbar}{2}\Delta\phi = \frac{dT}{dt}V + Q_i\frac{\partial V}{\partial x_i} + T\frac{\partial V}{\partial t} & \end{cases} \tag{4.28}$$

The (local) Lie symmetry group of Eq (4.23) results from product of exponentials of generators N.

When $x \in \mathbb{R}^3$ and $V = A = 0$, for instance, the symmetry group is 13 dimensional.

Now consider a classical Lagrangian $L(\omega, \dot{\omega}, t)$ associated with the deformed system whose Eq (3.11) reduces to the form (4.23), it is $L(\omega, \dot{\omega}, t) = \frac{1}{2}|\dot{\omega}|^2 - V(\omega, t)$ (The scalar potential V can be explicitly time-dependent without altering our arguments above). One version of the classical Theorem of Noether is the following:

$$v = T(t)\frac{\partial}{\partial t} + Q_i(\omega,t)\frac{\partial}{\partial \omega_i} + \left(\frac{dQ_i}{dt} - \dot{\omega}_i\frac{dT}{dt}\right)\frac{\partial}{\partial \dot{\omega}_i}$$

is called a divergence symmetry of L if there is a scalar field $\phi = \phi(\omega, t)$ such that

$$v(L) + L\frac{dT}{dt} = \frac{d\phi}{dt}. \tag{4.29}$$

When L admits such a divergence symmetry, then along each extremal of the classical action S_L,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\omega}_i}Q_i - \left(\frac{\partial L}{\partial \dot{\omega}_i}\dot{\omega}_i - L\right)T - \phi\right] = \qquad (4.30)$$
$$\frac{d}{dt}\left[\dot{\omega}_i Q_i - \left(\frac{1}{2}|\dot{\omega}|^2 + V(\omega)\right)T - \phi\right] = 0$$

for L as before. Equivalently, the expression between brackets is a constant of the motion of the system.

For instance, any conservative system (i.e with V time independent) admits $v = \frac{\partial}{\partial t}$ i.e $T = 1$, $Q = 0$, $\phi = 0$ and (4.30) reduces to the energy conservation.

The stochastic deformation of this Theorem, for the same class of systems is the

Stochastic Noether's Theorem [25, 26]

If the Lagrangian $\mathcal{L}(X, D_t X, t) = \frac{1}{2}|D_t X|^2 + V(X, t)$ admits a divergence symmetry of the form

$$T(t)\frac{\partial}{\partial t}\mathcal{L} + Q_i \frac{\partial}{\partial X_i}\mathcal{L} + \left(D_t Q_i - D_t X_i \frac{dT}{dt}\right)\frac{\partial \mathcal{L}}{\partial D_t X_i} + \mathcal{L}\frac{dT}{dt} = D_t \phi \qquad (4.31)$$

for any analytic T, Q, ϕ solving the "Determining equations" (4.28) then along any Bernstein diffusion $X(\cdot)$ critical for the action $J[X]$, almost surely

$$D_t(D_t X_i Q_i - hT - \phi)(X(t), t) = 0 \qquad (4.32)$$

where h is the energy function (4.11).

For instance, if V is time independent, $\mathcal{L}$ admits $T = 1$, $Q = \phi = 0$ and we recover our energy martingale.

The stochastic Noether Theorem is a theorem of structure, here, without which our deformation would be dynamically meaningless.

But let us observe that, from the start of this Section 4 only one filtration, the increasing one $\mathcal{P}_t, s < t < u$, has been used. As a result of this, the stochastic Euler-Lagrange equation (4.15), for instance, is certainly not invariant under time reversal in the sense defined in Section 3. So (4.15) cannot be the full dynamical characterization of processes respecting intrinsically this invariance.

But it is easy to find the solution of the puzzle. On the time interval $[s, u]$, the time reversal of the Action functional (4.7) is

$$E^{xt}\int_s^t \left(\frac{1}{2}|D_\tau^* X(\tau)|^2 + V(X(\tau))\right)d\tau + \int_s^t A \circ dX(\tau) \qquad (4.7^*)$$

where we have adopted the notation E^{xt} for a conditional expectation given the future configuration $X(t) = x$, $s < t$, and used the rule (3.23).

Calling $J^*[X]$ the functional (4.7*), we can look for its critical points among diffusions with fixed diffusion matrix, solving a backward (i.e $\mathcal{F}_t$) stochastic differential equation whose (backward) drift is unknown. Notice that, now, the Stratonovich integral in (4.7*) must be interpreted using the backward version of the relation (4.6) namely, according to Itô [14],

$$A \circ dX(\tau) = A\, d_* X(\tau) - \frac{\hbar}{2}\nabla \cdot A\, d\tau. \tag{4.6*}$$

This means that, with respect to $\mathcal{F}_\tau$, the Lagrangian of J^* is now represented by

$$\mathcal{L}^*(X(\tau), D_\tau^* X(\tau)) = \frac{1}{2}|D_\tau^* X(\tau)|^2 + V(X(\tau)) + \\ A(X(\tau)) D_\tau^* X(\tau) - \frac{\hbar}{2}\nabla \cdot A. \tag{4.1*}$$

Then, one checks in the same way as before, that the critical point of $J^*[X]$, in fact a minimum, is unique and that its backward drift is given in term of a positive solution of Eq (3.10) by the expression (3.15*):

$$D_\tau^* X = -\hbar \nabla \log \eta_t^*(X) - A(X).$$

The (backward) stochastic Euler-Lagrange equation it solves (ignoring boundary condition at $t > s$), is

$$D_\tau^* D_\tau^* X(\tau) = \nabla V(X(\tau)) + D_\tau^* X(\tau) \wedge \operatorname{rot} A - \frac{\hbar}{2} \operatorname{rot} \operatorname{rot} A. \tag{4.15*}$$

As a matter of fact, such a calculation is not even necessary. Indeed, as said before (cf (3.23)), $D_\tau \to -D_\tau^*$ and $A \to -A$ under time reversal. This means that both (3.15*) and (4.15*) are time reversed versions of their forward counterparts (3.15) and (4.15).

Since, in particular, (4.15) and (4.15*) provide different informations, associated with $\mathcal{P}_t$ and $\mathcal{F}_t$ respectively, about the same Bernstein diffusion, the complete, time-symmetric, dynamical equation of $X(\tau)$, $s \leqslant \tau \leqslant u$, is

$$\frac{1}{2}(D_\tau D_\tau X(\tau) + D_\tau^* D_\tau^* X(\tau)) = \\ \frac{1}{2}(D_\tau X(\tau) + D_\tau^* X(\tau)) \wedge \operatorname{rot} A(X(\tau)) + \nabla V(X(\tau)). \tag{4.33}$$

Let us stress that, now, this stochastic deformation of the classical Euler-Lagrange equation is an electromagnetic field, involving the deformed Lorentz force on the right hand side, is indeed invariant under time reversal, as it should. Using the relations , for $f \in C^2$,

$$\frac{d}{d\tau} E[f(X(\tau))] = E[D_\tau f(X(\tau))] = E[D_\tau^* f(X(\tau))]$$

following, for instance, from Dynkin formula (cf. also [53]), we can get closer to Feynman dynamical law (2.18) in taking the absolute expectation of (4.33):

$$\frac{d^2}{d\tau^2} E[X(\tau)] = E\left[\frac{1}{2}(D_\tau X(\tau) + D_\tau^* X(\tau)) \wedge \operatorname{rot} A(X(\tau)) + \nabla V(X(\tau))\right]. \tag{4.34}$$

Notice also that what plays the role of the time derivative $\dot{\omega}(\tau)$ in Feynman's law of motion (2.18) is now the average of the two drifts. whose behaviour under time reversal is the expected one, in contrast with each of them taken separately. Using the same method, it is easy to find the backward version of our Stochastic Noether Theorem, for instance, producing backward martingales of the system.

5 Computational and Geometric content

Let us start with some consequences of our Noether Theorem.

Although it is clear, for a member of the community of Geometric Mechanics (in particular) that Noether is the key to start a serious study of the dynamics, we shall try to show its interest also for the theory of stochastic processes itself.

Consider diffusions on the line, for simplicity, with $A = V = 0$ in Eq (3.13) or, equivalently a Lagrangian (4.1) reduced to

$$\mathcal{L}(X, D_t X) = \frac{1}{2}|D_t X|^2. \tag{5.1}$$

One verifies that $T = 2t$, $Q = x$, $\phi = 0$ solves the one dimensional version of the determining equations (4.28) for $V = 0$. So $N = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$ generates a one-parameter symmetry group:

$$(e^{\alpha N})(t, x, \eta) = (e^{\alpha}x, e^{2\alpha}t, \eta) = (t_\alpha, x_\alpha, \eta_\alpha). \tag{5.2}$$

This implies that if $\eta = \eta(x,t) > 0$ solves the free heat equation (4.23) for $V = 0$, i.e $\hbar\frac{\partial \eta}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2 \eta}{\partial x^2}$, so does

$$\eta_\alpha(x,t) = e^{-\alpha N}\eta = \eta(e^{-\alpha x}x, e^{-2\alpha}t). \tag{5.3}$$

Then define $h_\alpha(x,t) = \frac{\eta_\alpha}{\eta}(x,t)$. If $X(t)$ solves the (forward) SDE with drift (3.15)

$$dX(t) = \hbar\nabla \log \eta(X(t), t)dt + \hbar^{1/2}dW_t \tag{5.4}$$

and, therefore, the associated a.s Euler-Lagrange Eq (4.15),

$$D_t D_t X(t) = 0 \tag{5.5}$$

one checks that $h_\alpha(X(t), t)$ is a positive martingale:

$$D_t h_\alpha(X(t), t) = 0. \tag{5.6}$$

Eq (5.3) corresponds to a Scaling transformation of the starting process $X(t)$, namely

$$X^\alpha(t) = x_\alpha(X(t_\alpha), t(t_\alpha)) = e^\alpha X(e^{-2\alpha}t) \tag{5.7}$$

where $t(t_\alpha)$ denotes the inversion of the time parameter transformation in (5.2).

The drift of $X^\alpha(t)$ results of a Doob's h transform of $X(t)$ whose martingale is h_α. Denoting by B^α and B the associated drifts, we find [27], using the definition of h_α,

$$B^\alpha(x,t) = B(x,t) + \nabla \log h_\alpha(x,t) = \hbar \nabla \log \eta_\alpha(x,t). \tag{5.8}$$

In other words, all diffusions $X(t)$ solving Eq (5.5) enjoy the scaling transformation symmetry, interpreted here dynamically. The standard Wiener $X(t) = W_t$, whose $\eta_\alpha(x,t) = 1$, therefore $h_\alpha = 1$ and $B_\alpha = 0$, is only one of them: with $\varepsilon = e^{-2\alpha}$, we recover its usual scaling law $W^\varepsilon(t) = \varepsilon^{-1/2} W(\varepsilon t)$. This is useful, for instance, in the computation of first passage times of any diffusions solving Eq (5.5). A large collection of parabolic equations (with first order and potential terms) is, in fact, equivalent to the above free heat equation, in terms of symmetries [27], so that our argument is more general.

Almost sure dynamical equations like (4.14) or (4.17), together with our stochastic Noether Theorem provide new (geometrical) relations between familiar stochastic processes, impossible to anticipate without them. But, what may seem more surprising, they provide as well new informations about Quantum Mechanics in Hilbert space. For instance, the solutions $(T, Q, \phi) = (0, t, x)$ of the (one dimensional) determining equations (4.28) for $V = 0$, applied to the standard Wiener $X(t) = W_t$, show that the family of Brownian martingales correspond to a family of (time dependent) constant observables in $L^2(\mathbb{R})$ of the free particle. Cf. [23]. Although elementary, this observation had not been done before. Even better, a naive analytic continuation in time from the symmetries of the parabolic equation to the one of Schrödinger equation provides a Quantum Theorem of Noether richer that the one mentioned in Textbooks, even in elementary cases [29].

The geometrical content of our stochastic deformation is worth an investigation in itself. Consider, for instance, the deformation of the classical method of Characteristics. One of the most elegant representation of Hamilton-Jacobi (HJ) equation is due to E. Cartan and makes use of the following Ideal of differential forms [30] (We denote here by x_i what was ω_i, in Eq (4.9), to avoid confusions with differential forms):

$$I_{HJ} = \begin{cases} \omega = p^i dx_i - E\, d\tau + dS \equiv \omega_{PC} + dS \\ \Omega = dp^i dx_i - dE\, d\tau \\ \beta = \left(E - \frac{1}{2}|p|^2 - V(x)\right) dx_i\, d\tau \end{cases} \tag{5.9}$$

on a 9-dimensional space $(x_i, p^i, S, \tau, E) \quad i = 1, 2, 3$.

This is Cartan's representation for the elementary systems underlying Eq (4.30). To recover HJ equation itself, consider the $\mathbb{R}^4$ "solution submanifold" where the a priori independent variable S becomes a function $S(x,t)$ (This is called "Sectioning" and denoted by $\sim$) and then pullback all differential forms to zero ("Annuling"). Then $\tilde{\omega} = 0$ implies that $p = -\nabla S$ and $E = \partial_\tau S$. The condition $\tilde{\Omega} = 0$ is equivalent to the existence of a Lagrangian manifold. Finally, $\tilde{\beta} = 0$ is equivalent to the classical Hamilton-Jacobi itself. This representation shows that Hamilton-Jacobi framework is a Contact Geometry, defined on an odd dimensional space, here $\mathbb{R}^9$; ω is, in fact, a Contact form [27].

A symmetry generator of HJ equation should, therefore, became now a "contact Hamiltonian" vector field (sometimes called "Isovector")

$$N = N^\tau \frac{\partial}{\partial \tau} + N_i^x \frac{\partial}{\partial x_i} + N^S \frac{\partial}{\partial S} + N^E \frac{\partial}{\partial E} + N_i^p \frac{\partial}{\partial P^i} \tag{5.10}$$

whose coefficients must be chosen so that, denoting by $\mathcal{L}_N$ the Lie derivative, or variation, along N:

$$\mathcal{L}_N(I_{HJ}) \subseteq I_{HJ}. \tag{5.11}$$

The stochastic deformation of I_{HJ} is the one providing Hamilton-Jacobi Bellman equation (4.20):

$$I_{HJB} \begin{cases} \omega = P^i dX_i + E\, d\tau + dS \equiv \omega_{pc} + dS \\ \Omega = dP^i dX_i + dE\, d\tau \\ \beta = \left(E + \frac{1}{2}|P|^2 - V\right) dX_i d\tau + \frac{\hbar}{2} dP^i d\tau. \end{cases} \tag{5.12}$$

The only deformation term, in β, is responsible for the deformation term $\frac{\hbar}{2}\Delta S$ in Eq (4.20). Sectioning and annulling as before we find the Lagrangian integrability conditions:

$$\tilde{\omega} = 0 \Rightarrow P = -\nabla S, E = -\partial_\tau S. \tag{5.13}$$

The definition of Symmetries for HJB is the same as classically, i.e Eq (5.11), for the deformed ideal (5.12), and the calculation of the coefficients $N^\bullet$ of Eq (5.10) is a tiring exercise (cf. [27]). But it is quite rewarding:

Theorem ([27])**.** *Along any N-variation as before, I_{HJB} and the Lagrangian $\mathcal{L}$ satisfy the following invariance conditions*

$$\begin{aligned} &\textit{(1)} \quad \mathcal{L}_N(\omega_{PC}) = -dN^S \\ &\textit{(2)} \quad \mathcal{L}_N(\Omega) = 0 \\ &\textit{(3)} \quad \mathcal{L}_N(\mathcal{L}) + \mathcal{L}\frac{dN^\tau}{d\tau} = -D_\tau N^S. \end{aligned} \tag{5.14}$$

This Theorem seems purely algebraic but encodes a lot of informations about our stochastic deformation, resulting from the substitution of smooth classical paths $\tau \mapsto \omega(\tau)$ by Bernstein diffusion sample paths $\tau \mapsto X(\tau)$. Eq (1) means that Poincaré-Cartan 1-form is invariant up to a phase coefficient N^S. Eq (2) shows the invariance of the Symplectic form over the time-dependent or extended phase space (cotangent bundle). Eq (3) expresses the transformation of the integrant of the Action functional (4.7) under the contact Hamiltonian N on the extended phase space. It should be regarded as the deformation of the classical expression (4.29).

The proof of the Theorem shows that it is, in fact, sufficient to consider symmetry contact Hamiltonian of the form $N(\tau, x, S, E, P) = N^x(x,\tau)P +$

$N^\tau(x,\tau)E + N^S(x,\tau)$, so that $N^x = Q$, $N^\tau = T$ and $N^S = -\phi$ in the notations of the stochastic Noether Theorem, where T, Q and ϕ solve its Determining Equations. After sectioning on the solution submanifold (x,τ) where, according to (3.9), we have a probabilistic interpretation by plugging $x = X(\tau)$, $\tilde{P}$ and $\tilde{E}$ become respectively our drift and energy random variables.

The whole construction summarized before is preserved if the diffusions $X(\tau)$ lives on a (smooth, connected, complete) n-dimensional Riemannian manifold with metric gij. The simplest Hamiltonian in Eq (4.25) becomes

$$H = -\frac{\hbar^2}{2}\nabla^j\nabla_j + V(x) \tag{5.15}$$

where ∇_j denotes the covariant derivative with respect to the Lévi-Civita connection Γ^i_{jk} and the associated Hamilton-Jacobi-Bellman equation (4.20) becomes

$$\frac{\partial S}{\partial t} - \frac{1}{2}\|\nabla S\|^2 + \frac{\hbar}{2}\nabla^i\nabla_i S + V = 0. \tag{5.16}$$

Two new geometric aspects deserve to be mentioned. The first one is that (as stressed by K. Itô [31]) an additional term shows up in the drift (3.15)

$$D_t X^i = \hbar\nabla \log \eta_t(X) - \frac{\hbar}{2}\Gamma^i_{jk}g^{jk} \tag{5.17}$$

for $E_t dX^i dX^j = \hbar g^{ij} dt$.

On such a Riemannian manifold, an almost sure Euler-Lagrange equation like (4.15) (when $A = 0$) requires to define the time derivative of a vector field. Even in the classical, deterministic case, a notion of parallel transport is needed to do that.According to Itô [31], the stochastic deformation of the Lévi-Civita transport of the vector field Y would transform Eq (4.13) into

$$D_\tau Y^i = \frac{\partial Y^i}{\partial \tau} + \hbar\nabla^k \log \eta_t \nabla_k Y^i + \frac{\hbar}{2}\nabla^k\nabla_k Y^i. \tag{5.18}$$

But Itô also indicated other possible choices. The one needed for our purpose has been called "Damped parallel transport" in Stochastic Analysis (cf. [32]), and replaces the Laplace-Beltrami term of (5.18) by

$$(\Delta Y)^i = \nabla^k\nabla_k Y^i + R^i_k Y^k \tag{5.19}$$

where R^j_k denotes the Ricci tensor. Then

$$D_\tau Y^i = \frac{\partial Y^i}{\partial \tau} + \hbar\nabla^k \log \eta_t \nabla_k Y^i + \frac{\hbar}{2}(\Delta Y)^i. \tag{5.20}$$

The point is that to preserve for (5.16) the integrability condition according to which the gradient of (4.20) coincides with the Euler-Lagrange equation we need that $[\Delta, \nabla^i]S = 0$, a property not satisfied by $\nabla^k\nabla_k$. Then, with (5.20), the dynamical law and the Noether Theorem keep the same form as above [61].

6 Conclusions

In 1985-6, I named after Bernstein the reciprocal property suggested by him in the context of the 1931 observation (forgotten until then) of Schrödinger [6]. I was, in fact, so impressed by Bernstein interpretation of such processes as stochastic counterparts of critical trajectories of Hamilton's principle that I used as well the term "variational processes" [1]. Of course, the local Markov property reappeared during the seventies in relation with Statistical and Quantum Physics. But Schrödinger's observation and Bernstein's probabilistic suggestion were an extraordinary anticipation back then, of Feynman Path Integral approach, among other ideas.

There are more than one way to interpret Schrödinger original observation, expressed originally in a statistical mechanics perspective, i.e in entropic terms. Aware of the fierce fights regarding the physical interpretation of the new born quantum theory, Schrödinger was looking for a "classical" analogy where probabilities would play a similar but less debated role. In the seventies, B. Jamison [8] elaborated some aspects of the construction suggested by Bernstein, but missed the looked for relation with quantum theory (he was, for instance, using only the increasing $\mathcal{P}_t$ filtration and would not start his construction from a given Hamiltonian H). In any case, since the mid-eighties, Bernstein processes have reappeared in a multitude of contexts, pure and applied, and under different names.They have been called "Schrödinger processes" (following Jamison) by H. Föllmer [33] in 1988 and studied on their own in the entropic perspective [34], [35], [49] and [50]. A promising link has been established with Optimal transport in recent years. An excellent review of this connection can be found in [36]. In this context, the natural approach is indeed the one of statistical mechanics and the original variational problem in [6] is called "Schrödinger 's problem", not to be confused with Eq (3.18) referred to as the "Schrödinger system". "Schrödinger bridges" is also a terminology used for those processes. They can really be regarded as a generalization of usual bridges where, instead of two boundary Dirac distributions, we are now given two regular (nodeless) probability distributions. In recent studies on Wiener space, they have also proved to be quite natural tools [37].

Reciprocal Bernstein processes can also be characterized by an integration by parts formula, typical of Stochastic Analysis, but even when they are not Markovian [38]. It seems, indeed, that Feynman's symbolic approach was too limited to the Markovian class, not appropriate in some cases.

As it is clear from the first and last part of Section 4, we need 3 kinds of stochastic integrals for the complete description of Bernstein processes. A very general approach to stochastic integration with the same features (besides the one of [14]) due to Russo and Vallois is known: of [60]. It would be interesting to reconsider our construction with the tools described there.

The symmetry of such processes, in the sense of Noether Theorem, can also be of interest for other purposes [51].

Clearly, the approach chosen here can be regarded as a random version of Geometric Mechanics (cf. [23], [54]). In this context, one of the most interesting open problem is the notion of Integrability suited to the random dynamical systems resulting from our approach. Some aspects of it have been used in the Ideal of differential forms (5.12) but a lot more remains to be done (cf. [55]) in this field.

It would be interesting as well to understand the relations between our stochastic deformation and the (deterministic) deformation of characteristics for Hamiltonian PDEs inspired by B. Dubrovin [62]. One can, indeed, guess the existence of common features, for some particular PDEs.

Some probabilists would wonder why to study such a special class of Bernstein reciprocal processes. The first reason is that this class is not as small as it seems. We hope that we made clear that the key elements of their construction are independent of the form of the starting Hamiltonian H. Besides those like (3.25), for Lévy processes, we claim that it is always possible to time-symmetrize regular stochastic processes the way we did here.

The second reason is that if it is precisely because this class is special that it carries all the qualitative properties needed to construct stochastic dynamical theories. As mentioned in §2, Stochastic Analysis did not go at all in this direction. But, as suggested by Feynman Path Integral approach, this direction seems to be the most natural one as far as physical theories are concerned.

There are many fields, outside Mathematics, where this unorthodox way to approach stochastic dynamics is also natural. For instance, in Finance [39] or Econometry [40]. Image processing is also a promising domain of application of these ideas.

Of course, various problems of Statistical Physics can benefit from the use of such time reversible probability measures [41]. P.O. Kazinski, among many others, considers various classical models in this perspective [42]. He is also the author who introduced the expression "Stochastic Deformation". Other applications in Theoretical Physics include [43], [58].

Random walks on graphs are described by Markov chains, reversible in a narrower sense than the one intrinsic to Bernstein processes. It is likely that the methods used here will also be relevant in this area [59]. Interesting links with physics are explored in [58].

In Applied Mathematics, the relations of the variational component of the program summarized here and Stochastic Control Theory are, of course, striking. They strongly suggest that there are very few ways to deform systematically classical mechanics along diffusion processes. But those relations are still far from completely explored. It is remarkable, as mentioned before, that some investigations of the seventies and eighties, aiming at a deformation of the classical calculus of variations along diffusion processes, were able to obtain results consistent with our probabilistic reinterpretation of Feynman's approach. What U.G. Haussmann [20] calls the adjoint process, for instance, is basically our (forward) momentum process (4.8). Of course, those results were all expressed with respect to a single (increasing) filtration and, as such, were not directly appropriate to a time reversible dynamical framework.

A last comment about the Stochastic Deformation program. In the late sixties, V. Arnold proved that the Euler equation of an ideal incompressible fluid could be interpreted in a (Lagrangian and Hamiltonian) analogy with the motion of a rigid body [44]. The configuration space was, then, the group of volume preserving diffeomorphisms of the region occupied by the fluid. If we are not only interested by "dry water" (as Feynman was calling the fluid described by Euler equation [45]) then we have to deal with Navier-Stokes equation. The idea that Navier-Stokes equation corresponds to a stochastic deformation of the Euler one was introduced in [52] and has been considerably elaborated in recent years [46].

This means that the method of stochastic deformation can also be applied to some infinite dimensional dynamical systems.

Various books have already been published, where Bernstein reciprocal processes play a major role. We mention only two recent ones besides [28]: [47], [48].

We are convinced that, also on the applied side, those processes do not have only a curious past but also, indeed, a bright future.

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