Limit Behaviour of the Voter Model, Exclusion Process and Regenerative Chains

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A drunk man will eventually find his way home
but a drunk bird may get lost forever.
— S. Kakutani

This work is dedicated
to my officemate Driss Baraka
who passed away three years ago
and to my dad whom I miss...

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Abstract

Though the following topics seem unlinked, most of the tools used in this thesis are related to random walks and renewal theory.

After introducing the voter model, we consider the parabolic Anderson model with the voter model as catalyst. In GÄRTNER, DEN HOLLANDER and MAILLARD [44], the behaviour of the annealed Lyapunov exponents, i.e., the exponential growth rates of the successive moments of the reactant with respect to the catalyst, was investigated. It was shown that these exponents exhibit an interesting dependence on the dimension and on the diffusion constant. In Chapter 3 we address some questions left open in this paper by considering specifically when the Lyapunov exponents are the a priori maximal value.

Then, we use exclusion process techniques to show that the evolution of a perturbed threshold voter model is recurrent in the critical case. The key to our approach is to develop the ideas of BRAMSON and MOUNTFORD [9] : we exhibit a Lyapunov-Foster function for the discrete time version of the process. We also make a widespread use of coupling arguments.

Finally, using the regenerative scheme of COMETS, FERNÁNDEZ and FERRARI [19], we establish a functional central limit theorem for discrete time stochastic processes with summable memory decay. Furthermore, under stronger assumptions on the memory decay, we identify the limiting variance in terms of the process only. As applications, we define classes of binary autoregressive processes and power-law Ising chains for which the limit theorem is fulfilled.

Keywords : Markov process, voter model, exclusion process, regenerative chain, random walk, coupling, parabolic Anderson model.

Résumé

Les sujets abordés dans cette thèse peuvent sembler déconnectés les uns des autres mais en fait, chaque chapitre utilise abondamment la théorie du renouvellement et les propriétés des marches aléatoires.

Tout d'abord nous présentons le modèle parabolique d'Anderson muni, en tant que milieu aléatoire, du modèle du votant que nous introduisons au préalable. Dans GÄRTNER, DEN HOLLANDER et MAILLARD [44] est analysé le comportement des exposants de Lyapunov *annealed*, c'est-à-dire, le taux de croissance exponentiel des moments successifs de l'agent actif par rapport au catalyseur. Il y est prouvé que ces exposants présentent une intéressante dépendance en la dimension et la constante de diffusion. Dans le Chapitre 3 nous traitons quelques questions laissées en suspens dans le papier précédent en abordant spécifiquement le cas où les exposants de Lyapunov atteignent leur valeur maximale a priori.

Puis nous utilisons les techniques liées au processus d'exclusion afin de montrer que l'évolution d'un modèle du votant à seuil est récurrente dans le cas critique lorsque l'on perturbe le processus. La clef de notre approche est d'étendre les idées de BRAMSON et MOUNTFORD [9] : nous exposons une fonction de Lyapunov-Foster pour le processus discrétisé par rapport au temps. Des arguments de couplage seront également fréquemment utilisés.

Enfin, grâce au principe de régénération de COMETS, FERNÁNDEZ et FERRARI [19], nous établissons un théorème fonctionnel de la limite centrale pour les processus stochastiques à temps discret munis d'une décroissance de la mémoire sommable. Nous illustrons notre résultat à l'aide de deux exemples : les chaînes d'Ising *power-law* et les processus binaires autorégressifs.

Mots-clefs : processus de Markov, modèle du votant, processus d'exclusion, chaîne avec régénération, marche aléatoire, couplage, modèle parabolique d'Anderson.

Contents

Introduction

This thesis concerns Markov processes and related topics, mainly the voter model, parabolic Anderson model, exclusion process and infinite regenerative chains. Though these mathematical objects are somewhat esoteric, the analysis essentially rests on extensive use of random walk and renewal theory. We first briefly motivate these topics with historical references.

In 1827, the Scottish botanist Robert BROWN was studying the structure of pollen grains. He placed them in water to examine them and remarked that they were in motion. After careful experiments, he was convinced that the particles were moving by themselves, not because of external factor. He was also convinced that this motion was not due to the *essence of life* of particles and obtained the same kind of phenomenon with powder of stone. Although biologist, he gave a revolutionary physical explanation.

Independently, in 1906, the Russian mathematician Andrei A. MARKOV studied sequences of mutually dependent variables to find limit laws of probability. We owe him the extension of weak law of large numbers and central limit theorem for some sequences of dependent variables. In 1926, the Russian mathematician Sergei N. BERNSTEIN was the first to use the name *Markov Chain* to describe a sequence of variables where the future does not depend on the past, but only on the actual state. The work of BROWN was unknown by MARKOV as many other topics. As popular examples of Markov chains, we can cite some of the urn problems studied by Pierre-Simon LAPLACE, Daniel BERNOULLI, and Paul EHRENFESTS, random walks studied first by Viktor Y. BUNYAKOVSKY or the work of Louis BACHELIER who used the Brownian motion to evaluate stock options.

The work of BROWN was refined by Norbert WIENER in 1921. In 1933, Andrey N. KOLMOGOROV published a monograph building probability theory from fundamental axioms [51] and, in 1938, he wrote a paper laying the foundations of the theory of Markov random processes [52]. Using this theoretic formulation of a probabilistic measure, around 1950, Vilibald FELLER linked Markov processes and partial differential equations in a functional analysis framework.

Many books discuss this topic, see e.g., the works of Eugene B. DYNKIN [30], David A. FREEDMAN [35], Daniel REVUZ [69], Sean P. MEYN and Richard L. TWEEDIE [67], Thomas M. LIGGETT [60]. In Chapter 1, we recall definitions and main results about Markov chains and random walks.

Interacting particle systems

In this thesis, we will also be concerned by a specific type of Markov processes : interacting particle systems. These models were motivated by applied fields such as population genetics or statistical mechanics. However, it is widely accepted that the precursors are Frank L. SPITZER (see [73]) and, independently, Roland L. DOBRUSHIN (see [24] and [25]). This is a natural extension of the theory of Markov processes and is now one of the richest topics of probability. A typical interacting system consists of particles that evolve according to an interaction rule. Although the system as a whole is Markovian, it is not true for the evolution of an individual particle. Thus, while there are connections with Markov processes, most of the techniques used are new. Many models exist, the best known are *exclusion process, contact process, Ising model, growth model* and *voter model.* This last one will be our main subject of study.

The voter model first appeared in a paper of Peter CLIFFORD and Aidan SUDBURY in 1973 under the name *invasion process* (see [17]). In 1975, Richard A. HOLLEY and Thomas M. LIGGETT proved first results about this topic (see [47]). In 1979, Maury BRAMSON and David GRIFFEATH studied stationary distributions in dimensions greater than three (see [8]), and in 1986, J. Theodore COX and David GRIFFEATH analyzed the clustering in two dimensions (see [21]). Since then, this subject has been widely studied and new questions raised, as occupation time or interface shape for example. Have now a closer look on this model.

We consider each element of S, a countable set, as being a site occupied by persons who have to choose between 0 and 1 (we can see these as voting intention). At any time, each person knows what he wants, so we write $\xi(x,t)$ the opinion of the person x at time t. However, our voters are not self-confident and can be easily influenced by their neighbors. Actually, each voter periodically reassesses his mind by choosing randomly one of its neighbors and adopting his view. We write $p(x,y)$ the probability that the person x copies the mind of the person y and understand "periodically" by "waiting randomly an exponential time with parameter one".

More precisely, we see $\{\xi(t)\}_{t\geqslant 0}$ as a stochastic process homogeneous in time and such that each configuration $\xi(t) = \xi(\cdot,t)$ is in $\{0,1\}^{\mathbb{Z}^d}$. The transition mechanism is specified by a nonnegative function $c(x,\eta)$ with $x \in \mathbb{Z}^d$ and $\eta \in \{0,1\}^{\mathbb{Z}^d}$ which represents the rate at which the site x flips its state (from 0 to 1, or from 1 to 0) when the system is in state η. Thus the process ξ satisfies

$$\mathbb{P}_\eta(\xi(x,t) \neq \eta(x)) = c(x,\eta) + o(t)$$

for all site $x \in \mathbb{Z}^d$ and every configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$. Moreover, two sites won't change on the same time as we have

$$\mathbb{P}_\eta(\xi(x,t) \neq \eta(x) \text{ and } \xi(y,t) \neq \eta(y)) = o(t)$$

for all different sites $x, y \in \mathbb{Z}^d$ and every configuration $\eta \in \{0,1\}^{\mathbb{Z}^d}$.

A formal construction of this model and basic results will be given in Section 2. More details can be found in the book of Thomas M. LIGGETT [58]. However, it is important to note that we use the graphical representation to present these elements instead of using the standard infinitesimal description and harmonic functions found in [58]. In particular we propose an original proof of the basic convergence theorem (Corollary 1.13 p. 231 of [58]).

Remark that many variations of the standard voter model exists : three-state modified voter model, rebellious voter model, exclusion-voter model or threshold voter model. The last one will be focused on in Chapter 4. In the threshold voter model, a person will change his mind if at least half of his direct neighbors have a different voting intention. We will use exclusion process techniques to study this process in a critical case and we will consider a perturbed system, i.e., one of the vote is favored compared to the other one.

Parabolic Anderson model

In 1958, Philippe W. ANDERSON conceived the concept of electron localization. When the degree of disorder is too high in a solid, the diffusive motion of the electron is stopped (see [1]). Indeed, as the standard homogenization approach cannot cover many special cases, this result was of a great importance and is sometimes called Anderson localization. This conducts to the principle of intermittency, when there are irregular alternations of phases of apparently periodic and chaotic dynamics. The Cauchy problem for the spatially discrete heat equation with a random potential is a good and simple example exhibiting the effect of intermittency. This is often called parabolic Anderson model. A very good reference about this topic is the book of René A. CARMONA and Stanislav A. MOLCHANOV [11].

To be more precise, the parabolic Anderson model is the partial differential equation

$$\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t) + \dot{W}(x,t)u(x,t) \quad x \in \mathbb{R}^d, t \geqslant 0,$$

for the $\mathbb{R}$-valued random field u, Δ the Laplacian, $\dot{W}$ the space-time white noise and some non-random initial condition u_0. The interested reader will find more informations in the paper of Lorenzo BERTINI and Nicoletta CANCRINI [4] and references therein. This model is really important in physics as it is related to the KARDAR-PARISI-ZHANG equation [66].

We can also consider a discrete version of the previous equation, i.e.,

$$\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t) + \xi(x,t)u(x,t) \quad x \in \mathbb{Z}^d, t \geqslant 0,$$

with Δ the discrete Laplacian and ξ an i.i.d $\mathbb{R}$-valued random potential. The survey of Jürgen GÄRTNER and Wolfgang KÖNIG [45] is an interesting source for references.

When ξ takes non-negative integer values, we can interpret this equation as a population dynamics. Consider two types of particles : A, the catalyst and B the reactant. We then have the following properties

- A-particles evolve autonomously, according to a prescribed stationary dynamics given by the ξ-field, with $\xi(x,t)$ denoting the number of A-particles at site x at time t;
- B-particles perform independent random walks at rate $2d\kappa$ and split into two at rate $\gamma\xi(x,t)$;
- the initial configuration of B-particles is one particle everywhere (we made a particular choice for u_0).

Thus $u(x,t)$ can be seen as the average number of B-particles at site x at time t conditioned on the evolution of the A-particles. One of the interest of this model is that the two terms in the right-hand side of the equation are in competition : the diffusion of B-particle described by $\kappa\Delta$ tends to make u flat, while the branching of B-particles caused by A-particles described by ξ tends to make u irregular.

In the quenched situation, i.e. conditioned to the catalyst (u is defined almost surely with respect to the medium ξ), the behaviour of the intermittency is well known for many time-independent random potentials ξ (see the survey of Jürgen GÄRTNER and Wolfgang KÖNIG [45] for examples) with the use of subadditive arguments. However, for time-dependent random potential, it is not anymore the case. The intermittency is then indirectly studied by comparing the successive annealed Lyapunov exponents, i.e., the exponential growth rates of the successive moments of the solution with respect to the potential, more precisely,

$$\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log\big(\mathbb{E}(u(0,t)^p)\big)^{1/p}, \quad p \in \mathbb{N}, t > 0,$$

where $\mathbb{E}$ denotes expectation with respect to ξ.

If the catalyst is a family of independent simple random walks or a symmetric exclusion process, spectral techniques were used to study annealed Lyapunov exponents. However, the non-reversibility of the voter model implies that new techniques are needed and motivate the interest of this model. In the paper of Jürgen GÄRTNER, Frank DEN HOLLANDER and Grégory MAILLARD [44], it was shown that, when the catalyst is a voter model, these exponents exhibit an interesting dependence on the dimension and on the diffusion constant. In Chapter 3, we address some questions left open in [44] by considering specifically when the Lyapunov exponents are the a priori maximal value in terms of strong transience of the Markov process underlying the voter model.

Regenerative chains

These are discrete-time stochastic processes with infinite memory that are natural extensions of Markov chains when the associated process depends on its whole past. Such processes have been extensively studied (see e.g. the article of Roberto FERNÁNDEZ and Grégory MAILLARD [33] and references therein), but surprisingly very few is known about limit theorems. In

Chapter 5, we partially fill this gap by establishing a functional central limit theorem.

Historically, the first central limit theorems for regenerative chains have been established under strong ergodic assumptions by Harry COHN (1966) [18], Il'dar A. IBRAGIMOV and Yurii V. LINNIK (1971) [48] (see also Serban GRIGORESCU and Marius IOSIFESCU (1990) [41]). More recently, the empirical entropies of chains with exponential memory decay have been studied both in terms of their limit behaviour (Davide GABRIELLI, Antonio GALVES and Daniela GUIOL (2003) [36]) and their large deviations (Jean-René CHAZOTTES and Davide GABRIELLI (2005) [12]).

Limit theorems such as *Local Central Limit Theorem* and *Law of Iterated Logarithm* have been broadly studied for Markov chains (see e.g. the book of Sean P. MEYN and Richard TWEEDIE [67]). Regeneration methods, introduced by Kai Lai CHUNG (1967) [15] and refined by Xia CHEN (1999) [13], have been used to divide the Markov chain into independent random blocks in order to derive such limit theorems. In 2001, Roberto FERNÀNDEZ, Pablo A. FERRARI and Antonio GALVES proposed a perfect simulation using renewal (see [31]). This constitute a motivating challenge to extend the techniques of Chen's work to the non-Markovian case.

1 Prerequisites

In this chapter we gather the tools used for this work. This will help to fix the notation. After basic notions about stochastic processes and some references for interested reader, we give more details about Voter Model. In particular we propose a new way to show the basic convergence theorem from the book of Thomas M. LIGGET [58].

1.1 Basic definitions

A Markov chain $\Phi = \{\Phi_0, \Phi_1, \ldots\}$ is a stochastic process with an important property : the future of the process is independent of the past given only its present value. Each Φ_i is a random variable on a set E. This set is a general set equipped with a countably generated σ-field $\mathscr{B}(E)$. For example, when E is endowed with a locally compact, separable, metrizable topology, $\mathscr{B}(E)$ is typically the Borel σ-field. Because of the Markov property, the process Φ can be constructed with the one-step transition probabilities. We say that $P = \{P(x, A), x \in E, A \in \mathscr{B}(E)\}$ is a transition probability kernel if

1. for each $A \in \mathscr{B}(E)$, $P(\cdot, A)$ is a non-negative measurable function on E;
2. for each $x \in E, P(x, \cdot)$ is a probability measure on $\mathscr{B}(E)$.

Then, for μ a measure on $\mathscr{B}(E)$ and P a probability kernel, we can construct a probability measure $\mathbb{P}_\mu$ on $\mathscr{F} = \Pi_{i=0}^{\infty} \mathscr{B}(E_i)$ and a stochastic process $\Phi = \{\Phi_0, \Phi_1, \ldots\}$ on $\Omega = \Pi_{i=0}^{\infty} E_i$, measurable with respect to $\mathscr{F}$ (where E_i are copies of E) such that

1. $\mathbb{P}_\mu(B)$ is the probability of the event $\{\Phi \in B\}$ for any $B \in \mathscr{F}$,
2. for any measurable $A_i \in E_i$, $i \in \{0, \ldots, n\}$ and any n, we have

$$\mathbb{P}_\mu(\Phi_0 \in A_0, \ldots, \Phi_n \in A_n) = \int_{y_0 \in A_0} \cdots \int_{y_{n-1} \in A_{n-1}} \mu(dy_0) P(y_0, dy_1) \cdots P(y_{n-1}, A_n). \tag{1.1}$$

More details about this construction can be found in section 3.4 of the book of Sean P. MEYN and Richard L. TWEEDIE [67] or in the book of Daniel REVUZ [69].

In the particular situation where E is a countable set, a chain is said irreducible when there is a strictly positive probability to reach every site y starting from any state x, i.e., $\forall x, y \in E$,

$$\exists n \text{ such that } \mathbb{P}_{\delta_x}(\Phi_n = y) > 0 \tag{1.2}$$

for δ_x the Dirac measure. A chain is said aperiodic when the return to state x do not occur periodically, i.e., $\forall x \in E$,

$$\exists n \text{ such that } \mathbb{P}_{\delta_x}(\Phi_{n'} = x) > 0 \text{ for all } n' > n. \tag{1.3}$$

One of the most important example of Markov chain is the random walk, in particular the simple random walk. Let $X_1, X_2, \dots$ be i.i.d. random variables taking values in $\mathbb{R}^d$ and define $S_n = X_1 + \ldots + X_n$. S_n is called a random walk. In this model, the transition kernel is homogeneous in space, i.e., $p(x, y) = p(y - x)$. We can also define the symmetrized kernel by

$$p^{(s)}(x) = \frac{1}{2}(p(x) + p(-x)). \tag{1.4}$$

The simple random walk arises when X_1 is uniformly distributed among the nearest neighbors of the origin. This is said to be recurrent when the hitting time is finite almost surely for any site i, i.e.,

$$\mathbb{P}_{\delta_i}(\inf\{n > 0 : S_n = i\}) < \infty) = 1 \tag{1.5}$$

and transient otherwise. Moreover we have a criterion for the recurrence.

Theorem 1.1.1. *Let* $\varepsilon > 0$. S_n *is recurrent if and only if*

$$\int_{(-\varepsilon,\varepsilon)^d} \mathrm{Re}\frac{1}{1-\phi(\mathrm{t})}\,\mathrm{dt} = \infty \tag{1.6}$$

with $\phi(t) = \mathbb{E}\left(e^{itX_i}\right)$ the characteristic function of one step of the random walk. In particular, we see the classical theorem of Polya : in dimension $d = 1$ or $d = 2$, the simple random walk is recurrent while it is transient in dimension $d \geqslant 3$. We will not give details here and invite the reader to look at one of the books on this topic, for example, the one of Rick DURRETT [29].

We can now extend our previous definition to time continuous case. Again, let E be a general set equipped with a countably generated σ-field $\mathscr{B}(E)$. Define $C(E)$ the space of continuous bounded real-valued functions on E and endow it with the uniform norm

$$||f(x)|| = \sup_{x\in E} |f(x)| \tag{1.7}$$

to turn it into a Banach space. Let Ω the set of right continuous functions $\omega : [0,\infty) \to E$ with left limits (sometimes called càdlàg). Define τ_s the s-drift function, i.e. $\tau_s\omega(t) = \omega(t+s)$. Moreover write $\xi(t,\omega) = \omega(t)$ the process on realizations of Ω. Here, we set $\mathscr{F}$ the smallest

σ-algebra on Ω such that the mapping $\omega \to \omega(t)$ is measurable for each $t > 0$ and $\mathscr{F}_t$ the smallest σ-algebra on Ω relative to which all the mapping τ_s, for $s \leqslant t$, are measurable.

We call Markov process (or Markov jump process) the collection of probability measures $\{\mathbb{P}_\eta, \eta \in E\}$ on Ω such that

- the process $\xi(t) = \xi(t, \cdot)$ is adapted to the filtration $\{\mathscr{F}_t, t \geqslant 0\}$,
- $\mathbb{P}_\eta(\xi(0) = \eta) = 1$ for all $\eta \in E$,
- $\mathbb{E}_\eta(Y \circ \tau_s | \mathscr{F}_s) = \mathbb{E}_{\xi(s)}(Y) \quad \mathbb{P}_\eta$-a.s

for all $\eta \in E$ and all bounded measurable function Y on Ω. E_η is the expectation corresponding to P_η defined by

$$\mathbb{E}_\eta(Y) = \int_\Omega Y \, d\mathbb{P}_\eta \tag{1.8}$$

for any bounded measurable function Y on Ω.

Moreover, if the mapping

$$\eta \to \mathbb{E}_\eta \left(f(\xi(t)) \right) \tag{1.9}$$

is in $C(E)$ for all $f \in C(E)$, $t \geqslant 0$, then the Markov process is called a Feller process. We write

$$[S(t)f](\eta) = \mathbb{E}_\eta \left(f(\xi(t)) \right) \tag{1.10}$$

and call the family of linear operator $\{S(t), t \geqslant 0\}$ on $C(E)$ a semigroup. If a semigroup satisfies the Feller property, it ensures the existence of a unique Feller process (see for example the book of Robert BLUEMENTHAL and Ronald GETOOR [6] for more details). Moreover, the Hille-Yosida theorem gives a one-to-one correspondence between semigroups and Markov generators (see the book of Eugene B. DYNKIN [30]) and is used to construct the semigroup of a Feller process with it's infinitesimal description. More details can be found in Chapter 1 of the book of Thomas M. LIGGET [58].

We can now imagine a Markov process starting from an initial distribution instead of a fixed configuration. Let $\mathscr{P}$ denote the set of all probability measures on E, with the topology of weak convergence, i.e., $\mu_n \to \mu$ in $\mathscr{P}$ if and only if $\int f \, d\mu_n \to \int f \, d\mu$ for all f bounded (or all $f \in C(E)$ when E is compact). The measure

$$\mathbb{P}_\mu = \int_E \mathbb{P}_\eta \, \mu(d\eta) \tag{1.11}$$

describes the distribution of a Markov process with initial distribution μ. Then we define the corresponding expectation by

$$\mathbb{E}_\mu(f(\xi(t)) = \int S(t) f \, d\mu \tag{1.12}$$

for all $f \in C(E)$ and also define the measure $\mu S(t)$ by

$$\int f \, d(\mu S(t)) = \int S(t) f \, d\mu \tag{1.13}$$

for all $f \in C(E)$ which can be seen as the distribution at time t of the process starting with initial distribution μ.

When a measure $\mu \in \mathscr{P}$ satisfies $\mu = \mu S(t)$ we call it stationary and denote the set of all stationary measures of $\mathscr{P}$ by $\mathscr{P}^s$.

Define τ_x the x-drift function, i.e. $\tau_x \xi(y) = \xi(y+x)$ for all $y \in S$. When a measure $\mu \in \mathscr{P}$ satisfies $\mu(\tau_x \xi) = \mu(\xi)$ for all $\xi \in E$, we call it invariant and denote the set of all invariant measures of $\mathscr{P}$ by $\mathscr{P}^i$.

Moreover, we say that a measure $\mu \in \mathscr{P}$ is ergodic if every invariant set is trivial, i.e.,

$$\text{if, for all } x, \tau_x A = A \text{ then } \mu(A) = 1 \text{ or } 0. \tag{1.14}$$

A Markov process with semigroup $\{S(t), t \geqslant 0\}$ is said ergodic if

- $\mathscr{P}^s$ is a singleton,
- $\lim_{t \to \infty} \mu S(t) \in \mathscr{P}^s$ for all $\mu \in \mathscr{P}$.

Remark that when E is countable, an irreducible Markov process is ergodic if all its state are positive recurrent, i.e., the hitting time is finite almost surely for any site i,

$$\mathbb{P}_{\delta_i}(\inf\{t > 0 : X(t) = i\}) < \infty) = 1 \tag{1.15}$$

and the mean recurrence time is finite for any site i,

$$\mathbb{E}_{\delta_i}(\inf\{t > 0 : X(t) = i\}) < \infty. \tag{1.16}$$

For a general E, Markov processes can be very complex. However, one of the most studied class of Markov processes are random walks. In this situation, E is more simple, usually $\mathbb{Z}^d$. The main references about random walk on $\mathbb{Z}^d$ are the book of Franck SPITZER (see [74]) and the books of Gregory F. LAWLER ([54] and [56] with Vlada LIMIC).

In fact, this work is focused on another large class of Markov processes : interacting particle systems. In this particular situation, $E = W^S$ with W a compact metric space called phase space and S a countable set called site space. E is then compact and metrizable. More precisely, we take $S = \mathbb{Z}^d$ and $W = \{0, 1\}$. The elements of E (named configurations) will be denoted by Greek letters ξ or η and elements of $S = \mathbb{Z}^d$ (named sites) will be denoted by Latin letters x, y or z, finite subsets of S by capital letters A, B, etc. Note that the processes are denoted by $(\xi(t))_{t \geqslant 0}$ and the state of a site x at time t by $\xi(x, t)$. To lighten the notation, we will see configurations ξ as a subset of $\mathbb{Z}^d$ such that if $x \in \xi$ then the site x has state 1. Formally, ξ is identified by the support of the function $x \mapsto \xi(x) \in \{0,1\}^{\mathbb{Z}^d}$. In particular, for $A \subset \mathbb{Z}^d$, if $\xi \cap A \neq \emptyset$, it means that there exists some $x \in A$ such that $\xi(x) = 1$.

We can describe the local dynamics by a collection of transition measures $c_T(\eta, d\xi)$. For each $\eta \in E$ and finite $T \subset S$, $c_T(\eta, d\xi)$ is supposed to be a finite positive measure on W^T and a continuous mapping with respect to η (for the topology of weak convergence on W^T). We see $c_T(\eta, d\xi)$ as the rate at which a modification will appear in T for the configuration η. We now need some notation to give a criteria for ergodicity. For $x \in S$ and finite $T \subset S$, let

$$c_T(x) = \sup_{\eta_1,\eta_2 \in E} \left\{ ||c_T(\eta_1, d\xi) - c_T(\eta_2, d\xi)||_{\mathrm{TV}} : \eta_1(y) = \eta_2(y) \text{ for all } x \neq y \right\} \tag{1.17}$$

where $|| \cdot ||_{\mathrm{TV}}$ refers to the total variation norm of a measure on W^T, defined by

$$||\mathbb{P}_1 - \mathbb{P}_2||_{\mathrm{TV}} = \sup_{F \in \mathscr{F}} \{|\mathbb{P}_1(F) - \mathbb{P}_2(F)|\} \tag{1.18}$$

for $\mathbb{P}_1$ and $\mathbb{P}_2$ two probability measures on $(\Omega, \mathscr{F})$. Let

$$\gamma(u, x) = \sum_{T \ni u} c_T(x) \tag{1.19}$$

for $x \neq u$ and $\gamma(x, x) = 0$. Write $\xi_{x,0} = \{\xi : \xi(x) = 0\}$ (similarly write $\xi_{x,1}$) and $\xi_{\eta,x,0} = \{\xi : \xi(x) = 0, \xi(y) = \eta(y)\}$ (similarly $\xi_{\eta,x,1}$), then define

$$\epsilon = \inf_{x \in S} \inf_{\eta} (c_T(\xi_{\eta,x,0}, \xi_{x,0}) - c_T(\xi_{\eta,x,1}, \xi_{x,1})), \tag{1.20}$$

and

$$M = \sup_{u \in S} \sum_{T \ni u} \sum_{x \neq u} c_T(x) = \sup_{u \in S} \sum_{u} \gamma(u, x). \tag{1.21}$$

Theorem 1.1.2. *Assuming*

$$\sup_{u \in S} \sum_{T \ni u} c_T < \infty \text{ and } M < \infty, \tag{1.22}$$

the Dobrushin's criterion states that

$$\text{if } M < \epsilon, \text{ then the process is ergodic.} \tag{1.23}$$

See Theorem 4.1 of [58] for a proof.

We can now construct the operator

$$\Omega f(\eta) = \sum_{T} \int_{W^T} c_T(\eta, d\xi)(f(\eta_\xi) - f(\eta)) \tag{1.24}$$

where

$$\eta_\xi(x) = \begin{cases} \eta(x) & \text{if } x \notin T, \\ \xi(x) & \text{if } x \in T. \end{cases} \tag{1.25}$$

The closure of this operator is the generator of a Markov semigroup corresponding to a unique Feller process (see Theorem 1.5 of [58]).

Before detailing the Voter Model, we first recall notions about coupling.

1.2 Coupling

Coupling is the name given to the method used to compare two probability measures evolving on the same measurable space. This has been extensively used since early ’80 on many different topics such Markov chains, renewal theory, Harris chains, random walks, birth and death processes. We briefly present basic definition and examples. More details can be found in the book of Torgny LINDVALL [61] and the one of Hermann ÞÓRISSON [70].

Definition 1.2.1. A coupling of the probability measure $\mathbb{P}$ and $\mathbb{P}'$ on a measurable space $(E, \mathscr{B}(E))$ is a probability measure $\hat{\mathbb{P}}$ on $(E^2, \mathscr{B}(E)^2)$ such that

$$\mathbb{P}(A) = \hat{\mathbb{P}}(A, E) \qquad \text{and} \qquad \mathbb{P}'(A) = \hat{\mathbb{P}}(E, A) \tag{1.26}$$

for any $A \in \mathscr{B}(E)$.

We can also see the random element $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{\mathbb{P}}, (\hat{X}, \hat{X}'))$ in $(E^2, \mathscr{B}(E)^2)$ as a coupling between the random elements $(\Omega, \mathscr{F}, \mathbb{P}, X)$ and $(\Omega', \mathscr{F}', \mathbb{P}', X')$ in $(E, \mathscr{B}(E))$ such that

$$X \overset{\mathscr{D}}{=} \hat{X} \qquad \text{and} \qquad X' \overset{\mathscr{D}}{=} \hat{X}'. \tag{1.27}$$

Hence $\hat{\mathbb{P}}((\hat{X}, \hat{X}')^{-1})$ is a coupling of $\mathbb{P}(X^{-1})$ and $\mathbb{P}(X'^{-1})$ in the sens of 1.26. For simplicity we can then lighten the notation and refer to $(\hat{X}, \hat{X}')$ as a coupling of the random elements X and X'.

The previous definition is not really intuitive. That’s why we now give an easy example by considering a situation with two simple random walks X, X' on $\mathbb{Z}^d$. First consider the case $d = 1$. We begin with two independent walks X and X'. We can create a coupling $(\hat{X}, \hat{X}')$ by

$$\hat{X}(t) = X(t) \text{ and } \hat{X}'(t) = \begin{cases} X'(t) & \text{if } t < T, \\ X(t) & \text{if } t \geqslant T, \end{cases} \tag{1.28}$$

for $T = \min\{t : X(t) = X'(t)\}$. As the simple random walk is recurrent on $\mathbb{Z}$, it ensures that $\mathbb{P}(T < \infty) = 1$. This is the so-called Ornstein’s coupling. However, for $d \geqslant 3$, the simple random walk is not anymore recurrent and so, apparently, the Ornstein’s coupling fails. In fact, we can apply this coupling componentwise. $\hat{X}'$ evolves according to X' in the first coordinate and according to X in the other coordinates. When the first projections of X and X' meet, $\hat{X}'$ then evolves according to X' in the second coordinate and according to X in the other coordinates. Continue until all dimension are exhausted. This way, the coupling is successful with probability 1.

We can also extend this example in a more general case. Consider two irreducible random walks (not simple) X and X' evolving according to the same transition kernel $p(\cdot)$. Without loss of generality, suppose that $X(0) = 0$ and $X'(0) = x$. We now present a coupling $(\hat{X}, \hat{X}')$ so that $\hat{X}(t) = \hat{X}'(t)$ for all $t \geqslant T$ for some stopping time $T < \infty$. In particular this holds for Poisson processes.

Lemma 1.2.2. *Consider two random walks X, X' derived from an irreducible and recurrent kernel. Irrespective of the initial values of the walks, there exists a coupling so that for all large t, $X(t) = X'(t)$.*

Proof. Without loss of generality one may suppose that $X(0) = 0$ and $X'(0) = x$. Let $T = \inf\{t : X(t) = X'(t)\}$ for two independent copies of this random walk. Construct the coupling by

$$\hat{X}(t) = X(t) \text{ and } \hat{X}'(t) = \begin{cases} X'(t) & \text{if } t < T, \\ X(t) & \text{if } t \geqslant T. \end{cases} \tag{1.29}$$

Recurrence insures that $T < \infty$ a.s. and make the coupling effective. □

However these ideas extend to all random walks such that the symmetrization is irreducible

Proposition 1.2.3. *Consider two random walks X, X' derived from a kernel whose symmetrization is irreducible. Irrespective of the initial values of the walks, there exists a coupling so that for all large t, $X(t) = X'(t)$.*

Proof. Without loss of generality one may suppose that $X(0) = 0$ and $X'(0) = x$. Given the irreducibility condition there exist $\{u_1, \ldots, u_n\} \subset \mathbb{Z}^d$ a family of distinct possible jumps, i.e., $p(u_i) > 0$ for all i such that there exist $\{l_1, \ldots, l_n\} \subset \mathbb{Z}$ with $\sum_i^n l_i u_i = x$. We will then use these jumps to describe the random walk X. We can write

$$X(t) = \sum_{i=1}^{n} N_{u_i}(t) u_i + Y(t) \text{ a.s.} \tag{1.30}$$

with Y a random walk, N_{u_i} rate $p(u_i)$ Poisson processes, such that $N_{u_i}(0) = l_i$ and Y, N_{u_i} are independent. We can also describe X' in the same way with

$$X'(t) = \sum_{i=1}^{n} N'_{u_i}(t) u_i + Y(t) \text{ a.s.} \tag{1.31}$$

Now create a coupling $\hat{N}_{u_i}, \hat{N}'_{u_i}$ such that

$$\hat{N}_{u_i}(t) = N_{u_i}(t), \qquad \hat{N}'_{u_i}(t) = \begin{cases} N'_{u_i}(t) & \text{for } t < T_{u_i}, \\ N_{u_i}(t) & \text{for } t \geqslant T_{u_i}, \end{cases} \tag{1.32}$$

with $T_{u_i} = \inf\{t : N_{u_i}(t) = N'_{u_i}(t)\}$. the construction is granted by Lemma 1.2.2.

Finally, we have the coupling $\hat{X}(t) = \sum_{i=1}^{n} \hat{N}_{u_i}(t) u_i + Y(t)$ and $\hat{X}'(t) = \sum_{i=1}^{n} \hat{N}'_{u_i}(t) u_i + Y(t)$ with the property

$$\hat{X}(t) = \hat{X}'(t) \text{ for } t \geqslant \max\{T_{u_1}, \ldots, T_{u_n}\} \tag{1.33}$$

Then the coupling is effective. $\square$

2 Voter model

As layed in introduction, the voter model first appeared in a paper of Peter CLIFFORD and Aidan SUDBURY in 1973 under the name invasion process (see [17]) and in the work of Richard A. HOLLEY and Thomas M. LIGGETT in 1975 (see [47]). In this section we will recall the construction explained in the book of Thomas M. LIGGETT [58] that uses Markov generator. We then give a more probabilistic construction using the Harris representation, as in the notes of Richard DURRETT [27] and give a formal description of what we called the dual of this process : the coalescing random walks. We then use these elements to propose a new proof of the basic convergence theorem from [58].

2.1 Infinitesimal description

We define the operator Ω on the set of absolutely continuous functions f on E by

$$\Omega f(\eta) = \sum_{x\in S} c(x,\eta)(f(\eta_x) - f(\eta)), \quad \eta \in \{0,1\}^{\mathbb{Z}^d}, \tag{2.1}$$

where

$$\eta_x(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ 1-\eta(y) & \text{if } y = x, \end{cases} \tag{2.2}$$

and

$$c(x,\eta) = \begin{cases} \sum_{y\in S} p(y)\eta(y) & \text{if } \eta(x) = 0, \\ \sum_{y\in S} p(y)(1-\eta(y)) & \text{if } \eta(x) = 1, \end{cases} \tag{2.3}$$

with $p(\cdot)$ an aperiodic kernel (i.e. the corresponding random walk is aperiodic).

It is important to remark that we consider a process homogeneous in space and that one can replace $p(y)$ by $p(x,y)$ in the previous definition to describe a more general model. We made this choice because we want the mechanics of the voter model to be derived from random walks. Formally, we need S to be a regular lattice to make our definition consistent but as explained in the previous Chapter, we consider the special case $W = \{0,1\}$, $S = \mathbb{Z}^d$ and thus $E = \{0,1\}^{\mathbb{Z}^d}$.

The closure of the operator Ω is the generator of a Markov semigroup $S(t)$ (by Theorem of Hille-Yosida, see Theorem 2.9 of [58]) and then there is a unique Markov process corresponding to it (see Theorem 1.5 of [58]). In particular, this process verify

$$\mathbb{P}_\eta(\eta(x,t) \neq \eta(x)) = c(x,\eta)t + o(t) \tag{2.4}$$

as $t \to 0$ and

$$\mathbb{P}_\eta(\eta(x,t) \neq \eta(x), \eta(y,t) \neq \eta(y)) = o(t) \tag{2.5}$$

as $t \to 0$, for all $x, y \in S$ different and every $\eta \in E$. The construction with infinitesimal rates, operators and semigroups is well known. However, we chose a more probabilistic point of view.

2.2 Harris construction

For each $x \in \mathbb{Z}^d$, take an independant Poisson process N^x with rate 1 and write $\{T_n^x\}_{n\geqslant 1}$ the realization times. Then take a transition kernel of an irreducible random walk p that generates an i.i.d. sequence R_n^x for each site x with $\mathbb{P}\left(x + R_n^x = y\right) = p(y-x)$ for all $y \in \mathbb{Z}^d$. The idea is that the person on site x will change his mind at times T_n^x and adopts the opinion of the voter at site $x + R_n^x$ for each n. Another natural way to construct the process exists by taking, for each pair $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, an independent Poisson process $N^{x,y}$ with rate $p(y-x)$ and define the corresponding $\{T_n^{x,y}\}_{n\geqslant 1}$. These indicate how x change his opinion. We can easily remark that both constructions are equivalent. For simplicity in notations, we will use the second one.

A common way to represent this process graphically is made by drawing an arrow from $\left(y, T_n^{x,y}\right)$ to $\left(x, T_n^{x,y}\right)$ and write a $\times$ at $(x, T_n^{x,y})$ for every x and n. Then you can imagine water entering the bottom where the sites have value 1 and flowing the structure from bottom to top. The $\times$ are closed gates and the arrows are pipelines. For a fixed time t, each site that is wet, is considered to have value 1 at time t.

We say that there is a path from $(x,0)$ to (y,t) if there is a sequence of times $s_0 = 0 < s_1 < \ldots < s_m < s_{m+1} = t$ and spatial locations $x_0 = x, x_1, \ldots, x_{m-1}, x_m = y$ such that

1. for $i = 1, \ldots, m$ there is an arrow from x_{i-1} to x_i at times s_i,
2. there is no $\times$ at site x_i between times s_i and s_{i+1} for all $i = 0, \ldots, m$.

For example, there is a path from $(-2,0)$ to $(-1,t)$ on Figure 2.1 but none from $(1,0)$ to $(2,t)$.

The process related to the voter model is defined by $\{\xi(t)\}_{t\geqslant 0}$ with ξ as starting state (i.e.

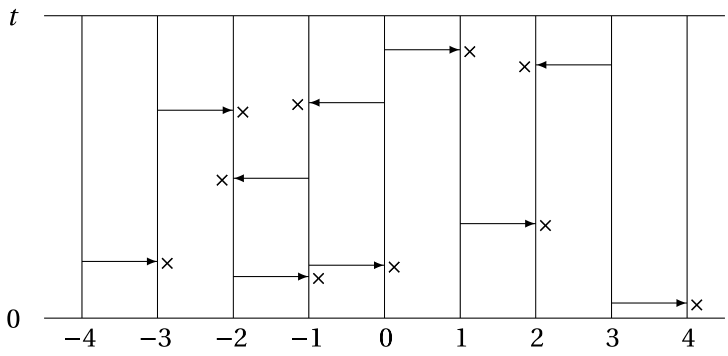


Figure 2.1: Harris representation in Z^1.

$\xi(0)=\xi$) and such that for all $x\in\mathbb{Z}^d$, we have $\xi(x,t)=1$ if and only if there is $y\in\mathbb{Z}^d$ such that there is a path from $(y,0)$ to (x,t) and $\xi(y,0)=1$.

2.3 Dual process

By construction, if there is a path from $(x,0)$ to (y,t), the value at site y at time t is the same as the value at site x at time 0. Moreover, for every site y, there exists an unique site x such that there is a path from $(x,0)$ to (y,t). This motivates the definition of a dual process. To construct it, reverse the direction of the arrows in the graphical representation and look at the time backward.

Besides, it is immediate that the evolution of the ancestor of the site x in the dual is a random walk with kernel p^*, the dual transition kernel of p defined by $p^*(y)=p(-y)$ for all $y\in\mathbb{Z}^d$. We denote these random walks $(X^x(t))_{t\geqslant 0}$ and the measure associated to this is denoted $\mathbb{P}_x$ with $\mathbb{E}_x$ the corresponding expectation. In particular we have

$$\xi(x,t)=1 \Leftrightarrow \xi(X^x(t),0)=1. \tag{2.6}$$

Moreover, for two different sites x and y, the random walks $(X^x(t))_{t\geqslant 0}$ and $(X^y(t))_{t\geqslant 0}$ evolve independently until they hit, then coalesce. For example, on Figure 2.2, random walks X^{-1} and X^1 coalesce very fast and $X^{-1}(t)=X^1(t)=-2$. Also remark that when the symmetrized transition kernel $p^{(s)}(x)=\frac{1}{2}(p(x)-p(-x))$ is recurrent, $\lim_{t\to\infty}\mathbb{P}(X^x(t)=X^y(t))=1$ and so $\lim_{t\to\infty}\mathbb{P}_\eta(\eta(x,t)=\eta(y,t))=1$ for any starting $\eta\in\{0,1\}^{\mathbb{Z}^d}$. Thus in this case we tend to a general consensus.

We also adopt the following notation. For any subset $A\subset\mathbb{Z}^d$, the family of ancestors is denoted by X^A, i.e.,

$$X^A(t)=\{X^x(t)\in\mathbb{Z}^d : x\in A\}. \tag{2.7}$$

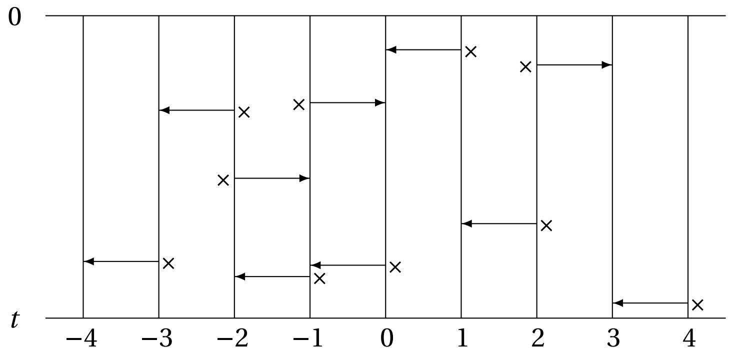


Figure 2.2: Dual of Figure 2.1.

Although there are perhaps non-distinct sites in $X^A(t)$, we see it as a subset of Z^d. With this remark in mind, we have, in particular,

$$\xi(x,t) = 1\,\forall x \in A \Leftrightarrow \xi(x,0) = 1\,\forall x \in X^A(t). \tag{2.8}$$

2.4 Stationary measures

We can remark that the voter model fails in Dobrushin criterion as $\epsilon = M = 1$. This situation is obvious as the voter Model is not ergodic. There are two trivial invariant measures : δ_0 and δ_1 (only 0 or only 1). The discussion above shows that if the underlying random walk has a recurrent symmetrization then these are the only (extremal) stationary measures. This raises the question of what are the stationary measures when we have transience.

Name $\mathscr{P}_e^i$ the set of extremal measures, i.e. the elements of $\mathscr{P}^i$ which cannot be written as a non trivial convex combination of elements of $\mathscr{P}^i$. We can easily show that the ergodic measures are also extremal (and vice versa) (see Corollary 4.14 from LIGGETT [58]).

We first show that there exists another stationary measure than the trivial two ones.

Proposition 2.4.1. *Denote by μ_α the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}^d}$, i.e., $\mu_\alpha(\eta(x) = 1) = \alpha$ independently for all $x \in \mathbb{Z}^d$. There exists a limiting measure*

$$\overline{\mu_\alpha} = \lim_{t\to\infty} \mu_\alpha S(t). \tag{2.9}$$

Proof. For A a finite subset of $\mathbb{Z}^d$, first remark that

$$\mu_\alpha S(t)(\xi : A \in \xi) = \mu_\alpha(\xi : \{X^A(t)\} \in \xi). \tag{2.10}$$

Recall that the expression $A \in \xi$ means that $\xi(x) = 1\,\forall x \in A$. As the random walks X^x are coalescing, the number of distinct points of $\{X^A(t)\}$ is decreasing, because of coalescing random walks, and as this is bounded from below by 1, there is a limit. Now take $x_1, \ldots, x_r \in$

$\mathbb{Z}^d$ and $i_1, \ldots, i_r \in \{0,1\}$ (for any r) and note that $\mu S(t)(\xi : \xi(x_1) = i_1, \ldots \xi(x_r) = i_r)$ can be written in terms of the form $\mu S(t)(\xi : A \subset \xi)$ with $A \subset \{x_1, \ldots, x_r\}$ using the inclusion-exclusion formula. As the collection of sets $\{\xi : \xi(x_1) = i_1, \ldots \xi(x_r) = i_r\}$ is an algebra (in fact the so-called cylinder algebra), it generates a unique smallest σ-algebra containing all these sets which is the $\sigma-$algebra we have on S. Using the Carathéodory's Extension Theorem, we show the existence of the limiting measure $\overline{\mu_\alpha}$. These notations will be used later : μ_α is the Bernoulli product measure and $\overline{\mu_\alpha}$ is its limit. □

We now use the prerequisite in Section 1.2 to show that every stationary measure will also be translation invariant.

Proposition 2.4.2. *Every stationary measure is also (translation) invariant.*

Proof. We will use the dual representation and therefore, the collection of coalescing random walks. Moreover, we will also take a system of random walks only coalescing until time R and then evolving independently. In order to proceed, we need new definitions and notations.

Consider the set of coalescing random walks $\{X^A\}$ starting from points of A. Let $c_t^A(k)$ be the probability of having exactly k distinct random walks at time t, with $k \leqslant |A|$ the cardinality of the set A. Write $c^A(k) = \lim_{t\to\infty} c_t^A(k)$. This limit exists as $c^A(k)$ is decreasing and bounded from below by 1. As explained before, define coupled processes depending on $R > 0$ such that the set $\{X_R^A\}$ evolves in the same way as $\{X^A\}$ for $t < R$ but then stops to coalesce and evolves independently.

We can choose R such that, for $\varepsilon > 0$ fixed,

$$\left|c_R^A(k) - c^A(k)\right| < \frac{\varepsilon}{|A|} \tag{2.11}$$

for all $1 \leqslant k \leqslant |A|$.

Now take $\mu \in \mathscr{P}^s$ a stationary measure. With similar arguments to those in the previous subsection, it is sufficient to show that

$$\mu(\xi : A \in \xi) = \mu(\xi : A + z \in \xi) \tag{2.12}$$

for every $z \in \mathbb{Z}^d$ (with $A + z$ the natural coset of A).

Consider any $A \in \mathbb{Z}^d$. As the measure μ is stationary, we have

$$\mu(\xi : A \in \xi) = \mu(\xi : X^A(t)) \in \xi). \tag{2.13}$$

Choose R sufficiently large such that Equation (2.11) is verified for A. Denote $X^{A,R}$ the collection of distinct (non coalesced) random walks after time R, starting from set A. Let $X^{A+z,R}$ evolve the exact same way as $X^{A,R}$. We can then make a coupling between $X^{A,R}$ and

$X^{A+z,R}$ componentwise (the first random walk of $X^{A,R}$ with the first of $X^{A+z,R}$ etc.) using the technique explained in Proposition 1.2.3. That way we have

$$\lim_{t\to\infty} \mathbb{P}(X^{A,R}(t) = X^{A+z,R}(t)) = 1. \tag{2.14}$$

For R sufficiently large, by (2.11), we have

$$\lim_{t\to\infty} \mathbb{P}(X^{A}(t) = X^{A+z}(t)) > 1-\varepsilon \tag{2.15}$$

and then, except with a small probability ε, we have that

$$\mu S(t)(\xi : A \in \xi) = \mu S(t)(\xi : A + z \in \xi) \tag{2.16}$$

for t sufficiently large. Invariance of μ concludes the proof. □

We are now ready for the main result of this Chapter.

2.4.1 Basic convergence theorem

We already saw that the voter model is not stationary because there are two trivial ergodic measures δ_0 and δ_1. It is important to give a characterization of the stationary measures. These are the only two when the symmetrized transition kernel is recurrent but when it is not anymore the case, there exist other stationary measures. These results are well known and a complete proof can be found in the book of Thomas M. LIGGETT [58].

Theorem 2.4.3. *Let $\overline{\mu}_\alpha$ be the limiting measure of a Bernoulli product measure (as stated in Proposition 2.4.1).*

- *If the symmetrized transition kernel is recurrent and $\mu \in \mathscr{P}^i$, then $\lim_{t\to\infty} \mu S(t) = \alpha\delta_0 + (1-\alpha)\delta_1$ where $\alpha = \mu\{\eta : \eta(x) = 1\}$;*
- *If the symmetrized transition kernel is transient and $\mu \in \mathscr{P}^i_e$, then $\lim_{t\to\infty} \mu S(t) = \overline{\mu}_\alpha$ where $\alpha = \mu\{\eta : \eta(x) = 1\}$.*

We now propose another way to prove the second part of this theorem. We choose a more probabilistic point of view which simplifies the proof in our opinion. We rather follow the philosophy of Rick DURRETT.

Take $\mu \in \mathscr{P}^i$ an invariant measure. By the ergodic decomposition theorem, we know there is a unique (up to a Borel isomorphism) Borel probability measure λ on $\mathscr{P}^i_e$ such that

$$\mu = \int_{\mathscr{P}^i_e} \nu \, d\lambda(\nu).$$

Many proofs of this theorem exist, for example, the one of Klaus SCHMIDT [71] or the one of Gustave CHOQUET [14].

We now prove the following theorem.

Theorem 2.4.4. *If $\mu \in \mathscr{P}_e^i$, then we have*

$$\lim_{t\to\infty} \mu S(t) = \overline{\mu}_\alpha \tag{2.17}$$

where

$$\alpha = \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{|x|_\infty \leqslant n} \eta(x) \quad \mu\text{-a.s.} \qquad \text{and} \qquad \overline{\mu}_\alpha = \lim_{t\to\infty} \mu_\alpha S(t)$$

with μ_α a Bernoulli distribution with parameter α.

Using the ergodic decomposition, we just need to show that $\lim_{t\to\infty} \mu S(t) = \overline{\mu}_\alpha$ if μ is ergodic and α, $\overline{\mu}_\alpha$ are defined as previously. Moreover, as the set of cylindric functions is dense in the set of continuous and bounded functions (remark that all continuous functions are bounded as the space is compact), we simply have to show

$$\mathbb{P}_\mu(\eta(x_1,t) = \ldots = \eta(x_r,t) = 1) \overset{t\to\infty}{\longrightarrow} \mathbb{P}_{\overline{\mu}_\alpha}(\eta(x_1) = \ldots = \eta(x_r) = 1) \tag{2.18}$$

for any $r \in \mathbb{N}$ and $x_1, \ldots, x_r \in \mathbb{Z}^d$.

Consider the same notations as in the previous subsection. It's immediate that the right hand side of equation (2.18) is equal to $\sum_{k=1}^r c^{x_1,\ldots,x_r}(k)\alpha^k$. For $\varepsilon > 0$ fixed, choose R such that (2.11) is verified.

Lemma 2.4.5 shows that the difference of behaviour between the two sets is small, more precisely, for any fixed ε, there exists some $R > 0$ such that,

$$\left|\mu\otimes\mathbb{P}\left(\eta\left(X^{x_1}(t)\right) = \ldots = \eta\left(X^{x_r}(t)\right) = 1\right) - \mu\otimes\mathbb{P}\left(\eta\left(X^{x_1,R}(t)\right) = \ldots = \eta\left(X^{x_r,R}(t)\right) = 1\right)\right| < \varepsilon \tag{2.19}$$

for all $t > 0$. For simplicity in notation, write $D_t(x_1,\ldots,x_r,R)$ for the left hand side of the previous inequality.

As Lemma 2.4.6 proves that

$$\lim_{t\to\infty} \mu\otimes\mathbb{P}\left(\eta\left(X^{x_1,R}(t)\right) = \ldots = \eta\left(X^{x_r,R}(t)\right) = 1\right) = \sum_{k=1}^r c_R^{x_1,\ldots,x_r}(k)\alpha^k, \tag{2.20}$$

Equation (2.11) and Lemma 2.4.5 show

$$\left|\lim_{t\to\infty} \mu\otimes\mathbb{P}\left(\eta\left(X^{x_1,R}(t)\right) = \ldots = \eta\left(X^{x_r,R}(t)\right) = 1\right) - \sum_{k=1}^r c^{x_1,\ldots,x_r}(k)\alpha^k\right| \leqslant \varepsilon + \frac{\varepsilon}{1-\alpha}. \tag{2.21}$$

This proved equation (2.18) as ε can be as small as we want.

Lemma 2.4.5. *For $D_t(x_1,\ldots,x_r,R)$ defined as in equation (2.19), we have*

$$D_t(x_1,\ldots,x_r,R) \leqslant \varepsilon \tag{2.22}$$

for R sufficiently large and for any $t > R$.

Proof. Let

$$\tau(x_1,\ldots,x_r,R) = \inf\{s > R : \exists i \neq j \text{ such that } X^{x_i}(R) \neq X^{x_j}(R), X^{x_i}(s) = X^{x_j}(s)\} \tag{2.23}$$

be the time of first coalescence after time R. By conditioning on the event $\tau \leqslant t$, then we have

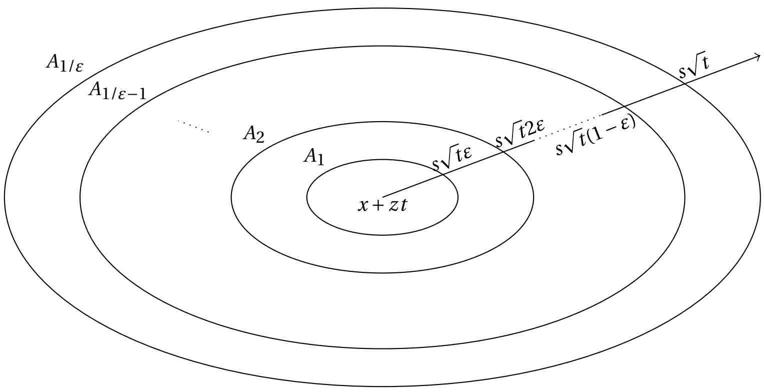
$$D_t(x_1,\ldots,x_r,R) \leqslant \mathbb{P}(\tau(x_1,\ldots,x_r,R) < t). \tag{2.24}$$

As $\mathbb{P}(\tau(x_1,\ldots,x_r,R) < t)$ is non increasing in R, we get the result. □

Lemma 2.4.6. *Take $X_t^{x_i,t,R}$ and $c_R^{x_1,\ldots,x_r}(k)$ defined as in proof of Theorem 2.4.4. Then we can show that*

$$\lim_{t\to\infty} \mu\otimes\mathbb{P}\left(\eta\big(X^{x_1,R}(t)\big) = \ldots = \eta\big(X^{x_r,R}(t)\big) = 1\right) = \sum_{k=1}^{r} c_R^{x_1,\ldots,x_r}(k)\alpha^k. \tag{2.25}$$

Proof. Take a finite range and aperiodic random walk with a Markov kernel p of mean z, and starting from site x. Then, we can apply a local central limit theorem to show that after time t the position of the random walk will be distributed by a multi-variable Gaussian law centered in $x+zt$, up to an error of order $O(t^{-(d+1)/2})$ (see Theorem 2.3.8 in [54]).



Construction of the annulus A_i.

Let A_i be the annulus centered in $x+zt$ with width ε as illustrated in the previous picture. Then we know that except a small probability of order $O(e^{-t})$, the random walk will be in one of the annulus A_i after a time t. Moreover, for ε sufficiently small, the distribution in the annulus can be considered uniform up to an error of order $O(\varepsilon t^{-d/2})$ (see Theorem 2.4.1 in [56]).

We can now cover every annulus with very small squares of size $\sqrt{t\varepsilon^m}$. Here m is just a big number chosen so that the probability that the random walk is not in any square is really small.

Recall that we supposed that $\mu \in \mathscr{P}^i$ and that

$$\alpha = \lim_{n\to\infty} \frac{1}{(2n+1)^d} \sum_{|x|_\infty \leqslant n} \eta(x) \quad \mu\text{-a.s.} \tag{2.26}$$

This means that for every $\delta > 0$, there exists a n_δ such that, for all $n > n_\delta$,

$$\mathbb{P}_\mu\left(\left|\frac{1}{(2n+1)^d} \sum_{|x|_\infty \leqslant n} \eta(x) - \alpha\right| > \delta\right) < \delta \tag{2.27}$$

and as μ is invariant we have, for all $n > n_\delta$ and all $y \in E$,

$$\mathbb{P}_\mu\left(\left|\frac{1}{(2n+1)^d} \sum_{|x-y|_\infty \leqslant n} \eta(x) - \alpha\right| > \delta\right) < \delta \tag{2.28}$$

That means that for each square, the density of 1' is close to α when t is sufficiently large.

More formally, first take s sufficiently large so that the probability that the random walk is outside an annulus of size $s\sqrt{t}$ is smaller than δ_1. Then, choose ε sufficiently small so that the error of supposing the distribution in an annulus is uniform is smaller than δ_2. Then, take m big enough so that the probability the random walk is not in a covering square is smaller than δ_3. Finally, take t big enough so that the error due to the local central limit theorem is smaller than δ_4 and so that the size of a square is bigger than n_{δ_5}. So the probability that the random walk has value 1 at time t is $\alpha \pm (\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5)$.

Suppose now that the random walk X is not finite range. We can then find X' and X'' such that X' is finite range and $X = X' + X''$. By space homogeneity, we have

$$\mu\otimes\mathbb{P}\left(\eta\left(X_t\right) = 1\right) = \sum_{x''\in\mathbb{Z}^d} \mu\otimes\mathbb{P}\left(\eta\left(x'' + X'_t\right) = 1 | X''_t = x''\right)\mathbb{P}\left(X''_t = x''\right) = \mu\otimes\mathbb{P}\left(\eta\left(X'''_t\right) = 1\right) \tag{2.29}$$

which is close to α for t sufficiently large. Here, X''' is the random walk evolving as X' but starting from 0.

As the different random walks are independent after time R, we then have,

$$\begin{aligned}
&\lim_{t\to\infty} \mu \otimes \mathbb{P}\big(\eta\big(X_t^{x_1,t,R}\big) = \ldots = \eta\big(X_t^{x_r,t,R}\big) = 1\big) \\
&\quad = \lim_{t\to\infty} \int \mathbb{P}\big(\eta\big(X_t^{x_1,t,R}\big) = \ldots = \eta\big(X_t^{x_r,t,R}\big) = 1 | \eta = \xi\big) \mu(d\xi) \\
&\quad = \int \sum_{k=1}^{r} c_R^{x_1,\ldots,x_r}(k) \alpha^k \mu(d\xi) \\
&\quad = \sum_{k=1}^{r} c_R^{x_1,\ldots,x_r}(k) \alpha^k.
\end{aligned} \tag{2.30}$$

□

3 Parabolic Anderson model

3.1 Model

As outlined in Introduction, the parabolic Anderson model is the partial differential equation

$$\frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + \gamma \xi(x,t) u(x,t) \quad x \in \mathbb{Z}^d, t \geqslant 0, \tag{3.1}$$

for the $\mathbb{R}$-valued random field u, with $\kappa \in [0,\infty)$ the diffusion constant, $\gamma \in [0,\infty)$ the coupling constant, Δ the discrete Laplacian, ξ the voter model starting from Bernoulli product measure μ_ρ with density $\rho \in (0,1)$ (with kernel $p(\cdot,\cdot) = p(\cdot)$ and $p^{(s)}(\cdot,\cdot)$ its symmetrization) and initial condition $u(x,0) = 1$ for all $x \in \mathbb{Z}^d$.

3.2 Lyapunov exponents

Our focus of interest will be on the p-th *annealed* Lyapunov exponent, defined by

$$\lambda_p = \lim_{t\to\infty} \frac{1}{t} \log\Big(\mathbb{E}_{\mu_\rho}\big([u(0,t)]^p\big)^{1/p}\Big), \qquad p \in \mathbb{N}, \tag{3.2}$$

which represents the exponential growth rate of the p-th moment of the solution of the equation (3.1). Note that λ_p also depends on the parameters κ, d, γ and ρ with the last two being fixed from now on. If the above limit exists, then, by Hölder's inequality, $\kappa \mapsto \lambda_p(\kappa)$ satisfies

$$\lambda_p(\kappa) \in [\rho\gamma, \gamma] \qquad \forall \kappa \in [0,\infty). \tag{3.3}$$

The behaviour of the annealed Lyapunov exponents with a voter model as catalyst has already been investigated by Jürgen GÄRTNER, Frank DEN HOLLANDER and Grégory MAILLARD [44], where it was shown that:

- the Lyapunov exponents defined in (3.2) exist;

- the function $\kappa \mapsto \lambda_p(\kappa)$ is globally Lipschitz outside any neighborhood of 0 and satisfies $\lambda_p(\kappa) > \rho\gamma$ for all $\kappa \in [0,\infty)$;
- the Lyapunov exponents satisfy the following dichotomy (see Figure 3.1):
  - when $1 \leqslant d \leqslant 4$, if $p(\cdot,\cdot)$ has zero mean and finite variance, then $\lambda_p(\kappa) = \gamma$ for all $\kappa \in [0,\infty)$;
  - when $d \geqslant 5$,
    - $\lim_{\kappa\to 0} \lambda_p(\kappa) = \lambda_p(0)$;
    - $\lim_{\kappa\to\infty} \lambda_p(\kappa) = \rho\gamma$;
    - if $p(\cdot,\cdot)$ has zero mean and finite variance, then $p \mapsto \lambda_p(\kappa)$ is strictly increasing for $\kappa \ll 1$.

The following questions were left open (see [44], Section 1.8):

(Q1) Does $\lambda_p < \gamma$ when $d \geqslant 5$ if $p(\cdot,\cdot)$ has zero mean and finite variance?

(Q2) Is there a full dichotomy in the behaviour of the Lyapunov exponents? Namely, $\lambda_p < \gamma$ if and only if $p^{(s)}(\cdot,\cdot)$ is strongly transient, i.e.,

$$\int_0^\infty t p_t^{(s)}(0,0)\, dt < \infty. \tag{3.4}$$

Since any transition kernel $p(\cdot,\cdot)$ in $d \geqslant 5$ satisfies $\int_0^\infty t\, p_t(0,0) dt < \infty$, a positive answer to (Q2) will also ensure a positive one to (Q1) in the particular case when $p(\cdot,\cdot)$ is symmetric. Theorems 3.3.2–3.3.4 in Section 3.3 give answers to question (Q2), depending on the symmetry of $p(\cdot,\cdot)$. A positive answer to (Q1), given in Theorem 3.3.1, can also be deduced from our proof of Theorem 3.3.2.

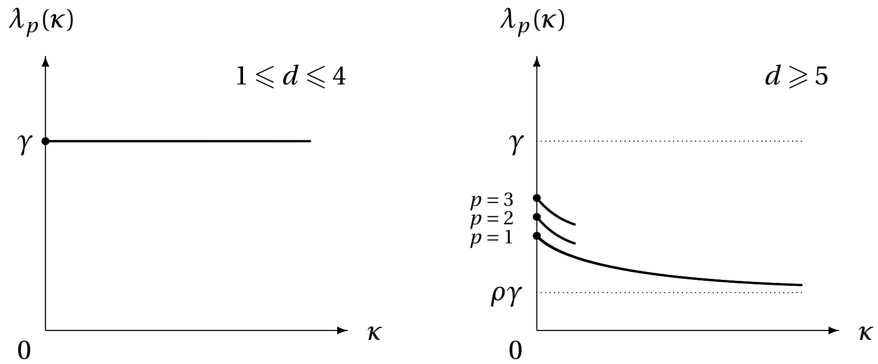


Figure 3.1: Dichotomy of the behaviour of $\kappa \mapsto \lambda_p(\kappa)$ when $p(\cdot,\cdot)$ has zero mean and finite variance.

By the Feynman-Kac formula, the solution of (3.1) reads

$$u(x,t) = \mathbb{E}_x\left(\exp\left[\gamma \int_0^t \xi\left(X^\kappa(s), t-s\right) ds\right]\right), \tag{3.5}$$

where $X^\kappa = (X^\kappa(t))_{t\geqslant 0}$ is a simple random walk on $\mathbb{Z}^d$ with step rate $2d\kappa$ and $\mathbb{E}_x$ denotes the expectation with respect to X^κ given $X^\kappa(0) = x$. This leads to the following representation of the Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty} \Lambda_p(t) \tag{3.6}$$

with

$$\Lambda_p(t) = \frac{1}{pt} \log\big(\mathbb{E}_{\mu_\rho} \otimes \mathbb{E}_0^{\otimes p}\big)\Big(\exp\Big[\gamma \int_0^t \sum_{j=1}^p \xi\big(X_j^\kappa(s), t-s\big)\, ds\Big]\Big), \tag{3.7}$$

where X_j^κ, $j = 1,\ldots,p$, are p independent copies of X^κ. In the above expression, the ξ and X^κ processes are evolving in time reversed directions. It is nevertheless possible to let them run in the same time evolution by using the following arguments. Let $\widetilde{\Lambda}_p(t)$ denote the ξ-*time-reversal* analogue of $\Lambda_p(t)$ defined by

$$\widetilde{\Lambda}_p(t) = \frac{1}{pt} \log\big(\mathbb{E}_{\mu_\rho} \otimes \mathbb{E}_0^{\otimes p}\big)\Big(\exp\Big[\gamma \int_0^t \sum_{j=1}^p \xi\big(X_j^\kappa(s), s\big)\, ds\Big]\Big) \tag{3.8}$$

and denote $\underline{\Lambda}_p(t) =$

$$\begin{aligned}
&\frac{1}{pt} \log \max_{x\in\mathbb{Z}^d} \big(\mathbb{E}_{\mu_\rho} \otimes \mathbb{E}_0^{\otimes p}\big)\Big(\exp\Big[\gamma \int_0^t \sum_{j=1}^p \xi\big(X_j^\kappa(s), t-s\big)\, ds\Big] \prod_{j=1}^p \delta_x\big(X_j^\kappa(t)\big)\Big)\\
&= \frac{1}{pt} \log \max_{x\in\mathbb{Z}^d} \big(\mathbb{E}_{\mu_\rho} \otimes \mathbb{E}_0^{\otimes p}\big)\Big(\exp\Big[\gamma \int_0^t \sum_{j=1}^p \xi\big(X_j^\kappa(s), s\big)\, ds\Big] \prod_{j=1}^p \delta_x\big(X_j^\kappa(t)\big)\Big),
\end{aligned}$$

where in the last line we reverse the time of the ξ-process by using that μ_ρ is shift-invariant and X_j^κ, $j = 1,\ldots,p$, are time-reversible. As noted in [22], Section 2.1, $\lim_{t\to\infty}[\Lambda_p(t) - \underline{\Lambda}_p(t)] = 0$ and, using the same argument, $\lim_{t\to\infty}[\widetilde{\Lambda}_p(t) - \underline{\Lambda}_p(t)] = 0$, after which we can conclude that

$$\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt} \log\big(\mathbb{E}_{\mu_\rho} \otimes \mathbb{E}_0^{\otimes p}\big)\Big(\exp\Big[\gamma \int_0^t \sum_{j=1}^p \xi\big(X_j^\kappa(s), s\big)\, ds\Big]\Big). \tag{3.9}$$

3.3 Main results

In what follows we give answers to questions (Q1) and (Q2) addressed in [44] concerning when the Lyapunov exponents are trivial, i.e., equal to their a priori maximal value γ.

Our first theorem gives a positive answer to (Q1). It will be proved in Section 3.4 as a consequence of the proof of Theorem 3.3.2.

Theorem 3.3.1. *If $d \geqslant 5$ and $p(\cdot,\cdot)$ has zero mean and finite variance, then $\lambda_p(\kappa) < \gamma$ for all $p \geqslant 1$ and $\kappa \in [0,\infty)$.*

Our two next theorems state that the full dichotomy in (Q2) holds in the case when $p(\cdot,\cdot)$ is symmetric (see Fig. 3.2). They will be proved in Section 3.4 and 3.5, respectively.

Theorem 3.3.2. *If $p(\cdot,\cdot)$ is symmetric and strongly transient, then $\lambda_p(\kappa) < \gamma$ for all $p \geqslant 1$ and $\kappa \in [0,\infty)$.*

Theorem 3.3.3. *If $p(\cdot,\cdot)$ is symmetric and not strongly transient, then $\lambda_p(\kappa) = \gamma$ for all $p \geqslant 1$ and $\kappa \in [0,\infty)$.*

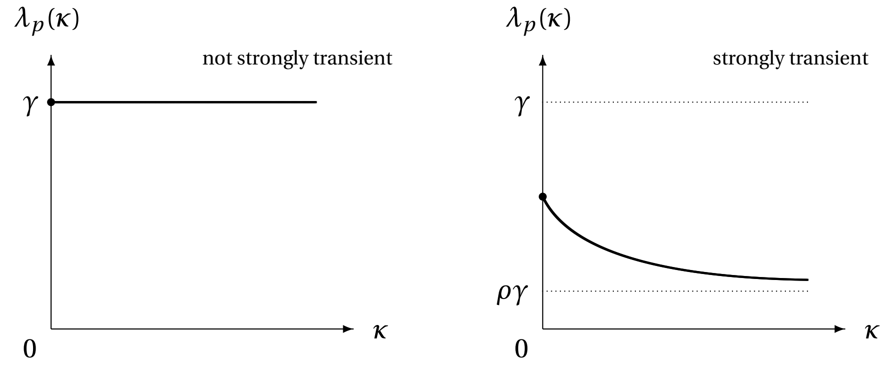


Figure 3.2: Full dichotomy of the behaviour of $\kappa \mapsto \lambda_p(\kappa)$ when $p(\cdot,\cdot)$ is symmetric.

A similar full dichotomy also holds for the case where ξ is symmetric exclusion process in equilibrium, between recurrent and transient $p(\cdot,\cdot)$ (see [42]).

Our fourth theorem shows that this full dichotomy only holds for symmetric transition kernels $p(\cdot,\cdot)$, ensuring that the assertion in (Q2) is not true in its full generality.

Theorem 3.3.4. *There exists $p(\cdot,\cdot)$ not symmetric with $p^{(s)}(\cdot,\cdot)$ not strongly transient such that $\lambda_p(\kappa) < \gamma$ for all $p \geqslant 1$ and $\kappa \in [0,\infty)$.*

In the strongly transient regime, the following problems remain open:

(a) $\lim_{\kappa\to 0} \lambda_p(\kappa) = \lambda_p(0)$;
(b) $\lim_{\kappa\to\infty} \lambda_p(\kappa) = \rho\gamma$;
(c) $p \mapsto \lambda_p(\kappa)$ is strictly increasing for $\kappa \ll 1$;
(d) $\kappa \mapsto \lambda_p(\kappa)$ is convex on $[0,\infty)$.

In [44], (a) and (b) were established when $d \geqslant 5$, and (c) when $d \geqslant 5$ and $p(\cdot,\cdot)$ has zero mean and finite variance. Their extension to the case when $p(\cdot,\cdot)$ is strongly transient remains open.

In what follows, we use generic notation $\mathbb{P}$ and $\mathbb{E}$ for probability and expectation whatever the corresponding process is (even for joint processes).

3.4 Proof of Theorems 3.3.1 and 3.3.2

We first give the proof of Theorem 3.3.2. Recall that the transition kernel associated to the voter model ξ is assumed to be symmetric. At the end of the section we will explain how to derive the proof of Theorem 3.3.1.

We have to show that $\lambda_p(\kappa) < \gamma$ for all $\kappa \in [0,\infty)$. In what follows we assume without loss of generality that $p = 1$, the extension to arbitrary $p \geqslant 1$ being straightforward. Our approach is to pick a *bad environment set* B_E associated to the ξ-process and a *bad random walk set* B_W associated to the random walk X^κ so that, for all $n \in \mathbb{N}$,

$$\begin{aligned}&\mathbb{E}\Big(\exp\Big[\gamma\int_0^n \xi(X^\kappa(s),s)\,ds\Big]\Big)\\&\qquad\leqslant \Big(\mathbb{P}(B_E)+\mathbb{P}(B_W)\Big)e^{\gamma n}+\mathbb{E}\Big(\mathbb{1}_{\{B_E^c\cap B_W^c\}}\exp\Big[\gamma\int_0^n \xi(X^\kappa(s),s)\,ds\Big]\Big)\end{aligned} \tag{3.10}$$

with, for some $0<\delta<1$,

$$\mathbb{P}(B_E)\leqslant e^{-\delta n}, \qquad \mathbb{P}(B_W)\leqslant e^{-\delta n}, \tag{3.11}$$

and,

$$\int_0^n \xi(X^\kappa(s),s)\,ds \leqslant n(1-\delta) \qquad \text{on } B_E^c\cap B_W^c. \tag{3.12}$$

Since, combining (3.10–3.12), we obtain

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\Big(\exp\Big[\gamma\int_0^n \xi(X^\kappa(s),s)\,ds\Big]\Big)<\gamma, \tag{3.13}$$

it is enough to prove (3.11) and (3.12).

The proof of (3.11) is given in Sections 3.4.1–3.4.3 below, and (3.12) will be obvious from our definitions of B_E and B_W.

3.4.1 Coarse-graining and skeletons

Write $\mathbb{Z}^d_{\mathrm{e}} = 2\mathbb{Z}^d$ and $\mathbb{Z}^d_{\mathrm{o}} = 2\mathbb{Z}^d+1$, where $1=(1,\dots,1)\in\mathbb{Z}^d$. We are going to use a *coarse-graining representation* defined by a *space-time block partition* B_y^j and a *random walk skeleton* $(y_i)_{i\geqslant 0}$. To that aim, for a fixed M, consider

$$B_y^j = \prod_{k=1}^d \big[(y_{[k]}-1)M,(y_{[k]}+1)M\big)\times\big[jM,(j+1)M\big)\subset\mathbb{Z}^d\times\mathbb{R}_+, \tag{3.14}$$

where $y_{[k]}$ is the kth coordinate of y, $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and

$$y \in \begin{cases} \mathbb{Z}^d_{\mathrm{e}} & \text{when } j \text{ is even,} \\ \mathbb{Z}^d_{\mathrm{o}} & \text{when } j \text{ is odd.} \end{cases} \tag{3.15}$$

Without loss of generality we can consider random walks trajectories on interval $[0, n]$ with $n \in \mathbb{N}$ multiple of M. Define the *M-skeleton set* by

$$\Xi = \left\{ \left(y_0, \ldots, y_{\frac{n}{M}} \right) \in (\mathbb{Z}^d)^{\frac{n}{M}+1} \colon\ y_{2k} \in \mathbb{Z}^d_{\mathrm{e}},\ y_{2k+1} \in \mathbb{Z}^d_{\mathrm{o}}\ \forall k \in \{0, \ldots, n/M\} \right\} \tag{3.16}$$

and the *M-skeleton set associated to a random walk X* by

$$\Xi(X) = \left\{ \left(y_0, \ldots, y_{\frac{n}{M}} \right) \in \Xi \colon\ X(kM) \in B^k_{y_k}\ \forall k \in \{0, \ldots, n/M\} \right\}. \tag{3.17}$$

In what follows, we will consider the M-skeleton $\Xi(X^\kappa)$, but, as X^κ starts from $0 \in \mathbb{Z}^d$, the first point of our M-skeleton will always be $y_0 := 0 \in \mathbb{Z}^d$ (see Fig. 3.3).

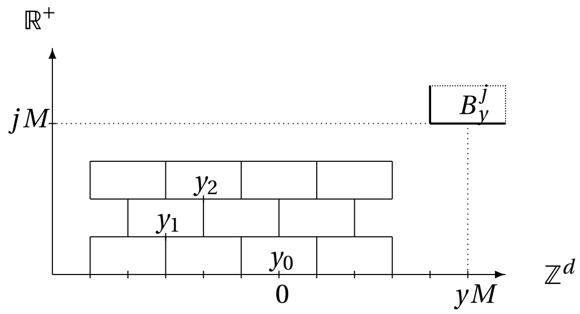


Figure 3.3: Illustration of a M-skeleton $(y_0, y_1, y_2, \ldots) \in \Xi$ and coarse-grained B^j_y blocks.

In the next lemma we prove that the number of M-skeletons not oscillating too much is at most exponential in n/M. For that, define

$$\Xi_A = \left\{ \left(y_0, \ldots, y_{\frac{n}{M}} \right) \in \Xi \colon\ \sum_{j=1}^{n/M} (\|y_j - y_{j-1}\|_\infty - 1) \leqslant \frac{n}{Md} \right\}, \tag{3.18}$$

the set of all M-skeletons that are *appropriate* where $\|\cdot\|_\infty$ is the standard l_∞ norm.

Lemma 3.4.1. *There exists some universal constant $K \in (1, \infty)$ such that, for any $n, M \in \mathbb{N}$,*

$$|\Xi_A| \leqslant K^{n/M}. \tag{3.19}$$

Proof. For any fixed $y_1 \in \mathbb{Z}^d$ and $N \in \mathbb{N}_0$, let

$$I(N) = \left| \left\{ y_2 \in \mathbb{Z}^d \colon\ \|y_1 - y_2\|_\infty - 1 = N \right\} \right| \tag{3.20}$$

be the number of elements of $\mathbb{Z}^d$ on the boundary of the cube of size $2N+3$ centered at y_1. For any $N \in \mathbb{N}$, we have $I(N) = (2N+3)^d - (2N+1)^d$ and $I(0) = 3^d - 1$, therefore, for any $N \in \mathbb{N}_0$,

$$I(N) \leqslant 3^d (N+1)^d. \tag{3.21}$$

Define, for any $N, k \in \mathbb{N}$,

$$I(N,k) = \left| \left\{ (y_j)_{0 \leqslant j \leqslant k} \in (\mathbb{Z}^d)^{k+1} : \sum_{j=1}^{k} (\| y_j - y_{j-1} \|_\infty - 1) = N \right\} \right| \tag{3.22}$$

the number of sequences in $(\mathbb{Z}^d)^{k+1}$ having size N. By (3.21) and (3.22), we have

$$\begin{aligned} I(N,k) &= \sum_{\substack{(N_1,\dots,N_k):\\ \sum_{i=1}^k N_i = N}} \left(\prod_{i=1}^{k} I(N_i) \right) \\ &\leqslant 3^{dk} \sum_{\substack{(N_1,\dots,N_k):\\ \sum_{i=1}^k N_i = N}} \left(\prod_{i=1}^{k} (N_i + 1) \right)^d \\ &\leqslant 3^{dk} \binom{k+N+1}{N} 2^{dN}, \end{aligned} \tag{3.23}$$

where, in the last line, we used that

$$\max_{\substack{(N_1,\dots,N_k):\\ \sum_{i=1}^k N_i = N}} \prod_{i=1}^{k} (N_i + 1) = 2^N. \tag{3.24}$$

Using (3.23) and the fact that $\Xi \subset \mathbb{Z}^d$, we obtain

$$\begin{aligned} |\Xi_A| &\leqslant \sum_{N=0}^{\lfloor \frac{n}{Md} \rfloor} I\left(N, \frac{n}{M}\right) \leqslant 3^{\frac{n}{M}} \sum_{N=0}^{\lfloor \frac{n}{Md} \rfloor} \binom{\frac{n}{M} + N + 1}{N} 2^{dN} \\ &\leqslant 3^{\frac{n}{M}} \sum_{N=0}^{\lfloor \frac{n}{Md} \rfloor} \binom{\frac{2n}{M}}{N} 2^{dN} \leqslant 3^{\frac{n}{M}} \sum_{N=0}^{\frac{2n}{M}} \binom{\frac{2n}{M}}{N} 2^{dN} \\ &= 3^{\frac{n}{M}} (2^d + 1)^{\frac{2n}{M}}, \end{aligned} \tag{3.25}$$

which ends the proof of the lemma. □

3.4.2 The bad environment set B_E

This section is devoted to the proof of the leftmost part of (3.11) for suitable set B_E defined below.

We say that an environment ξ is *good* w.r.t. an M-skeleton $(y_0,\cdots,y_{\frac{n}{M}})$ if we have

$$\left|\left\{0\leqslant j<\frac{n}{M}:\exists(x,jM)\in B^j_{y_j}\text{ s.t. }\xi(x,s)=0\ \forall s\in[jM,jM+1]\right\}\right|\geqslant\frac{n}{4M}. \tag{3.26}$$

Since we want the environment to be good w.r.t. all appropriate M-skeletons, we define the *bad environment set* as

$$B_E=\{\exists\text{ an }M\text{-skeleton}\in\Xi_A\text{ s.t. }\xi\text{ is not good w.r.t. it}\}. \tag{3.27}$$

In the next lemma we prove that for any fixed M-skeleton, the probability that ξ is not good w.r.t. it is at most exponentially small in n/M.

Lemma 3.4.2. *Take $(y_i)_{0\leqslant i\leqslant\frac{n}{M}}\in\Xi$ an M-skeleton. For M big enough, we have*

$$\mathbb{P}\Big(\xi\textit{ is not good w.r.t. }(y_i)_{0\leqslant i\leqslant\frac{n}{M}}\Big)\leqslant(4K)^{-n/M}, \tag{3.28}$$

where K is the universal constant defined in Lemma 3.4.1.

Therefore, combining Lemmas 3.4.1 and 3.4.2, we get

$$\mathbb{P}(B_E)\leqslant 4^{-n/M} \tag{3.29}$$

for M big enough, from which we obtain the leftmost part of (3.11).

Before proving Lemma 3.4.2, we first give an auxiliary lemma. In order to study the evolution of the voter model, we consider, as usual, the dual process, namely, a system of coalescing random walks that evolve backwards in time. However, we will consider random walks starting from different times. To that aim, define $(X^{x,t}(s))_{0\leqslant s\leqslant t}$ to be the random walk starting from x at time t (i.e., $X^{x,t}(0)=x$). From the graphical representation of the voter model, we can write $\xi(t)(x)=\xi(X^{x,t}(t),0)$, $x\in\mathbb{Z}^d$, and therefore the voter model can be expressed in terms of its initial configuration and a system of coalescing random walks. Two random walks $X^{x,s}$ and $X^{x',s'}$ with $s'<s$ meet if there exists $u\leqslant s'$ such that $X^{x',s'}(s'-u)=X^{x,s}(s-u)$. It is therefore the same to say that $X^{x,s}$ and $X^{x',s'}$ with $s'<s$ meet (in some appropriate time interval) if there exists $t\geqslant 0$ in this interval (by letting $t=s'-u$) such that

$$X^{x',s'}(t)=X^{x,s}(t+s-s'). \tag{3.30}$$

For convenience we will adopt this notation in the rest of the section.

Lemma 3.4.3. *Take two independent random walks $X^{x,s}$ and $X^{x',s'}$ with $s'<s$. Then the probability they ever meet is bounded above by*

$$\int_{s-s'}^{\infty}p_t(0,0)\,dt. \tag{3.31}$$

Proof. Consider the random variable

$$W = \int_0^\infty 1\left\{X^{x,s}(t) = X^{x',s'}(t+(s-s'))\right\} dt. \tag{3.32}$$

By symmetry, its expectation satisfies

$$\begin{aligned}\mathbb{E}(W) &= \int_0^\infty \mathbb{P}\left(X^{0,0}(2t+s-s') = x-x'\right) dt \\ &\leqslant \int_0^\infty \mathbb{P}\left(X^{0,0}(2t+s-s') = 0\right) dt \\ &= \frac{1}{2}\int_{s-s'}^\infty p_t(0,0)\, dt.\end{aligned} \tag{3.33}$$

Moreover, we have

$$\mathbb{E}(W \mid W>0) = \int_0^\infty p_{2t}(0,0)\, dt \geqslant \frac{1}{2} \tag{3.34}$$

and then, since $\mathbb{E}(W) = \mathbb{E}(W \mid W>0)\,\mathbb{P}(W>0)$, it follows that

$$\mathbb{P}(W>0) \leqslant 2\mathbb{E}(W) = \int_{s-s'}^\infty p_t(0,0)\, dt. \tag{3.35}$$

□

We are now ready to prove the Lemma 3.4.2.

Proof. Recall that $\xi(x,s) = \xi(X^{x,s}(s),0)$, where $\xi(0)$ is distributed according to a product Bernoulli law with density $\rho \in (0,1)$. We first consider any M-skeleton $(y_0,\dots,y_{n/M}) \in \Xi$ (even not appropriate). For each $0 \leqslant j < n/M$, we choose R sites $(x_1^j, jM),\dots,(x_R^j, jM) \in B_{y_j}^j$ such that

$$\mathbb{P}\left(\exists 0 \leqslant k,k' \leqslant R,\ k \neq k' \colon X^{x^j_k, jM}(s) = X^{x^j_{k'}, jM}(s) \text{ for some } s \in [0, jM]\right) \leqslant \epsilon \tag{3.36}$$

for $\epsilon \ll 1$ to be specified later (see Fig. 3.4). Remark that we first fix ϵ and R and then we choose M large enough so we can find these R sites. As we are in a strongly transient regime, we know that these points exist. If two such random walks hit each other, then we freeze all the random walks issuing from the corresponding block j.

For any $1 \leqslant j < \frac{n}{M}$, $1 \leqslant k \leqslant R$ for some $R>0$, we have

$$\begin{aligned}&\mathbb{E}\left(\sum_{j'=j+1}^{\frac{n}{M}-1}\sum_{k'=1}^{R} \mathbb{1}_{\left\{X^{x^{j'}_{k'}, j'M}(s+(j'-j)M) = X^{x^j_k, jM}(s) \text{ for some } s\in[0,jM]\right\}}\right) \\ &\qquad \leqslant R\sum_{j'=j+1}^{\frac{n}{M}-1}\int_{(j'-j)M}^\infty p_t(0,0)\, dt \leqslant R\int_M^\infty \frac{t}{M}\, p_t(0,0)\, dt,\end{aligned}$$

and therefore, summing over $1 \leqslant k \leqslant R$, we get

$$\mathbb{E}\left(\sum_{j'=j+1}^{\frac{n}{M}-1}\sum_{k,k'=1}^{R}\mathbb{1}_{\left\{X^{x_{k'}^{j'},j'M}(s+(j'-j)M)=X^{x_k^j,jM}(s)\text{ for some }s\in[0,jM]\right\}}\right)$$
$$\leqslant \frac{R^2}{M}\int_M^\infty t\,p_t(0,0)\,dt \leqslant \epsilon^2, \tag{3.37}$$

for M sufficiently large. Again, remark that we first fix ϵ and R, then we choose M large enough.

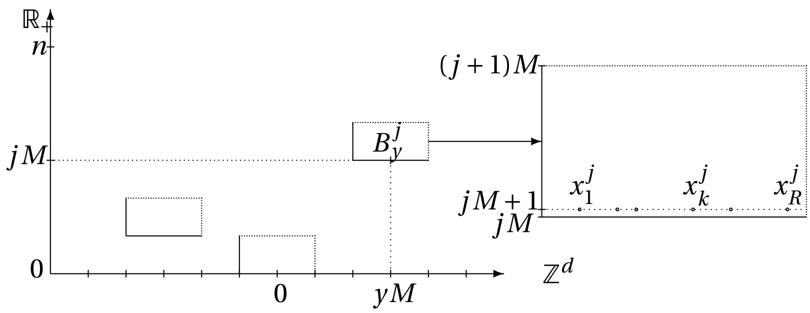


Figure 3.4: Illustration of sites (x_k^j, jM), $k \in \{1,\dots,R\}$, for a fixed M-skeleton.

For each j, we now define the filtration

$$\mathscr{F}_t^j = \sigma\left(X^{x_k^j,jM}(s)\colon 0 \leqslant s \leqslant t,\ 1 \leqslant k \leqslant R\right) \tag{3.38}$$

and the sub-martingale $Z^j(t) :=$

$$\mathbb{E}\left(\sum_{j'=j+1}^{\frac{n}{M}-1}\sum_{k,k'=1}^{R}\mathbb{1}_{\left\{X^{x_{k'}^{j'},j'M}(s+(j'-j))=X^{x_k^j,jM}(s)\text{ for some }s\in[0,t]\right\}}\ \Big|\ \mathscr{F}_t^j\right), \tag{3.39}$$

with the stopping time

$$\tau^j = \inf\left\{t \geqslant 0\colon Z^j(t) > \epsilon\right\}. \tag{3.40}$$

We freeze every random walk issuing from block j at time $jM \wedge \tau^j$. Using (3.37) and the Doob's inequality, we can see that

$$\mathbb{P}(\tau^j \leqslant jM) \leqslant \mathbb{P}\left(\sup_{0\leqslant t\leqslant jM} Z_t^j \geqslant \epsilon\right) \leqslant \frac{R^2}{M\epsilon}\int_M^\infty t p_t(0,0)\,dt \leqslant \epsilon. \tag{3.41}$$

Since, Z^j is a continuous sub-martingale except at jump times of one of the random walks

$X^{x_k^j, jM}$ and when a jump occurs, the increment is at most

$$R \sum_{j'=j+1}^{\frac{n}{M}-1} p_{(j'-j)M}(0,0) \leqslant \epsilon^2 \tag{3.42}$$

if M is big enough. Therefore, for all $0 \leqslant t \leqslant \tau^j$, we get

$$Z^j(t) < \epsilon + \epsilon^2 \leqslant 2\epsilon \qquad \mathbb{P}\text{-a.s.}. \tag{3.43}$$

Now we say that j is *good* if

- $\tau^j > jM$;
- the R random walks $X^{x_1^j, jM}, \cdots, X^{x_R^j, jM}$ do not meet;
- the random walks $X^{x_k^j, jM}$ do not hit any point $x_{k'}^{j'}$ during interval $[(j-j')M-1, (j-j')M]$ for $j' < j$;
- the random walks $X^{x_k^j, jM}$ do not meet $X^{x_{k'}^{j'}, j'M}$ for $j' < j$.

By (3.41), we know that the probability that the first condition does not occur is smaller than ϵ. By definition of the sites $x_1^j, \ldots, x_R^j$, we know that the probability that random walks issuing from the same block j at sites x_k^j, $k = 1, \ldots, R$, hit each other is smaller than ϵ (recall (3.36)). Moreover, the probability that the third condition does not occur is bounded from above by

$$R \sum_{j'=1}^{j-1} \int_{(j-j')M-1}^{(j-j')M} p_t(0,0)\, dt \tag{3.44}$$

which is as small as we want for M large because we are in a transient case. We still have to compute the probability that the fourth condition does not occur. Furthermore, we can see that two random walks issuing from the same block evolve independently until they meet, provided they do not meet a previous random walk. From the above considerations, we get

$$\mathbb{P}\left(j \text{ is not good} \mid \mathcal{G}^{j-1}\right) \leqslant \quad 3\epsilon + \sum_{j'=1}^{j-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \mid \mathcal{G}^{j'}\right), \tag{3.45}$$

where

$$\mathcal{G}^j = \sigma\left(X^{x_k^{j'}, j'M}(s) \colon 1 \leqslant k \leqslant R,\, 1 \leqslant j' \leqslant j,\, 0 \leqslant s \leqslant j'M \wedge \tau^{j'}\right). \tag{3.46}$$

Here, we recall that $X^{x_k^j, jM}$ meets $X^{x_{k'}^{j'}, j'M}$ (with $j' < j$) if we have

$$X^{x_k^j, jM}(s + (j - j')M) = X^{x_k^{j'}, j'M}(s) \text{ for some } s \in \left[0, \tau^{j'} \wedge j'M\right]. \tag{3.47}$$

By (3.43), we have, for all j' fixed,

$$\sum_{j=j'+1}^{\frac{n}{M}-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \;\middle|\; \mathcal{G}^{j'}\right) \leqslant 2\epsilon. \tag{3.48}$$

Summing over all $1 \leqslant j' \leqslant n/M - 2$, we get

$$\sum_{j'=1}^{\frac{n}{M}-2} \sum_{j=j'+1}^{\frac{n}{M}-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \;\middle|\; \mathcal{G}^{j'}\right) \leqslant \frac{n}{M} 2\epsilon, \tag{3.49}$$

and then, interchanging the sums, we arrive at

$$\sum_{j=2}^{\frac{n}{M}-1} \sum_{j'=1}^{j-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \;\middle|\; \mathcal{G}^{j'}\right) \leqslant \frac{n}{M} 2\epsilon. \tag{3.50}$$

Thus, there are at most $\lfloor \frac{n}{2M} \rfloor$ random positions j with the property

$$\sum_{j'=1}^{j-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \;\middle|\; \mathcal{G}^{j'}\right) \geqslant 4\epsilon, \tag{3.51}$$

and so at least $\lceil \frac{n}{2M} \rceil - 2$ random positions j have the property

$$\sum_{j'=1}^{j-1} \sum_{k,k'=1}^{R} \mathbb{P}\left(X^{x_k^j, jM} \text{ meets } X^{x_{k'}^{j'}, j'M} \;\middle|\; \mathcal{G}^{j'}\right) < 4\epsilon. \tag{3.52}$$

For these random positions j, we then have

$$\mathbb{P}\big(j \text{ is good} \mid \mathcal{G}^{j-1}\big) \geqslant 1 - 7\epsilon. \tag{3.53}$$

Using an elementary coupling, we have at least $\frac{n}{3M}$ positions that are good with probability bounded by

$$\mathbb{P}\left(Y \geqslant \frac{n}{3M}\right) \geqslant 1 - e^{-c(\epsilon)n/M} \tag{3.54}$$

for

$$Y \sim B\left(\left\lceil \frac{n}{2M} \right\rceil - 2, 1 - 7\epsilon\right) \tag{3.55}$$

and $c(\epsilon) \to \infty$ as $\epsilon \to 0$. Therefore, outside a small probability $e^{-c(\epsilon)n/M}$, for at least $\frac{n}{3M}$ positions j, we have that the random walks $X^{x_k^j, jM}$ are disjoint and so the values $\xi(X^{x_k^j, jM}(s), 0)$ are independent until time $s \leqslant jM$. Then, in this case, using the fact that $(\xi(x,0))_{x \in \mathbb{Z}^d}$ are i.i.d. according to a Bernoulli product measure with parameter ρ, we have that the number of positions j so that there exists $(x, jM) \in B_{y_j}^j$ with $\xi(x,s) = 0$ and $s \in [jM, jM+1]$ is at least $\frac{n}{4M}$

outside the probability, for some constant $c(R)$ (so that $c(R) \to \infty$ as $R \to \infty$),

$$\mathbb{P}\left(Y' \geqslant \frac{n}{12M}\right) \leqslant e^{-c(R)n/M} \tag{3.56}$$

with $Y' \sim B\left(\frac{n}{3M}, \left((1-e^{-1}(1-\rho))^R\right)\right)$, for R large enough.

Then, by equations (3.54) and (3.56), we have, for ϵ sufficiently small and R sufficiently large,

$$\mathbb{P}\left(\xi \text{ is not good w.r.t. } (y_i)_{0\leqslant i\leqslant \frac{n}{M}}\right) \leqslant e^{-c(\epsilon)n/M} + e^{-c(R)n/M}. \leqslant \left(\frac{1}{4K}\right)^{n/M}. \tag{3.57}$$

□

The proof of the leftmost part of (3.11) is now completed.

3.4.3 The bad random walk set B_W

This section is devoted to the proof of the rightmost part of (3.11).

We are now interested in the random walk X^κ. We are going to prove that $(X^\kappa(s))_{0\leqslant s\leqslant n}$ has an appropriate M-skeleton and touches enough zeros outside a probability event exponentially small in n (see Lemmas 3.4.4 and 3.4.5 below, respectively). To define the bad random set B_W announced in (3.5.11), we are going to define $B_W = B_{W_1} \cup B_{W_2}$, where the bad sets B_{W_1} and B_{W_2} correspond, respectively, to random walks trajectories X^κ which do not have appropriate M-skeleton and do not touch enough sites with value 0. To be more precise, define

$$B_{W_1} = \left\{(\Xi(X^\kappa) \notin \Xi_A\right\}. \tag{3.58}$$

B_{W_2} will be defined later, in (3.67). In the next lemma, we prove that the probability of B_{W_1} is exponentially small in n.

Lemma 3.4.4. *Take $(X^\kappa(s))_{0\leqslant s\leqslant n}$ and $\Xi(X^\kappa) = (y_0, \cdots, y_{n/M})$ the associated M-skeleton. Then, there exists a constant K' not depending on n such that*

$$\mathbb{P}(B_{W_1}) \leqslant e^{-K'n}. \tag{3.59}$$

Proof. In order to have the random walk moving from one block of the skeleton to a nonadjacent one, the random walk has to make at least M steps in the same direction. Keeping that in mind, define

$$Y_j(s) = X^\kappa(jM+s) - X^\kappa(jM) \tag{3.60}$$

and let

$$\tau_1^j = \inf\{s\colon \|Y_j(s)\|_\infty \geqslant M\}, \qquad \tau_i^j = \inf\left\{s > \tau_{i-1}^j\colon \|Y_j(s) - Y_j(\tau_{i-1}^j)\|_\infty \geqslant M\right\}. \tag{3.61}$$

Next, define

$$W_i^j = \mathbb{1}_{\left\{\{\tau_j^i < M\}\right\}} \tag{3.62}$$

and use an elementary coupling to have

$$\mathbb{P}(W_1^j = 1) \leqslant e^{-c/M} \quad \text{and} \quad \mathbb{P}(W_i^j = 1 | W_{i-1}^j = 1) \leqslant e^{-c(i)/M} \leqslant e^{-c/M} \tag{3.63}$$

for some constants c and $c(i)$ which verify $c(i) \geqslant c$. Therefore, we have that the number of jumps for the jth block is bounded above by the number of W_i^j equal to 1. Using a coupling we can see that this is bounded above by a geometric law with parameter $e^{-c/M}$. Now if we consider all the blocks, by elementary properties of geometric random variables, we have

$$\mathbb{P}(B_{W_1}) \leqslant \mathbb{P}\left(Y \geqslant \frac{n}{Md}\right) \leqslant e^{-c'n} \tag{3.64}$$

for $Y \sim B\left(\frac{n}{M}\left(1 + \frac{1}{d}\right), e^{-c/M}\right)$, some constant $c' > 0$, M being large and the proof is done. □

Lemma 3.4.4 proves the first part of the rightmost part of (3.11), namely the part concerned with bad set B_{W_1}. Now we look at the number of times X^κ stays on a site where the voter model has zero value. For that, define

$$\tau_{i+1} = \inf\left\{t > \tau_i + 1 : \exists x \in \mathbb{Z}^d \text{ s.t. } \|x - X^\kappa(t)\|_\infty \leqslant 2M, \xi(x,s) = 0 \ \forall s \in [t, t+1]\right\} \tag{3.65}$$

with $\tau_0 = 0$ and

$$k(M) = e^{-1/2} \inf_{\|x\|_\infty \leqslant 2M} \mathbb{P}\big(X^\kappa(1/2) = x \mid X^\kappa(0) = 0\big) \tag{3.66}$$

(remark that $k(M)$ does not depend on n and is strictly positive). Finally, we define

$$B_{W_2} = \left\{(X^\kappa, \xi) : \tau_{\lfloor \frac{n}{2M} \rfloor} \leqslant n - 1 \text{ and } \int_0^n \xi(X^\kappa(s), s)\, ds \geqslant n\left(1 - \frac{k(M)}{8M}\right)\right\} \tag{3.67}$$

as being the bad set corresponding to random walks trajectories X^κ which do not touch enough sites occupied by a zero configuration of the voter model. In the next lemma, we prove that such a set has an exponentially small probability in n.

Lemma 3.4.5. *There exists a constant $\delta > 0$ not depending on n, such that, for M big enough we have*

$$\mathbb{P}(B_{W_2}) \leqslant e^{-n\delta}. \tag{3.68}$$

Proof. Take any realization of ξ and for each time τ_i, define a random variable Y_i which takes value 1 if X^κ reaches a site with value zero at time $\tau_i + \frac{1}{2}$ and stays at this site until time $\tau_i + 1$, or takes value 0 otherwise. Remark that after having fixed n, we can choose the state of $\xi(t)$,

$t > n$, as we want, for example, full of zeros. Continue until $\tau_{\lfloor n/(2M)\rfloor}$ which is finite if ξ is well chosen after time n. Using the strong Markov property, for every $k_i \in \{0,1\}$, we see that

$$\mathbb{P}(Y_i = 1 | Y_j = k_j, j < i) = \mathbb{P}(Y_i = 1 | Y_{i-1} = k_{i-1}) \geqslant k(M)\,. \tag{3.69}$$

Then, it follows that $Y := \sum_{i=1}^{\lfloor n/(2M)\rfloor} Y_i$ is stochastically greater than Y' the binomial random variable $B\left(\frac{n}{2M}, k(M)\right)$. Moreover, if $\tau_{\lfloor n/(2M)\rfloor} \leqslant n-1$, we have

$$\int_0^n \xi(X(s),s)\,ds \leqslant n - \frac{1}{2}\,Y\,. \tag{3.70}$$

Hence, we get

$$\begin{aligned}
\mathbb{P}(B_{W_2}) &\leqslant \mathbb{P}\left(\tau_{\lfloor \frac{n}{2M}\rfloor} \leqslant n-1 \text{ and } n - \frac{1}{2}\,Y \geqslant n - \frac{nk(M)}{8M}\right) \\
&\leqslant \mathbb{P}\left(n - \frac{Y}{2} \geqslant n - \frac{nk(M)}{8M}\right) \\
&= \mathbb{P}\left(Y \leqslant \frac{nk(M)}{4M}\right) \\
&\leqslant e^{-cn}
\end{aligned}$$

for n sufficiently large and c a positive constant not depending on n. This result being shown for any realization of ξ (up to time n), this ends the proof. $\square$

Lemma 3.4.5 proves the second part of the rightmost part of (3.11), namely the part concerned with bad set B_{W_2}. To complete the proof of (3.12), it suffices to use the definition of B_E and $B_W = B_{W_1} \cup B_{W_2}$, and to remark that if B_E and B_{W_1} do not occur, then the first condition of B_{W_2}, namely $\tau_{\lfloor n/2M\rfloor} \leqslant n-1$, is satisfied and therefore the second must be violated.

3.4.4 Proof of Theorem 3.3.1

The proof of Theorem 3.3.1 can be deduced from the proof of Theorem 3.3.2. Without assuming that $p(\cdot,\cdot)$ is symmetric, it is enough to see that

- $\int_0^\infty t p_t(0,0)\,dt < \infty$, by local central limit theorem, and
- there is enough symmetry because there exists some $C > 0$ such that $p_t(x,0) \leqslant C p_t(0,0)$ for all $x \in \mathbb{Z}^d$ and $t \in [0,\infty)$. Therefore, Lemma 3.4.2 can still be applied.

From these two observations, the proof of Theorem 3.3.1 goes through the same lines as the one of Theorem 3.3.2.

3.5 Proof of Theorem 3.3.3

In this section we consider the Lyapunov exponents when the random walk kernel associated to the voter model noise is symmetric and also not strongly transient, that is

$$\int_0^\infty t p_t(0,0)\, dt = \infty. \tag{3.71}$$

We want to show that when $p(\cdot,\cdot)$ is symmetric and not strongly transient, then

$$\lambda_p(\kappa) \equiv \gamma \qquad \forall \kappa \in (0,\infty),\ \forall p \geqslant 1. \tag{3.72}$$

Since the result is easily seen for recurrent random walks, we can and will assume in the following that

$$\int_0^\infty p_t(0,0)\, dt < \infty. \tag{3.73}$$

Given the reasoning of [42], Section 3.1 and [44], Section 5.1 this result will follow from Proposition 3.5.1, below. Consider, in the graphical representation associated to the voter model ξ,

$$\chi(t) := \text{ number of distinct coalescing random walks produced on } \{0\} \times [0,t] \tag{3.74}$$

(This quantity is discussed in Bramson, Cox and Griffeath [7]).

Proposition 3.5.1. *Assume that $p(\cdot,\cdot)$, is symmetric and not strongly transient, then for any $\epsilon > 0$, we have that*

$$\lim_{t\to\infty} \mathbb{P}\big(\chi(t) \leqslant \epsilon t\big) = 1. \tag{3.75}$$

Before proving Proposition 3.5.1, we will first give the proof of Theorem 3.3.3.

Proof. From the graphical representation of the voter model and Proposition 3.5.1, we can see that for all $\delta > 0$ and $M < \infty$,

$$\mathbb{P}\Big(\xi(x,s) = 1\ \ \forall \|x\|_\infty \leqslant M,\ \forall s \in [0,t]\Big) \geqslant e^{-\delta t} \tag{3.76}$$

for all t sufficiently large (see [44], proof of Lemma 5.1). Thus, just as in [44], Section 5.1, we have for all $p \geqslant 1$,

$$\begin{aligned}
&\mathbb{E}([u(0,t)]^p) \\
&\qquad \geqslant e^{\gamma p t}\mathbb{P}\Big(\|X^\kappa(s)\|_\infty < M\ \ \forall s \in [0,t]\Big)^p \mathbb{P}\Big(\xi(x,s) = 1\ \ \forall \|x\|_\infty < M,\ \forall s \in [0,t]\Big) \\
&\qquad \geqslant e^{t(\gamma p - \delta - c(M)p)}
\end{aligned} \tag{3.77}$$

for $c(M) \to 0$ as $M \to \infty$. From this, it is immediate that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}([u(0,t)]^p)^{1/p} = \gamma. \tag{3.78}$$

□

To prove Proposition 3.5.1, we consider the following system of coalescing random walks

$$I = \{X^t : t \in P\} \tag{3.79}$$

for P a two sided, rate one Poisson process and X^t a random walk defined on $s \in [t,\infty)$, starting at 0 at time t (we could equally well consider a system of random walks indexed by $h\mathbb{Z}$ for some constant h). The coalescence is such that for $t < t' \in P$, X^t, $X^{t'}$ evolve independently until $T^{t,t'} = \inf\{s > t' : X^t(s) = X^{t'}(s)\}$, and then, for $s \geqslant T^{t,t'}$, $X^t(s) = X^{t'}(s)$.

We will be interested in the density or number of distinct random walks at certain times. To aid this line we will adopt a labelling procedure for the random walks, whereby effectively when two random walks meet for the first time, one of them (chosen at random) dies; in this optic the number of distinct random walks will be the number still alive. Our labeling scheme involves defining for each $t \in P$, the label process l^t_s for $s \geqslant t-$ (it will be helpful to be able to define $l^t_{t-} = t$, though since at time t there may well be other random walks present at the origin, it will not necessarily be the case that $l^t_t = t$). These processes will be defined by the following properties:

- if for $t \neq t' \in P$, $X^t(s) \neq X^{t'}(s)$, then $l^t_s \neq l^{t'}_s$;
- if $t_1, t_2, \ldots, t_r$ are elements of P, then at $s \geqslant \max\{t_1, \ldots, t_r\}$, if $X^{t_1}(s) = X^{t_2}(s) = \cdots = X^{t_r}(s)$, then $l^{t_1}_s = l^{t_2}_s = \cdots = l^{t_r}_s = u$ for some $u \in P$ with $X^{t_1}(s) = X^u(s)$;
- if for $t \neq t' \in P$, X^t meets $X^{t'}$ for the first time at s, then independently of past and future random walks or labeling decisions $l^t_s = l^{t'}_s = l^{t'}_{s-}$ with probability $\frac{1}{2}$ and with equal probability $l^t_s = l^{t'}_s = l^t_{s-}$;
- the process l^t_s can only change at moments where X^t meets a distinct random walk for the first time.

For $t \in P$, $s > t$, we say that t is *alive at time* s, if $l^t_s = t$; it *dies* at time s if $l^t_{s-} = t, l^t_s \neq t$. We say X^t, X^u *coalesce* at time s if this is the first time at which the two labels are equal. The following are easily seen:

- the events $A^t_s = \{l^t_s = t\}$ for $t \in P$ are decreasing in s;
- A^t_s depends only on the random motions of the coalescing random walks and on the labeling choices involving X^t;
- for $s > 0$, the number of independent random walks $X^t(s)$, $t \in P \cap [-n, 0]$ is simply equal to the number of distinct labels l^t_s, $t \in P \cap [-n, 0]$.

Let

$$c_0 = \lim_{s\to\infty} \mathbb{P}^t(A_s^t) \in [0,1], \tag{3.80}$$

according to palm measure, $\mathbb{P}^t$, for $t \in P$. We obtain easily:

Proposition 3.5.2.

$$\lim_{s\to\infty} \frac{1}{s} |\{\textit{distinct random walks } X^t(0) \colon t \in P \cap [-s,0)\}| = c_0 \qquad \textit{a.s.} \tag{3.81}$$

Proof. Using the definition of c_0 in (3.80) and ergodicity of the system we see that the limit is greater than c_0. Then, Lemma 3.5.3 gives the result. □

Lemma 3.5.3. *For c_0 as defined in (3.80), for each $\epsilon > 0$, there exists $R < \infty$ so that if we consider the finite system of coalescing random walks $(X^t)_{t\in(-R,0]\cap P}$, then with probability at least $1-\epsilon$ at time R there are less than $(c_0+\epsilon)R$ distinct random walks labels.*

Proof. By definition of c_0, for all $\epsilon > 0$ there exists a T_0 so that

$$\mathbb{P}^0\big(\text{ label 0 is alive at time } s\big) < c_0 + \frac{\epsilon}{100} \qquad \forall s \geqslant T_0. \tag{3.82}$$

Now pick R_1 so that

$$\mathbb{P}\big(\|X^0(s)\|_\infty \leqslant R_1 \;\; \forall s \in (0,T_0)\big) \geqslant 1 - \frac{\epsilon}{100}. \tag{3.83}$$

Therefore,

$$\begin{aligned} &\mathbb{P}^0\big(\text{ label 0 is not alive at time } s \geqslant T_0 \text{ and } \|X^0(s)\|_\infty \leqslant R_1 \;\; \forall s \in (0,T_0)\big) \\ &\quad \geqslant 1 - c_0 - \frac{2\epsilon}{100}. \end{aligned} \tag{3.84}$$

We then pick T_1 so that

$$\mathbb{P}\big(\exists t \in P \cap [-T_1,T_1]^{\mathrm{c}} \colon \|X^t(s)\|_\infty \leqslant R_1 \text{ for some } s \in (0,T_0)\big) < \frac{\epsilon}{100}. \tag{3.85}$$

Thus

$$\mathbb{P}^0\big(\exists t \in P \cap [-T_1,T_1] \setminus \{0\} \colon l_{R_1}^0 = t\big) \geqslant 1 - c_0 - \frac{\epsilon}{30}. \tag{3.86}$$

From the translation invariant property of the system and ergodicity if

$$\lambda_s := \Big|\Big\{t \in [-s,s] \cap P \colon X^t \text{ loses its label to a random walk } X^{t'} \text{ with } |t-t'| \leqslant T_1\Big\}\Big|, \tag{3.87}$$

then

$$\liminf_{s\to\infty} \frac{\lambda_s}{2s} \geqslant 1 - c_0 - \frac{\epsilon}{30} \qquad \text{a.s.} \tag{3.88}$$

The result now follows easily. □

Proposition 3.5.1 will be proven by showing:

Proposition 3.5.4. *If $p(\cdot,\cdot)$ is symmetric and not strongly transient, then $c_0 = 0$.*

The proof of Proposition 3.5.4 will work for any Poisson process rate, in particular for P having rate $M \gg 1$. The distinct random walks treated in Proposition 3.5.1 can be divided into those coalesced with a random walk from the system derived from P (and so by Proposition 3.5.4 of small "density") and those uncoalesced (also of small "density" if M is large). Thus Proposition 3.5.1 follows almost immediately from Proposition 3.5.4.

The argument for Proposition 3.5.4 is low level and intuitive. We argue by contradiction and suppose that $c_0 > 0$. From this we can deduce, loosely speaking, that after a certain time either a random walk has lost its original label, or it will keep it forever. We then introduce coupling on these random walks so that we may regard these random walks as essentially independent random walks starting at 0 (at different times). We then introduce convenient comparison systems so that we can analyze subsequent coalescences. We will use automatically, without reference, the following "obvious" result:

Lemma 3.5.5. *Consider two collections of coalescing random walks $\{Y^i\}$ and $\{(Y')^i\}$ for i in some index set. If the coalescence rule is weaker for the $\{(Y')^i\}$ system, in that if two walks $(Y')^i$ and $(Y')^j$ are permitted to coalesce at time t, then so are Y^i and Y^j, then there is a coupling of the two systems so that the weaker contains the stronger.*

We now fix $\epsilon > 0$ so that $\epsilon \ll c_0$ (by hypothesis $c_0 > 0$). We choose R according to Lemma 3.5.3 and divide up time into intervals $I_j = [jR, (j+1)R)$. We first consider the coalescing system where random walks X^t, $X^{t'}$, $t, t' \in P$, can only "coalesce" (or destroy a label t or t') if t, t' are in the same I_j interval. Thus we have a system of random walks that is invariant to time shifts by integer multiples of R. We now introduce a system of random walks Y^t, $t \in V := \cup_j \{[jR, (j+1)R) \cap jR + \frac{1}{c_0}\mathbb{Z}\}$. The random walks Y^t, $t \in \left[jR, (j+1)R\right)$ are not permitted to coalesce up until time $(j+1)R$ (at least) and will evolve independently of the system $(X^t)_{t\in\mathscr{P}}$ until time $(j+1)R$. We will match up the points in $V \cap I_j$, with those in $P \cap I_j \cap K$ in a maximal measurable way for $K = \{t \in P\colon \text{label } t \text{ survives to time } (j+1)R\}$.

Lemma 3.5.6. *Unmatched points in $\cup_j P \cap I_j \cap K$ and in V have density less than 2ϵ for R fixed sufficiently large.*

Remark: the system is not translation invariant with respect to all shifts but it possesses enough invariance for us to speak of densities.

We similarly have

Lemma 3.5.7. *Unmatched Y particles have density less than 2ϵ for R fixed sufficiently large.*

It is elementary that two random walks X, Z can be coupled so that for t sufficiently large $X(t) = Z(t)$. For given $\epsilon > 0$ we choose M_0 and then M_1 so that

$$\mathbb{P}\Big(\sup_{t\leqslant R} \|X(t)\|_\infty \geqslant M_0\Big) < \frac{\epsilon}{10} \tag{3.89}$$

and

$$\sup_{|z|\leqslant 2M_0} \mathbb{P}\Big(X_0, Z_z \text{ not coupled by time } M_1\Big) < \frac{\epsilon}{10}, \tag{3.90}$$

where X_0 and Z_z denote that the random walks X and Z start at point 0 and point z respectively.

We then (on interval I_j) couple systems Y and X by letting married pairs Y^t, $X^{t'}$, $t \in V$, $t' \in P \cap K$, evolve independently of other Y, X random walks so that they couple by time $M_1 + (j+1)R$ with probability at least $1 - \frac{3\epsilon}{10}$.

Thus we have two types of random walk labels, l^t, for the X system which are equal to t at time $t + R + M_1$: those for which the associated random walk was paired with a Y random walk and such that the random walks have coupled by time $t + R + M_1$ said to be *coupled* and the others, said to be *decoupled*. Similarly for the points in V associated to Y random walks. We note that the foregoing implies that the density of uncoupled labels is bounded by $2\epsilon + \frac{3\epsilon}{10} \leqslant 3\epsilon$. The point is that modulo this small density, we have an identification of the coalescing X random walks and the Y random walks.

We now try to show that enough Y particles will coalesce in a subsequent time interval to imply that there will be a significant decrease in surviving labels for the X system. To do this we must bear in mind that, essentially, it will be sufficient to show a decrease in the density of Y random walk labels definitely greater than ϵ. Secondly, as already noted, we will adopt a coalescence scheme that is a little complicated namely $Y^i, Y^{i'}$ in V can only "coalesce" at time $t \geqslant \max\{i, i'\}$ if

- $(t - i', t - i)$ are in some time set to be specified;
- for $i < i'$, $\frac{t-i'}{i'-i} \in \left(\frac{9}{10}, \frac{11}{10}\right)$.

We now begin to specify our coalescence rules for the random walk system $\{Y^i\}_{i\in V}$. The objective here will be to facilitate the necessary calculations. A first objective is to have coalescence of $Y^i, Y^{i'}$ at times $t > \max\{i, i'\}$ so that $p_t(0,0)$ is well behaved around $t - i$, $t - i'$. It follows from symmetry of the random walk that $t \mapsto p_t(0,0)$ is decreasing. The problem we address is that it is not immediate how to achieve bounds in the opposite direction. This is the purpose of the next result.

Lemma 3.5.8. *Consider positive $\{a_n\}_{n\geqslant 0}$ so that $\sum_{n=0}^\infty a_n = \infty$. For all $r \in \mathbb{Z}_+$, there exists a subsequence $\{a_{n_i}\}_{i\geqslant 0}$ so that*
(i) $\sum a_{n_i} = \infty$;
(ii) $a_{n_i} > \frac{1}{2} a_{n_i - r}$, $\forall i \geqslant 0$.

Proof. If $r > 1$ we may consider the r subsequences $\{a_{ri+j}\}_{i\geqslant 0}$ for $j \in \{0, 1, \cdots, r-1\}$. At least one of these must satisfy $\sum a_{ri+j} = \infty$ so, without loss of generality, we take $r = 1$.

Now we classify i as good or bad according to whether $a_i > a_{i-1}/2$ or not. This decomposes $\mathbb{Z}$ into intervals of bad sites, alternating with intervals of good sites. By geometric bounds, the sum of bad sites is bounded by the sum of the good a_i for which i is the right end point of a good interval. Thus we have

$$\sum_{i \text{ good}} a_i = \infty, \tag{3.91}$$

from which the result is immediate. □

Corollary 3.5.9. *For our symmetric kernel $p_t(0,0)$ we can find $n_i \uparrow \infty$ so that*
(i) $\sum_i \int_{2^{n_i}}^{2^{n_i+1}} t p_t(0,0)\, dt = \infty$;
(ii) $p_{2^{n_i-1}}(0,0) \leqslant 2^{12} p_{2^{n_i+3}}(0,0)$.

Proof. In Lemma 3.5.8 take

$$a_n = \int_{2^{n+3}}^{2^{n+4}} t p_t(0,0)\, dt \tag{3.92}$$

and take $r = 5$. Then by the monotonicity of $t \to p_t(0,0)$, we have

$$2^{2n_i+7} p_{2^{n_i+3}}(0,0) \geqslant a_{n_i} \geqslant \frac{1}{2} a_{n_i-5} \geqslant \frac{1}{2} 2^{2n_i-4} p_{2^{n_i-1}}(0,0). \tag{3.93}$$

□

We fix such a sequence $\{n_j\}_{j\geqslant 1}$ once and for all.

We assume, as we may, that $n_j < n_{j+1} - 4$ for all $j \geqslant 1$ and also assume again, as we may, that

$$\int_{2^{n_j}}^{2^{n_j+1}} t p_t(0,0)\, dt < \frac{\epsilon}{100} \tag{3.94}$$

for all $j \geqslant 1$. We are now ready to consider our coalescence rules. We choose $\epsilon \ll \alpha \ll 1$ (we will fully specify α later on but we feel it more natural to defer the technical relations). We then choose k_0 so that $2^{n_{k_0}} > R + M_1$ with R as in Lemma 3.5.3 and Lemma 3.5.6 and M_1 as in (3.90), and

$$k_1 := \inf\left\{k > k_0 \colon \sum_{j=k_0}^{k} \int_{2^{n_j}}^{2^{n_j+1}} t p_t(0,0)\, dt > \alpha\right\}. \tag{3.95}$$

We have coalescence between Y^i and $Y^{i'}$, for $i < i'$ only at $t \in [i' + 2^{n_j}, i' + 2^{n_j+1}]$, $j \in [k_0, k_1]$ if

(a) $\frac{t-i'}{i'-i} \in (9/10, 11/10)$;
(b) the interval of $t \in [i' + 2^{n_j}, i' + 2^{n_j+1}]$ satisfying (a) is of length at least 2.

We say (i, i') and (i', i) are in j and write $(i, i') \in j$ if the above relations hold. To show that sufficient coalescence occurs, we essentially use Bonferroni inequalities (see, e.g., [29], p. 21).

To aid our argument we introduce a family of independent (non coalescing) random walks $\{(Z^i(s))_{s\geqslant i}\}_{i\in V}$ such that for each $i \in V$, $Y^i(s) = Z^i(s)$ for $s \geqslant i$ such that $l_s^i = i$. In the following we will deal with random walks Y^0, Z^0, but lack of total translation invariance notwithstanding, it will be easy to see that all bounds obtained for these random walks remain valid for more general random walks Y^i, Z^i. For a given random walk Y^0, say, the probability that Y^0 is killed by Y^i (with i possible in the sense of the above rules) is in principle a complicated event given the whole system of coalescing random walks. Certainly the event

$$\{Z^0 \text{ meets } Z^i \text{ in appropriate time interval after first having met } Z^k\} \tag{3.96}$$

is easier to deal with than the corresponding Y event. From this point on we will shorten our phraseology by taking "Z^i hits Z^k" to mean that Z^i meets Z^k at a time t satisfying the conditions (a) and (b) above with respect to i, k.

For $(Z^0(s))_{s\geqslant 0}$ and $(Z^i(s))_{s\geqslant i}$ independent random walks each beginning at 0, we first estimate

$$\sum_i \mathbb{P}\big(Z^0 \text{ hits } Z^i\big). \tag{3.97}$$

This of course decomposes as

$$\sum_j \sum_{(0,i)\in j} \mathbb{P}\big(Z^0 \text{ hits } Z^i\big). \tag{3.98}$$

We fix j and consider $i > 0$ so that $(0, i) \in j$ (the case $i < 0$ is similar). That is the interval of times s with

$$(s-i)/i \in (9/10, 11/10), \ \ s \in [i + 2^{n_j}, i + 2^{n_j+1}] \tag{3.99}$$

is at least 2 in length: we note that for each $i \in \big(\frac{5}{4}2^{n_j}, \frac{7}{4}2^{n_j}\big)$ the relevant interval, $\frac{19}{10} i \leqslant s \leqslant \frac{21}{10} i$ is an interval of length greater than $\frac{5}{4}2^{n_j}\frac{1}{5} = \frac{2^{n_j}}{4}$.

Lemma 3.5.10. *There exists $c_2 \in (0,\infty)$ so that for any interval I of length at least 1 contained in $(1,\infty)$,*

$$\frac{1}{c_2}\int_I p_t(0,0)\,dt \leqslant \mathbb{P}\big(X^0(t) = 0 \textit{ for some } t \in I\big) \leqslant c_2 \int_I p_t(0,0)\,dt \tag{3.100}$$

Proof. Consider random variable $W = \int_a^{b+1} 1_{\{X^0(s)=0\}}\, ds$ for $I = [a,b]$. Then

$$\mathbb{E}(W) = \int_a^{b+1} \mathbb{P}(X^0(s)=0)\, ds = \int_a^{b+1} p_s(0,0)\, ds \leqslant 2\int_I p_s(0,0)\, ds, \tag{3.101}$$

by monotonicity of $p_s(0,0)$ and the fact that $b-a \geqslant 1$. But for $\tau := \inf\{s \in I\colon X^0(s)=0\}$ we have $\mathbb{E}(W|\mathscr{F}_\tau) \geqslant e^{-1}$ on $\{\tau < \infty\}$ so

$$\mathbb{P}(\tau<\infty) = \mathbb{P}\big(X^0(t) = 0 \text{ for some } t \in I\big) \leqslant \mathbb{E}(W)e \leqslant 2e\int_I p_s(0,0)ds. \tag{3.102}$$

Equally for $W' = \int_a^b 1_{\{X^0(s)=0\}}\, ds$, we have

$$\mathbb{E}(W'|\mathscr{F}_\tau) \leqslant \gamma = \int_0^\infty p_s(0,0)\, ds \quad \text{on } \{\tau<\infty\} \tag{3.103}$$

and so

$$\mathbb{P}(\tau<\infty) \geqslant \frac{\mathbb{E}(W')}{\gamma} = \frac{1}{\gamma}\int_a^b p_s(0,0)\, ds. \tag{3.104}$$

□

Proposition 3.5.11. *For some universal $c_3 \in (0,\infty)$,*

$$c_3^{-1} 2^{2n_j} p_{2^{n_j}}(0,0) \leqslant \sum_{(0,i)\in j} \mathbb{P}\big(Z^0 \textit{ hits } Z^i\big) \leqslant c_3 2^{2n_j} p_{2^{n_j}}(0,0). \tag{3.105}$$

Proof. We consider first the upper bound. There are less than 2^{n_j} relevant i. For such an i,

$$\mathbb{P}\big(Z^0 \text{ hits } Z^i\big) \leqslant \mathbb{P}\big(X^0(t) \text{ hits } 0 \text{ for some } t \in \big[a+2^{n_j}, a+3\cdot 2^{n_j}\big]\big) \tag{3.106}$$

for some $a \geqslant 0$. By monotonicity of $t \to p_t(0,0)$ and using Lemma 3.5.10, this is bounded by

$$c_2 \int_{2^{n_j}}^{3\cdot 2^{n_j}} p_s(0,0)\, ds \leqslant c_3 2^{n_j} p_{2^{n_j}}(0,0) \tag{3.107}$$

for some $c_3 > 0$. On the other side the number of $i \in \big(\frac{5}{4}2^{n_j}, \frac{7}{4}2^{n_j}\big)$ is greater than $c_1\frac{1}{3}2^{n_j}$ if R was fixed sufficiently large and for each such i, $(\frac{9}{10}i, \frac{11}{10}i) \subset [2^{n_j}, 2^{n_j+1}]$. Moreover, we have

$$\begin{aligned}\mathbb{P}\big(Z^0 \text{ hits } Z^i\big) &\geqslant \frac{1}{c_2}\int_{\frac{28}{10}i}^{\frac{32}{10}i} p_s(0,0)\, ds \geqslant \frac{1}{c_2}\frac{4}{10}\, i\, p_{2^{n_j+3}}(0,0) \geqslant \frac{1}{c_2} 2^{n_j-1} p_{2^{n_j+3}}(0,0) \\ &\geqslant \frac{2^{-13}}{c_2} 2^{n_j} p_{2^{n_j-1}}(0,0) \geqslant c_3^{-1} 2^{n_j} p_{2^{n_j}}(0,0),\end{aligned} \tag{3.108}$$

because of Lemma 3.5.10, Corollary 3.5.9 (by our choice of j), monotonicity of $t \to p_t(0,0)$ and possibly after increasing c_3. □

Thus, using that $j \in [k_0, k_1]$ (recall (3.95)), we have a universal c_4 such that

$$c_4\alpha \geqslant \sum_j \sum_{(0,i)\in j} \mathbb{P}\big(Z^0 \text{ hits } Z^i\big) \geqslant \frac{\alpha}{c_4}. \tag{3.109}$$

There are two issues to address

(a) to show that

$$\mathbb{P}\big(\exists j, \exists i \text{ so that } (0,i) \in j,\ Z^0 \text{ hits } Z^i\big) \tag{3.110}$$

is of the order α;

(b) to show that (a) holds with Z^0, Z^i replaced by our coalescing random walks Y^0, Y^i.

In fact both parts are resolved by the same calculation.

We consider the probability that random walk Z^0 is involved in a "3-way" collision with Z^i and $Z^{i'}$ either due to Z^0 hitting Z^i in the appropriate time interval and then hitting $Z^{i'}$, or Z^0 hitting Z^i and, subsequently Z^i hitting $Z^{i'}$. The first case is important to bound so that one can use simple Bonferroni bounds to get a lower bound on $\mathbb{P}(\exists i \text{ so that } Z^i \text{ hits } Z^0)$. The second is to take into account the fact that we are interested in the future coalescence of a given random walk Y^0. As already noted, we can couple the systems in the usual way so that for all t, $\cup_i\{Y^i_t\} \subseteq \cup_i\{Z^i_t\}$. The problem is that if for some i, Z^i hits Z^0 due to coalescence this do not necessarily imply that Y^i hits Y^0: if the Y^i particles coalesced with a $Y^{i'}$ before Z^i hits Z^0. Fortunately this event is contained in the union of events above over i, i'.

Proposition 3.5.12. *There exists universal constant K so that for all i and i' with* $(0,i') \in j'$

$$\mathbb{P}\big(Z^0 \textit{ hits } Z^i \textit{ and then } Z^{i'}\big) \leqslant K 2^{n_{j'}} p_{2^{n_{j'}}}(0,0)\mathbb{P}\big(Z^0 \textit{ hits } Z^i\big). \tag{3.111}$$

Proof. There are several cases to consider: $i < 0 < i'$, $i < i' < 0$, $i' < i < 0$, $i' < 0 < i$, $0 < i < i'$ and $0 < i' < i$. All are essentially the same so we consider explicitly $0 < i < i'$. We leave the reader to verify that the other cases are analogous. We choose j, j' so that $(0,i) \in j$ and $(0,i') \in j'$ (so necessarilly $j' \geqslant j$). We condition on T_j, $Z^i(T_j)(= Z^0(T_j))$, for

$$T_j := \inf\left\{s \in \left(\frac{19i}{10}, \frac{21i}{10}\right) \cap \left[i + 2^{n_j}, i + 2^{n_j+1}\right] : Z^i(s) = Z^0(s)\right\} < \infty. \tag{3.112}$$

With $x = Z^0(T_j)$ we have

$$\begin{aligned}\mathbb{P}&\left(\exists s' \geqslant T_j \in \left(\frac{19i'}{10}, \frac{21i'}{10}\right) \cap \left[i' + 2^{n_{j'}}, i' + 2^{n_{j'}+1}\right] : Z^{i'}(s') = Z^0(s') \;\Big|\; G^{0,i}\right)\\ &= \mathbb{P}\big(Z^0(t) = x \text{ for some } t \in I_j\big),\end{aligned} \tag{3.113}$$

where I_j is the image of the interval

$$\left[(i'+2^{n_{j'}})\vee T_j\vee\frac{19i'}{10},\, i'+2^{n_{j'}+1}\wedge\frac{21i'}{10}\right]\Big), \tag{3.114}$$

by the function $t\mapsto 2t-T_j-i'$, for $G^{0,i}=\sigma(Z^0(s),Z^i(s)\colon s\leqslant T_j)$. By elementary algebra this is less than

$$\mathbb{P}\left(Z^0(t)=x \text{ for } t\in\left(\frac{9i'}{10},\frac{16i'}{5}\right)\right), \tag{3.115}$$

but by arguing as in Lemma 3.5.10, this is bounded by

$$\begin{aligned} c_2\int_{\frac{9i'}{10}}^{\frac{16i'}{5}} p_s(0,x)\,ds \leqslant c_2\int_{\frac{9}{10}i'}^{\frac{16}{5}i'} p_s(0,0)\,ds \leqslant c_2\frac{23}{10}\,i' p_{2^{n_{j'}-1}}(0,0) \\ \leqslant c_2\frac{23}{9}2^{13}2^{n_{j'}}p_{2^{n_{j'}+3}}(0,0)\leqslant c'2^{n_{j'}}p_{2^{n_{j'}}}(0,0) \end{aligned} \tag{3.116}$$

for some universal constant c', where we use symmetry and monotonicity of $p_s(\cdot,\cdot)$ and Corollary 3.5.9, by the choice of our $n_{j'}$. So given that

$$\mathbb{P}(T_i<\infty)=\mathbb{P}\big(Z^0 \text{ hits } Z^i\big), \tag{3.117}$$

the desired bound is achieved. □

Corollary 3.5.13. *For α sufficiently small*

$$\mathbb{P}\big(\exists i\colon Z^0 \textit{ hits } Z^i\big)\geqslant\frac{\alpha}{2}. \tag{3.118}$$

Proof. By Bonferroni bounds, the desired probability is greater than

$$\sum_i \mathbb{P}\big(Z^0 \text{ hits } Z^i\big)-\sum_{i,i'}\mathbb{P}\big(Z^0 \text{ hits } Z^i \text{ and then } Z^{i'}\big)\geqslant\alpha-Kc_3^2\alpha^2\geqslant\frac{\alpha}{2} \tag{3.119}$$

if $\alpha\leqslant 1/(2Kc_3^2)$. □

We similarly show

Proposition 3.5.14. *There exists universal constant K so that for all i' and i with $(0,i)\in j$,*

$$\mathbb{P}\big(Z^0 \textit{ hits } Z^i \textit{ after } Z^i \textit{ hits } Z^{i'}\big)\leqslant K2^{n_j}p_{2^{n_j}}(0,0)\mathbb{P}\big(Z^i \textit{ hits } Z^{i'}\big). \tag{3.120}$$

This gives as a corollary

Corollary 3.5.15. *For the coalescing system $\{Y^i\}_{i\in V}$ provided α is sufficiently small,*

$$\mathbb{P}\big(Y^i \textit{ dies after time } R\big)\geqslant\frac{\alpha}{5}. \tag{3.121}$$

Proof. We have of course from the labeling scheme

$$
\begin{aligned}
&\mathbb{P}\big(Y^i \text{ dies after time } R\big)\\
&\qquad \geqslant \frac{1}{2}\mathbb{P}\big(Y^i \text{ hits } Y^{i'} \text{ in appropriate time interval for some } i'\big)\\
&\qquad \geqslant \frac{1}{2}\mathbb{P}\big(Z^i \text{ hits } Z^{i'} \text{ in appropriate time interval for some } i'\big)\\
&\qquad - \frac{1}{2}\mathbb{P}\big(Z^i \text{ hits } Z^{i'} \text{ in appropriate time interval for some } i' \text{ so that}\\
&\qquad\qquad\quad Z^{i'} \text{ hits some } Z^{i''} \text{ previously}\big)\\
&\qquad \geqslant \frac{\alpha}{4} - Kc_3^2\alpha^2 \geqslant \frac{\alpha}{5}
\end{aligned}
\tag{3.122}
$$

for $\alpha \leqslant 1/(20Kc_3^2)$. □

We can now complete the proof of Proposition 3.5.4 and hence that of Proposition 3.5.1. If we have $c_0 > 0$, then we can find $0 < \epsilon < \alpha/200$ and α so small that the relevant results above hold, in particular Corollary 3.5.15. Thus the density of Y's is reduced by at least $\alpha/5$. But by our choice of ϵ and Lemma 3.5.6, the density of X's is reduced by at least $\alpha/5 - 6\epsilon \geqslant \alpha/6 \geqslant 3\epsilon$ which is a contradiction with Proposition 3.5.2, because it would entail the density falling strictly below c_0.

3.6 Proof of Theorem 3.3.4

In what follows we assume, as in Section 3.4, that $p = 1$, the extension to arbitrary $p \geqslant 1$ being straightforward.

We begin by specifying the random walk $(X(t))_{t\geqslant 0}$ on $\mathbb{Z}^4$ defined by

$$X(t) = S(t) + e_1 N(t) \tag{3.123}$$

with $(S(t))_{t\geqslant 0}$ denoting a simple random walk on $\mathbb{Z}^4$, $(N(t))_{t\geqslant 0}$ a rate 1 Poisson process and $e_1 = (1,0,0,0)$ the first unit vector in $\mathbb{Z}^4$.

Thus our random walk $(X(t))_{t\geqslant 0}$ is highly transient but its symmetrization is a mean zero random walk and by the local central limit theorem, we have

$$\int_0^\infty t p_t^{(s)}(0,0)\,dt = \infty, \tag{3.124}$$

where $p_t(\cdot,\cdot)$ is the semigroup associated to $(X(t))_{t\geqslant 0}$.

It remains to show that $\lambda_p(\kappa) < \gamma$ for all $\kappa \in [0,\infty)$. Our approach is modeled on the proof of the first part of Theorem 3.3.2. We wish again to pick *bad environment set* B_E associated to the

ξ-process and *bad random walk set* B_W associated to the random walk X^κ so that

$$\begin{aligned}&\mathbb{E}\Big(\exp\Big[\gamma\int_0^n \xi(X^\kappa(s),s)\,ds\Big]\Big)\\&\qquad\leqslant \Big(\mathbb{P}(B_E)+\mathbb{P}(B_W)\Big)e^{\gamma n}+\mathbb{E}\Big(\mathbb{1}_{\{B_E^c\cap B_W^c\}}\exp\Big[\gamma\int_0^n \xi(X^\kappa(s),s)\,ds\Big]\Big)\end{aligned} \tag{3.125}$$

with, for some $0<\delta<1$,

$$\mathbb{P}(B_E)\leqslant e^{-\delta n},\qquad \mathbb{P}(B_W)\leqslant e^{-\delta n}, \tag{3.126}$$

and, automatically from the definition of B_E and B_W,

$$\int_0^n \xi(X^\kappa(s),s)\,ds\leqslant n(1-\delta)\qquad \text{on } B_E^c\cap B_W^c \tag{3.127}$$

(as in the proof of Theorem 3.3.2). Since, combining (3.125–3.127), we obtain

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\Big(\exp\Big[\gamma\int_0^n \xi(X^\kappa(s),s)\,ds\Big]\Big)<\gamma, \tag{3.128}$$

it is enough to prove (3.126). All of this has been done in the proof of Theorem 3.3.2 in a different situation. The major difference is that we need to modify the collection of skeletons used.

Lemma 3.6.1. *Let $X^\kappa(\cdot)$ be a speed κ simple random walk in four dimensions. Fix $M\in\mathbb{N}\setminus\{1\}$. There exists $c>0$ so that for M large and all n, outside of an e^{-cn} probability event, there exists $0\leqslant i_1<i_2<\cdots<i_{n/2M}\leqslant n$ so that*

$$X^\kappa_{(1)}(i_jM+kM)-X^\kappa_{(1)}(i_jM)>-\frac{kM}{2},\qquad j\in\{1,\cdots,n/(2M)\},\ k\geqslant 0, \tag{3.129}$$

where $(X^\kappa_{(1)}(t))_{t\geqslant 0}$ denotes the first coordinate of $(X^\kappa(t))_{t\geqslant 0}$.

Proof. Define

$$\sigma_1=\inf\big\{kM>0\colon X^\kappa_{(1)}(kM)\leqslant -kM/2\big\} \tag{3.130}$$

and recursively

$$\sigma_{i+1}=\inf\big\{kM>\sigma_i\colon X^\kappa_{(1)}(kM)-X^\kappa_{(1)}(\sigma_i)\leqslant -(kM-\sigma_i)/2\big\}. \tag{3.131}$$

Since the event

$$\{rM\leqslant\sigma_1<\infty\}\subset\cup_{k=r}^{\infty}\big\{X^\kappa_{(1)}(kM)\leqslant -kM/2\big\}, \tag{3.132}$$

we have easily that, for all r,

$$\mathbb{P}(rM\leqslant\sigma_1<\infty)\ \leqslant\ e^{-rMc} \tag{3.133}$$

for $c>0$ not depending on n or M. If we now define

$$\tau_1 = \inf\left\{kM>0\colon X^{\kappa}_{(1)}(jM)-X^{\kappa}_{(1)}(kM)>-(j-k)M/2 \ \forall j>k\right\} \tag{3.134}$$

and recursively

$$\tau_{i+1} = \inf\left\{kM>\tau_i\colon X^{\kappa}_{(1)}(jM)-X^{\kappa}_{(1)}(kM)>-(j-k)M/2 \ \forall j>k\right\}, \tag{3.135}$$

it is easily seen that

$$\begin{aligned}
\mathbb{P}(\tau_1 \geqslant rM) &\leqslant \mathbb{P}(\exists 1\leqslant k\leqslant r\colon rM\leqslant \sigma_k<\infty)\\
&\leqslant \sum_{k=1}^{r-1}\sum_{0<x_1<\cdots<x_k<r}\mathbb{P}\big(\sigma_i = x_iM \ \forall i\leqslant k, rM\leqslant \sigma_{k+1}<\infty\big)\\
&\leqslant \sum_{k=1}^{r-1}\sum_{0<x_1<\cdots<x_k<r} e^{-rMc}\leqslant e^{-rMc}2^r
\end{aligned} \tag{3.136}$$

which is less than $e^{-rMc/2}$ if M is fixed sufficiently large. We have

- $(\tau_{i+1}-\tau_i)_{i\geqslant 1}$ are i.i.d. (this follows from Kuczek's argument (see [53])).
- Provided M has been fixed sufficiently large for each integer $r\geqslant 1$, $\mathbb{P}(\tau_{i+1}-\tau_i\geqslant rM)\leqslant e^{-rMc/4}$. This follows from the fact that random variable $\tau_{i+1}-\tau_i$ is simply the random variable τ_1 conditioned on an event of probability at least $1/2$ (provided M was fixed large).

Thus by elementary properties of geometric random variables we have

$$\mathbb{P}(\tau_{n/2M}>n)\leqslant \mathbb{P}\left(Y\geqslant \frac{n}{2M}\right)\leqslant e^{-cn} \tag{3.137}$$

for $Y\sim B\left(\frac{n}{M}, e^{-cM/4}\right)$, $c>0$ and M large. This completes the proof of the lemma. □

Given that the path of the random walk satisfies the condition of this lemma, we call the (not uniquely defined) points $i_1, i_2,\cdots$ *regular* points.

Given this result, we consider the M-skeleton induced by the values $X^{\kappa}(jM)$, $0\leqslant j\leqslant n/M\}$, discretized via spatial cubes of length $M/8$ (rather than $2M$ as in the proof of Theorem 3.3.2). It is to be noted that if $(X^{\kappa}(t))_{0\leqslant t\leqslant n}$ satisfies the claim for Lemma 3.6.1 and $y_0 := 0, y_1, y_2,\cdots, y_{n/M}$ with $y_k\in\mathbb{Z}^4$, $0\leqslant k\leqslant n/M$, is its M-skeleton, namely,

$$X^{\kappa}(kM)\in C_{y_k} := \prod_{j=1}^{4}\left[y_k^{(j)}\frac{M}{8},(y_k^{(j)}+1)\frac{M}{8}\right), \qquad 0\leqslant k\leqslant n/M, \tag{3.138}$$

where $y_k^{(j)}$ denotes the j-th coordinate of y_k (we suppose without loss of generality that M is a multiple of 8). Then, by (3.129) and (3.138), we must have

$$y_{i_j}^{(1)} - 4k \leqslant y_{i_j+k}^{(1)} + 1. \tag{3.139}$$

In particular, we must have

$$y_{i_{j'}}^{(1)} - 4(i_j - i_{j'}) \leqslant y_{i_j}^{(1)} + 1 \qquad \forall i_{j'} < i_j. \tag{3.140}$$

In the following we modify the definition of appropriate skeletons by adding in the requirement that the skeleton must possess at least $n/2M$ indices $i_1, i_2, \ldots, i_{n/2M}$ with the corresponding y_{i_j} satisfying (3.140). We note that the resizing of the cubes makes the notion of acceptability a little more stringent but does not change the essentials.

Remark first that Lemma 3.4.5 is still valid in our new setting. Lemma 3.6.1 immediately gives that with this new definition, Lemma 3.4.4 remains true. Of course since this definition is more restrictive we have

$$|\Xi_A| \leqslant K^{n/M} \tag{3.141}$$

for K as in Lemma 3.4.1.

In fact in our program all that remains to do, that is in any substantive way different from the proof of Theorem 3.3.2, is to give a bound on the probability of B_E for appropriate B_E. This is the content of the lemma below (analogous to Lemma 3.4.2). Given this lemma, we can then proceed exactly as with the proof of Theorem 3.3.2.

Lemma 3.6.2. *For any skeleton $(y_k)_{0 \leqslant k \leqslant n/M}$ in Ξ_A, the probability that ξ is not good for $(y_k)_{0 \leqslant k \leqslant n/M}$, i.e.,*

$$\nexists \frac{n}{4M} \text{ indices } 1 \leqslant j \leqslant \frac{n}{M} : \xi(z,s) = 0 \ \forall s \in [jM, jM+1] \text{ for some } z \in C_{y_j}, \tag{3.142}$$

is less than $(4K)^{-n/M}$.

Proof. We note that proving the analogous result for Theorem 3.3.2, we did not need our skeleton to be in Ξ_A, the proof worked over any skeleton. For us however it is vital that our skeleton satisfies (3.140).

We consider a skeleton in Ξ_A. Let the first $n/2M$ regular points of our skeleton be $i_1, i_2, \cdots i_{n/2M}$. For each $1 \leqslant i_j \leqslant n/2M$, we choose R points

$$x_1^{i_j}, \ldots, x_R^{i_j} \in C_{y_{i_j}} \tag{3.143}$$

so spread out that for random walks $(X(t))_{t \geqslant 0}$ as in (3.123) beginning at the points $x_k^{i_j}$, $k = 1, \cdots, R$, the chance that two of them meet is less than $0 < \epsilon \ll 1$.

Now, consider $i_{j'} < i_j$ and the probability that a random walk starting at $(x_k^{i_j}, i_j M)$ meets a random walk starting at $(x_{k'}^{i_{j'}}, i_{j'} M)$ satisfies the following lemma.

Lemma 3.6.3. *For $i_{j'} < i_j$, there exits $K > 0$ such that*

$$\mathbb{P}\Big(X^{x_k^{i_j}, i_j M} \ \textit{meets} \ X^{x_{k'}^{i_{j'}}, i_{j'} M}\Big) \leqslant \frac{K}{M^2 (i_j - i_{j'})^2}. \tag{3.144}$$

Proof. The important point is that since our skeleton is in Ξ_A,

$$\left(x_{k'}^{i_{j'}}\right)^{(1)} \leqslant \left(x_k^{i_j}\right)^{(1)} + (i_j - i_{j'})\frac{M}{2} + \frac{M}{4}, \tag{3.145}$$

and so we have

$$\begin{aligned}
&\mathbb{P}\Big(X^{x_k^{i_j}, i_j M} \ \text{meets} \ X^{x_{k'}^{i_{j'}}, i_{j'} M}\Big) \\
&\quad \leqslant \mathbb{P}\Big(\Big(X^{x_k^{i_j}, i_j M}\Big)^{(1)} \big((i_j - i_{j'})M\big) \geqslant \big(x_k^{i_j}\big)^{(1)} + (i_j - i_{j'})\frac{3M}{4}, \\
&\qquad\qquad X^{x_k^{i_j}, i_j M} \ \text{meets} \ X^{x_{k'}^{i_{j'}}, i_{j'} M}\Big) \\
&\quad + \mathbb{P}\Big(\Big(X^{x_k^{i_j}, i_j M}\Big)^{(1)} \big((i_j - i_{j'})M\big) \leqslant \big(x_k^{i_j}\big)^{(1)} + (i_j - i_{j'})\frac{3M}{4}\Big).
\end{aligned} \tag{3.146}$$

By standard large deviations bounds,

$$\mathbb{P}\Big(\Big(X^{x_k^{i_j}, i_j M}\Big)^{(1)} \big((i_j - i_{j'})M\big) \leqslant \big(x_k^{i_j}\big)^{(1)} + (i_j - i_{j'})\frac{3M}{4}\Big) \leqslant e^{-CM(i_j - i_{j'})}, \tag{3.147}$$

for some universal $C \in (0, \infty)$. For the other term, we have the following lemma.

Lemma 3.6.4. *For two independent processes $X = (X(t))_{t \geqslant 0}$ and $Y = (Y(t))_{t \geqslant 0}$ with $X(0) = x \in \mathbb{Z}^4$ and $Y(0) = y \in \mathbb{Z}^4$, the probability that X ever meets Y is bounded by $K / \|x - y\|_\infty^2$.*

Proof. $X - Y$ is not exactly a simple random walk, but it is a symmetric random walk and so local central limit theorem gives appropriate random walks bounds (see, e.g., [56]). □

From this and inequality (3.145), we have

$$\begin{aligned}
&\mathbb{P}\Big(X^{x_k^{i_j}, i_j M} \text{meets } X^{x_{k'}^{i_{j'}}, i_{j'} M} \Big| \Big(X^{x_k^{i_j}, i_j M}\Big)^{(1)} \big((i_j - i_{j'})M\big) \geqslant \big(x_k^{i_j}\big)^{(1)} + (i_j - i_{j'})\frac{3M}{4}\Big) \\
&\quad \leqslant \frac{K}{M^2 (i_j - i_{j'})^2}.
\end{aligned} \tag{3.148}$$

□

Thus, for any R large but fixed, we can choose M so that for all skeletons in Ξ_A and each $i_{j'}$, we have

$$\begin{aligned}\sum_{i_j<i_{j'}}\sum_{k,k'}\mathbb{P}\Big(X^{x_k^{i_j},i_jM}\ \text{meets}\ X^{x_{k'}^{i_{j'}},i_{j'}M}\Big) &\leqslant \frac{R^2K}{M^2}\sum_{r=1}^{+\infty}\frac{1}{r^2}\\ &\leqslant \frac{R^2K'}{M^2}<\epsilon^2\end{aligned} \tag{3.149}$$

with M chosen sufficiently large, which is analogous to (3.37). From this point on, the rest follows as for the proof of Lemma 3.4.2. □

4 Asymmetric threshold-2 voter model

We saw previously that the standard voter model evolves as follows : the rate of change is proportional to the number of direct neighbours with opposite sign. In a threshold voter model, the rate is 1 or 0 depending on a threshold : if the number of direct neighbours exceeds or equals the threshold, the rate is 1, otherwise, it is 0. We define the operator Ω on the set of continuous functions f on E by

$$\Omega f(\eta) = \sum_{x \in S} c(x,\eta)(f(\eta_x) - f(\eta)) \tag{4.1}$$

where

$$\eta_x(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ -\eta(y) & \text{if } y = x, \end{cases} \tag{4.2}$$

and

$$c(x,\eta) = \begin{cases} 1 & \text{if } \sum_{|y-x|=1} \mathbb{1}_{\{\eta(y)\neq\eta(x)\}} \geqslant \widetilde{d}, \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

with $0 < \widetilde{d} < 2d$ the fixed threshold. Remark that sites take value in $\{-1,+1\}$. We made this choice to avoid confusion with exclusion process used later. This process was first studied by J. Theodore COX and Rick DURRETT [20]. We also recommend the book of Thomas M. LIGGETT [59] as reference.

If the threshold is strictly greater than d, it is clear that every unanimous hyper rectangle will remain unchanged forever, so the process will fixate fast and no consensus arises. However, when the threshold is equal to d, hyper rectangles will eventually disappear as their corners can change. Thus the study of the evolution in a quadrant becomes of high interest. In particular, this dynamics corresponds to a stochastic Ising model at zero temperature, for the Hamiltonian with uniform ferromagnetic interaction between nearest neighbours. For this model, it is known that when the system starts from a Bernoulli product measure with density

p of sites with value $+1$ and if p is sufficiently close to 1, then the system fixates in the sense that for almost every realization of the initial configuration and dynamical evolution, each site flips only finitely many times, reaching eventually the state $+1$ for all sites (see the paper of Luiz Renato G. FONTES, Roberto H. SCHONMANN and Vladas SIDORAVICIUS [34] for more details).

Thus, in the critical case (when $\widetilde{d} = d$) roughly if we have only a finite quantity of -1's, then we must tend to a consensus. This chapter consider whether this can be strongly perturbed : if we start with a small amount of -1's, must we go to all $+1$'s if a bias to -1's is created ? To answer this question, we are focused on the two dimensional case with the threshold equal to $d = 2$. Starting with the upper-right quadrant full of -1, spin $+1$ elsewhere, the dynamics of the model shows that the initial sites with value $+1$ are frozen and will never change. In particular, we have a nice interface between $+1$ and -1 that can be mapped to an exclusion process with infinitely many 1's for negative positions and infinitely many 0's for positive positions (more details are given in the next section). It is known that this process is transient : although the movement of one particle is symmetric, the big collection of 1's on the left acts like a wall and induces a bias towards the right. Remark that this result is consistent with the one in [34].

Hence we want to know if we can help the -1's to induce a bias towards the left that can outshine the right bias in order to have a recurrent process. We suggest to allow a flip from -1 to $+1$ with only one neighbour but sadly the interface become ugly and useless. However, we can construct an auxiliary process (see Section 4.2) with a nicer interface (see Section 4.3) and construct a Lyapunov-Foster function to show Theorem 4.5.1.

4.1 Threshold-2 voter model and standard exclusion process

Consider the usual threshold-2 voter model $\{\eta(t)\}_{t\geqslant 0}$ on $\{-1,1\}^{\mathbb{Z}^2}$ as defined in the book of Thomas M. LIGGETT [59]. As outlined in introduction, the flip rate $c(x,\eta)$ at a site x is determined by

$$c(x,\eta) = \begin{cases} 1 & \text{if } \sum_{|y-x|=1} \mathbb{1}_{\{\eta(y)\neq\eta(x)\}} \geqslant 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

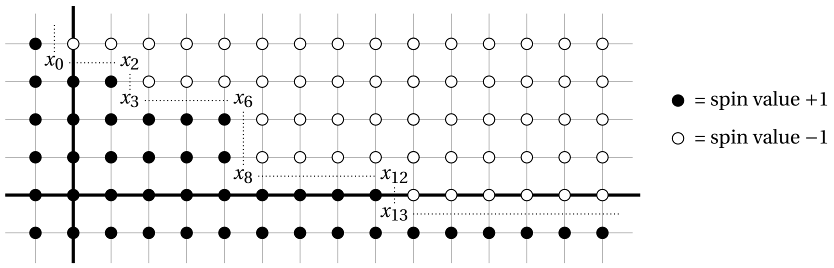
Write $\eta^{\max}$ for the configuration with spin -1 in $[0,\infty)^2$, spin 1 elsewhere, and suppose that $\eta(0) = \eta^{\max}$. It is clear that all sites with spin 1 must remain in this state for ever as they have at most one neighbour with value -1.

As outlined previously, the interface of the process $\{\eta(t)\}$ can be mapped to a standard exclusion process as explained in LIGGETT [58], pp. 411-412. We now give some details about this construction.

Starting from $\eta^{\max}$, there exists a finite integer R and a sequence $x_0, \cdots, x_R$ with

$$x_0 \in \left\{-\frac{1}{2}\right\} \times \left\{\mathbb{Z}_+ + \frac{1}{2}\right\} \qquad \text{and} \qquad x_R \in \left\{\mathbb{Z}_+ + \frac{1}{2}\right\} \times \left\{-\frac{1}{2}\right\}$$

and $x_i - x_{i-1} \in \{(1,0),(0,-1)\}$ for all $1 \leqslant i \leqslant R$ such that $\eta(x,t) = 1$ if and only if $x \leqslant x_i - \left(\frac{1}{2}, \frac{1}{2}\right)$ (in the natural partial order) for some i.



Interface's illustration.

We will call interface the set of $\{(x_{i-1}, x_i) : 1 \leqslant i \leqslant R\}$ extended from x_0 to $\left\{-\frac{1}{2}\right\} \times \infty$ and from x_R to $\infty \times \left\{-\frac{1}{2}\right\}$. We can thus map this into an element of

$$\Omega = \left\{ \varrho \in \{0,1\}^{\mathbb{Z}} : \sum_{x \leqslant 0} (1 - \varrho(x)) = \sum_{x>0} \varrho(x) < \infty \right\} \tag{4.5}$$

vertical unit segments corresponding to 1's, horizontal corresponding to 0's and shifting to get as many 0's on the right of origin as 1's on the left.

$$\ldots 1\,1\,1\,1\,0\,1\,0\,0\,1 \,|\, 1\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \ldots$$

Origin

Exclusion process corresponding to the previous interface.

With this description the configuration $\{\varrho(t)\}_{t \geqslant 0}$ corresponding to the interface of $\{\eta(t)\}_{t \geqslant 0}$ evolves as a symmetric nearest neighbour exclusion process beginning from the state

$$\varrho^{\max}(x) = \begin{cases} 1 & \text{if } x \leqslant 0, \\ 0 & \text{if } x > 0. \end{cases}$$

4.2 Model and notation

Consider now an asymmetric, spatially inhomogeneous variant of $\{\eta(t)\}_{t \geqslant 0}$, called $\{\xi(t)\}_{t \geqslant 0}$, starting from the same initial configuration $\eta^{\max}$ but such that the flip rates at sites having current value -1 in the upper right quadrant change ; it only needs one neighbour to flip instead of two as previously. Formally the flip rate at a site $x \in [0,\infty)^2$ is modified and now determined by

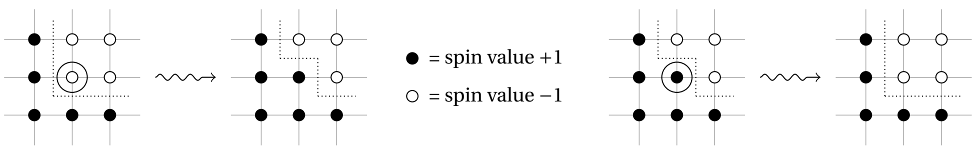
$$c(x,\xi) = \begin{cases} 1 & \text{if } \xi(x) = -1 \text{ and } \sum\limits_{|y-x|=1} \mathbb{1}_{\{\xi(y)\neq\xi(x)\}} \geqslant 2, \\ 1 & \text{if } \xi(x) = 1 \text{ and } \sum\limits_{|y-x|=1} \mathbb{1}_{\{\xi(y)\neq\xi(x)\}} \geqslant 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4.6}$$

and the flip rate at a site $x \in \left([0,\infty)^2\right)^c$ stays the same as for $\{\eta(t)\}_{t\geqslant 0}$. In particular, the $+1$'s outside of $[0,\infty)^2$ are frozen and spin value -1 will be helped in the upper right quadrant. It is clear from usual attractiveness considerations that (under the natural coupling) for all $t \geqslant 0$, we have $\eta(t) \geqslant \xi(t)$ under the natural partial order. Establishing stability for ξ will be a big step towards showing that even with a big amount of 1's, the -1's can make over.

A difficulty in analyzing the process $\{\xi(t)\}_{t\geqslant 0}$ comes from the bias towards the spin value -1 : the set of sites with spin -1 needs no longer to be connected and certainly there is no nice interface between $+1$'s and -1's. For this reason we will introduce a new Markov process $\{\lambda(t)\}_{t\geqslant 0}$ with almost the same behaviour but a nicer interface.

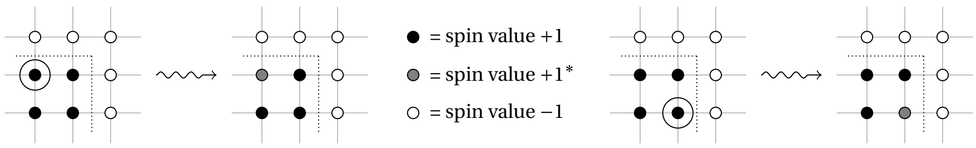
We forget to turn a $+1$ in -1 if it has only one neighbour -1 unless there is another -1 not too far away. In this case, the $+1$ turns in a new state called $+1^*$ (we can interpret this as a possible flip). In particular, we define a process $\{\lambda(t)\}_{t\geqslant 0}$ on $\{-1,+1^*,+1\}^{\mathbb{Z}^2}$ with 1's frozen outside $[0,\infty)^2$ and evolving in $[0,\infty)^2$ as follows :

- site x with spin value -1 or $+1^*$ will flip to 1 with a rate 1 or 0 according to whether at least two neighbours are of spin type 1 (see case 1).
- site x with spin value 1 will flip to -1 with a rate 1 or 0 according to whether at least two neighbours are of spin type -1 (see case 2).



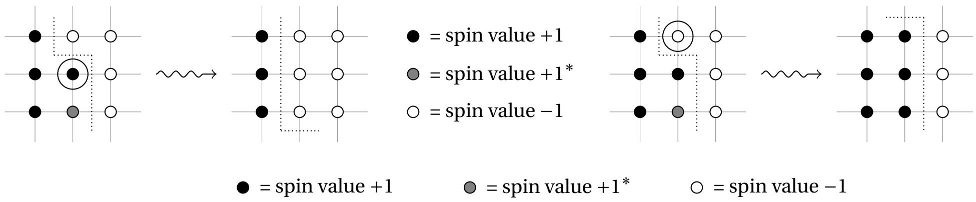
Cases 1 and 2 (the circle indicates the flipping site).

- when a site $x = (x_1, x_2)$ with spin value $+1$ has only one neighbour of type -1 :
  - if the sites (x_1, x_2+1) (the direct neighbour) and (x_1+2, x_2) (the 'not too far' neighbour) are with spin value -1, this will flip to 1^* with rate 1 (see case 3) ;
  - similarly, if the sites (x_1+1, x_2) and (x_1, x_2+2) are with spin value -1, this will flip to $+1^*$ with rate 1 (see case 4) ;
  - else, the flip rate is 0.



Cases 3 and 4.

- if at any time, a site with spin value $+1^*$ has at least two neighbours with spin type -1, it immediately turns to -1 (see case 5).
- if at any time, a site with spin value $+1^*$ is such that site (x_1+2, x_2) (or (x_1, x_2+2)) is not with spin value -1, it immediately turns to 1 (see case 6).

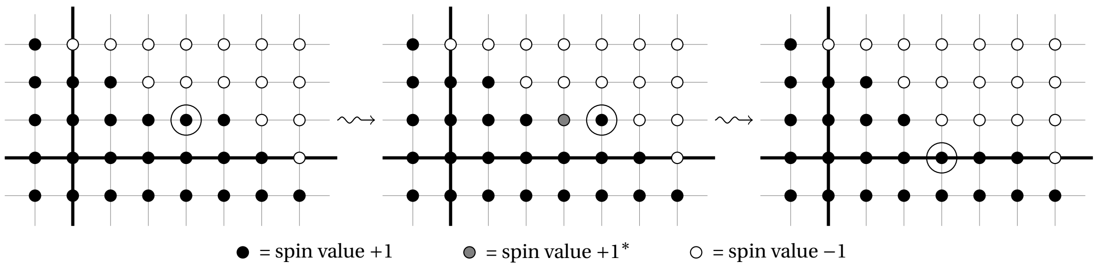


Cases 5 and 6.

We then define the collapse $\{\lambda^c(t)\}_{t\geqslant 0}$ of $\{\lambda(t)\}_{t\geqslant 0}$ by

$$\begin{cases} \lambda^c(x,t) = -1 & \text{if and only if } \lambda(x,t) = -1, \\ \lambda^c(x,t) = 1 & \text{if and only if } \lambda(x,t) \in \{+1, +1^*\}. \end{cases} \tag{4.7}$$

We can remark that under natural coupling, for all $t \geqslant 0$, we have $\eta(t) \geqslant \lambda^c(t) \geqslant \xi(t)$ under the natural partial order. Moreover, as with process $\{\eta(t)\}_{t\geqslant 0}$, there will be a down-right interface between 1's and -1's for $\{\lambda^c(t)\}_{t\geqslant 0}$.



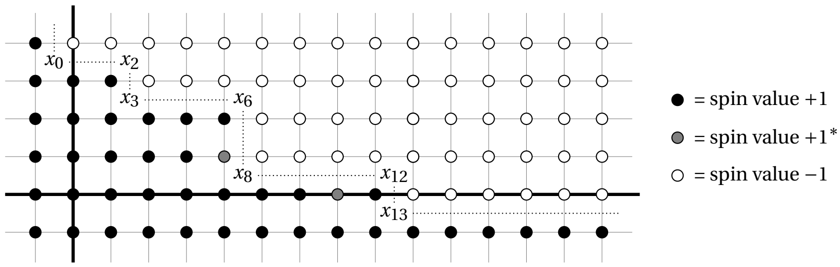
Evolution of $\{\lambda(t)\}_{t\geqslant 0}$: spin $(3,1)$ changes its value (case 3), then spin $(4,1)$ (case 2 & 5). Spin $(3,0)$ can't flip.

4.3 The interface

We now consider the motion from the point of view of the interface. Similarly to η, there exists a finite integer R and a sequence $x_0, \cdots, x_R$ with

$$x_0 \in \left\{-\frac{1}{2}\right\} \times \left\{\mathbb{Z}_+ + \frac{1}{2}\right\} \qquad \text{and} \qquad x_R \in \left\{\mathbb{Z}_+ + \frac{1}{2}\right\} \times \left\{-\frac{1}{2}\right\}$$

and $x_i - x_{i-1} \in \{(1,0),(0,-1)\}$ for all $1 \leqslant i \leqslant R$ such that $\lambda^c(x,t) = 1$ if and only if $x \leqslant x_i + \left(\frac{1}{2}, \frac{1}{2}\right)$ (in the natural partial order) for some i. Recall that $+1^*$ is replaced by $+1$ in λ^c.



Interface's illustration.

As we can remark on the previous illustration, the process corresponding to the interface of $\{\lambda^c(t)\}_{t\geqslant 0}$ will not be a Markov process due to the influence of $+1^*$ spins. However, using $\{\lambda(t)\}_{t\geqslant 0}$, it can be turned into a Markov process $\{\rho(t)\}_{t\geqslant 0}$ by enlarging the spin space from $\{0,1\}$ to $\{0,0^*,1,1^*\}$, where a horizontal site will generate a spin value 0^* if its nearest neighbour is a site with value $+1^*$ and a vertical site will generate a spin value 1^* if its nearest neighbour is a site with value $+1^*$.

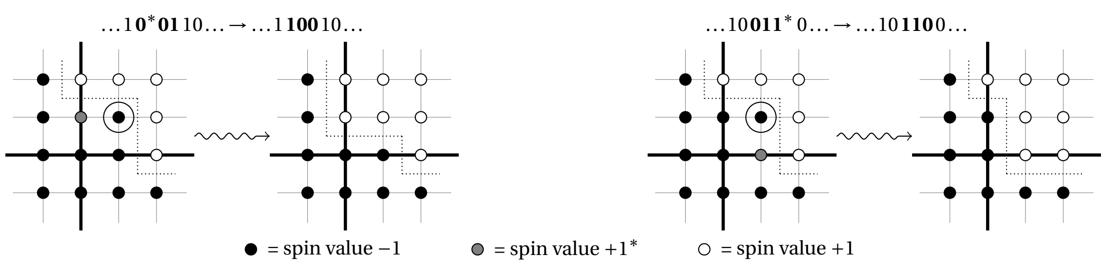
$$\ldots 1\,1\,1\,1\,0\,1\,0\,0\,1\,|\,1\,1^*\,0\,0\,0\,0^*\,1\,0\,0\;\;0\,0\,0\ldots$$

Origin

Auxiliary process corresponding to the previous interface.

Thus, if * modifications are disregarded, particles move according to simple exclusion but with some additions

- if $\rho(x-2,t_-)\rho(x-1,t_-)\rho(x,t_-) = 0^*01$ and the particle at x moves to site $x-1$, then the particle will instead move to site $x-2$, replacing the 0^*, that is $\rho(x-2,t)\rho(x-1,t)\rho(x,t) = 100$ (this corresponds to case 3 of Section 4.2),
- if $\rho(x-1,t_-)\rho(x,t_-)\rho(x+1,t_-) = 011^*$ and the particle at x moves to site $x-1$, then the particle at $x+1$ will also move, that is $\rho(x-1,t)\rho(x,t)\rho(x+1,t) = 110$ (this corresponds to case 4 of Section 4.2).



Links between $\{\lambda_t\}_{t\geqslant 0}$ and its perturbed exclusion process $\{\rho(t)\}_{t\geqslant 0}$.

Remark that these two additions are convenient because of their symmetry : if we permute 1's and 0's, we get an inversed process completely equivalent. This description is rather informal and will be formalized in the next section.

Rather than introduce more notation, we will adopt the convention that, for functions of spins, the * will be dropped, so for instance, $g(\rho(t,x))$ will equal $g(0)$ if $\rho(t,x) \in \{0,0^*\}$. In this way we also regard the process as a process on Ω defined in (4.5).

4.4 Harris system

We will make a widespread use of coupling arguments. To this end we will now explain how to generate our process $\{\rho(t)\}_{t\geqslant 0}$ with a family of independent rate one Poisson processes $N^{x,x+1}$, $x \in \mathbb{Z}$, representing potential exchange between site x and site $x+1$. The process evolves as follows (here time is written as subscript to lighten notation), when $t \in N^{x,x+1}$ and

- $\rho(x-1,t_-)\rho(x,t_-)\rho(x+1,t_-) = 011$, then $\rho(x-1,t)\rho(x,t)\rho(x+1,t) = 011^*$ (case 4),
- $\rho(x-1,t_-)\rho(x,t_-)\rho(x+1,t_-) = 011^*$, then $\rho(x-1,t)\rho(x,t)\rho(x+1,t) = 011$ (case 1),
- $\rho(x,t_-)\rho(x+1,t_-)\rho(x+2,t_-) = 001$, then $\rho(x,t)\rho(x+1,t)\rho(x+2,t) = 0^*01$ (case 3),
- $\rho(x,t_-)\rho(x+1,t_-)\rho(x+2,t_-) = 0^*01$, then $\rho(x,t)\rho(x+1,t)\rho(x+2,t) = 001$ (case 1),
- $\rho(x,t_-)\rho(x+1,t_-) = 10$, then $\rho(x,t)\rho(x+1,t) = 01$ (case 1),
- $\rho(x,t_-)\rho(x+1,t_-) = 10^*$, then $\rho(x,t)\rho(x+1,t) = 01$ (cases 1 & 6),
- $\rho(x,t_-)\rho(x+1,t_-) = 1^*0$, then $\rho(x,t)\rho(x+1,t) = 01$ (cases 1 & 6),
- $\rho(x-1,t_-)\rho(x,t_-)\rho(x+1,t_-) = 0^*01$, then $\rho(x-1,t)\rho(x,t)\rho(x+1,t) = 100$ (cases 2 & 5),
- $\rho(x,t_-)\rho(x+1,t_-)\rho(x+2,t_-) = 011^*$, then $\rho(x,t)\rho(x+1,t)\rho(x+2,t) = 110$ (cases 2 & 5),
- else, if $\rho(x,t_-)\rho(x+1,t_-) = 01$, then $\rho(x,t)\rho(x+1,t) = 10$ (case 2),
- else, nothing happens.

We can see that globally ρ evolves as a standard exclusion process with exception that sometimes, a 1 can make two steps to the left or a 0 can make two steps to the right. Remark that formally rates should be $\frac{1}{2}$ in order to have a jump rate of 1. To lighten the notations we decided to use this representation as results will stay unchanged.

4.5 Main result

We are now ready to state the main result of this chapter.

Theorem 4.5.1. *Define*

$$\Gamma = \left\{ \xi \in \{-1,+1\}^{\mathbb{Z}^2} : \xi(x) \equiv 1 \textit{ on } \left([0,\infty)^2\right)^{\mathrm{c}} \textit{ and } \sum_{x\in[0,\infty)^2} \mathbb{1}_{\{\xi(x)=+1\}} < \infty \right\}. \tag{4.8}$$

The process $\{\xi_t\}_{t\geqslant 0}$ defined in (4.6) is positive recurrent on Γ.

To prove this result, we will use the Markovian process $\{\lambda_t\}_{t\geqslant 0}$ and its collapse $\{\lambda_t^c\}_{t\geqslant 0}$ defined in (4.7) and prove

Theorem 4.5.2. *The process $\{\lambda_t^c\}_{t\geqslant 0}$ is positive recurrent on Γ.*

Recall that under the natural coupling, we have for all t

$$\eta_t \geqslant \lambda_t^c \geqslant \xi_t \tag{4.9}$$

under the natural partial order. Given the monotonicity and relation (4.9), Theorems 4.5.2 implies Theorem 4.5.1. From now on, therefore, we will not explicitly mention our original process $\{\xi(t)\}_{t\geqslant 0}$ and work instead with $\{\lambda(t)\}_{t\geqslant 0}$ and its associated process $\{\rho(t)\}_{t\geqslant 0}$.

The key to our approach is that of the paper of Maury BRAMSON and Thomas MOUNTFORD [9] : to establish the positive recurrence of $\{\rho(t)\}_{t\geqslant 0}$, it is sufficient to find a Lyapunov-Foster function for the discrete time Markov process $\{\rho(rN)\}_{r\geqslant 0}$ (for some large constant N fixed), i.e., we need to find a function f and a set A such that

$$\begin{cases} \forall \rho_0 \notin A, & \mathbb{E}_{\rho_0}(f(\rho(N))) \leqslant f(\rho_0) - 1, \\ \forall \rho_0 \in A, & \mathbb{E}_{\rho_0}(f(\rho(N))) < \infty. \end{cases} \tag{4.10}$$

More informations about the Foster's criterion can be found in the book of Sean P. MEYN and Richard L. TWEEDIE [67].

In the following section, we give the proof of our main result. In Section 4.7 we find a technical result, Proposition 4.7.1, that describes the behaviour of particles when the quantity is finite with a low density.

4.6 Proof of Theorem 4.5.2

Define the functions L, the leftmost 0, and R, the rightmost 1, i.e.,

$$L(\rho(t)) = \min\{x \in \mathbb{Z} : \rho(x,t) = 0\} \quad \text{and} \quad R(\rho(t)) = \max\{x \in \mathbb{Z} : \rho(x,t) = 1\}. \tag{4.11}$$

These functions are well defined as we consider configurations in Ω, defined in (4.5). First remark the following lemma.

Lemma 4.6.1. *Uniformly over all configurations $\rho_0 \in \Omega$, we have*

$$\mathbb{E}_{\rho_0}\left((R(\rho(N)) - R(\rho_0))_+\right) \leqslant N \tag{4.12}$$

and

$$\mathbb{E}_{\rho_0}\left((R(\rho(N)) - R(\rho_0))_+^2\right) \leqslant N^2 + N \tag{4.13}$$

Proof. We simply note that, for $0 \leqslant t \leqslant N$, $R(\rho(t)) - R(\rho_0)$ is dominated by the sum of the rightward jumps of $R(\rho(s))$ for s increasing up to t. Thus, $(R(\rho(t)) - R(\rho_0))_+$ is stochastically less than a Poisson random variable with rate N. $\square$

Remark that the same arguments applies to $(L(\rho_0) - L(\rho(N)))_+$, as the process is symmetric for 0's, to obtain

$$\mathbb{E}_{\rho_0}\left((L(\rho_0) - L(\rho(N)))_+\right) \leqslant N \tag{4.14}$$

and

$$\mathbb{E}_{\rho_0}\left((L(\rho_0) - L(\rho(N)))_+^2\right) \leqslant N^2 + N \tag{4.15}$$

Now consider the function $f = g + h : \Omega \to \mathbb{R}$ where

$$g(\rho) = \sum_{x \leqslant 0} -x(1-\rho(x)) + \sum_{x>0} x\rho(x) \tag{4.16}$$

and

$$h(\rho) = \sigma N\left((-L(\rho) - 5N)_+ + (R(\rho) - 5N)_+\right) \tag{4.17}$$

where the constant σ satisfies $0 < \sigma \ll 1$ and will be specified in Section 4.6.2 (see equations (4.47) and (4.58) for more details). Consider also the subset $A \subset \Omega$ such that

$$A = \{\rho \in \Omega : L(\rho) < -50N \text{ or } R(\rho) > 50N\}^{\mathrm{c}}. \tag{4.18}$$

Here $50N$ can be changed. We just need something big enough to validate the construction in Subsection 4.6.2, situation 2 when $\rho_0 \in \Omega_2$ that is defined later in (4.25).

As outlined previously, the objective is to show that f and A verify the Lyapunov-Foster's criterion (4.10). It is reasonably clear that the second property of (4.10) is satisfied. The following lemma gives a justification.

Lemma 4.6.2. *There exists $K < \infty$ so that uniformly over all configurations $\rho_0 \in A$, we have*

$$\mathbb{E}_{\rho_0}(f(\rho(N))) \leqslant K. \tag{4.19}$$

Proof. Take $X = R(\rho(N)) \vee (-L(\rho(N)))$ and remark that X is stochastically less than the maximum of two independent Poisson processes beginning with values $50N$. As we have $f(\rho(N)) \leqslant 2X^2 + 2\sigma NX$, the result is immediate. $\square$

But first, let's have a closer look on function g.

The quantity $g(\rho)$ jumps with jump size 1 or 2. If the * are disregarded, $g(\rho)$ has only jump size 1, a positive jump rate of

$$\sum_{x\in\mathbb{Z}} \mathbb{1}_{\{\rho(x)\rho(x+1)=10\}} \tag{4.20}$$

and a negative jump rate of

$$\sum_{x\in\mathbb{Z}} \mathbb{1}_{\{\rho(x-1)\rho(x)=01\}}. \tag{4.21}$$

Note that in Ω, (4.21) is always 1 less than (4.20). Now consider the *. For those x with $\rho(x-2)\rho(x-1)\rho(x) = 0^*01$, any jump of the site x on the left will be of order 2 in fact. Similarly, if $\rho(x-1)\rho(x)\rho(x+1) = 011^*$, when x jumps on the left, it will induce a jump of $g(\rho)$ with jump size 2. This motivates the following decomposition

$$g(\rho(t)) - g(\rho(0)) = M(t) + N(t) - A(t) \tag{4.22}$$

where

- $M(t)$ is a local martingale (given $\rho(0)$, it is easily seen to be a martingale) and $M(t)$ makes jumps of size 1 or -1, $M(t)$ can be seen as a time changed random walk,
- $N(t)$ is a rate 1 Poisson process,
- $A(t)$ is a jump process making only jump of size 1 with a jump rate $\sum_{x\in\mathbb{Z}} \mathbb{1}_{\{\rho(x,t)\in\{0^*,1^*\}\}}$.

Fix $M(0) = N(0) = A(0) = 0$. Thus we have

$$\mathbb{E}_{\rho_0}(g(\rho(N))) = g(\rho_0) + N - \mathbb{E}(A(N)). \tag{4.23}$$

The function g is a good candidate to pass the Lyapunov-Foster's criterion because $A(t)$ is usually increasing faster than $N(t)$. However, when the rightmost 1 and the leftmost 0 are isolated, g is not necessarly negative and fails to pass the criterion. That's why we add the h function, because in this situation, this decreases fast and then the criterion is verified. Thus we split $\Omega \setminus A$ into two parts corresponding to the two cases explained before : Ω_1 and Ω_2.

Let $I_L = \{L(\rho_0),\dots,L(\rho_0)+5N\}$ and $I_R = \{R(\rho_0)-5N,\dots,R(\rho_0)\}$. Consider $\delta \ll 1$ that will be fixed small enough later and set

$$\Omega_1 = \left\{\rho_0 \in \Omega : L(\rho) < -50N \text{ or } R(\rho_0) > 50N, \sum_{x\in I_L}(1-\eta(x)) < \delta N \text{ and } \sum_{x\in I_R}\eta(x) < \delta N\right\} \tag{4.24}$$

and

$$\Omega_2 = \left\{\rho_0 \in \Omega : L(\rho) < -50N \text{ or } R(\rho_0) > 50N \text{ and } \sum_{x\in I_L}(1-\eta(x)) \geqslant \delta N \text{ or } \sum_{x\in I_R}\eta(x) \geqslant \delta N\right\}. \tag{4.25}$$

4.6.1 Extreme points are isolated : when $\rho_0 \in \Omega_1$

In this situation, we need to have a finite speed propagation in order to control h.

Lemma 4.6.3. *Consider ϱ a process generated by the same Harris system as ρ and suppose that $\rho(x,0) = \varrho(x,0)$ for all $x \in \{\dots,-1,0\}$, then outside probability e^{-cN},*

$$\rho(x,N) = \varrho(x,N) \text{ for all } x \in \{\dots,-5N-1,-5N\} \tag{4.26}$$

for $c > 0$ not depending on N.

Proof. Define

$$D(t) = \inf\{x : \rho(x,t) \neq \varrho(x,t)\} \tag{4.27}$$

and remark that D is piecewise constant and jumps by at most 2 to the left, with a rate smaller than 2 by our construction. This means that $D(t)$ is dominated from below by $-2P(t)$ with P a rate 2 Poisson process. Then, we have

$$\mathbb{P}(D(N) \leqslant -5N) \leqslant \mathbb{P}(P(N) \geqslant 2.5). \tag{4.28}$$

Large deviations lead to the conclusion. □

As we are in an isolated situation, we can observe the behaviour of the rightmost 1. Proposition 4.7.1 in Section 4.7 insures that when there are at most δN particles,

$$\mathbb{P}(R(\rho(N)) - R(\rho_0) \leqslant \delta N) \leqslant e^{-cN} \tag{4.29}$$

for some constant $c > 0$ not depending on N. By exchanging 1's and 0's, we get a similar result for L. Then by Lemma 4.6.3, for some constants $c_1, c_2 > 0$, outside probability $2e^{-c_1 N} + 2e^{-c_2 N}$, we have

$$R(\rho(N)) < R(\rho_0) - \delta N \text{ and } L(\rho(N)) > L(\rho_0) + \delta N. \tag{4.30}$$

Let's call this event G to clean the further notations. We can now return to our function f. Recall that

$$\mathbb{E}_{\rho_0}(g(\rho(N))) \leqslant g(\rho_0) + N - \mathbb{E}_{\rho_0}(A(N)) \leqslant g(\rho_0) + N \tag{4.31}$$

by decomposition (4.22). When $\rho_0 \in \Omega_1$, $A(N)$ is really small and then we need the part h to verify the criterion. By previous result (4.30) and as $R(\rho_0) > 50N$ or $L(\rho_0) < 50N$, we have

$$E_{\rho_0}(h(\rho(N)) - h(\rho_0)) \leqslant E_{\rho_0}(h(\rho(N)) - h(\rho_0)|G^c)\mathbb{P}_{\rho_0}(G^c) - \sigma N \cdot 2\delta N \cdot (1 - 2e^{-c_1 N} - 2e^{-c_2 N}). \tag{4.32}$$

By Cauchy-Schwarz inequality and Lemma 4.6.1, we have

$$E_{\rho_0}(h(\rho(N)) - h(\rho_0)|G^c)\mathbb{P}_{\rho_0}(G^c) \leqslant \quad E_{\rho_0}((h(\rho(N)) - h(\rho_0))^2)^{\frac{1}{2}}\mathbb{P}_{\rho_0}(G^c)^{\frac{1}{2}} \\ \leqslant \quad \sigma N \cdot 2\sqrt{N^2+N} \cdot \sqrt{2e^{-c_1 N} + 2e^{-c_2 N}} \tag{4.33}$$

which leads to, for N sufficiently large and for some constant $c > 0$,

$$E_{\rho_0}(h(\rho(N)) - h(\rho_0)) \leqslant 2\sigma\delta c N^2 \tag{4.34}$$

with the help of equation (4.32). Thus, by equation (4.31), we get

$$\mathbb{E}_{\rho_0}(f(\rho(N))) \leqslant f(\rho_0) - 2\sigma\delta c N^2 + 2N \leqslant f(\rho_0) - 1 \tag{4.35}$$

for N large enough.

4.6.2 Mixed situation : when $\rho_0 \in \Omega_2$

As $\rho_0 \in \Omega_2$, we know that either the number of 1's in $\{R(\rho_0) - 5N, \dots, R(\rho_0)\}$ exceeds δN or the number of 0's in $\{L(\rho_0), \dots, L(\rho_0) + 5N\}$ exceeds δN. As both situations are symmetric, consider that the first one occurs.

In this case, we need to verify that h is increasing slower than g in order the criterion to be checked. We control $\mathbb{E}_{\rho_0}(h(\rho(N)) - h(\rho_0))$ via Lemma 4.6.1, obtaining the bound $2\sigma N^2$. Thus it suffices to show that $\mathbb{E}_{\rho_0}(A(N))$ is sufficiently positive as we have

$$\mathbb{E}_{\rho_0}(g(\rho(N))) \leqslant g(\rho_0) + N - \mathbb{E}_{\rho_0}(A(N)). \tag{4.36}$$

To study $A(\rho(N))$, we partition $\{L(\rho_0), \dots, R(\rho_0)\}$ into intervals of length between $\frac{\delta N}{4}$ and $\frac{\delta N}{2}$, named $I_1, \dots, I_m$ (for some m), so that $R(\rho_0) - 5N$ is endpoint of an interval. These partitions include intervals of three types, where either the proportion of 1's or 1^*'s exceeds $\frac{1-\delta^3}{4}$ (and the interval is said *positive*), either the proportion of 0's or 0^*'s exceeds $\frac{1-\delta^3}{4}$ (and the interval is said *negative*) or *intermediate* when the interval is neither positive nor negative. These values are chosen so that at least one of the intervals in I_R (resp. I_L) is not negative (resp. positive) when the quantity of 1's or 1^*'s (resp. 0's or 0^*'s) exceeds δN in I_R (resp. I_L).

We now consider two situations :

1. there is j so that I_j is not positive and I_{j+1} is not negative,
2. such a j does not exist.

In both cases, we adapt Proposition 4.7.1 (that will be proved in Section 4.7) to show that $A(N)$ will, with high probability, be at least of the order N^2.

Lemma 4.6.4. *Given $m < \delta N$ points $x_1 > \dots > x_m$ such that $\rho_0(x_i) \in \{1, 1'*\}$ for all i, there exist a family of motions $\{Y_1(t), \dots, Y_m(t)\}_{0 \leqslant t \leqslant N}$ so that*

- $Y_i(t) > Y_{i+1}(t)$ *for all* i *and* t,
- $\rho(Y_i(t), t) = 1$ *for all* i *and* t.

Moreover, $Y_i(t)$ *is right continuous, piecewise constant, making a finite number of jumps on* $t \in [0, N]$ *such that for all* t, $Y_i(t) - Y_i(t_-) \in \{-2, -1, 0, 1\}$ *and with probability at least* e^{-cN},

$$Y_i(N) < Y_i(0) - \delta N \quad \text{for all } i. \tag{4.37}$$

Proof. Take $Y_i(0) = x_i$ and let Y_i evolve freely, as in Proposition 4.7.1. While the particle is isolated, everything goes well. When the particle wants to move but is blocked by one of its neighbours to the left, we simply swap their positions. For instance, suppose that $Y_i(t_-) = x$ (which means that $\rho(x, t_-) = 1$), $\rho(x-1, t_-) = 1$ and $t \in N^{x,x-1}$, then we simply write that $Y_i(t) = x - 1$. By this way, Y_i evolves in the same way as X_i in Proposition 4.7.1. □

Remark that the analogous conclusion holds for 0's and 0^*'s.

Now consider situation 1. For some j, I_j has at least m 0's or 0^*'s and I_{j+1} has at least m 1's or 1^*'s for

$$m = \frac{\delta^4}{16} N. \tag{4.38}$$

Choose exactly m points in each interval. Let $Y_1(t), \ldots, Y_m(t)$ be the m motions associated to the 1's or 1^*'s in I_{j+1} and $Z_1(t), \ldots, Z_m(t)$ be the m motions associated to the 0's or 0^*'s in I_j. By construction, we have

$$\max\{Z_1(0), \ldots, Z_m(0)\} < \min\{Y_1(0), \ldots Y_m(0)\} \tag{4.39}$$

and by Lemma 4.6.4, we have

$$\max\{Y_1(N), \ldots, Y_m(N)\} < \min\{Z_1(N), \ldots Z_m(N)\}. \tag{4.40}$$

Thus choose $\lfloor \frac{m}{11} \rfloor$ subscripts from $\{1, \ldots, m\}$, named $i_1, \ldots, i_{\lfloor \frac{m}{11} \rfloor}$, so that $i_{k+1} - i_k \geqslant 10$ for all k. Repeat that operation to create the family $j_1, \ldots, j_{\lfloor \frac{m}{11} \rfloor}$ so that $j_k - j_{k+1} \geqslant 10$ for all k. By this way, we ensure that a motion can only correspond to one event G (described later). For $u, v \leqslant \lfloor \frac{m}{11} \rfloor$, we can now define the stopping time

$$T_{u,v} = \inf\{t : Y_{j_u}(t) - Z_{i_v}(t) \leqslant 2\} \tag{4.41}$$

and the time interval

$$I_{T_{u,v}} = [T_{u,v}, T_{u,v} + 1]. \tag{4.42}$$

Then consider the interval

$$I_{u,v} = \{Z_{i_v}(T_{u,v}) - 1, \dots, Y_{j_u}(T_{u,v}) + 1\} \tag{4.43}$$

and define the event

$$G_{u,v} = \{\text{no particle enters } I_{u,v} \text{ and there is a jump of size 2 during time interval } I_{T_{u,v}}\}. \tag{4.44}$$

Remark that the jump in event $G_{u,v}$ can be to left or to right. Although the probability of event $G_{u,v}$ can be deduced from the Harris system, we don't need the exact value and simply refer to it as $p_{u,v}$. Then, for all u, we have that $\sum_{v=1}^{\lfloor \frac{m}{11} \rfloor} \mathbb{1}_{\{G_{u,v}\}}$ is stochastically greater than a binomial $\text{Bin}(\lfloor \frac{m}{11} \rfloor, p_u)$ with $p_u = \min_v\{p_{u,v}\}$. Hence, by large deviations and as no jump of size 2 can correspond to two distinct (u, v), we have

$$\mathbb{P}\left(\exists u \leqslant \left\lfloor \frac{m}{11} \right\rfloor : \sum_{v=1}^{\lfloor \frac{m}{11} \rfloor} \mathbb{1}_{\{G_{u,v}\}} \leqslant \frac{\lfloor \frac{m}{11} \rfloor p}{2}\right) \leqslant \left\lfloor \frac{m}{11} \right\rfloor e^{-c \lfloor \frac{m}{11} \rfloor} \leqslant e^{-c' m} \tag{4.45}$$

for $p = \min_u\{p_u\}$, c, c', some constants and m sufficiently large. In particular, this means that, for some constants $c_1, c_2 > 0$, outside probability $2e^{-c_1 N} + e^{-c_2 N}$ we have

$$\mathbb{E}_{\rho_0}(A_N) \geqslant \frac{\delta^8 N^2}{11^2 \cdot 16^2} \cdot \left(\frac{p}{2}\right)^2. \tag{4.46}$$

Situation 1 is done if σ verifies

$$2\sigma - \frac{\delta^8}{11^2 \cdot 16^2} \cdot \left(\frac{p}{2}\right)^2 < 0. \tag{4.47}$$

Now consider situation 2. The rightmost non negative interval is necessary on the right of $R(\rho_0) - 5N$ because $\rho_0 \in \Omega_2$ as pointed previously. The interval immediately on its left cannot be non positive and has to be positive. Actually, the reasoning repeats and every interval on the left should be positive. We are then in a situation where we have a high density of particles and as the number of 1's at the right of the origin equals the amount of 0's at the left of origin, $L(\rho_0)$ has to be very far from origin. By the way, for δ sufficiently small, we can find $x \geqslant L(\rho_0)$ and $y \geqslant x + 25N$ such that the number of 0's is smaller than δN but greater than $\frac{\delta N}{2}$ on interval $\{x, \dots, y\}$. We now want to show that the amount of jumps of size 2 is at least of order N^2 in this interval with high probability so that $\mathbb{E}_{\rho_0}(A(N)) \geqslant cN^2$ for some constant $c > 0$.

Consider an auxiliary process ρ_{F} identically to ρ but where there are no 0's and no 0^*'s outside interval $\{x, x+1, \dots, y-1, y\}$. We are then in the situation of Proposition 4.7.1. Suppose that the number of particles is m (recall that $\delta N \geqslant m \geqslant \frac{\delta N}{2}$) and define $X_1, \dots X_m$ to be the motions associated to each particle. We can now consider a decomposition of $S_m(t) = \sum_{i=1}^{m} X_i(t)$

similar to (4.22)

$$S_F(t) - S_F(0) = M_F(t) + A_F(t) \tag{4.48}$$

with M_F a martingale and A_F a jump process simply counting the number of jumps of size 2 (on the right, as we are looking on 0's and 0^*'s). Note that $M_F(0) = A_F(0) = 0$. Remark that M_F is a random walk with rate smaller than m and then, for some constants $c_1, c_2 > 0$, we have

$$\mathbb{P}(|M_F(N)| > c_1 mN) \leqslant e^{-c_2 N}. \tag{4.49}$$

Proposition 4.7.1 shows that, for some constants $c_3 > 0$, we have

$$\mathbb{P}(S_F(N) < \delta N \cdot m) \leqslant e^{-c_3 t}. \tag{4.50}$$

Then, if we take δ sufficiently small for $c_1 > \frac{\delta}{2}$, then we have, for some constant $c > 0$,

$$\mathbb{P}(A_F(N)) < \frac{\delta N m}{2}) \leqslant e^{-cN}. \tag{4.51}$$

Now, simply remark that $m \geqslant \frac{\delta N}{2}$ and we have, for some constant $c > 0$,

$$\mathbb{P}(A_F(N) < \frac{\delta^2 N^2}{4}) \leqslant e^{-cN}. \tag{4.52}$$

We now need to control the contribution of 0's and 0^*'s in intervals $\{x, \ldots, x+5N\}$ and $\{y-5N, \ldots, y\}$. We say that a site is associated with a jump of size 2 if it is one of the two sites involved in the jump, i.e.,

- $t \in N^{2,3}$ and $\rho(t_-) = 0^*01$: we get $\rho(t) = 100$, sites 2 and 3 are associated with this jump,
- $t \in N^{1,2}$ and $\rho(t_-) = 011^*$: we get $\rho(t) = 110$, sites 1 and 2 are associated with this jump.

If we fix a particle x (a site with value 0 or 0^*, as we try to control them), it is then clear that the number of jumps associated to it is stochastically dominated by a Poisson process of rate 1 as we need that $t \in N^{x,x+1}$. In particular, if we write $K(t)$ for the number of associated jumps (until time t) for a fixed particle, there exists $c > 0$ so that

$$\mathbb{P}(K(N) > 2N) \leqslant e^{-cN} \tag{4.53}$$

by standard Poisson bounds. As we have less than a proportion $\frac{\delta^3}{4}$ of 0's in intervals $\{x, \ldots, x+5N\}$ and $\{y-5N, \ldots, y\}$, the contribution of 0's and 0^*'s in these intervals, until time N, is at most

$$\frac{\delta^3}{4} \cdot 5N \cdot 2N \cdot 2 = \delta^3 5N^2 \tag{4.54}$$

outside a probability of e^{-cN}, for some constant $c > 0$.

However, we did not consider interactions between particles in $\{x,\dots,x+5N\}$ and in $\{x+5N+1,\dots,x+10N\}$ that can modify the previous results. Then, repeat the reasoning of the previous paragraph but this time consider intervals $\{x,\dots,x+10N\}$ and $\{y-10N,\dots,y\}$ to compute equation (4.54). Lemma 4.6.3 ensures that, with high probability, particles in intervals $\{x,\dots,x+5N\}$ and $\{y-5N,\dots,y\}$ do not interact with particles in interval $\{x+10N+1,\dots,y-10N-1\}$. This shows that the contribution of 0's and 0^*'s in intervals $\{x,\dots,x+5N\}$ and $\{y-5N,\dots,y\}$ to A_{F} is at most

$$\delta^3 10N^2 \tag{4.55}$$

outside probability of e^{-cN}, for some constant $c>0$. This implies, due to equation (4.52), that the total contribution of 0's and 0^*'s in interval $\{x+5N+,\dots,y-5N-1\}$ is at least

$$\frac{\delta^2 N^2}{4} - \delta^3 10N^2 \geqslant \frac{\delta^2 N^2}{8} \tag{4.56}$$

for δ sufficiently small, outside probability e^{-cN} for some constant $c>0$.

Then, using the finite speed propagation (Lemma 4.6.3), we know that, outside a probability e^{-cN}, for some constant $c>0$, bound (4.56) is also true for the non-finite model ρ restricted to interval $\{x+5N+1,\dots,y-5N-1\}$. This implies that for some constant $c>0$, outside probability e^{-cN},

$$A(N) \geqslant \frac{\delta^2 N^2}{8}. \tag{4.57}$$

If σ verifies

$$2\sigma - \frac{\delta^2}{8} < 0, \tag{4.58}$$

it concludes the proof.

4.7 Technical results

In this Section, we consider the case where there is a finite quantity (of order N) of 0's close together surrounded by 1's. We may speak of the position $X_1(t)$ of the leftmost 0 of $\rho(t)$, $X_i(t)$ the i-leftmost of $\rho(t)$ and $L(t)$ the leftmost particle 0 of $\rho(t)$. By spatial homogeneity, we can fix $L(0)=0$. We decided to identify 0's as particles so that the bias is positive. As the model is symmetric, we get the same result for 1's in the opposite direction.

Formally we have

$$\sum_{i=0}^{\infty} \big(1-\rho(x,0)\big) < \delta N. \tag{4.59}$$

with δ very small, not depending on N. We wish to show that, outside an exponentially small probability (in N), for all $j \leqslant \delta N$, $X_i(N) - X_i(0)$ will be of order N. We will use the fact that X_i has a strictly positive drift to the right.

Proposition 4.7.1. *Consider ρ the process associated to λ^c with only a finite amount of particles with value 0. Take $X_i(t)$ to be the i-rightmost particle with value 0 of $\rho(t)$. There exists some constant δ_0 very small so that, for all $\delta < \delta_0$, if the quantity of particles with value 0 is less than δN, then there exist some constants $c_1, c_2 > 0$ not depending on N such that*

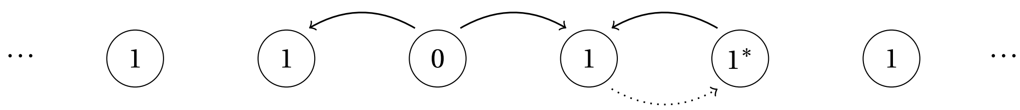
$$\mathbb{P}(X_i(N) - X_i(0) \leqslant \delta_0 N) \leqslant c_1 e^{-c_2 N} \quad \forall i. \tag{4.60}$$

We then define the following (non-Markov) motion, derived from the given Harris system. $\{Y(t)\}_{t\geqslant 0}$ will be a process on $\mathbb{Z}$ which moves in jumps of 1 to the left (or jumps of size -1) and jumps of 1 or 2 to the right. The process will jump from site x to site $x-1$ at time t if and only if $Y(t_-) = x$ and $t \in N^{x-1,x}$, it will jump rightwards if and only if $t \in N^{x,x+1}$. At such times it will jump a distance of 2 if within the relevant time interval, Y has not moved, there is a point in $N^{x+1,x+2}$ and no point in $N^{x+1,x+2}$ later. We can easily see that $\{Y(t)\}_{t\geqslant 0}$ is an asymmetric random walk with drift $\frac{1}{8}$. To be more precise, the transitions are given by

- -1 with probability $\frac{1}{2}$,
- $+1$ with probability $\frac{3}{8}$,
- $+2$ with probability $\frac{1}{8}$.

This process can be turned into a Markov process by the addition of an auxiliary process $\{\Delta(t)\}_{t\geqslant 0}$ taking values 0 or 1 such that there will be a jump of Y at time t of magnitude 2 if and only if $t \in N^{x,x+1}$ and $\Delta(t_-) = 1$. Process Δ will evolve as follows :

- $\Delta(0) = 0$;
- when $t \in N^{Y(t_-)+1,Y(t_-)+2}$, $\Delta(t) = 1$;
- if $\Delta(t_-) = 1$, then $\Delta(t) = 0$ when $t \in N^{Y(t_-)-1,Y(t_-)}$, $t \in N^{Y(t_-),Y(t_-)+1}$ or $t \in N^{Y(t_-)+1,Y(t_-)+2}$.



Motions in black set Δ to value 0 and dotted motions set Δ to value 1.

It is clear that $Y(t) \leqslant X_1(t)$ if $Y(0) = X_1(0)$. Moreover, it also provides a basis lower bound for $X_i(t)$. First define a function h by

$$h(1) = X_1(0) \qquad \text{and} \qquad h(i) = X_i(0) \wedge (h(i-1) - b) \tag{4.61}$$

for all appropriate i, with $b \geqslant 2$.

Then define inductively $\{Y_i(t)\}_{t\geqslant 0}$ evolving as Y and given Y_{i-1}, one constrains Y_i to always be bounded by $Y_{i-1}-2$. That is, if $Y_i(t_-) = Y_{i-1}(t_-)-2$ and Y_{i-1} jumps on the left, formally when $t \in N^{Y_{i-1}(t_-)-1,Y_{i-1}(t_-)}$, then Y_i must also do so at the same time and the auxiliary process $\Delta(t)$ is set to 0. Equally if $Y_i(t_-) = Y_{i-1}(t_-)-3$ or $Y_i(t_-) = Y_{i-1}(t_-)-2$ and Y_i jumps on the right, formally when $t \in N^{Y_i(t_-),Y_i(t_-)+1}$, we set $Y_i(t) = Y_{i-1}(t)-2$ and $\Delta(t)$ is reset to 0. We also fix the start of these processes such that $Y_i(0) = X_i(0) \wedge (X_{i-1}(0)-2)$ which means that we have $Y_i(t) \leqslant X_i(t)$ for all i and all time t. We will then show in Proposition 4.7.2 that Y_i (and so X_i) has a positive drift and use it to prove Theorem 4.7.1.

Proposition 4.7.2. *For b chosen large enough and $c>0$ small enough,*

$$\mathbb{E}\left(\int_0^\infty \exp\left(-c\left(Y_i(t)-h(i)-\frac{1}{16}\,t\right)\right)dt\right) \leqslant C \tag{4.62}$$

for appropriate C, for all i and all initial configuration ρ_0 satisfying condition (4.59).

The basic idea behind the inductive argument we will use to prove Proposition 4.7.2 is that, as long Y_i lies at distance 2 below Y_{i-1}, it will not be impeded in its movement. Since this occurs most of the time when the particle density is low, Y_i will move similarly to a continuous time random walk with drift almost $\frac{1}{8}$, the actual drift of Y. We will then proceed as in [9], section 5, and introduce a system of subprocesses when $Y_i(t_j)$ decreased quickly to control the effect Y_i has on Y_{i+1}.

The goal is to show that Y_i has a long term drift at least $\frac{1}{16}$. It's more convenient to shift coordinates in order to prove it. We set

$$\phi_{i-1}(t) = (Y_{i-1}(t)-2)-\frac{1}{16}\,t-h(i) \tag{4.63}$$

and

$$Z_i(t) = Y_i(t)-\frac{1}{16}\,t-h(i). \tag{4.64}$$

ϕ_{i-1} will act as a barrier on Z_i in a similar way than Y_{i-1} on Y_i. If we bound

$$\mathbb{E}\left(\int_0^\infty \exp(-cZ_i(t))dt\right), \tag{4.65}$$

we get the result for Y_i. The main problem here is that ϕ_i will not act as a line with positive slope but can decrease quickly which can cause Z_i to decrease. It means that we cannot simply use standard large deviation results. That's why we will define a system of subprocesses $\{Z_{i,j}(t)\}_{t\geqslant t_j}$, to control the effect of ϕ_{i-1} over Z_i. But first, we give some basic properties.

As $Y_i(t) \leqslant Y_{i-1}(t)-2$ for all $t>0$, we have

$$Z_i(t) \leqslant \phi_{i-1}(t) \quad \text{for all } t>0 \tag{4.66}$$

and by definition of h, we have $Z_i(0) \geqslant 0$ and

$$Z_i(t) = \phi_i(t) + h(i+1) - h(i) + 2 \leqslant \phi_i(t) - b + 2 \quad \text{for all } t > 0. \tag{4.67}$$

We now introduce stopping times $\{S_i(j)\}_{j\in\mathbb{N}}$ to control the decrease of ϕ_i. Fix $S_i(0) = 0$ and inductively set

$$S_i(j) = \lfloor S_i(j-1)+1 \rfloor \wedge \min\Big\{t > S_i(j-1) : \phi_{i-1}(t) \leqslant \phi_{i-1}\big(S_i(j-1)\big) - 1\Big\}. \tag{4.68}$$

We can remark that the amount ϕ_j can decrease between these times is bounded by 1 and $S_i(j) - S_i(j-1) \leqslant 1$ for all j. We are now ready to define the system of subprocesses $\{Z_{i,j}(t)\}_{t \geqslant t_j}$. At each time $S_i(j)$, we add a process $Z_{i,j}$ starting from

$$Z_{i,j}(t) = \phi_{i-1}\big(S_i(j)\big) \quad \text{for } t = S_i(j) \tag{4.69}$$

and evolving as Z_i (according to the same translated random walk) except

- $Z_{i,j}$ is not stopped by the barrier ϕ_{i-1},
- on the initial interval $\big[S_i(j), \lfloor S_i(j)+1 \rfloor\big)$, positive jumps of $Z_{i,j}$ are suppressed.

We will show in Lemma 4.7.3, that the processes $\{Z_{i,j}\}_{j\in\mathbb{N}}$ will provide a lower bound for Z_i.

Lemma 4.7.3. *Fix i and define*

$$U_i = \min\{t \geqslant 0 : Z_i(t) = \phi_{i-1}(t)\}. \tag{4.70}$$

For $t \geqslant U_i$, we have

$$Z_i(t) > \min_{j \geqslant 0} Z_{i,j}(t) - 1. \tag{4.71}$$

Proof. Suppose that $Z_i(T) = \phi_{i-1}(T)$ at a given time T. We claim that

$$Z_i(T) > Z_{i,j}(T) - 1 \quad \text{for some } j \geqslant 0. \tag{4.72}$$

1. If $T = S_i(j)$ for some j, then $Z_i(T) = Z_{i,j}(T)$;
2. Otherwise T is not an integer and there exist some j such that $S_i(j) \in [\lfloor T \rfloor, T)$ for which $\phi_{i-1}(T) > \phi_{i-1}(S_i(j)) - 1$ by definition (4.68). Since $Z_{i,j}$ cannot increase on $[S_i(j), T]$, this implies that

$$Z_{i,j}(T) - 1 \leqslant Z_{i,j}(S_i(j)) - 1 = \phi_{i-1}(S_i(j)) - 1 < \phi_{i-1}(T) = Z_i(T). \tag{4.73}$$

Hence, (4.72) holds.

After time T, the process $Z_{i,j}$ evolves according to the same law as Z_i except it does not take care about the barrier ϕ_{i-1} and positive jumps on $\big[S_i(j), \lfloor S_i(j)+1 \rfloor\big)$ are suppressed. So until the next time at which Z_i is restricted by the barrier ϕ_{i-1}, $Z_i(t) > Z_{i,j}(t) - 1$.

After a finite amount of time, Z_i has only a finite number of changes of state, and then attempts to cross ϕ_{i-1} only a finite number of times. We can also use induction to prove that (4.71) holds for all $t \geqslant U_i$. □

To prove Proposition 4.7.2, we have to analyze each $Z_{i,j}$. We also define the translated random walk Z with $Z(0) = 0$ and which evolves according to the same transition law as each $Z_{i,j}$ except that positive jump during interval $[0,1)$, instead of $\left[S_i(j), \lfloor S_i(j)+1\rfloor\right)$, are suppressed. On $[1,\infty)$, Z is a finite range random walk with positive drift and so $Z(t) \to \infty$ linearly off a large deviation set as $t \to \infty$.

Lemma 4.7.4. *For $c > 0$ chosen small enough, we have*

$$\mathbb{E}\left(\int_0^\infty e^{-cZ(t)}\,dt\right) < \infty. \tag{4.74}$$

Proof. Choose $a > 1$ and set $\tilde{Z}(t) = Z(t+a) - Z(a)$. It defines a random walk starting at 0 with no suppression of jumps. Write $z_x = \mathbb{E}\left(\int_0^x e^{-cZ(t)}\,dt\right)$. For $x \geqslant a$, we have,

$$z_x \leqslant z_a + \mathbb{E}\left(e^{-cZ(a)}\right) z_{x-a} \leqslant z_a + \mathbb{E}\left(e^{-cZ(a)}\right) z_x. \tag{4.75}$$

For $c > 0$ small enough, we have

$$\mathbb{E}\left(e^{-c\tilde{Z}(a-1)}\right) \leqslant \gamma^{a-1} \tag{4.76}$$

for $\gamma \in (0,1)$. So for large enough a, we get

$$\mathbb{E}\left(e^{-cZ(a)}\right) = \mathbb{E}\left(e^{-c\tilde{Z}(a-1)}\right)\mathbb{E}\left(e^{-cZ(1)}\right) \leqslant \gamma^{a-1}\mathbb{E}\left(e^{-cZ(1)}\right) < 1. \tag{4.77}$$

By 4.75 we have

$$z_x \leqslant \frac{z_a}{1 - \mathbb{E}\left(e^{-cZ(a)}\right)} \tag{4.78}$$

and then, letting $x \to \infty$ implies $z_\infty < \infty$. □

Fix i and take $j_0, j_1, j_2, \ldots$ the subscripts at which $S_i(j_n) = n$. We can then set $T_n = S_i(j_n - 1)$, the last stopping time before time n. Using this with lemma 4.7.4 gives a bound for the integrals of the moment generating function of $Z_{i,j}$ summed over j for which $S_i(j) \in [n-1, n)$, in terms of the moment generating function of $\phi_{i-1}(T_n)$.

Lemma 4.7.5. *For $c > 0$ chosen large enough,*

$$\mathbb{E}\left(\sum_{j=j_{n-1}}^{j_n-1} \int_0^\infty e^{-cZ_{i,j}(t)}\,dt\right) \leqslant C\,\mathbb{E}\left(e^{-c\phi_{i-1}(T_n)}\right) \tag{4.79}$$

for all n, $1 \leqslant i \leqslant N\delta$ and appropriate C.

Proof. Recall that Z is the translated random walk starting on $Z(0) = 0$ and which evolves according to the same transition law as each $Z_{i,j}$ except that positive jump during interval $[0,1)$, instead of $\big[S_i(j), \lfloor S_i(j)+1\rfloor\big)$, are suppressed. We then have

$$Z_{i,j}(t+S_i(j)) - Z_{i,j}(S_i(j)) \geqslant Z(t) \tag{4.80}$$

for all t and Z is independent of $\mathscr{F}_{S_i(j)}$. So we get

$$\begin{aligned}\mathbb{E}\left(\int_{S_i(j)}^{\infty} e^{-cZ_{i,j}(t)}\,dt\right) &= \mathbb{E}\left(\int_0^{\infty} e^{-cZ_{i,j}(t+S_i(j))}\,dt\right)\\ &= \mathbb{E}\left(e^{-cZ_{i,j}(S_i(j))}\int_0^{\infty} e^{-c\left(Z_{i,j}(t+S_i(j))-Z_{i,j}(S_i(j))\right)}\,dt\right)\\ &\leqslant \mathbb{E}\left(e^{-cZ_{i,j}(S_i(j))}\int_0^{\infty} e^{-cZ(t)}\,dt\right)\\ &= \mathbb{E}\left(\int_0^{\infty} e^{-cZ(t)}\,dt\right)\mathbb{E}\left(e^{-cZ_{i,j}(S_i(j))}\right).\end{aligned} \tag{4.81}$$

We know that $\phi_{i-1}(S_i(j+1)) \leqslant \phi_{i-1}(S_i(j)) - 1$ for stopping times $S_i(j)$ and $S_i(j+1)$ in interval $[n-1,n)$ and any n. It means that

$$\phi_{i-1}(S_i(j)) - \phi_{i-1}(S_i(j_n-1)) \geqslant (j_n-1) - j \tag{4.82}$$

for $j \in \{j_{n-1},\dots,j_n-1\}$ and then, using (4.69),

$$\mathbb{E}\left(e^{-cZ_{i,j}(S_i(j))}\right) \leqslant \mathbb{E}\left(e^{-c(j_n-1-j)}e^{-cZ_{i,j_n-1}(S_i(j_n-1))}\right) = e^{-c(j_n-1-j)}\mathbb{E}\left(e^{-cZ_{i,j_n-1}(T_n)}\right) \tag{4.83}$$

for such j. Lemma 4.7.4, (4.81) and (4.83) lead to

$$\mathbb{E}\left(\sum_{j=j_{n-1}}^{j_n-1}\int_{S_i(j)}^{\infty} e^{-cZ_{i,j}(t)}\,dt\right) \leqslant C\,\mathbb{E}\left(e^{-cZ_{i,j_n-1}(T_n)}\right) = C\,\mathbb{E}\left(e^{-c\phi_{i-1}(T_n)}\right). \tag{4.84}$$

□

We are now ready to prove Proposition 4.7.2 using the bound of the integral of the moment generating function of $Z_{i,j}$ given in Lemma 4.7.5 and the fact that these processes bound Z_i as stated in Lemma 4.7.3.

Proof. We will use induction to show that for i,

$$\mathbb{E}\left(\int_0^{\infty} e^{-cZ_i(t)}\,dt\right) \leqslant C \tag{4.85}$$

for large enough b, small enough $c > 0$ and appropriate C, which do not depend on i. Using this inequality and (4.64) prove (4.62).

As Y_1 is a finite range random walk with drift $\frac{1}{8}$, the case $i = 1$ is clear. Let say that it works for a certain constant C_1 appropriate and chose C such that $C = 2C_1$. Assume now that (4.85) holds for $i-1$. On $t < U_i$, Z_i is not blocked by the barrier and evolves as a translated random walk with the same transition law as Z_1. We denote $\bar{Z}$ the extended process coupled with Z_i before U_i and evolving as Z_1 after U_i. By Lemma 4.7.3,

$$Z_i(t) \geqslant \left(\min_{j\geqslant 0} Z_{i,j}(t) - 1\right) \wedge \bar{Z}(t) \tag{4.86}$$

for all t. Consequently,

$$\mathbb{E}\left(\int_0^\infty e^{-cZ_i(t)}\,dt\right) \leqslant e^c\,\mathbb{E}\left(\sum_{j\geqslant 0}\int_0^\infty e^{-cZ_{i,j}(t)}\,dt\right) + \mathbb{E}\left(\int_0^\infty e^{-c\bar{Z}(t)}\,dt\right). \tag{4.87}$$

Moreover, as Z_1 and $\bar{Z}$ have the same law and $\bar{Z}(0) \geqslant 0$,

$$\mathbb{E}\left(\int_0^\infty e^{-c\bar{Z}(t)}\,dt\right) \leqslant C_1 \tag{4.88}$$

and by Lemma (4.7.5),

$$\mathbb{E}\left(\sum_{j\geqslant 0}\int_0^\infty e^{-cZ_{i,j}(t)}\,dt\right) = \mathbb{E}\left(\sum_{n=1}^{\infty}\sum_{j=j_{n-1}}^{j_n-1}\int_0^\infty e^{-cZ_{i,j}(t)}\,dt\right) \leqslant C_2\,\mathbb{E}\left(\sum_{n=1}^{\infty} e^{-c\phi_{i-1}(T_n)}\right) \tag{4.89}$$

for small enough $c > 0$ and appropriate C_2 not depending on i. Take $P \sim \mathbb{P}(1)$ a Poisson rate 1 random variable. Using the space homogeneity of the jumps of ϕ_{i-1} and the fact that over interval $[n-1, n)$, the difference of supremum and infimum of ϕ_{i-1} is at most the number of jumps, we get

$$\begin{aligned}
&\mathbb{E}\left(\sum_{n=1}^{\infty}\exp\left(-c\inf_{t\in[n-1,n)}\phi_{i-1}(t)\right)\right)\\
&= \mathbb{E}\left(\sum_{n=1}^{\infty}\exp\left(c\sup_{t\in[n-1,n)}\phi_{i-1}(t) - c\inf_{t\in[n-1,n)}\phi_{i-1}(t)\right)\exp\left(-c\sup_{t\in[n-1,n)}\phi_{i-1}(t)\right)\right)\\
&\leqslant \sum_{n=1}^{\infty}\mathbb{E}\left(e^{cP}\right)\mathbb{E}\left(\exp\left(-c\sup_{t\in[n-1,n)}\phi_{i-1}(t)\right)\right)\\
&\leqslant (e^c - 1)\,\mathbb{E}\left(\int_0^\infty e^{-c\phi_{i-1}(t)}\,dt\right).
\end{aligned} \tag{4.90}$$

As $T_n \in [n-1, n)$, using (4.90), (4.67) and the induction hypothesis, we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{n=1}^{\infty} e^{-c\phi_{i-1}(T_n)}\right) &\leqslant \mathbb{E}\left(\sum_{n=1}^{\infty} \exp\left(-c \inf_{t\in[n-1,n)} \phi_{i-1}(t)\right)\right) \\
&\leqslant C_3\,\mathbb{E}\left(\int_0^{\infty} e^{-c\phi_{i-1}(t)}\,dt\right) \\
&\leqslant C_3\, e^{c(2-b)}\mathbb{E}\left(\int_0^{\infty} e^{-cZ_{i-1}(t)}\,dt\right) \\
&\leqslant 2C_1C_3\, e^{c(2-b)}
\end{aligned} \tag{4.91}$$

for appropriate C_3 not depending on i. Plugging (4.88), (4.89) and (4.91) into (4.87) implies that for $i > 1$,

$$\mathbb{E}\left(\int_0^{\infty} e^{cZ_i(t)}\,dt\right) \leqslant C_1\left(1 + 2C_2C_3\, e^{c(2-b)}\right) \leqslant 2C_1 = C \tag{4.92}$$

for b large enough, as C_1, C_2, C_3 and $c > 0$ do not depend on i. □

Theorem 4.7.1 can now be proved.

Proof. Recall there are at most $N\delta$ particles and write L for the leftmost particle. We first show that Proposition 4.7.2 implies

$$\mathbb{E}\left(\exp\left(\frac{c}{16}N - cbN\delta - cL(N)\right)\right) \leqslant C \tag{4.93}$$

for appropriate c, C and b.

Using the same arguments as in (4.90) for Z_i instead of ϕ_{i-1} and Proposition 4.7.2, we have

$$\mathbb{E}\left(\sum_{n=1}^{\infty} \exp\left(-c \inf_{t\in[n-1,n)} Z_i(t)\right)\right) \leqslant C \tag{4.94}$$

which implies

$$\mathbb{E}\left(e^{-cZ_i(t)}\right) \leqslant C \tag{4.95}$$

for an appropriate C. Remark that as $X_1(0) \geqslant 0$, $h(k) > -bk$ for all k, so as h is decreasing, for any k we have $h(k) > -b\delta N$. Then, choose $i = \delta N$ in (4.95) and do the substitution to get

$$\mathbb{E}\left(\exp\left(c\left(\frac{1}{16}t - bN\delta - L(t)\right)\right)\right) \leqslant C. \tag{4.96}$$

Fix time $t = N$ and you obtain (4.93).

Now we show that the leftmost particle will move at least almost $\frac{1}{16}N$ to the right, except for a small probability in N. Choose δ_0 as close as desired to $\frac{1}{16}$. We claim that

$$\mathbb{P}(L(N) \leqslant \delta_0 N) \leqslant Ce^{-cN} \tag{4.97}$$

for $c > 0$ and C appropriate. First remark that if b and δ_0 are fixed, we can choose δ small enough to have

$$\frac{1}{16} - \delta_0 > b\delta. \tag{4.98}$$

Write $\delta' = \frac{1}{16} - \delta_0 - b\delta > 0$. Using δ such that (4.98) holds, Markov's inequality and (4.93) inequality, we have

$$\begin{aligned}
\mathbb{P}(cL(N) \leqslant c\kappa N) &= \mathbb{P}\left(\frac{c}{16}N - cbN\delta - cL(N) \geqslant \frac{c}{16}N - cbN\delta - c\delta_0 N\right) \\
&\leqslant \mathbb{P}\left(e^{\frac{c}{16}N - cbN\delta - cL(N)} \geqslant e^{cN\delta'}\right) \\
&\leqslant e^{-cN\delta'}\mathbb{E}\left(e^{\frac{c}{16}N - cbN\delta - cL(N)}\right) \\
&\leqslant Ce^{-cN\delta'}.
\end{aligned} \tag{4.99}$$

By shift invariance, we then have

$$\mathbb{P}(L(N) - L(0) \leqslant \delta_0 N) \leqslant Ce^{c - \delta' N} \tag{4.100}$$

even if we don't suppose $L(0) = 0$. Now remark that X_i's are favored compared to L : this leads to (4.60). □

5 Infinite regenerative chains

In this chapter, we prove a general functional central limit theorem (Theorem 5.2.1) for chains satisfying the regenerative scheme introduced in COMETS, FERNÁNDEZ and FERRARI [19]. We give an explicit expression for the associated limiting variance depending both on the original and regenerative processes. As a corollary (Corollary 5.2.2), we give a more tractable condition on the memory decay of the chain under which Theorem 5.2.1 is fulfilled. We also give a regime of the memory decay for which the limiting variance can be expressed in terms on the original process only. As applications, we derive theorems for autoregressive binary processes and Ising chains (Propositions 5.3.1 and 5.3.2). We also give some remark about the law of the iterated logarithm in Section 5.5.

The chapter is organized as follows. In Section 5.1, we give some definitions and preliminaries. In Section 5.2, we state the main results. In Section 5.3 we introduce binary autoregressive processes and power-law Ising chains for which we give central limit theorems. Finally, Section 5.4 is devoted to the proofs.

5.1 Notation and preliminary definitions

We consider a measurable space $(E,\mathscr{E})$ where E is a finite alphabet and $\mathscr{E}$ is the discrete σ-algebra. We denote $(\Omega,\mathscr{F})$ the associated product measurable space with $\Omega = E^{\mathbb{Z}}$. For each $\Lambda \subset \mathbb{Z}$ we denote $\Omega_\Lambda = E^\Lambda$ and σ_Λ for the restriction of a configuration $\sigma \in \Omega$ to Ω_Λ, namely the family $(\sigma_i)_{i\in\Lambda} \in E^\Lambda$. Also, $\mathscr{F}_\Lambda$ will denote the sub-σ-algebra of $\mathscr{F}$ generated by cylinders based on Λ ($\mathscr{F}_\Lambda$-measurable functions are insensitive to configuration values outside Λ). When Λ is an interval, $\Lambda = [k,n]$ with $k,n \in \mathbb{Z}$ such that $k \le n$, we use the notation: $\omega_k^n = \omega_{[k,n]} = \omega_k,\ldots,\omega_n$, $\Omega_k^n = \Omega_{[k,n]}$ and $\mathscr{F}_k^n = \mathscr{F}_{[k,n]}$. For semi-intervals we denote also $\mathscr{F}_{\le n} = \mathscr{F}_{(-\infty,n]}$, etc. The concatenation notation $\omega_\Lambda\sigma_\Delta$, where $\Lambda\cap\Delta = \emptyset$, indicates the configuration on $\Lambda\cup\Delta$ coinciding with ω_i for $i \in \Lambda$ and with σ_i for $i \in \Delta$.

5.1.1 Chains

We start by briefly reviewing the well-known notions of chains in a shift-invariant setting. In this particular case, chains are also called g-measures (see [50]).

Definition 5.1.1. A *g-function* g is a probability kernel $g : \Omega_0 \times \Omega_{-\infty}^{-1} \to [0,1]$, i.e.,

$$\sum_{\omega_0 \in \Omega_0} g\big(\omega_0 | \omega_{-\infty}^{-1}\big) = 1, \qquad \omega_{-\infty}^{-1} \in \Omega_{-\infty}^{-1}. \tag{5.1}$$

The g-function g is:

1. continuous if the function $g\,(\omega_0 \mid \cdot)$ is continuous for each $\omega_0 \in \Omega_0$, i.e., for all $\epsilon > 0$, there exists $n \geqslant 0$ so that

$$|g\big(\omega_0 \mid \omega_{-\infty}^{-1}\big) - g\big(\sigma_0 \mid \sigma_{-\infty}^{-1}\big)| < \epsilon \tag{5.2}$$

   for all $\omega_{-\infty}^{0}, \sigma_{-\infty}^{0} \in \Omega_{-\infty}^{0}$ with $\omega_{-n}^{0} = \sigma_{-n}^{0}$,
2. bounded away form zero if $g\,(\omega_0 \mid \cdot) \geqslant c > 0$ for each $\omega_0 \in \Omega_0$,
3. regular if g is continuous and bounded away from zero.

Definition 5.1.2. A probability measure $\mathbb{P}$ on $(\Omega, \mathscr{F})$ is said to be *consistent* with a g-function g if $\mathbb{P}$ is shift-invariant and

$$\int h(\omega) g(x \mid \omega)\, \mathbb{P}(d\omega) = \int_{\{\omega_0 = x\}} h(\omega)\, \mathbb{P}(d\omega) \tag{5.3}$$

for all $x \in E$ and $\mathscr{F}_{\leqslant -1}$-measurable function h. The family of these measures will be denoted by $\mathscr{G}(g)$ and for each $\mathbb{P} \in \mathscr{G}(g)$, the process $(X_i)_{i \in \mathbb{Z}}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ will be called a *g-chain*.

Remark 5.1.3. In the consistency definition (5.3), $\mathbb{P}$ needs only to be defined on $(\Omega_{-\infty}^{0}, \mathscr{F}_{\leqslant 0})$. Because of its shift-invariance, $\mathbb{P}$ can be extended in a unique way to $(\Omega, \mathscr{F})$. That's why, without loss of generality, we can make no distinction between $\mathbb{P}$ on $(\Omega_{-\infty}^{0}, \mathscr{F}_{\leqslant 0})$ and its natural extension on $(\Omega, \mathscr{F})$.

5.1.2 Regeneration

The following regeneration result is due to COMETS, FERNÁNDEZ and FERRARI [19] (see Theorem 4.1, Corollary 4.3 and Proposition 5.1). It will be the starting point of our analysis.

Theorem 5.1.4 (Comets & al. (2002))**.** *Let g be a regular g-function such that*

$$\prod_{k \geqslant 0} a_k > 0 \quad \textit{with} \quad a_k = \inf_{\sigma_{-k}^{-1} \in \Omega_{-k}^{-1}} \sum_{\xi_0 \in E} \inf_{\omega_{-\infty}^{-k-1} \in \Omega_{-\infty}^{-k-1}} g\big(\xi_0 \mid \sigma_{-k}^{-1} \omega_{-\infty}^{-k-1}\big) \tag{5.4}$$

with the convention $\sigma_0^{-1} = \emptyset$.

Then

1. *there exists a unique probability measure* $\mathbb{P}$ *consistent with* g,
2. *there exists a shift-invariant renewal process* $(T_i)_{i\in\mathbb{Z}}$ *with renewal distribution*

$$\mathbb{P}(T_{i+1} - T_i \geqslant M) = \rho_M, \qquad M > 0,\ i \neq 0 \tag{5.5}$$

   with ρ_M *the probability of return to the origin at epoch* M *of the Markov chain on* $\mathbb{N}\cup\{0\}$ *starting at time zero at the origin with transition probabilities*

$$\begin{cases} p(k,k+1) = a_k, \\ p(k,0) = 1 - a_k, \\ p(k,j) = 0 \text{ otherwise} \end{cases} \tag{5.6}$$

   and such that

$$T_0 \leqslant 0 < T_1. \tag{5.7}$$

3. *the random blocks* $\{(X_j :\ T_i \leqslant j < T_{i+1})\}_{i\in\mathbb{Z}}$, *where* $(X_i)_{i\in\mathbb{Z}}$ *on* $(\Omega, \mathscr{F}, \mathbb{P})$ *is the associated* g*-chain, are independent and, except for* $i = 0$, *identically distributed.*
4. $1 \leqslant \mathbb{E}(T_2 - T_1) = \sum_{i=1}^{\infty} \rho_i < \infty$.

5.2 Main Results

We are now ready to state the main results of this chapter as outlined in introduction.

Theorem 5.2.1. *Let* $(X_i)_{i\in\mathbb{Z}}$ *be a regular* g*-chain satisfying the regeneration assumption (5.4) and* $f : E \to \mathbb{R}$ *be a function such that*

$$\mathbb{E}\big(f(X_0)\big) = 0. \tag{5.8}$$

Then,

$$\frac{1}{\sqrt{n}} S_n := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} f(X_i) \longrightarrow \mathcal{N}\big(0, \sigma^2\big) \quad \textit{in distribution} \tag{5.9}$$

with

$$0 \leqslant \sigma^2 = \frac{\mathbb{E}\left(\left(\sum_{i=T_1}^{T_2-1} f(X_i)\right)^2\right)}{\mathbb{E}(T_2 - T_1)} < \infty. \tag{5.10}$$

Furthermore, if $\mathbb{E}((T_2 - T_1)^2) < \infty$, *then*

$$\sigma^2 = \mathbb{E}\Big(f^2(X_0)\Big) + 2\sum_{i\geqslant 1} \mathbb{E}\Big(f(X_0) f(X_i)\Big) < \infty. \tag{5.11}$$

Corollary 5.2.2. *Let* $(X_i)_{i\in\mathbb{Z}}$ *be a regular* g*-chain such that there exist* $a \in \mathbb{R}$, $b \in [1, \infty)$, $C > 0$

and $K \geqslant 2$ such that for all $k \geqslant K$

$$a_k \geqslant 1 - C\frac{(\log(k))^a}{k^b}. \tag{5.12}$$

Let $f : E \to \mathbb{R}$ satisfying (5.8).

1. *If $a \in \mathbb{R}$ and $b > 1$, or $a < -1$ and $b = 1$, then (5.9) is satisfied with σ^2 defined by (5.10).*
2. *If $a \in \mathbb{R}$ and $b > 2$, or $a < -1$ and $b = 2$, then (5.9) is satisfied with σ^2 defined by (5.11).*

5.3 Applications

5.3.1 Binary autoregressive processes

The binary version of autoregressive processes is mainly used in statistics and econometrics. It describes binary responses when covariates are historical values of the process (see the book of Peter MCCULLAGH and John A. NELDER (1989) [65], Section 4.3, for more details).

In what follows, we consider the example that was introduced previously in [19]. For the alphabet $E = \{-1, +1\}$, consider θ_0 a real number and $(\theta_k : k \geqslant 1)$ an absolutely summable real sequence. Let $q : \mathbb{R} \to (0,1)$ be a function strictly increasing and continuously differentiable. Assume that $g(\cdot \mid \omega_{-\infty}^{-1})$ is the Bernoulli law on $\{-1,+1\}$ with parameter $q(\theta_0 + \sum_{k\geqslant 1}\theta_k\omega_{-k})$, that is,

$$g\big(+1 \mid \omega_{-\infty}^{-1}\big) = q\Big(\theta_0 + \sum_{k\geqslant 1}\theta_k\omega_{-k}\Big) = 1 - g\big(-1 \mid \omega_{-\infty}^{-1}\big). \tag{5.13}$$

Denote

$$r_k = \sum_{m>k} |\theta_m|, \qquad k \geqslant 0. \tag{5.14}$$

Proposition 5.3.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a regular g-chain with g defined by (5.13) and*

$$f(x) = x - \mathbb{E}(X_0). \tag{5.15}$$

1. *If $\sum_{k\geqslant 0} r_k^2 < \infty$, then $|\mathscr{G}(g)| = 1$.*
2. *If $\sum_{k\geqslant 0} r_k < \infty$, then the central limit theorem is satisfied with σ^2 defined by (5.10) for f defined by (5.15).*
3. *If there exists $K \geqslant 0$ such that $r_k \leqslant C(\log(k))^a/k^b$, $k \geqslant K$, with $C > 0$, ($a \in \mathbb{R}$ and $b > 2$) or ($a < -1$ and $b = 2$), then the central limit theorem is satisfied with σ^2 defined by (5.11) for f defined by (5.15).*

5.3.2 Power-law Ising chain

For the usual Ising (Gibbs) model, a central limit theorem is well known (see the paper of Charles M. NEWMAN [68]). In the chain context, since the relationship between one dimensional Gibbs measures and chains, discussed in [32], does not allow to easily interpret Gibbs results in a chain setting, the problem becomes relevant. In what follows, we give central limit theorem for power-law Ising chains.

For the alphabet $E = \{-1, 1\}$, consider the power-law Ising chain defined by

$$g\big(\omega_0 \mid \omega_{-\infty}^{-1}\big) = \frac{\exp\Big[-\sum\limits_{k=-\infty}^{-1} \phi_k\big(\omega_{-\infty}^{0}\big)\Big]}{\sum\limits_{\sigma_0\in E}\exp\Big[-\sum\limits_{k=-\infty}^{-1} \phi_k\big(\omega_{-\infty}^{-1}\sigma_0\big)\Big]} \tag{5.16}$$

with

$$\phi_k\big(\omega_{-\infty}^{0}\big) = -\beta\frac{1}{|k|^p}\,\omega_0\,\omega_k, \qquad k \leqslant -1, \beta > 0, p > 1. \tag{5.17}$$

Proposition 5.3.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a regular g-chain with g defined by (5.16–5.17) and*

$$f(x) = x. \tag{5.18}$$

1. *If $p > 3/2$, then $|\mathcal{G}(g)| = 1$.*
2. *If $p > 2$, then the central limit theorem is satisfied with σ^2 defined by (5.10) for f defined by (5.18).*
3. *If $p > 3$, then the central limit theorem is satisfied with σ^2 defined by (5.11) for f defined by (5.18).*

5.4 Proofs

5.4.1 Proof of Theorem 5.2.1

Take $(T_i)_{i\in\mathbb{Z}}$ as given by Theorem 5.1.4 (ii) and write

$$S_n = \sum_{i=0}^{n-1} f(X_i) = \sum_{i=0}^{T_1-1} f(X_i) + \sum_{i=T_1}^{T_{i(n)}-1} f(X_i) + \sum_{i=T_{i(n)}}^{n-1} f(X_i) \tag{5.19}$$

with

$$i(n) = \begin{cases} \max\{k \geqslant 1 \colon T_k < n\} & \text{if } T_1 < n,\\ 0 & \text{otherwise.}\end{cases} \tag{5.20}$$

Lemma 5.4.1. *Under (5.4), both $n^{-1/2}\sum_{i=0}^{T_1-1} f(X_i)$ and $n^{-1/2}\sum_{i=T_{i(n)}}^{n-1} f(X_i)$ tend to zero in prob-*

ability.

Proof. For any $K > 0$, we obviously have

$$\mathbb{P}\left(\Big|\sum_{i=0}^{T_1-1} f(X_i)\Big| > K\sqrt{n}\right) \leqslant \mathbb{P}\left(T_1 \sup_{x\in E}\big|f(x)\big| > K\sqrt{n}\right). \tag{5.21}$$

Therefore,

$$\sup_{n\geqslant 1} \mathbb{P}\left(\Big|\sum_{i=0}^{T_1-1} f(X_i)\Big| > K\sqrt{n}\right) \leqslant \mathbb{P}\left(T_1 > \frac{K\sqrt{n}}{M}\right) \tag{5.22}$$

with

$$M = \sup_{x\in E}\big|f(x)\big|. \tag{5.23}$$

On the other hand, by the shift-invariance of $(T_i)_{i\in\mathbb{Z}}$,

$$\begin{aligned}
\mathbb{P}\left(\Big|\sum_{i=T_{i(n)}}^{n-1} f(X_i)\Big| > K\sqrt{n}\right) &\leqslant \mathbb{P}\left(\sum_{i=T_{i(n)}}^{T_{i(n)+1}-1} \big|f(X_i)\big| > K\sqrt{n}\right) \\
&\leqslant \mathbb{P}\left((T_{i(n)+1} - T_{i(n)}) \sup_{x\in E}\big|f(x)\big| > K\sqrt{n}\right) \\
&= \mathbb{P}\left((T_1 - T_0) \sup_{x\in E}\big|f(x)\big| > K\sqrt{n}\right),
\end{aligned} \tag{5.24}$$

where we used that $i(0) = 0$. Therefore,

$$\sup_{n\geqslant 1} \mathbb{P}\left(\Big|\sum_{i=T_{i(n)}}^{n-1} f(X_i)\Big| > K\sqrt{n}\right) \leqslant \mathbb{P}\left(T_1 - T_0 > \frac{K\sqrt{n}}{M}\right), \tag{5.25}$$

where M is defined by (5.23). Noticing that, under condition (5.4), both $\mathbb{P}(T_1 > K\sqrt{n}/M)$ and $\mathbb{P}(T_1 - T_0 > K\sqrt{n}/M)$ tend to zero as n goes to infinity, we can conclude that both $n^{-1/2}\sum_{i=0}^{T_1-1} f(X_i)$ and $n^{-1/2}\sum_{i=T_{i(n)}}^{n-1} f(X_i)$ tend to zero in probability. □

Proposition 5.4.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a regular g-chain satisfying (5.4) and $f : E \to \mathbb{R}$. Then the following statements are equivalent:*

(i) $S_n/\sqrt{n} \longrightarrow \mathcal{N}(0,\sigma^2)$ *in distribution for*

$$\sigma^2 = \frac{\mathbb{E}\left(\left(\sum_{i=T_1}^{T_2-1} f(X_i)\right)^2\right)}{\mathbb{E}(T_2 - T_1)}, \tag{5.26}$$

where T_1 and T_2 are defined in(5.5–5.7);

(ii) $(S_n/\sqrt{n})_{n\geqslant 0}$ *is bounded in probability;*

(iii) $\mathbb{E}\left(\sum_{i=T_1}^{T_2-1} f(X_i)\right) = 0$ *and* $\mathbb{E}\left(\left(\sum_{i=T_1}^{T_2-1} f(X_i)\right)^2\right) < \infty$.

Proof. The direction (i) $\Rightarrow$ (ii) is trivial, so we only have to show (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i).

To prove (ii)$\Rightarrow$ (iii), we first remark that equation (5.19), Lemma 5.4.1 and assertion (ii) imply that $n^{-1/2}\sum_{i=T_1}^{T_{i(n)}-1} f(X_i)$ is bounded in probability. Then, by the converse of the central limit theorem for real i.i.d. sequences (see e.g. the book of Michel LEDOUX and Michel TALAGRAND [57], Section 10.1), we must have (iii).

To prove (iii) $\Rightarrow$ (i), we first see that

$$\frac{i(n)}{n} \longrightarrow \frac{1}{\mathbb{E}(T_2 - T_1)} \quad \text{a.s.,} \tag{5.27}$$

which follows from Theorem 5.5.2 of [16] and the fact that $(T_{i+1} - T_i)_{i>0}$ is an i.i.d. process. Let us denote

$$\xi_k = \sum_{i=T_k}^{T_{k+1}-1} f(X_i) \quad \text{and} \quad e(n) = \left\lfloor \frac{n}{\mathbb{E}(T_2 - T_1)} \right\rfloor. \tag{5.28}$$

Thanks to the Lemma 5.4.1, to prove (i), it is enough to show that

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{i(n)-1} \xi_k \longrightarrow \mathcal{N}\left(0, \frac{\mathbb{E}(\xi_1^2)}{\mathbb{E}(T_2 - T_1)}\right) \quad \text{in law.} \tag{5.29}$$

This follows from the standard central limit theorem result

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{e(n)} \xi_k \longrightarrow \mathcal{N}\left(0, \frac{\mathbb{E}(\xi_1^2)}{\mathbb{E}(T_2 - T_1)}\right) \quad \text{in law,} \tag{5.30}$$

and

$$\frac{1}{\sqrt{n}} \left(\sum_{k=1}^{i(n)-1} \xi_k - \sum_{k=1}^{e(n)} \xi_k \right) \longrightarrow 0 \quad \text{in probability.} \tag{5.31}$$

To prove the latter, for any $\epsilon > 0$, let

$$K' = \left\lfloor \frac{\epsilon^3}{2\big(1 + \mathbb{E}(\xi_k^2)\big)} \right\rfloor. \tag{5.32}$$

First, remark that

$$\begin{aligned} &\mathbb{P}\left(\left| \sum_{k=1}^{i(n)-1} \xi_k - \sum_{k=1}^{e(n)} \xi_k \right| \geqslant \sqrt{n}\epsilon \right) \\ &\qquad \leqslant 2\,\mathbb{P}\left(\max_{j \leqslant nK'} \left| \sum_{k=1}^{j} \xi_k \right| \geqslant \sqrt{n}\epsilon \right) + \mathbb{P}\Big(|i(n) - 1 - e(n)| \geqslant nK' \Big). \end{aligned} \tag{5.33}$$

Then, use the Kolmogorov's maximal inequality, (iii) and (5.32), to get

$$\mathbb{P}\Big(\max_{j\leqslant nK'}\Big|\sum_{k=1}^{j}\xi_k\Big|\geqslant\sqrt{n}\epsilon\Big)\leqslant\frac{1}{n\epsilon^2}\operatorname{Var}\Big(\sum_{k=1}^{nK'}\xi_k\Big)=\frac{K'\,\mathbb{E}(\xi_k^2)}{\epsilon^2}\leqslant\frac{\epsilon}{2}. \tag{5.34}$$

Since

$$\lim_{n\to\infty}\mathbb{P}\Big(|i(n)-1-e(n)|\geqslant nK'\Big)=0, \tag{5.35}$$

it follows from (5.33–5.34) that

$$\lim_{n\to\infty}\mathbb{P}\Big(\Big|\sum_{k=1}^{i(n)-1}\xi_k-\sum_{k=1}^{e(n)}\xi_k\Big|\geqslant\sqrt{n}\epsilon\Big)<\epsilon. \tag{5.36}$$

□

We now give the proof of Theorem 5.2.1.

Proof. First, using the shift-invariance of $\mathbb{P}$, we see that for all $n>1$

$$\begin{aligned}\frac{\mathbb{E}(S_n^2)}{n}&=\frac{1}{n}\Big(\mathbb{E}\Big(\sum_{i=0}^{n-1}f^2(X_i)\Big)+2\,\mathbb{E}\Big(\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}f(X_i)f(X_j)\Big)\Big)\\&=\mathbb{E}\Big(f^2(X_0)\Big)+\frac{2}{n}\sum_{i=1}^{n-1}(n-i)\,\mathbb{E}\Big(f(X_0)f(X_i)\Big).\end{aligned} \tag{5.37}$$

Then, under assumptions (5.4) and (5.8), Theorem 1 in the paper of Xavier BRESSAUD, Roberto FERNÁNDEZ, and Antonio GALVES [10] insures that

$$\sum_{i\geqslant1}\mathbb{E}\Big(f(X_0)f(X_i)\Big)\leqslant C\sum_{i\geqslant1}\rho_i<\infty, \tag{5.38}$$

for some $C>0$. Therefore, it follows from Kronecker's lemma that

$$\lim_{n\to\infty}\frac{\mathbb{E}(S_n^2)}{n}=\mathbb{E}\Big(f^2(X_0)\Big)+2\sum_{i\geqslant1}\mathbb{E}\Big(f(X_0)f(X_i)\Big)<\infty. \tag{5.39}$$

In particular, Chebyshev's inequality implies that

$$\Big(\frac{S_n}{\sqrt{n}}\Big)_{n\geqslant1} \tag{5.40}$$

is bounded in probability. Then Proposition 5.4.2 concludes the proof of (5.9-5.10).

To show (5.11), we use first Proposition 5.4.2 (iii) to get

$$\mathbb{E}\Big(\Big[\sum_{k=1}^{i(n)+1}\xi_k\Big]^2\Big)=\mathbb{E}\Big(\sum_{k=1}^{i(n)+1}\xi_k^2\Big)=\mathbb{E}\big(\xi_k^2\big)\,\mathbb{E}\big(i(n)+1\big), \tag{5.41}$$

where the rightmost equality follows from the Wald's Equation and the fact that $i(n)+1$ is a stopping time w.r.t. $(\xi_k)_{k\geqslant 1}$. Since

$$\lim_{n\to\infty} \frac{\mathbb{E}\big(i(n)\big)}{n} = \frac{1}{\mathbb{E}(T_2 - T_1)} \tag{5.42}$$

(see Theorem 5.5.2 in the book of Kai Lai CHUNG [16]), (5.10) and (5.41) give

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\bigg(\Big[\sum_{k=1}^{i(n)+1} \xi_k\Big]^2\bigg) = \sigma^2. \tag{5.43}$$

But, we have

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\Big(\max_{k\leqslant n} \xi_k^2\Big) = 0 \tag{5.44}$$

(see p. 90 in the book of Kai Lai CHUNG [15] for a proof) and therefore

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\big(\xi_k^2\big) = 0, \qquad k \in \{i(n), i(n)+1\}. \tag{5.45}$$

Hence, by (5.43) and (5.45), we finally have

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\bigg(\Big[\sum_{k=1}^{i(n)-1} \xi_k\Big]^2\bigg) = \sigma^2. \tag{5.46}$$

Now, if we show that

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\bigg(\Big[\sum_{k=1}^{T_1-1} f(X_k)\Big]^2\bigg) = 0 \tag{5.47}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\bigg(\Big[\sum_{k=T_{i(n)}}^{n-1} f(X_k)\Big]^2\bigg) = 0, \tag{5.48}$$

then, using Theorem 5.1.4 (iii), (5.19) and (5.46), we can conclude the proof of (5.11).

The proofs of (5.47–5.48) are similar, we will prove (5.48) only. To that aim, we first remark that

$$k\mathbb{P}(T_1 - T_0 \geqslant k) \to 0 \quad \text{as} \quad k \to \infty, \tag{5.49}$$

because $\mathbb{P}(T_1 - T_0 \geqslant k)$ is decreasing to 0 and

$$\sum_{k\geqslant 0} \mathbb{P}(T_1 - T_0 \geqslant k) = \mathbb{E}(T_1 - T_0) = \frac{\mathbb{E}([T_2 - T_1]^2)}{\mathbb{E}(T_2 - T_1)} < \infty, \tag{5.50}$$

where the rightmost equality uses that $\mathbb{P}(T_1 - T_0 = k) = k\mathbb{P}(T_2 - T_1 = k)/\mathbb{E}(T_2 - T_1)$ (see e.g. the book of Gregory F. LAWLER [55], Chapter 6, Section 6.2, equality (6.10)). Therefore, recalling

(5.23) and using the shift-invariance of $\mathbb{P}$, we have

$$\begin{aligned}\frac{1}{n}\mathbb{E}\Big(\Big[\sum_{k=T_{i(n)}}^{n-1} f(X_k)\Big]^2\Big) &\leqslant \frac{M^2}{n}\mathbb{E}\Big([n-T_{i(n)}]^2\Big)\\ &= \frac{M^2}{n}\sum_{k=1}^{n} k^2\,\mathbb{P}\big(n-T_{i(n)}=k\big)\\ &= \frac{M^2}{n}\sum_{k=1}^{n}(2k-1)\mathbb{P}\big(n-T_{i(n)}\geqslant k\big)\\ &\leqslant \frac{M^2}{n}\sum_{k=1}^{n}(2k-1)\,\mathbb{P}(T_1-T_0\geqslant k),\end{aligned} \tag{5.51}$$

which in view of (5.49), goes to zero as n tends to infinity. □

5.4.2 Proof of Corollary 5.2.2

Proof. To prove (i), it suffices to see that when $(a_k)_{k\geqslant 0}$ satisfies (5.12), then ($a\in\mathbb{R}$ and $b>1$) or ($a<-1$ and $b=1$) if and only if $\sum_{k\geqslant 0}(1-a_k)<\infty$ which is equivalent to $\prod_{k\geqslant 0}a_k>0$. Therefore, applying Theorem 5.2.1 first part, we get the result.

To prove (ii), we first denote

$$\tilde{a}_k = 1 - C\,\frac{(\log(k))^a}{k^b} \tag{5.52}$$

and the associated $\tilde{\rho}_k$, defined by the analogous of (5.6). Since $b>1$, Proposition 5.5 (iv) in Fernández, Ferrari and Galves [31] gives that there exists some constant $C_1>0$ so that

$$\tilde{\rho}_k \leqslant C_1(1-\tilde{a}_k), \qquad k\geqslant 0. \tag{5.53}$$

Therefore, using that $a_k\geqslant \tilde{a}_k$ implies $\rho_k\leqslant\tilde{\rho}_k$, we have

$$\mathbb{E}\big((T_2-T_1)^2\big) = \sum_{k\geqslant 0} k\rho_k \leqslant C_1\sum_{k\geqslant 0} k(1-\tilde{a}_k), \tag{5.54}$$

which is finite if and only if ($a\in\mathbb{R}$ and $b>2$) or ($a<-1$ and $b=2$). Then, applying Theorem 5.2.1 second part, we get the result. □

5.4.3 Proof of Proposition 5.3.1

Proof. Define the variation by

$$\mathrm{var}_k = \sup\Big\{|g\big(\omega_0\mid\omega_{-\infty}^{-1}\big)-g\big(\sigma_0\mid\sigma_{-\infty}^{-1}\big)|:\ \omega_{-\infty}^{0},\sigma_{-\infty}^{0}\in\Omega_{-\infty}^{0},\ \omega_{-k}^{0}=\sigma_{-k}^{0}\Big\}, \quad k\geqslant 0. \tag{5.55}$$

Then, because $|E| = 2$, we have for any $k \geqslant 0$ (with the convention $\sigma_0^{-1} = \emptyset$)

$$\begin{aligned} a_k &= \inf\left\{g\big(1 \mid \omega_{-\infty}^{-k-1}\sigma_{-k}^{-1}\big) + g\big(-1 \mid \xi_{-\infty}^{-k-1}\sigma_{-k}^{-1}\big) : \sigma_{-k}^{-1} \in \Omega_{-k}^{-1}, \omega_{-\infty}^{-k-1}, \xi_{-\infty}^{-k-1} \in \Omega_{-\infty}^{-k-1}\right\} \\ &= 1 - \sup\left\{-g\big(1 \mid \omega_{-\infty}^{-k-1}\sigma_{-k}^{-1}\big) + g\big(1 \mid \xi_{-\infty}^{-k-1}\sigma_{-k}^{-1}\big) : \sigma_{-k}^{-1} \in \Omega_{-k}^{-1}, \omega_{-\infty}^{-k-1}, \xi_{-\infty}^{-k-1} \in \Omega_{-\infty}^{-k-1}\right\} \\ &= 1 - \mathrm{var}_k. \end{aligned} \tag{5.56}$$

Therefore,

$$a_k = 1 - \sup\left\{q\left(\theta_0 + \sum_{j=1} k\theta_j\sigma_{-j} + r_k\right) - q\left(\theta_0 + \sum_{j=1}^{k}\theta_j\sigma_{-j} - r_k\right) : \sigma_{-k}^{-1} \in \Omega_{-k}^{-1}\right\}. \tag{5.57}$$

Because q is continuously differentiable on a compact set, there exists $C > 0$ such that

$$\mathrm{var}_k \leqslant C r_k, \qquad k \geqslant 0. \tag{5.58}$$

To prove (i), it suffices to remark that

$$\sum_{k\geqslant 0} \mathrm{var}_k^2 < \infty, \tag{5.59}$$

which is a tight uniqueness criteria in terms on the variation (see the paper of Anders JOHANSSON and Anders ÖBERG [49] and the one of Noam BERGER, Christopher E. HOFFMAN and Vladas SIDORAVICIUS [3]).

To show (ii), we simply note that

$$\mathbb{E}\big(f(X_0)\big) = \mathbb{E}\big(X_0 - \mathbb{E}(X_0)\big) = 0 \tag{5.60}$$

and that

$$\sum_{k\geqslant 0} r_k < \infty \quad \Longrightarrow \quad \prod_{k\geqslant 0} a_k > 0. \tag{5.61}$$

Finally, to prove part (iii), it suffices first to combine (5.56) and (5.58) to get

$$a_k \geqslant 1 - C r_k, \qquad k \geqslant 0, \tag{5.62}$$

and then to apply Corollary 5.2.2 (ii).

□

5.4.4 Proof of Proposition 5.3.2

Proof. We need the following well-known bound, whose proof can be found in Appendix of [64] (we follow the idea of Lemma V.1.4 in the paper of Barry SIMON [72]).

Lemma 5.4.3. *Let g be a g-function satisfying (5.16) with $\sum_{k=-\infty}^{-1}|\phi_k| < \infty$ and h be a $\mathscr{F}_0$-measurable function. Then, for any $\omega_{-\infty}^{-1}, \sigma_{-\infty}^{-1} \in \Omega_{-\infty}^{-1}$,*

$$\begin{aligned}\Big|\sum_{\omega_0\in E} h(\omega_0)\Big(g\big(\omega_0 \mid \omega_{-\infty}^{-1}\big) - g\big(\omega_0 \mid \sigma_{-\infty}^{-1}\big)\Big)\Big| \\ \leqslant \sup_{x\in E}|h(x)| \sup_{\omega_0\in E}\Big|\sum_{k=-\infty}^{-1}\Big[\phi_k\big(\omega_{-\infty}^{0}\big) - \phi_k\big(\sigma_{-\infty}^{-1}\omega_0\big)\Big]\Big|.\end{aligned} \tag{5.63}$$

Applying the previous lemma for $h \equiv 1$, we obtain that, for any $\omega_{\leq i-1}, \sigma_{\leq i-1} \in \Omega_{\leq i-1}$,

$$\Big|g\big(\omega_0 \mid \omega_{-\infty}^{-1}\big) - g\big(\omega_0 \mid \sigma_{-\infty}^{-1}\big)\Big| \leqslant |\beta| \sup_{\omega_0\in E}\Big|\sum_{k=-\infty}^{-1}\frac{1}{|k|^p}\big(\omega_0\omega_k - \omega_0\sigma_k\big)\Big|, \tag{5.64}$$

from which

$$\operatorname{var}_k \leqslant 2|\beta|\sum_{j=-\infty}^{-k-1}\frac{1}{|j|^p} \leqslant 2|\beta|\frac{1}{k^{p-1}}, \qquad k \geqslant 1, \tag{5.65}$$

is an immediate consequence.

To prove (i), it suffices to see that (5.65) with $p > 3/2$ implies the validity of (5.59).

To prove (ii) and (iii), we first remark that similarly to the gibbsian setting, it can be easily checked that $\mathbb{E}(X_0) = 0$ and therefore under (5.18), (5.8) is fulfilled. Then, we combine (5.56) and (5.65) to get

$$a_k \geqslant 1 - 2|\beta|\frac{1}{k^{p-1}}, \quad k \geqslant 1. \tag{5.66}$$

Thus, the results are direct consequences of Corollary 5.2.2 (i) and (ii). □

5.5 Law of the iterated logarithm

Now that we gave a central limit theorem, it seems logical to study the magnitude of the fluctuation of the chain. In fact, as we considered a finite alphabet, a law of the iterated logarithm is easy to deduce from the proof of Theorem 5.2.1.

Theorem 5.5.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a regular g-chain satisfying the regeneration assumption (5.4) and $f : E \to \mathbb{R}$ be a function such that*

$$\mathbb{E}\big(f(X_0)\big) = 0. \tag{5.67}$$

Then,

$$\frac{1}{\sqrt{n \ln \ln n}} S_n = \frac{1}{\sqrt{n \ln \ln n}} \sum_{i=0}^{n-1} f(X_i) \longrightarrow \sqrt{2}\sigma \quad a.s. \tag{5.68}$$

with

$$0 \leqslant \sigma^2 = \frac{\mathbb{E}\left(\left(\sum_{i=T_1}^{T_2-1} f(X_i)\right)^2\right)}{\mathbb{E}(T_2 - T_1)} < \infty. \tag{5.69}$$

Furthermore, if $\mathbb{E}((T_2 - T_1)^2) < \infty$, *then*

$$\sigma^2 = \mathbb{E}\Big(f^2(X_0)\Big) + 2 \sum_{i \geqslant 1} \mathbb{E}\Big(f(X_0) f(X_i)\Big) < \infty. \tag{5.70}$$

Proof. Recall that E is a finite alphabet. In this situation, convergence in probability and convergence almost sure are equivalent. Using the regenerative scheme in the exact same way than Theorem 5.2.1, we can then show that the first and last part of sequence will vanish. We then simply apply the law of iterated logarithm for an i.i.d. sequence to conclude the proof. □

It is now natural to extend these results to large deviations. It is quite clear that the last part of the sum (between $T_{i(n)}$ and n) will behave as the first random block. Sadly, we cannot immediately conclude as in Theorem 5.5.1. We need to refine hypothesis on the behaviour of the first block, between T_1 and T_0. We can see in the paper of Hacène DJELLOUT and Arnaud GUILLIN [23] that regeneration is again the key to answer this question for a Markov process. We are convinced that this technique should work in our situation and make it an interesting forthcoming study.

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