Lie Group and Lie Algebra Variational Integrators for Flexible Beam and Plate in R^3

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Résumé

Le but de cette thèse est de développer des intégrateurs variationnels synchrones ou bien asynchrones, qui puissent être utilisés comme des outils pour étudier des structure complexes composées de plaques et de poutres soumises à de grandes déformations et sous contraintes.

Les modèles de poutre et de plaque sont les modèles géométriquement exacts, dont l'espace de configuration sont des groupes de Lie. Ils sont adaptés à la modélisation d'objets soumis à de grandes déformations, où l'énergie de déformation élastique choisie convient pour les types de matériaux correspondant à notre domaine d'étude (isotropes, ou composites).

Les travaux de J. E. Marsden, de ses doctorants et post-doctorants, ont servi de base pour développer des intégrateurs variationnels, qui sont symplectiques et conservent parfaitement les symétries. En outre, par une bonne discrétisation, l'objectivité des modèles de poutre et de plaque étudiés est conservée.

L'idée qui sous-tend ce travail est de tirer profit des propriétés de ces intégrateurs pour définir la position d'équilibre des structures, que l'on ne connait généralement pas à priori, ainsi que pour déterminer les contraintes, tout en conservant les invariants de la structure.

Parallèlement à la résolution de cette problématique, nous poursuivons la démarche de J.E. Marsden qui consiste à poser les bases d'une mécanique discrète, avec ses théorèmes, ses axiomes, ses définitions qui ont la même valeur que les lois de la mècanique des milieux continus mais pour un domaine discret. C'est à dire que les trajectoires discrètes d'un mouvement obtenues par ces intégrateurs variationnels vérifient ces lois discrètes.

Mots clés : poutre, plaque, intégrateurs mécaniques, principe d'Hamilton, conservation des symétries, mécanique discrète, réduction, intégrateur asynchrone

Abstract

The purpose of this thesis is to develop variational integrators synchronous or asynchronous, which can be used as tools to study complex structures composed of plates and beams subjected to large deformations and stress.

We consider the geometrically exact models of beam and plate, whose configuration spaces are Lie groups. These models are suitable for modeling objects subjected to large deformations, where the stored energy chosen is adapted for the types of materials used in our field (isotropic or composite).

The work of J. E. Marsden, and of his doctoral and post-doctoral students, were the basis for the development of variational integrators which are symplectic and perfectly preserve symmetries. Furthermore, discrete mechanical systems with symmetry can be reduced. In addition, by a "good discretization", the strain measures are unchanged by superposed rigid motion.

The idea behind this work is to take advantage of the properties of these integrators to define the equilibrium position of structures, which are generally unknown, as well as to determine the constraints, while preserving the invariants of the structure.

Along with solving these problems, we continue to develop the ideas of J.E. Marsden who laid the foundations of discrete mechanics, with its theorems, axioms, and definitions, which parallel those in continuum mechanics but for a discrete domain. That is, the discrete trajectories of a motion obtained by variational integrators satisfy these discrete laws.

Keywords : beam, plate, mechanical integrators, variational principles, conservation properties, discrete mechanics, symmetry, reduction, asynchronous

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Moreover I thank my co-authors Sigrid Leyendecker, Julien Nembrini, and Sina Ober-Blöbaum, all of whom have contributed markedly to my enjoyment of research. In addition I would like to thank Mathieu Desbrun for his valuable suggestions and help along the way.

Finally, this work would not have been possible without the understanding of my daughter Phénissia.

Introduction

This work can be seen either in the context of solving complex problems of applied mechanics, or, equally well in the perspective of the development of the theory of discrete mechanics.

The first point of view is ontologically linked to the origin of the thesis project, that is, the desire to explore complex forms composed of multiple singular points and free forms with multiple contact points. The aim is to find the equilibrium position as well as to calculate the stress to which the material is subjected. Indeed, these forms are so complex and flexible that we cannot guess at first glance their equilibrium positions.

The second perspective is directly related to the properties enjoyed by the mathematical objects we develop, namely, the conservation of symmetries and the study of the statics and the dynamics of the material. The mathematical objects with which we work are worth studying and naturally fit the objects under study.

This thesis further develops the subject of Lie group and Lie algebra variational integrators as it applies to exact models of beams and plates subject to large deformations. Moreover, additional topics connected with discrete mechanics will be developed since they are needed in our development.

The point of view taken in this thesis is that we do not discretize the equations but the problem itself. This is achieved by discretizing space and time in the setup of the of the studied object. We will use variational principles for all the problems that we considered and hence the goal is to discretize the variational principle in order to get discrete equations of motion. Then, we shall formulate theorems in this discrete setting that are analogues of the classical continuous time and space statements.

A very successful and well developed technique in numerical analysis is the finite element method. It uses a simplicial decomposition of the given domain and discretizes the local law of the continuous problem. Thus, for many important problems, especially long time simulations for conservative systems, the development of stable finite element methods remains extremely challenging or even out of reach, the underlying geometric or variational structures of the simulated continuous systems being often destroyed. We believe that this problem can be circumvented by the use of variational integrators. The geometric formulation of the continuous theory is used to guide the development of discrete analogues of the geometric structure, such as discrete conservation laws, discrete (multi)symplectic forms, and discrete variational principles.

The past years have seen major developments in discrete variational mechanics and corresponding numerical integrators. The theory of discrete variational mechanics has its roots in the optimal control literature of the 1960's. The variational view of discrete mechanics and its numerical implementation has been developed in the past ten years mainly by Jerrold Marsden of Caltech, his students, postdocs, and collaborators.

Discrete mechanics was born as a result of the interplay of classical theoretical mechanics, numerical analysis, and computer science. It has become increasingly important in concrete applications as different as the modelisation of specific physical systems, animation, computer vision and graphics, image processing, shocks between elastic solids, atmospheric and oceanographic simulations of Lagrangian coherent states, spacecraft mission design, and many others. Remarkably, to our knowledge, there is no major application of these discrete mechanics techniques to civil engineering. In particular, we are not aware of any application of discrete mechanics to the study of surfaces formed by plates tied by multi-edges and exhibiting sharp corners.

Understanding and controlling many physical systems typically requires numerical simulations of dynamics that occurs over a wide range of time and space scales.

Recent years have seen an explosive growth of discrete mechanics, discrete exterior calculus, and corresponding integrators preserving various geometric structures. There has been a growing realization that stability of numerical methods can be obtained by methods which are compatible with these structures in the sense that many discrete variational integrators are symplectic-momentum methods, that is, they preserve the symplectic structure on phase space and momentum maps arising from the symmetries of the system (see e.g. Marsden, and West [90]).

A large number of mechanical systems in nature are governed by Hamilton's variational principle. The basic idea of underlying discrete variational integrators is to discretize the variational principle rather than discretizing the equations of motion themselves which is the standard approach taken by the finite element method.

Furthermore, a well-known result Ge, and Marsden [35] states that integrators with fixed time step typically cannot simultaneously preserve energy, the symplectic structure, and all conserved quantities. But one can still achieve this if one uses time step adaptive schemes as in Kane, Marsden, and Ortiz [56] and Lew, Marsden, Ortiz, and West [69] who developed the theory of AVIs based on the introduction of spacetime discretization allowing different time steps for different elements in a given finite element.

The first mathematical model expressed in terms of discrete mechanics uses discrete variational integrators; see Marsden, and West [90]. However, in order to have the flexibility to focus on parts of the phase portrait where dynamics is more complicated, asynchronous variational integrators (AVIs) have proved to be more effective. These integrators are based, as mentioned earlier, on the introduction of spacetime discretization allowing different time steps for different elements in a finite element mesh along with the derivation of time integration algorithms in the context of discrete mechanics, i.e., the algorithm is given by a spacetime version of the discrete Euler-Lagrange (DEL) equations of a discrete version of Hamilton's principle.

The advantage of these discrete variational integrators is that they preserve the symplectic structure (a classical property of mechanical systems), and preserve momenta for systems with symmetry, have excellent energy behavior (even with some dissipation added), and allow the usage of different time steps at different points. These properties significantly enhance the efficiency of these algorithms. We shall use discrete variational integrators in the study of beams and shells.

The second mathematical model used to study thin-shells is based on the mathematical formulation of three dimensional elasticity as developed, for example, in Marsden, and Hughes [83]. This approach tightly links elasticity theory with geometric mechanics and symplectic geometry (see, e.g., Abraham, and Marsden [1]).

Unfortunately, our knowledge of nonlinearly elastic, laminated, or composite materials, their dynamics, and their behavior near corners is very limited. Standard demonstrations of the utility of a given rod or shell theory for effectively approximating a limited number of problems should not lead to the impression that all of these problems have good numerical simulations. Much recent work on rod and shell theory has been motivated by developments in numerical analysis and computational techniques.

Advantages of the discrete mechanics point of view. The finite element method is an important computational tool to study the dynamics and the statics of beams and plates. However, even with significant advances in error control, convergence and stability of these finite approximations, the invariant geometric structures can be lost. For example, in a finite element approximation of the motion of the free rigid body, one can gain or lose momentum and thereby fail to preserve fundamental geometric and topological structures underlying the continuous model. The main problem with this method is that it discretizes the differential equations of continuum mechanics in order to obtain a position, a discrete trajectory, a moment, or other relevant quantities relevant to the motion of the system. It is not at all sure that the solutions thus obtained satisfy some of the fundamental properties of the continuum mechanical model.

A key point of this thesis is to work both on a discrete theory of mechanics and to use these results to study beams and plates. That is, as soon as one uses a variational integrator, the theoretical results are checked, something that is far from being trivial since this work involves, for example, reduction theory, an indispensable tool in the study of stability of relative equilibria, and multisymplectic theory, where one replaces the discrete time point with a mesh in spacetime thus allowing different time steps for different elements of the mesh when asynchronous variational integrators (AVI) are used. Thus, in this thesis we continue modestly the work begun by J. Marsden and his PhD students, that is, to develop the general theory of discrete mechanics.

From theory to algorithm. One chooses a configuration space Q with coordinates $\{q^j\}$ that describes the configuration of the system under study. The discrete version of the tangent bundle TQ of the configuration space Q is $Q\times Q$. Given an a priori choice of time interval Δt_0, a point $(q^0, q^1) \in Q \times Q$ corresponds to a tangent vector at q. Given a smooth Lagrangian $L : TQ \to \mathbb{R}$, usually the kinetic minus the potential energy, one associates to it a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$ and a discrete action functional

$$S_d = \sum_{j=0}^{N-1} L_d(q^j, q^{j+1}, \Delta t_j).$$

The discrete variational principle states that $\delta S_d = 0$, which means that one seeks sequences $\{q^j\}_{k\in\mathbb{N}}$ for which the functional S_d is stationary under variations of q^j with fixed endpoints q^0 and q^N. As in the smooth case, the discrete variational principle leads to the discrete Euler-Lagrange equations

$$D_2L_d(q^{j-1}, q^j, \Delta t_{j-1}) + D_1L_d(q^j, q^{j+1}, \Delta t_j) = 0,$$

where D_k denotes the kth partial derivative, $k = 1, 2$. In this way an update rule $(q^{j-1}, q^j) \to (q^j, q^{j+1})$ is obtained; this is the variational integrator.

If one uses time step adaptive schemes as in Kane, Marsden, and Ortiz [56] we obtain a variational integrator for conservative mechanical systems that are symplectic, energy, and momentum conserving. Indeed, whatever the choice of the discrete Lagrangian, for the non-dissipative and non-forced systems, variational integrators are symplectic and conserve the symmetries. The symplectic nature of the integrator is given by the conservation of the discrete two-form $\Omega_d = D_1D_2L_d\, dq^j \wedge dq^{j+1}$ on $Q\times Q$, which appears as integrand in the boundary terms of the discrete variational principle when endpoints are allowed to vary. Moreover, the energy behavior is remarkably stable in the conservative case, as proved by Hairer, Lubich and Wanner [41].

In describing the dynamic response of elastic bodies under loading, one begins by selecting a reference configuration $\mathcal{B} \subset \mathbb{R}^3$ of the body at initial time t_0. The motion of the body is described by the deformation mapping $\varphi : B \to \mathbb{R}^3$. Let $\mathcal{T}$ be a triangulation of $\mathcal{B}$. A key observation underlying the formulation of variational integrators is that, owing to the extensive character of the Lagrangian, the following element-by-element additive decomposition holds:

$$L = \sum_{K\in\mathcal{T}} L_K,$$

where L_K is the contribution of the element $K \in \mathcal{T}$ to the total Lagrangian L.

Another key feature is the existence of asynchronous variational integrators where the elements K and nodes a defining the triangulation of the body are updated asynchronously in time; each element K carries its own set of time steps Θ_K, which induces a set of time steps Θ_a for each node a. The discrete Euler-Lagrange equations $D_2L_d(\mathbf{x}_a^{j-1}, \mathbf{x}_a^j) + D_1L_d(\mathbf{x}_a^j, \mathbf{x}_a^{j+1}) = 0$ are applied to each node a, where $\mathbf{x}_a^j$ is the position of the node at time $t_a^j \in \Theta_a$. One obtains an update rule associated to this node and thus the discrete trajectories of the system.

Method used to study complex structures. The principle is to consider the studied object as a system oscillating about the equilibrium position under the influence of its load. After applying a certain kind of dissipation which conserve the symmetries, we get equilibrium position. Then we obtain the strain and the stress for the obtained deformation.

The objects studied are the exact nonlinear models of beam and plate of Simo, where the space configuration is a Lie group, that is $SE(3)$ or $SO(3)\times\mathbb{R}^3$. For the spatial discretization of this model we take into account the developments in [24] to obtain perfect objectivity.

We will develop Lie group and Lie algebra variational integrators for a given classical discrete Lagrangian, i.e., it equals kinetic minus potential energy. These algorithms are obtained by forming a discrete version of Hamilton's variational principle. For dissipative or forced systems, one uses the Lagrange-d'Alembert principle.

Variational integrators exhibit remarkable properties. For non-dissipative and non-forced problems, no matter the choice of the discrete Lagrangian, they are symplectic and momentum conserving. Moreover, with a "good " dissipation, the momentum maps are conserved. In addition, variational integrators have remarkably good energy behavior (see Hairer, Lubich, and Wanner [41]).

Organization of the thesis. The thesis consists of nine chapters.

In the first chapter, the theory of discrete mechanics is reviewed and the necessary background is developed. In the second and third chapters, we present two simple examples, the spherical pendulum and the spring pendulum, in order to familiarize the reader with variational integrators. We also compare two different time discretizations.

Chapter four is devoted to the numerical study of the Simó beam model. We develop several Lie group variational integrators, with two different time discretisations, both for synchronous and asynchronous integrators. In chapter five, we develop a discrete version of affine Euler-Poincaré equations, extending discrete Euler-Poincaré equations for semi-direct products to the case of an affine representation of the Lie group configuration space on the vector space. This yields a variational integrator for beams. Associated to this theory, a discrete Lie-Poisson reduction for semi-direct products is also developed. In chapter six, we develop a discrete Lie algebra variational integrator motivated by the fact that, if applicable, these integrators are easier to implement than the Lie group variational integrators. We apply it to the Simó beam model in this chapter and to the Simó plate model in chapter seven. In this second example, we need to handle also a natural holonomic constraint inherent to the model.

In chapter eight, we address the problem of dissipation. We construct a specific discrete model of dissipation such that energy is dissipated but angular momentum is conserved. We also establish a discrete affine Euler-Poincaré reduction with forces. This theory is applied to beam and plate models.

Up to this point, all mechanical systems considered had as configuration space a Lie group, possibly infinite dimensional. Chapter nine addresses the general problem and is devoted to the discretization of the reduction process for mechanical systems whose configuration space is a general manifold. In this context, the standard continuous theory uses in an essential manner a connection on a principal bundle. Thus, we introduce the discrete mechanical connection which enables us to split the discrete trajectory into its horizontal and vertical parts, thereby obtaining a pair of discrete Lagrange-Poincaré equations. This also allows us to study the stability of the motion and, in particular, to dissociate mechanical instabilities from instabilities due to the implementation. Examples of splitting of discrete trajectories are given.

Chapter 1

History and background

1.1 History

During the last decade, major developments have been done in the area of discrete variational mechanics and their corresponding numerical integrators. The theory of discrete variational mechanics has its roots in the optimal control literature of the 1960's. For instance systems described by non-linear difference equations, by Jordan and Polak [54], maximum principle by Hwang and Fan [47], discrete calculus of variations by Cadzow [20]. In addition, studies relevant to the discrete mechanics began in the 1970's : discrete time systems by Cadzow [21], invariance properties of the discrete Lagrangian by Logan [77], discrete Lagrangian systems with symmetries by Maeda [79; 80; 81], time discretization by Lee [63].

This theory was then developed in a systematic way. A formulation of the discrete Hamilton's principle, discrete symplectic form, discrete momentum map and Noether theorem were given by Wendlandt and Marsden [115; 116], and the time step adaptation in order to get symplectic-energy-momentum preserving variational integrators by Kane, Marsden and Ortiz [56]. Discrete analogues of Euler-Poincaré and Lie-Poisson reduction theory with discrete Lagrangian were developed by Marsden, Pekarsky and Shkoller [86], a discretization of the Lagrange d'Alembert principle as well as a variational formulation of dissipation by Kane, Marsden, Ortiz and West [57], long time behaviour of symplectic methods by Hairer and Lubitch [40] backward error analysis by Benettin and Giorgilli [5], Hairer [38], Hairer and Lubitch [39], Reich [98]. And to conclude this period, Marsden and West [90], gave an important review of integration algorithms for finite dimensional mechanical systems, that are based on the discrete variational principle.

From this time, based on different variational formulations (e.g. Lagrange, Hamilton, Lagrange-d'Alembert, Hamilton-Pontryagin, etc.), variational integrators have been developed in various fields :

• The integrators have been extended to non smooth framework by Kane, Repetto, Ortiz and Marsden [58], by Fetecau, Marsden, Ortiz and West [30], and by Pandolfi, Kane, Marsden, and Ortiz [96].

• The theory of Lagrangian mechanics on Lie groups, with discrete Lagrangian reduction, discrete Euler-Poincaré equations, and semi-direct product was developed by Bobenko and Suris [11; 12], and by Marsden, Pekarsky and Shkoller [87]. Thereby Lee, Leok and McClamroch studied variational approach on the Lie group of rigid bodies configurations, for example, under their mutual gravity in [64].

• In multisymplectic geometry, Marsden, Patrick and Shkoller [85] have investigated a spacetime multisymplectic formulation. And a new class of asynchronous variational integrators (AVI) for non-linear elastodynamics has been introduced by Lew, Marsden, Ortiz and West [71; 70].

• A Lie-Poisson integrator for Lie-Poisson Hamiltonian system was developed by Ma and Rowley [78].

• In stochastic mechanics a discrete Lagrangian theory for stochastic Hamiltonian system has been exhibited by Bou-Rabee and Owhadi [14].

• In order to solve optimal control problems for mechanical systems, Ober-Blöbaum, Junge and Marsden [94] proposed optimization algorithm, which lets the discrete solution directly inherit characteristic structural properties from the continuous one. Furthermore Kobilarov and Marsden [60] constructed necessary conditions for optimal trajectories, with mechanical systems on Lie groups.

• To study mechanical systems with holonomic and non holonomic constraints, where there are abundance of important models, Kobilarov, Marsden, and Sukhatme [59] proposed a vertical and horizontal splitting of the variational principle with non-holonomic constraints. And, with holonomic constraints, using the discrete null space method, Leyendecker, Marsden, and Ortiz [74], as well as Leyendecker, Ober-Blöbaum, Marsden, and Ortiz [76] have eliminated the constraint forces and reduced the system to its minimal dimension.

• Multiscale systems with fast variables which have a computational cost determined by slow variables were examined by Tao, Owhadi, and Marsden [114].

As a consequence of these developments, variational integrators have become increasingly important in concrete applications such as animation, computer vision and graphics, image processing, shocks between elastic solids, atmospheric and oceanographic simulations of Lagrangian coherent states, spacecraft mission design.

In particular, we mention the works of Gawlik, Mullen, Pavlov, Marsden, and Desbrun [31], and those of Pavlov, Mullen, Tong, Kanso, Marsden and Desbrun [97] in fluid mechanics; that of Ryckman and Lew [103] in contact problems; and one of Bergou, Wardezky, Robinson, Audoly and Grinspun [6] in computer science.

However, these new tools have not yet been fully explored in the context engineering sciences and this work aims to contribute in this direction.

1.2 Discrete Lagrangian mechanics

In this section we briefly review some basic facts about discrete Lagrangian mechanics, following Marsden, and West [90].

Let Q be the configuration manifold of a mechanical system. Suppose that the dynamics of this system is described by the Euler-Lagrange equations associated to a Lagrangian $L : TQ \to \mathbb{R}$ defined on the tangent bundle of the configuration manifold Q. Recall that these equations characterize the critical curves of the action functional associated to L, namely

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \Leftrightarrow \delta \int_0^T L(q(t), \dot{q}(t))dt = 0,$$

for variations of the curve vanishing at the endpoints. Recall that the Legendre transform associated to L is the mapping $\mathbb{F}L : TQ \to T^*Q$ that associates to a velocity its corresponding conjugate momentum, where T^*Q denotes the cotangent bundle of Q. It is locally given by $(q, \dot{q}) \mapsto (q, \frac{\partial L}{\partial \dot{q}})$.

Symmetries of the systems are given by Lie group actions $\Phi : G \times Q \to Q$, $(g, q) \mapsto \Phi_g(q)$ under which the Lagrangian is invariant. In this case, the Noether theorem guarantees that the associated momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$, given by

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle \quad \alpha_q \in T^*Q, \;\; \xi \in \mathfrak{g} \tag{1.2.1}$$

is a conserved quantity, where $\mathfrak{g}$ denotes the Lie algebra of the Lie group G, $\mathfrak{g}^*$ its dual, and the vector field ξ_Q on Q is the infinitesimal generator of the action associated to $\xi \in \mathfrak{g}$, that is,

$$\xi_Q(q) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Phi_{\exp(\varepsilon\xi)}(q),$$

where $\exp : \mathfrak{g} \to G$ is the exponential map of the Lie group G.

Discrete Euler-Lagrange equations. We shall now recall the discrete version of this approach (see e.g. [90]). Suppose that a time step Δt has been fixed, denote by $\{t_j = j\Delta t \mid j = 0, ..., N\}$ the sequence of time, and by $q_d : \{t_j\}_{j=0}^N \to Q$, $q_d(t_j) = q^j$ a discrete curve. Let $L_d : Q \times Q \to \mathbb{R}$, $L_d = L_d(q^j, q^{j+1})$ be a discrete Lagrangian which we think of as approximating the action integral of L along the curve segment between q^j and q^{j+1}, that is, we have

$$L_d(q^j, q^{j+1}) \approx \int_{t^j}^{t^{j+1}} L(q(t), \dot{q}(t))dt,$$

where $q(t^j) = q^j$ and $q(t^{j+1}) = q^{j+1}$. The *discrete Euler-Lagrange equations* are obtained by applying the discrete Hamilton's principle to the discrete action

$$\mathfrak{S}_d(q_d) = \sum_{j=0}^{N-1} L_d(q^j, q^{j+1}).$$

The resulting equations

$$D_2L_d(q^{j-1},q^j)+D_1L_d(q^j,q^{j+1})=0, \quad \text{for } j=1,...,N-1. \tag{1.2.2}$$

are called the *discrete Euler-Lagrange equations*.

The *discrete Legendre transforms* $\mathbb{F}^+L_d, \mathbb{F}^-L_d : Q\times Q \to T^*Q$ associated to L_d are defined by

$$\begin{aligned}\mathbb{F}^+L_d(q^j,q^{j+1}) &:= D_2L_d(q^j,q^{j+1}) \in T^*_{q^{j+1}}Q\\ \mathbb{F}^-L_d(q^j,q^{j+1}) &:= -D_1L_d(q^j,q^{j+1}) \in T^*_{q^j}Q,\end{aligned} \tag{1.2.3}$$

so that the discrete Euler-Lagrange equation can be equivalently written as

$$\mathbb{F}^+L_d(q^{j-1},q^j)=\mathbb{F}^-L_d(q^j,q^{j+1}), \quad \text{for } j=1,...,N-1. \tag{1.2.4}$$

If both discrete Legendre transforms are locally isomorphisms (for nearby q^j and q^{j+1}), then we say that L_d is *regular*.

When the discrete Lagrangian L_d is regular, the discrete Euler-Lagrange equations define a well-defined *discrete Lagrangian evolution operator*

$$X_{L_d}: Q\times Q\to (Q\times Q)\times(Q\times Q), \quad X_{L_d}(q^{j-1},q^j)=\left((q^{j-1},q^j),(q^j,q^{j+1})\right),$$

and a well-defined *discrete Lagrangian flow*

$$F_{L_d}: Q\times Q\to Q\times Q, \quad F_{L_d}(q^{j-1},q^j)=(q^j,q^{j+1}).$$

Similarly as in the continuous case, the *discrete Lagrangian one forms* $\Theta^+_{L_d}$ and $\Theta^-_{L_d}$ on $Q\times Q$ are obtained by pulling-back the canonical one-form Θ on T^*Q via the Legendre transform, that is

$$\Theta^{\pm}_{L_d}=\left(\mathbb{F}^{\pm}L_d\right)^*\Theta, \tag{1.2.5}$$

where we recall that Θ is defined by $\left\langle\Theta(\alpha_q),w_{\alpha_q}\right\rangle=\left\langle T\pi_Q(w_{\alpha_q}),\alpha_q\right\rangle$, with $\pi_Q: T^*Q\to Q$ the cotangent bundle projection. We thus have the local formulas

$$\begin{aligned}\Theta^+_{L_d}\left(q^j,q^{j+1}\right)&=D_2L_d(q^j,q^{j+1})dq^{j+1},\\ \Theta^-_{L_d}\left(q^j,q^{j+1}\right)&=-D_1L_d(q^j,q^{j+1})dq^j,\end{aligned} \tag{1.2.6}$$

where $\Theta^+_{L_d}\left(q^j,q^{j+1}\right)\in T^*_{q^{j+1}}Q$ and $\Theta^-_{L_d}\left(q^j,q^{j+1}\right)\in T^*_{q^j}Q$. Note that $\mathbf{d}L_d=\Theta^+_{L_d}-\Theta^-_{L_d}$ so that $\mathbf{d}\Theta^+_{L_d}=\mathbf{d}\Theta^-_{L_d}$. Thus there only one single *discrete Lagrangian symplectic two form* $\Omega_{L_d}:=-\mathbf{d}\Theta^+_{L_d}=-\mathbf{d}\Theta^-_{L_d}$ and we have

$$\Omega_{L_d}=\left(\mathbb{F}^{\pm}L_d\right)^*\Omega, \tag{1.2.7}$$

where $\Omega=-\mathbf{d}\Theta$ is the canonical symplectic form on T^*Q, and where both $\mathbb{F}^+L_d$ and $\mathbb{F}^-L_d$ can be used to define Ω_{L_d}.

A map $f: Q\times Q\to Q\times Q$ is said to be a *special discrete symplectic map* if $f^*\Theta^-_{L_d}=\Theta^-_{L_d}$ and $f^*\Theta^+_{L_d}=\Theta^+_{L_d}$. It is called a *discrete symplectic map* if $f^*\Omega_{L_d} = \Omega_{L_d}$. For example, the discrete Lagrangian flow F_{L_d} is a discrete symplectic map:

$$(F_{L_d})^*\Omega_{L_d} = \Omega_{L_d}.$$

The *discrete Hamiltonian map* $\tilde{F}_{L_d} : T^*Q \to T^*Q$ is defined by $\tilde{F}_{L_d} := \mathbb{F}^{\pm}L_d \circ F_{L_d} \circ (\mathbb{F}^{\pm}L_d)^{-1}$, where F_{L_d} is the discrete Lagrangian flow. The fact that the discrete Hamiltonian map can be equivalently defined with either discrete Legendre transform is a consequence of the fact that the following diagram commute.

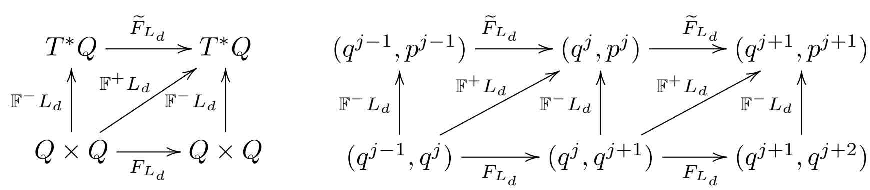


Figure 1.2.1: Properties of the discrete Legendre transforms and discrete flows

Discrete Lagrangian systems with symmetries. Let Φ be a group action of a Lie group G on Q with the infinitesimal generator $\xi_Q(q)$ associated to the Lie algebra element $\xi \in \mathfrak{g}$. There is a naturally induced action on $Q \times Q$ given by

$$\Phi_g^{Q\times Q}(q^j, q^{j+1}) := \left(\Phi_g(q^j), \Phi_g(q^{j+1})\right),$$

with the infinitesimal generator

$$\xi_{Q\times Q}(q^j, q^{j+1}) = \left(\xi_Q(q^j), \xi_Q(q^{j+1})\right).$$

Given a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$ (not necessarily G-invariant), the *discrete Lagrangian momentum maps* $\mathbf{J}^+_{L_d}, \mathbf{J}^-_{L_d} : Q \times Q \to \mathfrak{g}^*$ are defined by

$$\begin{aligned}
\left\langle \mathbf{J}^+_{L_d}(q^j, q^{j+1}), \xi \right\rangle &= \left\langle \Theta^+_{L_d}(q^j, q^{j+1}), \xi_{Q\times Q}(q^j, q^{j+1}) \right\rangle \\
&= \left\langle \mathbb{F}^+L_d(q^j, q^{j+1}), \xi_Q(q^{j+1}) \right\rangle \\
\left\langle \mathbf{J}^-_{L_d}(q^j, q^{j+1}), \xi \right\rangle &= \left\langle \Theta^-_{L_d}(q^j, q^{j+1}), \xi_{Q\times Q}(q^j, q^{j+1}) \right\rangle \\
&= \left\langle \mathbb{F}^-L_d(q^j, q^{j+1}), \xi_Q(q^j) \right\rangle.
\end{aligned} \tag{1.2.8}$$

Note that we have

$$\mathbf{J}^{\pm}_{L_d} = \left(\mathbb{F}^{\pm}L_d\right)^* \mathbf{J},$$

where $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is the *cotangent lift momentum map* given by $\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle$.

It is important to note that if the discrete curve $\{q^j\}_{j=0}^N$ verifies the discrete Euler-Lagrange then we have the equality

$$\mathbf{J}^+_{L_d}(q^{j-1}, q^j) = \mathbf{J}^-_{L_d}(q^j, q^{j+1}), \quad \text{for all } j = 1, ..., N-1. \tag{1.2.9}$$

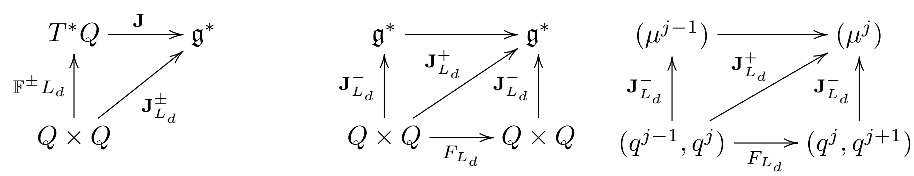


Figure 1.2.2: On the left: the definition of the discrete momentum maps. Two diagrams on the right: illustration of the equality (1.2.9)

When G acts on $Q \times Q$ by special discrete symplectic maps, that is, if $(\Phi_g^{Q\times Q})^*\Theta^{\pm}_{L_d} = \Theta^{\pm}_{L_d}$, then the discrete Lagrangian momentum maps are G-equivariant, that is,

$$\mathbf{J}^{+}_{L_d} \circ \Phi_g^{Q\times Q} = \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^{+}_{L_d},$$
$$\mathbf{J}^{-}_{L_d} \circ \Phi_g^{Q\times Q} = \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^{-}_{L_d}.$$

This happens for example if the discrete Lagrangian L_d is G-invariant, since in this case $\Phi_g^{Q\times Q}$ is a special discrete symplectic map. Moreover, in this case the two momentum maps coincide: $\mathbf{J}^{+}_{L_d} = \mathbf{J}^{-}_{L_d}$, and therefore, from (1.2.9) we obtain the discrete Noether's theorem.

1.2.1 Theorem (Discrete Noether's theorem) *Consider a given discrete Lagrangian system $L_d : Q \times Q \to \mathbb{R}$ which is invariant under the lift of the left action $\Phi : G \times Q \to Q$. Then the corresponding discrete Lagrangian momentum map $\mathbf{J}_{L_d} : Q \times Q \to \mathfrak{g}^*$ is a conserved quantity of the discrete Lagrangian map $F_{L_d} : Q \times Q \to Q \times Q$, that is, $\mathbf{J}_{L_d} \circ F_{L_d} = \mathbf{J}_{L_d}$.*

1.3 Energy computation

Review of geometric mechanics. Given the Lagrangian $L : TQ \to \mathbb{R}$ we define the action $A : TQ \to \mathbb{R}, \quad v_q \mapsto \langle \mathbb{F}L(v_q), v_q\rangle$ with $v_q = (q, \dot{q})$, and the energy by $E = A - L$.

The Lagrangian L is said hyperregular if the Legendre transform $\mathbb{F}L$ is a diffeomorphism. Then we have the following theorem

1.3.1 Theorem *The hyperregular Lagrangians L on TQ and hyperregular Hamiltonian H on T^*Q correspond in a bijective manner : H is constructed from L by means of $H = E \circ (\mathbb{F}L)^{-1}$, and L from H by means of $L = A - E = A - H \circ (\mathbb{F}H)^{-1}$, where $\mathbb{F}H : T^*Q \to T^{**}Q \approx TQ$ is the fiber derivative of $H : T^*Q \to \mathbb{R}$.*

Thus, in this case, we can calculate the energy of the system, at time t, both using the energy E as well as the Hamiltonian H with the same result.

But with the discrete configuration $Q\times Q$ we cannot define a discrete action. On the other hand we can add the discrete kinetic energy with the discrete potential energy and obtain in someway a discrete energy. However if the discrete

Lagrangian is regular, we can go from the discrete structure in time on $Q \times Q$ to the continuous structure in time on TQ by the discrete Legendre transform and we obtain the discrete energy E_d or the discrete Hamiltonian H_d.

In this case if we want to do this process properly, it seems necessary to define the general framework in terms of chain complexes.

Chain complexes. Let a set $S = \{a_0, ..., a_p\}$ of $p+1$ independent points in $\mathbb{R}^n$. The geometric *p-simplex* in $\mathbb{R}^n$ is the set of all points of the p-dimensional hyperplane H^p, containing S, for which the barycentric coordinates with respect to S are all non-negative. The p-simplex, with an ordering of its vertices, is denoted $\Delta^p = \langle a_0, ..., a_p \rangle$.

We obtain a geometric Δ-*complex* X by quotienting a collection of disjoint simplices identified by some faces via homeomorphisms preserving the ordering of vertices. The Δ-complex are denoted *simplicial complex* when simplices are uniquely determined by their vertices.

Then we consider a particular set of morphisms $\sigma_i : \Delta^p \to X$, for all $p \in \{0, ..., n\}$, which is the orientation of the faces of the simplexes with respect to each other in X, such that

$$\sigma_i(\Delta^p) = \Delta_i^p.$$

A p-dimensional chain on the Δ-complex X with coeficients in a group G is a function c_p on the oriented p-simplexes of X with values in the group G such that if $c_p(\Delta_i^p) = g_i$, then $c_p(-\Delta_i^p) = -g_i$. The collection of all such p-dimensional chains on X is denoted a *p-chain complex* $C_p(X, G)$.

This p-chain complex $C_p(X, G)$ provided with the boundary operator $\partial_p : \Delta_p(X) \to \Delta_{p-1}(X)$, which verify $\partial_{p-1} \circ \partial_p = 0$, forms an object of the *category of the chain complexes* <u>*Ch*</u>. Where the morphisms $\varphi_p : C_p(X_1, G) \to C_p(X_2, G)$ of this category are the morphisms of abelian groups such that the commutative relation $\partial_p(\varphi_p(c_p)) = \varphi_{p-1}(\partial_p(c_p))$ holds for each chain $c_p \in C_p(X_1, G)$. We have the commutative diagram in Figure (1.3.1).

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\partial_{p+1}} & C_p(X_2, \mathbb{R}) & \xrightarrow{\partial_p} & C_{p-1}(X_2, \mathbb{R}) & \xrightarrow{\partial_{p-1}} & \cdots \\
 & & \uparrow{\scriptstyle \varphi_p} & & \uparrow{\scriptstyle \varphi_{p-1}} & & \\
\cdots & \xrightarrow{\partial_{p+1}} & C_p(X_1, \mathbb{R}) & \xrightarrow{\partial_p} & C_{p-1}(X_1, \mathbb{R}) & \xrightarrow{\partial_{p-1}} & \cdots
\end{array}$$

Figure 1.3.1: Chain complex <u>*Ch*</u>

1.3.1 Discrete energy

Let a configuration $Q : \mathcal{C}^\infty(\mathcal{D}, G)$, where $\mathcal{D}$ is a compact with piecewise boundary, and G a given Lie group. Let an hyper-regular Lagrangian $L : TQ \to \mathbb{R}$. We assume that there is a *G-invariant Riemannian metric* γ on the configuration space Q, and that the Lagrangian is of the form

$$L(q,\dot{q}) = \frac{1}{2}\gamma(\dot{q},\dot{q}) - V(q),$$

for a potential $V : Q \to \mathbb{R}$. The associated Legendre transform $\mathbb{F}L : TQ \to T^*Q$ becomes in this case

$$\langle \mathbb{F}L(v_q), w_q\rangle = \gamma(v_q, w_q).$$

After spatial discretization of $\mathcal{D}$, we obtain a set of oriented simplices Δ_i^n with nodes a, that we will denote for simplification by K. Given the configuration $q_a \in Q$ at nodes $a \in K$, we get by interpolation an hyper-regular Lagrangian $L_K : TQ \to \mathbb{R}$ on K as

$$L_K(q_K, \dot{q}_K) := \sum_{a\in K} \frac{1}{2}\gamma_a(\dot{q}_a, \dot{q}_a) - \mathbb{V}_K(q_K),$$

where $q_K = \{q_a\}_{a\in K}$. Such that we have the approximation

$$\int_{\mathcal{D}} \left(\frac{1}{2}\gamma(\dot{q},\dot{q}) - V(q)\right) dV \approx \sum_{K\in\mathcal{T}}\sum_{a\in K} \frac{1}{2}\gamma_a(\dot{q}_a, \dot{q}_a) - \mathbb{V}_K(q_K).$$

The set of simplices K correctly assembled, together with the Lagrangian L_K taking values in $\mathbb{R}$, forms a n-chain complex $\mathcal{T} \in C_n(\mathcal{D}, \mathbb{R})$, where the morphisms are the maps which take an oriented simplicial complex at time t^j and brings it at time t^{j+1}, and which may be continuous or not.

Then we apply a temporal discretization by constructing an increasing sequence of times $\{t^j = j\Delta t \mid j = 0,,N\} \subset \mathbb{R}$ from the time-step Δt, and obtain the discrete regular Lagrangian $L_d(q_K^j, q_K^{j+1}) : Q \times Q \to \mathbb{R}$ which is a time discretization of $\int_{t^j}^{t^{j+1}} L_K(q_K, \dot{q}_K)dt$, with $q_K^j = q_K(t^j)$, and $q_K^{j+1} = q_K(t^{j+1})$,

$$L_d(q_K^j, q_K^{j+1}) \approx \int_{t^j}^{t^{j+1}} L_K(q_K, \dot{q}_K)dt.$$

In such a way that L_d as well as L_K are defined on $\mathcal{T}$. Furthermore, because of the discrete regularity of L_d, there exists a local isomorphism $(\mathbb{F}^- L_d)^\sharp : \{q^j\} \times Q \to T_{q^j}Q$, and another one $(\mathbb{F}^+ L_d)^\sharp : Q \times \{q^{j+1}\} \to T_{q^{j+1}}Q$, where $\sharp : T^*Q \to TQ$ is the inverse of the index lowering operator $\flat : T_qQ \ni v \mapsto \langle v, \cdot\rangle \in T_q^*Q$. See diagram (1.3.1).

$$\mathbb{R} \xleftarrow{L_d} Q \times Q \xrightarrow{\mathbb{F}^\pm L_d} T^*Q \underset{\flat}{\overset{\sharp}{\rightleftarrows}} TQ \xrightarrow{L_K} \mathbb{R} \tag{1.3.1}$$

Thus we can define an energy $E_d = A - L_K$ on TQ at time t^j for a simplex

K, to be

$$
\begin{aligned}
E_d & \left(q_K^j, \left(\mathbb{F}^- L_d^j\right)_K^\sharp\right) \\
&= \sum_{a \in K} \left\langle \mathbb{F} L_K \left(\left(\mathbb{F}^- L_d^j\right)_a^\sharp\right), \left(\mathbb{F}^- L_d^j\right)_a^\sharp \right\rangle - L_K \left(q_K^j, \left(\mathbb{F}^- L_d^j\right)_K^\sharp\right) \\
&= \frac{1}{2} \sum_{a \in K} \gamma_a \left(\left(\mathbb{F}^- L_d^j\right)_a^\sharp, \left(\mathbb{F}^- L_d^j\right)_a^\sharp\right) + \mathbb{V}_K(q_K^j),
\end{aligned}
\tag{1.3.2}
$$

with

$$
\left(q_K^j, \left(\mathbb{F}^- L_d^j\right)_K^\sharp\right) = \left\{ \left(q_a^j, \left(\mathbb{F}^- L_d^j\right)_a^\sharp\right) \right\}_{a \in K},
$$

with $\mathbb{F}^\pm L_d^j := \mathbb{F}^\pm L_d(q^j, q^{j+1})$. Moreover an energy, at time t^{j+1}, can be equivalently defined with $\mathbb{F}^+ L_d^j$.

1.3.2 Discrete Hamiltonian

We know that a variational integrator on $Q \times Q$ preserves the discrete symplectic form $\Omega_d = (\mathbb{F}^\pm L_d)^* \Omega$, where Ω is the canonical two-form on T^*Q. Then the discrete Hamiltonian flow $\widetilde{F}_{L_d} = \mathbb{F}^+ L_d \circ (\mathbb{F}^- L_d)^{-1}$ will preserve the pushforwards of these structures (see Marsden and West [90]).

Therefore the discrete Hamiltonian flow

$$
(q^j, \mathbb{F}^- L_d(q^j, q^{j+1})) \mapsto (q^{j+1}, \mathbb{F}^+ L_d(q^j, q^{j+1}))
$$

is symplectic with respect to the Poisson bracket $\{\cdot, \cdot\}$ on T^*Q. And it is possible to define an Hamiltonian function $H_d = E_d \circ (\mathbb{F} L_K)^{-1}$ on T^*Q such that

$$
\begin{aligned}
H_d & \left(q_K^j, \left(\mathbb{F}^- L_d^j\right)_K\right) \\
&= \sum_{a \in K} \left\langle \left(\mathbb{F}^- L_d^j\right)_a, \ (\mathbb{F} L_K)^{-1} \left(\mathbb{F}^- L_d^j\right)_a \right\rangle - L_K \left(q_K^j, \left(\mathbb{F}^- L_d^j\right)_K^\sharp\right) \\
&= \frac{1}{2} \sum_{a \in K} \left\langle \left(\mathbb{F}^- L_d^j\right)_a, \ (\mathbb{F} L_K)^{-1} \left(\mathbb{F}^- L_d^j\right)_a \right\rangle + \mathbb{V}_K(q_K^j).
\end{aligned}
\tag{1.3.3}
$$

An Hamiltonian function, at time t^{j+1}, can be equivalently defined with $\mathbb{F}^+ L_d^j$. Moreover we know by (1.3.1) that the discrete energy and the discrete Hamiltonian have the same value for a given discrete Legendre transform $\mathbb{F}^\pm L_d^j$.

Almost-conservation of energy. The main feature of the numerical scheme $(q^{j-1}, q^j) \mapsto (q^j, q^{j+1})$ given by solving the discrete Euler-Lagrange equations is that the associated scheme $(q^j, p^j) \mapsto (q^{j+1}, p^{j+1})$ induced on the phase space T^*Q through the discrete Legendre transform defines a *symplectic integrator*. Here we supposed that the discrete Lagrangian L_d is regular, that is, both discrete Legendre transforms $\mathbb{F}^+ L_d, \mathbb{F}^- L_d : Q \times Q \to T^*Q$ are locally isomorphisms (for nearby q^j and q^{j+1}). The symplectic character of the integrator is obtained by showing that the scheme $(q^{j-1}, q^j) \mapsto (q^j, q^{j+1})$ preserves the discrete symplectic two-forms $\Omega^{\pm}_{L_d} := (\mathbb{F}^{\pm}L_d)^*\Omega_{can}$, where Ω_{can} is the canonical symplectic form on T^*Q, so that $(q^j, p^j) \mapsto (q^{j+1}, p^{j+1})$ preserves Ω_{can} and is therefore symplectic, see Marsden, and West [90], Lew, Marsden, Ortiz, and West [70].

It is known, see Hairer, Lubich, and Wanner [41], that given a Hamiltonian H, a symplectic integrator for H is exactly solving a modified Hamiltonian system for a Hamiltonian $\bar{H}$ which is close to H. So the discrete trajectory has all the properties of a conservative Hamiltonian system, such as conservation of the energy $\bar{H}$. The same conclusion holds on the Lagrangian side for variational integrators (see e.g. Lew, Marsden, Ortiz, and West [70]). This explains why energy is approximately conserved for variational integrators, and typically oscillates about the true energy. We refer to Hairer, Lubich, and Wanner [41] for a detailed account and a full treatment of backward error analysis for symplectic integrators.

1.3.3 Discrete energy on $Q \times Q$ and extended variational principle

We can also calculate the energy on $Q \times Q$, by extending the configuration in time, in the framework of the multysymplectic geometry, (see Marsden, Patrick, and Shkoller [85], Lew, Marsden, Ortiz and West [69; 70] among other papers). The discrete Lagrangian is now defined on $Q \times Q \times \mathbb{R}$

$$L_d(q^j, q^{j+1}, t^{j+1} - t^j) : Q \times Q \times \mathbb{R} \to \mathbb{R}.$$

Then we extend the discrete variational principle using positions and time, and we get a system of two equations

$$\begin{aligned} D_2L_d(q^{j-1}, q^j, t^j - t^{j-1}) + D_1L_d(q^j, q^{j+1}, t^{j+1} - t^j) = 0, \\ D_3L_d(q^{j-1}, q^j, t^j - t^{j-1}) - D_3L_d(q^j, q^{j+1}, t^{j+1} - t^j) = 0. \end{aligned}$$

The second equation means that following energy E_d^j defined at time t^j is conserved

$$E_d^j = -D_3L_d(q^j, q^{j+1}, t^{j+1} - t^j). \tag{1.3.4}$$

This discrete energy generally represents the sum of discrete kinetic and potential energy. And we note that this definition is more general than previous ones as it does not require the discrete regularity of L_d.

1.4 Discrete forced Lagrangian systems

To integrate discrete Lagrangian with discrete external forcing it is possible to extend the discrete variational framework to include forcing, as was done in Marsden, and West [90]. In presence of an external force field, given by a fiber preserving map $F : TQ \to T^*Q$, Hamilton's principle is replaced by the *Lagrange-d'Alembert principle*

$$\delta \int_0^T L(q(t), \dot{q}(t))dt + \int_0^T F(q(t), \dot{q}(t)) \cdot \delta q \, dt = 0,$$

where $F(q, \dot{q}) \cdot \delta q$ is the virtual work done by the force field F with a virtual displacement δq. This principle yields the *Lagrange-d'Alembert equations*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F(q, \dot{q})$$

Review of discrete forced Lagrangian systems. As with other discrete structures, there are two discrete Lagrangian forces

$$F_d^\pm : Q \times Q \to T^*Q,$$

which are fiber preserving in the sense that

$$\pi_q \circ F_d^\pm = \pi_Q^\pm,$$

where $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection and $\pi_Q^\pm : Q \times Q \to Q$ are defined by

$$\pi_Q^-(q^j, q^{j+1}) = q^j \quad \text{and} \quad \pi_Q^+(q^j, q^{j+1}) = q^{j+1}.$$

Thus, in local coordinates, we have

$$F_d^-(q^j, q^{j+1}) = \left(q^j, F_d^-(q^j, q^{j+1})\right) \quad \text{and} \quad F_d^+\left(q^j, q^{j+1}\right) = \left(q^{j+1}, F_d^+(q^j, q^{j+1})\right)$$

We now recall from Marsden and West [90] the discrete Lagrange-d'Alembert principle.

1.4.1 Theorem (Discrete Lagrange-d'Alembert principle) *Let $L_d : Q \times Q \to \mathbb{R}$ be a discrete Lagrangian, and consider the discrete Lagrangian forces $F_d^\pm : Q \times Q \to T^*Q$. Then the following are equivalent:*

(i) *The discrete curve $\{q^j\}$ satisfies the discrete Euler-Lagrange equations for L_d with forcing:*

$$D_2 L_d(q^{j-1}, q^j) + D_1 L_d(q^j, q^{j+1}) + F_d^+(q^{j-1}, q^j) + F_d^-(q^j, q^{j+1}) = 0,$$

for all $j = 1, ..., N-1$.

(ii) *The discrete Lagrange d'Alembert principle*

$$\delta \sum_{j=0}^{N-1} L_d(q^j, q^{j+1}) + \sum_{j=0}^{N-1} \left[F_d^-(q^j, q^{j+1}) \cdot \delta q^j + F_d^+(q^j, q^{j+1}) \cdot \delta q^{j+1}\right] = 0, \tag{1.4.1}$$

holds for variations δq^j with fixed endpoints $\delta q^0 = \delta q^N = 0$.

Note that in the discrete Lagrange-d'Alembert principle, the two discrete forces F_d^+ an F_d^- combine to give a single one-form $F_d : Q \times Q \to T^*(Q \times Q)$ given by

$$F_d(q^j, q^{j+1}) \cdot (\delta q^j, \delta q^{j+1}) = F_d^+(q^j, q^{j+1}) \cdot \delta q^{j+1} + F_d^-(q^j, q^{j+1}) \cdot \delta q^j.$$

The forced discrete Euler-Lagrange equations implicitly define the *forced discrete Lagrangian map* $F_{L_d} : Q \times Q \to Q \times Q$.

Although in the continuous case we used the standard Legendre transform for systems with forcing, in the discrete case it is necessary to take the forced discrete Legendre transforms to be

$$\begin{aligned} \mathbb{F}^{F^-} L_d(q^j, q^{j+1}) &= \left(q^j, -D_1 L_d(q^j, q^{j+1}) + F_d^-(q^j, q^{j+1})\right) \\ \mathbb{F}^{F^+} L_d(q^j, q^{j+1}) &= \left(q^{j+1}, D_2 L_d(q^j, q^{j+1}) + F_d^+(q^j, q^{j+1})\right). \end{aligned} \tag{1.4.2}$$

As in (1.2.4), the discrete Euler-Lagrange equations with forces can be equivalently written as

$$\mathbb{F}^{F+} L_d(q^{j-1}, q^j) = \mathbb{F}^{F-} L_d(q^j, q^{j+1}), \quad \text{for } j = 1, ..., N-1.$$

The *forced discrete Hamiltonian map* is defined by

$$\tilde{F}_{L_d} := \mathbb{F}^{F^\pm} L_d \circ F_{L_d} \circ \left(\mathbb{F}^{F^\pm} L_d\right)^{-1}.$$

We thus have $(q^{j+1}, p^{j+1}) = \tilde{F}_{L_d}(q^j, p^j)$ where

$$p^j = -D_1 L_d(q^j, q^{j+1}) - F_d^-(q^j, q^{j+1}) \quad \text{and} \quad p^{j+1} = D_2 L_d(q^j, q^{j+1}) + F_d^+(q^j, q^{j+1}).$$

Discrete forced Noether's theorem. Consider an action $\Phi : G \times Q \to Q$, and let $L_d : Q \times Q \to \mathbb{R}$ be a discrete Lagrangian. In the presence of forcing, the discrete momentum maps are defined by

$$\begin{aligned} \left\langle \mathbf{J}_{L_d}^{F^+}(q^j, q^{j+1}), \xi \right\rangle &= \left\langle \mathbb{F}^{F^+} L_d(q^j, q^{j+1}), \xi_Q(q^{j+1}) \right\rangle \\ \left\langle \mathbf{J}_{L_d}^{F^-}(q^j, q^{j+1}), \xi \right\rangle &= \left\langle \mathbb{F}^{F^-} L_d(q^j, q^{j+1}), \xi_Q(q^j) \right\rangle. \end{aligned}$$

Note that these expressions recover (1.2.8), when the forces are zero. If the discrete force F_d is orthogonal to the group action, so that $\langle F_q, \xi_{Q\times Q}\rangle = 0$, for all $\xi \in \mathfrak{g}$, then we have

$$\mathbf{J}_{L_d}^{F^+} = \mathbf{J}_{L_d}^{F^-}, \tag{1.4.3}$$

and we denote by $\mathbf{J}_{L_d}^F : Q \times Q \to \mathfrak{g}^*$ this unique map. With this notation, we have the following result, which is the discrete analog of Theorem 8.1.1.

1.4.2 Theorem (Discrete forced Noether's theorem) *Let $\Phi : G \times Q \to Q$ be an action and let $L_d : Q \times Q \to \mathbb{R}$ be a G-invariant discrete Lagrangian system with discrete forces $F_d^+, F_d^- : Q \times Q \to T^*Q$, such that $\langle F_d, \xi_{Q\times Q}\rangle = 0$, for all $\xi \in \mathfrak{g}$. Then the discrete momentum map $\mathbf{J}_{L_d}^F : Q \times Q \to \mathfrak{g}^*$ will be preserved by the discrete Lagrangian map, so that $\mathbf{J}_{L_d}^F \circ F_{L_d} = \mathbf{J}_{L_d}^F$.*

Chapter 2

Spherical pendulum

Introduction

The spherical pendulum is composed of a single mass which is fixed to a pivot point. This is a simple example where the potential W_{ext} is completely determined by the external gravitational field and the symmetry is about the vertical axis. During the motion, the mass spins around the vertical axis while oscillating between two parallel circles of the sphere.

Thus we develop a geometric variational discretization especially well-suited for systems on Lie groups. Based on Moser, and Veselov [93], discrete Euler-Lagrange equations for systems on Lie groups, and the associated discrete Lagrangian reductions have been carried out in Bobenko, and Suris [11; 12], Marsden, Pekarsky, and Shkoller [86], and further developed in Lee [66] and applied to many examples. These integrators are referred to as *Lie group variational integrators*. See also Iserles, Munthe-Kaas, Nørsett, and Zanna [50] for a related approach for solving differential equations on Lie groups.

With this simple example we highlight the properties of variational integrators. That is, we observe the conservation of symmetries (i.e., of the momentum maps) and the almost constant behavior of the total energy in time. Moreover, with respect to the work of Lee [66], we paid special attention to the discretization of the Lagrangian, in particular, to the speed (2.2.8) that remains in the Lie algebra after discretization.

One of the beautiful and important things about discrete mechanics is that it permits to switch from the discrete to the continuous world by the Legendre transform. This is what we did in writing the Hamiltonian equations from the discrete variational point of view.

2.1 Lie group variational integrator

We recall briefly the Lie group variational integrator presented in Bobenko and Suris [11].

2.1.1 Lie group variational integrator

Let a configuration space $Q = \mathcal{C}^\infty(\mathcal{D}, G) \ni g$ which is a Lie group, and where $\mathcal{D}$ is a compact domain. We suppose that G acts on Q by left translation, which is a smooth mapping

$$\Phi : G \times Q \to Q \quad (h, g) \mapsto \Phi(h, g) =: hg, \tag{2.1.1}$$

such that (i) for all $g \in Q$, $\Phi(e, g) = g$, where e is the identity element of G, and (ii) for every $h_1, h_2 \in G$, $\Phi(h_1, \Phi(h_2, g)) = \Phi(h_1 h_2, g)$ for all $g \in Q$.

Let $L : TQ \to \mathbb{R}$ be a smooth regular Lagrangian defined on the tangent bundle TQ to the Lie group Q.

From now on, by convenience, we denote in this subsection the configuration space by G instead of Q. And, for convenience, we translate the vector $\dot{g} \in T_g G$ to $(g, g^{-1}\dot{g}) \in G \times T_e G$, by left trivialization as described in Bobenko and Suris [11]. We recall that $T_e G =: \mathfrak{g}$ is the Lie algebra of the Lie group G.

By vector bundle isomorphism we trivialized the Lagrange function L by pull-back through $G \times \mathfrak{g} \ni (g, g^{-1}\dot{g}) \overset{\sim}{\longmapsto} \dot{g} \in TG$, which induces $\mathcal{L} : G \times \mathfrak{g} \to \mathbb{R}$ defined by

$$\mathcal{L}(g, \xi) := L(g, \dot{g}), \quad g^{-1}\dot{g} := \xi. \tag{2.1.2}$$

Given an interval of time $[0, T]$, define the path space to be

$$\mathcal{C}(G) = \mathcal{C}([0, T], G) = \{g : [0, T] \to G \mid g \text{ is a } C^2 \text{ curve}\}.$$

Then a curve $g \in \mathcal{C}(G)$ is said to be a solution of the Euler-Lagrange equations in terms of $\mathcal{L}$, if g satisfies

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \xi}\right) - \mathrm{ad}^*_\xi \frac{\partial \mathcal{L}}{\partial \xi} = g^{-1}\frac{\partial \mathcal{L}}{\partial g}.$$

It is important to note that by trivialization we carry $\dot{g} \in T_g G$ to $g^{-1}\dot{g} \in \mathfrak{g}$, which is expressed in body coordinates (see Abraham and Marsden [1]).

By spatial and temporal discretization we get, on the interval of time $[t^j, t^{j+1}]$, the discrete trivialized Lagrangian $\mathcal{L}_d^j := \mathcal{L}_d(g^j, f^j) : G \times G \to \mathbb{R}$, with $f^j = (g^j)^{-1} g^{j+1}$.

Let $\mathcal{C}_d(G) = \{g_d : \{t^j\}_{j=0}^N \to G, \quad t^j \mapsto g^j := g(t^j)\}$ be the discrete path space. The discrete action map $\mathfrak{S}_d(\mathcal{L}_d) : C_d(Q) \to \mathbb{R}$ is defined by

$$\mathfrak{S}_d(\mathcal{L}_d) := \sum_{j=0}^{N-1} \mathcal{L}_d(g^j, f^j).$$

Let g_ε^j be a deformation of g^j in $\mathcal{C}_d(G)$, such that $g_\varepsilon^0 = g^0$ and $g_\varepsilon^N = g^N$ for any ε in an open interval $]-\lambda, \lambda[$ and $g_0^j = g^j$ for all $j = 0, 1, \ldots, N.$. Let

$$\delta g^j := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} g_\varepsilon^j \in T_{g^j} G$$

be the corresponding variation $\delta g^j = g^j\eta^j$, and $\delta f^j = -\eta^j f^j + f^j\eta^{j+1}$ where $\eta^j \in \mathfrak{g}$. The endpoints are fixed when we have $\eta^0 = \eta^N = 0$.

We compute the derivative of the discrete action map

$$\begin{aligned}\mathbf{d}\mathfrak{S}(\mathcal{L}_d)\cdot(\delta g_d, \delta f_d) &= \frac{d}{d\epsilon}\bigg|_{\varepsilon=0}\sum_{k=0}^{N-1}\mathcal{L}_d\left(g_\varepsilon^j, (g_\varepsilon^j)^{-1}g_\varepsilon^{j+1}\right)\\ &= \sum_{k=0}^{N-1} D_{g^j}\mathcal{L}_d(g^j,f^j)\cdot\delta g^j + D_{f^j}\mathcal{L}_d(g^j,f^j)\cdot\delta f^j,\end{aligned}$$

denoted $\delta\mathfrak{S}_d$ for simplification. The calculation gives

$$\begin{aligned}&\sum_{j=0}^{N-1} T_I^*L_{g^j}(D_{g^j}\mathcal{L}_d^j)\cdot\eta^j - \mathrm{Ad}^*_{(f^j)^{-1}}\Big(T_I L^*_{f^j}(D_{f^j}\mathcal{L}_d^j)\Big)\cdot\eta^j + T_I^*L_{f^j}(D_{f^j}\mathcal{L}_d^j)\cdot\eta^{j+1}\\ &= \sum_{j=1}^{N-1}\Big\{T_I^*L_{f^{j-1}}(D_{f^{j-1}}\mathcal{L}_d^{j-1}) - \mathrm{Ad}^*_{(f^j)^{-1}}\Big(T_I^*L_{f^j}(D_{f^j}\mathcal{L}_d^j)\Big) + \ T_I^*L_{g^j}(D_{g^j}\mathcal{L}_d^j)\Big\}\cdot\eta^j\\ &\qquad + \Big\{T_I^*L_{g^0}(D_{g^0}\mathcal{L}_d^0) - \mathrm{Ad}^*_{(f^0)^{-1}}\left(T_I L^*_{f^0}(D_{f^0}\mathcal{L}_d^0)\right)\Big\}\cdot\eta^0\\ &\qquad + \ T_I^*L_{f^{N-1}}(D_{f^{N-1}}\mathcal{L}_d^{N-1})\cdot\eta^N.\end{aligned}$$

A discrete path $g_d \in \mathcal{C}_d(Q)$ is said a solution of the discrete Euler-Lagrange equations if for all variations $\delta g_d \in T_{g_d}\mathcal{C}_d(Q)$ we have

$$T_I^*L_{f^{j-1}}(D_{f^{j-1}}\mathcal{L}_d^{j-1}) - \mathrm{Ad}^*_{(f^j)^{-1}}\Big(T_I^*L_{f^j}(D_{f^j}\mathcal{L}_d^j)\Big) + \ T_I^*L_{g^j}(D_{g^j}\mathcal{L}_d^j) = 0\ . \tag{2.1.3}$$

The discrete Lagrangian one-forms $\Theta^{\pm}_{L_d} : G\times G \to G\times\mathfrak{g}^*$ are

$$\begin{aligned}\Theta^+_{\mathcal{L}_d}(g^j,f^j) &= \Big(T_I^*L_{f^j}(D_{f^j}\mathcal{L}_d^j)\Big)\,\mu^{j+1},\\ \Theta^-_{\mathcal{L}_d}(g^j,f^j) &= \Big(-T_I^*L_{g^j}(D_{g^j}\mathcal{L}_d^j) + \mathrm{Ad}^*_{(f^j)^{-1}}\Big(T_I L^*_{f^j}(D_{f^j}\mathcal{L}_d^j)\Big)\Big)\,\mu^j.\end{aligned} \tag{2.1.4}$$

The discrete Legendre transforms $\mathbb{F}\mathcal{L}_d^{\pm} : G\times G \to G\times\mathfrak{g}^*$ have the expressions

$$\begin{aligned}\mathbb{F}^+\mathcal{L}_d(g^j,f^j) &= \Big(g^j, T_I^*L_{f^j}(D_{f^j}\mathcal{L}_d^j)\Big)\\ \mathbb{F}^-\mathcal{L}_d(g^j,f^j) &= \Big(g^{j+1}, \mathrm{Ad}^*_{(f^j)^T}\Big(T^*L_{f^j}(D_{f^j}\mathcal{L}_d^j)\Big) - T_I^*L_{g^j}(D_{g^j}\mathcal{L}_d^j)\Big).\end{aligned}$$

And the discrete Euler-Lagrange equations (2.1.3) can be written as

$$\mathbb{F}^-\mathcal{L}_d(g^j,f^j) = \mathbb{F}^+\mathcal{L}_d(g^{j-1},f^{j-1}).$$

2.1.2 Discrete momentum map

Given the left Lie group action Φ as defined in (2.1.1), and $\xi \in \mathfrak{g}$, the infinitesimal generator $\xi_Q(g)$ of the left translation of G on itself is given by

$$\xi_G(g) = (g, \xi g). \tag{2.1.5}$$

And the infinitesimal generator $\xi_{G\times G} \to T(G\times G)$ to be

$$\xi_{G\times G}(g^j, f^j) = (g^j, \xi g^j, f^j, 0).$$

2.1.1 Remark We note that $\xi = \xi_G(g^j)(g^j)^{-1} \in \mathfrak{g}$ represent the infinitesimal generator $\xi_G(g^j)$ in space coordinates, which is due to the right translation by g^j, and $J_{\mathcal{L}_d}(g^j, f^j)$ is the space discrete momentum map.

Given variations $(\delta g^j, \delta f^j)$ and the discrete Lagrangian one-forms $\Theta^{\pm}_{L_d}$ already given in (2.1.4), we have

$$\begin{aligned}
\Theta^+_{\mathcal{L}_d}(g^j, f^j)\cdot(\delta g^j, \delta f^j) &= \left\langle \mathbb{F}^+\mathcal{L}_d(g^j, f^j),\ \eta^{j+1}\right\rangle \\
&= \left\langle T_I^* L_{f^j}(D_{f^j}\mathcal{L}_d^j),\ (f^j)^{-1}\delta f^j + \mathrm{Ad}_{(f^j)^{-1}}(g^j)^{-1}\delta g^j \right\rangle, \\
\Theta^-_{L_d}(g^j, f^j)\cdot(\delta g^j, \delta f^j) &= \left\langle \mathbb{F}^-\mathcal{L}_d(g^j, f^j),\ \eta^{j}\right\rangle \\
&= \left\langle \mathrm{Ad}^*_{(f^j)^T}\left(T^* L_{f^j}(D_{f^j}\mathcal{L}_d^j)\right) - T_I^* L_{g^j}(D_{g^j}\mathcal{L}_d^j),\ (g^j)^{-1}\delta g^j \right\rangle,
\end{aligned}$$

Then by pairing $\Theta^{\pm}_{\mathcal{L}_d}(g^j, f^j)$ and $\xi_{Q\times Q}(g^j, f^j)$ we obtain

$$\begin{aligned}
\mathbf{J}^+_{\mathcal{L}_d}(g^j, f^j)\cdot\xi &= \left\langle T_I^* L_{f^j}(D_{f^j}\mathcal{L}_d^j),\ \mathrm{Ad}_{(g^{j+1})^{-1}}\xi \right\rangle, \\
\mathbf{J}^-_{\mathcal{L}_d}(\Lambda^j, F^j)\cdot\xi &= \left\langle \mathrm{Ad}^*_{(f^j)^{-1}}\left(T^* L_{f^j}(D_{f^j}\mathcal{L}_d^j)\right) - T_I^* L_{g^j}(D_{g^j}\mathcal{L}_d^j),\ \mathrm{Ad}_{(g^j)^{-1}}\xi \right\rangle,
\end{aligned}$$

Thus the discrete momentum maps $J^{\pm}_{\mathcal{L}_d} : G\times G \to \mathfrak{g}$ are given by

$$\begin{aligned}
\mathbf{J}^+_{\mathcal{L}_d}(g^j, f^j) &= \mathrm{Ad}^*_{(g^{j+1})^{-1}}\left(T_I^* L_{f^j}(D_{f^j}\mathcal{L}_d^j)\right), \\
\mathbf{J}^-_{\mathcal{L}_d}(g^j, f^j) &= \mathrm{Ad}^*_{(g^j)^{-1}}\left(\mathrm{Ad}^*_{(f^j)^{-1}}\left(T^* L_{f^j}(D_{f^j}\mathcal{L}_d^j)\right) - T_I^* L_{g^j}(D_{g^j}\mathcal{L}_d^j)\right),
\end{aligned} \tag{2.1.6}$$

which are the expression of the discrete momentum maps in space coordinates. The discrete momentum maps in body coordinates, denoted $\mathbf{\Pi}^{\pm}_{\mathcal{L}_d}(g^j, f^j) \in \mathfrak{g}^*$, are given by the coadjoint action

$$\begin{aligned}
\mathbf{\Pi}^+_{\mathcal{L}_d}(g^j, f^j) &:= \mathrm{Ad}^*_{g^{j+1}} J^+_{\mathcal{L}_d}(g^j, f^j), \\
\mathbf{\Pi}^-_{\mathcal{L}_d}(g^j, f^j) &:= \mathrm{Ad}^*_{g^{j}} J^-_{\mathcal{L}_d}(g^j, f^j).
\end{aligned} \tag{2.1.7}$$

And the discrete Euler-Lagrange equations (2.1.3) can be expressed in terms of discrete body momentum maps $\mathbf{\Pi}^{\pm}$, as

$$\mathbf{\Pi}^+_{\mathcal{L}_d}(g^{j-1}, f^{j-1}) = \mathbf{\Pi}^-_{\mathcal{L}_d}(g^j, f^j).$$

Discrete Lagrange-d'Alembert equations. As shown in Marsden, and West [90], in the presence of discrete Lagrangian forces $F_d^{\pm} : G\times G \to T^*G$ (which are fiber preserving maps), it is possible to define a discrete Lagrange-d'Alembert equation. With the discrete trivialized Lagrangian $\mathcal{L}_d : G\times G \to \mathbb{R}$ on a Lie group G, we define the discrete Lagrangian forces as follows

$$\begin{aligned}
\mathcal{F}_d^-(g^j, f^j) &:= F_d^-(g^j, g^{j+1}) \in T^*_{g^j}G, \\
\mathcal{F}_d^+(g^{j+1}, f^j) &:= F_d^+(g^j, g^{j+1}) \in T^*_{g^{j+1}}G.
\end{aligned}$$

The discrete Lagrange principle for Lie group variational integrator becomes in this case

$$\delta \sum_{j=0}^{N-1} L_d(g^j, f^j) + \sum_{j=0}^{N-1} \left(\mathcal{F}_d^-(g^j, f^j) \cdot \delta g^j + \mathcal{F}_d^+(g^{j+1}, f^j) \cdot \delta g^{j+1} \right) = 0,$$

for all variations $\delta g^j = g^j \eta^j$, with $\eta^j \in \mathfrak{g}$ vanishing at endpoints. Taking into account the variation $\delta f^j = -\eta^j f^j + f^j \eta^{j+1}$ and isolating the quantities η^j, we obtain the discrete Lagrange-d'Alembert equations

$$\begin{aligned} (g^j)^{-1} D_{g^j} \mathcal{L}_d^j - \mathrm{Ad}^*_{(f^j)^{-1}} \left((f^j)^{-1} D_{f^j} \mathcal{L}_d^j \right) + (f^{j-1})^{-1} D_{f^{j-1}} \mathcal{L}_d^{j-1} \\ + (g^j)^{-1} \mathcal{F}_d^+(g^j, f^{j-1}) + (g^j)^{-1} \mathcal{F}_d^-(g^j, f^j) = 0 \end{aligned} \tag{2.1.8}$$

with $g^j = g^{j-1} f^{j-1}$ for fixed endpoint conditions, that is $\eta^0 = \eta^N = 0$.

2.2 Simple spherical pendulum

2.2.1 Geometric mechanics

The configuration space of the simple pendulum, we consider, is the special orthogonal Lie group $Q = SO(3)$. [1] We suppose that $SO(3)$ acts on Q by left action.

The gravitational potential of a pendulum with length ℓ, for a mass m, is $mg \langle \Lambda \boldsymbol{\rho}, \mathbf{E}_3 \rangle$, where $\Lambda \in SO(3)$, $\mathbf{E}_3$ is the unitary vector which indicate the vertical, and $\boldsymbol{\rho} = \ell \mathbf{E}_3$. We denote by $\omega \in \mathbb{R}^3$ the convective variable $\widehat{\omega} := \Lambda^{-1} \dot{\Lambda}$, where we use the standard Lie algebra isomorphism, the *hat map*, $\widehat{\ } : (\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\,,\,])$ given by

$$\Omega = \begin{pmatrix} k^1 \\ k^2 \\ k^3 \end{pmatrix} \mapsto \widehat{\Omega} := \begin{pmatrix} 0 & -k^3 & k^2 \\ k^3 & 0 & -k^1 \\ -k^2 & k^1 & 0 \end{pmatrix}. \tag{2.2.1}$$

As seen in (2.1.2) the Lagrange function $L : TSO(3) \to \mathbb{R}$ is trivialized by pull-back, through $SO(3) \times \mathfrak{so}(3) \ni (\Lambda, \omega) \mapsto (\Lambda, \dot{\Lambda}) \in TSO(3)$. Thus the trivialized Lagrangian $\mathcal{L} : SO(3) \times \mathfrak{so}(3) \to \mathbb{R}$ is given by

$$\begin{aligned} \mathcal{L}(\Lambda, \Lambda^{-1}\dot{\Lambda}) &= \frac{1}{2} \int_{\mathcal{B}} \|\widehat{\omega} \boldsymbol{\rho}\|^2 dm + mg \langle \Lambda \boldsymbol{\rho}, \mathbf{E}_3 \rangle \\ &= \frac{1}{2} \omega^T J \omega + mg \langle \Lambda \boldsymbol{\rho}, \mathbf{E}_3 \rangle, \end{aligned} \tag{2.2.2}$$

where the Riemannian metric is

$$\langle\langle \widehat{\omega}, \widehat{\gamma} \rangle\rangle = \omega^T J \gamma \quad \text{with} \quad \widehat{\omega}, \widehat{\gamma} \in \mathfrak{so}(3), \quad \text{and} \quad J\gamma \in \mathfrak{so}^*(3).$$

We note that the Lagrangian L is S^1-invariant with respect to the vertical axis. The Lie algebra $\mathfrak{g} \cong \mathbb{R}$ of $G = S^1$ is the set of vectors along $\mathbf{E}_3$.

[1] It is also possible to consider that the configuration space is $Q = S^2$ thus defined in [82].

Equations of motions Given an interval of time $[0,T]$ and the action map $\mathfrak{S}(\mathcal{L}) : \mathcal{C}([0,T], SO(3)) \to \mathbb{R}$ to be

$$\mathfrak{S}(\mathcal{L})(\Lambda) = \int_0^T \mathcal{L}(\Lambda,\omega)dt.$$

Given $\Lambda \in SO(3)$, $\widehat{\omega} \in \mathfrak{so}(3)$, $\widehat{\eta} \in \mathfrak{so}(3)$, where $\omega \in \mathbb{R}^3$ and $\eta \in \mathbb{R}^3$, the variations $\delta\Lambda$, $\delta\widehat{\omega}$ and $\delta\omega$ are

$$\delta\Lambda = \Lambda\widehat{\eta}, \qquad \delta\omega = \dot{\eta} + \omega \times \eta.$$

Computing the variation of the action map $\mathfrak{S}(\mathcal{L})$ gives

$$\begin{aligned}\delta\mathfrak{S}(\mathcal{L})(\Lambda) &= \int_0^T \left(J\omega\delta\omega + mg\ell\left\langle \mathbf{E}_3, \delta\Lambda\mathbf{E}_3\right\rangle\right)dt \\ &= \int_0^T \left((J\omega)\times\omega - J\dot{\omega} - 2mg\ell\left(\left(\mathbf{E}_3\mathbf{E}_3^T\Lambda\right)^{(A)}\right)^{\vee}\right)\cdot\eta dt \\ &\quad + (J\omega)\cdot\eta\Big|_0^T,\end{aligned}$$

where $^{\vee}$ is the inverse map of the hat map $\widehat{\ }$ as defined in (2.2.1).

Thus the Euler-Lagrange equations for stationary values of the action integral $\delta\mathfrak{S} = 0$ are

$$(J\omega)\times\omega - \frac{d}{dt}(J\omega) - 2mg\ell\left(\left(\mathbf{E}_3\mathbf{E}_3^T\Lambda\right)^{(A)}\right)^{\vee} = 0, \tag{2.2.3}$$

Momentum map. The Lagrangian one-form on $SO(3)\times\mathfrak{so}(3)$ is

$$\Theta_{\mathcal{L}}(\Lambda,\omega) = (J\omega)\mu,$$

which verifies

$$\left\langle \Theta_{\mathcal{L}}(\Lambda,\omega), (\delta\Lambda,\delta\omega)\right\rangle = \left\langle \mathbb{F}\mathcal{L}(\Lambda,\omega),\eta\right\rangle = \left\langle \mathbb{F}\mathcal{L}(\Lambda,\omega), \Lambda^{-1}\delta\Lambda\right\rangle,$$

where $\mathbb{F}\mathcal{L} : SO(3)\times\mathfrak{so}(3) \to SO(3)\times\mathfrak{so}^*(3)$, $\quad (\Lambda,\omega)\mapsto(\Lambda,J\omega)$, is the Legendre transform.

For $\widehat{\xi} \in \mathfrak{so}(3)$, the infinitesimal generator $\xi_Q(\Lambda)$ of the left action Φ is $\xi_Q(\Lambda) = \widehat{\xi}\Lambda$, as seen in (2.1.5). Thus the infinitesimally equivariant momentum map $\mathbf{J}_{\mathcal{L}} : SO(3)\times\mathfrak{so}(3) \to \mathfrak{g}^*$ associated to the S^1-invariance, is given by

$$\left\langle \mathbf{J}_{\mathcal{L}}(\Lambda,\omega),\xi\right\rangle = \left\langle \mathbb{F}\mathcal{L}(\Lambda,\omega), \Lambda^{-1}\xi_{SO(3)}(\Lambda)\right\rangle,$$

with $\xi_{SO(3)}(\Lambda) = \widehat{\theta\mathbf{E}_3}\Lambda$, where $\theta\in\mathbb{R}$. Thus we obtain

$$\mathbf{J}_{\mathcal{L}}(\Lambda,\omega) = \left\langle \Lambda J\omega, \mathbf{E}_3\right\rangle \in \mathbb{R}.$$

2.2.2 First temporal discretization

The trajectory of the pendulum describes the motion of a single point which is the center of gravity of the mass m. Thus the system should not be discretized spatially, but only temporally.

Let $\mathcal{C}_d(SO(3)) = \{\Lambda_d : \{t^j\}_{j=0}^N \to SO(3), \quad t^j \mapsto \Lambda^j := \Lambda(t^j)\}$ the discrete path space. We apply to the velocity $\omega(\Lambda, t)$ the following temporal discretization

$$\mathfrak{so}(3) \ni \widehat{\omega}^j \approx (\Lambda^j)^T \left(\frac{\Lambda^{j+1} - \Lambda^j}{\Delta t}\right) = \frac{F^j - I_3}{\Delta t} \notin \mathfrak{so}(3), \tag{2.2.4}$$

where $F^j = (\Lambda^j)^T \Lambda^{j+1}$. (We note that $\frac{F^j - I_3}{\Delta t}$ is not an element of the Lie algebra $\mathfrak{so}(3)$, while $\omega^j \in \mathfrak{so}(3)$.)

Thus we obtain the discretized trivialized Lagrangian defined on the interval of time $\Delta t = t^{j+1} - t^j$

$$\begin{aligned}
\mathcal{L}_d(\Lambda^j, F^j) &= K(F^j) - V(\Lambda^j, F^j) \\
&= \frac{1}{2\Delta t}\mathrm{Tr}[(F^j - I_3)J_d(F^j - I_3)^T] + \Delta t\ mg\ell \left\langle \Lambda^j \mathbf{E}_3, \mathbf{E}_3 \right\rangle \\
&= \frac{1}{\Delta t}\mathrm{Tr}[(I_3 - F^j)J_d] + \Delta t\ mg\ell (\mathbf{E}_3)^T \Lambda^j \mathbf{E}_3,
\end{aligned}$$

where J_d is the non-standard inertia matrix which verify

$$\widehat{J\omega} = \widehat{\omega} J_d + J_d \widehat{\omega},$$

and where we used the following properties

$$\mathrm{Tr}\left[F_a^j J_d (F_a^j)^T\right] = \mathrm{Tr}\left[J_d (F_a^j)^T F_a^j\right] = \mathrm{Tr}\left[J_d\right] \quad \text{and} \quad \mathrm{Tr}[J_d (F_a^j)^T] = \mathrm{Tr}[F_a^j J_d].$$

Variational integrator

A discrete path $\Lambda_d \in \mathcal{C}_d(SO(3))$ is said to be a solution of the discrete Euler-Lagrange equations if we have (2.1.3).

In order to obtain the discrete Euler-Lagrange equation it is necessary to achieve a number of intermediate calculations. Before computing these equations concretely, we recall that we identify the dual space $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ via the natural pairing of $\mathbb{R}^3$, i.e.

$$\langle \widehat{\mathbf{v}}, \widehat{\mathbf{w}} \rangle := \mathbf{v} \cdot \mathbf{w} = \frac{1}{2}\ \mathrm{Tr}\left(\widehat{\mathbf{v}}^T \widehat{\mathbf{w}}\right).$$

Given $F^j \in SO(3)$ and its variation $\delta F^j = F^j \widehat{\xi} \in T_{F^j} SO(3)$, we have

$$\begin{aligned}
D_{F^j} \mathcal{L}_d^j \cdot F^j \xi &= -\frac{1}{\Delta t}\ \mathrm{Tr}\left[\delta F^j\ J_d\right] \\
&= -\frac{1}{\Delta t}\ \mathrm{Tr}\left[J_d\ F^j \widehat{\xi}\right] \\
&= -\frac{1}{\Delta t}\ \mathrm{Tr}\left[\frac{1}{2}\left(J_d\ F^j - (F^j)^T J_d\right) \widehat{\xi}\right].
\end{aligned}$$

$$T_I^* L_{F^j}(D_{F^j}\mathcal{L}_d^j) = \frac{1}{\Delta t}\left(J_d\ F^j - (F^j)^T J_d\right)^\vee \in \mathbb{R}^3 \simeq \mathfrak{so}(3),$$

and

$$\mathrm{Ad}^*_{(F^j)^{-1}}\left(T_I^* L_{F^j} D_{F^j}\mathcal{L}_d^j\right) = \frac{1}{\Delta t}\left(F^j J_d - J_d(F^j)^T\right)^\vee$$

The derivative of L_d^j with respect to Λ^j is

$$\begin{aligned} D_{\Lambda^j}\mathcal{L}_d^j \cdot \delta\Lambda^j &= \Delta t\ mg\ell\ \mathrm{Tr}[\delta\Lambda^j \mathbf{E}_3(\mathbf{E}_3)^T] \\ &= \Delta t\ mg\ell\ \mathrm{Tr}[\mathbf{E}_3(\mathbf{E}_3)^T\Lambda^j\widehat{\xi}] \\ &= -2\Delta t\ mg\ell\left(\left(\mathbf{E}_3(\mathbf{E}_3)^T\Lambda^j\right)^{(A)}\right)^\vee \cdot \xi \end{aligned}$$

$$T_I^* L_{\Lambda^j}(D_{\Lambda^j}\mathcal{L}_d^j) = -2\Delta t\ mg\ell\left(\left(\mathbf{E}_3(\mathbf{E}_3)^T\Lambda^j\right)^{(A)}\right)^\vee.$$

Thus the discrete Euler-Lagrange equations of the discrete trivialized Lagrangian are

$$\begin{aligned} &\frac{1}{\Delta t}\left(J_d F^{j-1} - (F^{j-1})^T J_d\right)^\vee - \frac{1}{(\Delta t)}\left(F^j J_d - J_d(F^j)^T\right)^\vee \\ &\qquad = 2\Delta t\ mg\ell\ \left((\mathbf{E}_3\mathbf{E}_3^T\Lambda^j)^{(A)}\right)^\vee. \end{aligned}$$

We note that the first line of the previous equation divided by Δt is the discrete equivalence of the term $(J\omega)\times\omega - J\dot{\omega}$ in (2.2.3). And the second line concerning the contribution of the potential energy is unchanged with respect to continuous formulation.

However the first line, that can be written as follows

$$\frac{2}{(\Delta t)^2}\left(\left((J_d F^{j-1})^{(A)}\right)^\vee + \left(\left(J_d(F^j)^T\right)^{(A)}\right)^\vee\right),$$

is too symmetrical to really bring up the term $(J\omega)\times\omega$. Therefore, in what follows, we will discretize time in a different way.

Discrete momentum map

Given (2.1.6) the discrete momentum maps $\mathbf{J}^\pm_{L_d} : SO(3)\times SO(3) \to \mathfrak{so}(3)^*$ associated to the Lie group action Φ to be

$$\mathbf{J}^+_{\mathcal{L}_d}(\Lambda^j, F^j) = \mathrm{Ad}^*_{(\Lambda^{j+1})^T}\left(\frac{1}{\Delta t}\left(J_d\ F^j - (F^j)^T J_d\right)\right)^\vee,$$

$$\mathbf{J}^-_{\mathcal{L}_d}(\Lambda^j, F^j) = \mathrm{Ad}^*_{(\Lambda^j)^T}\left(\frac{1}{\Delta t}\left(F^j J_d - J_d(F^j)^T\right) + 2\Delta t\ mg\ell\ \left(\mathbf{E}_3\mathbf{E}_3^T\Lambda^j\right)^{(A)}\right)^\vee,$$

which are the expression of the discrete momentum maps in space coordinates. The body discrete momentum maps are

$$\begin{aligned} \mathbf{\Pi}^+_{\mathcal{L}_d}(\Lambda^j, F^j) &:= \mathrm{Ad}^*_{\Lambda^{j+1}} J^+_{L_d}(\Lambda^j, F^j) \\ \mathbf{\Pi}^-_{\mathcal{L}_d}(\Lambda^j, F^j) &:= \mathrm{Ad}^*_{\Lambda^j} J^-_{L_d}(\Lambda^j, F^j), \end{aligned}$$

where $\mathbf{\Pi}^{\pm}_{\mathcal{L}_d}(\Lambda^j, F^j) \in \mathfrak{so}(3)^*$ are more precisely the angular momentum maps.

The discrete momentum maps associated to the symmetry group S^1 are

$$\mathbf{J}^{+}_{\mathcal{L}_d}(\Lambda^j, F^j) = \left\langle \Lambda^{j+1}\left(\frac{1}{\Delta t}\left(J_d\ F^j - (F^j)^T J_d\right)\right)^{\vee}, \mathbf{E}_3 \right\rangle,$$

$$\mathbf{J}^{-}_{\mathcal{L}_d}(\Lambda^j, F^j) = \left\langle \Lambda^{j}\left(\frac{1}{\Delta t}\left(F^j J_d - J_d (F^j)^T\right) + 2\Delta t\ mg\ell\ \left(\mathbf{E}_3 \mathbf{E}_3^T \Lambda^j\right)^{(A)}\right)^{\vee}, \mathbf{E}_3 \right\rangle.$$

Discrete energy defined on $SO(3) \times SO(3)$**.** The discrete energy $E_d^j : SO(3) \times SO(3) \to \mathbb{R}$ at time t^j, seen as the sum of the discrete kinetic energy and the discrete potential energy, to be

$$E_d^j(\Lambda^j, F^j) = \frac{1}{(\Delta t)^2}\mathrm{Tr}[(I_3 - F^j)J_d] - mg\ell(\mathbf{E}_3)^T \Lambda^j \mathbf{E}_3. \tag{2.2.5}$$

Discrete Hamiltonian on trivialized cotangent bundle $SO(3) \times \mathfrak{so}(3)^*$**.** By cotangent lift, we trivialize the cotangent bundle $T^*SO(3)$

$$SO(3)\times\mathfrak{so}(3)^* \ni \left(g^j, T^*L_{g^j}\mathbb{F}^- L_d(g^j, g^{j+1})\right) \mapsto \left(g^j, \mathbb{F}^- L_d(g^j, g^{j+1})\right) \in T^*SO(3).$$

We take into account the results of Bobenko and Suris [11], where it is proved that

$$T^*L_{g^j}\mathbb{F}^- L_d(g^j, g^{j+1}) = \mathbb{F}^+\mathcal{L}_d(g^{j-1}, (g^{j-1})^{-1} g^j),$$

and, that the discrete Hamiltonian flow $(g^j, \mathbb{F}^-\mathcal{L}_d(g^j, f^j)) \mapsto (g^{j+1}, \mathbb{F}^+\mathcal{L}_d(g^j, f^j))$ is symplectic on $SO(3) \times \mathfrak{so}(3)^*$ with respect to the following Poisson bracket

$$\{f, h\} = -\left\langle T^*L_g(D_g f), D_\mu h\right\rangle + \left\langle T^*L_g(D_g h), D_\mu f\right\rangle + \left\langle \mu, [D_\mu f, D_\mu h]\right\rangle, \tag{2.2.6}$$

for any C^1 functions $f, h : SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R}$.

Thus we can define the discrete Hamilton function $\mathcal{H}_d : SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R}$ such that

$$\begin{aligned}\mathcal{H}_d\left(g^j, \left(\mathbb{F}^-\mathcal{L}_d^j\right)\right) &= \left\langle \left(\mathbb{F}^-\mathcal{L}_d^j\right),\ (\mathbb{F}\mathcal{L})^{-1}\left(\mathbb{F}^-\mathcal{L}_d^j\right)\right\rangle - \mathcal{L}\left(g^j, \left(\mathbb{F}^-\mathcal{L}_d^j\right)^{\sharp}\right)\\ &= \frac{1}{2}\left\langle \left(\mathbb{F}^-\mathcal{L}_d^j\right),\ (\mathbb{F}\mathcal{L})^{-1}\left(\mathbb{F}^-\mathcal{L}_d^j\right)\right\rangle + \mathbb{V}(g^j),\end{aligned}$$

and, a Hamiltonian, at time t^{j+1}, can be equivalently defined with $\mathbb{F}^+\mathcal{L}_d^j$.

Thus, for a spherical pendulum, we obtain a discrete Hamiltonian $\mathcal{H}_d : SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R}$, as

$$\mathcal{H}_d(\Lambda^j, (\mathbf{\Pi}^j_{\mathcal{L}_d})^-) = \frac{1}{2}\left((\mathbf{\Pi}^j_{\mathcal{L}_d})^-\right)^T (J)^{-1}(\mathbf{\Pi}^j_{\mathcal{L}_d})^- - mg\ell(\mathbf{E}_3)^T\Lambda^j\mathbf{E}_3. \tag{2.2.7}$$

where $\mathbf{\Pi}^-_{\mathcal{L}_d}(\Lambda^j, F^j)$ is the body angular momentum map, defined in (2.1.7).

Given Hamiltonian function $\mathcal{H}_d$, and the Poisson bracket (2.2.6). The following formula

$$\dot{f} = \{f, \mathcal{H}_d\} = X_{\mathcal{H}_d}[f], \quad \text{for all } f : SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R},$$

which is the directional derivative of f along the flow of $X_{\mathcal{H}_d}$, allows to get the Hamilton's equations. If we let f by each of the canonical coordinate in turn, we obtain

$$\begin{aligned}
\{\Lambda^j,\ \mathcal{H}_d\} &= -\Lambda^j D_{(\mathbf{\Pi}^j_{\mathcal{L}_d})^-} \mathcal{H}_d(\Lambda^j, (\mathbf{\Pi}^j_{\mathcal{L}_d})^-), \\
\{\mathbf{\Pi}^j_{\mathcal{L}_d})^-,\ \mathcal{H}_d\} &= (\Lambda^j)^{-1} D_{\Lambda^j} \mathcal{H}_d(\Lambda^j, (\mathbf{\Pi}^j_{\mathcal{L}_d})^-).
\end{aligned}$$

Thus the Hamilton's equations are

$$\begin{aligned}
\dot{\Lambda} &= -\Lambda(J)^{-1}\Pi, \\
\dot{\mathbf{\Pi}} &= 2mg\ell(\Lambda)^{-1} \left(\left(\mathbf{E}_3 \mathbf{E}_3^T \Lambda\right)^{(A)} \right)^{\vee}.
\end{aligned}$$

Initial conditions

In many case, it seems easier to initialize the simulation by prescribing the position $q(t^0)$ and the speed $\dot{q}(t^0)$ in the tangent bundle $T_{q(t^0)}Q$ to the configuration space Q, instead of initialize the simulation by giving two successive positions q^0 and q^1 at time t^0 and t^1 in the discrete setting $Q \times Q$.

Thus, for the spherical pendulum, we take into account the initial continuous speeds $\widehat{\omega}(t^0)$, in body configuration, at time t^0 and at point $\Lambda^0 = \Lambda(t^0)$.

To incorporate the boundary conditions into the discrete description, we take into account the following relationship at time t^0 when the trivialized Lagrangian $\mathcal{L}$ and its discretize version $\mathcal{L}_d$ are regular. We have the formula

$$\mathbb{F}^- \mathcal{L}_d(\Lambda^0, F^0) = \mathbb{F}\mathcal{L}(\Lambda(t^0), \widehat{\omega}(t^0)),$$

represented by

$$SO(3) \times \{\Lambda^0\} \xrightarrow{\mathbb{F}^- \mathcal{L}_d} SO(3) \times \mathfrak{so}(3)^* \xleftarrow{\mathbb{F}\mathcal{L}} SO(3) \times \mathfrak{so}(3),$$

which can be expressed as follows for the spherical pendulum

$$\begin{aligned}
&\left(\frac{1}{\Delta t} \left(F^0 J_d - J_d (F^0)^T\right) + 2\Delta t\ mg\ell\ \left(\mathbf{E}_3 \mathbf{E}_3^T \Lambda^0\right)^{(A)} \right)^{\vee} = J\omega(t^0), \\
\iff \quad & \left(F^0 J_d - J_d (F^0)^T\right) = \Delta t\ \widehat{J\omega(t^0)} - 2(\Delta t)^2 mg\ell\ \left(\mathbf{E}_3 \mathbf{E}_3^T \Lambda^0\right)^{(A)}.
\end{aligned}$$

And the initial energy E_0, corresponding to the initial conditions $(\Lambda(t^0), \widehat{\omega}(t^0))$, is calculated from the continuous Lagrangian as defined in (2.2.2). Its value is

$$E_0 = \frac{1}{2}(\omega(t^0))^T J \omega(t^0) - mg\ell(\mathbf{E}_3)^T \Lambda(t^0) \mathbf{E}_3.$$

2.2.3 Alternative temporal discretization

Instead of a linear interpolation in time between t^j and t^{j+1}, as we did in (2.2.4), we now consider an interpolation preserving the group $SO(3)$. We thus consider the interpolation

$$\Lambda(t) := \Lambda^j \exp\left(\frac{t - t^j}{t^{j+1} - t^j}\widehat{\Psi}^j\right),$$

between Λ^j and Λ^{j+1}, where $\exp(\widehat{\Psi}^j) = (\Lambda^j)^T\Lambda^{j+1} = F^j$. As a consequence, we get the following approximations of $\widehat{\omega}$ at time t^j:

$$\mathfrak{so}(3) \ni \widehat{\omega}^j = \left(\Lambda^j\right)^T \dot{\Lambda}^j \approx \frac{\widehat{\Psi}^j}{t^{j+1} - t^j} \in \mathfrak{so}(3). \tag{2.2.8}$$

Now the discrete Lagrangian can be written, on the interval of time $\Delta t = t^{j+1} - t^j$, as follows

$$\mathcal{L}_{d2}(\Lambda^j, F^j) = \frac{1}{2\Delta t}(\Psi^j)^T J\Psi^j + \Delta t\ mg\ell(\mathbf{E}_3)^T\Lambda^j\mathbf{E}_3.$$

Variational integrator

It is particularly convenient (and computationally efficient) to approximate the exponential by the Cayley transform, i.e. $\widehat{\Psi}^j = \mathrm{cay}^{-1}(F^j) = 2(F^j - I)(F^j + I)^{-1}$. To obtain discrete Euler-Lagrange equations (2.1.3), we calculate the variations of $\widehat{\Psi}^j$ and F^j, to be

$$\delta\widehat{\Psi}^j = \delta\ \mathrm{cay}^{-1}(F^j) = (2I - \widehat{\Psi}^j)\delta F^j(F^j + I)^{-1}, \quad \text{and} \quad \delta F^j = F^j\widehat{\xi}.$$

We recall that $\left(J_d\widehat{\Psi}^j\right)^{(A)} = \frac{1}{2}\widehat{J\Psi^i}$, and we perform intermediate calculations

$$\begin{aligned}
D_{F^i}\mathcal{L}^j_{d2}\cdot\delta F^j &= \frac{1}{\Delta t}\,\mathrm{Tr}\left(\delta\widehat{\Psi}^j J_d(\widehat{\Psi}^j)^T\right) = -\frac{1}{2}\frac{1}{\Delta t}\,\mathrm{Tr}\left(\delta\widehat{\Psi}^j\widehat{J\Psi^j}\right)\\
&= -\frac{1}{2\Delta t}\,\mathrm{Tr}\left((F^j + I)^{-1}\widehat{J\Psi^j}(2I - \widehat{\Psi}^j)\delta F^j\right)\\
&= -\frac{1}{2\Delta t}\,\mathrm{Tr}\left(\left((F^j + I)^{-1}\widehat{J\Psi^j}(2I - \widehat{\Psi}^j)F^j\right)^{(A)}\widehat{\xi}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
T_I^*L_{F^j}(D_{F^j}\mathcal{L}^j_{d2}) &= \frac{1}{\Delta t}\left(\left((F^j + I)^{-1}\widehat{J\Psi^j}(2I - \widehat{\Psi}^j)F^j\right)^{(A)}\right)^{\vee},\\
\mathrm{Ad}^*_{(F^j)^T}\left(T_I^*L_{F^j}(D_{F^j}\mathcal{L}^j_{d2})\right) &= \frac{1}{\Delta t}\left(\left(F^j(F^j + I)^{-1}\widehat{J\Psi^j}(2I - \widehat{\Psi}^j)\right)^{(A)}\right)^{\vee}.
\end{aligned}$$

The derivative of $\mathcal{L}^j_{d2}$ with respect to Λ^j is unchanged with respect to the first time discretization. The discrete Euler-Lagrange equations are then

$$\frac{1}{\Delta t}\left(\left((F^{j-1}+I)^{-1}\widehat{J\Psi^{j-1}}(2I-\widehat{\Psi}^{j-1})F^{j-1}\right)^{(A)}\right)^{\vee}$$
$$-\frac{1}{\Delta t}\left(\left(F^{j}(F^{j}+I)^{-1}\widehat{J\Psi^{j}}(2I-\widehat{\Psi}^{j})\right)^{(A)}\right)^{\vee}$$
$$=2\Delta t\; mg\ell\;\left(\left(\mathbf{E}_3\mathbf{E}_3^T\Lambda^j\right)^{(A)}\right)^{\vee}.$$

2.2.1 Remark Given $F^j=\operatorname{cay}(\widehat{\Psi}^j)$ which is comutative we observe that

$$F^j=\left(I-\frac{\widehat{\Psi}^j}{2}\right)^{-1}\left(I+\frac{\widehat{\Psi}^j}{2}\right)=\left(I+\frac{\widehat{\Psi}^j}{2}\right)\left(I-\frac{\widehat{\Psi}^j}{2}\right)^{-1},$$
$$(F^j+I)^{-1}=\left(\left(I-\frac{\widehat{\Psi}^j}{2}\right)^{-1}\left(I+\frac{\widehat{\Psi}^j}{2}\right)+I\right)^{-1}=\frac{1}{2}\left(I-\frac{\widehat{\Psi}^j}{2}\right),$$
$$F^j(F^j+I)^{-1}=\frac{1}{2}\left(I+\frac{\widehat{\Psi}^j}{2}\right).$$

Then we get

$$\widehat{b}=\left(F^{j}(F^{j}+I)^{-1}\widehat{J\Psi^{j}}(2I-\widehat{\Psi}^{j})\right)^{(A)}$$
$$=\widehat{J\Psi^j}+\frac{1}{2}\left(\Psi^j\times J\Psi^j\right)^{\wedge}+\frac{1}{4}\left((J\Psi^j)^T\Psi^j\right)\widehat{\Psi}^j,$$

which may be written equivalently as

$$B(\Psi^j)=J\Psi^j+\frac{1}{2}\left(\Psi^j\times J\Psi^j\right)+\frac{1}{4}\Psi^j\left((J\Psi^j)\cdot\Psi^j\right)-b=0,$$

with the Jacobian $DA(\Psi^j)$ given by

$$DB(\Psi^j)=J+\frac{1}{2}\widehat{\Psi}^jJ-\frac{1}{2}\widehat{J\Psi}^j+\frac{1}{4}(\Psi^j\cdot(J\Psi^j))I+\frac{1}{2}\Psi^j(\Psi^j)^TJ.$$

Discrete momentum map

Given the infinitesimal generator $\xi_Q(\Lambda^j)=(\Lambda^j,\widehat{\xi}\Lambda^j)$, the discrete Lagrangian momentum maps $J^{\pm}_{\mathcal{L}_d}:SO(3)\times SO(3)\to\mathfrak{so}(3)^*$, in spatial coordinate, from

(2.1.6), become

$$\mathbf{J}^+_{\mathcal{L}_{d2}}(\Lambda^j, F^j) = \mathrm{Ad}^*_{(\Lambda^{j+1})^T}\left(\frac{1}{\Delta t}\left((F^j+I)^{-1}\widehat{J\Psi^j}(2I-\widehat{\Psi}^j)F^j\right)^{(A)}\right)^{\vee},$$

$$\mathbf{J}^-_{\mathcal{L}_{d2}}(\Lambda^j, F^j) = \mathrm{Ad}^*_{(\Lambda^{j})^T}\left(\frac{1}{\Delta t}\left(F^j(F^j+I)^{-1}\widehat{J\Psi^j}(2I-\widehat{\Psi}^j)\right)^{(A)}\right.$$
$$\left. + 2\Delta t\; mg\ell\;\left(\mathbf{E}_3\mathbf{E}_3^T\Lambda^j\right)^{(A)}\right)^{\vee}.$$

In body coordinates, the discrete angular momentum map $\mathbf{\Pi}^{\pm}_{L_d}$ are defined in the same way as for the previous discretization, that is as (2.1.7).

Total energy

Considering that discrete energy $E^j_d : SO(3)\times\mathfrak{so}(3)\to\mathbb{R}$, at time t^j, is simply the kinetic energy plus the potential energy as in (2.2.5), then we get

$$E^j_{d2} = \frac{1}{2(\Delta t)^2}(\Psi^j)^T J\Psi^j - mg\ell(\mathbf{E}_3)^T\Lambda^j\mathbf{E}_3, \qquad \text{with} \quad \widehat{\Psi}^j = \mathrm{cay}^{-1}(F^j).$$

And, as in (2.2.7), we can define a discrete Hamiltonian $\mathcal{H}_{d2} : SO(3)\times\mathfrak{so}(3)^*\to\mathbb{R}$, as follows

$$\mathcal{H}_{d2}(\Lambda^j, (\mathbf{\Pi}^j_{\mathcal{L}_{d2}})^-) = \frac{1}{2}\left((\mathbf{\Pi}^j_{\mathcal{L}_{d2}})^-\right)^T (J)^{-1}(\mathbf{\Pi}^j_{\mathcal{L}_{d2}})^- - mg\ell(\mathbf{E}_3)^T\Lambda^j\mathbf{E}_3.$$

Initial conditions

As we take into account the initial continuous speeds $\widehat{\omega}(t^0)$, in body configuration, at time t^0 and at point $\Lambda^0=\Lambda(t^0)$, and given the temporal discretization as defined in (2.2.8), we obtain just the value of the discrete speed at time t^0, that is

$$\widehat{\Psi}^0 = \Delta t\,\widehat{\omega}(t^0).$$

2.3 Example

The physical constants for the 3D pendulum are chosen as

$$m = 1 \text{ kg}, \quad \ell = 0.3 \text{ m}, \quad J = \text{diag}[0.13, 0.28, 0.17] \text{ kgm}^2.$$

And the initial conditions are chosen as

$$\Lambda^0 = I, \quad \omega(t^0) = [4.14, 4.14, 4.14], \quad \text{with time step } \Delta t = 0.01.$$

About the results. The momentum is perfectly preserved with the two discretizations. On the other hand we observe that the trajectory obtained by the second time discretization is more symmetrical than the first discretization, as seen in Fig.(2.3). In this vein, it would be interesting to compare these trajectories with the one obtained with a multisymplectic variational integrator, that is when adapting the time step for a perfect conservation of energy.

Otherwise the behavior of the energy is very good. Moreover we note that there is a particular phenomenon of symmetry for E_d and H_d. So that if one averages the two values we get almost a constant. (See Fig. (6.4).)

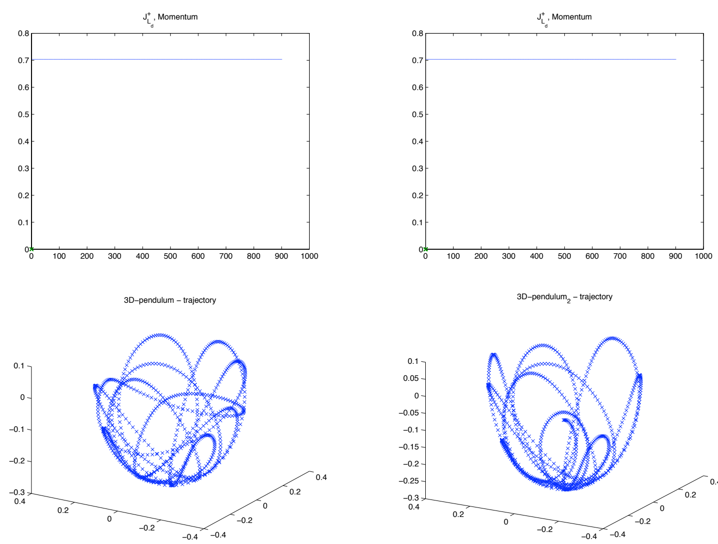


Figure 2.3.1: On the left : the first discretization, on the right : the second discretization. From top to bottom : a) spatial momentum map $J^-_{L_d}$; b) trajectory after 9s. With time step $\Delta t = 0.01$.

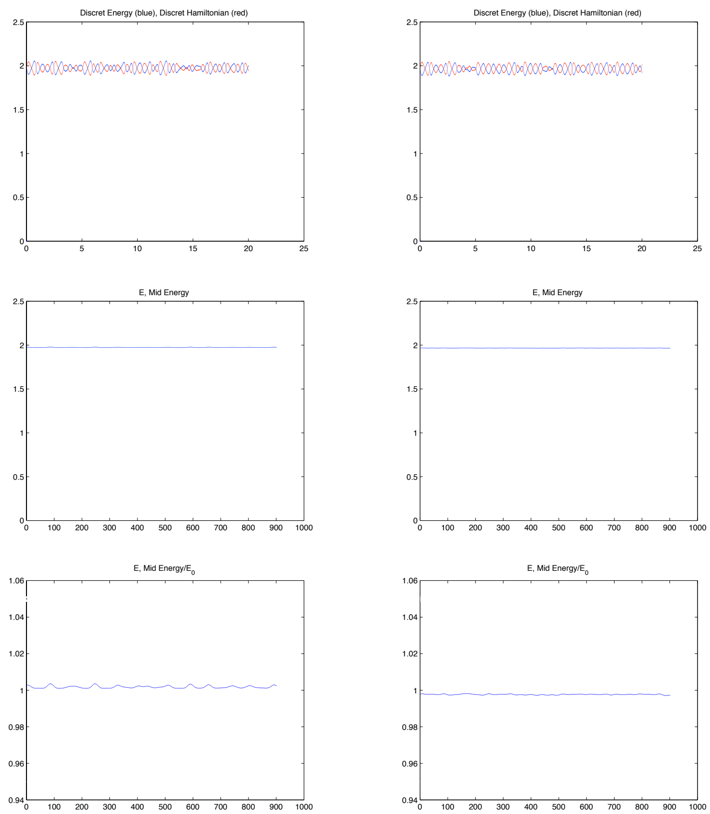


Figure 2.3.2: On the left : the first discretization, on the right : the second discretization. From top to bottom : a) discrete Energy on $SO(3) \times SO(3)$ (2.2.5) and discrete Hamiltonian on $SO(3) \times \mathfrak{so}(3)^*$ (2.2.7) after 20s; b) average of the discrete energy and the discrete Hamiltonian; c) ratio of the average energy with initial energy E_0. With time step $\Delta t = 0.01$.

Chapter 3

Spring pendulum

Introduction

In this chapter, we consider the spring pendulum where the mass is attached to the point pivot by a spring. This time, the potential energy of the spring pendulum is not only given by the exterior gravitational field, giving rise to the exterior gravitational potential W_{ext}, but also of an internal potential

$$W_{spring} = \frac{1}{2}k\Delta\mathbf{x}^2,$$

where $\Delta\mathbf{x}$ is the elongation of the spring, and k is the spring constant. This internal energy is similar to the energy due to the elongation of the beam

$$W_{elongation} = \frac{1}{2}EA\left(\frac{\partial\phi_x}{\partial x}\right)^2,$$

where $\partial\phi_x/\partial x$ is the longitudinal strain of the beam in a point x of the line of centroids, E is the Young modulus, and A is the cross-sectional area.

For physicists, the behavior of the elastic spherical pendulum is quite similar to the molecular oscillations of CO_2 and with 2nd harmonic generation in nonlinear laser optics. (See Holms [45].)

As for the spherical pendulum, the configuration space is a Lie group, so we develop a Lie group variational integrator to study the spring pendulum. This example is slightly more complicated than the spherical pendulum, because the configuration space changes from $SO(3)$ to $SO(3)\times\mathbb{R}^3$. Moreover, this model is fitted with the internal energy W_{spring}. We shall use a Lie group variational integrator to simulate the motion of the spring pendulum.

As in the case of the spherical pendulum, we ensure that the discretization of the speed (2.2.8) remains in the Lie algebra after discretization.

3.1 Pendulum attached to the origin by a very stiff spring

3.1.1 Geometric mechanics

In this section we consider the direct product $SO(3) \times \mathbb{R}^3$ of Lie groups, with group multiplication and inversion given by

$$(\Lambda_1, \boldsymbol{\phi}_1)(\Lambda_2, \boldsymbol{\phi}_2) = (\Lambda_1\Lambda_2, \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2), \qquad (\Lambda, \boldsymbol{\phi})^{-1} = \left(\Lambda^{-1}, -\boldsymbol{\phi}\right).$$

The configuration of the spring pendulum is completely determined by the maps Λ and ϕ in the configuration space

$$Q = C^{\infty}\left([0,\ell], SO(3) \times \mathbb{R}\right) \ni (\Lambda, \phi),$$

with the standard metric. We denote $\mathbf{r}(s,t) = \Lambda(t)\boldsymbol{\phi}(s,t)$, such that $\boldsymbol{\phi}(s) = \phi(s)\mathbf{E}_3$, where $\mathbf{E}_3$ is an unitary vector in $\mathbb{R}^3$, as the vertical direction, and $\phi(0,t) = 0$ for all t.

We denote ω the skew matrix $\omega = \Lambda^{-1}\dot{\Lambda}$, and $\boldsymbol{\phi}_\ell = \boldsymbol{\phi}(\ell)$. The kinetic energy to be

$$\begin{aligned} T(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) &= \frac{1}{2}m\langle \dot{\Lambda}\boldsymbol{\phi}_\ell + \Lambda\dot{\boldsymbol{\phi}}_\ell,\ \dot{\Lambda}\boldsymbol{\phi}_\ell + \Lambda\dot{\boldsymbol{\phi}}_\ell\rangle \\ &= \frac{1}{2}m\left(\|\omega\boldsymbol{\phi}_\ell\|^2 + \|\dot{\boldsymbol{\phi}}_\ell\|^2 + 2\langle\omega\boldsymbol{\phi}_\ell, \dot{\boldsymbol{\phi}}_\ell\rangle\right) =: \mathcal{T}(\Lambda, \phi, \omega, \dot{\phi}). \end{aligned}$$

To see this, note that

$$\left\langle \omega\boldsymbol{\phi}_\ell, \dot{\boldsymbol{\phi}}_\ell \right\rangle = \mathrm{Tr}\left[\omega\boldsymbol{\phi}_\ell(\dot{\boldsymbol{\phi}}_\ell)^T\right] = (\phi_\ell\dot{\phi}_\ell)\mathrm{Tr}\left[\omega\mathbf{E}_3\mathbf{E}_3^T\right] = 0.$$

Let Π_{int} is the elastic internal energy, and Π_{ext} is the gravitational potential for a mass m. The potential energy may be written as

$$\Pi_{int}(\Lambda, \phi) + \Pi_{ext}(\Lambda, \phi) = \int_0^\ell \frac{1}{2}EA(\|\boldsymbol{\phi}'(s)\| - 1)^2 ds - mg\left\langle \Lambda\boldsymbol{\phi}_\ell, \mathbf{E}_3\right\rangle.$$

The trivialized Lagrangian of the beam is given by

$$\begin{aligned} \mathcal{L}(\Lambda, \phi, \omega, \dot{\phi}) &= \frac{1}{2}\left(\omega^T\left(\left(\frac{\phi_\ell}{\ell}\right)^2 J\right)\omega + m\|\dot{\boldsymbol{\phi}}_\ell\|^2\right) - \Pi_{int}(\Lambda, \boldsymbol{\phi}) - \Pi_{ext}(\Lambda, \boldsymbol{\phi}) \\ &= \frac{1}{2}m\left(\mathrm{Tr}\left[\widehat{\omega}\boldsymbol{\phi}_\ell(\widehat{\omega}\boldsymbol{\phi}_\ell)^T\right] + \|\dot{\boldsymbol{\phi}}_\ell\|^2\right) - \Pi_{int}(\Lambda, \boldsymbol{\phi}) - \Pi_{ext}(\Lambda, \boldsymbol{\phi}), \end{aligned} \tag{3.1.1}$$

where $\hat{\omega} \in \mathfrak{so}(3)$, and J is the spherical pendulum inertia matrix of length ℓ. We note that $\left(\left(\frac{\phi_\ell}{\ell}\right)^2 J\right)$ represents the inertia value of a pendulum with respect to its length ϕ_ℓ. The Riemannian metric in $(\Lambda, \mathbf{x}) \in SO(3) \times \mathbb{R}^3$ is given by

$$\langle\langle(\omega, \mathbf{U}), (\gamma, \mathbf{V})\rangle\rangle_{(\Lambda, \mathbf{x})} = \omega^T\left(\left\|\frac{\mathbf{x}}{\ell}\right\|^2 J\right)\gamma + \mathbf{U}^T m\mathbf{V},$$

with $(\omega, \mathbf{U}),\ (\gamma,\ \mathbf{V}) \in \mathfrak{so}(3)\times\mathbb{R}^3$, and $(J\omega, m\mathbf{V}) \in \mathfrak{so}^*(3)\times\mathbb{R}^3$. The Legendre transform is thus

$$\mathbb{F}\mathcal{L}(\Lambda,\phi,\omega,\dot{\phi}) = \begin{pmatrix} \left(\left(\frac{\phi_\ell}{\ell}\right)^2 J\right)\omega \\ m\dot{\boldsymbol{\phi}}_\ell \end{pmatrix}$$

We note that the trivialized Lagrangian may be expressed in $\mathbb{R}^3$ as

$$\mathcal{L}(\Lambda,\phi,\omega,\dot{\phi}) = \frac{1}{2}m\left(\|\omega\times\boldsymbol{\phi}_\ell\|^2 + \|\dot{\boldsymbol{\phi}}_\ell\|^2\right) - \Pi_{int}(\Lambda,\phi) - \Pi_{ext}(\Lambda,\phi).$$

Euler-Lagrange equations For the interval of time $[0,T]$ we get the Euler-Lagrange equations by the condition

$$\delta\mathfrak{S}(\mathcal{L})(\Lambda,\phi) = \int_0^T \mathcal{L}(\Lambda,\phi,\omega,\dot{\phi})dt = 0,$$

where the variation $(\delta\Lambda,\delta\phi)$ is over smooth curves in Q with fixed endpoints $\delta\Lambda(0) = \delta\Lambda(T) = 0$, and $\delta\boldsymbol{\phi}(0) = \delta\boldsymbol{\phi}(T) = 0$. Given $\Lambda \in SO(3)$, $\widehat{\omega} \in \mathfrak{so}(3)$, $\widehat{\eta} \in \mathfrak{so}(3)$, where we use the standard Lie algebra isomorphism, the *hat map*, $\widehat{\ } : \left(\mathbb{R}^3,\times\right) \to (\mathfrak{so}(3),[\,,])$ defined in (2.2.1).

Given the variations $\delta\Lambda$, $\delta\widehat{\omega}$ and $\delta\omega$

$$\delta\Lambda = \Lambda\widehat{\eta}, \qquad \delta\widehat{\omega} = \dot{\widehat{\eta}} + \widehat{\omega}\widehat{\eta} - \widehat{\eta}\widehat{\omega}, \qquad \text{or} \quad \delta\omega = \dot{\eta} + \omega\times\eta, \tag{3.1.2}$$

we compute

$$\delta\mathfrak{S}(\mathcal{L})(\Lambda,\phi) = \int_0^T \Bigg\{ m\mathrm{Tr}\left[\delta\widehat{\omega}\boldsymbol{\phi}_\ell(\widehat{\omega}\boldsymbol{\phi}_\ell)^T\right] + m\langle\widehat{\omega}\boldsymbol{\phi}_\ell, \widehat{\omega}\delta\boldsymbol{\phi}_\ell\rangle + m\langle\dot{\boldsymbol{\phi}}_\ell, \delta\dot{\boldsymbol{\phi}}_\ell\rangle$$
$$- \int_0^\ell EA\frac{(\|\boldsymbol{\phi}'\|-1)}{\|\boldsymbol{\phi}'\|}\boldsymbol{\phi}'\cdot\delta\boldsymbol{\phi}'ds \ + mg\mathrm{Tr}\left[\delta\Lambda\boldsymbol{\phi}_\ell\mathbf{E}_3^T\right] + mg\left\langle\Lambda\delta\boldsymbol{\phi}_\ell, \mathbf{E}_3\right\rangle \Bigg\}dt.$$

By taking into account the formulas for $\delta\widehat{\omega}, \delta\Lambda$ defined in (3.1.2). We isolate the quantities $\eta, \delta\boldsymbol{\phi}$ by integrating by parts and using the following notations

$$\langle\widehat{\mathbf{v}},\widehat{\mathbf{w}}\rangle := \mathbf{v}\cdot\mathbf{w} = \frac{1}{2}\ \mathrm{Tr}\left(\widehat{\mathbf{v}}^T\widehat{\mathbf{w}}\right),$$

we obtain

$$\delta\mathfrak{S}(\mathcal{L}) = \int_0^T \Bigg\{ 2m\frac{d}{dt}\left(\boldsymbol{\phi}_\ell(\widehat{\omega}\boldsymbol{\phi}_\ell)^T\right)^{(A)}\cdot\widehat{\eta} - 2m\left(\boldsymbol{\phi}_\ell(\widehat{\omega}\boldsymbol{\phi}_\ell)^T\widehat{\omega}\right)^{(A)}\cdot\widehat{\eta}$$
$$+ m\widehat{\omega}^T\widehat{\omega}\boldsymbol{\phi}_\ell\cdot\delta\boldsymbol{\phi}_\ell - m\frac{d}{dt}(\dot{\boldsymbol{\phi}}_\ell)\cdot\delta\boldsymbol{\phi}_\ell + \int_0^\ell\left(EA\frac{(\|\boldsymbol{\phi}'\|-1)}{\|\boldsymbol{\phi}'\|}\boldsymbol{\phi}'\right)'\cdot\delta\boldsymbol{\phi}ds$$
$$- 2mg\left(\boldsymbol{\phi}_\ell\mathbf{E}_3^T\Lambda\right)^{(A)}\cdot\widehat{\eta} + mg\Lambda^T\mathbf{E}_3\cdot\delta\boldsymbol{\phi}_\ell - \left[EA\frac{(\|\boldsymbol{\phi}'\|-1)}{\|\boldsymbol{\phi}'\|}\boldsymbol{\phi}'\cdot\delta\boldsymbol{\phi}\right]_0^\ell \Bigg\}dt$$
$$+ 2m\left\langle\left((\widehat{\omega}\boldsymbol{\phi}_\ell)\boldsymbol{\phi}_\ell^T\right)^{(A)},\widehat{\eta}\right\rangle\Bigg|_0^T + m\langle\dot{\boldsymbol{\phi}}_\ell,\delta\boldsymbol{\phi}_\ell\rangle\Big|_0^T.$$

And we get the Euler-Lagrange equations

$$\begin{cases} \left(\dfrac{d}{dt}\left(\dot{\boldsymbol{\phi}}_\ell\right) + \omega \times (\omega \times \boldsymbol{\phi}_\ell) - g\Lambda^T \mathbf{E}_3\right) \cdot \mathbf{E}_3 = 0, \\ \dfrac{d}{dt}\left(\boldsymbol{\phi}_\ell \times (\omega \times \boldsymbol{\phi}_\ell)\right) + (\omega \times (\boldsymbol{\phi}_\ell \times \omega)) \times \boldsymbol{\phi}_\ell - g\boldsymbol{\phi}_\ell \times \Lambda^T \mathbf{E}_3 = 0, \\ \left(EA\dfrac{(\|\boldsymbol{\phi}'\| - 1)}{\|\boldsymbol{\phi}'\|}\boldsymbol{\phi}'\right)' = 0, \\ (\|\boldsymbol{\phi}'\| - 1)\boldsymbol{\phi}'|_{s=0} = 0, \qquad (\|\boldsymbol{\phi}'\| - 1)\boldsymbol{\phi}'|_{s=\ell} = 0, \end{cases}$$

Momentum map. The group of rotation $SO(3)$ acts on Q by the left translation Φ as

$$\Phi_R : Q \to Q, \quad \Phi_R(\Lambda, \boldsymbol{\phi}_\ell) = (R\Lambda, \boldsymbol{\phi}_\ell), \quad \text{where } R \in SO(3).$$

Given $\xi \in \mathfrak{so}(3)$, the corresponding infinitesimal generator $\xi_{TQ} : (SO(3) \times \mathbb{R}) \times (\mathfrak{so}(3) \times \mathbb{R}) \to T(SO(3) \times \mathbb{R}) \times T(\mathfrak{so}(3) \times \mathbb{R}^3)$, associated to the action Φ is given by

$$\xi_{TQ}((\Lambda, \phi), (\omega, \dot{\phi})) = \left((\Lambda, \boldsymbol{\phi}_\ell), (\widehat{\xi}\Lambda, 0); (\omega, \dot{\boldsymbol{\phi}}_\ell), (0, 0)\right).$$

The Lagrangian momentum map $\mathbf{J}_\mathcal{L} : (SO(3) \times \mathbb{R}) \times (\mathfrak{so}(3) \times \mathbb{R}) \to \mathfrak{so}(3)^*$ is defined as

$$\mathbf{J}_\mathcal{L}((\Lambda, \phi), (\omega, \dot{\phi})) = 2m\Lambda\left(\left((\widehat{\omega}\boldsymbol{\phi}_\ell)\boldsymbol{\phi}_\ell^T\right)^{(A)}\right)^\vee,$$

or in $\mathbb{R}^3$

$$\mathbf{J}_\mathcal{L}((\Lambda, \phi), (\omega, \dot{\phi})) = m\Lambda\left(\boldsymbol{\phi}_\ell \times (\widehat{\omega}\boldsymbol{\phi}_\ell)\right).$$

The spring pendulum is invariant under the action of S^1 with respect to the vertical axis. The momentum map associated to this symmetry is

$$\mathbf{J}_\mathcal{L}((\Lambda, \phi), (\omega, \dot{\phi})) = m\Lambda\left(\boldsymbol{\phi}_\ell \times (\widehat{\omega}\boldsymbol{\phi}_\ell)\right) \cdot \mathbf{E}_3 \in \mathbb{R}.$$

Including external forces. In the presence of external forces, assumed to be fiber preserving maps $F_L : TQ \to T^*Q$, we modify Hamilton's principle to the Lagrange-d'Alembert principle, where one seeks curves satisfying

$$\delta \int_0^T L(\Lambda, \phi, \dot{\Lambda}, \dot{\phi})dt + \int_0^T F_L(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) \cdot (\delta\Lambda, \delta\phi)\, dt = 0,$$

with fixed end-points.

3.1.2 Lie group variational integrator

Spatial discretization. We discretize the string by a one-dimensional line element. The unstreched length of element is ℓ. A natural coordinate is defined by

$$\phi(s) = (1 - \frac{s}{\ell})\phi_0 + \frac{s}{\ell}\phi_\ell, \quad \text{where } s \in [0, \ell], \tag{3.1.3}$$

with $\phi_0 = 0$ and $\phi_\ell = \phi(\ell)$. The spatially discretized Lagrangian is obtained by inserting the variable considered in (3.1.3) in the continuous Lagrangian (3.1.1). We obtain

$$\mathcal{L}_{sp}(\Lambda, \phi_\ell, \omega, \dot{\phi}_\ell) = \frac{1}{2}m\left(\|\widehat{\omega}\boldsymbol{\phi}_\ell\|^2 + \|\dot{\boldsymbol{\phi}}_\ell\|^2\right) - \left\{\frac{\ell}{2}EA\left(\frac{\phi_\ell}{\ell} - 1\right)^2 - mg\Lambda\boldsymbol{\phi}_\ell \cdot \mathbf{E}_3\right\}. \tag{3.1.4}$$

Temporal discretization. For a given time step $\Delta t \in \mathbb{R}$, we construct the increasing sequence of time $\{t^j = j\Delta t \mid j = 0, ..., N\}$. The discrete time evolution of the center of mass of the pendulum is given by the discrete curve $\{(\Lambda^j, \boldsymbol{\phi}_1^j) \mid t^j \in \Theta\}$ in $SO(3) \times \mathbb{R}^3$. We consider the following interpolations over the time interval $[t^j, t^{j+1}]$:

$$\Lambda_h(t) := \Lambda^j \exp\left(\frac{t}{\Delta t}\widehat{\Psi}^j\right) \in SO(3),$$
$$\boldsymbol{\phi}_h(t) := \boldsymbol{\phi}_\ell^j + \frac{t}{\Delta t}\Delta\boldsymbol{\phi}_\ell^j \in \mathbb{R}^3,$$

where

$$\Delta\boldsymbol{\phi}_\ell^j := \boldsymbol{\phi}_\ell^{j+1} - \boldsymbol{\phi}_\ell^j \quad \text{and} \quad \exp(\widehat{\Psi}^j) := (\Lambda^j)^T\Lambda^{j+1}.$$

Note that we consistently have $\Lambda_h(\Delta t) = \Lambda^{j+1}$, and $\boldsymbol{\phi}_h(\Delta t) = \boldsymbol{\phi}_\ell^{j+1}$. We know by [24] that

$$(\Lambda_h(t))^{-1}\dot{\Lambda}_h(t) = \frac{\Psi^j}{\Delta t}, \qquad \text{for all } t \in [0, \Delta t].$$

Thus we get the following approximations of $\widehat{\omega} = (\Lambda)^T \dot{\Lambda}$ and $\dot{\boldsymbol{\phi}}$, at time t^j:

$$\mathfrak{so}(3) \ni \widehat{\omega}^j \approx \frac{\widehat{\Psi}^j}{\Delta t} \in \mathfrak{so}(3),$$
$$\mathbb{R}^3 \ni \dot{\boldsymbol{\phi}}_\ell^j \approx \frac{\Delta\boldsymbol{\phi}_\ell^j}{\Delta t} \in \mathbb{R}^3.$$

The temporal discretization, on the interval of time Δt, of the trivialized Lagrangian $\mathcal{L}_{sp}(\Lambda, \phi_\ell, \omega, \dot{\phi}_\ell)$, defined in (3.1.4), to be

$$\mathcal{L}_d(\Lambda^j, \phi_\ell^j, \Psi^j, \Delta\phi_\ell^j) = \frac{1}{2\Delta t}\left(\frac{\phi_\ell}{\ell}\right)^2 \operatorname{Tr}\left[\widehat{\Psi}^j J_d(\widehat{\Psi}^j)^T\right] + \frac{1}{2\Delta t}m(\Delta\phi_\ell^j)^2$$
$$- \Delta t\left\{\frac{\ell}{2}EA\left(\frac{|\phi_\ell^j|}{\ell} - 1\right)^2 - mg\phi_\ell^j\mathbf{E}_3^T\Lambda^j\mathbf{E}_3\right\}, \tag{3.1.5}$$

where $J_d = \frac{1}{2}\mathrm{Tr}(J)I_3 - J$ is the non standard inertia matrix, defined in terms of the inertia matrix J of a 3D-pendulum of length ℓ and mass m. We note that the discrete Lagrangian $\mathcal{L}_d^j$ may be written concisely like

$$\mathcal{L}_d(\Lambda^j, \phi_\ell^j, \Psi^j, \Delta\phi_\ell^j) = \frac{1}{2}m\Delta t\langle \mathbf{m}^j, \mathbf{m}^j\rangle - \Delta t\left\{\frac{\ell}{2}EA\left(\frac{|\phi_\ell^j|}{\ell} - 1\right)^2 - mg\langle\Lambda^j\boldsymbol{\phi}_\ell^j, \mathbf{E}_3\rangle\right\},$$

where

$$\mathbf{m}^j = \frac{\Delta\boldsymbol{\phi}_\ell^j}{\Delta t} + \frac{\widehat{\Psi}^j}{\Delta t}\boldsymbol{\phi}_\ell^j = \frac{\Delta\boldsymbol{\phi}_\ell^j}{\Delta t} + \frac{\Psi^j}{\Delta t}\times\boldsymbol{\phi}_\ell^j,$$

since we have

$$\left\langle\widehat{\Psi}^j\boldsymbol{\phi}_\ell^j, \Delta\boldsymbol{\phi}_\ell^j\right\rangle = \mathrm{Tr}\left[\widehat{\Psi}^j\boldsymbol{\phi}_\ell^j(\Delta\boldsymbol{\phi}_\ell^j)^T\right] = (\phi_\ell^j\Delta\phi_\ell^j)\mathrm{Tr}\left[\widehat{\Psi}^j\mathbf{E}_3\mathbf{E}_3^T\right] = 0.$$

Lie group variational integrator of the spring pendulum We approximate the exponential map by the Cayley transform for efficiency. We define $(F^j, H^j) \in SO(3)\times\mathbb{R}^3$ as follows

$$(\Lambda^j, \phi_\ell^j) \quad \text{and} \quad (F^j, H^j) := (\Lambda^j, \phi_\ell^j)^T(\Lambda^{j+1}, \phi_\ell^{j+1}) = \left((\Lambda^j)^T\Lambda^{j+1}, (\phi_\ell^{j+1} - \phi_\ell^j)\right).$$

For all the variations $\delta F^j = F^j\widehat{\xi}$ and $\delta\Lambda^j = \Lambda^j\widehat{\eta}$, with $\xi, \eta \in \mathbb{R}^3$. And given the variation of $\Psi^j = \mathrm{cay}^{-1}(F^j)$

$$\delta\widehat{\Psi}^j = \delta\ \mathrm{cay}^{-1}(F^j) = (2I - \widehat{\Psi}^j)\delta F^j(F^j + I)^{-1}, \quad \text{and} \quad \left(J_d\widehat{\Psi}^j\right)^{(A)} = \frac{1}{2}\widehat{J\Psi^j},$$

we obtain

$$\begin{aligned}
D_{F^j}\mathcal{L}_d^j\cdot\delta F^j &= -\frac{1}{2\Delta t}\left(\frac{\phi_\ell^j}{\ell}\right)^2\mathrm{Tr}\left[\left((F^j + I)^{-1}\widehat{J\Psi^j}(2I - \widehat{\Psi}^j)F^j)\right)^{(A)}\widehat{\xi}\right],\\
D_{\Lambda^j}\mathcal{L}_d^j\cdot\delta\Lambda^j &= \Delta t\ mg\phi_\ell^j\mathrm{Tr}\left[\left(\mathbf{E}_3\mathbf{E}_3^T\Lambda^j\right)^{(A)}\widehat{\eta}\right],\\
D_{\Delta\phi_\ell^j}\mathcal{L}_d^j &= \frac{m}{\Delta t}\Delta\phi_\ell^j,\\
D_{\phi_\ell^j}\mathcal{L}_d^j &= \frac{1}{\Delta t}\phi_\ell^j\mathrm{Tr}\left[\widehat{\Psi}^jJ_d(\widehat{\Psi}^j)^T\right]\\
&\quad - \Delta t\left\{\mathrm{sgn}(\phi_\ell^j)EA\left(\frac{|\phi_\ell^j|}{\ell} - 1\right) + mg\mathbf{E}_3^T\Lambda^j\mathbf{E}_3\right\}.
\end{aligned}$$

By (2.1.3), we get the discrete Euler-Lagrange equations, for $j = 1, ..., N-1$

$$\begin{cases} \frac{1}{\Delta t}\left(\frac{\phi_\ell^j}{\ell}\right)^2 \left(\left(F^j(F^j+I)^{-1}\widehat{J\Psi^j}(2I-\widehat{\Psi}^j)\right)^{(A)}\right)^\vee \\ -\frac{1}{\Delta t}\left(\frac{\phi_\ell^{j-1}}{\ell}\right)^2 \left(\left((F^{j-1}+I)^{-1}\widehat{J\Psi^{j-1}}(2I-\widehat{\Psi}^{j-1})F^{j-1}\right)^{(A)}\right)^\vee \\ \qquad = -2\Delta t\ mg\phi_\ell^j\left(\left(\mathbf{E}_3\mathbf{E}_3^T\Lambda^j\right)^{(A)}\right)^\vee, \\ \frac{m}{\Delta t}\Delta\phi_\ell^j - \frac{m}{\Delta t}\Delta\phi_\ell^{j-1} - \frac{1}{\Delta t}\phi_\ell^j \mathrm{Tr}\left[\widehat{\Psi}^j J_d(\widehat{\Psi}^j)^T\right] \\ \qquad = -\Delta t\left\{\mathrm{sgn}(\phi_\ell^j)EA\left(\frac{|\phi_\ell^j|}{\ell}-1\right) - mg\mathbf{E}_3^T\Lambda^j\mathbf{E}_3\right\}. \end{cases} \tag{3.1.6}$$

Remark about the computation. From equation (3.1.6) part 1, we consider the following equation

$$\widehat{b} = \left(\frac{\phi_\ell^j}{\ell}\right)^2 \left(F^j(F^j+I)^{-1}\widehat{J\Psi^j}(2I-\widehat{\Psi}^j)\right)^{(A)},$$

which may be written equivalently as following

$$B(\Psi^j) = J\Psi^j + \frac{1}{2}\left(\Psi^j \times (J\Psi^j)\right) + \frac{1}{2}\left\langle \Psi^j, (J\Psi^j)\right\rangle \frac{\Psi^j}{2} - \frac{\ell^2}{(\phi_1^j)^2} b = 0. \tag{3.1.7}$$

with the Jacobian $DA(\Psi^j)$ given by

$$DB(\Psi^j) = J + \frac{1}{2}\widehat{\Psi^j}J - \frac{1}{2}\widehat{J\Psi^j} + \frac{1}{2}\left\langle \frac{\Psi^j}{2}, (J\Psi^j)\right\rangle I + \frac{1}{2}\Psi^j(\Psi^j)^T J.$$

Thus to solve (3.1.7) we use Newton iterations.

Discrete momentum map. For $\xi = \theta\widehat{\mathbf{E}}_3 \in \mathfrak{so}(3)$ and the vertical S^1-symmetry associated. By (2.1.6) we obtain the spatial discrete momentum maps

$$\begin{aligned} &\mathbf{J}^+_{\mathcal{L}_d}\left((\Lambda^j, \phi_\ell^j), (F^j, \Delta\phi_\ell^j)\right) \\ &\quad = \left\langle \Lambda^{j+1}\left(\frac{1}{\Delta t}\left(\frac{\phi_\ell^j}{\ell}\right)^2 \mathbb{A}\right)^\vee, \mathbf{E}_3\right\rangle, \\ &\mathbf{J}^-_{\mathcal{L}_d}\left((\Lambda^j, \phi_\ell^j), (F^j, \Delta\phi_\ell^j)\right) \\ &\quad = \left\langle \Lambda^j\left(2\Delta t\ mg\phi_\ell^j\left(\mathbf{E}_3\mathbf{E}_3^T\Lambda^j\right)^{(A)} + \frac{1}{\Delta t}\left(\frac{\phi_\ell^j}{\ell}\right)^2 \mathbb{B}\right)^\vee, \mathbf{E}_3\right\rangle. \end{aligned}$$

where

$$\mathbb{A} = \left((F^j + I)^{-1} \widehat{J\Psi^j} (2I - \widehat{\Psi}^j) F^j) \right)^{(A)},$$
$$\mathbb{B} = \left(F^j (F^j + I)^{-1} \widehat{J\Psi^j} (2I - \widehat{\Psi}^j)) \right)^{(A)}$$

Discrete energy. For the given trivialized Lagrangian (3.1.1), the discrete energy at time t^{j+1}, as defined in (1.3.2), to be

$$\mathcal{E}_d \left((\Lambda^j, \phi_\ell^j), \mathbb{F}^+ \mathcal{L}_d^j \right) = \frac{(\phi_\ell^j)^2}{2\ell^2} \frac{\left((\mathbb{A})^\vee \right)^T}{\Delta t} J^{-1} \frac{(\mathbb{A})^\vee}{\Delta t} + \frac{1}{2} m \frac{(\Delta \phi_\ell^j)^T}{\Delta t} \frac{\Delta \phi_\ell^j}{\Delta t} + \left(\frac{\ell}{2} EA \left(\frac{|\phi_\ell^j|}{\ell} - 1 \right)^2 - mg \phi_\ell^j \mathbf{E}_3^T \Lambda^j \mathbf{E}_3 \right).$$

Discrete forcing. As described in (2.1.8), there exists a discrete version of the Lagrange-d'Alembert equations for discrete exterior forces $\mathcal{F}_d^\pm$.

Given the discrete Lagrangian $\mathcal{L}_d$ as defined in (3.1.5), we choose the fiber preserving forces as follows: for a given torque $\mathfrak{M}$ and force $\mathfrak{F}$, we set

$$\mathcal{F}_d^+ ((\Lambda^{j+1}, \phi_\ell^{j+1}), (F^j, H^j)) = (0, 0)$$
$$\mathcal{F}_d^- ((\Lambda^j, \phi_\ell^j), (F^j, H^j)) = \Delta t \left(\mathfrak{M}_a^- ((\Lambda^j, \mathbf{x}^j), (F^j, H^j)), \mathfrak{F}_a^- ((\Lambda^j, \phi_\ell^j), (F^j, H^j)) \right) \in T^*_{(\Lambda^j, \phi_\ell^j)} S0(3) \times \mathbb{R}.$$

3.2 Example

The physical constants for the spring pendulum are chosen as

$$m = 1 \text{ kg}, \quad \ell = 0.3 \text{ m}, \quad J = \frac{1}{(0.3)^2} \text{diag}[0.13, 0.28, 0.17] \text{ kgm}^2$$
$$A = (0.01)^2 \text{ m}^2.$$

We choose for the Young's modulus E the values $10^5, 10^7$ and 10^9. Which corresponds to the transition from a soft to a stiff spring, where the coefficient of the equivalent spring is defined as $k = \frac{EA}{\ell} = \frac{10^{-3}}{3} E$. The initial conditions are

$$(\Lambda^0, \phi_\ell^0) = (I, -\ell), \quad \omega_0 = 10^{-2} \cdot [4.14, 4, 14, 4.14], \quad \dot{\phi}_\ell^0 = 0.$$

We implement the system of equations (3.1.6) obtained via a Lie group integrator. For the first equation we have an implicit update to determine Λ^{j+1} at time t^{j+1}. For the second equation, we have an explicit update to determine ϕ_ℓ^{j+1} at time t^{j+1}. In order to solve the equation $F^j = (\Lambda^j)^{-1} \Lambda^{j+1} \in SO(3)$ we used a Newton iteration scheme, as described in (3.1.7).

About the results. When we reduce the time-step we get more informations about the trajectory, which are lost when the time-step increase.

In Fig.(3.2.1), we note the perfect conservation of the momentum map, that is, the conservation of the symmetry with respect to the vertical axis.

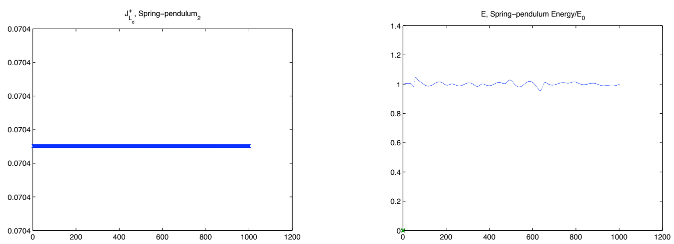


Figure 3.2.1: On the left : the discrete momentum map. On the right the discrete energy with respect to the initial energy. Young modulus $= 10^5$, time-step $=0.005$.

We observe that, as the spring becomes stiffer, the trajectory becomes identical to that of the spherical pendulum, and energy becomes more stable as seen in Fig.(3.2.2) and Fig.(3.2.3).

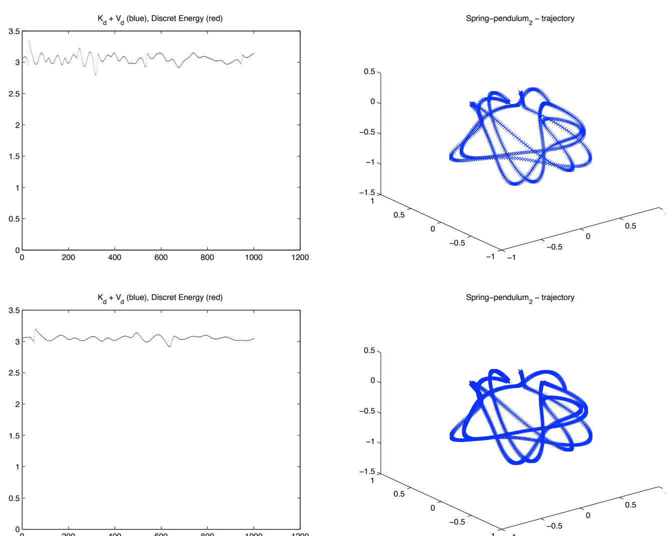


Figure 3.2.2: On the left : the discrete kinetic plus discrete potential (blue) and discrete Energy (red), on the right : the trajectory. From top to bottom : a) Young modulus $E = 10^5$, and time step $\Delta t = 0.01$; b) Young modulus $E = 10^5$, and time step $\Delta t = 0.005$.

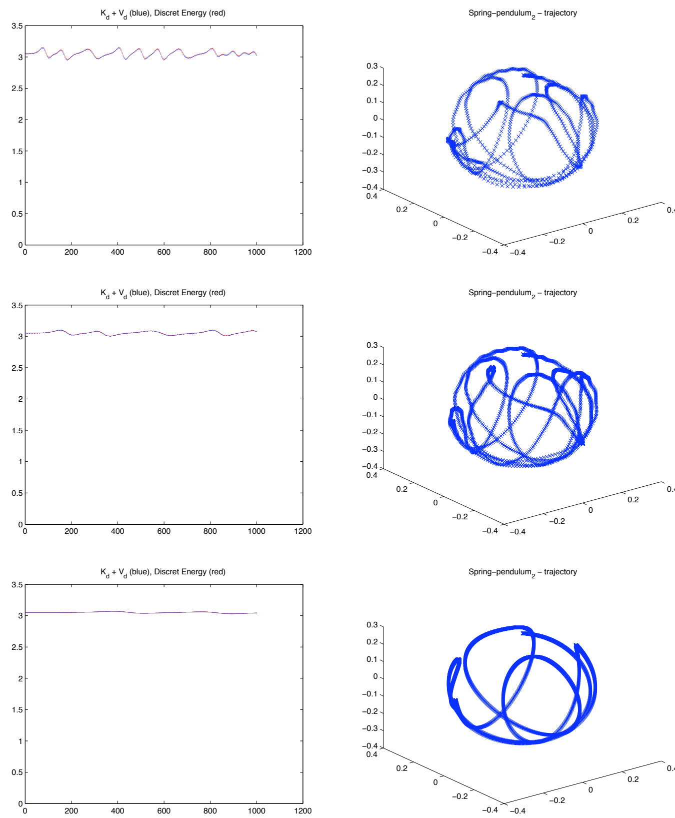


Figure 3.2.3: On the left : the discrete kinetic plus discrete potential (blue) and discrete Energy (red), on the right : the trajectory. From top to bottom : a) Young modulus $E = 10^7$, and time step $\Delta t = 0.01$;b) Young modulus $E = 10^7$, and time step $\Delta t = 0.005$; c) Young modulus $E = 10^9$, and time step $\Delta t = 0.002$.

Conclusion We have applied discrete mechanics and Lie group theory to an elastodynamics problem to develop of a symplectic variational integrator capable of conserving the symmetries and the energy in the absence of dissipation. This is impossible to achieve with classical methods.

The great advantage of this integrator is its symplectic nature. The consequences associated with non-symplecticity, are particularly devastating for a pendulum, because they cannot capture the periodic nature of the motion.

Chapter 4

Lie group variational integrator of geometrically exact beam dynamics

Introduction

The goal of this chapter is to derive a structure preserving integrator for geometrically exact beam dynamics. We use the Lagrangian variational formulation of the continuous problem to obtain a Lie group variational integrator that preserves the symmetries and symplectic structure at the discrete level. In addition, the algorithm exhibits almost-perfect energy conservation. The geometrically exact theory of beam dynamics was developed in Simo [107], Simo, Marsden, and Krishnaprasad [110]. This approach generalizes the formulation originally developed by Reissner [99; 100] for the plane static problem to the fully 3-dimensional dynamical case. It can be regarded as a convenient parametrization of a three-dimensional extension of the classical Kirchhoff-Love rod model due to Antmann [2]. The equations of motion of geometrically exact beams are obtained by applying Hamilton's principle to the Lagrangian (kinetic minus potential energy) defined in material representation and expressed uniquely in terms of convective variables (velocities and strains). In this paper, we take advantage of this geometric approach to deduce a discrete variational principle in convective representation, thereby obtaining a structure preserving integrator. The discretization is done both spatially and temporally in manner that preserves the geometric Lie group structure of the problem.

We derive a numerical scheme for the geometrically exact theory of beams by using variational integrators Marsden, West [90]. These integrators are based on a discrete variational formulation of the underlying system, e.g. based on a discrete version of Hamilton's principle for conservative mechanical systems. The resulting integrators given by the discrete Euler-Lagrange equations are symplectic and momentum-preserving and have an excellent long-time energy behavior.

In the case of the beam, the configuration Lie group is infinite dimensional. It contains the parametrization of the centroid line together with the orientation of cross-sections. In order to apply Lie group variational integrator to this case, we first spatially discretize the problem by preserving the Lie group structure.

Modelling geometrically exact beams as a special Cosserat continuum (see e.g. Antman [3]) has been the basis for many finite element formulations starting with the work of Simo [107]. The formulation of the beam dynamics as Lagrangian system immediately raises the question of the representation of the rotational degrees of freedom and their kinematics, which can on the one hand be treated by a local parametrization of the the Lie group $SO(3)$ or, on the other hand, by using a redundant configuration variable which is subject to constraints.

Many current semi-discrete beam formulations avoid the introduction of constraints by using rotational degrees of freedom, see e.g. Jelenic [51], Ibrahimbegović, and Mamouri [49]. However, it has been shown by Crisfield, and Jelenic [24], that the interpolation of non-commutative finite rotations bears the risk of destroying the objectivity of the strain measures in the semi-discrete model. This can be circumvented by the introduction of a director triad, which is constrained to be orthonormal in each node of the central line of the beam, thus it forms the columns of the rotational matrix. The spatial interpolation of the director triad leads to objective strain measures in the spatially discretised configuration (even though the interpolated directors might fail to be orthonormal). This idea is independently developed in Romero, and Armero [101] and Betsch, and Steinmann [8]. Romero [102] offers an overview on the effects of different interpolation techniques concerning frame invariance and the appearance of singularities. Furthermore, this subject is elaborated in Betsch, Menzel, and Stein [7], Ibrahimbegović, Frey, and Kozar [48], Jelenić, and Crisfield [52; 53], Bottasso, Borri, and Trainelli [13]. The constrained formulation is particularly popular when the beam is part of a multibody system, where further constraints representing the connection to other (rigid or flexible) components are naturally present. One formulation that is popular is the so called absolute nodal coordinates formulation based on works like Shabana [104], Shabana, and Yacoub [106]. Recently, Lie group formulations are becoming more and more important in multibody dynamics, see e.g. Brüls, and Cardona [17], and Brüls, Cardona, and Arnold [18]. To the author's knowledge, none of the present works on beam dynamics simulations uses a discrete dynamics approach which is variational both in time and in space. However, Jung, Leyendecker, Linn, and Ortiz [55] derives a purely static discrete equilibrium for Cosserat beams using a discrete variational principle in space.

4.1 Lagrangian dynamics of a beam in $\mathbb{R}^3$

4.1.1 Basic kinematics of a beam

We review below the kinematic description of a beam in the ambient space $\mathbb{R}^3$ following Simo [107], see also Simo, Marsden, and Krishnaprasad [110]. The static version of the beam model summarized below goes back essentially to Reissner [100] who modified the classical Kirchhoff-Love model to account for shear deformation.

The configuration of a beam is defined by specifying the position of its line of centroids by means of a map $\phi : [0, L] \to \mathbb{R}^3$ and the orientation of cross-sections at points $\phi(S)$ by means of a moving basis $\{\mathbf{d}_1(S), \mathbf{d}_2(S), \mathbf{d}_3(S)\}$ (sometimes called directors) attached to the cross section. The orientation of the moving basis is described with the help of an orthogonal transformation $\Lambda : [0, L] \to SO(3)$ such that

$$\mathbf{d}_I(S) = \Lambda(S)\mathbf{E}_I, \quad I = 1, 2, 3,$$

where $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a fixed basis referred to as the *material frame*. The configuration of the beam is thus completely determined by the maps ϕ and Λ in the configuration space

$$Q = C^\infty\left([0, L], SO(3) \times \mathbb{R}^3\right) \ni \Phi = (\Lambda, \phi).$$

If boundary conditions are imposed, then they need to be included in this configuration space. For example at $S = 0$, one can consider the boundary conditions $\phi(0) = 0$, $\Lambda(0) = Id$, that is, the point $\phi(0)$ of the centroid line is fixed (e.g stays at the origin) and the cross-section at the point $\phi(0)$ is fixed. One can also add the $\phi'(0) = \lambda\mathbf{E}_3$, for an arbitrary $\lambda > 0$, which means that the centroid line at $\phi(0)$ is orthogonal to the plan defined by $\mathbf{E}_1$ and $\mathbf{E}_2$. At the other extremity $S = L$ similar boundaries conditions can be imposed.

Figure 4.1.1: Illustration of the boundary conditions: $\Lambda(0) = Id,\ \phi(0) = 0$ (left), and $\Lambda(0) = Id,\ \phi(0) = 0,\ \phi'(0) = \mathbf{E}_3$ (right).

Suppose that the beam is in the configuration determined by $(\Lambda, \phi) \in Q$ and that its cross section is given by a compact subset $\mathcal{A} \subset \mathbb{R}^2$ with smooth boundary, then the set occupied by the beam is

$$\mathcal{B} = \left\{ X \in \mathbb{R}^3 \;\middle|\; X = \phi(S) + \sum_{\alpha=1}^{2} \xi^\alpha \Lambda(S)\mathbf{E}_\alpha, \text{ with } \left(\xi^1, \xi^2, S\right) \in \mathcal{A} \times [0, L] \right\}.$$

For simplicity, we assume that $\phi(S)$ is passing through the center of mass of the cross section $\mathcal{A}$.

The time-evolution of the beam is described by a curve $\Phi(t) = (\Lambda(t), \phi(t)) \in Q$, in the configuration space. The *material velocity* V_Φ is defined by

$$V_\Phi(S,t) = \frac{d}{dt}\Phi(S,t) = \left(\dot{\Lambda}(S,t), \dot{\phi}(S,t)\right),$$

and thus belongs to the tangent space $T_\Phi Q$ to Q at Φ. Before defining the convective angular and linear velocities, we first recall some notations concerning the Lie algebra of $SO(3)$.

Notations. We denote by $\mathfrak{so}(3)$ the Lie algebra of $SO(3)$ consisting of skew symmetric matrices endowed with the Lie bracket $[A,B] = AB - BA$. The adjoint representation of $SO(3)$ on its Lie algebra is given by by $\operatorname{Ad}_\Lambda A = \Lambda A \Lambda^{-1}$, where $A \in \mathfrak{so}(3)$ and $\Lambda \in SO(3)$. We have the identity $\operatorname{Ad}_A \widehat{\Omega} = \widehat{A\Omega}$. ♦

The *convected angular velocity* and *convected linear velocity* are the $\mathbb{R}^3$-valued map $\omega, \gamma : [0,L] \to \mathbb{R}^3$ defined by

$$\widehat{\omega} := \Lambda^T \dot{\Lambda} \quad \text{and} \quad \gamma := \Lambda^T \dot{\phi}. \tag{4.1.1}$$

4.1.2 Kinetic energy

The kinetic energy is found by integrating the kinetic energy of the material points over the whole body. Given $\mathcal{D} = [0,L] \times \mathcal{A}$, we have

$$\begin{aligned} T(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) &= \frac{1}{2}\int_{\mathcal{D}} \left\|\dot{\phi} + \xi^1 \dot{\mathbf{d}}_1 + \xi^2 \dot{\mathbf{d}}_2\right\|^2 \rho(S,\xi^1,\xi^2) dS d\xi^1 d\xi^2 \\ &= \frac{1}{2}\int_{\mathcal{D}} \left\|\dot{\phi} + \Lambda\widehat{\omega}\left(\xi^1\mathbf{E}_1 + \xi^2\mathbf{E}_2\right)\right\|^2 \rho(S,\xi^1,\xi^2) dS d\xi^1 d\xi^2 \\ &= \frac{1}{2}\int_{\mathcal{D}} \left[\left\|\dot{\phi}\right\|^2 + \left\|\widehat{\omega}\left(\xi^1\mathbf{E}_1 + \xi^2\mathbf{E}_2\right)\right\|^2\right] \rho(S,\xi^1,\xi^2) dS d\xi^1 d\xi^2, \end{aligned}$$

where $\rho(S)$ is the mass density and where we used the fact that the mid-line ϕ passes through the center of mass, i.e.

$$\iint_{\mathcal{A}} \left(\xi^1\mathbf{E}_1 + \xi^2\mathbf{E}_2\right) \rho(S,\xi^1,\xi^2) d\xi^1 d\xi^2 = 0.$$

For simplicity, we assume that $\rho(S) = \rho_0$ is a constant. Using the relation $\widehat{A}B = -\widehat{B}A$ we get

$$T(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) = \frac{1}{2}\int_0^L \left[M\left\|\dot{\phi}\right\|^2 + \omega^T J \omega\right] dS \tag{4.1.2}$$

where $M = \rho_0 \times \operatorname{area}(\mathcal{A})$ is the distributed loads per unit length, and J is the inertia matrix in the body fixed frame defined as

$$J = -\int_{\mathcal{A}} \rho_0 \left(\widehat{\xi^1\mathbf{E}_1 + \xi^2\mathbf{E}_2}\right)^2 d\xi^1 d\xi^2.$$

Note that the kinetic energy can also we written explicitly in terms of $\widehat{\omega}$ as

$$T(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) = \frac{1}{2} \int_0^L \left[M \left\| \dot{\phi} \right\|^2 + \operatorname{Tr}\left(\widehat{\omega}^T J_d \widehat{\omega} \right) \right] dS \tag{4.1.3}$$

where the (non-standard) inertia matrix J_d is defined by

$$J_d := \int_{\mathcal{A}} \rho_0 \left(\xi^1 \mathbf{E}_1 + \xi^2 \mathbf{E}_2 \right) \left(\xi^1 \mathbf{E}_1 + \xi^2 \mathbf{E}_2 \right)^T d\xi^1 d\xi^2.$$

Note that the kinetic energy T is $SO(3)$-left invariant, since

$$(\Psi\Lambda)^T \partial_t (\Psi\Lambda) = \Lambda^T \partial_t \Lambda, \quad \text{and} \quad (\Psi\Lambda)^T \partial_t (\Psi\phi) = \Lambda^T \partial_t \phi, \quad \text{for all } \Psi \in SO(3).$$

4.1.3 Potential energy

The potential energy is given by the sum of interior potential energy (bending energy) and exterior potential energy (gravitational energy and energy created by external force and torque).

Bending energy. Given a configuration $(\Lambda, \phi) \in Q$, the *deformation gradient* is defined as

$$F(S,t) = \Phi'(S,t) := \left(\Lambda'(S,t), \phi'(S,t) \right), \quad \text{where} \quad (.)' := \frac{\partial}{\partial S}.$$

As in [107], we use the convective variables $\Omega, \Gamma : [0, L] \to \mathbb{R}^3$ defined by

$$\widehat{\Omega} := \Lambda^T \Lambda' \quad \text{and} \quad \Gamma := \Lambda^T \phi'. \tag{4.1.4}$$

The bending energy is assumed to depend on the deformation gradient only through the quantities Ω and Γ, that is, we have

$$\Pi_{int}(\Lambda, \phi) = \int_0^L \Psi_{int}(\Omega, \Gamma) dS,$$

where $\Psi_{int}(\Omega, \Gamma)$ is the stored energy function.

We assume that the unstressed state is undeformed. That is, we have $\phi'(S, t = 0) = \mathbf{E}_3$ and $\Lambda(S, t = 0) = Id$, for all $S \in [0, L]$. Also by considering that the thickness of the rod is small compared to its length, and that the material is homogeneous and isotropic, we can interpret, as in Simo, and Vu-Quoc [111] and Dichmann, Li, and Maddocks [26], the stored energy by the following quadratic model

$$\Psi_{int}(\Omega, \Gamma) = \frac{1}{2} \left((\Gamma - \mathbf{E}_3)^T \quad \Omega^T \right) \operatorname{Diag}\left(GA_1 \;\; GA_2 \;\; EA \;\; EI_1 \;\; EI_2 \;\; GJ \right) \begin{pmatrix} \Gamma - \mathbf{E}_3 \\ \Omega \end{pmatrix},$$

where the elastic coeficients are $GA_1, GA_2, EA, EI_1, EI_2, GJ$, with $A_1 = A_2 = A$, $J = I_1 + I_2$. Here $A = \text{area}(\mathcal{A})$ is the cross-sectional area of the rod, I_1

and I_2 are the principal moments of inertia of the cross-section while J is its polar moment of inertia, E is Young's modulus, $G = E/[2(1+\nu)]$ is the shear modulus, and ν is Poisson's ratio.

With this stored energy function the internal energy may be written as

$$\Pi_{int}(\Lambda, \phi) = \frac{1}{2}\int_0^L \left[(\Gamma - \mathbf{E}_3)^T \mathbf{C}_1 (\Gamma - \mathbf{E}_3) + \Omega^T \mathbf{C}_2 \Omega\right] dS, \tag{4.1.5}$$

where we defined the matrices

$$\mathbf{C}_1 := \mathrm{Diag}\,(GA_1 \;\; GA_2 \;\; EA) \quad \text{and} \quad \mathbf{C}_2 := \mathrm{Diag}\,(EI_1 \;\; EI_2 \;\; GJ)\,.$$

Note that the internal energy is invariant under the left action of elements of $SE(3)$.[1]

Exterior potential energy. We consider the potential energy

$$\Pi_{ext}(\phi) = \int_0^L \langle \mathbf{q}, \phi \rangle \, dS, \tag{4.1.6}$$

created by the exterior conservative forces $\mathbf{q}$ per unit length.

Stresses. The stresses along the beam are defined by

$$\mathbf{n} := \frac{\partial \Psi_{int}}{\partial \Gamma} = \mathbf{C}_1(\Gamma - \mathbf{E}_3), \tag{4.1.7}$$

where the $\mathbf{E}_1$- and $\mathbf{E}_2$-components are the shear stresses and the $\mathbf{E}_3$-component is the stretch stress. The momenta along the beam are defined by

$$\mathbf{m} := \frac{\partial \Psi_{int}}{\partial \Omega} = \mathbf{C}_2 \Omega, \tag{4.1.8}$$

where the $\mathbf{E}_1$- and $\mathbf{E}_2$-components are the bending momenta, with respect to the principal axes of the cross-section, and the $\mathbf{E}_3$-component is the torsional moment.

4.1.4 Equation of motion

We now derive the equations of motion by computing the Euler-Lagrange equations associated to the Lagrangian of the beam $L : TQ \to \mathbb{R}$ given by

$$\begin{aligned} L(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) =& \frac{1}{2}\int_0^L \left[M \left\|\dot{\phi}\right\|^2 + \omega^T J \omega\right] dS \\ & - \frac{1}{2}\int_0^L \left[(\Gamma - \mathbf{E}_3)^T \mathbf{C}_1 (\Gamma - \mathbf{E}_3) + \Omega^T \mathbf{C}_2 \Omega\right] dS - \int_0^L \langle \mathbf{q}, \phi \rangle \, dS, \end{aligned} \tag{4.1.9}$$

[1] For all rigid motions of (Λ, ϕ) given by the transformation $(\widetilde{\Lambda}, \widetilde{\phi}) = (R\Lambda, \mathbf{v} + R\phi)$, where $R \in SO(3)$, and $\mathbf{v} \in \mathbb{R}^3$, we have $\Pi_{int}(\widetilde{\Lambda}, \widetilde{\phi}) = \Pi_{int}(\Lambda, \phi)$, since

$$\left(\widetilde{\Lambda}\right)^T \left(\widetilde{\Lambda}\right)' = \Lambda^T \Lambda', \quad \text{and} \quad \left(\widetilde{\Lambda}\right)^T \left(\widetilde{\phi}\right)' = \Lambda^T \phi', \quad \text{for all } R \text{ and } \mathbf{v}.$$

where we recall that $\widehat{\omega} = \Lambda^T \dot{\Lambda}$, $\widehat{\Omega} = \Lambda^T \Lambda'$ and $\Gamma = \Lambda^T \phi'$.

The Euler-Lagrange equations are obtained by applying Hamilton's principle to the action

$$\mathfrak{S}(\Lambda, \phi) = \int_{t_0}^{t_1} L\left(\Lambda(t), \phi(t), \dot{\Lambda}(t), \dot{\phi}(t)\right) dt.$$

Consider variations $\varepsilon \mapsto (\Lambda_\varepsilon, \phi_\varepsilon)$ of the curves (Λ, ϕ) with fixed endpoints. The infinitesimal variations are denoted by

$$\delta\Lambda = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Lambda_\varepsilon, \quad \delta\phi = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \phi_\varepsilon$$

and vanish at the endpoints. Since the Lagrangian is expressed in terms of the auxiliary variables ω, Ω, and Γ it is useful to compute the variations $\delta\omega$, $\delta\Omega$ and $\delta\Gamma$ induced by the variations $\delta\Lambda$ and $\delta\phi$. A direct computation shows that we have

$$\begin{aligned} \delta\omega &= \dot{\eta} + \omega \times \eta = \dot{\eta} + \widehat{\omega}\eta \\ \delta\Omega &= \eta' + \Omega \times \eta = \eta' + \widehat{\Omega}\eta \\ \delta\Gamma &= \Lambda^T \delta\phi' + \Gamma \times \eta, \end{aligned} \tag{4.1.10}$$

where $\eta = \Lambda^T \delta\Lambda$. We compute

$$\begin{aligned} \delta\mathfrak{S} = \int_{t_0}^{t_1} \Bigg[& \int_0^L \left(M\dot{\phi}^T \left(\delta\dot{\phi}\right) + \omega^T J \delta\omega \right) dS - \int_0^L \left((\Gamma - \mathbf{E}_3)^T \mathbf{C}_1 \delta\Gamma + \Omega^T \mathbf{C}_2 \delta\Omega \right) dS \\ & - \int_0^L \mathbf{q}^T \delta\phi \, dS \Bigg] dt. \end{aligned}$$

Taking into account the expressions for $\delta\omega$, $\delta\Gamma$, $\delta\Omega$ in (4.1.10), we isolate the quantities η, $\delta\phi$ by integrating by parts and obtain

$$\begin{aligned} \int_{t_0}^{t_1} \Bigg[& \int_0^L \left(-M\ddot{\phi}^T \delta\phi + \left(-\dot{\omega}^T J + \omega^T J \widehat{\omega} \right) \eta \right) dS \\ & + \int_0^L \left((\Gamma - \mathbf{E}_3)^T \mathbf{C}_1 \Lambda^T \right)' \delta\phi \, dS - \left[(\Gamma - \mathbf{E}_3)^T \mathbf{C}_1 \Lambda^T \delta\phi \right]_0^L \\ & - \int_0^L \left((\mathbf{C}_1(\Gamma - \mathbf{E}_3) \times \Gamma)^T + \Omega^T \mathbf{C}_2 \widehat{\Omega} - (\Omega^T \mathbf{C}_2)' \right) \eta dS - \left[\Omega^T \mathbf{C}_2 \eta \right]_0^L - \int_0^L \mathbf{q}^T \, \delta\phi dS \Bigg] dt. \end{aligned}$$

We thus obtain the Euler-Lagrange equations

$$\begin{cases} J\dot{\omega} + \omega \times J\omega + \mathbf{C}_1(\Gamma - \mathbf{E}_3) \times \Gamma - \Omega \times \mathbf{C}_2 \Omega - \mathbf{C}_2 \Omega' = 0 \\ M\ddot{\phi} - (\Lambda \mathbf{C}_1(\Gamma - \mathbf{E}_3))' + \mathbf{q} = 0 \end{cases} \tag{4.1.11}$$

with boundary conditions

$$\begin{cases} (\Gamma - \mathbf{E}_3)|_{S=0} = 0 \\ (\Gamma - \mathbf{E}_3)|_{S=L} = 0 \\ \Omega(0) = \Omega(L) = 0. \end{cases} \tag{4.1.12}$$

The condition $(\Gamma - \mathbf{E}_3)|_{S=0} = 0$ means that the mid-line remains orthogonal to the cross section at point $\phi(0)$ at all time. The condition $\Omega(0) = 0$ means there is no bending or torsion at the end boundary.

We note that the equations (4.1.11) can be written as follows

$$\begin{cases} J\dot{\omega} + \omega \times J\omega + \mathbf{n} \times \Gamma - \Omega \times \mathbf{m} - \mathbf{m}' = 0, \\ M\ddot{\phi} - \Lambda(\Omega \times \mathbf{n}) - \Lambda\mathbf{n}' + \mathbf{q} = 0, \end{cases}$$

where $\mathbf{m}$ and $\mathbf{n}$ are the stresses and the momenta as defined in (4.1.7) and (4.1.8). These are the statements of balance of angular, mass and linear momentum in the convective description, as in Simo, Marsden, and Krishnaprasad [110].

Including external forces and torques. In presence of external forces $F : TQ \to T^*Q$ the equations of motion are given by the Lagrange-d'Alembert principle

$$\delta \int_{t_0}^{t_1} L\left(\Lambda(t), \phi(t), \dot{\Lambda}(t), \dot{\phi}(t)\right) dt + \int_{t_0}^{t_1} F\left(\Lambda(t), \phi(t), \dot{\Lambda}(t), \dot{\phi}(t)\right) \cdot (\delta\Lambda(t), \delta\phi(t))\, dt = 0.$$

Writing the force as

$$F\left(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}\right) = \left(\mathfrak{M}\left(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}\right), \mathfrak{F}\left(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}\right)\right) \tag{4.1.13}$$

we get the forced Euler-Lagrange equations

$$\begin{cases} J\dot{\omega} + \omega \times J\omega + \mathbf{C}_1(\Gamma - \mathbf{E}_3) \times \Gamma - \Omega \times \mathbf{C}_2\Omega - \mathbf{C}_2\Omega' = \Lambda^{-1}\mathfrak{M} \\ M\ddot{\phi} - (\Lambda\mathbf{C}_1(\Gamma - \mathbf{E}_3))' + \mathbf{q} = \mathfrak{F} \end{cases}$$

We observe that these equations are the Euler-Lagrange equations with forcing term added. Note that different kinds of forcing is possible like dead loads, configuration-dependent follower forces or velocity-dependent dissipative forces.

4.2 Lie group variational integrator for the beam

4.2.1 The Lie group structure and trivialization

The goal of this section is to develop a Lie group variational integrator for the beam. This can be done by identifying the configuration space Q of the beam with the infinite dimensional Lie group $G = C^\infty([0, L], SE(3))$, with group multiplication given by pointwise multiplication in the group $SE(3)$, that is

$$(\Lambda_1, \phi_1)(\Lambda_2, \phi_2) = (\Lambda_1\Lambda_2, \phi_1 + \Lambda_1\phi_2).$$

Recall that the inverse of an element is $(\Lambda, \phi)^{-1} = \left(\Lambda^{-1}, -\Lambda^{-1}\phi\right)$ and that the tangent lift of left translation reads $(\Lambda_1, \phi_1)(\dot{\Lambda}_2, \dot{\phi}_2) = \left(\Lambda_1\dot{\Lambda}_2, \Lambda_1\dot{\phi}_2\right)$, so that the convective velocity can be written as

$$(\hat{\omega}, \gamma) = (\Lambda, \phi)^{-1}(\dot{\Lambda}, \dot{\phi}). \tag{4.2.1}$$

It is important to observe that, in this setting, if boundary conditions have to be imposed on the configuration space, they have to preserve the group structure. For example, both boundary conditions considered in Fig. 4.1.1 preserve the group structure of G.

Trivialized Euler-Lagrange equations on Lie groups. When we consider below the spatial discretization of the Lagrangian, it will be convenient to refer to its trivialized expression.

Recall that, given a Lagrangian $L = L(g,\dot{g}) : TG \to \mathbb{R}$ defined on the tangent bundle of a Lie group G, its trivialized expression $\mathcal{L} = \mathcal{L}(g,\xi) : G \times \mathfrak{g} \to \mathbb{R}$ is defined by

$$\mathcal{L}(g,\xi) := L(g,\dot{g}), \quad \text{where} \quad \dot{g} := g\xi.$$

The Euler-Lagrange equations for L, written in terms of $\mathcal{L}$, read

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \xi}\right) - \mathrm{ad}^*_\xi \frac{\partial \mathcal{L}}{\partial \xi} = g^{-1}\frac{\partial \mathcal{L}}{\partial g},$$

as a direct computation shows. In the G-invariant case, we have $\partial \mathcal{L}/\partial g = 0$, so that these equations recover the Euler-Poincaré equations, see [89].

The trivialized Lagrangian of the beam. In the case of the beam, the Lie group is $G = C^\infty([0,L], SE(3))$, therefore, we have $g = (\Lambda, \phi)$ and $\xi = g^{-1}\dot{g} = (\Lambda,\phi)^{-1}(\dot{\Lambda},\dot{\phi}) = (\widehat{\omega},\gamma)$ and the trivialized expression associated to the Lagrangian (4.1.9) reads

$$\mathcal{L}(\Lambda,\phi,\widehat{\omega},\gamma) = \frac{1}{2}\int_0^L M\left\|\gamma\right\|^2 dS + \frac{1}{2}\int_0^L \omega^T J\omega \; dS - \Pi_{int}(\Lambda,\phi) - \Pi_{ext}(\phi). \tag{4.2.2}$$

4.2.2 Spatial discretization

Discretization of the variables. We decompose the interval $[0,L]$ into N subintervals $K = [S_a, S_{a+1}]$ of length $l_K = S_{a+1} - S_a$. We denote by $\mathcal{T}$ the set of all elements K that subdivide the interval $[0,L]$. The configuration of the beam at the node a is given by $\Lambda_a := \Lambda(S_a)$ and $\mathbf{x}_a = \phi(S_a)$.

Given the configurations $(\Lambda_a, \mathbf{x}_a)$ and $(\Lambda_{a+1}, \mathbf{x}_{a+1})$ of the beam at the nodes a and $a+1$, we consider the following interpolations over the subinterval K

$$\Lambda_h(S)|_K := \Lambda_a \exp\left(\frac{S}{l_K}\widehat{\psi}_a\right) \qquad \text{and} \qquad \phi_h(S)|_K := \mathbf{x}_a + \frac{S}{l_K}\Delta\mathbf{x}_a, \tag{4.2.3}$$

where $S \in [0, l_K]$, and

$$\Delta\mathbf{x}_a := \mathbf{x}_{a+1} - \mathbf{x}_a \quad \text{and} \quad \exp(\widehat{\psi}_a) := \Lambda_a^T\Lambda_{a+1}. \tag{4.2.4}$$

Note that we consistently have $\Lambda_h(l_K) = \Lambda_{a+1}$ and $\mathbf{x}_h(l_K) = \mathbf{x}_{a+1}$. This interpolation was considered by Crisfield, and Jelenic [24] in order to obtain a spatial discretization that preserves the objectivity of the strain measures Ω and Γ [2]. Note that, for simplicity, we parametrize the element K using $S \in [0, l_K]$ instead of $S \in [S_a, S_{a+1}]$.

The associated convected variables $\widehat{\omega}_h(S), \gamma_h(S), \widehat{\Omega}_h(S)$, and $\Gamma_h(S)$ are obtained by using the approximations $\phi_h(S)$ and $\Lambda_h(S)$ instead of the original variables $\phi(S)$ and $\Lambda(S)$ in their definitions. We thus get

$$\begin{aligned}
\widehat{\omega}_h(S) &= \Lambda_h(S)^T \dot{\Lambda}_h(S), \\
\gamma_h(S) &= \Lambda_h(S)^T \dot{\phi}_h(S) = \Lambda_h(S)^T \left(\dot{\mathbf{x}}_a + S \Delta \dot{\mathbf{x}}_a / l_K \right), \\
\widehat{\Omega}_h(S) &= \Lambda_h(S)^T \Lambda_h'(S) = \widehat{\psi}_a / l_K, \\
\Gamma_h(S) &= \Lambda_h(S)^T \phi_h'(S) = \Lambda_h(S)^T \Delta \mathbf{x}_a / l_K.
\end{aligned} \tag{4.2.5}$$

Note that by considering $S = 0$ and $S = l_K$, we obtain that at each node the relation (4.2.1) between the material and convected velocities is preserved, that is,

$$\omega_a = \Lambda_a^T \dot{\Lambda}_a, \;\; \omega_{a+1} = \Lambda_{a+1}^T \dot{\Lambda}_{a+1} \quad \text{and} \quad \gamma_a = \Lambda_a^T \dot{\mathbf{x}}_a, \;\; \gamma_{a+1} = \Lambda_{a+1}^T \dot{\mathbf{x}}_{a+1}.$$

We use the notation $\Lambda_K = (\Lambda_a, \Lambda_{a+1})^T$, $\mathbf{x}_K = (\mathbf{x}_a, \mathbf{x}_{a+1})^T$, and similarly for $\dot{\Lambda}_K$, $\dot{\mathbf{x}}_K$, ω_K, γ_K, to denote the variables associated to an element K with nodes a and $a+1$.

The boundary conditions in Fig. 4.1.1 are given by $\Lambda_{a_0} = Id$, $\mathbf{x}_{a_0} = 0$, or $\Lambda_{a_0} = Id$, $\mathbf{x}_{a_0} = 0$, $\Delta \mathbf{x}_{a_0} = \lambda \mathbf{E}_3$, $\lambda > 0$.

The discrete Lagrangian. The spatially discretized Lagrangian is obtained by inserting the variables considered in (4.2.5) and Λ_h, Φ_h in the continuous Lagrangian (4.2.2) and by considering approximations.

For the kinetic energy, we make the following approximations on an element K of length l_K:

$$\frac{1}{2} \int_0^{l_K} M \|\gamma_h(S)\|^2 dS \approx \frac{l_K}{4} M \left(\|\gamma_a\|^2 + \|\gamma_{a+1}\|^2 \right),$$
$$\frac{1}{2} \int_0^{l_K} \left(\omega_h(S)^T J \omega_h(S) \right) dS \approx \frac{l_K}{4} \left(\omega_a^T J \omega_a + \omega_{a+1}^T J \omega_{a+1} \right).$$

Concerning the potential energy, the expression obtained by using Λ_h and ϕ_h instead of Λ and ϕ is denoted by

$$V_K(\Lambda_K, \mathbf{x}_K) := \int_K \mathcal{V}_h(S) dS,$$

[2]Consider a rigid motion of (Λ, ϕ) given by the transformation $(\widetilde{\Lambda}, \widetilde{\phi}) = (R\Lambda, \mathbf{v} + R\phi)$, where $R \in SO(3)$, and $\mathbf{v} \in \mathbb{R}^3$. Since $(\widetilde{\Lambda}_a)^T \widetilde{\Lambda}_{a+1} = \Lambda_a^T \Lambda_{a+1}$ and $\widetilde{\Lambda}_a^T \Delta \widetilde{\mathbf{x}}_a = \Lambda_a^T \Delta \mathbf{x}_a$, the strain measures are unchanged by this transformation.

where $\mathcal{V}_h(S) := \Psi_{int}\left(\Lambda_h(S), \phi_h(S)\right) + \Psi_{ext}(\phi_h(S))$, see (4.1.5) and (4.1.6). We approximate the potential energy V_K with the expression $\mathbb{V}_K$ defined by

$$\begin{aligned}\mathbb{V}_K(\Lambda_K, \mathbf{x}_K) &:= \frac{l_K}{2}\left(\mathcal{V}_h(0) + \mathcal{V}_h(l_K)\right)\\ &= \frac{l_K}{4}\left[\left(\Lambda_a^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1 \left(\Lambda_a^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right)\right.\\ &\quad \left.+ \left(\Lambda_{a+1}^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1 \left(\Lambda_{a+1}^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right) + \frac{2}{(l_K)^2}(\psi_a)^T \mathbf{C}_2 \psi_a\right]\\ &\quad + \frac{l_K}{2}\left\langle \mathbf{q}_a, \mathbf{x}_a \right\rangle + \frac{l_K}{2}\left\langle \mathbf{q}_{a+1}, \mathbf{x}_{a+1} \right\rangle,\end{aligned} \tag{4.2.6}$$

where we recall that $\Delta \mathbf{x}_a = \mathbf{x}_{a+1} - \mathbf{x}_a$, and $\widehat{\psi}_a = \exp^{-1}(\Lambda_a^T \Lambda_{a+1})$. In this last expression, the exponential map can be approximated by the Cayley transform $\mathrm{cay} : \mathfrak{g} \to G$ defined by

$$\Lambda = \mathrm{cay}\left(\widehat{\Omega}\right) = \left(I - \widehat{\Omega}/2\right)^{-1}\left(I + \widehat{\Omega}/2\right)$$

with inverse

$$\widehat{\Omega} = \mathrm{cay}^{-1}(\Lambda) = 2(\Lambda - I)(\Lambda + I)^{-1}.$$

As a consequence, the spatially discretized Lagrangian $L_K : TSE(3)^2 \to \mathbb{R}$ and its trivialized form $\mathcal{L}_K : SE(3)^2 \times \mathfrak{se}(3)^2 \to \mathbb{R}$, over an element K of length l_K, are given by

$$\begin{aligned}L_K(\Lambda_K, \mathbf{x}_K, \dot{\Lambda}_K, \dot{\mathbf{x}}_K) &= \frac{l_K}{4} M\left(\|\dot{\mathbf{x}}_a\|^2 + \|\dot{\mathbf{x}}_{a+1}\|^2\right)\\ &\quad + \frac{l_K}{4}\left(\omega_a^T J \omega_a + \omega_{a+1}^T J \omega_{a+1}\right)\\ &\quad - \mathbb{V}_K(\mathbf{x}_K, \Lambda_K) = \mathcal{L}_K(\Lambda_K, \mathbf{x}_K, \widehat{\omega}_K, \gamma_K).\end{aligned} \tag{4.2.7}$$

The spatial discrete Lagrangian L of the total system is obtained by summing over all the elements K, that is $L = \sum_{K \in \mathcal{T}} L_K$. Assuming that all elements K have initially the same undeformed length l_K and taking care of boundary notes a_0 and a_N, we get

$$\begin{aligned}L\left((\Lambda_a, \mathbf{x}_a, \dot{\Lambda}_a, \dot{\mathbf{x}}_a)_{a \in \mathcal{N}}\right) &= \sum_{a \in int(\mathcal{N})}\left(\frac{l_K}{2} M \|\dot{\mathbf{x}}_a\|^2 + \frac{l_K}{2} \omega_a^T J \omega_a\right)\\ &\quad + \sum_{a \in \partial\mathcal{N}}\left(\frac{l_K}{4} M \|\dot{\mathbf{x}}_a\|^2 + \frac{l_K}{4} \omega_a^T J \omega_a\right)\\ &\quad - \sum_{K \in \mathcal{T}} \mathbb{V}_K(\mathbf{x}_K, \Lambda_K),\end{aligned} \tag{4.2.8}$$

where $\mathcal{N}$ denotes the set of all nodes, $\partial\mathcal{N} = \{a_0, a_N\}$ is the set of boundary nodes, and $int(\mathcal{N}) = \{a_1, ..., a_{N-1}\}$ denotes the set of internal nodes.

Discrete stresses. Similar to with (4.1.7) and (4.1.8), the discrete stresses along the beam are defined by

$$\mathbf{n}_h := \frac{\partial \mathcal{V}_h}{\partial \Gamma_h} = \mathbf{C}_1(\Gamma_h - \mathbf{E}_3).$$

Given an element K, the associated discrete stress is defined by

$$\mathbf{n}_K := \frac{1}{2}(\mathbf{n}_h(0)+\mathbf{n}_h(l_K)) = \frac{1}{2}\left(\mathbf{C}_1\left(\Lambda_a\frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right) + \mathbf{C}_1\left(\Lambda_{a+1}\frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right)\right) \tag{4.2.9}$$

The discrete momenta along the beam are defined by

$$\mathbf{m}_h := \frac{\partial \mathcal{V}_h}{\partial \Omega_h} = \mathbf{C}_2\Omega_h.$$

As before, the discrete momenta associated with K read

$$\mathbf{m}_K := \frac{1}{2}(\mathbf{m}_h(0) + \mathbf{m}_h(l_K)) = \mathbf{C}_2\psi_a/l_K. \tag{4.2.10}$$

4.2.3 Lie group variational integrators

The Lie group variational integrators have already been presented in §(2.1.1), but in a slightly different way. In particular we calculated the discrete momentum map using the discrete one forms $\theta^{\pm}_{L_d}$.

Discrete Euler-Lagrange equations on Lie groups

Lie group variational integrators, originated in the work of Moser, and Veselov [93], were developed in Bobenko, and Suris [11; 12], Marsden, Pekarsky, and Shkoller [86] for the numerical treatment of mechanical systems on finite dimensional Lie groups, by using a discrete analogue of Lagrangian reduction. These methods were further developed and exploited in Lee [66]. In this approach, given a Lagrangian $L : TG \to \mathbb{R}$ defined on the tangent bundle TG of a Lie group G, the discrete trivialized Lagrangian $\mathcal{L}_d(g^j, f^j) : G \times G \to \mathbb{R}$ is defined such that the following approximation holds:

$$\mathcal{L}_d(g^j, f^j) \approx \int_{t^j}^{t^{j+1}} L(g(t), \dot{g}(t))dt,$$

were $g(t)$ is the solution of the Euler-Lagrange equations with $g(t^j) = g^j$ and $g(t^{j+1}) = g^{j+1} = g^j f^j$. For simplicity we use the notation $\mathcal{L}^j_d := \mathcal{L}_d(g^j, f^j)$. The discrete Euler-Lagrange equations are obtained by extremizing the discrete action functional

$$\mathfrak{S}_d(g_d) = \sum_{j=0}^{N-1} \mathcal{L}_d(g^j, f^j)$$

over variations of the discrete curve g^j, $j = 0, ..., N$ with fixed endpoints g^0, g^N. Denoting these variations by $\delta g^j = g^j \eta^j$, we have $\delta f^j = -\eta^j f^j + f^j \eta^{j+1}$ and the associated discrete Euler-Lagrange equations read

$$(f^{j-1})^{-1}(D_{f^{j-1}}\mathcal{L}_d^{j-1}) - \mathrm{Ad}^*_{(f^j)^{-1}}\left((f^j)^{-1}(D_{f^j}\mathcal{L}_d^j)\right) + (g^j)^{-1}(D_{g^j}\mathcal{L}_d^j) = 0 \ , \tag{4.2.11}$$

with $g^j = g^{j-1} f^{j-1}$,

see Proposition 3.2 in Bobenko, and Suris [11]. We review below the derivation of these equations in the more general context of the discrete Lagrange-d'Alembert equations. Note that given (g^{j-1}, f^{j-1}), we obtain $g^j = g^{j-1} f^{j-1}$ from the second equation, and we solve the first equation to find f^j. This yields a discrete-time flow map $(g^{j-1}, f^{j-1}) \in G \times G \mapsto (g^j, f^j) \in G \times G$, $j = 1, ..., N$.

Notations for Lie groups. Left and right multiplication by $g \in G$ are denoted by $L_g, R_g : G \to G$, $L_g(f) = gf$, $R_g(f) = fg$. The tangent lifted actions $TL_g, TR_g : TG \to TG$ are sometimes denoted as $gv_f := TL_g(v_f)$ and $v_f g := TR_g(v_f)$ for simplicity, where $v_f \in TG$. Similarly, the cotangent lifted actions $T^*L_{g^{-1}}, T^*R_{g^{-1}} : T^*G \to T^*G$ is denoted by

$$g\alpha_f := T^*L_{g^{-1}}(\alpha_f) \quad \text{and} \quad \alpha_f g := T^*R_{g^{-1}}(\alpha_f), \quad \alpha_f \in T^*G,$$

for simplicity. This notation is used in (4.2.11).

Discrete Legendre transforms. Recall that there are two discrete Legendre transforms $\mathbb{F}^{\pm}L_d : G \times G \to T^*G$ associated to a discrete Lagrangian $L_d(g^j, g^{j+1})$, see (1.2.3). We write the Legendre transforms in terms of the discrete Lagrangian $\mathcal{L}_d(g^j, f^j)$ used for Lie group variational integrators, by using the following relation between L_d and $\mathcal{L}_d$, namely,

$$\mathcal{L}_d(g^j, f^j) = L_d(g^j, g^{j+1}) \quad \text{with} \quad g^{j+1} = g^j f^j. \tag{4.2.12}$$

A direct computation using (1.2.3) and (4.2.12), together with the (left) trivialization $T^*G \simeq G \times \mathfrak{g}^*$ of the cotangent bundle, yields the expressions $\mathbb{F}^{\pm}\mathcal{L}_d : G \times G \to G \times \mathfrak{g}^*$ given by

$$\mathbb{F}^+\mathcal{L}_d^j = (g^j f^j, (\pi^j)^+) \quad \text{and} \quad \mathbb{F}^-\mathcal{L}_d^j = (g^j, (\pi^j)^-), \tag{4.2.13}$$

where $\pi^j_{\pm}$ are the *discrete body momenta* defined by

$$(\pi^j)^- = -(g^j)^{-1} D_{g^j}\mathcal{L}_d^j + \mathrm{Ad}^*_{(f^j)^{-1}}\left((f^j)^{-1} D_{f^j}\mathcal{L}_d^j\right) \quad \text{and} \quad (\pi^j)^+ = (f^j)^{-1} D_{f^j}\mathcal{L}_d^j, \tag{4.2.14}$$

see [12]

Similarly to (1.2.4), we note that the discrete Euler-Lagrange equation (4.2.11) can be written in terms of the Legendre transforms as

$$\mathbb{F}^+\mathcal{L}_d^{j-1} = \mathbb{F}^-\mathcal{L}_d^j, \qquad \text{i.e.} \qquad g^{j-1} f^{j-1} = g^j \quad \text{and} \quad (\pi^{j-1})^+ = (\pi^j)^-. \tag{4.2.15}$$

Discrete momentum mappings and subgroup actions. Recall from (1.2.8) that given a discrete Largangian $L_d(q^j, q^{j+1})$ and a Lie group action of G on Q, two discrete momentum maps $\mathbf{J}^{\pm}_{L_d} : Q \times Q \to \mathfrak{g}^*$ can be defined.

In the present case, we shall choose a subgroup H of G and consider the action of H on G by left translation. Using the relation (4.2.12) and the expression (1.2.8), one computes that the discrete momentum maps $\mathbf{J}^{\pm}_{\mathcal{L}_d} : G \times G \to \mathfrak{h}^*$ associated to the discrete Lagrangian $\mathcal{L}_d(g^j, f^j)$ are given by

$$\mathbf{J}^{+}_{\mathcal{L}_d}(g^j, f^j) = i^* \left(\mathrm{Ad}^*_{(g^{j+1})^{-1}} \left((f^j)^{-1} D_{f^j} \mathcal{L}^j_d \right) \right),$$
$$\mathbf{J}^{-}_{\mathcal{L}_d}(g^j, f^j) = i^* \left(-\mathrm{Ad}^*_{(g^j)^{-1}} \left((g^j)^{-1} D_{g^j} \mathcal{L}^j_d \right) + \mathrm{Ad}^*_{(g^{j+1})^{-1}} \left((f^j)^{-1} D_{f^j} \mathcal{L}^j_d \right) \right),$$

where $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the dual map to the Lie algebra inclusion $i : \mathfrak{h} \to \mathfrak{g}$. We note that we have the relations

$$\mathbf{J}^{+}_{\mathcal{L}_d}(g^j, f^j) = i^* \left(\mathrm{Ad}^*_{(g^{j+1})^{-1}} (\pi^j)^+ \right) \quad \text{and} \quad \mathbf{J}^{-}_{\mathcal{L}_d}(g^j, f^j) = i^* \left(\mathrm{Ad}^*_{(g^j)^{-1}} (\pi^j)^- \right) \tag{4.2.16}$$

between the discrete momentum maps and the discrete Legendre transforms and that the discrete Euler-Lagrange equations imply the relation

$$\mathbf{J}^{+}_{\mathcal{L}_d}(g^{j-1}, f^{j-1}) = \mathbf{J}^{-}_{\mathcal{L}_d}(g^j, f^j) \tag{4.2.17}$$

in $\mathfrak{h}^*$. The quantities $\mathrm{Ad}^*_{(g^{j+1})^{-1}}((\pi^j)^+)$ and $\mathrm{Ad}^*_{(g^j)^{-1}}((\pi^j)^-)$ are referred to as the *discrete spatial momenta*.

We note that in the special case $H = G$, the relation (4.2.17) is not only implied by the discrete Euler-Lagrange equations, but is equivalent to them.

If the discrete Lagrangian $\mathcal{L}$ is H-invariant, then the two momentum maps coincide: $\mathbf{J}^{+}_{\mathcal{L}_d} = \mathbf{J}^{-}_{\mathcal{L}_d} =: \mathbf{J}_{\mathcal{L}_d}$, and (4.2.17) yields the *discrete Noether theorem*

$$\mathbf{J}_{\mathcal{L}_d}(g^{j-1}, f^{j-1}) = \mathbf{J}_{\mathcal{L}_d}(g^j, f^j). \tag{4.2.18}$$

Example: $G = SE(3)$. We compute the relation (4.2.16) for the Lie group $SE(3)$ because of its importance in beam dynamics. We identify the Lie algebra $\mathfrak{se}(3) = \mathfrak{so}(3) \circledS \mathbb{R}^3$ of $SE(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$ by using the hat map (2.2.1). Via this identification, the adjoint action reads

$$\mathrm{Ad}_{(\Lambda, \phi)}(\mathbf{u}, \mathbf{v}) = (\Lambda \mathbf{u}, \Lambda \mathbf{v} + \phi \times \Lambda \mathbf{u}).$$

Identifying the dual space $\mathfrak{se}(3)^*$ with $\mathbb{R}^3 \times \mathbb{R}^3$ via the usual pairing on $\mathbb{R}^3$, the coadjoint action reads

$$\mathrm{Ad}^*_{(\Lambda, \phi)^{-1}}(\mathbf{m}, \mathbf{n}) = (\Lambda \mathbf{m} + \phi \times \Lambda \mathbf{n}, \Lambda \mathbf{n}). \tag{4.2.19}$$

The discrete body momenta $(\pi^j)^{\pm}$ read

$$(\pi^j)^- = ((\Pi^j)^-, (\Gamma^j)^-) \quad \text{and} \quad (\pi^j)^+ = ((\Pi^j)^+, (\Gamma^j)^+),$$

where $(\Pi^j)^\pm$ are the discrete angular momenta and $(\Gamma^j)^\pm$ are the discrete linear momenta. Using the notations $g^j = (\Lambda^j, \mathbf{x}^j)$, $f^j = (F^j, H^j) \in SE(3)$, the relations (4.2.16) read

$$\begin{aligned}
\mathbf{J}^+_{\mathcal{L}_d}((\Lambda^j, \mathbf{x}^j), (F^j, H^j)) &= \mathrm{Ad}^*_{(\Lambda^{j+1}, \mathbf{x}^{j+1})^{-1}}((\Pi^j)^+, (\Gamma^j)^+) \\
&= (\Lambda^{j+1}(\Pi^j)^+ + \mathbf{x}^{j+1} \times \Lambda^{j+1}(\Gamma^j)^+,\ \Lambda^{j+1}(\Gamma^j)^+), \\
\mathbf{J}^-_{\mathcal{L}_d}((\Lambda^j, \mathbf{x}^j), (F^j, H^j)) &= \mathrm{Ad}^*_{(\Lambda^j, \mathbf{x}^j)^{-1}}((\Pi^j)^-, (\Gamma^j)^-) \\
&= (\Lambda^j(\Pi^j)^- + \mathbf{x}^j \times \Lambda^j(\Gamma^j)^-, \Lambda^j(\Gamma^j)^-).
\end{aligned}$$

Symplecticity of the properties of the discrete flow

As we already recalled, the numerical scheme $(g^{j-1}, g^j) \mapsto (g^j, g^{j+1})$ given by the discrete Euler-Lagrange equations yields a symplectic integrator $(g^j, p^j) \mapsto (g^{j+1}, p^{j+1})$ on T^*G by using the discrete Legendre transforms $\mathbb{F}^\pm L_d(g^j, g^{j+1})$. This implies the same property for Lie group variational integrators, namely, the numerical scheme $(g^j, \mu^j) \mapsto (g^{j+1}, \mu^{j+1})$ induced on $G \times \mathfrak{g}^*$ by using the discrete Legendre transforms $\mathbb{F}^\pm \mathcal{L}_d(g^j, f^j)$ is symplectic relative to the trivialized canonical symplectic form on $G \times \mathfrak{g}^*$, see [11].

As a consequence, the Hamiltonian $h : G \times \mathfrak{g}^* \to \mathbb{R}$ obtained from the continuous trivialized Lagrangian $\mathcal{L} : G \times \mathfrak{g} \to \mathbb{R}$ via the Legendre transform is approximately conserved, that is, the sequence $h(g^j, \mu^j)$, $j = 0, ..., N$ oscillates about the true value of the Hamiltonian.

Discrete Lagrange-d'Alembert equations on Lie groups

As recalled in the introduction, the discrete Lagrange-d'Alembert principle is formulated with the help of discrete Lagrangian forces $F_d^\pm = F^\pm(g^j, g^{j+1}) : G \times G \to T^*G$. In the case of Lie group variational integrators, we reformulate $F_d^\pm$ in terms of g^j and f^j as follows. We define the discrete forces $\mathcal{F}_d^\pm : G \times G \to T^*G$ by

$$\begin{aligned}
\mathcal{F}_d^-(g^i, f^i) &:= F_d^-(g^j, g^{j+1}) \in T^*_{g^j} G \\
\mathcal{F}_d^+(g^{i+1}, f^i) &:= F_d^+(g^j, g^{j+1}) \in T^*_{g^{j+1}} G
\end{aligned}$$

where $g^{j+1} = g^j f^j$. From (1.4.1) and using these definitions, we deduce that the discrete Lagrange-d'Alembert principle for Lie group variational integrators is

$$\delta \sum_{j=0}^{N-1} \mathcal{L}_d(g^j, f^j) + \sum_{j=0}^{N-1} \left[\mathcal{F}_d^-(g^j, f^j) \cdot \delta g^j + \mathcal{F}_d^+(g^{j+1}, f^j) \cdot \delta g^{j+1} \right] = 0,$$

for all variations δg^j vanishing at endpoints. We now derive the stationarity condition. Defining $\eta^j := (g^j)^{-1} \delta g^j$, we compute that the induced variation of $f^j = (g^j)^{-1} g^{j+1}$ is

$$\delta f^j = -\eta^j f^j + f^j \eta^{j+1}.$$

Taking into account of the formulas for the δg^j, and δf^j, we isolate the quantities η^j, and obtain

$$\begin{aligned}\delta\mathfrak{S}_d &= \sum_{j=0}^{N-1} D_{g^j}\mathcal{L}_d^j\cdot\delta g^j + D_{f^j}\mathcal{L}_d^j\cdot\delta f^j + \mathcal{F}_d^-(g^j,f^j)\cdot\delta g^j + \mathcal{F}_d^+(g^{j+1},f^j)\cdot(\delta g^j f^j + g^j\delta f^j)\\ &= \sum_{j=1}^{N-1}\Big\{(g^j)^{-1}D_{g^j}\mathcal{L}_d^j - \mathrm{Ad}^*_{(f^j)^{-1}}\left((f^j)^{-1}D_{f^j}\mathcal{L}_d^j\right) + (f^{j-1})^{-1}D_{f^{j-1}}\mathcal{L}_d^{j-1}\\ &\qquad + (g^j)^{-1}\mathcal{F}_d^+(g^j,f^{j-1}) + (g^j)^{-1}\mathcal{F}_d^-(g^j,f^j)\Big\}\cdot\eta^j = 0,\end{aligned}$$

where we use the fixed endpoint condition, that is $\eta^0=\eta^N=0$. We thus get the discrete Lagrange-d'Alembert equations

$$\begin{aligned}&(g^j)^{-1}D_{g^j}\mathcal{L}_d^j - \mathrm{Ad}^*_{(f^j)^{-1}}\left((f^j)^{-1}D_{f^j}\mathcal{L}_d^j\right) + (f^{j-1})^{-1}D_{f^{j-1}}\mathcal{L}_d^{j-1}\\ &\qquad\qquad + (g^j)^{-1}\mathcal{F}_d^+(g^j,f^{j-1}) + (g^j)^{-1}\mathcal{F}_d^-(g^j,f^j) = 0\\ &\text{with }\ g^j = g^{j-1}f^{j-1}.\end{aligned}$$

From (1.4.2), we obtain that the forced discrete Legendre transforms $\mathbb{F}^{\mathcal{F}\pm}\mathcal{L}_d : G\times G\to G\times\mathfrak{g}^*$ are

$$\mathbb{F}^{\mathcal{F}+}\mathcal{L}_d^j = \left(g^{j+1},(\pi_{\mathcal{F}}^j)^+\right)\quad\text{and}\quad \mathbb{F}^{\mathcal{F}-}\mathcal{L}_d^j = \left(g^j,(\pi_{\mathcal{F}}^j)^-\right),$$

where $\pi^j_{\mathcal{F}\pm}$ are the *discrete body momenta in presence of forces* defined by

$$\begin{aligned}(\pi_{\mathcal{F}}^j)^+ &:= (\pi^j)^+ + (g^jf^j)^{-1}\mathcal{F}_d^+(g^jf^j,f^j) = (f^j)^{-1}D_{f^j}\mathcal{L}_d^j + (g^jf^j)^{-1}\mathcal{F}_d^+(g^jf^j,f^j)\\ (\pi_{\mathcal{F}}^j)^- &:= (\pi^j)^- - (g^j)^{-1}\mathcal{F}_d^-(g^j,f^j)\\ &= -(g^j)^{-1}D_{g^j}\mathcal{L}_d^j + \mathrm{Ad}^*_{(f^j)^{-1}}\left((f^j)^{-1}D_{f^j}\mathcal{L}_d^j)\right) - (g^j)^{-1}\mathcal{F}_d^-(g^j,f^j).\end{aligned}$$

As in (4.2.15), the discrete Lagrange-d'Alembert equations can be equivalently written as

$$\mathbb{F}^{\mathcal{F}+}\mathcal{L}_d^{j-1} = \mathbb{F}^{\mathcal{F}-}\mathcal{L}_d^j,\qquad\text{i.e.}\qquad g^{j-1}f^{j-1}=g^j\quad\text{and}\quad(\pi_{\mathcal{F}}^{j-1})^+ = (\pi_{\mathcal{F}}^j)^-.$$

4.2.4 Lie group variational integrator for the beam

Time discretization

Using the same notation as before, given a node a, the discrete time evolution of this node is given by the discrete curve $(\Lambda_a^j,\mathbf{x}_a^j), j=0,...,N$ in $SE(3)$. The discrete variables g^j and $f^j=(g^j)^{-1}g^{j+1}$ associated to this node are $(\Lambda_a^j,\mathbf{x}_a^j)$ and

$$(F_a^j,H_a^j) := (\Lambda_a^j,\mathbf{x}_a^j)^T(\Lambda_a^{j+1},\mathbf{x}_a^{j+1}) = \left((\Lambda_a^j)^T\Lambda_a^{j+1},(\Lambda_a^j)^T(\mathbf{x}_a^{j+1}-\mathbf{x}_a^j)\right),$$

where, in the last equality, we used multiplication in $SE(3)$. We denote the time-step by $\Delta t = t^j - t^{j-1}$, supposed to be of uniform size.

In terms of these variables (F_a^j, H_a^j), we make the following approximations.

$$\begin{aligned}\widehat{\omega}_a^j &= \left(\Lambda_a^j\right)^T \dot{\Lambda}_a^j \approx \left(\Lambda_a^j\right)^T \left(\frac{\Lambda_a^{j+1} - \Lambda_a^j}{\Delta t}\right) = \frac{F_a^j - I_3}{\Delta t},\\ \gamma_a^j &= \left(\Lambda_a^j\right)^T \dot{\mathbf{x}}_a^j \approx \left(\Lambda_a^j\right)^T \left(\frac{\mathbf{x}_a^{j+1} - \mathbf{x}_a^j}{\Delta t}\right) = \frac{H_a^j}{\Delta t}.\end{aligned} \tag{4.2.20}$$

With this approximation, the kinetic energy due to rotation, at a node $a \in K$, reads

$$\begin{aligned}\frac{l_K \Delta t}{4} \operatorname{Tr}\left(\widehat{\omega}_a^j J_d (\widehat{\omega}_a^j)^T\right) &\approx \frac{l_K}{4\Delta t} \operatorname{Tr}\left((F_a^j - I_3) J_d (F_a^j - I_3)^T\right)\\ &= \frac{l_K}{2\Delta t} \operatorname{Tr}\left((I_3 - F_a^j) J_d\right),\end{aligned}$$

where we use (4.1.3) and the following properties

$$\operatorname{Tr}\left(F_a^j J_d (F_a^j)^T\right) = \operatorname{Tr}\left(J_d (F_a^j)^T F_a^j\right) = \operatorname{Tr}\left(J_d\right) \quad \text{and} \quad \operatorname{Tr}(J_d (F_a^j)^T) = \operatorname{Tr}(F_a^j J_d).$$

The discrete Lagrangian L_K^j approximating the action of the Lagrangian L_K in (4.2.7) during the time step Δt is therefore

$$\mathcal{L}_K^j = \sum_{a \in K} \left\{ \frac{l_K}{4} \frac{M \|H_a^j\|^2}{\Delta t} + \frac{l_K}{2} \frac{\operatorname{Tr}((I_3 - F_a^j) J_d)}{\Delta t} \right\} - \Delta t \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j\right). \tag{4.2.21}$$

The discrete action sum which approximates the continuous action over the time interval $[0, T]$ is computed as follows

$$\begin{aligned}\mathfrak{S}_d((\Lambda_d, \mathbf{x}_d)) &= \sum_{K \in \mathcal{T}} \sum_{1 \le j < N} \mathcal{L}_K^j\\ &= \sum_{a \neq a_0, a_N} \sum_{j=0}^{N-1} \left\{ \frac{l_K}{2} \frac{M \|H_a^j\|^2}{\Delta t} + l_K \frac{\operatorname{Tr}((I_3 - F_a^j) J_d)}{\Delta t} \right\}\\ &\quad + \sum_{j=0}^{N-1} \left\{ \frac{l_K}{4} \frac{M \|H_{a_0}^j\|^2}{\Delta t} + \frac{l_K}{2} \frac{\operatorname{Tr}((I_3 - F_{a_0}^j) J_d)}{\Delta t} \right\}\\ &\quad + \sum_{j=0}^{N-1} \left\{ \frac{l_K}{4} \frac{M \|H_{a_N}^j\|^2}{\Delta t} + \frac{l_K}{2} \frac{\operatorname{Tr}((I_3 - F_{a_N}^j) J_d)}{\Delta t} \right\}\\ &\quad - \sum_{K \in \mathcal{T}} \sum_{1 \le j < N} \Delta t \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j\right).\end{aligned} \tag{4.2.22}$$

Lie group variational integrator

The discrete evolution is obtained by applying the discrete Hamilton's principle to the discrete action (4.2.22). Equivalently this consists in computing the discrete Euler-Lagrange equations for each node a. From (4.2.11), we get the following systems of discrete Euler-Lagrange equations

$$\begin{aligned} T_e^* L_{(F_a^{j-1}, H_a^{j-1})} \left(D_{F_a^{j-1}} \mathcal{L}_a^{j-1}, \quad D_{H_a^{j-1}} \mathcal{L}_a^{j-1} \right) \\ - \mathrm{Ad}^*_{(F_a^j, H_a^j)^{-1}} T_e^* L_{(F_a^j, H_a^j)} \left(D_{F_a^j} \mathcal{L}_a^j, \ D_{H_a^j} \mathcal{L}_a^j \right) \\ + \ T_e^* L_{(\Lambda_a^j, \mathbf{x}_a^j)} \left(D_{\Lambda_a^j} \mathcal{L}_a^j, \ D_{\mathbf{x}_a^j} \mathcal{L}_a^j \right) = 0, \end{aligned} \tag{4.2.23}$$

for all $a \in \mathcal{N}$, where $\mathcal{L}_a^j$ denotes the dependence of the discrete action $\mathfrak{S}_d$ on $(\Lambda_a^j, \mathbf{x}_a^j, F_a^j, H_a^j)$, similar for $\mathcal{L}_a^{j-1}$. Recall that we denote by $\mathcal{N}$ the set of all nodes, by $\partial\mathcal{N} = \{a_0, a_N\}$ the set of boundary nodes, and by $int(\mathcal{N}) = \{a_1, ..., a_{N-1}\}$ the set of internal nodes. The equations are slightly different for $a \in int(\mathcal{N})$ and $a \in \partial\mathcal{N}$. Indeed, for $a \in int(\mathcal{N})$ the discrete Lagrangian $\mathcal{L}_a^j$ is

$$\mathcal{L}_a^j = \frac{l_K}{2} \frac{M \|H_a^j\|^2}{\Delta t} + l_K \frac{\mathrm{Tr}((I_3 - F_a^j) J_d)}{\Delta t} - \sum_{K \ni a} \Delta t \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j \right), \tag{4.2.24}$$

whereas, for a boundary node $a \in \partial\mathcal{N}$, it reads

$$\mathcal{L}_a^j = \frac{l_K}{4} \frac{M \|H_a^j\|^2}{\Delta t} + \frac{l_K}{2} \frac{\mathrm{Tr}((I_3 - F_a^j) J_d)}{\Delta t} - \sum_{K \ni a} \Delta t \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j \right). \tag{4.2.25}$$

Note that in (4.2.24) the sum in the last term involves two spatial elements K, whereas in (4.2.25) the sum involves only one subinterval.

4.2.1 Remark (Duality pairing) Before computing these equations concretely, we recall that we identify the dual space $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ via the natural pairing of $\mathbb{R}^3$, i.e.

$$\langle \widehat{\mathbf{v}}, \widehat{\mathbf{w}} \rangle := \mathbf{v} \cdot \mathbf{w} = \frac{1}{2} \ \mathrm{Tr} \left(\widehat{\mathbf{v}}^T \widehat{\mathbf{w}} \right), \tag{4.2.26}$$

where $\widehat{\ }: \mathbb{R}^3 \to \mathfrak{so}(3)$ is the hat map defined in (2.2.1). Recall that the tangent space at $\Lambda \in SO(3)$ reads $T_\Lambda SO(3) = \{\Lambda\xi \mid \xi \in \mathfrak{so}(3)\}$. We identify the cotangent space $T_\Lambda^* SO(3)$ with $T_\Lambda SO(3)$ using the left-invariant pairing introduced in (4.2.26), i.e.

$$\langle \alpha_\Lambda, V_\Lambda \rangle := \left\langle \Lambda^{-1}\alpha_\Lambda, \Lambda^{-1} V_\Lambda \right\rangle = \frac{1}{2} \mathrm{Tr} \left((\Lambda^{-1}\alpha_\Lambda)^T \Lambda^{-1} V_\Lambda \right) = \frac{1}{2} \mathrm{Tr} \left(\alpha_\Lambda^T V_\Lambda \right).$$

With this identification, the cotangent lift of left translation $T^* L_{(\Lambda,\phi)} : T^*_{(\Lambda,\phi)} G \to T_e^* G$ reads

$$(\Lambda, \phi)^{-1}(\alpha_\Lambda, (\phi, \mathbf{v})) = T^* L_{(\Lambda,\phi)}(\alpha_\Lambda, (\phi, \mathbf{v})) = \left(\Lambda^T \alpha_\Lambda, \Lambda^T \mathbf{v} \right) \in \mathfrak{se}(3). \qquad \blacklozenge \tag{4.2.27}$$

Discrete Euler-Lagrange equations for an internal node. We now compute the discrete Euler-Lagrange equations (4.2.23) for an internal node $a \in$

$int(\mathcal{N})$. For $\xi \in \mathbb{R}^3$, we have

$$T_I^* L_{F_a^j} D_{F_a^j} \mathcal{L}_a^j \cdot \xi = \left\langle D_{F_a^j} \mathcal{L}_a^j, F_a^j \widehat{\xi} \right\rangle = \frac{-l_K}{\Delta t} \operatorname{Tr}\left(F_a^j \widehat{\xi} J_d \right) = \frac{-l_K}{\Delta t} \operatorname{Tr}\left(J_d F_a^j \widehat{\xi} \right)$$
$$= \frac{-l_K}{\Delta t} \operatorname{Tr}\left(\frac{1}{2} \left(J_d \; F_a^j - (F_a^j)^T J_d \right) \widehat{\xi} \right).$$

Thus, for a node $a \in int(\mathcal{N})$, using (4.2.26), we get

$$T_I^* L_{F_a^j} D_{F_a^j} \mathcal{L}_a^j = \frac{l_K}{\Delta t} \left(J_d F_a^j - (F_a^j)^T J_d \right)^\vee \in \mathbb{R}^3,$$

where $^\vee : \mathfrak{so}(s) \to \mathbb{R}^3$ is the inverse of the hat map. The derivative of $\mathcal{L}_a^j$ with respect to H_a^j is

$$D_{H_a^j} \mathcal{L}_a^j = \frac{M l_K}{\Delta t} H_a^j,$$

Thus, denoting $e = (I, 0)$, using (4.2.27) and (4.2.19), we obtain

$$T_e^* L_{(F_a^j, H_a^j)} \left(D_{F_a^j} \mathcal{L}_a^j, D_{H_a^j} \mathcal{L}_a^j \right) = \left(\frac{l_K}{\Delta t} \left(J_d F_a^j - (F_a^j)^T J_d \right)^\vee, \frac{l_K M}{\Delta t} (F_a^j)^T H_a^j \right)$$

and

$$\operatorname{Ad}^*_{(F_a^j, H_a^j)^{-1}} T_e^* L_{(F_a^j, H_a^j)} \left(D_{F_a^j} \mathcal{L}_a^j, D_{H_a^j} \mathcal{L}_a^j \right) = \left(\frac{l_K}{\Delta t} \left(F_a^j J_d - J_d (F_a^j)^T \right)^\vee, \frac{l_K M}{\Delta t} H_a^j \right).$$

The derivatives of $\mathcal{L}_a^j$ with respect to Λ_a^j and $\mathbf{x}_a^j$ are, respectively,

$$D_{\Lambda_a^j} \mathcal{L}_a^j = -\Delta t \sum_{K \ni a} D_{\Lambda_a^j} \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j \right)$$
$$D_{\mathbf{x}_a^j} \mathcal{L}_a^j = -\Delta t \sum_{K \ni a} D_{\mathbf{x}_a^j} \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j \right),$$

so that, by (4.2.27), we get

$$T_e^* L_{(\Lambda_a^j, \mathbf{x}_a^j)} \left(D_{(\Lambda_a^j, \mathbf{x}_a^j)} \mathcal{L}_K^j \right) = -\Delta t \sum_{K \ni a} \left((\Lambda_a^j)^T D_{\Lambda_a^j} \mathbb{V}_K (\Lambda_K^j, \mathbf{x}_K^j), (\Lambda_a^j)^T D_{\mathbf{x}_a^j} \mathbb{V}_K (\Lambda_K^j, \mathbf{x}_K^j) \right),$$

where $D_{\Lambda_a^j} \mathbb{V}_K (\Lambda_K^j, \mathbf{x}_K^j) \in T^*_{\Lambda_a^j} SO(3) \simeq T_{\Lambda_a^j} SO(3)$ and $(\Lambda_a^j)^T D_{\Lambda_a^j} \mathbb{V}_K (\Lambda_K^j, \mathbf{x}_K^j) \in \mathfrak{so}(3)$.

Putting together the computations we made above, we obtain that (4.2.23) is equivalent to the two equations

$$\begin{cases} \frac{l_K}{\Delta t} \left(J_d F_a^{j-1} - (F_a^{j-1})^T J_d \right)^\vee - \frac{l_K}{\Delta t} \left(F_a^j J_d - J_d (F_a^j)^T \right)^\vee \\ \qquad = \Delta t \sum_{K \ni a} \left((\Lambda_a^j)^T D_{\Lambda_a^j} \mathbb{V}_K (\Lambda_K^j, \mathbf{x}_K^j) \right)^\vee \\ \frac{M l_K}{\Delta t} (F_a^{j-1})^T \; H_a^{j-1} - \frac{M l_K}{\Delta t} H_a^j = \Delta t \sum_{K \ni a} (\Lambda_a^j)^T D_{\mathbf{x}_a^j} \mathbb{V}_K (\Lambda_K^j, \mathbf{x}_K^j). \end{cases} \tag{4.2.28}$$

The second equation can be equivalently written as

$$\frac{Ml_K}{\Delta t}\Delta\mathbf{x}_a^{j-1} - \frac{Ml_K}{\Delta t}\Delta\mathbf{x}_a^j = \Delta t \sum_{K\ni a} D_{\mathbf{x}_a^j}\mathbb{V}_K(\Lambda_K^j, \mathbf{x}_K^j),$$

where $\Delta\mathbf{x}_a^j := \mathbf{x}_a^{j+1} - \mathbf{x}_a^j$ (not to be confused with $\Delta\mathbf{x}_a$ defined in (4.2.4)).

Discrete Euler-Lagrange equations for a boundary node. They can be computed in the same way, by using the discrete Lagrangian (4.2.25) instead of (4.2.24). The resulting system can be obtained from the system (4.2.28) by multiplying the left hand side by $1/2$ and bearing in mind that the sum involves only one element for boundary nodes.

Computation of the potential terms. We now compute explicitly the terms $D_{\Lambda_a}\mathbb{V}_K$ and $D_{\mathbf{x}_a}\mathbb{V}_K$ due to the potential energy $\mathbb{V}_K$ given in (4.2.6). Note that two situations can occur for a fixed node a. Either K is the element whose right node is a or K is the element whose left node is a. For the computation below, we fix an element K and denote by a its left node and by $a+1$ its right node. Recall that the variable ψ_a in (4.2.6) is given by $\widehat{\psi}_a = \exp^{-1}(\Lambda_a^T\Lambda_{a+1})$. This expression is approximated using the Cayley transformation, i.e. we write $\widehat{\psi}_a = \mathrm{cay}^{-1}(\Lambda_a^T\Lambda_{a+1})$. For $\delta\Lambda_a = \Lambda_a\widehat{\xi} \in T_{\Lambda_a}SO(3)$, we have

$$\begin{aligned} D_{\Lambda_a}\widehat{\psi}_a \cdot \delta\Lambda_a &= 2(\delta\Lambda_a)^T\Lambda_{a+1}(\Lambda_a^T\Lambda_{a+1} + I)^{-1} \\ &\qquad - 2(\Lambda_a^T\Lambda_{a+1} - I)(\Lambda_a^T\Lambda_{a+1} + I)^{-1}\delta\Lambda_a^T\Lambda_{a+1}(\Lambda_a^T\Lambda_{a+1} + I)^{-1} \\ &= \left(\widehat{\psi}_a - 2I\right)\widehat{\xi}\left(I + \Lambda_{a+1}^T\Lambda_a\right)^{-1}. \end{aligned}$$

So we get

$$\begin{aligned} D_{\Lambda_a}\mathbb{V}_K \cdot \delta\Lambda_a &= \frac{l_K}{2}\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1\delta\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K} + \frac{1}{l_K}\psi_a^T\mathbf{C}_2 D_{\Lambda_a}\psi_a \cdot \delta\Lambda_a \\ &= -\frac{1}{2}\operatorname{Tr}\left(\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T\mathbf{C}_1\widehat{\xi}\Lambda_a^T\Delta\mathbf{x}_a\right) - \frac{1}{2l_K}\operatorname{Tr}\left(\widehat{\mathbf{C}_2\psi_a}(D_{\Lambda_a}\widehat{\psi}_a \cdot \delta\Lambda_a)\right) \\ &= -\frac{1}{2}\operatorname{Tr}\left(\left(\Lambda_a^T\Delta\mathbf{x}_a\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T\mathbf{C}_1\right)^{(A)}\widehat{\xi}\right) \\ &\qquad - \frac{1}{2l_K}\operatorname{Tr}\left(\left(\left(I + \Lambda_{a+1}^T\Lambda_a\right)^{-1}\widehat{\mathbf{C}_2\psi_a}\left(\widehat{\psi}_a - 2I\right)\right)^{(A)}\widehat{\xi}\right). \end{aligned}$$

Then, using the identities

$$-\frac{1}{2}\operatorname{Tr}\left(M^A\widehat{\xi}\right) = (M^{(A)})^\vee \cdot \xi \qquad \text{and} \qquad \left((\mathbf{v}\mathbf{w}^T)^{(A)}\right)^\vee = \frac{1}{2}\mathbf{w}\times\mathbf{v}, \quad (4.2.29)$$

for all $\xi, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and 3×3 matrices M, we get

$$\begin{aligned}(\Lambda_a^T D_{\Lambda_a} \mathbb{V}_K)^\vee =& \frac{1}{2}\mathbf{C}_1 \left(\Lambda_a^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right) \times \Lambda_a^T \Delta\mathbf{x}_a \\ &+ \frac{1}{l_K}\left(\left(\left(I + \Lambda_{a+1}^T \Lambda_a\right)^{-1} \widehat{\mathbf{C}_2\psi_a}\left(\widehat{\psi}_a - 2I\right)\right)^{(A)}\right)^\vee .\end{aligned}$$

Now, given $\widehat{\psi}_a = \text{cay}^{-1}(\Lambda_a^T \Lambda_{a+1})$, for $\delta\Lambda_{a+1} = \Lambda_{a+1}\widehat{\xi} \in T_{\Lambda_{a+1}} SO(3)$, we have

$$\begin{aligned}D_{\Lambda_{a+1}} \widehat{\psi}_a \cdot \delta\Lambda_{a+1} =& 2\Lambda_a^T \delta\Lambda_{a+1}(\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ &- 2(\Lambda_a^T \Lambda_{a+1} - I)(\Lambda_a^T\Lambda_{a+1} + I)^{-1}\Lambda_a^T \delta\Lambda_{a+1}(\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ =& (2I - \widehat{\psi}_a)\Lambda_a^T \Lambda_{a+1}\widehat{\xi}(\Lambda_a^T\Lambda_{a+1} + I)^{-1}.\end{aligned}$$

So we get

$$\begin{aligned}D_{\Lambda_{a+1}} \mathbb{V}_K \cdot \delta\Lambda_{a+1} =& \frac{1}{2}\left(\Lambda_{a+1}^T \frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1 \delta\Lambda_{a+1}^T \Delta\mathbf{x}_a + \frac{1}{l_K}\psi_a^T \mathbf{C}_2 D_{\Lambda_{a+1}}\psi_a \cdot \delta\Lambda_{a+1} \\ =& -\frac{1}{2}\operatorname{Tr}\left(\left(\Lambda_{a+1}^T \frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1 \widehat{\xi}\Lambda_{a+1}^T \Delta\mathbf{x}_a\right) - \frac{1}{2l_K}\operatorname{Tr}\left(\widehat{\mathbf{C}_2\psi_a}(D_{\Lambda_{a+1}}\widehat{\psi}_a \cdot \delta\Lambda_{a+1})\right) \\ =& -\frac{1}{2}\operatorname{Tr}\left(\left(\Lambda_{a+1}^T \Delta\mathbf{x}_a \left(\Lambda_{a+1}^T \frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1\right)^{(A)} \widehat{\xi}\right) \\ &- \frac{1}{2l_K}\operatorname{Tr}\left(\left((\Lambda_a^T\Lambda_{a+1} + I)^{-1}\widehat{\mathbf{C}_2\psi_a}(2I - \widehat{\psi}_a)\Lambda_a^T\Lambda_{a+1}\right)^{(A)}\widehat{\xi}\right),\end{aligned}$$

which shows, by using (4.2.29), that

$$\begin{aligned}(\Lambda_{a+1}^T D_{\Lambda_{a+1}} \mathbb{V}_K)^\vee =& \frac{1}{2}\mathbf{C}_1 \left(\Lambda_{a+1}^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right) \times \Lambda_{a+1}^T \Delta\mathbf{x}_a \\ &+ \frac{1}{l_K}\left(\left((\Lambda_a^T\Lambda_{a+1} + I)^{-1}\widehat{\mathbf{C}_2\psi_a}(2I - \widehat{\psi}_a)\Lambda_a^T\Lambda_{a+1}\right)^{(A)}\right)^\vee .\end{aligned}$$

The derivatives of $\mathbb{V}_K$ with respect to $\mathbf{x}_a$ and $\mathbf{x}_{a+1}$ are, respectively,

$$\begin{aligned}D_{\mathbf{x}_a}\mathbb{V}_K \cdot \delta\mathbf{x}_a =& \frac{1}{2}\left(\Lambda_a^T \frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1(-\Lambda_a^T \delta\mathbf{x}_a) \\ &+ \frac{1}{2}\left(\Lambda_{a+1}^T \frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1(-\Lambda_{a+1}^T\delta\mathbf{x}_a) + \frac{l_K}{2}\left\langle \mathbf{q}_a, \delta\mathbf{x}_a\right\rangle\end{aligned}$$

$$\begin{aligned}D_{\mathbf{x}_{a+1}}\mathbb{V}_K \cdot \delta\mathbf{x}_{a+1} =& \frac{1}{2}\left(\Lambda_a^T \frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1(\Lambda_a^T \delta\mathbf{x}_{a+1}) \\ &+ \frac{1}{2}\left(\Lambda_{a+1}^T \frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T \mathbf{C}_1(\Lambda_{a+1}^T\delta\mathbf{x}_{a+1}) + \frac{l_K}{2}\left\langle \mathbf{q}_{a+1}, \delta\mathbf{x}_{a+1}\right\rangle .\end{aligned}$$

By making use of all the above computations, we can now write explicitly the discrete Euler-Lagrange equations for the beam.

Summary of the discrete Euler-Lagrange equations. Discrete Euler-Lagrange equations for rotations:

(i) Interior nodes $a \notin \{a_0, a_N\}$

$$\begin{aligned}&\frac{l_K}{\Delta t}\left(J_d F_a^{j-1} - (F_a^{j-1})^T J_d\right)^\vee - \frac{l_K}{\Delta t}\left(F_a^j J_d - J_d (F_a^j)^T\right)^\vee \\ &= \Delta t \Bigg\{ \frac{1}{2}\mathbf{C}_1\left(\Lambda_a^T \frac{\Delta \mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3\right) \times \Lambda_a^T \Delta \mathbf{x}_{a-1} + \frac{1}{2}\mathbf{C}_1\left(\Lambda_a^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right) \times \Lambda_a^T \Delta \mathbf{x}_a \\ &\qquad + \frac{1}{l_K}\left(\left(\left(I + \Lambda_{a+1}^T \Lambda_a\right)^{-1} \widehat{\mathbf{C}_2 \psi_a}\left(\widehat{\psi}_a - 2I\right)\right)^{(A)}\right)^\vee \\ &\qquad + \frac{1}{l_K}\left(\left((\Lambda_{a-1}^T \Lambda_a + I)^{-1} \widehat{\mathbf{C}_2 \psi_{a-1}} (2I - \widehat{\psi}_{a-1}) \Lambda_{a-1}^T \Lambda_a\right)^{(A)}\right)^\vee \Bigg\}\Bigg|_{t=t^j}.\end{aligned} \tag{4.2.30}$$

(ii) Left node $a = a_0$

$$\begin{aligned}&\frac{l_K}{2\Delta t}\left(J_d F_{a_0}^{j-1} - (F_{a_0}^{j-1})^T J_d\right)^\vee - \frac{l_K}{2\Delta t}\left(F_{a_0}^j J_d - J_d (F_{a_0}^j)^T\right)^\vee \\ &\qquad = \Delta t \Bigg\{ \frac{1}{2}\mathbf{C}_1\left(\Lambda_a^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3\right) \times \Lambda_a^T \Delta \mathbf{x}_a \\ &\qquad\qquad + \frac{1}{l_K}\left(\left(\left(I + \Lambda_{a+1}^T \Lambda_a\right)^{-1} \widehat{\mathbf{C}_2 \psi_a}\left(\widehat{\psi}_a - 2I\right)\right)^{(A)}\right)^\vee \Bigg\}\Bigg|_{t=t^j}.\end{aligned} \tag{4.2.31}$$

(iii) Right node $a = a_N$

$$\begin{aligned}&\frac{l_K}{2\Delta t}\left(J_d F_{a_N}^{j-1} - (F_{a_N}^{j-1})^T J_d\right)^\vee - \frac{l_K}{2\Delta t}\left(F_{a_N}^j J_d - J_d (F_{a_N}^j)^T\right)^\vee \\ &\qquad = \Delta t \Bigg\{ \frac{1}{2}\left[\mathbf{C}_1\left(\Lambda_a^T \frac{\Delta \mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3\right) \times \Lambda_a^T \Delta \mathbf{x}_{a-1}\right] \\ &\qquad\qquad + \frac{1}{l_K}\left(\left((\Lambda_{a-1}^T \Lambda_a + I)^{-1} \widehat{\mathbf{C}_2 \psi_{a-1}} (2I - \widehat{\psi}_{a-1}) \Lambda_{a-1}^T \Lambda_a\right)^{(A)}\right)^\vee \Bigg\}\Bigg|_{t=t^j}.\end{aligned} \tag{4.2.32}$$

Discrete Euler-Lagrange equations for positions:

(i) Interior nodes $a \notin \{a_0, a_N\}$

$$\begin{aligned}
&\frac{l_K M}{\Delta t}\Delta\mathbf{x}_a^j - \frac{l_K M}{\Delta t}\ \Delta\mathbf{x}_a^{j-1} \\
&\quad = \Delta t\Bigg\{\frac{1}{2}\Lambda_a\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right) - \frac{1}{2}\Lambda_{a-1}\mathbf{C}_1\left(\Lambda_{a-1}^T\frac{\Delta\mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3\right) \\
&\qquad + \frac{1}{2}\Lambda_{a+1}\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right) - \frac{1}{2}\Lambda_a\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3\right) \\
&\qquad - l_K\mathbf{q}_a\Bigg\}\Bigg|_{t=t^j}.
\end{aligned} \tag{4.2.33}$$

(ii) Left node $a = a_0$

$$\begin{aligned}
&\frac{l_K M}{2\Delta t}\Delta\mathbf{x}_{a_0}^j - \frac{l_K M}{2\Delta t}\ \Delta\mathbf{x}_{a_0}^{j-1} \\
&\quad = \Delta t\Bigg\{\frac{1}{2}\Lambda_a\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right) \\
&\qquad + \frac{1}{2}\Lambda_{a+1}\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right) - \frac{l_K}{2}\mathbf{q}_{a_0}\Bigg\}\Bigg|_{t=t^j}.
\end{aligned} \tag{4.2.34}$$

(iii) Right node $a = a_N$

$$\begin{aligned}
&\frac{l_K M}{2\Delta t}\Delta\mathbf{x}_{a_N}^j - \frac{l_K M}{2\Delta t}\ \Delta\mathbf{x}_{a_N}^{j-1} \\
&\quad = \Delta t\Bigg\{ - \frac{1}{2}\Lambda_{a-1}\mathbf{C}_1\left(\Lambda_{a-1}^T\frac{\Delta\mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3\right) \\
&\qquad - \frac{1}{2}\Lambda_a\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3\right) - \frac{l_K}{2}\mathbf{q}_{a_N}\Bigg\}\Bigg|_{t=t^j}.
\end{aligned} \tag{4.2.35}$$

Note that the equations for translation and rotation are fully decoupled for the derived scheme. The later equations can be solved explicitly for the unknown translation, while an iterative method is necessary to solve for the unknown rotation (see Section (4.3) on examples for further details).

4.2.2 Remark (Discrete versus continuous) We compare the discrete equations of motion to the continuous equations (4.1.11). Given $F_a^j = (\Lambda_a^j)^T\Lambda_a^{j+1}$ as a relative rotation of cross-section associated to node a between times t^j and t^{j+1} we note that the first line of equations (4.2.30)–(4.2.32) divided by the time step Δt is the discrete analog of the term $J\dot{\omega} + \omega \times J\omega$ of (4.1.11). This is consistent with the analog term arising in the discrete Euler-Lagrange equations for rigid bodies obtained by Lie group variational integrators, see [65]. The right hand side of (4.2.30)–(4.2.32) corresponds to the contribution of the discrete potential force at time t^j. By comparing with (4.1.11), we observe that the right hand side is the discrete analog of the potential term in (4.1.11). The same holds for a comparison of (4.2.33)–(4.2.35) to the second equation in (4.1.11).

Discrete body momenta and Legendre transforms

In the case of the beam, the discrete momenta read

$$(\pi^j)^{\pm} = ((\pi^j_{a_0})^{\pm}, ..., (\pi^j_{a_N})^{\pm}),$$

where $(\pi^j_a)^{\pm}$ are the discrete body momenta corresponding to the node $a \in \mathcal{N}$. Each of these momenta reads $(\pi^j_a)^{\pm} = ((\Pi^j_a)^{\pm}, (\Gamma^j_a)^{\pm})$, where $(\Pi^j_a)^{\pm}$ are the *discrete body angular momenta* and $(\Gamma^j_a)^{\pm}$ are the *discrete linear body momenta*. From (4.2.14) we know that these discrete momenta are given by

$$\begin{aligned}(\pi^j_a)^- &= ((\Pi^j_a)^-, (\Gamma^j_a)^-)\\ &= -T^*_e L_{(\Lambda^j_a, \mathbf{x}^j_a)} \left(D_{\Lambda^j_a} \mathcal{L}^j_a,\ D_{\mathbf{x}^j_a} \mathcal{L}^j_a \right) + \mathrm{Ad}^*_{(F^j_a, H^j_a)^{-1}} T^*_e L_{(F^j_a, H^j_a)} \left(D_{F^j_a} \mathcal{L}^j_a,\ D_{H^j_a} \mathcal{L}^j_a \right)\\ (\pi^j_a)^+ &= ((\Pi^j_a)^+, (\Gamma^j_a)^+)\\ &= T^*_e L_{(F^{j-1}_a, H^{j-1}_a)} \left(D_{F^{j-1}_a} \mathcal{L}^{j-1}_a,\quad D_{H^{j-1}_a} \mathcal{L}^{j-1}_a \right).\end{aligned}$$

Their concrete expression is easily obtained from the computations made above in §4.2.4. In particular, as we already mentioned, there are some slight differences between the formulas for interior nodes and for boundary nodes.

The discrete Euler-Lagrange equations (4.2.30)–(4.2.32) can be equivalently written as

$$(\Pi^{j-1}_a)^+ = (\Pi^j_a)^-, \quad a \in \mathcal{N}$$

while the discrete Euler-Lagrange equations (4.2.33)–(4.2.35) can be equivalently written as

$$(\Gamma^{j-1}_a)^+ = (\Gamma^j_a)^-, \quad a \in \mathcal{N}.$$

Recall from (4.2.13) that the expressions of the momenta appear in the discrete Legendre transforms, whose a-component read

$$\begin{aligned}(\mathbb{F}^- \mathcal{L}^j_d)_a &= ((\Lambda^j_a, \mathbf{x}^j_a), ((\Pi^j_a)^-, (\Gamma^j_a)^-))\\ (\mathbb{F}^+ \mathcal{L}^j_d)_a &= ((\Lambda^j_a, \mathbf{x}^j_a)(F^j_a, H^j_a), ((\Pi^j_a)^+, (\Gamma^j_a)^+))\end{aligned}$$

Invariance and discrete momentum maps

From the expression (4.2.21) of the discrete Lagrangian of the beam, we obtain that it is H-invariant if and only if the potential $\mathbb{V}_K$ is H-invariant. From (4.2.6), we see that when the conservative force $\mathbf{q}$ is absent, the Lagrangian is left-$SE(3)$-invariant under the action Φ given by

$$\Phi_{(A, \mathbf{v})} \left((\Lambda^j_a, \mathbf{x}^j_a)_{a \in \mathcal{N}} \right) = (A\Lambda^j_a, \mathbf{v} + A\mathbf{x}^j_a)_{a \in \mathcal{N}}.$$

Note that this action is the left translation by the subgroup $SE(3) \subset SE(3)^N$ (diagonal inclusion). The Lie algebra inclusion $i : \mathfrak{se}(3) \to \mathfrak{se}(3)^N$ reads $i(\Omega, \mathbf{v}) = (\Omega, \mathbf{v})_{a\in\mathcal{N}}$, and its dual map is given by

$$i^* \left((\Pi_a, \Gamma_a)_{a\in\mathcal{N}}\right) = \sum_{a\in\mathcal{N}} (\Pi_a, \Gamma_a) \in \mathfrak{se}(3)^*.$$

Using the general formula (4.2.16) relating the discrete momentum maps for left translation by subgroups and the discrete body momenta, together with the formula (4.2.19) for the coadjoint action for $SE(3)$, we get

$$\begin{aligned}\mathbf{J}^+_{\mathcal{L}_d}((\Lambda^j, \mathbf{x}^j), (F^j, H^j)) &= i^* \left(\left(\mathrm{Ad}^*_{(\Lambda^{j+1}_a, \mathbf{x}^{j+1}_a)^{-1}} ((\Pi^j_a)^+, (\Gamma^j_a)^+) \right)_{a\in\mathcal{N}} \right) \\ &= i^* \left((\Lambda^{j+1}_a (\Pi^j_a)^+ + \mathbf{x}^{j+1}_a \times \Lambda^{j+1}_a (\Gamma^j_a)^+, \Lambda^{j+1}_a (\Gamma^j_a)^+)_{a\in\mathcal{N}} \right) \\ &= \left(\sum_{a\in\mathcal{N}} \Lambda^{j+1}_a (\Pi^j_a)^+ + \mathbf{x}^{j+1}_a \times \Lambda^{j+1}_a (\Gamma^j_a)^+, \sum_{a\in\mathcal{N}} \Lambda^{j+1}_a (\Gamma^j_a)^+ \right).\end{aligned}$$

Similarly, we get

$$\mathbf{J}^-_{\mathcal{L}_d}((\Lambda^j, \mathbf{x}^j), (F^j, H^j)) = \left(\sum_{a\in\mathcal{N}} \Lambda^j_a (\Pi^j_a)^- + \mathbf{x}^j_a \times \Lambda^j_a (\Gamma^j_a)^-, \sum_{a\in\mathcal{N}} \Lambda^j_a (\Gamma^j_a)^- \right).$$

By the general theory developed earlier, these momentum maps coincide since the discrete Lagrangian is $SE(3)$-invariant.

The discrete Noether theorem (4.2.18) ensures that when the discrete Euler-Lagrange equations (4.2.30)–(4.2.35) are fulfilled, then $\mathbf{J}_{\mathcal{L}_d}$ is conserved in $\mathfrak{se}(3)^*$, i.e.

$$\mathbf{J}_{\mathcal{L}_d}((\Lambda^j, \mathbf{x}^j), (F^j, H^j)) = \mathbf{J}_{\mathcal{L}_d}((\Lambda^{j-1}, \mathbf{x}^{j-1}), (F^{j-1}, H^{j-1})).$$

We denote by

$$(\mathbf{J}^j_{ang}, \mathbf{J}^j_{lin}) := \mathbf{J}_{\mathcal{L}_d}((\Lambda^j, \mathbf{x}^j), (F^j, H^j)) \tag{4.2.36}$$

the discrete angular and linear momentum map.

In general, the presence of external forces breaks the $SE(3)$ symmetry. For example if we consider the gravity force $\mathbf{q}_a = -m_a g \mathbf{E}_3$, then the discrete Lagrangian is S^1-invariant under the S^1-action

$$\Phi_\theta \left((\Lambda^j_a, \mathbf{x}^j_a)_{a\in\mathcal{N}} \right) = (\exp(\theta \widehat{\mathbf{E}}_3) \Lambda^j_a, \exp(\theta \widehat{\mathbf{E}}_3) \mathbf{x}^j_a)_{a\in\mathcal{N}}.$$

In this case, the Lie algebra inclusion $i : \mathbb{R} \to \mathfrak{se}(3)^N$ reads $i(\theta) = (\theta \widehat{\mathbf{E}}_3, 0)_{a\in\mathcal{N}}$, and its dual map is given by

$$i^* \left((\Pi_a, \Gamma_a)_{a\in\mathcal{N}}\right) = \mathbf{E}_3 \cdot \sum_{a\in\mathcal{N}} \Pi_a \in \mathbb{R}.$$

Applying the same formulas as above, the discrete momentum maps are

$$\begin{aligned}\mathbf{J}^+_{\mathcal{L}_d}((\Lambda^j, \mathbf{x}^j), (F^j, H^j)) &= \mathbf{E}_3 \cdot \sum_{a\in\mathcal{N}} \Lambda^{j+1}_a (\Pi^j_a)^+ + \mathbf{x}^{j+1}_a \times \Lambda^{j+1}_a (\Gamma^j_a)^+ \\ \mathbf{J}^-_{\mathcal{L}_d}((\Lambda^j, \mathbf{x}^j), (F^j, H^j)) &= \mathbf{E}_3 \cdot \sum_{a\in\mathcal{N}} \Lambda^j_a (\Pi^j)^-_a + \mathbf{x}^j_a \times \Lambda^j_a (\Gamma^j_a)^-.\end{aligned} \tag{4.2.37}$$

As above, these two momentum maps coincide and the discrete Noether theorem ensures its conservation.

Approximate energy conservation

The spatially discretized energy is given by the Hamiltonian H associated to the Lagrangian (4.2.8) via Legendre transformation. We work with the trivialized expression of H given by

$$\begin{aligned}\mathcal{H}\left((\Lambda_a, \mathbf{x}_a, \Pi_a, \Gamma_a)_{a\in\mathcal{N}}\right) = &\sum_{a\in int(\mathcal{N})}\left(\frac{1}{2l_K M}\|\Gamma_a\|^2 + \frac{1}{2l_K}(J^{-1}\Pi_a)^T\Pi_a\right)\\ &+ \sum_{a\in\partial\mathcal{N}}\left(\frac{1}{l_K M}\|\Gamma_a\|^2 + \frac{1}{l_K}(J^{-1}\Pi_a)^T\Pi_a\right) \quad (4.2.38)\\ &+ \sum_{K\in\mathcal{T}} \mathbb{V}_K(\mathbf{x}_K, \Lambda_K).\end{aligned}$$

Initial conditions

Suppose that the initial configuration of the continuous system on G is given by $(g(0), \xi(0)) \in G \times \mathfrak{g}$. In order to solve the discrete Euler-Lagrange equations

$$g^{j-1}f^{j-1} = g^j \quad \text{and} \quad (\pi^{j-1})^+ = (\pi^j)^-,$$

we have to initialize them by choosing g^0 and f^0. Given the initial conditions $(g(0), \xi(0))$ we define $g^0 := g(0)$, $(\pi^0)^+ =: \frac{\partial\mathcal{L}}{\partial\xi}(g(0), \xi(0))$, where $\mathcal{L}$ is the continuous Lagrangian, and f^0 is defined by solving the equation

$$(f^0)^{-1}D_{f^0}\mathcal{L}_d(g^0, f^0) = \frac{\partial\mathcal{L}}{\partial\xi}(g(0), \xi(0)),$$

where the left term of the equation is defined in (4.2.14).

4.2.5 Including external torques and forces

As we mentioned in §4.2.3, external forces can be incorporated in the variational integrator by using the discrete Lagrange-d'Alembert variational principle. In the case of the beam, external forces are given by expressions F of the form given in (4.1.13). A spatial discretization yields expressions $F((\Lambda_a, \mathbf{x}_a, \dot{\Lambda}_a, \dot{\mathbf{x}}_a)_{a\in\mathcal{N}})_a$ at each node. The time integral of the virtual work

$$\int_0^T \sum_{a\in\mathcal{N}} F((\Lambda_a, \mathbf{x}_a, \dot{\Lambda}_a, \dot{\mathbf{x}}_a)_{a\in\mathcal{N}})_a \cdot (\delta\Lambda_a, \delta\mathbf{x}_a)dt$$

done by these forces in the Lagrange-d'Alembert principle is then approximated via temporal discretization by an appropriate choice of the expressions

$\mathcal{F}_d^+((\Lambda^{j+1}, \mathbf{x}^{j+1}), (F^j, H^j))_a$ and $\mathcal{F}_d^-((\Lambda^j, \mathbf{x}^j), (F^j, H^j))_a$. Here for simplicity we restrict ourselves to a one-point quadrature by choosing

$$\begin{aligned}\mathcal{F}_d^+((\Lambda^{j+1}, \mathbf{x}^{j+1}), (F^j, H^j))_a &= (0,0)\\ \mathcal{F}_d^-((\Lambda^j, \mathbf{x}^j), (F^j, H^j))_a &= \Delta t\left(\mathfrak{M}_a^-((\Lambda^j, \mathbf{x}^j), (F^j, H^j)), \mathfrak{F}_a^-((\Lambda^j, \mathbf{x}^j), (F^j, H^j))\right)\\ &\in T^*_{(\Lambda_a^j, \mathbf{x}_a^j)} SE(3)\end{aligned}$$

According to what was recalled in §4.2.3, in presence of external forces, the discrete body momenta $(\Pi_a^j)^\pm$ and $(\Gamma_a^j)^\pm$ are modified as follows

$$\begin{aligned}(\Pi_{\mathcal{F},a}^j)^+ &= (\Pi_a^j)^+\\ (\Gamma_{\mathcal{F},a}^j)^+ &= (\Gamma_a^j)^+\\ (\Pi_{\mathcal{F},a}^j)^- &= (\Pi_a^j)^- - \Delta t(\Lambda_a^j)^{-1}\mathfrak{M}_a^-((\Lambda^j, \mathbf{x}^j), (F^j, H^j))\\ (\Gamma_{\mathcal{F},a}^j)^- &= (\Gamma_a^j)^- - \Delta t(\Lambda_a^j)^{-1}\mathfrak{F}_a^-((\Lambda^j, \mathbf{x}^j), (F^j, H^j)).\end{aligned} \tag{4.2.39}$$

The discrete Lagrange-d'Alembert principle yields the equations

$$(\Pi_{\mathcal{F},a}^{j-1})^+ = (\Pi_{\mathcal{F},a}^j)^- \quad \text{and} \quad (\Gamma_{\mathcal{F},a}^{j-1})^+ = (\Gamma_{\mathcal{F},a}^j)^-,$$

or, using (4.2.39), by

$$\begin{aligned}(\Pi_a^{j-1})^+ &= (\Pi_a^j)^- - \Delta t(\Lambda_a^j)^{-1}\mathfrak{M}_a^-((\Lambda^j, \mathbf{x}^j), (F^j, H^j))\\ (\Gamma_a^{j-1})^+ &= (\Gamma_a^j)^- - \Delta t(\Lambda_a^j)^{-1}\mathfrak{F}_a^-((\Lambda^j, \mathbf{x}^j), (F^j, H^j)).\end{aligned}$$

In absence of external forces, one recovers the discrete Euler-Lagrange equations (4.2.30)–(4.2.35).

4.3 Examples

4.3.1 Beam with deformed initial configuration

As a first example we consider a geometrically exact beam lying in the $(\mathbf{E}_1, \mathbf{E}_3)$-plane with an initial deformation as depicted in the first picture of Fig 4.3.1. We choose the following parameters: beam length $L = 0.5$, mass density $\varrho = 1000$, square cross-section with edge length $a = 0.05$, Poisson ratio $\nu = 0.35$, and Young's modulus $E = 5 \cdot 10^7$.

For the numerical simulation a constant time step $\Delta t = 10^{-4}$ and an equidistant spatial discretization of the central line of the beam by 22 beam elements are chosen. We consider this problem without potential or external forces, such that $\Pi_{ext}(\phi) = 0$ and $F = 0$ (in (4.1.13)).

We implemented the equations (4.2.30)–(4.2.35), obtained via the Lie group integrator approach. For equations (4.2.33)–(4.2.35) we have an explicit update to determine $\mathbf{x}_a$ at time $j+1$, while for (4.2.30)–(4.2.32), one has to solve an implicit expression of the form

$$(FJ - JF^T) = \widehat{\mathbf{a}}.$$

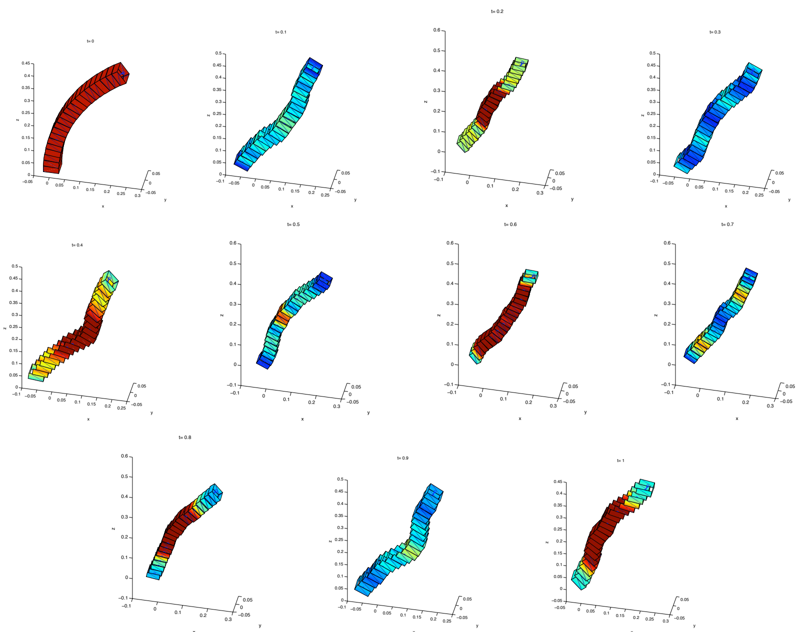


Figure 4.3.1: Beam with deformed initial configuration: snapshots of the motion and deformation at $t \in \{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$.

In order to solve this equation for $F \in SO(3)$, (the vector $\mathbf{a}$ and the symmetric matrix J being given), we use a Newton iteration method based on the Cayley transformation as described in Lee [66] (§3.3.8).

Snapshots of the motion and deformation of the spatially discretized beam are given in Figures 4.3.1 through consecutive congurations for a total simulation time $T = 1$. The elements of the beam are coloured by a linear interpolation of the sum of the norms of the stress resultants $\|\mathbf{n}_K\| + \|\mathbf{m}_K\|$ in the elements $K = 1, \ldots, 22$, as defined in (4.2.9), (4.2.10).

Fig. 4.3.2 illustrates the structure preserving properties of the variational Lie group integrator. Since the scheme is symplectic, the total energy of the beam (on the left) is not exactly preserved but numerically bounded leading to good longtime energy behavior of the simulation scheme. The plotted energy is obtained by evaluating the Hamiltonian (4.2.38) on the solution $(\Lambda_a^j, \mathbf{x}_a^j, \Pi_a^j, \Gamma_a^j)_{a \in \mathcal{N}}$ of the discrete Euler-Lagrange equations (4.2.30)-(4.2.35).

As expected from the discrete Noether theorem (§4.2.4), the two components (angular and linear, see (4.2.36)) of the discrete momentum map $\mathbf{J}_{\mathcal{L}_d}$ associated with $SE(3)$-invariance are preserved up to numerical accuracy, as shown in the middle and right plots of Fig. 4.3.2. Note that all momenta are zero in the presented case.

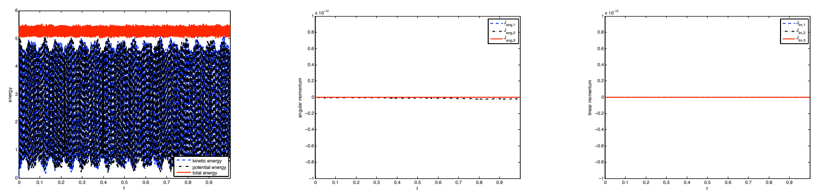


Figure 4.3.2: Beam with deformed initial configuration: evolution of energy (left), angular momentum (middle), linear momentum (right).

4.3.2 Beam with concentrated masses

To demonstrate the performance of the derived Lie group variational integrator including forces via the discrete Lagrange-d'Alembert principle, we consider a geometrically exact beam with a concentrated mass m at the middle node and concentrated masses M at the boundary nodes and with a three-dimensional loading acting on the concentrated masses (as depicted in Fig. 4.3.3).

This is a standard benchmark example which has been previously addressed e.g. in [9] with slightly different loading. The beam is initially aligned along the $\mathbf{E}_3$-axis and undeformed. For this problem, the following parameters are used: beam length $L = 2$, concentrated masses $M = 10$ and $m = 1$, mass density $\varrho = 1000$, square cross-section with edge length $a = 0.01$, Poisson ratio $\nu = 0.35$, and Young's modulus $E = 5 \cdot 10^{10}$. The temporally bounded external

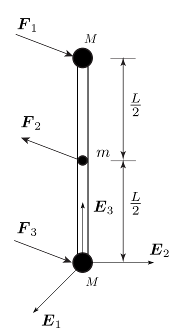


Figure 4.3.3: Beam with concentrated masses.

loading has the form

$$\mathbf{F}_\kappa(t) = f(t)\mathbf{P}_k \qquad \text{for} \quad k = 1, 2, 3$$

with

$$\begin{aligned} \mathbf{P}_1 = \mathbf{P}_3 &= -1.0\mathbf{E}_1 + 1.6\mathbf{E}_2 - 1.2\mathbf{E}_3 \\ \mathbf{P}_2 &= 1.0\mathbf{E}_1 - 1.6\mathbf{E}_2 + 1.2\mathbf{E}_3 \end{aligned}$$

and the function

$$f(t) = \begin{cases} 100\left(1 - \cos\left(\dfrac{2\pi t}{T_{\text{load}}}\right)\right) & \text{for} \quad t \leq T_{\text{load}} \\ 0 & \text{for} \quad t > T_{\text{load}} \end{cases}$$

for $T_{\text{load}} = 0.1$. No other external loads are present in this example. Furthermore, the beam's initial translational velocity is linearly distributed as

$$\dot{\phi}(S,0) = \varphi(S)\mathbf{P}, \quad \mathbf{P} = \frac{1}{20}(1.0\mathbf{E}_1 + 2.0\mathbf{E}_2 + 3.0\mathbf{E}_3)$$

with the function

$$\varphi(S) = \begin{cases} 5.5 - 11.0S & \text{for} \quad S \leq L/2 \\ -14.5 + 11.0S & \text{for} \quad S > L/2 \end{cases}$$

and the initial rotational velocity $\Omega(S,0)$ is zero.

The simulation is based on a constant time step $\Delta t = 10^{-5}$ and an equidistant spatial discretization of the central line of the beam by 22 beam elements. Snapshots of consecutive congurations for a total simulation time $T = 0.3$ are shown in Fig. 4.3.4.

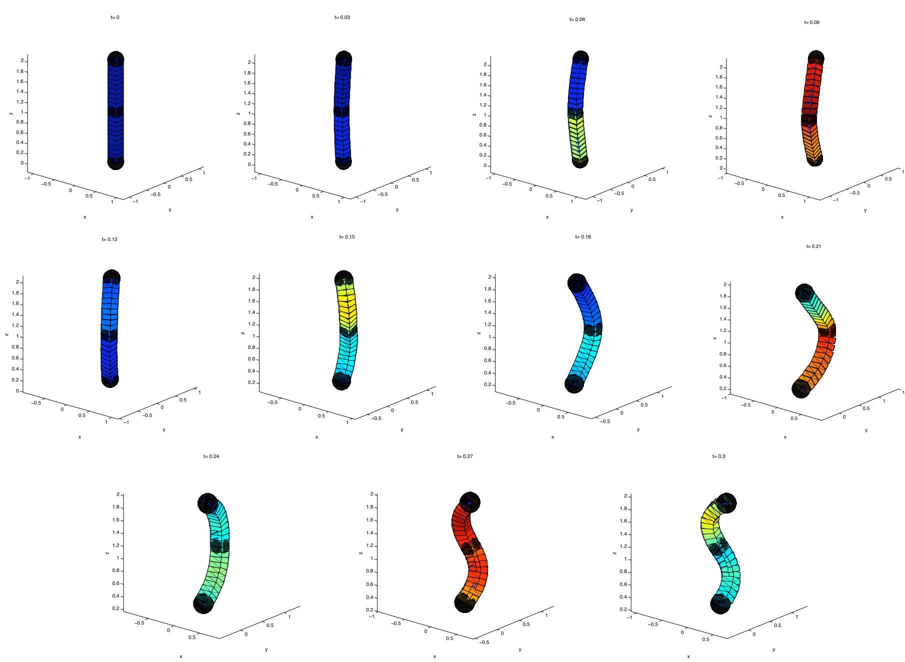


Figure 4.3.4: Beam with concentrated masses: snapshots of the motion and deformation at $t \in \{0, 0.03, 0.06, 0.09, 0.12, 0.15, 0.18, 0.21, 0.24, 0.27, 0.3\}$.

For a comparison, the same problem is simulated using an energy-momentum preserving time stepping scheme with finite elements in space as described in

Leyendecker, Betsch, and Steinmann [73]. For further energy-momentum conserving simulations of geometrically exact beam dynamics using a finite element space discretization see e. g. Romero, and Armero [101].

In Fig. 4.3.5 the energy and the angular momentum, and in Fig. 4.3.6, the linear momentum (bottom) of the beam are depicted for the two methods, the variational Lie group integrator and the energy-momentum time stepping scheme with finite elements in space.

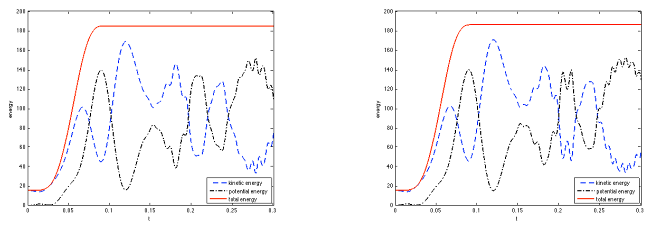


Figure 4.3.5: Beam with concentrated masses: evolution of energy. Left: Variational Lie group integrator. Right: Energy-momentum time stepping scheme with finite elements in space.

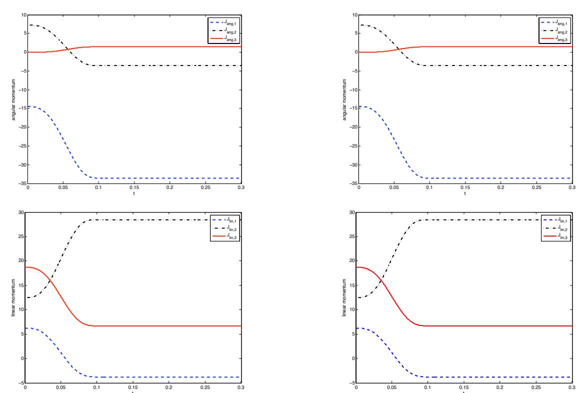


Figure 4.3.6: Beam with concentrated masses: evolution of angular momentum (top) linear momentum (bottom). Left: Variational Lie group integrator. Right: Energy-momentum time stepping scheme with finite elements in space.

Both methods provide very similar results for the evolution of these quantities: After the external loads vanish at $T_{\text{load}} = 0.1$, the total energy and all components of the angular and the linear momentum are conserved.

The evolution of the stress resultants $\mathbf{n}_K$ (shear stresses and stretch) and $\mathbf{m}_K$ (bending moment and torsional moment) in the spatial elements 1, 12 and 22 is depicted in Fig. 4.3.7. Again, the results obtained by the variational Lie group integrator (left) nicely coincide with the benchmark solution obtained by the energy-momentum time stepping scheme with finite elements (right).

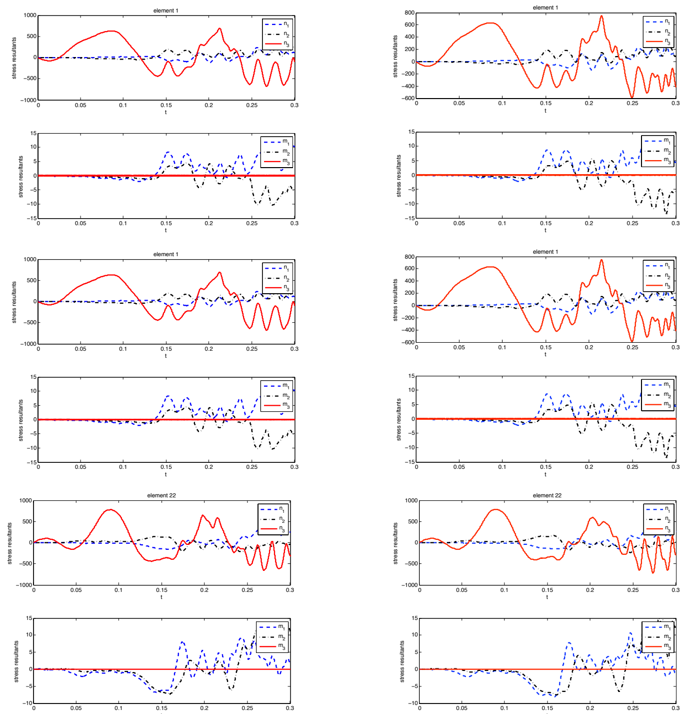


Figure 4.3.7: Beam with concentrated masses: evolution of two shear stresses and stretch (top), two bending momenta and torsional moment (bottom) in elements $1, 12, 22$. Left: Variational Lie group integrator. Right: Energy-momentum time stepping scheme with finite elements in space.

4.3.1 Remark The implementation and testing were performed by Leyendecker (University of Erlangen-Nuremberg) and Ober-Blöbaum (University of Paderborn).

4.4 Alternative temporal discretization

4.4.1 Lie group variational integrator for the beam

We now present an alternative temporal discretization of the variables $\Lambda_a(t)$, (denoted discretization $d2$). Instead of a linear interpolation in time between Λ_a^j and Λ_a^{j+1}, as we did in (4.2.4), we now consider an interpolation preserving the group $SO(3)$. More precisely, we use the temporal analog of the spatial discretization we used in (4.2.3) to preserve objectivity. We thus consider the interpolation

$$\left(\Lambda_a^j \exp\left(\frac{t-t^j}{\Delta t}\widehat{\Psi}_a^j\right),\ \mathbf{x}_a^j + \frac{t-t^j}{\Delta t}\Delta\mathbf{x}_a^j\right),$$

between $(\Lambda_a^j, \mathbf{x}_a^j)$ and $(\Lambda_a^{j+1}, \mathbf{x}_a^{j+1})$, where $\exp(\widehat{\Psi}_\alpha^j) = (\Lambda_a^j)^T \Lambda_a^{j+1} = F_a^j$. Note that we didn't change the discretization of the variable $\mathbf{x}_a(t)$. As a consequence, we get the following approximations of $\widehat{\omega}_a$ and γ_a, at time t^j:

$$\begin{aligned} \mathfrak{so}(3) \ni \widehat{\omega}_a^j &= \left(\Lambda_a^j\right)^T \dot{\Lambda}_a^j \approx \frac{\widehat{\Psi}_a^j}{\Delta t} \in \mathfrak{so}(3), \\ \mathbb{R}^3 \ni \gamma_a^j &= \left(\Lambda_a^j\right)^T \dot{\mathbf{x}}_a^j \approx \frac{(\Lambda_a^j)^T \Delta\mathbf{x}_a^j}{\Delta t} = \frac{H_a^j}{\Delta t} \in \mathbb{R}^3. \end{aligned} \tag{4.4.1}$$

The discrete Lagrangian L_K^j approximating the action of the Lagrangian $\mathcal{L}_K$ in (4.2.7) during the interval $[t^j, t^{j+1}]$, over elements K of length l_K, is therefore

$$\mathcal{L}_K^j = \sum_{a\in K}\left\{\frac{l_K}{4}\frac{M\left\|H_a^j\right\|^2}{\Delta t} + \frac{l_K}{4}\frac{(\Psi_a^j)^T J\Psi_a^j}{\Delta t}\right\} - \Delta t\ \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right),$$

where we took into account the potential energy $\mathbb{V}_K(\Lambda_K, \mathbf{x}_K)$ associated to the element K.

Then the discrete action is as follows

$$\begin{aligned} S_{d2} = &\sum_{a\neq a_0, a_N}\ \sum_{1\le j<N}\left\{\frac{l_K}{2}\frac{M\left\|H_a^j\right\|^2}{\Delta t} + \frac{l_K}{2}\frac{(\Psi_a^j)^T J\Psi_a^i}{\Delta t}\right\} \\ &+ \sum_{1\le j<N}\left\{\frac{l_K}{4}\frac{M\left\|H_{a_0}^j\right\|^2}{\Delta t} + \frac{l_K}{4}\frac{(\Psi_{a_0}^j)^T J\Psi_{a_0}^j}{\Delta t}\right\} \\ &+ \sum_{1\le j<N}\left\{\frac{l_K}{4}\frac{M\left\|H_{a_N}^j\right\|^2}{\Delta t} + \frac{l_K}{4}\frac{(\Psi_{a_N}^i)^T J\Psi_{a_N}^j}{\Delta t}\right\} \\ &- \sum_{K\in\mathcal{T}}\ \sum_{1\le j<N}\Delta t\ \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right). \end{aligned}$$

The discrete Lagrangians $\mathcal{L}_a^j$ associated to nodes a are different for $a \in int(\mathcal{N})$ and $a \in \partial\mathcal{N}$. Indeed, for $a \in int(\mathcal{N})$ the discrete Lagrangian $\mathcal{L}_a^j$ is

$$\mathcal{L}_a^j = \frac{l_K}{2}\frac{M\|H_a^j\|^2}{\Delta t} + \frac{l_K}{2}\frac{(\Psi_a^j)^T J\Psi_a^i}{\Delta t} - \sum_{K\ni a}\Delta t\mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right),$$

whereas, for a boundary node $a \in \partial\mathcal{N}$, it reads

$$\mathcal{L}_a^j = \frac{l_K}{4}\frac{M\|H_a^j\|^2}{\Delta t} + \frac{l_K}{4}\frac{(\Psi_a^j)^T J\Psi_a^i}{\Delta t} - \sum_{K\ni a}\Delta t\mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right).$$

Then using the formulas

$$\delta\widehat{\Psi}_a^j = \delta\ \mathrm{cay}^{-1}(F_a^j) = (2I - \widehat{\Psi}_a^j)\delta F_a^j(F_a^j + I)^{-1} \quad \text{and} \quad \left(J_d\widehat{\Psi}_a^j\right)^{(A)} = \frac{1}{2}\widehat{J\Psi_a^j}$$

and denoting $\delta F_a^j = F_a^j \widehat{\xi}$, we have

$$\begin{aligned} D_{F_a^j} \mathcal{L}_a^j \cdot \delta F_a^j &= \frac{l_K}{2\Delta t} \operatorname{Tr}\left(\delta \widehat{\Psi}_a^j J_d (\widehat{\Psi}_a^j)^T \right) = -\frac{l_K}{4\Delta t} \operatorname{Tr}\left(\delta \widehat{\Psi}_a^j \widehat{J \Psi_a^j} \right) \\ &= -\frac{l_K}{4\Delta t} \operatorname{Tr}\left((F_a^j + I)^{-1} \widehat{J \Psi_a^j} (2I - \widehat{\Psi}_a^j) \delta F_a^j \right) \\ &= -\frac{l_K}{4\Delta t} \operatorname{Tr}\left(\left((F_a^j + I)^{-1} \widehat{J \Psi_a^j} (2I - \widehat{\Psi}_a^j) F_a^j \right)^{(A)} \widehat{\xi} \right). \end{aligned}$$

So, using (4.2.26), we get

$$T_I^* L_{F_a^j} D_{F_a^j} \mathcal{L}_a^j = \frac{l_K}{2\Delta t} \left(\left((F_a^j + I)^{-1} \widehat{J \Psi_a^j} (2I - \widehat{\Psi}_a^j) F_a^j \right)^{(A)} \right)^{\vee}.$$

With respect to H_a^j we have

$$D_{H_a^j} \mathcal{L}_a^j = M \frac{l_K}{2\Delta t} \; H_a^j,$$

so, denoting $e = (I, 0)$, using (4.2.27) and (4.2.19), we obtain

$$\begin{aligned} & T_e^* L_{(F_a^j, H_a^j)} \left(D_{F_a^j} \mathcal{L}_a^i, \quad D_{H_a^j} \mathcal{L}_a^j \right) \\ & = \left\{ \frac{l_K}{2\Delta t} \left(\left((F_a^j + I)^{-1} \widehat{J \Psi_a^j} (2I - \widehat{\Psi}_a^j) F_a^j \right)^{(A)} \right)^{\vee}, \; M \frac{l_K}{2\Delta t} (F_a^j)^T \; H_a^j \right\} \end{aligned}$$

and

$$\begin{aligned} & \operatorname{Ad}^*_{(F_a^j, H_a^j)^{-1}} \left(T_I^* L_{F_a^j} D_{F_a^j} \mathcal{L}_a^j, \; T_0^* L_{H_a^j} D_{H_a^j} \mathcal{L}_a^j \right) \\ & = \left\{ \frac{l_K}{2\Delta t} \left(\left(F_a^j (F_a^j + I)^{-1} \widehat{J \Psi_a^j} (2I - \widehat{\Psi}_a^j) \right)^{(A)} \right)^{\vee}, \; M \frac{l_K}{2\Delta t} \; H_a^j \right\} \end{aligned}$$

We note that $F_a^j (F_a^j + I)^{-1} = (I + (F_a^j)^T)^{-1}$. Then given $D_{\Lambda_a} \mathbb{V}_K, D_{\Lambda_{a+1}} \mathbb{V}_K$, $D_{\mathbf{x}_a} \mathbb{V}_K$, and $D_{\mathbf{x}_{a+1}} \mathbb{V}_K$ already calculated we can now write explicitly the discrete Euler-Lagrange equations for the beam.

Summary of the discrete Euler-Lagrange equation. Discrete Euler-Lagrange equations for rotations :

(i) $a \notin \{a_0, a_N\}$

$$
\begin{aligned}
&\frac{l_K}{\Delta t}\left(\left((I+F_a^{j-1})^{-1}\widehat{J\Psi_a^{j-1}}(2I-\widehat{\Psi}_a^{j-1})F_a^{j-1}\right)^{(A)}\right)^{\vee} \\
&\qquad -\frac{l_K}{\Delta t}\left(\left((I+(F_a^j)^T)^{-1}\widehat{J\Psi_a^j}(2I-\widehat{\Psi}_a^j)\right)^{(A)}\right)^{\vee} \\
&= \Delta t\Bigg\{\frac{1}{2}\mathbf{C}_1\left(\Lambda_a^T\frac{\mathbf{\Delta x}_{a-1}}{l_K}-\mathbf{E}_3\right)\times\Lambda_a^T\mathbf{\Delta x}_{a-1}+\frac{1}{2}\mathbf{C}_1\left(\Lambda_a^T\frac{\mathbf{\Delta x}_a}{l_K}-\mathbf{E}_3\right)\times\Lambda_a^T\mathbf{\Delta x}_a \\
&\qquad +\frac{1}{l_K}\left(\left(\left(I+\Lambda_{a+1}^T\Lambda_a\right)^{-1}\widehat{\mathbf{C}_2\psi_a}\left(\widehat{\psi}_a-2I\right)\right)^{(A)}\right)^{\vee} \\
&\qquad +\frac{1}{l_K}\left(\left((\Lambda_{a-1}^T\Lambda_a+I)^{-1}\widehat{\mathbf{C}_2\psi_{a-1}}(2I-\widehat{\psi}_{a-1})\Lambda_{a-1}^T\Lambda_a\right)^{(A)}\right)^{\vee}\Bigg\}\Bigg|_{t=t^j}.
\end{aligned}
\tag{4.4.2}
$$

(ii) $a = a_0$

$$
\begin{aligned}
&\frac{l_K}{2\Delta t}\left(\left((I+F_a^{j-1})^{-1}\widehat{J\Psi_a^{j-1}}(2I-\widehat{\Psi}_a^{j-1})F_a^{j-1}\right)^{(A)}\right)^{\vee} \\
&\qquad -\frac{l_K}{2\Delta t}\left(\left((I+(F_a^j)^T)^{-1}\widehat{J\Psi_a^j}(2I-\widehat{\Psi}_a^j)\right)^{(A)}\right)^{\vee} \\
&= \Delta t\Bigg\{\frac{1}{2}\mathbf{C}_1\left(\Lambda_a^T\frac{\mathbf{\Delta x}_a}{l_K}-\mathbf{E}_3\right)\times\Lambda_a^T\mathbf{\Delta x}_a \\
&\qquad +\frac{1}{l_K}\left(\left(\left(I+\Lambda_{a+1}^T\Lambda_a\right)^{-1}\widehat{\mathbf{C}_2\psi_a}\left(\widehat{\psi}_a-2I\right)\right)^{(A)}\right)^{\vee}\Bigg\}\Bigg|_{t=t^j}.
\end{aligned}
\tag{4.4.3}
$$

(iii) $a = a_N$

$$
\begin{aligned}
&\frac{l_K}{2\Delta t}\left(\left((I+F_a^{j-1})^{-1}\widehat{J\Psi_a^{j-1}}(2I-\widehat{\Psi}_a^{j-1})F_a^{j-1}\right)^{(A)}\right)^{\vee} \\
&\qquad -\frac{l_K}{2\Delta t}\left(\left((I+(F_a^j)^T)^{-1}\widehat{J\Psi_a^j}(2I-\widehat{\Psi}_a^j)\right)^{(A)}\right)^{\vee} \\
&= \Delta t\Bigg\{\frac{1}{2}\left[\mathbf{C}_1\left(\Lambda_a^T\frac{\mathbf{\Delta x}_{a-1}}{l_K}-\mathbf{E}_3\right)\times\Lambda_a^T\mathbf{\Delta x}_{a-1}\right] \\
&\qquad +\frac{1}{l_K}\left(\left((\Lambda_{a-1}^T\Lambda_a+I)^{-1}\widehat{\mathbf{C}_2\psi_{a-1}}(2I-\widehat{\psi}_{a-1})\Lambda_{a-1}^T\Lambda_a\right)^{(A)}\right)^{\vee}\Bigg\}\Bigg|_{t=t^j}.
\end{aligned}
\tag{4.4.4}
$$

We consistently observe that the term $J\dot{\omega} + \omega \times J\omega$ and its spatial analog $\mathbf{C}_2\Omega' + \Omega \times \mathbf{C}_2\Omega$ appearing in (4.1.11) are now discretized in the same way.

Discrete Euler-Lagrange equations for positions, are unchanged with respect to the first discretization. That is, they are (4.2.33), (4.2.34), and (4.2.35).

Some remarks about the computations

From equation (4.4.2), (4.4.3), (4.4.4) we need to consider the expression

$$\left((I + F^T)^{-1}\widehat{J\Psi}(2I - \widehat{\Psi})\right)^{(A)} = \widehat{\mathbf{a}},$$

where

$$F = \operatorname{cay}(\widehat{\Psi}) = \left(I - \frac{\widehat{\Psi}}{2}\right)^{-1}\left(I + \frac{\widehat{\Psi}}{2}\right).$$

Defining $A := \frac{\widehat{\Psi}}{2}$, we can write the left hand side as

$$\begin{aligned}
4\left((I + (I - A)(I + A)^{-1})^{-1}\widehat{J\mathbf{A}}(I - A)\right)^{(A)} &= 2\left((I + A)\widehat{J\mathbf{A}}(I - A)\right)^{(A)} \\
&= 2(I + A)\widehat{J\mathbf{A}}(I - A) \\
&= 2\left(\widehat{J\mathbf{A}} + [A, \widehat{J\mathbf{A}}] - A\widehat{J\mathbf{A}}A\right) \\
&= 2\left(\widehat{J\mathbf{A}} + \widehat{\mathbf{A} \times J\mathbf{A}} + (\mathbf{A} \cdot J\mathbf{A})\widehat{\mathbf{A}}\right)
\end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^3$ is such that $\widehat{\mathbf{A}} = A$. We have $\widehat{J\mathbf{A}} = 2(J_d A)^{(A)}$.

So the initial equation reads

$$2\left(J\mathbf{A} + \mathbf{A} \times J\mathbf{A} + (\mathbf{A} \cdot J\mathbf{A})\mathbf{A}\right) = \mathbf{a},$$

or in terms of Ψ,

$$J\Psi + \frac{1}{2}\Psi \times J\Psi + \frac{1}{4}(\Psi \cdot J\Psi)\Psi = \mathbf{a},$$

which may be written equivalently as the non-linear equation

$$A(\Psi) = J\Psi + \frac{1}{2}\left(\Psi \times J\Psi\right) + \frac{1}{4}\Psi\left((J\Psi) \cdot \Psi\right) - a = 0,$$

with the Jacobian $DA(\Psi)$ given by

$$DA(\Psi) = J + \frac{1}{2}\widehat{\Psi}J - \frac{1}{2}\widehat{J\Psi} + \frac{1}{4}(\Psi \cdot (J\Psi))I + \frac{1}{2}\Psi(\Psi)^T J.$$

4.4.2 Discrete body momenta and Legendre transforms

To calculate the momentum and the Legendre transformation, with or without constraints, we use the same results as with the first discretization, that is (4.2.4) to obtain the discrete body momenta when there are no constraints, and (4.2.39) with constraints.

4.5 Asynchronous Lie group variational integrator (AVI)

We develop a particular Lie group variational integrator in the multisymplectic field-theoretic setting. One replaces the discrete time points with spacetime discretization. There are many references for the multisymplectic formalism (e.g. Gotay, Isenberg, Marsden, and Montgomery [36]). The discrete multisymplectic variational view of continuum mechanics was developed in Marsden, Patrick, and Shkoller [85], in Bridges, and Reich [16], and in Marsden, Pekarsky, Shkoller, and West [88]. Then after the class of asynchronous variational integrators (AVI) for non-linear elastodynamics was described in Lew, Marsden, Ortiz, and West [69; 70] in order to allow different time steps for different elements K in the mesh. (They emphasized potential computational savings for problems with localized singularities.)

Engineering researchers already explored spacetime discretization (e.g., Masud and Hughes [91]) instead of working just in space with fixed time steps. After the development of AVIs, Lew in his thesis [68] illustrates their performance through complex multiphysics problems involving multiple timescales. More recently Ryckman and Lew [103] employed these algorithms in order to study impact and contact problems, taking advantage of asynchronous time stepping.

We now derive the discrete equations of motions for the beam, by combining the tools of two variational integrators: *asynchronous variational integrators* appropriate for continuum systems and *Lie group variational integrators* appropriate for mechanical systems defined on Lie groups. We review below some basic facts needed from these two approaches.

4.5.1 Multisymplectic geometry

In continuum mechanics, the configuration is a mapping from a reference configuration $\mathcal{B} \subset \mathbb{R}^n$ to an ambient space $\mathcal{S} = \mathbb{R}^n$ which defines the configuration at each time t, in the time interval $[0, T]$.

In a multisymplectic space-time formulation one considers the base space $\mathcal{X} := \mathbb{R} \times \mathcal{B}$ which is defined to be spacetime, and we define the *configuration space* $Y := \mathcal{X} \times \mathcal{S}$, where the configuration bundle Y is a fiber bundle over χ. With the projection map $\pi_{\mathcal{X}Y} : Y \to \mathcal{X}$, and the section $\varphi : \mathcal{X} \to Y$, such that $\pi_{\mathcal{X}Y} \circ \varphi = Id$.

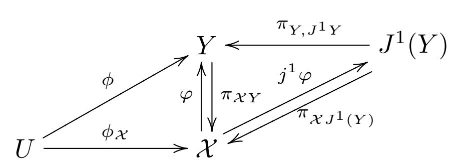


Figure 4.5.1: Jet bundle $J^1(Y)$ over Y

Where φ maps a time t and a material position X to the corresponding deformed position $x = \varphi(t, X)$. We note that it is the classic deformation maping.

The configuration of the system is specified by a map $\phi : \mathcal{U} \to Y$, where $\mathcal{U}$ is an open subset of $\mathcal{X}$, with the given base space configuration $\phi_{\mathcal{X}} : U \to \mathcal{X}$, which verify $\varphi = \phi \circ (\phi_{\mathcal{X}})^{-1}$, or $\pi_{\mathcal{X}Y} \circ \phi = \phi_{\mathcal{X}}$.

Given the configuration space Y, we construct the *jet bundle* $J^1 Y$ over Y with fibers over $y = \pi_{\mathcal{X}Y}^{-1}(t, X) \in Y_{(t,X)}$ consisting of linear maps $\gamma : T_{(t,X)}\mathcal{X} \to T_y Y$ satisfying

$$T_{\pi_{\mathcal{X}Y}} \circ \gamma = Id.$$

And we define the jet extension of φ by $j^1\varphi : \mathcal{X} \mapsto (X, T_{(t,X)}\varphi)$, which is a section of $J^1(Y)$ regarded as a bundle over $\mathcal{X}$, that is it verifies

$$\pi_{\mathcal{X}J^1(Y)} \circ j^1\varphi = I_d.$$

We call $j^1\varphi$ the *first jet* of φ. In the terminology of the motion described by the deformation mapping φ, the components of the first jet of the section φ are

$$j^1\varphi(t, X) = ((t, X), \varphi(t, X), \dot{\varphi}(t, X), F(X)),$$

where $F(X)$ is the gradient deformation. And the Lagrangian L is now defined on the jet bundle $J^1(Y) \to \mathcal{X}$, with image $L(j^1\varphi(X)) \in \mathbb{R}$.

AVIs

The idea is to replace the infinite dimensional configuration space by a finite dimensional configuration space with a basis; the latter is the frame of the algorithm. (See [69; 70] for more details.)

In discrete mechanics, we are given a fixed reference mesh $\mathcal{T}$ in $\mathcal{B}$. Associated to the nodes of the mesh, we defined the *nodal base space* $\mathcal{X}_d$ of points in $\mathcal{X}$ and the *elemental base space* $\mathcal{E}_d$ which encode the connectivity between sets of nodes and elements $K \in \mathcal{T}$. Thus we get the *discrete base space configuration* $\phi_{d,\mathcal{X}} = \{\mathcal{X}_d, \mathcal{E}_d\}$,

$$\begin{cases} \mathcal{X}_d := \{X_a^i = (t_a^i, X_a) \mid a \in \mathcal{T},\ 1 \leqslant i \leqslant n_a\}, \\ \mathcal{E}_d := \left\{E_K^j = \{X_a^i \mid a \in K,\ t_a^i \in \Theta_K^j\} \mid K \in \mathcal{T},\ 1 \leqslant j < n_K\right\}, \end{cases}$$

where $X_a^i = (t_a^i, X_a)$ is the position of node a at time t_a^i in the base space and for each element K of the mesh, Θ_K and Θ_K^j are defined as

$$\Theta_K = \{t_0 = t_K^1 < ... < t_K^{n_K - 1} < t_K^{n_K}\}$$
$$\Theta_K^j = \bigcup_{K' \in \mathcal{T} \ \mid \ K' \cap K \neq \emptyset} \left\{\Theta_{K'} \cap [t_K^j, t_K^{j+1}]\right\}.$$

We denote the entire time set by

$$\Theta = \bigcup_{K \in \mathcal{T}} \Theta_K. \tag{4.5.1}$$

In addition, we assume that there exists a map

$$\mathcal{E}_d \to \mathcal{X}, \qquad E_K^j \mapsto \mathcal{X}_{E_K^j} = [t_K^j, t_K^{j+1}] \times K,$$

where $\mathcal{X}_{E_K^j}$ is the *elemental subsets* of $\mathcal{X}$, as in Figure 4.5.2. Thus we get a mesh of space-time which discretizes the base space $\mathcal{X}$ in a finite number of elemental subsets $\mathcal{X}_E$.

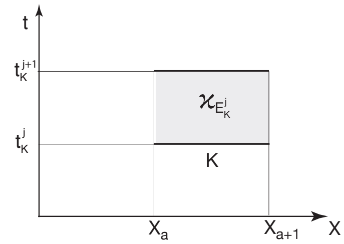


Figure 4.5.2: Elemental subsets $\mathcal{X}_{E_K^j}$

The discrete configuration ϕ_d consists of a base-space configuration $\phi_{d,\mathcal{X}}$ and a section of the discrete configuration bundle Y_d, where Y_d is defined to be the fiber bundle over $\mathcal{X}_d$, with the fiber over $X_a^i \in \mathcal{X}_d$ being the configuration bundle fiber $Y_{X_a^i}$. Then we can specify the *discrete jet extension* as

$$j^1\phi_d(E_K^j) = \left(E_K^j\ ,\ \{\mathbf{x}_a^i \in \mathbb{R}\times\mathcal{S}\ |\ X_a^i \in E_K^j\}\right),$$

where $\mathbf{x}_a^i$ is the position of X_a^i after deformation. The discrete Lagrangian is defined on J^1Y_d as

$$L_d\left(j^1\phi_d\left(E_K^j\right)\right) \approx \int_{\mathcal{X}_{E_k^j}} L\left(j^1\varphi\right) d^{n+1}X.$$

where φ is the exact solution of the Euler-Lagrange equations for L over the elemental subset $\mathcal{X}_E$. And the approximation of the continuous action integral over E_K^j is

$$\mathfrak{S}_d(\phi_d) = \sum_{E_K^j\in\mathcal{E}_d} L_d\left(j^1\phi_d\left(E_K^j\right)\right),$$

to which we apply the discrete Hamilton principle $d\mathfrak{S}_d(\phi_d)\cdot\delta\phi_d = 0$, for all $\delta\phi_d$ with zero boundary components.

In order to simplify the notation we will define a few expressions. That is, for a given node a, the ordered nodal time set for node a is denoted

$$\Theta_a := \bigcup_{\{K\in\mathcal{T}|K\ni a\}} \Theta_K = \left\{t_0 = t_a^1 \leq ... \leq t_a^{N_a-1} \leq t_a^{N_a}\right\}$$

which contains all the time steps associated to the node a, that is, all the time steps associated to the elements $K\in\mathcal{T}$ containing the node a. (See Fig. (4.5.3).)

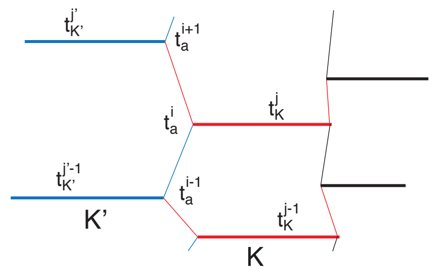


Figure 4.5.3: The ordered nodal time Θ_a

We denote by

$$\Xi = \left\{ \mathbf{x}_a^i \mid a \in K, i = 1, ..., N_a, K \in \mathcal{T} \right\}$$

the set of all nodal coordinates defining the discrete trajectory. And by

$$\Xi_K^j = \left\{ \mathbf{x}_a^i, \; a \in K, \; t_a^i \in \Theta_K^j \right\},$$

the set of discrete trajectories of nodes of K, on the interval of times $[t^j, t^{j+1}]$.

The informations given by the variables (Ξ, Θ) are the same that those contained in ϕ_d. So the variations of (Ξ, Θ) are equivalent to the variations of ϕ_d. And the total discrete action is then defined by

$$\mathfrak{S}_d(\Xi, \Theta) := \sum_{K \in \mathcal{T}} \sum_{1 \leq j < N_K} L_K^j \left(\Xi_K^j \right). \tag{4.5.2}$$

The discrete evolution is obtained by applying discrete Hamilton's principle to this action for each node a, which we denote by

$$D_a^i \mathfrak{S}_d(\Xi, \Theta) = 0, \quad \text{for } a \in \mathcal{T} \text{ and } t_a^i \in \Theta_a. \tag{4.5.3}$$

In other words, we are considering the discrete Euler-Lagrange equations for the discrete curve $\{\mathbf{x}_a^i \mid t_a^i \in \Theta_a\}$.

Until now, the time steps have been chosen arbitrarily. It is possible to choose Θ in such a way that the discrete energy is preserved by the discrete dynamics. This can be achieved by imposing the condition

$$D_K^j \mathfrak{S}_d = 0, \tag{4.5.4}$$

for all $K \in \mathcal{T}$ and all $t_K^j \in \Theta_K$, where $D_K^j \mathfrak{S}_d$ indicates the partial derivative of the discrete action with respect to the elemental time t_K^j. The equation (4.5.4) expresses a balance transfer between the different subsystems.

4.5.2 Asynchronous Lie group variational integrator for the beam

Using the same notations as before, given a node a, the discrete time evolution of this node is given by the discrete curve $\{(\Lambda_a^i, \mathbf{x}_a^i) \mid t_a^i \in \Theta_a\}$ in $SE(3)$. Since we want to apply a Lie group variational integrator, the discrete variables g^i and $f^i = (g^i)^{-1} g^{i+1}$ associated to this node are $(\Lambda_a^i, \mathbf{x}_a^i)$ and

$$(F_a^i, H_a^i) := (\Lambda_a^i, \mathbf{x}_a^i)^T (\Lambda_a^{i+1}, \mathbf{x}_a^{i+1}) = \left((\Lambda_a^i)^T \Lambda_a^{i+1}, (\Lambda_a^i)^T (\mathbf{x}_a^{i+1} - \mathbf{x}_a^i)\right),$$

where, in the last equality, we use $SE(3)$ multiplication.

First discretization

In terms of these variables (F_a^i, H_a^i), we make the same approximations as in (4.2.20). The discrete trivialized Lagrangian $\mathcal{L}_K^j$ approximating the action of the Lagrangian $\mathcal{L}_K$ in (4.2.7) during the interval $[t_K^j, t_K^{j+1}]$ is therefore

$$\mathcal{L}_K^j = \sum_{a \in K} \sum_{t_K^j \leqslant t_a^i < t_K^{j+1}} \left\{ \frac{l_K}{4} \frac{M \left\| H_a^i \right\|^2}{\left(t_a^{i+1} - t_a^i\right)} + \frac{l_K}{2} \frac{\operatorname{Tr}\left[(I_3 - F_a^i) J_d\right]}{\left(t_a^{i+1} - t_a^i\right)} \right\} - \left(t_K^{j+1} - t_K^j\right) \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j\right).$$

Where $\mathcal{L}_K^j$ depends on the nodal coordinates Ξ_K^j. We assume that $t_K^j \neq t_{K'}^{j'}$, for any pair of elements K and K'.

Using formula (4.5.2), we compute the discrete action sum (see Lew, Marsden, Ortiz, and West [70] eq. (45)), which approximates the continuous action over the time interval $[0, T]$ as follows

$$\begin{aligned}
\mathfrak{S}_d &= \sum_{K \in \mathcal{T}} \sum_{1 \leq j < N_K} \mathcal{L}_K^j \\
&= \sum_{a \neq a_0, a_N} \sum_{i=0}^{N_a - 1} \left\{ \frac{l_K}{2} \frac{M \left\| H_a^i \right\|^2}{\left(t_a^{i+1} - t_a^i\right)} + l_K \frac{\operatorname{Tr}\left[(I_3 - F_a^i) J_d\right]}{\left(t_a^{i+1} - t_a^i\right)} \right\} \\
&\qquad + \sum_{i=0}^{N_{a_0} - 1} \left\{ \frac{l_K}{4} \frac{M \left\| H_{a_0}^i \right\|^2}{\left(t_{a_0}^{i+1} - t_{a_0}^i\right)} + \frac{l_K}{2} \frac{\operatorname{Tr}\left[(I_3 - F_{a_0}^i) J_d\right]}{\left(t_{a_0}^{i+1} - t_{a_0}^i\right)} \right\} \\
&\qquad + \sum_{i=0}^{N_{a_N} - 1} \left\{ \frac{l_K}{4} \frac{M \left\| H_{a_N}^i \right\|^2}{\left(t_{a_N}^{i+1} - t_{a_N}^i\right)} + \frac{l_K}{2} \frac{\operatorname{Tr}\left[(I_3 - F_{a_N}^i) J_d\right]}{\left(t_{a_N}^{i+1} - t_{a_N}^i\right)} \right\} \\
&\quad - \sum_{K \in \mathcal{T}} \sum_{1 \leq j < N_K} \left(t_K^{j+1} - t_K^j\right) \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j\right),
\end{aligned} \tag{4.5.5}$$

where a_0 and a_N are the boundary nodes. Moreover, we used the hypothesis $t_K^j \neq t_{K'}^{j'}$, for any pair of elements K and K'.

The discrete evolution is obtained by applying discrete Hamilton's principle to $\mathfrak{S}_d$ for each node a. In other words, we are considering the discrete Euler-Lagrange equations for the discrete curve $\{(\Lambda_a^i, \mathbf{x}_a^i) \mid t_a^i \in \Theta_a\}$. From (4.5.3), we get the following systems of discrete Euler-Lagrange equations

$$\begin{aligned} T_e^* L_{(F_a^{i-1}, H_a^{i-1})} \left(D_{F_a^{i-1}} \mathcal{L}_a^{i-1}, \quad D_{H_a^{i-1}} \mathcal{L}_a^{i-1} \right) \\ - \operatorname{Ad}^*_{(F_a^i, H_a^i)^{-1}} T_e^* L_{(F_a^i, H_a^i)} \left(D_{F_a^i} \mathcal{L}_a^i, \ D_{H_a^i} \mathcal{L}_a^i \right) \\ + \ T_e^* L_{(\Lambda_a^i, \mathbf{x}_a^i)} \left(D_{\Lambda_a^i} \mathcal{L}_a^i, \ D_{\mathbf{x}_a^i} \mathcal{L}_a^i \right) = 0, \end{aligned} \tag{4.5.6}$$

for all $a \in \mathcal{T}$, where $\mathcal{L}_a^i$ denotes the dependence of $\mathfrak{S}_d$ on $(\Lambda_a^i, \mathbf{x}_a^i, F_a^i, H_a^i)$, similarly for $\mathcal{L}_a^{i-1}$. The equations are different for interior nodes, and boundary nodes a_0, a_N.

Given the discrete action sum (4.5.5) we can define the discrete Lagrangian $\mathcal{L}_a^i$ at node a and at time t_a^i as

(i) Interior nodes $a \notin \{a_0, a_N\}$

$$\begin{aligned} \mathcal{L}_a^i = \frac{l_K}{2} M \frac{\|H_a^i\|^2}{(t_a^{i+1} - t_a^i)} + l_K \frac{\operatorname{Tr}\left[(I_3 - F_a^i) J_d\right]}{(t_a^{i+1} - t_a^i)} \\ - \left(t_K^{j+1} - t_K^j\right) \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j\right)\Big|_{t_K^j = t_a^i} \end{aligned}$$

(ii) Boundaries nodes $a \in \{a_0, a_N\}$

$$\begin{aligned} \mathcal{L}_a^i = \frac{l_K}{4} M \frac{\|H_a^i\|^2}{(t_a^{i+1} - t_a^i)} + \frac{l_K}{2} \frac{\operatorname{Tr}\left[(I_3 - F_a^i) J_d\right]}{(t_a^{i+1} - t_a^i)} \\ - \left(t_K^{j+1} - t_K^j\right) \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j\right)\Big|_{t_K^j = t_a^i} \end{aligned}$$

where in the potential term we choose the unique element K containing a and such that $t_K^j = t_a^i$.

Let a single element K. Putting together the computations we made in section §(4.2.4), the system of discrete Euler-Lagrange equations (4.5.6) is equivalent to : :

(i) Rotation of the node a

$$\begin{aligned} \frac{l_K}{2\left(t_a^i - t_a^{i-1}\right)} \left(J_d F_a^{i-1} - (F_a^{i-1})^T J_d\right)^\vee - \frac{l_K}{2\left(t_a^{i+1} - t_a^i\right)} \left(F_a^i J_d - J_d (F_a^i)^T\right)^\vee \\ = \left(t_K^{j+1} - t_K^j\right) \left((\Lambda_a^i)^T D_{\Lambda_a^i} \mathbb{V}_K(\Lambda_K^j, \mathbf{x}_K^j)\right)^\vee \Big|_{t_a^i = t_K^j}, \end{aligned}$$

(ii) Displacement of the node a

$$\begin{aligned} M \frac{l_K}{2(t_a^i - t_a^{i-1})} (F_a^{i-1})^T \ H_a^{i-1} - M \frac{l_K}{2(t_a^{i+1} - t_a^i)} \ H_a^i \\ = \left(t_K^{j+1} - t_K^j\right) (\Lambda_a^i)^T D_{\mathbf{x}_a^i} \mathbb{V}_K(\Lambda_K^j, \mathbf{x}_K^j)\Big|_{t_a^i = t_K^j}. \end{aligned}$$

Note that the previous equation can be equivalently written as

$$M\frac{l_K}{2(t_a^i - t_a^{i-1})}\Delta\mathbf{x}_a^{i-1} - M\frac{l_K}{2(t_a^{i+1} - t_a^i)}\,\Delta\mathbf{x}_a^i = \left(t_K^{j+1} - t_K^j\right) D_{\mathbf{x}_a^j}\mathbb{V}_K(\Lambda_K^j, \mathbf{x}_K^j)\Big|_{t_a^i = t_K^j},$$

where $\Delta\mathbf{x}_a^i = \mathbf{x}_a^{i+1} - \mathbf{x}_a^i$ (not to be confused with $\Delta\mathbf{x}_a$ defined in (4.2.4)). This component of the discrete Euler-Euler Lagrange equations of our AVI consistently recover the discrete-Euler Lagrange equations derived in Lew, Marsden, Ortiz, and West [69], equ. (31). We already compute explicitly the components $D_{\Lambda_a}\mathbb{V}_K \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*$ and $D_{\mathbf{x}_a}\mathbb{V}_K \in \mathbb{R}^3$ due to the potential energy $\mathbb{V}_K$ given in (4.2.6).

Note that two situations can occur for a fixed node a. Either K is the element whose right node is a or K is the element whose left node is a. This depends on which element K satisfies $t_K^j = t_a^i$.

Discrete Euler-Lagrange equation with respect to matrix of rotation. From the preceding results, we obtain that the discrete Euler-Lagrange equations associated to rotation read

(i) Node $a \neq a_0$ on the left of K

$$\frac{l_K}{\left(t_a^i - t_a^{i-1}\right)}\left(J_d F_a^{i-1} - (F_a^{i-1})^T J_d\right)^\vee - \frac{l_K}{\left(t_a^{i+1} - t_a^i\right)}\left(F_a^i J_d - J_d (F_a^i)^T\right)^\vee$$
$$= \left(t_K^{j+1} - t_K^j\right)\left\{\frac{1}{2}\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)\times\Lambda_a^T\Delta\mathbf{x}_a\right.$$
$$\left.+\frac{1}{l_K}\left(\left(\left(I + \Lambda_{a+1}^T\Lambda_a\right)^{-1}\widehat{\mathbf{C}_2\psi_a}\left(\widehat{\psi}_a - 2I\right)\right)^{(A)}\right)^\vee\right\}\Bigg|_{t_a^i = t_K^j},$$

(ii) Node $a+1 \neq a_N$ on the right of K

$$\frac{l_K}{\left(t_{a+1}^i - t_{a+1}^{i-1}\right)}\left(J_d F_{a+1}^{i-1} - (F_{a+1}^{i-1})^T J_d\right)^\vee - \frac{l_K}{\left(t_{a+1}^{i+1} - t_{a+1}^i\right)}\left(F_{a+1}^i J_d - J_d (F_{a+1}^i)^T\right)^\vee$$
$$= \left(t_K^{j+1} - t_K^j\right)\left\{\frac{1}{2}\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)^T\times\Lambda_{a+1}^T\Delta\mathbf{x}_a\right.$$
$$\left.+\frac{1}{l_K}\left(\left((\Lambda_a^T\Lambda_{a+1} + I)^{-1}\widehat{\mathbf{C}_2\psi_a}(2I - \widehat{\psi}_a)\Lambda_a^T\Lambda_{a+1}\right)^{(A)}\right)^\vee\right\}\Bigg|_{t_{a+1}^i = t_K^j}.$$

(iii) Node $a = a_0$

$$\frac{l_K}{2\left(t_a^i - t_a^{i-1}\right)}\left(J_d F_a^{i-1} - (F_a^{i-1})^T J_d\right)^\vee - \frac{l_K}{2\left(t_a^{i+1} - t_a^i\right)}\left(F_a^i J_d - J_d (F_a^i)^T\right)^\vee$$
$$= \left(t_K^{j+1} - t_K^j\right)\left\{\frac{1}{2}\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K} - \mathbf{E}_3\right)\times\Lambda_a^T\Delta\mathbf{x}_a\right.$$
$$\left.+\frac{1}{l_K}\left(\left(\left(I + \Lambda_{a+1}^T\Lambda_a\right)^{-1}\widehat{\mathbf{C}_2\psi_a}\left(\widehat{\psi}_a - 2I\right)\right)^{(A)}\right)^\vee\right\}\Bigg|_{t_a^i = t_K^j},$$

(vi) Node $a+1 = a_N$

$$\frac{l_K}{2\left(t_{a+1}^{i}-t_{a+1}^{i-1}\right)}\left(J_d F_{a+1}^{i-1}-(F_{a+1}^{i-1})^T J_d\right)^{\vee}-\frac{l_K}{2\left(t_{a+1}^{i+1}-t_{a+1}^{i}\right)}\left(F_{a+1}^{i} J_d - J_d (F_{a+1}^{i})^T\right)^{\vee}$$
$$=\left(t_K^{j+1}-t_K^{j}\right)\left\{\frac{1}{2}\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)^T\times\Lambda_{a+1}^T\Delta\mathbf{x}_a\right.$$
$$\left.\left.+\frac{1}{l_K}\left(\left((\Lambda_a^T\Lambda_{a+1}+I)^{-1}\widehat{\mathbf{C}_2\psi_a}(2I-\widehat{\psi}_a)\Lambda_a^T\Lambda_{a+1}\right)^{(A)}\right)^{\vee}\right\}\right|_{t_{a+1}^{i}=t_K^{j}}.$$

Discrete Euler-Lagrange equation with respect to position. Similarly the discrete Euler-Lagrange equation yield

(i) Node $a \neq a_0$ on the left of K

$$\frac{l_K M}{\left(t_a^{i}-t_a^{i-1}\right)}\Delta\mathbf{x}_a^{i-1}-\frac{l_K M}{\left(t_a^{i+1}-t_a^{i}\right)}\Delta\mathbf{x}_a^{i}$$
$$=\left(t_K^{j+1}-t_K^{j}\right)\left\{\frac{1}{2}(-\Lambda_a)\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)\right.$$
$$\left.\left.+\frac{1}{2}(-\Lambda_{a+1})\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)+\frac{l_K}{2}\mathbf{q}+\frac{l_K}{2}\mathbf{N}\right\}\right|_{t_a^{i}=t_K^{j}},$$

(ii) Node $a+1 \neq a_N$ on the right of K

$$\frac{l_K M}{\left(t_{a+1}^{i}-t_{a+1}^{i-1}\right)}\Delta\mathbf{x}_{a+1}^{i-1}-\frac{l_K M}{\left(t_{a+1}^{i+1}-t_{a+1}^{i}\right)}\Delta\mathbf{x}_{a+1}^{i}$$
$$=\left(t_K^{j+1}-t_K^{j}\right)\left\{\frac{1}{2}(\Lambda_a)\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)\right.$$
$$\left.\left.+\frac{1}{2}(\Lambda_{a+1})\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)+\frac{l_K}{2}\mathbf{q}+\frac{l_K}{2}\mathbf{N}\right\}\right|_{t_{a+1}^{i}=t_K^{j}}.$$

(iii) Node $a = a_0$

$$\frac{l_K M}{2\left(t_a^{i}-t_a^{i-1}\right)}\Delta\mathbf{x}_a^{i-1}-\frac{l_K M}{2\left(t_a^{i+1}-t_a^{i}\right)}\Delta\mathbf{x}_a^{i}$$
$$=\left(t_K^{j+1}-t_K^{j}\right)\left\{\frac{1}{2}(-\Lambda_a)\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)\right.$$
$$\left.\left.+\frac{1}{2}(-\Lambda_{a+1})\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)+\frac{l_K}{2}\mathbf{q}+\frac{l_K}{2}\mathbf{N}\right\}\right|_{t_a^{i}=t_K^{j}},$$

(vi) Node $a+1=a_N$

$$
\begin{aligned}
&\frac{l_K M}{2\left(t^i_{a+1}-t^{i-1}_{a+1}\right)}\Delta\mathbf{x}^{i-1}_{a+1}-\frac{l_K M}{2\left(t^{i+1}_{a+1}-t^{i}_{a+1}\right)}\;\Delta\mathbf{x}^{i}_{a+1}\\
&\qquad=\left(t^{j+1}_K-t^j_K\right)\left\{\frac{1}{2}(\Lambda_a)\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)\right.\\
&\qquad\left.\left.+\frac{1}{2}(\Lambda_{a+1})\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)+\frac{l_K}{2}\mathbf{q}+\frac{l_K}{2}\mathbf{N}\right\}\right|_{t^i_{a+1}=t^j_K}.
\end{aligned}
$$

4.5.3 Energy conservation

The explicit equation of energy conservation (4.5.4) for element K and time $t^j_K=t^i_a$ may be regarded as a set of conditions determining Θ, as defined in (4.5.1). They constitute a set of discrete Euler-Lagrange equations

$$
D^j_K\mathfrak{S}_d=0, \tag{4.5.7}
$$

which indicate the partial derivative of the discrete action with respect to the elemental time $t^j_K\in\Theta$, and reads

(i) Node $a\notin\{a_0,a_N\}$

$$
\begin{aligned}
&\sum_{a\in K}\left\{\frac{l_K}{2}M\frac{\left\|H^i_a\right\|^2}{\left(t^{i+1}_a-t^i_a\right)^2}+l_K\,\mathrm{Tr}\left[\left(\frac{(F^i_a-I_3)J_d(F^i_a-I_3)^T}{(t^{i+1}_a-t^i_a)^2}\right)\right]\right\}+\mathbb{V}^j_K\\
&=\sum_{a\in K}\left\{\frac{l_K}{2}M\frac{\left\|H^{i-1}_a\right\|^2}{\left(t^{i}_a-t^{i-1}_a\right)^2}+l_K\,\mathrm{Tr}\left[\left(\frac{(F^{i-1}_a-I_3)J_d(F^{i-1}_a-I_3)^T}{(t^{i}_a-t^{i-1}_a)^2}\right)\right]\right\}+\mathbb{V}^{j-1}_K.
\end{aligned}
$$

(ii) Nodes $a_b\in\{a_0,a_N\}$ and $a\notin\{a_0,a_N\}$

$$
\begin{aligned}
&\frac{l_K}{4}M\frac{\left\|H^i_{a_b}\right\|^2}{\left(t^{i+1}_{a_b}-t^i_{a_b}\right)^2}+\frac{l_K}{2}\,\mathrm{Tr}\left[\left(\frac{(F^i_{a_b}-I_3)J_d(F^i_{a_b}-I_3)^T}{(t^{i+1}_{a_b}-t^i_{a_b})^2}\right)\right]\\
&\frac{l_K}{2}M\frac{\left\|H^i_a\right\|^2}{\left(t^{i+1}_a-t^i_a\right)^2}+l_K\,\mathrm{Tr}\left[\left(\frac{(F^i_a-I_3)J_d(F^i_a-I_3)^T}{(t^{i+1}_a-t^i_a)^2}\right)\right]+\mathbb{V}^j_K\\
&=\frac{l_K}{4}M\frac{\left\|H^{i-1}_{a_b}\right\|^2}{\left(t^{i}_{a_b}-t^{i-1}_{a_b}\right)^2}+\frac{l_K}{2}\,\mathrm{Tr}\left[\left(\frac{(F^{i-1}_{a_b}-I_3)J_d(F^{i-1}_{a_b}-I_3)^T}{(t^{i}_{a_b}-t^{i-1}_{a_b})^2}\right)\right]\\
&\frac{l_K}{2}M\frac{\left\|H^{i-1}_a\right\|^2}{\left(t^{i}_a-t^{i-1}_a\right)^2}+l_K\,\mathrm{Tr}\left[\left(\frac{(F^{i-1}_a-I_3)J_d(F^{i-1}_a-I_3)^T}{(t^{i}_a-t^{i-1}_a)^2}\right)\right]+\mathbb{V}^{j-1}_K.
\end{aligned}
$$

These equations allow us to calculate implicitly the value of t^{i+1}_a when we know values of t^i_a and t^{i-1}_a for all $a\in K$, (that is the value of $t^{j'+1}_{K'}$ in Fig. (4.5.3)).

The global energy balance between the initial and final configuration is given by the equation

$$E_d = \sum_K D_K^0 \mathfrak{S}_d = -\sum_K D_K^{N_K} \mathfrak{S}_d.$$

as the discrete Euler-Lagrange equations (4.5.7) are verified for $j = 1, ..., N_K - 1$.

Second discretization

With the discretization as defined in (4.4.1), the discrete Lagrangian $\mathcal{L}_K^j$ approximating the action of the Lagrangian $\mathcal{L}_K$ in (4.2.7) during the interval $[t_K^j, t_K^{j+1}]$, over elements K of length l_K, is therefore

$$\mathcal{L}_K^j = \sum_{a\in K} \sum_{t_K^j \leqslant t_a^i \leqslant t_K^{j+1}} \left\{ \frac{l_K}{4} M \left(t_a^{i+1} - t_a^i\right) \left\| \frac{H_a^i}{t_a^{i+1} - t_a^i} \right\|^2 + \frac{l_K}{4} \frac{(\Psi_a^i)^T J \Psi_a^i}{\left(t_a^{i+1} - t_a^i\right)} \right\} - \left(t_K^{j+1} - t_K^j\right) \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j\right).$$

And we obtain discrete Euler-Lagrange equations as previously. But we do not develop these calculations here. This is a simple repetition of what we did earlier.

4.6 Example (AVI)

Parameters of the beam : length $L = 2$, mass density $\varrho = 1000$, square cross-section with edge length $a = 1 \cdot 10^{-2}$, Poisson ratio $\nu = 0.35$, and Young's modulus $E = 5 \cdot 10^7$, for the following test:

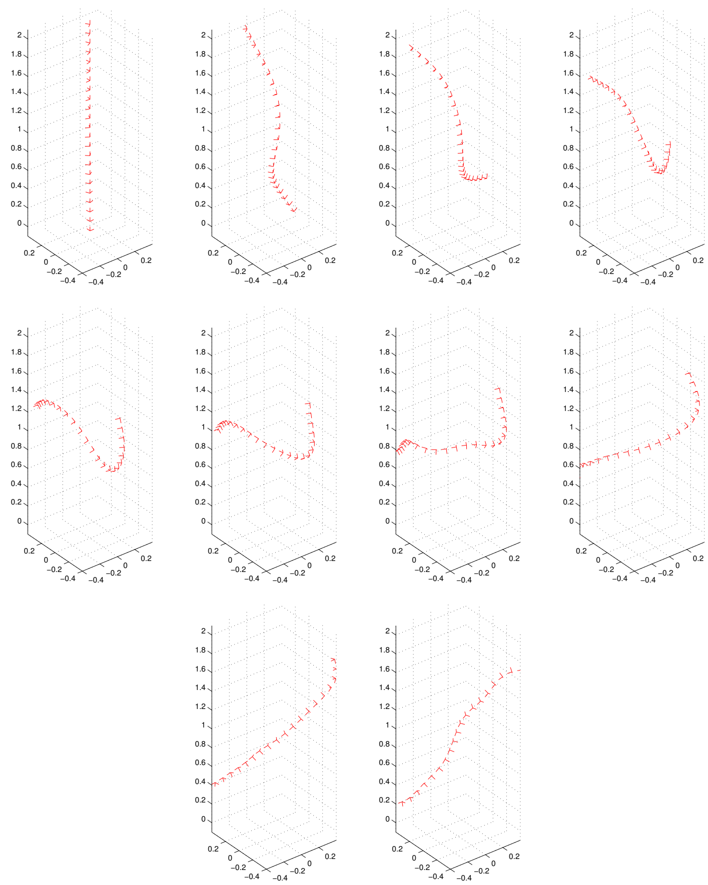


Figure 4.6.1: Motion of a beam with initial torsion and stretch, with initial velocities, without gravity.

4.6.1 Remark The implementation, in progress, is performed by Leitz (University of Erlangen-Nuremberg), Leyendecker (University of Erlangen-Nuremberg), and Ober-Blöbaum (University of Paderborn).

The integrator already works well and allows the passage from synchronous to asynchronous time-stepping without difficulties. The quality of the results is the same as those obtained in §4.3 for a regular mesh. The purpose of the actual testing is to determine what will be the results for an irregular mesh and if it performs well when the speeds have significant differences.

Chapter 5

Discrete affine Euler-Poincaré equations

Introduction

Reduction is an important tool to study many aspects of mechanical systems with symmetry. Indeed, apart from the computational simplification afforded by reduction, reduction also is an interesting way to identify invariant subsystems.

The affine Euler-Poincaré reduction is concerned with some important themes. Namely, the semi-direct product of a group G with a vector space V, where the construction of a semi-direct product involves a Lie group representation, secondly the one-cocycle $c \in \mathcal{F}(G, V^*)$ and the associated affine representation $\theta : G \to GL(V^*)$, and finally the reduction which may be Euler-Poincaré or Lie-Poisson.[1] (We can cite as a reference Marsden, Misiołek, Ortega, Perlmutter, and Ratiu [84].)

The theory of affine Euler-Poincaré that brings together these three themes was developed in Gay-Balmaz, and Ratiu [33] for fluid mechanics, and in Ellis, Gay-Balmaz, Holm, Putkaradze, and Ratiu [28] for charged molecular strands.

Given a left G-invariant Lagrangian $L : TG \to \mathbb{R}$ on TG, the reduced Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$ is defined on the Lie algebra $\mathfrak{g}$. And the evolution of the variable $\xi \in \mathfrak{g}$ is determined by the famous Euler-Poincaré equations

$$\frac{d}{dt}\frac{\delta \ell}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi}.$$

For example, given a rigid body, with $G = SO(3)$ as space of configuration, an element $\Lambda \in SO(3)$ defines the configuration of the body. For a rigid body without gravity in motion, with velocity $\dot{\Lambda}$ and inertia J, the Euler-Poincaré equations are

$$J\dot{\omega} = J\omega \times \omega,$$

[1] It is interesting to know that the link between Lie's work on the Lie-Poisson bracket and the Poincaré work on the Euler-Poincaré equations took nearly one century to improve.

where $\omega = \Lambda^{-1}\dot{\Lambda}$ is the body angular velocity.

But for the beam the gravity breaks the $SO(3)$ symmetry. The potential energy is only invariant under rotations S^1 about vertical axis $\mathbf{E}$. In this case it is more interesting to consider the Lagrangian $L : TG \times V^* \to \mathbb{R}$ defined on $TG \times V^*$, where V^* is the space of linearly advected quantities such as strain (Ω, Γ) or the direction $\chi = \Lambda^{-1}\mathbf{E}$.

If the Lagrangian $L : TG \times V^* \to \mathbb{R}$ is left G-invariant under the natural action $(v_h, a) \mapsto (g\, v_h, g\, a)$ where $g, h \in G$, $v_h \in TG$, $a \in V^*$, then we take into account the Lagrangian semi-direct product theory. (See Holm, Marsden, and Ratiu [46]), for example, with the heavy top, or with a compressible fluid associated to a right G-invariant Lagrangian.

But if the Lagrangian $L : TG \times V^* \to \mathbb{R}$ is left G-invariant under the affine action $(v_h, a) \mapsto (g\, v_h, \theta_g\, a) = (g\, v_h, g\, a + c(g))$ where $g, h \in G$, $v_h \in TG$, $a \in V^*$, and $c \in \mathcal{F}(G, V^*)$ is a one-cocycle, then we can consider the affine Euler-Poincaré theory. What is the situation encountered with the charged molecular strands, or with the geometrically exact beam.

In this chapter we develop the discrete affine Euler-Poincaré theory in order to obtain a Lie group invariant discrete Lagrangian and discrete reduction. This is one of the interesting paths that can be taken in the direction of the discrete Lagrange-Poincaré equations. The continuous Lagrange-Poincaré equations are discussed in Cendra, Marsden, and Ratiu [22] in order to study stability of relative equilibria. Nevertheless, the discrete theory still has a long way to go because in this area, nothing has been done so far.

5.1 Affine Euler-Poincaré reduction

Representations and affine representations. Let $\rho : G \to GL(V)$ be a *left Lie group representation* of a Lie group G on a vector space V. We will denote simply by

$$gu = \rho(g)(u) \quad \text{and} \quad ga = \rho(g^{-1})^*(a), \quad g \in G, \;\; u \in V, \;\; a \in V^*$$

this representation and the associated contragredient representation of G on V^*, respectively.

Let $\rho' : \mathfrak{g} \to \mathrm{End}(V)$ be the induced *Lie algebra representation* of the Lie algebra $\mathfrak{g}$ of G on V. For simplicity, we will denote by

$$\xi v = \rho'(\xi)v = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\xi)v, \quad \text{and}$$

$$\xi a = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \exp(\varepsilon\xi)a, \quad \text{with } \; \xi \in \mathfrak{g}, \;\; v \in V, \;\; a \in V^*,$$

this representation and the associated representation on V^*, respectively. Recall that we have the formula $\rho'([\xi, \nu]) = \rho'(\xi)\rho'(\nu) - \rho'(\nu)\rho'(\xi)$.

Given a left representation of G on V and a left group one-cocycle $c \in \mathcal{F}(G, V^*)$ satisfying

$$c(gh) = c(g) + gc(h), \tag{5.1.1}$$

we consider the associated *affine left representation* $\theta : G \to GL(V^*)$ defined by

$$\theta_g(a) := ga + c(g), \quad g \in G, \ \ a \in V^*. \tag{5.1.2}$$

The associated infinitesimal generator is

$$\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(t\xi)}(a) = \xi a + \mathbf{d}c(\xi), \tag{5.1.3}$$

where $\mathbf{d}c = T_e c : \mathfrak{g} \to V^*$ is the differential of c at the neutral element $e \in G$.

Given a particular value $a_{ref} \in V^*$, we denote by

$$G^c_{a_{ref}} := \{g \in G \mid \theta_g(a_{ref}) = a_{ref}\}\,. \tag{5.1.4}$$

the isotropy group of a_{ref} with respect to the affine action θ.

Semidirect products. Given a left representation of G on V, we denote by $S = G \circledS V$ the associated semidirect product whose group multiplication and inverse are

$$(g_1, u_1)(g_2, u_2) = (g_1 g_2, u_1 + g_1 u_2), \quad (g, v)^{-1} = (g^{-1}, -g^{-1}v). \tag{5.1.5}$$

We denote by $\mathfrak{s} = \mathfrak{g} \circledS V$ the Lie algebra of S, with Lie bracket

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1)\,.$$

The adjoint and coadjoint representation $\mathrm{Ad} : S \times \mathfrak{s} \to \mathfrak{s}$ and $\mathrm{Ad}^* : S \times \mathfrak{s}^* \to \mathfrak{s}^*$ are respectively given by

$$\begin{aligned} \mathrm{Ad}_{(g,v)}(\xi, u) &= (\mathrm{Ad}_g \xi, gu - (\mathrm{Ad}_g \xi)v)\,, \quad \text{and} \\ \mathrm{Ad}^*_{(g,v)}(\mu, a) &= \left(\mathrm{Ad}^*_g(\mu - v \diamond a), g^{-1}a\right), \end{aligned}$$

where $\diamond : V \times V^* \to \mathfrak{g}^*$ is the bilinear map defined by

$$\langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle \xi a, v \rangle_V\,.$$

Note that this map can be rewritten as $v \diamond a = (\rho'_v)^*(a)$, where $\rho'_v : \mathfrak{g} \to V$ is defined by $\rho'_v(\xi) = \rho'(\xi)(v)$, and $(\rho'_v)^* : V^* \to \mathfrak{g}^*$ is its dual map.

The associated infinitesimal adjoint and coadjoint representations $\mathrm{ad} : \mathfrak{s} \times \mathfrak{s} \to \mathfrak{s}$ and $ad^* : \mathfrak{s} \times \mathfrak{s}^* \to \mathfrak{s}^*$ are

$$\begin{aligned} \mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}_{(\exp(\varepsilon\xi_1), \varepsilon v_1)}(\xi_2, v_2) \\ &= ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1) = [(\xi_1, v_1), (\xi_2, v_2)] \end{aligned}$$

and

$$\mathrm{ad}^*_{(\xi,u)}(\mu, a) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathrm{Ad}^*_{(\exp(\varepsilon\xi), \varepsilon u)}(\mu, a) = \left(\mathrm{ad}^*_\xi \mu - u \diamond a, -\xi a\right).$$

5.1.1 Affine Euler-Poincaré equations

Consider a function $L : TG \times V^* \to \mathbb{R}$ which is left G-invariant under the affine action given by

$$\begin{aligned} G \times (TG \times V^*) &\to (TG \times V^*), \\ (g, (v_h, a)) &\mapsto (gv_h, \theta_g(a)) = (gv_h,\ ga + c(g)), \end{aligned} \tag{5.1.6}$$

where $g, h \in G$, $v_h \in T_hG$, $a \in V^*$, and gv_h denotes the tangent lifted action of G on TG.

Given $a_{ref} \in V^*$, we define the Lagrangian $L_{a_{ref}} : TG \to \mathbb{R}$, by $L_{a_{ref}}(v_g) := L(v_g, a_{ref})$. Then $L_{a_{ref}}$ is left invariant under the lift to TG of the left action of the isotropy group $G^c_{a_{ref}}$ on G.

By G-invariance, L induces a function $\ell : \mathfrak{g} \times V^* \to \mathbb{R}$ defined by

$$\ell\left(g^{-1}v_g, \theta_{g^{-1}}(a)\right) = L(v_g, a), \quad \forall\ g \in G, \quad v_g \in T_gG, \quad a \in V^*,$$

where $g^{-1}v_g \equiv (e, g^{-1}\dot{g})$.

Given a curve $g(t) \in G$ and $a_{ref} \in V^*$, we consider $\xi(t) := g(t)^{-1}\dot{g}(t) \in \mathfrak{g}$, and define the curve $a(t) \in V^*$ as the unique solution of the following affine differential equation

$$\dot{a} = -\xi a - \mathbf{d}c(\xi), \tag{5.1.7}$$

with the initial condition

$$a(0) = \theta_{g(0)^{-1}}(a_{ref}) = g(0)^{-1}a_{ref} + c\left(g(0)^{-1}\right), \quad \text{for} \quad g(0) \in G.$$

Recalling (5.1.3), the solution of (5.1.7) can then be written as

$$a(t) = \theta_{g(t)^{-1}}(a_{ref}) = g(t)^{-1}a_{ref} + c\left(g(t)^{-1}\right). \tag{5.1.8}$$

Without loss of generality, we always consider $g(0) = e$, so that $a(0) = a_{ref}$.

Using the preceding notation, we now recall the following Theorem from [33].

5.1.1 Theorem (Affine Euler-Poincaré reduction for semidirect product) *The following are equivalent.*

(i) *With a_{ref} held fixed, Hamilton's variational principle*

$$\delta \int_{t_0}^{t_1} L_{a_{ref}}(g, \dot{g})dt = 0,$$

holds for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

(ii) *$g(t)$ satisfies the Euler-Lagrange equations for $L_{a_{ref}}$ on G.*

(iii) *The constrained variational principle*

$$\delta \int_{t_0}^{t_1} \ell(\xi, a)dt = 0$$

holds on $\mathfrak{g} \times V^*$*, upon using variations of the form*

$$\delta\xi = \frac{\partial \eta}{\partial t} + [\xi, \eta], \qquad \delta a = -\eta a - \mathbf{d}c(\eta),$$

where $\eta(t) \in \mathfrak{g}$ *vanishes at the endpoints.*

(iv) *The* ***affine Euler-Poincaré equations*** *hold on the submanifold* $\mathfrak{g} \times V^*$ *:*

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} + \frac{\delta \ell}{\delta a} \diamond a - \mathbf{d}c^T\left(\frac{\delta \ell}{\delta a}\right).$$

Proof. See [33].

5.1.2 Affine reduction for fixed parameter

In many situations, such as the molecular strand in [28], there is no explicit expression for the G-invariant Lagrangian $L : TG \times V^* \to \mathbb{R}$. Only the expression of $L_{a_{ref}} : TG \to \mathbb{R}$ for a particular value of $a_{ref} \in V^*$ is known. However, as we recall below, the reduction process described previously can still be implemented in this case.

Let us consider a Lagrangian $L_{a_{ref}} : TG \to \mathbb{R}$ which is $G^c_{a_{ref}}$-invariant. The reduced Lagrangian ℓ associated to $L_{a_{ref}}$ is now only defined on $\mathfrak{g} \times \mathcal{O}^c_{a_{ref}} \subset \mathfrak{g} \times V^*$, where $\mathcal{O}^c_{a_{ref}} := \{\theta_g(a_{ref}) \mid g \in G\}$ is the G-orbit of a_{ref}, whose tangent space at a is

$$T_a\mathcal{O}^c_{a_{ref}} = \{\mathbf{d}c(\eta) + \eta a \mid \eta \in \mathfrak{g}\}.$$

As before, ℓ is defined by

$$\ell : \mathfrak{g} \times \mathcal{O}^c_{a_{ref}} \subset \mathfrak{g} \times V^* \to \mathbb{R}, \quad \ell(\xi, a) := \ell(g^{-1}v_g, \theta_{g^{-1}}(a_{ref})) := L_{a_{ref}}(v_g).$$

Given the $G^c_{a_{ref}}$-invariant Lagrangian $L_{a_{ref}}$ and the reduced Lagrangian $\ell(\xi, a)$, it is possible to state

5.1.2 Theorem *Let* a_{ref} *be a fixed element in* V^* *and* $g(t)$ *be a curve in* G *with* $g(0) = e$*. Define the curves* $\xi(t) = g(t)^{-1}\dot{g}(t) \in \mathfrak{g}$ *and* $a(t) := \theta_{g(t)^{-1}}(a_{ref}) \in V^*$*. Then the following are equivalent.*

(i) *With* a_{ref} *held fixed, Hamilton's variational principle*

$$\delta \int_{t_0}^{t_1} L_{a_{ref}}(g, \dot{g})dt = 0$$

holds for variations $\delta g(t)$ *of* $g(t)$ *vanishing at the endpoints.*

(ii) $g(t)$ *satisfies the Euler-Lagrange equations for* $L_{a_{ref}}$ *on* G*.*

(iii) *The constrained variational principle*

$$\delta \int_{t_0}^{t_1} \ell(\xi, a)dt = 0$$

holds on $\mathfrak{g} \times \mathcal{O}^c_{a_{ref}} \subset \mathfrak{g} \times V^*$*, upon using variations of the form*

$$\delta\xi = \frac{\partial \eta}{\partial t} + [\xi, \eta], \quad \delta a = -\eta a - \mathbf{d}c(\eta),$$

where $\eta(t) \in \mathfrak{g}$ *vanishes at the endpoints.*

(iv) *Extending* ℓ *arbitrarily to* $\mathfrak{g} \times V^*$*, the affine Euler-Poincaré equations hold on the submanifold* $\mathfrak{g} \times \mathcal{O}^c_{a_{ref}} \subset \mathfrak{g} \times V^*$ *:*

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} + \frac{\delta \ell}{\delta a} \diamond a - \mathbf{d}c^T\left(\frac{\delta \ell}{\delta a}\right). \tag{5.1.9}$$

We refer to [28] p. 43–44 for the proof.

5.2 Material and convective Lagrangian dynamics of a beam in $\mathbb{R}^3$.

5.2.1 Deformation expressed relative to the inertial frame.

As seen in section §(4.1.1), the configuration of a beam is defined by specifying the position of its curve of centroids by means of a map $\phi : [0, L] \to \mathbb{R}^3$ and the orientation of cross-sections at points $\phi(S)$ by means of a moving basis $\{\mathbf{d}_1(S), \mathbf{d}_2(S), \mathbf{d}_3(S)\}$ attached to the cross section. The orientation of the moving basis is described with the help of an orthogonal transformation $\Lambda : [0, L] \to SO(3)$ such that

$$\mathbf{d}_I(S) = \Lambda(S)\mathbf{E}_I, \quad I = 1, 2, 3,$$

where $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ is a ed basis referred to as the *material frame*.

The configuration of the beam is thus completely determined by the maps (Λ, ϕ) so that the configuration space can be identified with the Lie group $G = \mathcal{F}([0, L], SE(3))$ of smoth $SE(3)$-valued function on $[0, L]$, where $SE(3)$ is the semi-direct product $SO(3) \circledS \mathbb{R}^3$ of the Lie group $SO(3)$ with the left representation space $\mathbb{R}^3$. The Lie group $SE(3)$ is fitted with a group multiplication and inversion as given by (5.1.5).

5.2.2 Description of the variables and functions involved.

Consider the Lie algebra $\mathfrak{g} = \mathcal{F}([0, L], \mathfrak{se}(3))$ of the Lie group $G = \mathcal{F}([0, L], SE(3))$. We define the dual vector space $V^* := \Omega^1([0, L], \mathfrak{se}(3)) \oplus \mathcal{F}([0, L], \mathbb{R}^3) \oplus \mathcal{F}([0, L], \mathbb{R}^3)$ consisting of pairs formed by smooth $\mathfrak{se}(3)$-valued one-forms on $[0, L]$, that is a one form with values in the Lie algebra $\mathfrak{se}(3)$ of $SE(3)$, and $\mathbb{R}^3$-valued functions on $[0, L]$. The elements of group G are denoted by (Λ, ϕ), elements of $\mathfrak{g}$ are denoted $(\widehat{\omega}, \gamma)$, and elements of V^* are denoted by $a = (\widehat{\Omega}, \Gamma, \rho, \chi)$, such that

$$\begin{aligned}
\widehat{\omega} &= \Lambda^{-1}\dot{\Lambda} \in \mathcal{F}([0,L], \mathfrak{so}(3)) \\
\gamma &= \Lambda^{-1}\dot{\phi} \in \mathcal{F}([0,L], \mathbb{R}^3) \\
\widehat{\Omega} &= \Lambda^{-1}\Lambda' \in \Omega^1([0,L], \mathfrak{so}(3)) \\
\Gamma &= \Lambda^{-1}\phi' \in \Omega^1([0,L], \mathbb{R}^3) \\
\rho &= \Lambda^{-1}\phi \in \mathcal{F}([0,L], \mathbb{R}^3) \\
\chi &= \Lambda^{-1}\mathbf{E}_1 \in \mathcal{F}([0,L], \mathbb{R}^3),
\end{aligned} \tag{5.2.1}$$

where χ allows us to use the affine Euler-Poincaré theory at fixed parameter. The reasons that lead us to define the new variable χ will become apparent at the end of the energy description of the beam.

The variables which represent position $\rho(S,t)$, deformation gradients $(\widehat{\Omega}(S,t), \Gamma(S,t))$, and velocities $(\widehat{\omega}(S,t), \gamma(S,t))$, as well as the new variable $\chi(S,t)$, are all viewed by an observer who rotates with the beam at (S,t).

We present below the Lagrangian function of the beam. We will then show how this Lagrangian and the equations in convective representation can be obtained by the affine Euler-Poincaré process described above. This amounts to identify the appropriate space V^* of advected quantities, the appropriate affine action of G on it, as well as the appropriate reference variable a_{ref}.

5.2.3 Kinetic energy

The kinetic energy of the beam was already defined in (4.1.2) and (4.1.3).

The kinetic energy due to rotation may be noted by

$$T_{rot}(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) = \frac{1}{2}\langle\langle \omega, \omega \rangle\rangle$$

where $\langle\langle , \rangle\rangle$ is the left invariant inertia metric given at the identity. The inertia tensor matrix $J : \mathfrak{g} \to \mathfrak{g}^*$ such that $J\omega \in \mathbb{R}^3 \cong \mathfrak{so}(3)^*$ can be seen as the image by the Legendre transform of $\omega \in \mathbb{R}^3 \cong \mathfrak{so}(3)$, with $\langle\langle \omega, b \rangle\rangle = J\omega \cdot b = \langle J\omega, b \rangle$ for all $b \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$, where $\cdot$ is the dot product on $\mathbb{R}^3$, and $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{so}(3)^*$ and $\mathfrak{so}(3)$ (see [89] p488). $J\omega$ is the local *body angular momentum* density noted

$$\mathbf{\Pi} = J\omega = \frac{\delta T_{rot}}{\delta \omega},$$

which is expressed in the body frame $\{\mathbf{d}_1(S), \mathbf{d}_2(S), \mathbf{d}_3(S)\}$. Then the Hamiltonian notation for kinetic energy due to rotation may be noted $\frac{1}{2}\langle\langle \mathbf{\Pi}, J^{-1}\mathbf{\Pi} \rangle\rangle$.

5.2.4 Potential energy

The potential energy is given by the sum of interior potential energy (bending energy) and exterior potential energy (gravitational energy and energy created by external force and torque).

Bending energy

Given a configuration $(\Lambda, \phi) \in G$, the *deformation gradient* is defined as

$$F(S,t) = \Phi'(S,t) := (\Lambda'(S,t), \phi'(S,t)), \quad \text{where} \quad (.)' := \frac{\partial}{\partial S}.$$

As in [107], we use the maps $\Omega, \Gamma : [0, L] \to \mathbb{R}^3$ defined by

$$\left(\widehat{\Omega}, \Gamma\right) := \left(\Lambda^T \Lambda', \Lambda^T \phi'\right).$$

The bending energy is assumed to depend on the deformation gradient only through the quantity Ω and Γ, that is, we have

$$\Pi_{int}(\Omega, \Gamma) = \int_0^L \Psi_{int}(\Omega, \Gamma) dS,$$

where $\Psi_{int}(\Gamma, \Omega)$ is the stored energy function, as defined in (4.1.5).

Exterior potential energy

We consider exterior energy created by exterior load

$$\Pi_{ext}(\phi) = \int_0^L \langle \mathbf{q}, \phi \rangle \, dS,$$

where $\mathbf{q} = -\mathbf{E}_1$ are distributed loads per unit length. In this form $\Pi_{ext}(\phi)$ is not invariant under the left action of elements of $SO(3)$. And in order to apply the framework of Euler-Poincaré theory, we interpret $\mathbf{q}$ as a reference value, i.e. a variable encoded in the quantity $a_{ref} \in V^*$ of the abstract theory. This lead to the definition of a new convective variable. Such an approach is standard in the Euler-Poincaré description of symmetry breaking in systems such as heavy tops, compressible fluids, or nematic particles, see [46], [34]. We thus rewrite the exterior energy as

$$\Pi_{ext}(\phi) = \int_0^L \left\langle \Lambda^{-1}\mathbf{q}, \Lambda^{-1}\phi \right\rangle dS = -q \int_0^L \langle \chi, \rho \rangle \, dS =: \Pi_{ext}(\rho, \chi),$$

where we introduced the new convective variable $\chi = \Lambda^{-1}\mathbf{E}_1$.

Summing all the expressions obtained above, we can rewrite the Lagrangian in terms of the convective variables $(\omega, \gamma, \Omega, \Gamma, \rho, \chi)$ defined in (5.2.1) as

$$L(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) = T(\omega, \gamma) - \Pi_{int}(\Omega, \Gamma) - \Pi_{ext}(\rho, \chi) \;, \tag{5.2.2}$$

which is a left G-invariant Lagrangian. In the next paragraph we show how this Lagrangian can be interpreted as a reduced Lagrangian in the sense of the affine Euler-Poincaré theory. Then we will be able to apply theorem and get the equations of motion by applying the affine Euler-Poincaré reduction.

5.2.5 Advected variables and affine action for the beam

We here describe the geometry of the beam by using the formalism developed for the molecular strand in [28].

Given the dual vector space $V^* := \Omega^1([0,L], \mathfrak{se}(3)) \oplus \mathcal{F}([0,L], \mathbb{R}^3))$, we consider the representation of G on V^* defined by

$$(\Lambda, \phi) \cdot (\Omega, \Gamma, \rho) = (\mathrm{Ad}_{(\Lambda,\rho)}(\Omega, \Gamma), \Lambda\rho)$$

and the group one-cocycle $c \in \mathcal{F}(SE(3), V^*)$ given by

$$c(\Lambda, \phi) := \left((\Lambda, \phi) \left((\Lambda, \phi)^{-1} \right)', -\phi \right).$$

Note that we have

$$\begin{aligned} c((\Lambda, \phi)^{-1}) &= \left((\Lambda, \phi)^{-1} (\Lambda, \phi)', \Lambda^{-1}\phi \right) \\ &= \left(\Lambda^{-1}\Lambda', \Lambda^{-1}\phi', \Lambda^{-1}\phi \right) = (\Omega, \Gamma, \rho). \end{aligned}$$

If we chose the reference value

$$a_{ref} = (\Omega_{ref}, \Gamma_{ref}, \rho_{ref}) = (0, 0, 0),$$

then, from (5.1.8), the unique solution $a(t) = (\Omega(t), \Gamma(t), \rho(t)) \in V^*$ of the advection equation (5.1.7) is given by

$$(\Omega(t), \Gamma(t), \rho(t)) = \theta_{(\Lambda,\phi)^{-1}}(0, 0, 0) = \left(\Lambda^{-1}\Lambda', \Lambda^{-1}\phi', \Lambda^{-1}\phi \right)$$

and thus recovers the convective variables Ω, Γ, ρ.

In our situation, to solve the problem of the symmetry broken by gravity, we add the variable $\chi = \Lambda^{-1}\mathbf{E}_1$. This variable can be incorporated in our previous formalism by enlarging the representation space as $V^* := \Omega^1([0,L], \mathfrak{se}(3)) \oplus \mathcal{F}([0,L], \mathbb{R}^3) \oplus \mathcal{F}([0,L], \mathbb{R}^3)$ and considering the left representation

$$(\Omega, \Gamma, \rho, \chi) \mapsto (\Lambda, \phi) \cdot (\Omega, \Gamma, \rho, \chi) = \left(\mathrm{Ad}_{(\Lambda,\phi)}(\Omega, \Gamma), \Lambda\rho, \Lambda\chi \right), \tag{5.2.3}$$

and the cocycle

$$c((\Lambda, \phi)^{-1}) = \left((\Lambda, \phi)^{-1}(\Lambda, \phi)', \Lambda^{-1}\phi, 0 \right) = (\Omega, \Gamma, \rho, 0). \tag{5.2.4}$$

If the chosen reference value is

$$a_{ref} = (\Omega_{ref}, \Gamma_{ref}, \rho_{ref}, \chi_{ref}) = (0, 0, 0, \mathbf{E}_1), \tag{5.2.5}$$

the curve $a(t) = \theta_{g(t)^{-1}}(a_{ref})$, as defined in (5.1.8), is now given by

$$a(t) = (\Lambda, \phi)^{-1} \cdot (0, 0, 0, \mathbf{E}_1) + c\left((\Lambda, \phi)^{-1} \right) = \left(\Lambda^{-1}\Lambda', \Lambda^{-1}\phi', \Lambda^{-1}\phi, \Lambda^{-1}\mathbf{E}_1 \right).$$

and thus recovers the convective variables $(\Omega, \Gamma, \rho, \chi)$.

The isotropy group of a_{ref} is

$$\begin{aligned} G^c_{a_{ref}} &= \left\{ (\Lambda, \phi) \in G \mid \theta_{(\Lambda,\phi)}(0,0,0,\mathbf{E}_1) = (\Lambda, \phi) \cdot (0,0,0,\mathbf{E}_1) + c((\Lambda,\phi)) = (0,0,0,\mathbf{E}_1) \right\} \\ &= \left\{ (\Lambda, \phi) \in G \mid \Lambda \in \mathrm{R}_{E_1},\ c(\Lambda,\phi) = 0 \right\}, \end{aligned}$$

where R_{E_1} denotes the group of rotations around the axis given by $\mathbf{E}_1$. The orbit of a_{ref} under the affine action of G is

$$\mathcal{O}^c_{a_{ref}} = \left\{ \theta_{(\Lambda,\phi)}(0,0,0,\mathbf{E}_1) \mid (\Lambda,\phi) \in G \right\} = \left\{ \left((\Lambda,\phi) \left((\Lambda,\phi)^{-1} \right)', -\phi,\ \Lambda \mathbf{E}_1 \right) \right\}.$$

Thus the expression of L obtained in (5.2.2) and the reduced Lagrangian $\ell : \mathfrak{g} \times \mathcal{O}^c_{a_{ref}} \to \mathbb{R}$ obtained by affine Euler-Poincaré reduction with cocycle (5.2.4) and reference value (5.2.5) can be written by

$$\begin{aligned} L_{a_{ref}}(v_g) &= \ell \left(g^{-1} v_g, \theta_{g^{-1}}(a_{ref}) \right) \\ &= \frac{1}{2} \int_0^L \left[M \|\gamma\|^2 + \mathrm{Tr} \left(\omega J_d\ \omega^T \right) \right] dS \\ &\quad - \frac{1}{2} \int_0^L \left[\Omega^T \mathbf{C}_1 \Omega + (\Gamma - \mathbf{E}_3)^T \mathbf{C}_2 (\Gamma - \mathbf{E}_3) \right] dS \\ &\quad + q \int_0^L \langle \rho, \chi \rangle \, dS. \end{aligned} \tag{5.2.6}$$

5.2.6 Equations of motion

First derivation: variational principle

The reduced energy Lagrangian $\ell : \mathfrak{g} \times \mathcal{O}_{a_{ref}} \subset \mathfrak{g} \times V^* \to \mathbb{R}$ is a functional of the variables $(\omega, \gamma, \Omega, \Gamma, \rho, \chi)$, and the stationary action principle $\delta \mathfrak{S} = 0$ holds with $\mathfrak{S} = \int_0^T \ell(\omega, \gamma, \Omega, \Gamma, \rho, \chi) dt$ on time interval $[0, T]$, where variations $\delta(\omega, \gamma)$ and $\delta(\Omega, \Gamma, \rho, \chi)$ vanish at endpoints.

Consider variations

$$\varepsilon \mapsto (\Lambda_\varepsilon, \phi_\varepsilon)$$

of the curves (Λ, ϕ) with fixed endpoints. The infinitesimal variations are denoted by

$$\delta\phi = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \phi_\varepsilon, \quad \delta\Lambda = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \Lambda_\varepsilon\ .$$

Then it is useful to compute the variations $\delta\omega$, $\delta\gamma$, $\delta\Omega$, $\delta\Gamma$, $\delta\rho$ and $\delta\chi$ induced by the variations $\widehat{\Sigma} = \Lambda^{-1} \delta\Lambda$ and $\Psi = \Lambda^{-1} \delta\phi$. A direct computation shows, [28] p.26, that we have

$$\begin{aligned} \delta\omega &= \omega \times \Sigma + \dot{\Sigma} \\ \delta\gamma &= \gamma \times \Sigma + \omega \times \Psi + \dot{\Psi} \\ \delta\Omega &= \Omega \times \Sigma + \Sigma' \\ \delta\Gamma &= \Gamma \times \Sigma + \Psi' + \Omega \times \Psi \\ \delta\rho &= \rho \times \Sigma + \Psi \end{aligned} \tag{5.2.7}$$

and calculation of $\delta\chi$ gives us

$$\delta\chi = -\Lambda^{-1}\delta\Lambda\Lambda^{-1}\mathbf{E}_1 = \chi \times \Sigma \, . \tag{5.2.8}$$

The variation of the action $\mathfrak{S}$ is

$$\delta\mathfrak{S} = \int_0^T \left[\left\langle \frac{\delta \ell}{\delta\omega}, \delta\omega \right\rangle + \left\langle \frac{\delta \ell}{\delta\gamma}, \delta\gamma \right\rangle + \left\langle \frac{\delta \ell}{\delta\Omega}, \delta\Omega \right\rangle + \left\langle \frac{\delta \ell}{\delta\Gamma}, \delta\Gamma \right\rangle \right.$$
$$\left. + \left\langle \frac{\delta \ell}{\delta\rho}, \delta\rho \right\rangle + \left\langle \frac{\delta \ell}{\delta\chi}, \delta\chi \right\rangle \right] dt \ = \ 0 \, ,$$

where $\langle \ , \ \rangle$ represents the L^2 pairing in the beam variables. By substitution of the constrained variations obtained in (5.2.7) and (5.2.8) we get

$$\delta\mathfrak{S} = \int_0^T \left[\left\langle \frac{\delta \ell}{\delta\omega}, \omega \times \Sigma + \dot{\Sigma} \right\rangle + \left\langle \frac{\delta \ell}{\delta\gamma}, \gamma \times \Sigma + \omega \times \Psi + \dot{\Psi} \right\rangle + \left\langle \frac{\delta \ell}{\delta\Omega}, \Omega \times \Sigma + \Sigma' \right\rangle \right.$$
$$\left. + \left\langle \frac{\delta \ell}{\delta\Gamma}, \Gamma \times \Sigma + \Psi' + \Omega \times \Psi \right\rangle + \left\langle \frac{\delta \ell}{\delta\rho}, \rho \times \Sigma + \Psi \right\rangle + \left\langle \frac{\delta \ell}{\delta\chi}, \chi \times \Sigma \right\rangle \right] dt.$$

Integration by parts yields

$$\delta\mathfrak{S} = \int_0^T \left[\left\langle \frac{\delta \ell}{\delta\omega} \times \omega - \frac{\partial}{\partial t}\frac{\delta \ell}{\delta\omega}, \ \Sigma \right\rangle + \left\langle \frac{\delta \ell}{\delta\gamma} \times \gamma, \ \Sigma \right\rangle + \left\langle \frac{\delta \ell}{\delta\gamma} \times \omega - \frac{\partial}{\partial t}\frac{\delta \ell}{\delta\gamma}, \ \Psi \right\rangle \right.$$
$$+ \left\langle \frac{\delta \ell}{\delta\Omega} \times \Omega - \frac{\partial}{\partial S}\frac{\delta \ell}{\delta\Omega}, \ \Sigma \right\rangle + \left\langle \frac{\delta \ell}{\delta\Gamma} \times \Gamma, \ \Sigma \right\rangle + \left\langle \frac{\delta \ell}{\delta\Gamma} \times \Omega - \frac{\partial}{\partial S}\frac{\delta \ell}{\delta\Gamma}, \ \Psi \right\rangle$$
$$\left. + \left\langle \frac{\delta \ell}{\delta\rho} \times \rho, \ \Sigma \right\rangle + \left\langle \frac{\delta \ell}{\delta\rho}, \ \Psi \right\rangle + \left\langle \frac{\delta \ell}{\delta\chi} \times \chi, \ \Sigma \right\rangle \right] dt \, ,$$

Imposing $\delta\mathfrak{S} = 0$ and collecting the terms proportional to Σ, we get

$$\left(\frac{\partial}{\partial t} + \omega\times\right)\frac{\delta \ell}{\delta\omega} + \left(\frac{\partial}{\partial S} + \Omega\times\right)\frac{\delta \ell}{\delta\Omega} + \rho \times \frac{\delta \ell}{\delta\rho} + \Gamma \times \frac{\delta \ell}{\delta\Gamma} + \gamma \times \frac{\delta \ell}{\delta\gamma} + \chi \times \frac{\delta \ell}{\delta\chi} = 0 \, . \tag{5.2.9}$$

Collecting now the terms proportional to Ψ, we find

$$\left(\frac{\partial}{\partial t} + \omega\times\right)\frac{\delta \ell}{\delta\gamma} + \left(\frac{\partial}{\partial S} + \Omega\times\right)\frac{\delta \ell}{\delta\Gamma} - \frac{\delta \ell}{\delta\rho} = 0 \, . \tag{5.2.10}$$

The equations of motion are obtained by inserting the functional derivatives of the reduced Lagrangian $\ell(\omega, \gamma, \Omega, \Gamma, \rho, \chi)$ given in (5.2.6).

Second derivation: the affine Euler-Poincaré equations.

Recall that given the left G-invariant Lagrangian $L_{a_{ref}} : TG \to \mathbb{R}$, the reduced Lagrangian ℓ appearing in the affine Euler-Poincare equations (5.1.9) is defined on $\mathfrak{g} \times \mathcal{O}^c_{a_{ref}} \subset \mathfrak{g} \times V^*$ by

$$\ell\left(g^{-1}v_g, \theta_{g^{-1}}(a_{ref})\right) = L_{a_{ref}}(v_g).$$

For the beam the Lagrangian $L_{a_{ref}}$ is known for a particular value $a_{ref} = (0,0,0,\mathbf{E}_1)$ in V^*. Let us identify all the relevant objects that appear in the affine Euler-Poincaré equations.

We have

$$\begin{aligned} G &= \mathcal{F}([0,L], SE(3)) \ni (\Lambda, \phi) \\ \mathfrak{g} &= \mathcal{F}([0,L], \mathfrak{se}(3)) \ni (\omega, \gamma) \\ V^* &= \Omega^1([0,L], \mathfrak{se}(3)) \oplus \mathcal{F}([0,L], \mathbb{R}^3) \oplus \mathcal{F}([0,L], \mathbb{R}^3) \ni (\Omega, \Gamma, \rho, \chi). \end{aligned}$$

The representation space V with dual V^* is

$$V = \mathfrak{X}([0,L], \mathbb{R}^3 \times \mathbb{R}^3) \oplus \mathcal{F}([0,L], \mathbb{R}^3) \oplus \mathcal{F}([0,L], \mathbb{R}^3) \ni (u, w, f, h).$$

The left representation of the Lie group G on $(\Omega, \Gamma, \rho, \chi) \in V^*$ was given in (5.2.3). The left representation of the Lie algebra $\mathfrak{g}$ on $(\Omega, \Gamma, \rho, \chi) \in V^*$ is calculated as follows

$$(\Omega, \Gamma, \rho, \chi) \mapsto \left.\frac{d}{d\epsilon}\right|_{t=0} (\exp(\epsilon\omega), \epsilon\gamma)(\Omega, \Gamma, \rho, \chi) = (\mathrm{ad}_{(\omega,\gamma)}(\Omega, \Gamma), \omega \times \rho, \omega \times \chi)$$

We already know, see p. 51 in [28], that

$$(u, w, f) \diamond (\Omega, \Gamma, \rho) = (\mathrm{ad}^*_\Omega u + w \diamond \Gamma + f \diamond \rho, -\Omega \times w),$$

so it remains to compute the expressions $h \diamond \chi$, $\mathrm{ad}^*_\Omega u$, $w \diamond \Gamma$, and $f \diamond \rho$. For a given $(\omega, \gamma) \in \mathfrak{g}$, we have

$$\langle h \diamond \chi, (\omega, \gamma) \rangle_{\mathfrak{g}} = -\langle (\omega, \gamma)\chi, h \rangle_V = -\langle \omega \times \chi, h \rangle_V = \int_0^L ((h \times \chi) \cdot \omega)\, dS,$$

which proves that $h \diamond \chi = (h \times \chi, 0)$. We have $\mathrm{ad}^*_\Omega u = u \times \Omega$, see e.g. [89] p. 454. Given $\xi \in \mathfrak{g}$ we can let $\langle w \diamond \Gamma, \xi \rangle_{\mathfrak{g}} = -\langle \xi \times \Gamma, w \rangle_V = -(\Gamma \times w) \cdot \xi$, then we get $w \diamond \Gamma = w \times \Gamma$ and, similarly, $f \diamond \rho = f \times \rho$. This shows that

$$\begin{aligned} (u, w, f, h) \diamond (\Omega, \Gamma, \rho, \chi) &= (\mathrm{ad}^*_\Omega u + w \diamond \Gamma + f \diamond \rho + h \times \chi, -\Omega \times w), \quad (5.2.11) \\ &= (u \times \Omega + w \times \Gamma + f \times \rho + h \times \chi, -\Omega \times w) \end{aligned}$$

Given an element of the group one-cocycle $c \in \mathcal{F}(G, V^*)$ as in (5.2.4), then $\mathbf{d}c : \mathfrak{g} \to V^*$ is given by $\mathbf{d}c(\omega, \gamma) = (-\mathbf{d}\omega, -\mathbf{d}\gamma, -\gamma, 0)$. Concerning $\mathbf{d}c^T : V \to \mathfrak{g}^*$, we have $\left\langle \mathbf{d}c^T(u, w, f, h), (\omega, \gamma) \right\rangle_{\mathfrak{g}} = \langle \mathbf{d}c(\omega, \gamma), (u, w, f, h) \rangle_V = \int_0^L (-\mathbf{d}\omega u - \mathbf{d}\gamma w - \gamma f) dS$, which yields

$$\mathbf{d}c^T(u, w, f, h) = (\mathrm{div}(u), \mathrm{div}(w) - f). \quad (5.2.12)$$

Given $\xi = (\omega, \gamma)$, $a = (\Omega, \Gamma, \rho, \chi)$, and since $\mathrm{ad}^* \mapsto -\times$, and $\diamond \mapsto \times$, we are then able to calculate the affine Euler-Poincaré equations by using the equalities

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \ell}{\partial \xi} &= \left(\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \omega},\ \frac{\partial}{\partial t}\frac{\delta \ell}{\delta \gamma}\right)\\
\operatorname{ad}^*_\xi \frac{\delta \ell}{\delta \xi} &= \left(-\omega \times \frac{\delta \ell}{\delta \omega} - \gamma \times \frac{\delta \ell}{\delta \gamma},\ -\omega \times \frac{\delta \ell}{\delta \gamma}\right)\\
\frac{\delta \ell}{\delta a} \diamond a &= \left(-\Omega \times \frac{\delta \ell}{\delta \Omega} + \frac{\delta \ell}{\delta \Gamma} \times \Gamma + \frac{\delta \ell}{\delta \rho} \times \rho + \frac{\delta \ell}{\delta \chi} \times \chi,\ -\Omega \times \frac{\delta \ell}{\delta \Gamma}\right)\\
\mathbf{d}c^T \left(\frac{\delta \ell}{\delta a}\right) &= \left(\frac{\partial}{\partial S}\left(\frac{\delta \ell}{\delta \Omega}\right),\ \frac{\partial}{\partial S}\left(\frac{\delta \ell}{\delta \Gamma}\right) - \frac{\delta \ell}{\delta \rho}\right).
\end{aligned}$$

Inserted in (5.1.9), these equations recover the dynamical equations (5.2.9) and (5.2.10) of the beam.

5.3 Discrete affine Euler-Poincaré reduction

In this section we generalize the discrete Euler-Poincaré equations of Marsden, Pekarsky, and Shkoller [86] to the case of semidirect products, with an affine action.

5.3.1 Review of the discrete Euler-Poincaré equations

Consider a Lagrangian $L : TG \to \mathbb{R}$ defined on the tangent bundle of a Lie group G. Suppose that a time step h has been fixed, and let $L_d : G \times G \to \mathbb{R}$, $L_d = L_d(g^j, g^{j+1})$ be a discrete Lagrangian which we think of as approximating the action integral along the curve segment between g^j and g^{j+1}. The discrete Euler-Lagrange equations are obtained by applying the discrete Hamilton's principle to the discrete action

$$\mathfrak{S}_d(g_d) = \sum_{j=0}^{N-1} L_d\left(g^j, g^{j+1}\right).$$

The resulting equations are

$$D_2 L_d(g^{j-1}, g^j) + D_1 L_d(g^j, g^{j+1}) = 0, \quad \text{for all } j = 1, ..., N-1.$$

see e.g. [90].

Discrete Euler-Poincaré reduction

The approach developed in Marsden, Pekarsky, and Shkoller [86] is the following. If the Lagrangian $L : TG \to \mathbb{R}$ is G-invariant, the discrete Lagrangian $L_d : G \times G \to \mathbb{R}$ is chosen such that it inherits this symmetry, namely, it is required to be G-invariant under the diagonal left action of G on $G \times G$. The discrete reduction is implemented by considering the quotient map $\pi_d : G \times G \to G$, $\pi_d(g, h) = g^{-1}h$ and the corresponding reduced Lagrangian $\ell_d : G \to \mathbb{R}$ defined by

$$\ell_d\left((g^j)^{-1} g^{j+1}\right) = L_d\left(g^j, g^{j+1}\right),$$

so that the reduced action reads

$$\mathfrak{S}_d(f_d) = \sum_{j=0}^{N-1} \ell(f^j).$$

The reduced Hamilton's principle applied to s_d yields the *discrete Euler-Poincaré equations*

$$-T^*R_{f^j}\left(D_{f^j}\ell_d(f^j)\right) + T^*L_{f^{j-1}}\left(D_{f^{j-1}}\ell_d(f^{j-1})\right) = 0, \quad \text{for all } j = 1, ..., N-1, \tag{5.3.1}$$

where $D_{f^j}\ell_d(f^j) \in T^*_{f^j}G$ denote the derivative of ℓ_d.

Poisson property of the discrete Euler-Poincaré flow

Let $\Omega_{L_d} := (\mathbb{F}^{\pm}L_d)^*\Omega$ be the discrete Lagrangian symplectic form associated to a discrete Lagrangian L_d on $G \times G$. Then the discrete Lagrangian flow $F_{L_d} : G \times G \to G \times G$ defined by $F_{L_d}(g^{j-1}, g^j) = (g^j, g^{j+1})$ is symplectic relative to Ω_{L_d}. In particular, it is Poisson relative to the Poisson bracket

$$\{F, E\}_{G \times G} = \Omega_{L_d}(X_F, X_E),$$

on $G \times G$, where F, E are any C^1 function on $G \times G$, and X_F, X_E are there Hamiltonian vector fields, satisfying $i_{X_F}\Omega_{L_d} = dF$.

Let $\{f, g\}_G$ be the reduced Poisson bracket associated to the quotient map $\pi_d : G \times G \to G$, $\pi_d(g, h) = g^{-1}h$, and let $F_{\ell_d} : G \to G$ be the discrete Euler-Poincaré flow defined by $F_{\ell_d}(f^j) = f^{j+1}$. Then, as shown in [86], $F_{\ell_d} : G \to G$ is a Poisson map relative to the Poisson structure $\{f, g\}_G$.

The associated Lie-Poisson algorithm

Recall that the solution of the Lie-Poisson equations read $\mu(t) = \mathrm{Ad}^*_{g(t)}\,\mu_0 \in \mathfrak{g}^*$, where $g(t) \in G$ is the evolution of the system in the configuration group. In the discrete case, given a discrete path $\{g^j\}_{j=0}^N$, solution of the discrete Euler-Lagrange equations on $G \times G$, we can construct the discrete path $\mu^j = \mathrm{Ad}^*_{g^j}\,\mu_0$ and we have

$$\mu^{j+1} = \mathrm{Ad}^*_{g^{j+1}}\,\mu_0 = \mathrm{Ad}^*_{g^j f^j}\,\mu_0 = \mathrm{Ad}^*_{f^j}\,\mathrm{Ad}^*_{g^j}\,\mu_0 = \mathrm{Ad}^*_{f^j}\,\mu^j.$$

Thus, the discrete Euler-Poincaré integrator $\{f^j\}_{j=0}^{N-1}$ provide a Lie-Poisson integrator

$$\mu^{j+1} = \mathrm{Ad}^*_{f^j}\,\mu^j$$

that preserves the $(-)$ Lie-Poisson structure on $\mathfrak{g}^*$. This Lie-Poisson integrator recovers the Moser-Veselov equations for generalized rigid-body dynamics on $SO(n)$, see [86].

Given a discrete Lagrangian L_d, the discrete Legendre transforms $\mathbb{F}^{\pm}L_d : G \times G \to T^*G$ are defined by

$$\begin{aligned}\mathbb{F}^+L_d(g^j, g^{j+1}) \cdot \delta g^{j+1} &= D_2L_d(g^j, g^{j+1}) \cdot \delta g^{j+1},\\ \mathbb{F}^-L_d(g^j, g^{j+1}) \cdot \delta g^j &= D_1L_d(g^j, g^{j+1}) \cdot \delta g^j.\end{aligned}$$

Moreover, L_d being G-invariant , the discrete Legendre transforms are G-equivariant maps, that is,

$$D_2 L_d(hg^j, hg^{j+1}) \cdot h\delta g^{j+1} = D_2 L_d(g^j, g^{j+1}) \cdot \delta g^{j+1}$$
$$D_1 L_d(hg^j, hg^{j+1}) \cdot h\delta g^j = D_1 L_d(g^j, g^{j+1}) \cdot \delta g^j,$$

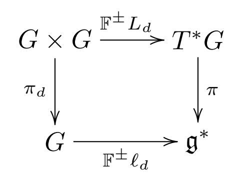
for all $h \in G$. Thus, there are natural discrete quotient maps

$$\mathbb{F}^{\pm}\ell_d : G \to \mathfrak{g}^*, \quad \text{defined by } \mathbb{F}^{\pm}\ell_d\left([g^j, g^{j+1}]_G\right) = \left[\mathbb{F}^{\pm}L_d(g^j, g^{j+1})\right]_G,$$

that is,

$$\mathbb{F}^{\pm}\ell_d(f^j) = \left[\mathbb{F}^{\pm}L_d(e, f^j)\right]_G.$$

We have thus the following commutative diagram where



$$\pi_d : G \times G \to (G \times G)/G \cong G, \qquad \pi : T^*G \to T^*G/G \cong \mathfrak{g}^*.$$

see [87].

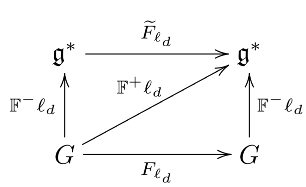
If the discrete curve $\{g^j\}$ is a solution of the discrete Euler-Lagrange equations, we have $\mathbb{F}^+L_d(g^{j-1}, g^j) = \mathbb{F}^-L_d(g^j, g^{j+1})$ and hence

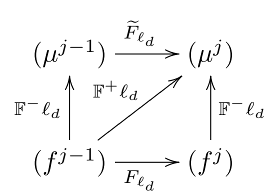
$$\mathbb{F}^+\ell_d(f^{j-1}) = \mathbb{F}^-\ell_d(f^j).$$

The corresponding discrete Hamiltonian flow is $\tilde{F}_{\ell_d} = \mathbb{F}^{\pm}\ell_d \circ F_{\ell_d} \circ (\mathbb{F}^{\pm}\ell_d)^{-1}$ and reads

$$\mu^{j+1} = \tilde{F}_{\ell_d}(\mu^j) = \mathrm{Ad}^*_{f^j}\, \mu^j.$$

As we see on the diagram we can use $\mathbb{F}^+\ell_d$ as well $\mathbb{F}^-\ell_d$ to define the discrete Hamiltonian flow $\tilde{F}_{\ell_d}$





It therefore recovers the above Lie-Poisson integrator. Furthermore by Theorem 3.2 in Marsden, Pekarsky, and Shkoller [87], we know that the Poisson structure on the Lie group G obtained by reduction of the Lagrange symplectic form Ω_{L_d} on $G \times G$ via π_d coincides with the Poisson structure on G obtained by the pull-back of the Lie-Poisson structure Ω_{ℓ_d} on $\mathfrak{g}^*$ by the Legendre transformation $\mathbb{F}\ell_d$.

Discrete Euler-Poincaré and Lie group variational integrator. Recall that the discrete trivialized Lagrangian for Lie group variational integrator is a map $\mathcal{L}_d : G \times G \to \mathbb{R}$ such that $\mathcal{L}_d(g^j, f^j)$ is an approximation of the action integral over a single time step and with boundary conditions g^j and $g^j f^j$. Discrete Hamilton's principle applied to the action

$$\mathfrak{S} = \sum_{j=0}^{N-1} \mathcal{L}_d(g^j, f^j)$$

yields the discrete time Euler-Lagrange equations

$$\begin{aligned} & T_e^* L_{f^{j-1}} \left(D_{f^{j-1}} \mathcal{L}_d^{j-1} \right) - T_e^* R_{f^j} \left(D_{f^j} \mathcal{L}_d^j \right) + T_e^* L_{g^j} \left(D_{g^j} \mathcal{L}_d^j \right) = 0, \\ & g^{j+1} = g^j f^j \quad \text{for all } j = 1, ..., N-1 \end{aligned} \tag{5.3.2}$$

We can go from the discrete Lagrangian $L_d(g^j, g^{j+1})$ of the discrete Euler-Poincaré approach to the discrete Lagrangian $\mathcal{L}_d(g^j, f^j)$ of the Lie group variational approach, by the relation

$$L_d(g^j, g^{j+1}) = \mathcal{L}_d \left(g^j, (g^j)^{-1} g^{j+1} \right).$$

We now compute the discrete Legendre transforms of $\mathcal{L}_d$ induced by the discrete Legendre transforms $\mathbb{F}^{\pm} L_d$.

$$\begin{aligned} & \mathbb{F}^- L_d(g^j, g^{j+1}) \cdot \delta g^j + \mathbb{F}^+ L_d(g^j, g^{j+1}) \cdot \delta g^{j+1} \\ & \quad = - \left\langle D_{g^j} L_d(g^j, g^{j+1}), \delta g^j \right\rangle + \left\langle D_{g^{j+1}} L_d(g^j, g^{j+1}), \delta g^{j+1} \right\rangle \\ & \quad = - \left\langle D_{g^j} \mathcal{L}_d(g^j, f^j), \delta g^j \right\rangle + \left\langle D_{f^j} \mathcal{L}_d(g^j, f^j), \delta f^j \right\rangle \\ & \quad = - \left\langle D_{g^j} \mathcal{L}_d(g^j, f^j), \delta g^j \right\rangle + \Big\langle D_{f^j} \mathcal{L}_d(g^j, f^j), \\ & \qquad\qquad \left(-(g^j)^{-1} \delta g^j f^j + f^j (g^{j+1})^{-1} \delta g^{j+1} \right) \Big\rangle \\ & \quad = - \left\langle D_{g^j} \mathcal{L}_d(g^j, f^j) - T^* L_{(g^j)^{-1}} T^* R_{f^j} D_{f^j} \mathcal{L}_d(g^j, f^j), \delta g^j \right\rangle \\ & \qquad\qquad + \left\langle T^* L_{(g^j)^{-1}} D_{f^j} \mathcal{L}_d(g^j, f^j), \delta g^{j+1} \right\rangle, \end{aligned}$$

then

$$\begin{aligned} & \mathbb{F}^+ L_d(g^j, g^{j+1}) = T^* L_{(g^j)^{-1}} D_{f^j} \mathcal{L}_d(g^j, f^j) \in T^*_{g^{j+1}} G \\ & \mathbb{F}^- L_d(g^j, g^{j+1}) = - D_{g^j} \mathcal{L}_d(g^j, f^j) + T^* L_{(g^j)^{-1}} T^* R_{f^j} D_{f^j} \mathcal{L}_d(g^j, f^j) \in T^*_{g^j} G. \end{aligned}$$

We get the discrete Hamiltonian map $\widetilde{F}_{\mathcal{L}_d} : G \times \mathfrak{g}^* \to G \times \mathfrak{g}^*$ by $\widetilde{F}_{\mathcal{L}_d} = \mathbb{F}^{\pm} \mathcal{L}_d \circ F_{\mathcal{L}_d} \circ (\mathbb{F}^{\pm} \mathcal{L}_d)^{-1}$ and the following commutative diagrams

$$\begin{array}{ccc} G \times \mathfrak{g}^* & \xrightarrow{\widetilde{F}_{\mathcal{L}_d}} & G \times \mathfrak{g}^* \\ {\scriptstyle \mathbb{F}^- \mathcal{L}_d} \uparrow & \nearrow {\scriptstyle \mathbb{F}^+ \mathcal{L}_d} & \uparrow {\scriptstyle \mathbb{F}^- \mathcal{L}_d} \\ G \times G & \xrightarrow[F_{\mathcal{L}_d}]{} & G \times G \end{array} \qquad \begin{array}{ccc} (g^{j-1}, \mu^{j-1}) & \xrightarrow{\widetilde{F}_{\mathcal{L}_d}} & (g^j, \mu^j) \\ {\scriptstyle \mathbb{F}^- \mathcal{L}_d} \uparrow & \nearrow {\scriptstyle \mathbb{F}^+ \mathcal{L}_d} & \uparrow {\scriptstyle \mathbb{F}^- \mathcal{L}_d} \\ (g^{j-1}, f^{j-1}) & \xrightarrow[F_{\mathcal{L}_d}]{} & (g^j, f^j) \end{array}$$

One observes that L_d is G-invariant under the diagonal action if and only if $\mathcal{L}_d$ does not depend on g^j, in which case, it recovers the reduced discrete Euler-Poincaré Lagrangian ℓ_d. Consistently, in this case the discrete Euler-Lagrange equation (5.3.2) recovers the discrete Euler-Poincaré equations (5.3.1).

5.3.2 Discrete affine Euler-Poincaré reduction

Consider the increasing sequence of times $\{t^j = hj \mid j = 0, ..., N\} \subset \mathbb{R}$ with time-step h, and define discrete path spaces

$$\mathcal{C}_d(G) = \{g_d : \{t^j\}_{k=0}^N \to G\}, \quad \text{and} \quad \mathcal{C}_d(V^*) = \{a_d : \{t^j\}_{j=0}^N \to V^*\}. \quad (5.3.3)$$

We identify discrete trajectories $g_d \in \mathcal{C}_d(G)$ and $a_d \in \mathcal{C}_d(V^*)$ with their images $g_d = \{g^j\}_{j=0}^N$ and $a_d = \{a^j\}_{k=0}^N$, where $g^j = g_d(t^j)$ and $a_j = a_d(t^j)$.

Let $L : TG \times V^* \to \mathbb{R}$ be a G-invariant function under the action (5.1.6) and a given $a_{ref} \in V^*$, as considered in § 5.1.1. Let $L_d : G \times G \times V^* \to \mathbb{R}$, be a function such that the discrete Lagrangian

$$L_{d,a_{ref}}(g^j, g^{j+1}) := L_d\left(g^j, g^{j+1}, a_{ref}\right)$$

is an approximation of the action integral of the original Lagrangian $L_{a_{ref}}$ along the curve segment between g^j and g^{j+1}. We assume that L_d is left invariant under the discrete affine action

$$g \cdot (g^j, g^{j+1}, a^j) := \left(gg^j, gg^{j+1}, \theta_g(a^j)\right), \quad (5.3.4)$$

where θ_g is the affine representation defined in (5.1.2). Similarly the continuous case, the discrete Lagrangian $L_{d,a_{ref}}$ is left invariant under the lift to $G \times G$ of the left action of the isotropy group $G^c_{a_{ref}}$ on G, see (5.1.4).

Then quotient map associated to the action (5.3.4) is chosen to be

$$\begin{aligned} \pi : G \times G \times V^* &\to (G \times G \times V^*)/G \cong G \times V^*, \\ \left(g^j, g^{j+1}, a_{ref}\right) &\mapsto \left((g^j)^{-1}g^{j+1}, \theta_{(g^j)^{-1}}(a_{ref})\right), \end{aligned}$$

so that the reduced discrete function $\ell_d : G \times V^* \to \mathbb{R}$, induced by the G-invariant function $L_d : G \times G \times V^* \to \mathbb{R}$ is given by

$$\ell_d^j := \ell_d\left(f^j, a^j\right) = L_d(g^j, g^{j+1}, a_{ref}), \quad g^j, g^{j+1} \in G, \;\; a^j \in V^*,$$

where $f^j \in G$ and $a^j \in V^*$ verify the relations

$$f^j = (g^j)^{-1}g^{j+1} \in G, \quad \text{for all } j = 0, ..., N-1.$$

and

$$a^j = \theta_{(g^j)^{-1}}(a_{ref}) \in V^*, \quad \text{for all } j = 0, ..., N-1.$$

Note that a_{ref} is not fixed here, it is an arbitrary element in V^*.

5.3.1 Theorem (Discrete affine Euler-Poincaré reduction) *With the preceding notation the following are equivalent.*

(i) *With $a_{ref} \in V^*$ held fixed, discrete Hamilton's variational principle*

$$\delta \sum_{j=0}^{N-1} L_{d,a_{ref}}(g^j, g^{j+1}) = 0, \tag{5.3.5}$$

holds, for variations $\delta g^0 = \delta g^N = 0$.

(ii) *The discrete path $\{g^j\}_{j=0}^N$ satisfies the discrete Euler-Lagrange equations*

$$D_2 L_{d,a_{ref}}(g^{j-1}, g^j) + D_1 L_{d,a_{ref}}(g^j, g^{j+1}) = 0, \quad \text{for all } j = 1, ..., N-1 \tag{5.3.6}$$

for $L_{d,a_{ref}}$ on $G \times G$.

(iii) *The constrained discrete variational principle*

$$\delta \sum_{j=0}^{N-1} \ell_d(f^j, a^j) = 0, \tag{5.3.7}$$

holds on $G \times V^*$ using variations of f^j and a^j of the form

$$\delta f^j = T_e L_{f^j} \left(\mathrm{Ad}_{(f^j)^{-1}} \eta^j + \eta^{j+1} \right), \qquad \delta a^j = -\eta^j a^j - \mathbf{d}c(\eta^j)$$

where $\{\eta^j\}_{j=0}^N$ is a sequence in $\mathfrak{g}$ satisfying $\eta^0 = \eta^N = 0$.

(iv) *The discrete path $\{f^j\}_{j=0}^{N-1}$ satisfies the* ***discrete affine Euler-Poincaré equations***

$$\begin{aligned} &- \mathrm{Ad}^*_{(f^j)^{-1}} T_e^* L_{f^j} \left(D_{f^j} \ell_d^j \right) + T_e^* L_{f^{j-1}} \left(D_{f^{j-1}} \ell_d^{j-1} \right) + D_{a^j} \ell_d^j \diamond a^j \\ &\quad - \mathbf{d}c^T \left(D_{a^j} \ell_d^j \right) = 0, \end{aligned} \tag{5.3.8}$$

for all $j = 1, ..., N-1$, and where ℓ_d is extended arbitrarily to $\mathfrak{g} \times V^*$.

Proof. The equivalence of (**i**) and (**ii**) is true in general, see e.g. Marsden, and West [90]. Next we show the equivalence of (**iii**) and (**iv**). The variation of f^j is computed as follows

$$\begin{aligned} \delta f^j &= -(g^j)^{-1} \delta g^j (g^j)^{-1} g^{j+1} + (g^j)^{-1} \delta g^{j+1} \\ &= -\eta^j f^j + (g^j)^{-1} g^{j+1} (g^{j+1})^{-1} \delta g^{j+1} \\ &= T_e L_{f^j} \left(- \mathrm{Ad}_{(f^j)^{-1}} \eta^j + \eta^{j+1} \right), \end{aligned} \tag{5.3.9}$$

where $\eta^j := (g^j)^{-1} \delta g^j$. The variation of a^j is computed, by using the cocycle property (5.1.1), as follows

$$\begin{aligned} \delta a^j &= -(g^j)^{-1} \delta g^j (g^j)^{-1} a_{ref} + \mathbf{d}c(-(g^j)^{-1} \delta g^j (g^j)^{-1}) \\ &= -\eta_j (g^j)^{-1} a_{ref} - \mathbf{d}c(\eta^j) - \eta_j c((g^j)^{-1}) \\ &= -\eta^j a^j - \mathbf{d}c(\eta^j). \end{aligned} \tag{5.3.10}$$

The discrete affine Euler-Poincaré equations are obtained by applying discrete Hamilton's principle to the action $\mathfrak{S}_d(f_d, a_d) = \sum_0^{N-1} \ell_d(f^j, a^j)$ relative to the constrained variations computed above. We have

$$\begin{aligned}
\delta\mathfrak{S}_d(f_d, a_d) &= \sum_{j=0}^{N-1} \left(\left\langle D_{f^j} \ell_d^j \,, \delta f^j \right\rangle + \left\langle D_{a^j} \ell_d^j \,, \delta a^j \right\rangle \right) \\
&= \sum_{j=0}^{N-1} \left(\left\langle T_e^* L_{f^j} \left(D_{f^j} \ell_d^j \right), - \operatorname{Ad}_{(f^j)^{-1}} \eta^j + \eta^{j+1} \right\rangle + \left\langle D_{a^j} \ell_d^j \,, -\eta^j a^j - \mathbf{d}c(\eta^j) \right\rangle \right) \\
&= \sum_{j=0}^{N-1} \left(\left\langle - \operatorname{Ad}^*_{(f^j)^{-1}} T_e^* L_{f^j} \left(D_{f^j} \ell_d^j \right), \eta^j \right\rangle + \left\langle T_e^* L_{f^j} \left(D_{f^j} \ell_d^j \right), \eta^{j+1} \right\rangle \right. \\
&\qquad \left. + \left\langle D_{a^j} \ell_d^j \diamond a^j, \eta^j \right\rangle - \left\langle \mathbf{d}c^T \left(D_{a^j} \ell_d^j \right), \eta^j \right\rangle \right) \\
&= \sum_{j=1}^{N-1} \left\langle - \operatorname{Ad}^*_{(f^j)^{-1}} T_e^* L_{f^j} \left(D_{f^j} \ell_d^j \right) + T_e^* L_{f^{j-1}} D_{f^{j-1}} \ell_d^{j-1} \right. \\
&\qquad \left. + D_{a^j} \ell_d^j \diamond a^j - \mathbf{d}c^T \left(D_{a^j} \ell_d^j \right), \eta^j \right\rangle ,
\end{aligned}$$

where in the last equality we used the endpoint conditions $\eta^0 = \eta^N = 0$. Since this should hold for any η^j, we obtain the equations (5.3.8).

Finally we show that (**i**) and (**iii**) are equivalent. By G-invariance of the discrete function $L_d : G \times G \times V^* \to \mathbb{R}$ and using the definition $a^j = \theta_{(g^j)^{-1}}(a_{ref})$, it follows that the discrete actions (5.3.5) and (5.3.7) are equal. Therefore, it suffices to show that all variations δg^j of g^j, with $\delta g^0 = \delta g^N = 0$, induce and are induced by the constrained variations δf^j of f^j with $\eta^0 = \eta^N = 0$. By the computations made in (5.3.9) and (5.3.10), it is clear the the variations δg^j induce the constrained variations δf^j and δa^i, where $\eta^j = (g^j)^{-1} \delta g^j$. The endpoint conditions $\delta g^0 = \delta g^N = 0$ imply $\eta^0 = \eta^N = 0$. Conversely, given the constrained variations δf^j, we define $\delta g^j := g^j \eta^j$, and observe that this yields arbitrary variations of g^j with the endpoint conditions $\delta g^0 = \delta g^N = 0$. From $\delta a^j = -\eta^j a^j - \mathbf{d}c(\eta^j)$, which is the variation of $a^j = \theta_{(g^j)^{-1}}(a_{ref})$, it follows that the variation of $\theta_{g^j}(a^j) = a_{ref}$ vanishes, which is consistent with the dependance of $L_{a_{ref}}$ only on g^j and g^{j+1}. ■

Discrete affine Euler-Poincaré reduction and Lie group variational integrators. Recall that the discrete Lagrangian of the affine Euler-Poincaré approach is

$$L_{d,a_{ref}}(g^j, g^{j+1}) = L_d(g^j, g^{j+1}, a_{ref}),$$

and the reduced discrete Lagrangian is defined by

$$\ell_d(f^j, a^j) = \ell_d((g^j)^{-1} g^{j+1}, \theta_{(g^0)^{-1}} a_{ref}) = L_d(g^j, g^{j+1}, a_{ref}).$$

It is related to the discrete Lagrangian $\mathcal{L}_d$ of the Lie group variational integrator approach by $L_d(g^j, g^{j+1}, a_{ref}) = \mathcal{L}_d(g^j, (g^j)^{-1} g^{j+1}, a_{ref})$, so that ℓ_d and

$\mathcal{L}_d$ are related by

$$\ell_d(f^j, \theta_{(g^j)^{-1}} a_{ref}) = \mathcal{L}_{d,a_{ref}}(g^j, f^j).$$

The derivative of $\mathcal{L}_{d,a_{ref}}$ with respect to g^j is thus given by

$$\begin{aligned} D_{g^j}\mathcal{L}_{d,a_{ref}} \cdot \delta g^j &= D_{a^j}\ell_d \cdot \delta\left(\theta_{(g^j)^{-1}} a_{ref})\right) = D_{a^j}\ell_d \cdot \left(\xi a^j + \mathbf{d}c(\xi)\right) \\ &= \left\langle -D_{a^j}\ell_d \diamond a^j + \mathbf{d}c^T\left(D_{a^j}\ell_d\right), \xi \right\rangle \\ &= \left\langle g^j\left(-D_{a^j}\ell_d \diamond a^j + \mathbf{d}c^T\left(D_{a^j}\ell_d\right)\right), \delta g^j \right\rangle, \qquad \xi = (g^j)^{-1}\delta g^j. \end{aligned}$$

which proves that

$$D_{g^j}\mathcal{L}_{d,a_{ref}} = g^j\left(-D_{a^j}\ell_d \diamond a^j + \mathbf{d}c^T\left(D_{a^j}\ell_d\right)\right).$$

Inserting this expression into (5.3.2), we get

$$T_e^*L_{f^{j-1}}\left(D_{f^{j-1}}\ell_d^{j-1}\right) - T_e^*R_{f^j}\left(D_{f^j}\ell_d^j\right) + D_{a^j}\ell_d \diamond a^j - \mathbf{d}c^T\left(D_{a^j}\ell_d\right) = 0$$

which recovers the discrete affine Euler-Poincaré equations.

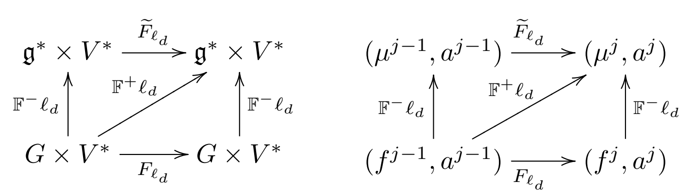
Discrete Legendre transforms. A direct computation, using (1.2.3), yields the expression $\mathbb{F}^{\pm}\ell_d : G \times V^* \to \mathfrak{g}^* \times V^*$ of the discrete Legendre transforms, given by

$$\begin{aligned} \mathbb{F}^+\ell_d(f^j, a^j) &= \left(T_e^*L_{f^j}\left(D_{f^j}\ell_d^j\right), a^{j+1}\right), \\ \mathbb{F}^-\ell_d(f^j, a^j) &= \left(\mathrm{Ad}^*_{(f^j)^{-1}} T_e^*L_{f^j}\left(D_{f^j}\ell_d^j\right) - D_{a^j}\ell_d^j \diamond a^j + \mathbf{d}c^T\left(D_{a^j}\ell_d^j\right), a^j\right). \end{aligned} \tag{5.3.11}$$

The reduced Hamiltonian $h : \mathfrak{g}^* \times V^* \to \mathbb{R}$ is defined as $h := H\big|_{\mathfrak{g}^* \times V^*}$ where $H : T^*G \times V^* \to \mathbb{R}$ is a left invariant function under the affine action (see Ellis, Gay-Balmaz, Holm, Putkaradze, and Ratiu [28]). We get the discrete Hamiltonian map $\tilde{F}_{\ell_d} : \mathfrak{g}^* \times V^* \to \mathfrak{g}^* \times V^*$

$$\tilde{F}_{\ell_d} = \mathbb{F}^{\pm}\ell_d \circ F_{\ell_d} \circ (\mathbb{F}^{\pm}\ell_d)^{-1},$$

and the following diagrams



5.3.3 Discrete affine Euler-Poincaré reduction for fixed parameter

As in §5.1.2, we now suppose that the expression of the Lagrangian $L_{a_{ref}}$ is only known for a particular fixed parameter $a_{ref} \in V^*$. As above, we consider a discrete function $L_{d,a_{ref}} : G \times G \to \mathbb{R}$, approximating the action integral of the original Lagrangian $L_{a_{ref}}$ along the curve segment between g^j and g^{j+1}. We assume that $L_{d,a_{ref}}$ is left invariant under the affine action of the isotropy subgroup $G^c_{a_{ref}}$.

As in the continuous case, the reduced discrete function ℓ_d associated to $L_{d,a_{ref}}$ is now only defined on $G \times \mathcal{O}^c_{a_{ref}} \subset G \times V^*$, where $\mathcal{O}^c_{a_{ref}} := \{\theta_g(a_{ref}) \mid g \in G\}$ is the G-orbit of a_{ref}. It is given by

$$\ell_d^j = \ell_d\left(f^j, a^j\right) = L_d(g^j, g^{j+1}, a_{ref}), \quad g^j, g^{j+1} \in G, \;\; a^j \in V^*,$$

where $f^j = (g^j)^{-1} g^{j+1} \in G$ and $a^j \theta_{(g^j)^{-1}}(a_{ref}) \in V^*$, for all $j = 0, ..., N-1$.

Given the $G^c_{a_{ref}}$-invariant Lagrangian $L_{d,a_{ref}}$ and the reduced function $\ell_d(\xi, a)$, it is possible to state the following result, whose proof follows the same steps as that of Theorem 5.3.1.

5.3.2 Theorem (Discrete affine Euler-Poincaré for fixed parameter) *Let a_{ref} be a fixed element in V^* and let $\{g^j\}_{j=0}^N$ be a discrete trajectory in G with $g^0 = e$. Define the discrete trajectories $f^j = (g^j)^{-1} g^{j+1} \in G$ and $a^j = \theta_{(g^j)^{-1}}(a_{ref}) \in V^*$, for $j = 0, ..., N-1$. Then the following are equivalent*

(i) *With $a_{ref} \in V^*$ held fixed, discrete Hamilton's variational principle*

$$\delta \sum_{j=0}^{N-1} L_{d,a_{ref}}(g^j, g^{j+1}) = 0,$$

holds, for variations $\delta g^0 = \delta g^N = 0$.

(ii) *The discrete path $\{g^j\}_{j=0}^N$ satisfies the discrete Euler-Lagrange equations*

$$D_2 L_{d,a_{ref}}(g^{j-1}, g^j) + D_1 L_{d,a_{ref}}(g^j, g^{j+1}) = 0, \quad \text{for all } j = 1, ..., N-1$$

for $L_{d,a_{ref}}$ on $G \times G$.

(iii) *The constrained discrete variational principle*

$$\delta \sum_{j=0}^{N-1} \ell_d(f^j, a^j) = 0,$$

holds on $G \times \mathcal{O}^c_{a_{ref}} \subset G \times V^*$ using variations of f^j and a^j of the form

$$\delta f^j = T_e L_{f^j}\left(\mathrm{Ad}_{(f^j)^{-1}} \eta^j + \eta^{j+1}\right), \qquad \delta a^j = -\eta^j a^j - \mathbf{d}c(\eta^j)$$

where $\{\eta^j\}_{j=0}^N$ is a sequence in $\mathfrak{g}$ satisfying $\eta^0 = \eta^N = 0$.

(iv) *The discrete path $\{f^j\}_{j=0}^{N-1}$ satisfies the **discrete affine Euler-Poincaré equations** on $G \times \mathcal{O}^c_{a_{ref}} \subset G \times V^*$:*

$$\begin{aligned} &- \mathrm{Ad}^*_{(f^j)^{-1}} T_e^* L_{f^j}\left(D_{f^j} \ell_d^j\right) + T_e^* L_{f^{j-1}}\left(D_{f^{j-1}} \ell_d^{j-1}\right) + D_{a^j} \ell_d^j \diamond a^j \\ &\quad - \mathbf{d}c^T\left(D_{a^j} \ell_d^j\right) = 0, \end{aligned} \tag{5.3.12}$$

for all $j = 1, ..., N-1$.

Discrete Legendre transforms. The Hamiltonian $H_{a_{ref}}$ counterpart of the Lagrangian $L_{a_{ref}}$ is only known for a particular fixed parameter $a_{ref} \in V^*$. And the reduced Hamiltonian $h_{a_{ref}}$, at fixed parameter, is only defined on the submanifold $\mathfrak{g}^* \times \theta^c_{a_{ref}} \subset \mathfrak{g}^* \times V^*$.

The discrete Legendre transform $\mathbb{F}^{\pm}\ell_d$ is given by (5.3.11), with $(f^j, a^j) \in G \times \theta^c_{a_{ref}}$ and image $\mathbb{F}^{\pm}\ell_d(f^j, a^j)$ in $\mathfrak{g}^* \times \theta^c_{a_{ref}}$.

5.4 Hamiltonian approach

This section (5.4) concerning Hamiltonian and Lie-Poisson is in progress. It must be supplemented by additional studies.

5.4.1 The affine Lie-Poisson algorithm

Let $S := G \circledS V$ be the semi-direct product. The lift of the left translation of S on T^*S induces the affine G-left action of G on $T^*G \times V^*$. Consider a Hamiltonian function $H : T^*G \times V^* \to \mathbb{R}$ which is left invariant under the affine G-action $(\alpha_g, a) \mapsto (h\,\alpha(t), \theta_h(a))$, for all $g, h \in G, \alpha_g \in T^*_g G$, and $a \in V^*$.

If $a_{ref} \in V^*$, we define the Hamiltonian $H_{a_{ref}} : T^*G \to \mathbb{R}$, by $H_{a_{ref}}(\alpha_g) := H(\alpha_g, a_{ref})$.

Given the G-invariance of H, the reduced Hamiltonian $h : \mathfrak{g}^* \times V^*$ is defined as

$$h(g^{-1}\alpha_g, \theta_{g^{-1}}(a)) = H(\alpha_g, a).$$

Then, as proved in [33], we have the following theorem

5.4.1 Theorem *For $\alpha(t) \in T^*_{g(t)}G$, $g(0) = e$, and $\mu(t) := T^*L_{g(t)}(\alpha(t)) \in \mathfrak{g}^*$ the following are equivalent*

(i) *$\alpha(t)$ satisfies Hamilton's equations for $H_{a_{ref}} \in T^*G$, with initial condition $\alpha(0) = \mu_0 \in T^*_e G$.*

(ii) *$(\mu(t), a(t)) := (g(t)^{-1}\alpha(t), \theta_{g(t)^{-1}}(a_{ref})) \in g^* \times V^*$, is a solution of the affine Lie-Poisson equations on $\mathfrak{s}^*$*

$$\begin{aligned}\frac{\partial}{\partial t}(\mu, a) &= \mathrm{ad}^*_{\left(\frac{\delta h}{\delta \mu}, \frac{\delta h}{\delta a}\right)}(\mu, a)\\ &= \left(\mathrm{ad}^*_{\frac{\delta h}{\delta \mu}}\mu - \frac{\delta h}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right), -\frac{\delta h}{\delta \mu}a - \mathbf{d}c\left(\frac{\delta h}{\delta \mu}\right)\right) \qquad (5.4.1)\end{aligned}$$

with initial conditions $(\mu(0), a(0)) = (\mu_0, a_{ref})$. The associated Lie-Poisson bracket of two functions $f, k : \mathfrak{s}^* \to \mathbb{R}$, on the semi-direct product Lie algebra $\mathfrak{s}^*$ is

$$\begin{aligned}\{f, k\}(\mu, a) = &-\left\langle \mu\,, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right]\right\rangle - \left\langle a\,, \frac{\delta f}{\delta a}\frac{\delta k}{\delta \mu} - \frac{\delta k}{\delta a}\frac{\delta f}{\delta \mu}\right\rangle\\ &-\left\langle \mathbf{d}c\left(\frac{\delta f}{\delta \mu}\right), \frac{\delta k}{\delta a}\right\rangle + \left\langle \mathbf{d}c\left(\frac{\delta k}{\delta \mu}\right), \frac{\delta f}{\delta a}\right\rangle.\end{aligned}$$

Moreover, in [33], they note that the affine Lie-Poisson equations for the reduced Hamiltonian h on $\mathfrak{s}^*$ are equivalent to the affine Euler-Poincaré equations (5.1.9) for the reduced Lagrangian ℓ on $\mathfrak{g} \times V^*$ together with the affine advection equation $\dot{a} + a\xi + \mathbf{d}c(\xi) = 0$.

For the left invariant system, the solution of the Lie-Poisson equations (5.4.1) read

$$(\mu(t), a(t)) = (g(t), v(t))(\mu_0, a_0), \qquad \mu(0) = \mu_0,\ a(0) = a_0,$$

where $(g(t), v(t))(\mu_0, a_0)$ is the affine coadjoint action of S on $\mathfrak{s}^*$ (see Ellis, Gay-Balmaz, Holm, Putkaradze, and Ratiu [28]), defined by

$$(g(t), v(t))(\mu_0, a_0) := \mathrm{Ad}^*_{(g(t),v(t))^{-1}}(\mu_0, a_0) + \sigma((g(t), v(t))^{-1}),$$

with the left group one-cocycle $\sigma : S \to (\mathfrak{g} \circledS V)^*$, which verify

$$\sigma(g(t), v(t)) = (v(t) \diamond c(g(t)) - \mathbf{d}c^T(v(t)),\ c(g(t)).$$

And where the trajectory $(g(t), v(t)) \in S$ is the evolution of the system, i.e. solution of the Euler-Lagrange equations defined on TS.

In the discrete case we obtain a Lie-Poisson integrator that preserves the $(-)$ Lie-Poisson structure on $\mathfrak{s}^*$, by constructing the discrete path

$$(\mu^j, a^j) = (g^j, v^j)(\mu_0, a_0),$$

where the discrete curve (g^j, v^j) is a solution of the discrete Euler-Lagrange equations on $S \times S$. Unfortunately we do not know these equations, and it seems that we cannot construct a discrete Lie-Poisson algorithm as it was done in Marsden, Pekarsky, and Shkoller [86].

5.4.2 Discrete Hamiltonian flow

If the discrete curve $\{g^j\}$ is a solution of the discrete Euler-Lagrange equations (5.3.6) on $G \times G$, and if the discrete Lagrangian is regular, then the discrete affine Euler-Poincaré equations, on $G \times V^*$, are simply

$$\mathbb{F}^+\ell_d(f^{j-1}, a^{j-1}) = \mathbb{F}^-\ell_d(f^j, a^j),$$

as seen in (5.3.11), where $\mathbb{F}^\pm\ell_d : G \times V^* \to \mathfrak{g}^* \times V^*$ are the discrete Legendre transforms of $\ell_d : G \times V^* \to \mathbb{R}$. And the corresponding discrete Hamiltonian flow, $\tilde{F}_{\ell_d} = \mathbb{F}^\pm\ell_d \circ F_{\ell_d} \circ (\mathbb{F}^\pm\ell_d)^{-1}$, reads

$$(\mu^{j+1}, a^{j+1}) = \tilde{F}_{\ell_d}(\mu^j, a^j).$$

5.4.3 Poisson property of the discrete affine Euler-Poincaré flow at fixed parameter.

It is worth noting that discrete Lagrangian L_d is left G-invariant, that is

$$L_d(h\, g^j, h\, g^{j+1}, \theta_h(a^j)) = L_d(g^j, g^{j+1}, a^j),$$

for all $g^j, g^{j+1}, h \in G$, and $a^j \in V^*$, while the discrete Lagrangian $L_{d,a_{ref}}$ is only $G^c_{a_{ref}}$-invariant, that is

$$L_{d,a_{ref}}(hg^j, hg^{j+1}) = L_{d,a_{ref}}(g^j, g^{j+1}),$$

for all $g^j, g^{j+1} \in G$, and $h \in G^c_{a_{ref}}$. Given the discrete hamilton's variational principle, for the discrete Lagrangian $L_{d,a_{ref}}$ at fixed parameter $a_{ref} \in V^*$, we obtain the Euler-Lagrange equations for $L_{d,a_{ref}}$ on $G \times G$ as well as the discrete symplectic form $\Omega_{L_{d,a_{ref}}}$ given in coordinate expression by

$$\Omega_{L_{d,a_{ref}}} = \frac{\partial^2 L_{d,a_{ref}}}{\partial g^j_\alpha \partial g^{j+1}_\beta} dg^j_\alpha \wedge dg^{j+1}_\beta,$$

which is related to the Hamiltonian momentum map $\Omega_{H_{a_{ref}}}$ by pullback under the fiber derivative

$$\Omega_{L_{d,a_{ref}}} = \left(\mathbb{F}^{\pm} L_{d,a_{ref}}\right)^* \Omega_{H_{a_{ref}}}.$$

We may associate a Poisson structure $\{\cdot,\cdot\}_{G\times G}$, which verify

$$\{F, E\}_{G\times G} = \Omega_{L_{d,a_{ref}}}(X_F, X_E),$$

on $G \times G$. Where F, E are any C^1 function on $G \times G$, and X_F, X_E are there Hamiltonian vector fields, satisfying $i_{X_F}\Omega_{L_{d,a_{ref}}} = dF$.

For the $G^c_{a_{ref}}$-invariant discrete Lagrangian $L_{a_{ref}}$, the quotient map associated to the discrete reduction is given by

$$\pi_d : G \times G \to (G \times G)/G^c_{a_{ref}}.$$

Thus we obtain a Poisson structure $\{\cdot,\cdot\}$, on $(G \times G)/G^c_{a_{ref}}$, by the relation

$$\{f, h\} \circ \pi_d = \{f \circ \pi_d, h \circ \pi_d\}_{G\times G},$$

where f, h are any C^1 function on $(G \times G)/G^c_{a_{ref}}$.

5.4.4 The associated Lie-Poisson structure at fixed parameter.

Let $S := G \circledS V$ be the semi-direct product. The lift of the left translation of S on T^*S induces the affine G-left action of G on $T^*G \times V^*$. Consider a $G^c_{a_{ref}}$-invariant Hamiltonian $H_{a_{ref}} : T^*G \to \mathbb{R}$, defined only for a fixed value $a_{ref} \in V^*$. In particular we do not know the expression of H_a for other values of $a \in V^*$.

The reduced Hamiltonian $h : \mathfrak{g}^* \times \mathcal{O}^c_{a_{ref}} \to \mathbb{R}$ is defined on the submanifold $\mathfrak{g}^* \times \mathcal{O}^c_{a_{ref}}$ of $\mathfrak{s}^*$, as

$$h(g^{-1}\alpha_g, \theta_{g^{-1}}(a_{ref})) = H(\alpha_g, a_{ref}).$$

The reduced motion evolves on an affine coadjoint orbit.

Then, as proved in Ellis, Gay-Balmaz, Holm, Putkaradze, and Ratiu [28], we have the following theorem

5.4.2 Theorem *Assuming the previous hypothesis.*

(i) *Let $\alpha(t) \in T^*_{g(t)}G$ be a solution of Hamilton's equations associated to $H_{a_{ref}}$ with initial condition $\mu_0 \in T^*_eG = \mathfrak{g}^*$. Then*

$$(\mu(t), a(t)) := (g(t)^{-1}\alpha(t), \theta_{g(t)^{-1}}(a_{ref})) \in \mathfrak{s}^*$$

is the integral curve of the Hamiltonian vector field X_h on the affine coadjoint orbit $(\mathcal{O}^\sigma_{(\mu_0,a_{ref})}, \omega^-)$ with initial condition (μ_0, a_0). Conversely, given $\mu_0 \in \mathfrak{g}^* = T^*_eG$, the solution $\alpha(t)$ of the Hamiltonian system associated to $H_{a_{ref}}$ is reconstructed from the solution $(\mu(t), a(t))$ of $X_h \in \mathfrak{X}(\mathcal{O}^\sigma_{(\mu_0,a_{ref})})$ with initial condition (μ_0, a_0) by setting $\alpha(t) = g(t)\mu(t)$, where $g(t)$ is the unique solution of the differential equation $\dot{g}(t) = g(t)\frac{\delta h}{\delta\mu(t)}$ with initial condition $g(0) = e$.

(ii) *Extending h arbirarily to $\mathfrak{s}^*$, Hamilton equations on $(\mathcal{O}^\sigma_{(\mu_0,a_{ref})}, \omega^-)$ can be written as*

$$\frac{\partial}{\partial t}(\mu, a) = \left(\mathrm{ad}^*_{\frac{\delta h}{\delta\mu}}\mu - \frac{\delta h}{\delta a} \diamond a + \mathbf{d}c^T\left(\frac{\delta h}{\delta a}\right), -\frac{\delta h}{\delta\mu}a - \mathbf{d}c\left(\frac{\delta h}{\delta\mu}\right)\right),$$

with initial conditions $(\mu(0), a(0)) = \left(\mu_0,\ g(0)^{-1}a_{ref} + c(g(0)^{-1})\right)$. The associated Lie-Poisson bracket is as defined in theorem (5.4.1) *.*

5.4.3 Remark It is worth noting that the solution $(\mu, a) \in \mathfrak{s}^*$ evolves on the affine coadjoint orbit $(\mathcal{O}^\sigma_{(\mu_0,a_{ref})}, \omega^-)$, for any $G^c_{a_{ref}}$-invariant Hamiltonian $H_{a_{ref}}$.

5.4.5 Discrete Hamiltonian flow at fixed parameter.

If the discrete curve $\{g^j\}$ is a solution of the discrete Euler-Lagrange equations, the discrete affine Euler-Poincaré at fixed parameter is simply

$$\mathbb{F}^+\ell_d(f^{j-1}, a^{j-1}) = \mathbb{F}^-\ell_d(f^j, a^j),$$

where $\mathbb{F}^\pm\ell_d$ are the discrete Legendre transforms of ℓ_d. If the discrete Lagrangian is regular, the corresponding discrete Hamiltonian flow is $\tilde{F}_{\ell_d} = \mathbb{F}^\pm\ell_d \circ F_{\ell_d} \circ (\mathbb{F}^\pm\ell_d)^{-1}$, reads

$$(\mu^{j+1}, a^{j+1}) = \tilde{F}_{\ell_d}(\mu^j, a^j).$$

5.5 Variational integrator

The reduced Lagrangian $\ell\left(g^{-1}\dot{g}, \theta_{g^{-1}}(a_{ref})\right)$ was defined in (5.2.6).

Spatial discretization We discretize the interval $[0, L]$ by N elements, such that for one element K of length l_K with two nodes a, and $a+1$, an element $g \in G$ may be approximated by

$$g_h(S) = (\Lambda_h(S), \phi_h(S)) = \left(\Lambda_a \exp\left(\frac{S}{l_K}\widehat{\psi}_a\right),\ \mathbf{x}_a + \frac{S}{l_K}\Delta\mathbf{x}_a\right),$$

where $R : G\times G \to G$ is the right translation map. We note that $g_h(0) = (\Lambda_a, \mathbf{x}_a)$ and $g_h(1) = (\Lambda_{a+1}, \mathbf{x}_{a+1})$. As a consequence the variables in the spatially discretized Lagrangian are the vector rotations matrix $\Lambda_K = (\Lambda_a,\ \Lambda_{a+1})^T$ in a and $a+1$, the vector positions $\mathbf{x}_K = (\mathbf{x}_a,\ \mathbf{x}_{a+1})^T$ of the nodes a and $a+1$, and $g_K = (\Lambda_K, \mathbf{x}_K)$.

We know by Crisfield, and Jelenic [24] p. 1133 and p. 1137 that this spatial discretization provides objective strain mesure, whereas with a linear interpolation of the rotational vector we lose the objectivity.

The variables $\widehat{\omega}_h(S), \widehat{\gamma}_h(S), \Omega_h(S), \Gamma_h(S), \rho_h(S)$ and $\chi_h(S)$ are obtained by using the approximations Λ_h and ϕ_h instead of the continuous variables Λ and ϕ. We thus have

$$
\begin{aligned}
\widehat{\omega}_h(S) &= \Lambda_h(S)^T \dot{\Lambda}_h(S) \in C^\infty([0,l_K], \mathfrak{so}(3)),\\
\gamma_h(S) &= \Lambda_h(S)^T \dot{\phi}_h(S) \in C^\infty([0,l_K], \mathbb{R}^3),\\
\widehat{\Omega}_h(S) &= \Lambda_h(S)^T \Lambda_h'(S) = \widehat{\psi}_a / l_K \in C^\infty([0,l_K], \mathfrak{so}(3)),\\
\Gamma_h(S) &= \Lambda_h(S)^T \phi_h'(S) = \Lambda_h(S)^T \Delta\mathbf{x}_a / l_K \in C^\infty([0,l_K], \mathbb{R}^3),\\
\rho_h(S) &= \Lambda_h(S)^T \left(\mathbf{x}_a + \frac{S}{l_K}\Delta\mathbf{x}_a\right) \in C^\infty([0,l_K], \mathbb{R}^3)\\
\chi_h(S) &= \Lambda_h(S)^T \mathbf{E}_1 \in C^\infty([0,l_K], \mathbb{R}^3).
\end{aligned}
\tag{5.5.1}
$$

Concerning the potential energy, the expression obtained by using Λ_h and ϕ_h instead of Λ and ϕ reads

$$
V_K(a_K) := \int_0^{l_K} f(S) dS, \tag{5.5.2}
$$

where $a_K = (\widehat{\Omega}_K, \Gamma_K, \rho_K, \chi_K)$, with

$$
\widehat{\Omega}_K = (\widehat{\Omega}_a, \widehat{\Omega}_{a+1}), \quad \Gamma_K = (\Gamma_a, \Gamma_{a+1}), \quad \rho_K = (\rho_a, \rho_{a+1}), \quad \chi_K = (\chi_a, \chi_{a+1}),
$$

and

$$
f(S) := \frac{1}{2}\left[(\Gamma_h(S) - \mathbf{E}_3)^T \mathbf{C}_1 (\Gamma_h(S) - \mathbf{E}_3) + \Omega_h(S)^T \mathbf{C}_2 \Omega_h(S)\right] + q \left\langle \rho_h(S), \chi_h(S) \right\rangle .
$$

For the kinetic energy, we make the following approximations on an element K of length l_K:

$$
\begin{aligned}
\frac{1}{2}\int_0^{l_K} M\|\gamma_h(S)\|^2 dS &\approx \frac{l_K}{4} M\left(\|\gamma_a\|^2 + \|\gamma_{a+1}\|^2\right),\\
\frac{1}{2}\int_0^{l_K} \left(\omega_h(S)^T J \omega_h(S)\right) dS &\approx \frac{l_K}{4}\left(\omega_a^T J\omega_a + \omega_{a+1}^T J \omega_{a+1}\right).
\end{aligned}
$$

Therefore the discretized Lagrangian $\ell_K : (\mathfrak{g} \times V^*)^2 \to \mathbb{R}$ is

$$
\begin{aligned}
\ell_K(g_K^{-1}\dot{g}_K, a_K) = {}& \frac{l_K}{4} M\left(\|\gamma_a\|^2 + \|\gamma_{a+1}\|^2\right)\\
&+ \frac{l_K}{4}\left(\operatorname{Tr}\left(\widehat{\omega}_a J_d (\widehat{\omega}_a)^T\right) + \operatorname{Tr}\left(\widehat{\omega}_{a+1} J_d (\widehat{\omega}_{a+1})^T\right)\right) + V_K(a_K),
\end{aligned}
\tag{5.5.3}
$$

where $(\widehat{\omega}_K,\ \gamma_K) = (\Lambda_K, \mathbf{x}_K)^{-1}(\dot{\Lambda}_K,\ \dot{\mathbf{x}}_K)$.

Temporal discretization. Given discrete path spaces $\mathcal{C}_d(G)$ and $\mathcal{C}_d(V^*)$ as defined in (5.3.3). Given a node a, the discrete time evolution of this node is given by the discrete curve

$$(g_a^j, a_a^j) = \left(\Lambda_a^j, \mathbf{x}_a^j, \Omega_a^j, \Gamma_a^j, \rho_a^j, \chi_a^j\right) \in SE(3) \times V^*.$$

The discrete variable $f^j = (g^j)^{-1} g^{j+1}$ associated to this node is

$$(F_a^j, H_a^j) := (\Lambda_a^j, \mathbf{x}_a^j)^T (\Lambda_a^{j+1}, \mathbf{x}_a^{j+1}) = \left((\Lambda_a^j)^T \Lambda_a^{j+1}, (\Lambda_a^j)^T (\mathbf{x}_a^{j+1} - \mathbf{x}_a^j)\right).$$

We denote the time-step by $\Delta t = t^j - t^{j-1}$, supposed to be of uniform size. In terms of these variables (F_a^j, H_a^j), we make, as in section (4.2.4) the following approximations.

$$\widehat{\omega}_a^j = \left(\Lambda_a^j\right)^T \dot{\Lambda}_a^j \approx \left(\Lambda_a^j\right)^T \left(\frac{\Lambda_a^{j+1} - \Lambda_a^j}{\Delta t}\right) = \frac{F_a^j - I_3}{\Delta t},$$
$$\gamma_a^j = \left(\Lambda_a^j\right)^T \dot{\mathbf{x}}_a^j \approx \left(\Lambda_a^j\right)^T \left(\frac{\mathbf{x}_a^{j+1} - \mathbf{x}_a^j}{\Delta t}\right) = \frac{H_a^j}{\Delta t}.$$

The discrete reduced Lagrangian $\ell_K\left(f_K^j, a_K^j\right)$ approximating the action of the Lagrangian ℓ_K in (5.5.3) during the time step Δt

$$\ell_K^j = \ell_K(f_K^j, a_K^j) \approx \int_{t^j}^{t^{j+1}} \ell_K(g_K(t)^{-1}\dot{g}_K(t), a_K(t))dt,$$

is therefore

$$\ell_K^j = \sum_{a \in K} \left\{ \frac{l_K}{4} \frac{M\|H_a^j\|^2}{\Delta t} + \frac{l_K}{2} \frac{\mathrm{Tr}((I_3 - F_a^j)J_d)}{\Delta t} \right\} - \Delta t\, V_K\left(a_K^j\right).$$

The discrete action sum which approximates the continuous action over the time interval $[0, T]$ is computed as follows

$$\begin{aligned}
\mathfrak{S}_d((\Lambda_d, \mathbf{x}_d, a_d)) &= \sum_{K \in \mathcal{T}} \sum_{1 \le j < N} \ell_K^j \\
&= \sum_{a \neq a_0, a_N} \sum_{j=0}^{N-1} \left\{ \frac{l_K}{2} \frac{M\|H_a^j\|^2}{\Delta t} + l_K \frac{\mathrm{Tr}((I_3 - F_a^j)J_d)}{\Delta t} \right\} \\
&\quad + \sum_{j=0}^{N-1} \left\{ \frac{l_K}{4} \frac{M\|H_{a_0}^j\|^2}{\Delta t} + \frac{l_K}{2} \frac{\mathrm{Tr}((I_3 - F_{a_0}^j)J_d)}{\Delta t} \right\} \\
&\quad + \sum_{j=0}^{N-1} \left\{ \frac{l_K}{4} \frac{M\|H_{a_N}^j\|^2}{\Delta t} + \frac{l_K}{2} \frac{\mathrm{Tr}((I_3 - F_{a_N}^j)J_d)}{\Delta t} \right\} \\
&\quad - \sum_{K \in \mathcal{T}} \sum_{1 \le j < N} \Delta t V_K\left(a_K^j\right).
\end{aligned}$$

Variational integrator. Recall that given a discrete reduced Lagrangian $\ell_{d,a_{ref}}(f^j, a^j)$ defined on $G \times \mathcal{O}^c_{a_{ref}} \subset G \times V^*$, where a_{ref} is a fixed element in V^*, the discrete-time affine Euler-Poincaré equations for each node $a \in \mathcal{N}$ are given by (5.3.12), where $\mathcal{N}$ is the set of all nodes. Thus we get the following systems of discrete affine Euler-Lagrange equations

$$\begin{aligned}-\operatorname{Ad}^*_{(F^j_a,H^j_a)^{-1}} T^*_e L_{(F^j_a,H^j_a)}\left(D_{(F^j_a,H^j_a)}\ell^j_a\right) + D_{a^j_a}\ell^j_a \diamond a^j_a \\ -\mathbf{d}c^T\left(D_{a^j_a}\ell^j_a\right) + T^*_e L_{(F^{j-1}_a,H^{j-1}_a)}\left(D_{(F^{j-1}_a,H^{j-1}_a)}\ell^{j-1}_a\right) = 0,\end{aligned}$$

where ℓ^j_a denotes the dependence of the discrete action $\mathfrak{S}_d$ on $(\Lambda^j_a, \mathbf{x}^j_a, F^j_a, H^j_a, a^j_a)$, similar for ℓ^{j-1}_a. We denote by $\partial\mathcal{N} = \{a_0, a_N\}$ the set of boundary nodes, and by $int(\mathcal{N}) = \{a_1, ..., a_{N-1}\}$ the set of internal nodes. For $a \in int(\mathcal{N})$ the discrete Lagrangian ℓ^j_a is

$$\ell^j_a = \frac{l_K}{2}\frac{M\|H^j_a\|^2}{\Delta t} + l_K\frac{\operatorname{Tr}((I_3 - F^j_a)J_d)}{\Delta t} - \sum_{K\ni a}\Delta t\, V_K\left(a^j_K\right),$$

whereas, for a boundary node $a \in \partial\mathcal{N}$, it reads

$$\ell^j_a = \frac{l_K}{4}\frac{M\|H^j_a\|^2}{\Delta t} + \frac{l_K}{2}\frac{\operatorname{Tr}((I_3 - F^j_a)J_d)}{\Delta t} - \sum_{K\ni a}\Delta t\, V_K\left(a^j_K\right).$$

So, for a node a, using (4.2.26), we get

$$T^*_I L_{F^j_a} D_{F^j_a}\ell^j_a = \frac{l_K}{2\Delta t}\left(J_d F^j_a - (F^j_a)^T J_d\right)^\vee \in \mathbb{R}^3 \simeq \mathfrak{so}(3)^*.$$

The derivative of ℓ^j_a with respect to H^j_a is

$$D_{H^j_a}\ell^j_a = M\frac{l_K}{2\Delta t}\, H^j_a,$$

so, denoting $e = (I, 0)$, using (4.2.27) and (4.2.19), we obtain

$$\begin{aligned}T^*_e L_{(F^j_a,H^j_a)}&\left(D_{F^j_a}\ell^j_a,\ \ D_{H^j_a}\ell^j_a\right)\\ &= \left(\frac{1}{2\Delta t}\left(J_d F^j_a - (F^j_a)^T J_d\right)^\vee,\ M\frac{1}{2\Delta t}(F^j_a)^T\, H^j_a\right)\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ad}^*_{(F^j_a,H^j_a)^{-1}}&\left(T^*_I L_{F^j_a} D_{F^j_a}\ell^j_a,\ T^*_0 L_{H^j_a} D_{H^j_a}\ell^j_a\right)\\ &= \left(\frac{1}{2\Delta t}\left(F^j_a J_d - J_d(F^j_a)^T\right)^\vee,\ \frac{M}{2\Delta t}\, H^j_a\right).\end{aligned}$$

The tangent map $D_{a^j_a}\ell^j_a \in T^*_{a^j_a}\mathcal{O}^c_{a_{ref}}$ is defined by

$$D_{a^j_a}\ell^j_a = \left(D_{\Omega^j_a}\ell^j_a,\ D_{\Gamma^j_a}\ell^j_a,\ D_{\rho^j_a}\ell^j_a,\ D_{\chi^j_a}\ell^j_a\right).$$

In order to compute $D_{a_a^j}\ell_a^j$, we approximate the expression V_K of the potential energy (5.5.2) by

$$\begin{aligned}V_K(a_K) \approx \mathbb{V}_K(a_K) &:= \frac{l_K}{2}\left(f(0)+f(l_K)\right)\\ &= \frac{l_K}{4}\left[(\Gamma_a-\mathbf{E}_3)^T\mathbf{C}_1(\Gamma_a-\mathbf{E}_3)+(\Gamma_{a+1}-\mathbf{E}_3)^T\mathbf{C}_1(\Gamma_{a+1}-\mathbf{E}_3)\right.\\ &\left.+(\Omega_a)^T\mathbf{C}_2\Omega_a+(\Omega_{a+1})^T\mathbf{C}_2\Omega_{a+1}\right]+\frac{l_K}{2}q\left\langle\rho_a,\chi_a\right\rangle+\frac{l_K}{2}q\left\langle\rho_{a+1},\chi_{a+1}\right\rangle.\end{aligned}$$

Thus we get

$$\begin{aligned}D_{\Omega_a^j}\ell_a^j &= -\frac{l_K}{2}\,\Delta t\,\mathbf{C}_2\,\Omega_a^j,\\ D_{\Gamma_a^j}\ell_a^j &= -\frac{l_K}{2}\,\Delta t\,\mathbf{C}_1\left(\Gamma_a^j-\mathbf{E}_3\right),\\ D_{\rho_a^j}\ell_a^j &= -\frac{l_K}{2}\,\Delta t\,q\,\chi_a^j,\\ D_{\chi_a^j}\ell_a^j &= -\frac{l_K}{2}\,\Delta t\,q\,\rho_a^j.\end{aligned}$$

Given $a_a^j=\left(\Omega_a^j,\ \Gamma_a^j,\ \rho_a^j,\ \chi_a^j\right)\in V^*$, and relations (5.2.11), we obtain

$$\begin{aligned}&\left(D_{\Omega_a^j}\ell_a^j,\ D_{\Gamma_a^j}\ell_a^j,\ D_{\rho_a^j}\ell_a^j,\ D_{\chi_a^j}\ell_a^j\right)\diamond\left(\Omega_a^j,\ \Gamma_a^j,\ \rho_a^j,\ \chi_a^j\right)\\ &=\left(D_{\Omega_a^j}\ell_a^j\times\Omega_a^j+D_{\Gamma_a^j}\ell_a^j\times\Gamma_a^j+D_{\rho_a^j}\ell_a^j\times\rho_a^j+D_{\chi_a^j}\ell_a^j\times\chi_a^j,\ -\Omega_a^j\times D_{\Gamma_a^j}\ell_a^j\right).\end{aligned}$$

In order to obtain the value of $\mathbf{d}c^T$ we choose ℓ_K instead of ℓ_a. Indeed we need to calculate the divergence, with respect to the element K. By relation (5.2.12) we obtain

$$\begin{aligned}&\mathbf{d}c^T\left(\left(D_{\Omega_a^j}\ell_K^j,\ D_{\Gamma_a^j}\ell_K^j,\ D_{\rho_a^j}\ell_K^j,\ D_{\chi_a^j}\ell_K^j\right)\right)\\ &\qquad=\left(\left(\operatorname{div}\left(D_{\Omega_h^j}\ell_K^j\right),\ \operatorname{div}\left(D_{\Gamma_h^j}\ell_K^j\right)-D_{\rho_h^j}\ell_K^j\right)\right)_a.\end{aligned}$$

By (5.5.1) we know that $\Omega_h(S)$ is constant on K, as well as $D_{\Omega_h}\ell_K^j$. Thus $\operatorname{div}\left(D_{\Omega_h^j}\ell_K^j\right)=0$. And we express $\operatorname{div}\left(D_{\Gamma_h^j}\ell_K^j\right)$ as follows

$$\begin{aligned}\operatorname{div}\left(D_{\Gamma_h^j}\ell_K^j\right) &\approx -\Delta t\int_0^{l_K}\mathbf{C}_1\frac{d}{dS}(\Gamma_h(S)-\mathbf{E}_3)dS\\ &= -\Delta t\,\mathbf{C}_1\left(\Gamma_{a+1}^j-\Gamma_a^j\right).\end{aligned}$$

We get

$$\begin{aligned}&\mathbf{d}c^T\left(\left(D_{\Omega_a^j}\ell_K^j,\ D_{\Gamma_a^j}\ell_K^j,\ D_{\rho_a^j}\ell_K^j,\ D_{\chi_a^j}\ell_K^j\right)\right)\\ &\qquad\approx\left(0,\ -\Delta t\,\mathbf{C}_1\left(\Gamma_{a+1}^j-\Gamma_a^j\right)+\frac{l_K}{2}\,\Delta t\,q\,\chi_a^j\right).\end{aligned}$$

Summary of the discrete affine Euler-Poincaré equations. Discrete affine Euler-Poincaré equations for rotations :

(i) Interior nodes $a \notin \{a_0, a_N\}$

$$\begin{aligned}
&\frac{l_K}{\Delta t}\left(J_d F_a^{j-1} - (F_a^{j-1})^T J_d\right)^\vee - \frac{l_K}{\Delta t}\left(F_a^j J_d - J_d (F_a^j)^T\right)^\vee \\
&= \frac{1}{2}\Delta t \Bigg\{ \mathbf{C}_1 \left((\Lambda_a)^T \frac{\Delta \mathbf{x}_{a-1}}{l_K} - E_3 \right) \times (\Lambda_a)^T \Delta \mathbf{x}_{a-1} + \frac{1}{l_K}\mathbf{C}_2 \psi_{a-1} \times \psi_{a-1} \\
&\qquad + \mathbf{C}_1 \left((\Lambda_a)^T \frac{\Delta \mathbf{x}_a}{l_K} - E_3 \right) \times (\Lambda_a)^T \Delta \mathbf{x}_a + \frac{1}{l_K}\mathbf{C}_2 \psi_a \times \psi_a \Bigg\}\Bigg|_{t=t^j}.
\end{aligned}$$

(ii) Left node $a = a_0$

$$\begin{aligned}
&\frac{l_K}{2\Delta t}\left(J_d F_a^{j-1} - (F_a^{j-1})^T J_d\right)^\vee - \frac{l_K}{2\Delta t}\left(F_a^j J_d - J_d (F_a^j)^T\right)^\vee \\
&= \frac{1}{2}\Delta t \Bigg\{ \mathbf{C}_1 \left((\Lambda_a)^T \frac{\Delta \mathbf{x}_a}{l_K} - E_3 \right) \times (\Lambda_a)^T \Delta \mathbf{x}_a + \frac{1}{l_K}\mathbf{C}_2 \psi_a \times \psi_a \Bigg\}\Bigg|_{t=t^j}.
\end{aligned}$$

(iii) Right node $a = a_N$

$$\begin{aligned}
&\frac{l_K}{2\Delta t}\left(J_d F_a^{j-1} - (F_a^{j-1})^T J_d\right)^\vee - \frac{l_K}{2\Delta t}\left(F_a^j J_d - J_d (F_a^j)^T\right)^\vee \\
&= \frac{1}{2}\Delta t \Bigg\{ \mathbf{C}_1 \left((\Lambda_a)^T \frac{\Delta \mathbf{x}_{a-1}}{l_K} - E_3 \right) \times (\Lambda_a)^T \Delta \mathbf{x}_{a-1} + \frac{1}{l_K}\mathbf{C}_2 \psi_{a-1} \times \psi_{a-1} \Bigg\}\Bigg|_{t=t^j}.
\end{aligned}$$

Discrete affine Euler-Poincaré equations for positions :

(i) Interior nodes $a \notin \{a_0, a_N\}$

$$\begin{aligned}
&\frac{l_K M}{\Delta t} H_a^{j-1} - \frac{l_K M}{\Delta t}\, H_a^j \\
&= \Delta t \Bigg\{ \frac{1}{2}\mathbf{C}_1 \left((\Lambda_a)^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3 \right) \times \psi_a - \mathbf{C}_1 \left((\Lambda_{a+1})^T - (\Lambda_a)^T \right) \frac{\Delta \mathbf{x}_a}{l_K} \\
&\quad \frac{1}{2}\mathbf{C}_1 \left((\Lambda_a)^T \frac{\Delta \mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3 \right) \times \psi_{a-1} - \mathbf{C}_1 \left((\Lambda_a)^T - (\Lambda_{a-1})^T \right) \frac{\Delta \mathbf{x}_{a-1}}{l_K} \\
&\quad + l_K (\Lambda_a)^T \mathbf{q} \Bigg\}\Bigg|_{t=t^j}.
\end{aligned}$$

(ii) Left node $a = a_0$

$$\frac{l_K M}{2\Delta t} H_a^{j-1} - \frac{l_K M}{2\Delta t}\ H_a^j$$
$$= \Delta t \Bigg\{ \frac{1}{2}\mathbf{C}_1 \left((\Lambda_a)^T \frac{\Delta \mathbf{x}_a}{l_K} - \mathbf{E}_3 \right) \times \psi_a - \mathbf{C}_1 \left((\Lambda_{a+1})^T - (\Lambda_a)^T \right) \frac{\Delta \mathbf{x}_a}{l_K}$$
$$+ \frac{l_K}{2} (\Lambda_a)^T \mathbf{q} \Bigg\} \Bigg|_{t=t^j} .$$

(ii) Right node $a = a_N$

$$\frac{l_K M}{2\Delta t} H_a^{j-1} - \frac{l_K M}{2\Delta t}\ H_a^j$$
$$= \Delta t \Bigg\{ \frac{1}{2}\mathbf{C}_1 \left((\Lambda_a)^T \frac{\Delta \mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3 \right) \times \psi_{a-1} - \mathbf{C}_1 \left((\Lambda_a)^T - (\Lambda_{a-1})^T \right) \frac{\Delta \mathbf{x}_{a-1}}{l_K}$$
$$+ \frac{l_K}{2} (\Lambda_a)^T \mathbf{q} \Bigg\} \Bigg|_{t=t^j} .$$

5.6 Alternative temporal discretization

As we did in section § (4.4), we can consider an alternative discretization to (4.2.20). If we consider that, for a node $a \in K$ and $t \in [t^j, t^{j+1}]$, the trajectory of an element $g \in G$ may be approximated by

$$g_{a,d}(t) = \left(\Lambda_a^j \exp\left(\frac{t - t^j}{\Delta t} \widehat{\Psi}_a^i \right),\ \mathbf{x}_a^j + \frac{t - t^j}{\Delta t} (\mathbf{x}_a^{j+1} - \mathbf{x}_a^j) \right),$$

with $\exp(\widehat{\Psi}_a^j) = (\Lambda_a^j)^T \Lambda_a^{j+1}$, then the approximations of $\widehat{\omega}_a^j$ and γ_a^j, in node a at time t^j, are

$$\mathfrak{so}(3) \ni \widehat{\omega}_a^j = \left(\Lambda_a^j\right)^T \dot{\Lambda}_a^j \approx \frac{\widehat{\Psi}_a^j}{\Delta t} \in \mathfrak{so}(3),$$
$$\mathbb{R}^3 \ni \gamma_a^j = \left(\Lambda_a^j\right)^T \dot{\mathbf{x}}_a^j \approx \left(\Lambda_a^j\right)^T \frac{\mathbf{x}_a^{j+1} - \mathbf{x}_a^j}{\Delta t} \in \mathbb{R}^3.$$

Thus the discrete Lagrangian ℓ_K^j is

$$\ell_K^j = \sum_{a \in K} \left\{ \frac{l_K}{4} \frac{M \|H_a^j\|^2}{\Delta t} + \frac{l_K}{4\,(\Delta t)} \mathrm{Tr}\left[\widehat{\Psi}_a^j J_d (\widehat{\Psi}_a^j)^T \right] \right\} - \Delta t\, V_K \left(a_K^j \right),$$

and we can obtain a different version of the discrete affine Euler-Poincaré equations for the geometrically exact model of beam.

Chapter 6

Lie algebra variational integrator of geometrically exact beam dynamics

Introduction

This chapter develops a Lie algebra variational integrator, which analyzes the deformations of the geometrically exact model of a beam introduced and described in chapter 4.

Take the configuration space of the beam to be $Q := C^\infty([0,\ell], SE(3))$, the space of smooth curves defined on the closed interval $[0,\ell]$ with values in the special Euclidean Lie group $SE(3)$. The Lie group expresses crucial geometric attributes of the underlying system (see, e.g., Iserles, Munthe-Kaas, Nørsett, and Zanna [50]). Thus, the configurations of the beam are completely defined by an element $(\Lambda, \phi) \in SE(3)$ specifying the rotation Λ of the cross-section and the position ϕ of the mid-line.

For the given Lie group $G = SE(3)$, the Lagrangian $L : TQ \to \mathbb{R}$ of the beam studied in this paper, has the form

$$L(g, \dot{g}) = \frac{1}{2}\gamma(\dot{g}, \dot{g}) - V(g),$$

where γ is a *G-invariant Riemannian metric* on the configuration space Q, and $V : Q \to \mathbb{R}$ is the G-invariant potential energy. Then, pushing forward L by left trivialization $(g, \dot{g}) \mapsto (g, g^{-1}\dot{g})$ gives the trivialized Lagrangian

$$\mathcal{L}(g, \xi) := L(g, \dot{g}), \quad \dot{g} := g\xi,$$

which is consistent with the convected representation.

In this chapter we discretize spatially the interval $[0,\ell]$ by a set $\mathcal{T}$ of N simplexes K with nodes a, while the objectivity strain measure is preserved (frame-indiference). This is how to maintain the strains invariant to the superposed rigid body rotation (see Crisfield, and Jelenic [24]). On an element K

we approximate the Lagrangian L_K by the trapezoidal rule. With the notation $g_K := (g_a, g_{a+1})^T$, and $\xi_K := (\xi_a, \xi_{a+1})^T$ for the variables associated to an element K with nodes a and $a+1$, we obtain

$$\mathcal{L}_K(g_K, \xi_K) = \frac{l_K}{4}\left\{\xi_a, \xi_a) + \gamma(\xi_{a+1}, \xi_{a+1})\right\} - \mathbb{V}_K(g_K).$$

Next, we discretize temporally, and approximate the convected velocities $\xi_a = g_a^{-1}\dot{g}_a$, at each node by elements in the Lie algebra $\mathfrak{g} = \mathfrak{se}(3)$. We obtain the discrete Lagrangian $\mathscr{L}_K^j : G \times \mathfrak{g} \to \mathbb{R}$ approximating the action of the trivialized Lagrangian $\mathcal{L}_K : G \times \mathfrak{g} \to \mathbb{R}$ over the interval $[t^j, t^{j+1}]$, for elements K of length l_K.

By applying the discrete Hamilton variational principle, we get the discrete Euler-Lagrange equations. The associated discrete evolution operator is

$$F : G \times \mathfrak{g} \to G \times \mathfrak{g}, \qquad (g^{j-1}, \xi^{j-1}) \mapsto (g^j, \xi^j),$$

where $\tau(\xi^j) = (g^j)^{-1}g^{j+1}$. Choose a smooth map $\tau : \mathfrak{g} \to G$ with $\tau(0) = e$; for example, τ may be exponential map or the Cayley transform. We note that, given ξ^j as the average velocity between g^j and g^{j+1}, τ is an approximation of the flow of the dynamics (see Kobilarov, and Marsden [60]). The important point is that the integrator is expressed in terms of the right logarithmic derivative $\mathrm{d}^R\tau$ of τ for the given left action (see Iserles, Munthe-Kaas, Nørsett, and Zanna [50] and Bou-Rabee, and Marsden [15]). This is a linear map, which is easy to invert, as it is the inverse of a matrix. Thus we obtain an integrator which has the properties of all the variational integrators and is numerically efficient.

6.1 Lagrangian dynamics of a beam in $\mathbb{R}^3$

To set the stage, we recall that the dynamics of a beam in $\mathbb{R}^3$ was review in (4.1) following the classical paper Simo [107]; see also Simo, Marsden, and Krishnaprasad [110]. This approach generalizes to the fully 3-dimensional dynamical case the formulation originally developed by Reissner [99] for the plane static problem. It can be regarded as a convenient parametrization of a three-dimensional extension of the classical Kirchhoff-Love rod model due to Antman [2].

6.2 Lie Algebra variational integrator for the beam

6.2.1 Lie group structure

In this section we develop an asynchronous Lie group variational integrator for the beam. To do this, we identity the configuration space Q of the beam with the infinite dimensional Lie group $G = C^\infty([0, L], SE(3))$; the group operation is given by pointwise multiplication in the group $SE(3)$, i.e.,

$$(\Lambda_1, \phi_1)(\Lambda_2, \phi_2) = (\Lambda_1\Lambda_2, \phi_1 + \Lambda_1\phi_2),$$

where $\Lambda \in C^\infty([0,L], SO(3))$ and $\phi \in C^\infty([0,L], \mathbb{R}^3)$. The identity element in G is the constant map $(Id, 0)$ and the inverse is given by $(\Lambda,\phi)^{-1} = \left(\Lambda^{-1}, -\Lambda^{-1}\phi\right)$. The tangent lift of left translation has the expression

$$(\Lambda_1,\phi_1)(\dot{\Lambda}_2,\dot{\phi}_2) = \left(\Lambda_1\dot{\Lambda}_2, \Lambda_1\dot{\phi}_2\right),$$

where $(\dot{\Lambda}_2,\dot{\phi}_2) \in T_{(\Lambda_2,\phi_2)}G$. Thus, the convective velocity is given by

$$(\widehat{\omega},\gamma) = (\Lambda,\phi)^{-1}(\dot{\Lambda},\dot{\phi}).$$

It is important to observe that, in this setting, if boundary conditions have to be imposed on the configuration space, they have to preserve the group structure. For example, both boundary conditions considered in Fig. 4.1.1 preserve the group structure of G.

Trivialized Euler-Lagrange equations on Lie groups. We briefly recall the expression of the trivialized Euler-Lagrange equations on the tangent bundle of a Lie group G. Let $L : TG \to \mathbb{R}$ be a smooth Lagrangian defined on the tangent bundle TG to a Lie group G. The push forward of L by the left trivialization vector bundle isomorphism $TG \ni v_g \stackrel{\sim}{\longmapsto} (g, g^{-1}v_g) \in G\times\mathfrak{g}$ yields the smooth function $\mathcal{L} : G\times\mathfrak{g} \to \mathbb{R}$ defined by

$$\mathcal{L}(g,\xi) := L(g,\dot{g}), \quad \dot{g} := g\xi.$$

The classical Euler-Lagrange equations are obtained by applying Hamilton's Principle to the action defined by L and a given interval $[t_0,t_1]$ for variations of curves keeping the endpoints at $t=t_0$ and $t=t_1$ fixed. Consequently, pushing everything forward by the left trivialization isomorphism, gives the constrained variational principle

$$0 = \delta\int_{t_0}^{t_1}\mathcal{L}(g,\xi)dt = \int_{t_0}^{t_1}\left(\left\langle\frac{\partial\mathcal{L}}{\partial g},\delta g\right\rangle + \left\langle\frac{\partial\mathcal{L}}{\partial\xi},\delta\xi\right\rangle\right)dt,$$

for all variations $\delta g : [t_0,t_1] \to TG$ and $\delta\xi : [t_0,t_1] \to \mathfrak{g}$ satisfying $\delta g(t_0) = 0$, $\delta g(t_1) = 0$, $\delta\xi = \dot{\eta} + \operatorname{ad}_\xi\eta$, where $\eta = g^{-1}\delta g$, so that $\eta(t_0) = \eta(t_1) = 0$ (see [89] p.438). Integration by parts yields hence

$$0 = \delta\int_{t_0}^{t_1}\mathcal{L}(g,\xi)dt = \int_{t_0}^{t_1}\left(\left\langle g^{-1}\frac{\partial\mathcal{L}}{\partial g} + \operatorname{ad}^*_\xi\frac{\partial\mathcal{L}}{\partial\xi},\eta\right\rangle - \left\langle\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\xi}\right),\eta\right\rangle\right)dt,$$

for all $\eta \in \mathfrak{g}$. Hence the Euler-Lagrange equations in terms of $\mathcal{L}$ read

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\xi}\right) - \operatorname{ad}^*_\xi\frac{\partial\mathcal{L}}{\partial\xi} = g^{-1}\frac{\partial\mathcal{L}}{\partial g}. \tag{6.2.1}$$

In our example, we take $G = C^\infty([0,L], SE(3))$.

The trivialized Lagrangian. In the case of the beam under study, the trivialized Lagrangian $\mathcal{L}(g, \xi)$ can be written as

$$\mathcal{L}(\Lambda, \phi, \widehat{\omega}, \gamma) = \frac{1}{2}\int_0^L M \left\|\gamma\right\|^2 dS + \frac{1}{2}\int_0^L \omega^T J\omega \; dS - \Pi_{int}(\Lambda, \phi) - \Pi_{ext}(\phi),$$

on $G \times \mathfrak{g} = C^\infty([0, L], SE(3)) \times C^\infty([0, L], \mathfrak{se}(3))$.

6.2.2 Spatial discretization

We return to the original beam problem whose equations of motion are (4.1.11) with boundary conditions (4.1.12). Recall that the configuration space is $Q = G = C^\infty([0, L], SE(3))$.

The discretization of the variables $(\Lambda, \phi) \in SE(3)$, and of the associated convected variables $(\widehat{\omega}, \gamma), (\widehat{\Omega}, \Gamma) \in \mathfrak{se}(3)$ is described in section § (4.2.2).

The spatially discretized Lagrangian $L_K : TSE(3)^2 \to \mathbb{R}$ and its trivialized form $\mathcal{L}_K : SE(3)^2 \times \mathfrak{se}(3)^2 \to \mathbb{R}$ over an element K of length l_K, are given by

$$\begin{aligned} L_K(\Lambda_K, \mathbf{x}_K, \dot{\Lambda}_K, \dot{\mathbf{x}}_K) = & \frac{l_K}{4} M \left(\|\dot{\mathbf{x}}_a\|^2 + \|\dot{\mathbf{x}}_{a+1}\|^2 \right) \\ & + \frac{l_K}{4} \left(\omega_a^T J \omega_a + \omega_{a+1}^T J \omega_{a+1} \right) \\ & - \mathbb{V}_K(\mathbf{x}_K, \Lambda_K) = \mathcal{L}_K(\Lambda_K, \mathbf{x}_K, \widehat{\omega}_K, \gamma_K), \end{aligned} \tag{6.2.2}$$

where $\mathbb{V}_K$ was defined in (4.2.6). The spatial discrete Lagrangian L_d of the total system is obtained by summing over all the elements K, i.e., $L_d = \sum_{K \in \mathcal{T}} L_K$.

6.2.3 Discrete Euler-Lagrange equations on Lie groups

Exponential map derivative. In this subsection we present a variational integrator for mechanics on Lie groups based on the paper of [50] that uses the right trivialized derivative of the exponential map, also known as the right logarithmic derivative. We will later apply this variational integrator to the beam.

If $G \times M \to M$ is a smooth left action and $\tau : M \to G$ is a smooth map, its *right logarithmic derivative* at $m \in M$ is the linear map defined by

$$\mathrm{d}^R\tau(m) := T_{\tau(m)} R_{\tau(m)^{-1}} \circ T_m \tau : T_m M \to \mathfrak{g}. \tag{6.2.3}$$

Thus, if $t \mapsto m(t)$ is a smooth curve in M, we have

$$\frac{d}{dt}\tau\left(m(t)\right) = \left(\mathrm{d}^R\tau(m(t)) \cdot \dot{m}(t)\right) \tau(m(t)) \in T_{\tau(m(t))}G.$$

Let us apply these formulas to the exponential map $\exp : \mathfrak{g} \to G$ which has the additional advantage that the right logarithmic derivative is known explicitly. Thus if $\xi \in \mathfrak{g}$, we have

$$\mathrm{d}^R \exp(\xi) = T_{\exp\xi} R_{\exp(-\xi)} \circ T_\xi \exp = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\,\mathrm{ad}_\xi^n = \frac{e^{\mathrm{ad}_\xi} - I}{\mathrm{ad}_\xi} : \mathfrak{g} \to \mathfrak{g}, \tag{6.2.4}$$

a linear map from $\mathfrak{g}$ to itself. Therefore, if $t \mapsto \xi(t)$ is a smooth curve in $\mathfrak{g}$, we have

$$\frac{d}{dt}\exp(\xi(t)) = \left(\mathrm{d}^R \exp(\xi(t)) \cdot \xi'(t)\right) \exp(\xi(t)) \in T_{\exp \xi(t)} G.$$

There are similar considerations for the left logarithmic derivative by simply replacing the series in (6.2.4) by the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \operatorname{ad}_\xi^n = \left(I - e^{-\operatorname{ad}_\xi}\right) / \operatorname{ad}_\xi$.

This right logarithmic derivative of τ was used also in [15] to develop a variety of integrators of variational partitioned Runge-Kutta type for Lie groups. In control theory, Kobilarov, and Marsden [60] developed a structure preserving variational integrator to actuate a system, based on the rigid body model, to move from its current state to a desired state with minimum control effort or time.

From (6.2.3), writing $\tau^{-1} \circ \tau = id$, we get the *inverse right logarithmic derivative* of τ

$$T_{\tau(\xi)}\tau^{-1} = \left(\mathrm{d}^R \tau(\xi)\right)^{-1} \circ T_{\tau(\xi)} R_{\tau(\xi)^{-1}} : T_{\tau(\xi)} G \to \mathfrak{g}. \tag{6.2.5}$$

The integrator we present is developed to treat numerically mechanical systems on finite dimensional Lie groups.

Explicit-implicit integrator

Let $L : TG \to \mathbb{R}$ be a Lagrangian defined on the tangent bundle TG of a Lie group G and let $\tau : \mathfrak{g} \to G$ be a map with $\tau(0) = e$. We assume that τ is a C^2-diffeomorphism in a neighborhood of the origin. The discrete Lagrangian $\mathscr{L}_d : G \times \mathfrak{g} \to \mathbb{R}$ is defined as an approximation of the action functional over one time step, namely, we have

$$\mathscr{L}_d(g^j, \xi^j) \approx \int_{t^j}^{t^{j+1}} L(g(t), \dot{g}(t)) dt,$$

where $g(t)$ is the unique solution of the Euler-Lagrange equations such that $g(t^j) = g^j$ and $g(t^{j+1}) = g^{j+1}$ and where

$$\tau(\Delta t\, \xi^i) = (g^j)^{-1} g^{j+1}. \tag{6.2.6}$$

We assume that the time step is small enough so that $(g^j)^{-1} g^{j+1}$ is in a neighborhood of the identity element of G where the map τ is a diffeomorphism.

In our applications, the Lagrangian is always of the classical form kinetic minus potential energy, where the kinetic energy is G-invariant, so that we can write the discrete Lagrangian as

$$\mathscr{L}_d(g^j, \xi^j) = K(\xi^j) - V(g^j), \tag{6.2.7}$$

We now compute the variation $\delta\xi^j$ induced by variations of g^j. Defining $\eta^j := (g^j)^{-1}\delta g^j$ and $f^j := (g^j)^{-1} g^{j+1}$, we have

$$\begin{aligned}
\Delta t \delta \xi^j &\overset{(6.2.6)}{=} T_{f^j}\tau^{-1}\left(\delta f^j\right) \overset{(6.2.5)}{=} \left(\mathrm{d}^R \tau(\Delta t\, \xi^j)\right)^{-1} \left(\left(\delta f^j \tau(\Delta t\, \xi^j)^{-1}\right)\right) \\
&= \left(\mathrm{d}^R \tau(\Delta t\, \xi^j)\right)^{-1} \left[\left(-(g^j)^{-1}\delta g^j (g^j)^{-1} g^{j+1} + (g^j)^{-1}\delta g^{j+1}\right)(f^j)^{-1}\right] \\
&= \left(\mathrm{d}^R \tau(\Delta t\, \xi^j)\right)^{-1} \left(-\eta^j + \operatorname{Ad}_{\tau(\Delta t\, \xi^j)} \eta^{j+1}\right).
\end{aligned} \tag{6.2.8}$$

The discrete Euler-Lagrange equations are obtained by applying the discrete Hamilton's principle to $\mathscr{L}_d$. Taking into account that $\eta^0 = \eta^N = 0$, we get

$$\begin{aligned}
\delta\mathfrak{S}_d &= \sum_{j=0}^{N-1} D_{g^j}\mathscr{L}_d(g^j,\xi^j)\cdot\delta g^j + D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\cdot\delta\xi^j \\
&= \sum_{j=0}^{N-1} D_{g^j}\mathscr{L}_d(g^j,\xi^j)\cdot g^j\eta^j \\
&\qquad + \frac{1}{\Delta t}D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\cdot\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\left(-\eta^j + \mathrm{Ad}_{\tau(\Delta t\,\xi^j)}\,\eta^{j+1}\right) \\
&= \sum_{j=1}^{N-1}\Bigg((g^j)^{-1}\left(D_{g^j}\mathscr{L}_d(g^j,\xi^j)\right) - \frac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\right) \\
&\qquad + \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^{j-1}}\mathscr{L}_d(g^{j-1},\xi^{j-1})\right)\Bigg)\cdot\eta^j.
\end{aligned}$$

Thus, the discrete Euler-Lagrange equations are

$$\begin{aligned}
&(g^j)^{-1}\left(D_{g^j}\mathscr{L}_d(g^j,\xi^j)\right) - \frac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\right) \\
&\quad + \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^{j-1}}\mathscr{L}_d(g^{j-1},\xi^{j-1})\right) = 0, \qquad (6.2.9) \\
&\text{with}\quad (g^j)^{-1}g^{j+1} = \tau(\Delta t\,\xi^j).
\end{aligned}$$

In (6.2.9), for a given pair (g^{j-1},ξ^{j-1}), we obtain $g^j = g^{j-1}\tau(\xi^{j-1})$ from the second equation, and we solve the first equation to find ξ^j. This yields a discrete-time flow map $(g^{j-1},\xi^{j-1}) \to (g^j,\xi^j)$, and this process is repeated. The discrete Euler-Lagrange equations may thus be written as

$$\begin{cases}
\mu^j - \mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\mu^{j-1} = (g^j)^{-1}\left(D_{g^j}\mathscr{L}_d(g^j,\xi^j)\right) \\
\mu^j = \dfrac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\right) \\
g^{j+1} = g^j\tau(\Delta t\,\xi^j).
\end{cases} \qquad (6.2.10)$$

In the context of Lie algebra variational integrators, the discrete Legendre transforms $\mathbb{F}^{\pm}\mathscr{L}_d : G\times\mathfrak{g} \to G\times\mathfrak{g}^*$ are given by

$$\begin{aligned}
\mathbb{F}^+\mathscr{L}_d(g^j,\xi^j) &= \left(g^{j+1},\ \mathrm{Ad}^*_{\tau(\Delta t\,\xi^j)}\mu^j\right) \\
\mathbb{F}^-\mathscr{L}_d(g^j,\xi^j) &= \left(g^j, -(g^j)^{-1}\left(D_{g^j}\mathscr{L}_d(g^j,\xi^j)\right) + \mu^j\right)
\end{aligned}$$

We note that equation (6.2.9) can be written in terms of the Legendre transform as

$$\mathbb{F}^+\mathscr{L}_d(g^{j-1},\xi^{j-1}) = \mathbb{F}^-\mathscr{L}_d(g^j,\xi^j),$$

as in (1.2.4).

The infinitesimal generator of left multiplication on G for $\zeta \in \mathfrak{g}$ has the expression $\zeta_G(g^j) = \zeta g^j$. In the context of Lie algebra variational integrators, the discrete Lagrangian momentum maps $\mathbf{J}^{\pm}_{\mathscr{L}_d} : G \times \mathfrak{g} \to \mathfrak{g}^*$ are defined by

$$\left\langle \mathbf{J}^{+}_{\mathscr{L}_d}(g^j, \xi^j), \zeta \right\rangle = \left\langle \mathbb{F}^{+}\mathscr{L}_d(g^j, \xi^j), \left(g^{j+1}\right)^{-1} \zeta_G(g^{j+1}) \right\rangle \tag{6.2.11}$$

$$\left\langle \mathbf{J}^{-}_{\mathscr{L}_d}(g^j, \xi^j), \zeta \right\rangle = \left\langle \mathbb{F}^{-}\mathscr{L}_d(g^j, \xi^j), \left(g^{j}\right)^{-1} \zeta_G(g^{j}) \right\rangle . \tag{6.2.12}$$

These definitions are adapted from the relations (1.2.8) to the case of Lie algebra integrators. Thus we get

$$J^{+}_{\mathscr{L}_d}(g^j, \xi^j) = \mathrm{Ad}^*_{(g^{j+1})^{-1}} \left(\mathrm{Ad}^*_{\tau(\Delta t\, \xi^j)} \mu^j \right) = \mathrm{Ad}^*_{(g^j)^{-1}}\, \mu^j ,$$
$$J^{-}_{\mathscr{L}_d}(g^j, \xi^j) = \mathrm{Ad}^*_{(g^j)^{-1}} \cdot \left(\mu^j - (g^j)^{-1} \left(D_{g^j} \mathscr{L}_d(g^j, \xi^j) \right) \right).$$

Note that equation (6.2.9) can also be written in terms of the spatial discrete Lagrangian momentum maps $\mathbf{J}^{\pm}_{\mathscr{L}_d}$ as

$$\mathrm{Ad}^*_{g^j} \mathbf{J}^{+}_{\mathscr{L}_d}(g^{j-1}, \xi^{j-1}) = \mathrm{Ad}^*_{g^j} \mathbf{J}^{-}_{\mathscr{L}_d}(g^j, \xi^j).$$

6.2.1 Remark Recall that for mechanical systems on Lie groups the spatial and body momenta associated to a momentum $\alpha_g \in T^*G$ are respectively given by $\pi_S = \alpha_g g^{-1}$ and $\pi_B = g^{-1}\alpha_g$, so the coadjoint representation Ad^*_g maps the spatial momentum to the body momentum (see Abraham and Marsden [1]). Note also that the momentum map associated to left invariance reads $\mathbf{J}_L : T^*G \to \mathfrak{g}^*$, $\mathbf{J}(\alpha_g) = \alpha_g g^{-1}$ and thus coincides with the spatial momentum. We refer to Demoures, Gay-Balmaz, Leyendecker, Ober-Blöbaum, Ratiu, and Weinand [25] for a more detailed study of these relationships in the discrete formulation and its connection with geometric integrators.

6.2.2 Remark Recall that if the Lagrangian $L : TG \to \mathbb{R}$ is left G-invariant, then the Euler-Lagrange equations reduce to the Euler-Poincaré equations, as one notes from (6.2.1) by inserting $\partial\mathcal{L}/\partial g = 0$. Similarly, in the discrete case, if $\mathscr{L}_d$ is a left invariant Lagrangian, that is, if $\mathscr{L}_d(g^i, \xi^i) = \mathscr{L}_d(\xi^i)$, then (6.2.10) is a discrete approximation of the Euler-Poincaré equations.

Discrete Lagrange-d'Alembert equations

In a similar way with the continuous case, external forces can be incorporated in the dynamics by replacing the discrete Hamilton's principle with the discrete Lagrange-d'Alembert principle (see Marsden and West [90]). For Lie algebra integrators, the discrete Lagrange-d'Alembert principle reads

$$\delta \sum_{j=0}^{N-1} \mathscr{L}_d(g^j, \xi^j) + \sum_{j=0}^{N-1} \left[\mathcal{F}^-_d(g^j, \xi^j) \cdot \delta g^j + \mathcal{F}^+_d(g^{j+1}, \xi^j) \cdot \delta g^{j+1} \right] = 0,$$

for all variations δg^j with $\delta g^0 = \delta g^N = 0$, where $\mathcal{F}^-_d(g^j, \xi^j) \in T^*_{g^j}G$ and $\mathcal{F}^+_d(g^{j+1}, \xi^j) \in T^*_{g^{j+1}}G$ are the discrete external Lagrangian forces. These discrete forces are chosen in such a way that the second term in the variational principle is an approximation of the virtual work done by the force field in the continuous case. Using the notation $\eta^j = (g^j)^{-1}\delta g^j$, we have

$$\begin{aligned}
\delta g^{j+1} &= \delta g^j \tau(\Delta t\,\xi^j) + g^j \delta(\tau(\Delta t\,\xi^j))\\
&= g^j \eta^j \tau(\Delta t\,\xi^j) + g^j \left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\cdot \Delta t\,\delta\xi^j\right)\tau(\Delta t\,\xi^j)\\
&\overset{(6.2.8)}{=} g^j \left(\mathrm{Ad}_{\tau(\Delta t\,\xi^j)}\,\eta^{j+1}\right)\tau(\Delta t\,\xi^j),\\
\Delta t\,\delta\xi^j &= \left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\left(-\eta^j + \mathrm{Ad}_{\tau(\Delta t\,\xi^j)}\eta^{j+1}\right).
\end{aligned}$$

Then, since $\eta^0 = \eta^N = 0$, we get

$$\begin{aligned}
\delta\mathfrak{S}_d = \sum_{j=0}^{N-1} &\Big(D_{g^j}\mathscr{L}_d(g^j,\xi^j) + \mathcal{F}_d^-(g^j,\xi^j)\Big)\cdot\delta g^j\\
&+ D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\cdot\delta\xi^j + \mathcal{F}_d^+(g^{j+1},\xi^j)\cdot\delta g^{j+1}\\
= \sum_{j=0}^{N-1} &\Big(D_{g^j}\mathscr{L}_d(g^j,\xi^j) + \mathcal{F}_d^-(g^j,\xi^j)\Big)\cdot g^j\eta^j\\
&+ \frac{1}{\Delta t}D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\cdot\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\Big(-\eta^j + \mathrm{Ad}_{\tau(\Delta t\,\xi^j)}\eta^{j+1}\Big)\\
&+ \mathcal{F}_d^+(g^{j+1},\xi^j)\cdot\Big(g^j\left(\mathrm{Ad}_{\tau(\Delta t\,\xi^j)}\,\eta^{j+1}\right)\tau(\Delta t\,\xi^j)\Big)\\
= \sum_{j=1}^{N-1} &\Big\{(g^j)^{-1}\left(D_{g^j}\mathscr{L}_d(g^j,\xi^j)\right) + (g^j)^{-1}\left(\mathcal{F}_d^-(g^j,\xi^j)\right)\\
&- \frac{1}{\Delta t}\Big(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\Big)^*\left(D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\right)\\
&+ \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\Big(\left(\mathrm{d}^R\tau(\Delta t\,\xi^{j-1})\right)^{-1}\Big)^*\left(D_{\xi^{j-1}}\mathscr{L}_d(g^{j-1},\xi^{j-1})\right)\\
&+ (g^j)^{-1}\left(\mathcal{F}_d^+(g^j,\xi^{j-1})\right)\Big\}\cdot\eta^j.
\end{aligned}$$

Thus, the discrete Lagrange d'Alembert equations are

$$\begin{aligned}
&(g^j)^{-1}\left(D_{g^j}\mathscr{L}_d(g^j,\xi^j)\right) + (g^j)^{-1}\left(\mathcal{F}_d^-(g^j,\xi^j)\right)\\
&\quad - \frac{1}{\Delta t}\Big(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\Big)^*\left(D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\right)\\
&\quad\quad + \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\Big(\left(\mathrm{d}^R\tau(\Delta t\,\xi^{j-1})\right)^{-1}\Big)^*\left(D_{\xi^{j-1}}\mathscr{L}_d(g^{j-1},\xi^{j-1})\right)\\
&\quad\quad\quad + (g^j)^{-1}\left(\mathcal{F}_d^+(g^j,\xi^{j-1})\right) = 0,
\end{aligned}$$

with $g^{j+1} = g^j\tau(\Delta t\,\xi^j)$.

They may be conveniently written as

$$\begin{cases}
\mu^j - \mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\mu^{j-1} = (g^j)^{-1}\left(D_{g^j}\mathscr{L}_d(g^j,\xi^j)\right)\\
\qquad\qquad + (g^j)^{-1}\left(\mathcal{F}_d^-(g^j,\xi^j)\right) + (g^j)^{-1}\left(\mathcal{F}_d^+(g^j,\xi^{j-1})\right)\\
\mu^j = \dfrac{1}{\Delta t}\Big(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\Big)^*\left(D_{\xi^j}\mathscr{L}_d(g^j,\xi^j)\right)\\
g^{j+1} = g^j\tau(\Delta t\,\xi^j).
\end{cases} \tag{6.2.13}$$

When forces are present, one has to incorporate them in the discrete Legendre transforms and the discrete momentum maps, as explained in Marsden and West [90]. In the context of Lie algebra variational integrators, the forced discrete Legendre transforms $\mathbb{F}^{f\pm}L_d : G \times \mathfrak{g} \to G \times \mathfrak{g}^*$ are

$$\mathbb{F}^{f+}L_d(g^j, \xi^j) = \Big(g^{j+1}, \mathrm{Ad}^*_{\tau(\Delta t\,\xi^j)}\mu^j + (g^{j+1})^{-1}\left(\mathcal{F}_d^+(g^{j+1}, \xi^j)\right)\Big)$$
$$\mathbb{F}^{f-}L_d(g^j, \xi^j) = \Big(g^j, \mu^j - (g^j)^{-1}D_{g^j}\mathscr{L}_d(g^j, \xi^j) + (g^j)^{-1}\mathcal{F}_d^-(g^j, \xi^j)\Big).$$

The forced discrete Lagrangian momentum maps $\mathbf{J}^{f\pm}_{\mathscr{L}_d} : G \times \mathfrak{g} \to \mathfrak{g}^*$, for $g^j \in G$ and $\xi^j \in \mathfrak{g}$, are given, in terms of the discrete Legendre transform, by (6.2.11) and (6.2.12). Thus we get

$$\mathbf{J}^{f+}_{\mathscr{L}_d}(g^j, \xi^j) = \mathrm{Ad}^*_{(g^{j+1})^{-1}}\Big(\mathrm{Ad}^*_{\tau(\Delta t\,\xi^j)}\mu^j + (g^{j+1})^{-1}\left(\mathcal{F}_d^+(g^{j+1}, \xi^j)\right)\Big),$$
$$\mathbf{J}^{f-}_{\mathscr{L}_d}(g^j, \xi^j) = \mathrm{Ad}^*_{(g^j)^{-1}}\Big(\mu^j - (g^j)^{-1}D_{g^j}\mathscr{L}_d(g^j, \xi^j) + (g^j)^{-1}\mathcal{F}_d^-(g^j, \xi^j)\Big).$$

Implicit-implicit integrator

In this approach, the discrete Lagrangian is evaluated on the couple (g^{j+1}, ξ^j) instead of (g^j, ξ^j), where ξ^j is still given by $\Delta t\,\xi^j = \tau^{-1}((g^j)^{-1}g^{j+1})$. In the context of discrete Lagrangian of the form (6.2.7), we now have

$$\mathscr{L}_d(g^{j+1}, \xi^j) = K(\xi^j) - V(g^{j+1}),$$

so that the only difference with the previous case is that the potential energy is evaluated at g^{j+1} instead of g^j. As we shall see below, this has important consequences; for example, the integrator that will be developed below is totally implicit.

The discrete Euler-Lagrange equations are obtained by applying the discrete Hamilton's principle to $\mathscr{L}_d$. As before, we get

$$\begin{aligned}
\delta\mathfrak{S}_d &= \sum_{j=0}^{N-1} D_{g^{j+1}}\mathscr{L}_d(g^{j+1}, \xi^j)\cdot\delta g^{j+1} + D_{\xi^j}\mathscr{L}_d(g^{j+1}, \xi^j)\cdot\delta\xi^j \\
&= \sum_{j=0}^{N-1}\Big(D_{g^{j+1}}\mathscr{L}_d(g^{j+1}, \xi^j)\cdot g^{j+1}\eta^{j+1} \\
&\qquad + \frac{1}{\Delta t}D_{\xi^j}\mathscr{L}_d(g^{j+1}, \xi^j)\cdot\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\left(-\eta^j + \mathrm{Ad}_{\tau(\Delta t\,\xi^j)}\eta^{j+1}\right)\Big) \\
&= \sum_{j=1}^{N-1}\Big\{(g^j)^{-1}D_{g^j}\mathscr{L}_d(g^j, \xi^{j-1}) - \frac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^* D_{\xi^j}\mathscr{L}_d(g^{j+1}, \xi^j) \\
&\qquad + \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^{j-1})\right)^{-1}\right)^* D_{\xi^{j-1}}\mathscr{L}_d(g^j, \xi^{j-1})\Big\}\cdot\eta^j.
\end{aligned}$$

The discrete Euler-Lagrange equations are thus given by

$$(g^j)^{-1} D_{g^j} \mathscr{L}_d(g^j, \xi^{j-1}) - \frac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\, \xi^j)\right)^{-1}\right)^* D_{\xi^j}\mathscr{L}_d(g^{j+1}, \xi^j) \\ + \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\, \xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\Delta t\, \xi^{j-1})\right)^{-1}\right)^* D_{\xi^{j-1}}\mathscr{L}_d(g^j, \xi^{j-1}) = 0,$$

and may thus be written as

$$\begin{cases} \mu^j - \mathrm{Ad}^*_{\tau(\Delta t\, \xi^{j-1})}\mu^{j-1} = (g^j)^{-1} D_{g^j}\mathscr{L}_d(g^j, \xi^{j-1}) \\ \mu^j = \dfrac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\, \xi^j)\right)^{-1}\right)^* D_{\xi^j}\mathscr{L}_d(g^j\tau(\Delta t\, \xi^j), \xi^j) \\ g^{j+1} = g^j\tau(\Delta t\, \xi^j). \end{cases}$$

The discrete Legendre transforms $\mathbb{F}^{\pm}\mathscr{L}_d : G\times\mathfrak{g}\to G\times\mathfrak{g}^*$ are given by

$$\mathbb{F}^+\mathscr{L}_d(g^{j+1}, \xi^j) = \left(g^{j+1}, (g^{j+1})^{-1}D_{g^{j+1}}\mathscr{L}_d(g^{j+1}, \xi^j) + \mathrm{Ad}^*_{\tau(\Delta t\, \xi^j)}\mu^j\right), \\ \mathbb{F}^-\mathscr{L}_d(g^{j+1}, \xi^j) = \left(g^j, \mu^j\right).$$

Given (6.2.11) and (6.2.12), the discrete Lagrangian momentum maps $\mathbf{J}^{\pm}_{\mathscr{L}_d} : G\times\mathfrak{g}\to\mathfrak{g}^*$, for $g^{j+1}\in G$ and $\xi^j\in\mathfrak{g}$ are

$$\begin{cases} \mathbf{J}^+_{\mathscr{L}_d}(g^{j+1}, \xi^j) = \mathrm{Ad}^*_{(g^{j+1})^{-1}}\left((g^{j+1})^{-1}D_{g^{j+1}}\mathscr{L}_d(g^{j+1}, \xi^j) + \mathrm{Ad}^*_{\tau(\Delta t\, \xi^j)}\mu^j\right), \\ \mathbf{J}^-_{\mathscr{L}_d}(g^{j+1}, \xi^j) = \mathrm{Ad}^*_{(g^j)^{-1}}\mu^j. \end{cases}$$

6.2.4 Lie algebra variational integrator for the beam

In this subsection we shall present a variational integrator for the beam, by applying the previous approach to the spatially discretized Lagrangian (6.2.2).

Time discretization

Recall that the Lagrangian (6.2.2) is defined on the tangent bundle

$$TSE(3)^{N+1} \ni (\Lambda_a, \mathbf{x}_a, \dot{\Lambda}_a, \dot{\mathbf{x}}_a)_{a\in\mathcal{N}}.$$

In the time discretized case, the discrete time evolution of a node a is given by the discrete curve $\{(\Lambda_a^j, \mathbf{x}_a^j) \mid t^j = j\Delta t\}$ in $SE(3)$ and hence the discrete Lagrangian is defined on $SE(3)^{N+1}\times\mathfrak{se}(3)^{N+1}$.

The discrete Lagrangian $\mathscr{L}_K^j$ approximating the action of the Lagrangian L_K in (4.2.7) over the interval $[t^j, t^{j+1}]$, for elements K of length l_K, is

$$\mathscr{L}_K^j = \Delta t\,\frac{l_K}{4}\sum_{a\in K}\left\{M\left\|\gamma_a^j\right\|^2 + (\omega_a^j)^T J\omega_a^j\right\} \\ - \Delta t\,\mathbb{V}_K\left(\Lambda_K^j, \Lambda_K^{j+1}, \mathbf{x}_K^j, \mathbf{x}_K^{j+1}\right). \qquad (6.2.14)$$

where we define (ω_a^j, γ_a^j) as

$$\begin{aligned}\Delta t\, \xi_a^j := \Delta t\,(\omega_a^j, \gamma_a^j) &= \tau^{-1}\left((g_a^j)^{-1} g_a^{j+1}\right) \\ &= \tau^{-1}\left((\Lambda_a^j)^{-1}\Lambda_a^{j+1}, (\Lambda_a^j)^{-1}\Delta \mathbf{x}_a^j\right). \qquad (6.2.15)\end{aligned}$$

In this formula, in $\mathbb{V}_K$, we keep the values of both variables at the times t^j and t^{j+1}. As we shall see later, we will develop two algorithms, one for which only $\mathbb{V}_K(\Lambda_K^j, \mathbf{x}_K^j)$ is needed and the other where only $\mathbb{V}_K(\Lambda_K^{j+1}, \mathbf{x}_K^{j+1})$ is needed. Of course, one can imagine other algorithms, such as ones using the midpoint rule, where all four variables intervene.

The discrete action, which approximates the continuous action over the time interval $[0, T]$, is therefore given by

$$\begin{aligned}\mathfrak{S}_d &= \sum_{K\in\mathcal{T}} \sum_{j=1}^{N-1} \mathscr{L}_K^j \\ &= \Delta t\, \frac{l_K}{4} \sum_{K\in\mathcal{T}} \sum_{j=1}^{N-1} \sum_{a\in K} \left\langle \mathbb{J}\xi_a^j, \xi_a^j \right\rangle - \Delta t \sum_{K\in\mathcal{T}} \sum_{j=1}^{N-1} \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right) \\ &= \Delta t\, \frac{l_K}{2} \sum_{a\neq a_0, a_N} \sum_{j=0}^{N-1} \left\langle \mathbb{J}\xi_a^j, \xi_a^j \right\rangle \\ &\quad + \Delta t\, \frac{l_K}{4} \sum_{j=0}^{N-1} \left\langle \mathbb{J}\xi_{a_0}^j, \xi_{a_0}^j \right\rangle + \Delta t\, \frac{l_K}{4} \sum_{j=0}^{N-1} \left\langle \mathbb{J}\xi_{a_N}^j, \xi_{a_N}^j \right\rangle \\ &\quad - \Delta t \sum_{K\in\mathcal{T}} \sum_{j=1}^{N-1} \mathbb{V}_K\left(\Lambda_K^j, \Lambda_K^{j+1}, \mathbf{x}_K^j, \mathbf{x}_K^{j+1}\right).\end{aligned}$$

where $\mathbb{J} : \mathfrak{se}(3) \to \mathfrak{se}(3)^*$ is the linear operator which has the matrix

$$\mathbb{J} = \begin{pmatrix} J & 0 \\ 0 & M\mathbf{I}_3 \end{pmatrix}.$$

Note that in the kinetic energy the sum is over the nodes a, whereas in the potential energy the sum is over the elements $K \in \mathcal{T}$. We have also isolated the terms corresponding to the boundaries.

Explicit-implicit integrator

By evaluating the potential term $\mathbb{V}_K$ at time t_j, in the discrete Lagrangian (6.2.14), the dependence of the discrete action $\mathfrak{S}_d$ on $(\Lambda_a^j, \mathbf{x}_a^j, \omega_a^j, \gamma_a^j)$ reads

(i) Interior nodes $a \notin \{a_0, a_N\}$

$$\mathscr{L}_a^j = \Delta t\, \frac{l_K}{2} \left\langle \mathbb{J}\xi_a^j, \xi_a^j \right\rangle - \Delta t \sum_{K\ni a} \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right), \qquad (6.2.16)$$

(ii) Boundary nodes $a \in \{a_0, a_N\}$

$$\mathscr{L}_a^j = \Delta t \, \frac{l_K}{4} \left\langle \mathbb{J}\xi_a^j, \xi_a^j \right\rangle - \Delta t \sum_{K \ni a} \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j \right). \tag{6.2.17}$$

where in the potential term we choose the elements K containing a.

We now compute the discrete Euler-Lagrange equations (6.2.10). Given $\xi_a^j \in \mathfrak{se}(3)$, as defined in (6.2.15) we have

(i) Interior nodes $a \notin \{a_0, a_N\}$

$$D_{\xi_a^j} \mathscr{L}_a^j = \Delta t \, l_K \begin{pmatrix} J\omega_a^j \\ M\gamma_a^j \end{pmatrix},$$

(i) Boundary nodes $a \in \{a_0, a_N\}$

$$D_{\xi_a^j} \mathscr{L}_a^j = \Delta t \, \frac{l_K}{2} \begin{pmatrix} J\omega_a^j \\ M\gamma_a^j \end{pmatrix}.$$

Next, we have to compute $D_{\Lambda_a}\mathbb{V}_K \in \mathbb{R}^3$ and $D_{\mathbf{x}_a}\mathbb{V}_K \in \mathbb{R}^3$ with $\mathbb{V}$ given in (4.2.6). It was already done in (4.2.4).

For the algorithms presented below, the map $\tau : \mathfrak{se}(3) \to SE(3)$ is the Cayley transform. The variables of these integrators are $g^j = (\Lambda^j, \mathbf{x}^j) \in SE(3)$ and $\xi^j = (\omega_a^j, \gamma_a^j) \in \mathfrak{se}(3)$.

Discrete Euler-Lagrange equations for the left boundary a_0 of the beam.

$$\begin{cases} -\Delta t \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} - \mu_0^j + \mathrm{Ad}^*_{\tau(\Delta t \, \xi_0^{j-1})} \mu_0^{j-1} = 0, \\ \mu_0^j = \dfrac{l_K}{2} \left(\left(\mathrm{d}^R\tau(\Delta t \, \xi_0^j) \right)^{-1} \right)^* \begin{pmatrix} J\omega_{a_0}^j \\ M\gamma_{a_0}^j \end{pmatrix}, \\ g_0^{j+1} = g_0^j \tau(\Delta t \, \xi_0^j), \end{cases} \tag{6.2.18}$$

with

$$\begin{aligned} U_0 = {} & \frac{1}{2}\mathbf{C}_1 \left(\Lambda_0^T \frac{\Delta \mathbf{x}_0}{l_K} - \mathbf{E}_3 \right) \times \Lambda_0^T \Delta \mathbf{x}_0 \\ & + \frac{1}{l_K} \left(\left(\left(I + \Lambda_1^T \Lambda_0 \right)^{-1} \widehat{\mathbf{C}_2 \psi_0} \left(\widehat{\psi}_0 - 2I \right) \right)^{(A)} \right)^{\vee} \Bigg|_{t=t^j}, \\ V_0 = {} & (\Lambda_0)^{-1} \left\{ \frac{1}{2}(-\Lambda_0)\mathbf{C}_1 \left(\Lambda_0^T \frac{\Delta \mathbf{x}_0}{l_K} - \mathbf{E}_3 \right) \right. \\ & \left. + \frac{1}{2}(-\Lambda_1)\mathbf{C}_1 \left(\Lambda_1^T \frac{\Delta \mathbf{x}_0}{l_K} - \mathbf{E}_3 \right) + \frac{l_K}{2}\mathbf{q} \right\} \Bigg|_{t=t^j}, \end{aligned}$$

where Λ_0, ψ_0 are the values at a_0, and Λ_1 at a_1. Moreover, $\psi_0 = \mathrm{cay}^{-1}(\Lambda_0^T \Lambda_1)$ and $\Delta \mathbf{x}_0 = \mathbf{x}_1 - \mathbf{x}_0$.

Discrete Euler-Lagrange equations for the right boundary a_N of the beam.

$$\begin{cases} -\Delta t \begin{pmatrix} U_N \\ V_N \end{pmatrix} - \mu_N^j + \mathrm{Ad}^*_{\tau(\Delta t\, \xi_N^{j-1})} \mu_N^{j-1} = 0, \\ \mu_N^j = \dfrac{l_K}{2} \left(\left(\mathrm{d}^R \tau(\Delta t\, \xi_N^j) \right)^{-1} \right)^* \begin{pmatrix} J\omega^j_{a_N} \\ M\gamma^j_{a_N} \end{pmatrix}, \\ g_N^{j+1} = g_N^j \tau(\Delta t\, \xi_N^j), \end{cases} \tag{6.2.19}$$

with

$$\begin{aligned} U_N = &\frac{1}{2}\mathbf{C}_1 \left(\Lambda_N^T \frac{\Delta \mathbf{x}_{N-1}}{l_K} - \mathbf{E}_3 \right) \times \Lambda_N^T \Delta \mathbf{x}_{N-1} \\ &+ \frac{1}{l_K} \left(\left((\Lambda_{N-1}^T \Lambda_N + I)^{-1} \widehat{\mathbf{C}_2 \psi_{N-1}} (2I - \widehat{\psi}_{N-1}) \Lambda_{N-1}^T \Lambda_N \right)^{(A)} \right)^{\vee} \Bigg|_{t=t^j}, \\ V_N = &(\Lambda_N)^{-1} \left\{ \frac{1}{2} (\Lambda_{N-1}) \mathbf{C}_1 \left(\Lambda_{N-1}^T \frac{\Delta \mathbf{x}_{N-1}}{l_K} - \mathbf{E}_3 \right) \right. \\ &\left. + \frac{1}{2} (\Lambda_N) \mathbf{C}_1 \left(\Lambda_N^T \frac{\Delta \mathbf{x}_{N-1}}{l_K} - \mathbf{E}_3 \right) + \frac{l_K}{2} \mathbf{q} \right\} \Bigg|_{t=t^j}, \end{aligned}$$

where Λ_N is the value at a_N, and $\Lambda_{N-1}, \psi_{N-1}, \Delta \mathbf{x}_{N-1}$ at a_{N-1}. Moreover, $\psi_{N-1} = \mathrm{cay}^{-1}(\Lambda_{N-1}^T \Lambda_N)$ and $\Delta \mathbf{x}_{N-1} = \mathbf{x}_N - \mathbf{x}_{N-1}$.

Discrete Euler-Lagrange equations for any node $a \notin \{a_0, a_N\}$.

$$\begin{cases} -\Delta t \begin{pmatrix} U_a \\ V_a \end{pmatrix} - \mu_a^j + \mathrm{Ad}^*_{\tau(\Delta t\, \xi_a^{j-1})} \mu_a^{j-1} = 0, \\ \mu_a^j = l_K \left(\left(\mathrm{d}^R \tau(\Delta t\, \xi_a^j) \right)^{-1} \right)^* \begin{pmatrix} J\omega^j_a \\ M\gamma^j_a \end{pmatrix}, \\ g_a^{j+1} = g_a^j \tau(\Delta t\, \xi_a^j), \end{cases} \tag{6.2.20}$$

with

$$\begin{aligned} U_a = &\frac{1}{2}\mathbf{C}_1 \left(\Lambda_a^T \frac{\Delta \mathbf{x}_{a-1}}{l_K} - \mathbf{E}_3 \right) \times \Lambda_a^T \Delta \mathbf{x}_{a-1} + \frac{1}{2}\mathbf{C}_1 \left(\Lambda_a^T \frac{\Delta \mathbf{x}_{a}}{l_K} - \mathbf{E}_3 \right) \times \Lambda_a^T \Delta \mathbf{x}_{a} \\ &+ \frac{1}{l_K} \left(\left((\Lambda_{a-1}^T \Lambda_a + I)^{-1} \widehat{\mathbf{C}_2 \psi_{a-1}} (2I - \widehat{\psi}_{a-1}) \Lambda_{a-1}^T \Lambda_a \right)^{(A)} \right)^{\vee} \\ &+ \frac{1}{l_K} \left(\left(\left(I + \Lambda_{a+1}^T \Lambda_a \right)^{-1} \widehat{\mathbf{C}_2 \psi_a} \left(\widehat{\psi}_a - 2I \right) \right)^{(A)} \right)^{\vee} \Bigg|_{t=t^j}, \end{aligned}$$

$$
\begin{aligned}
V_a =&(\Lambda_a)^{-1}\left\{\frac{1}{2}(\Lambda_{a-1})\mathbf{C}_1\left(\Lambda_{a-1}^T\frac{\Delta\mathbf{x}_{a-1}}{l_K}-\mathbf{E}_3\right)+\frac{1}{2}(-\Lambda_a)\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)\right.\\
&+\frac{1}{2}(\Lambda_a)\mathbf{C}_1\left(\Lambda_a^T\frac{\Delta\mathbf{x}_{a-1}}{l_K}-\mathbf{E}_3\right)+\frac{1}{2}(-\Lambda_{a+1})\mathbf{C}_1\left(\Lambda_{a+1}^T\frac{\Delta\mathbf{x}_a}{l_K}-\mathbf{E}_3\right)\\
&\left.+l_K\mathbf{q}\right\}\Big|_{t=t^j},
\end{aligned}
$$

where $\widehat{\psi}_a=\mathrm{cay}^{-1}(\Lambda_a^T\Lambda_{a+1})$, $\widehat{\psi}_{a-1}=\mathrm{cay}^{-1}(\Lambda_{a-1}^T\Lambda_a)$, $\Delta\mathbf{x}_{a-1}=\mathbf{x}_a-\mathbf{x}_{a-1}$, and $\Delta\mathbf{x}_{a-1}=\mathbf{x}_a-\mathbf{x}_{a-1}$.

Discrete momentum maps. We consider the action of $SO(3)$ on $SE(3)$ given by $R\cdot(\Lambda^j,\mathbf{x}^j):=(R\Lambda^j,R\mathbf{x}^j)$. In particular, we admit the S^1 symmetry with respect to the orientation of the gravity. Then the infinitesimal generator for a given $\zeta=\theta\widehat{\mathbf{E}}_3\in\mathfrak{so}(3)$ is

$$
\zeta_{SE(3)\times\mathfrak{se}(3)}((\Lambda^j,\mathbf{x}^j),(\omega^j,\gamma^j))=\left((\Lambda^j,\mathbf{x}^j,\zeta\Lambda^j,\zeta\mathbf{x}^j),(\omega^j,\gamma^j,0,0)\right).
$$

Then the discrete momentum map $\mathbf{J}^+_{\mathscr{L}_d}$ as describe in (4.2.37) (chapter 4) is :

$$
\begin{aligned}
&\mathbf{J}^+_{\mathscr{L}_d}((\Lambda^j,\mathbf{x}^j),(\omega^j,\gamma^j))\\
&=\mathbf{E}_3\cdot\sum_{a\in\{a_0,a_N\}}\frac{l_K}{2}\,\mathrm{Ad}^*_{((\Lambda^j,\mathbf{x}^j))^{-1}}\cdot\left(\left(\left(\mathrm{d}^R\tau\left(\Delta t(\omega^j,\gamma^j)\right)\right)^{-1}\right)^*\begin{pmatrix}J\omega_a^j\\ M\gamma_a^j\end{pmatrix}\right)\\
&\quad+\mathbf{E}_3\cdot\sum_{a\notin\{a_0,a_N\}}l_K\,\mathrm{Ad}^*_{((\Lambda_a^j,\mathbf{x}_a^j))^{-1}}\cdot\left(\left(\left(\mathrm{d}^R\tau\left(\Delta t(\omega_a^j,\gamma_a^j)\right)\right)^{-1}\right)^*\begin{pmatrix}J\omega_a^j\\ M\gamma_a^j\end{pmatrix}\right).
\end{aligned}
$$

By discrete Noether Theorem we know that discrete momentum is a conserved quantity of the discrete Lagrangian map, that is

$$
\mathbf{J}^+_{\mathscr{L}_d}(g^j,\xi^j)=\mathbf{J}^-_{\mathscr{L}_d}(g^j,\xi^j).
$$

Discrete Lagrange d'Alembert equations

Given the discrete Lagrangian $\mathscr{L}_a$ as define in (6.2.16) and (6.2.17), we choose the fiber preserving forces as following

$$
\begin{aligned}
&\mathcal{F}^+_{a,d}((\Lambda_a^{j+1},\mathbf{x}_a^{j+1}),(\omega_a^j,\gamma_a^j))=(0,0)\\
&\mathcal{F}^-_{a,d}((\Lambda_a^j,\mathbf{x}_a^j),(\omega_a^j,\gamma_a^j))\\
&\qquad=\Delta t\,\left(\mathbf{M}_a^-((\Lambda_a^j,\mathbf{x}_a^j),(\omega_a^j,\gamma_a^j)),\ \mathbf{F}_a^-((\Lambda_a^j,\mathbf{x}_a^j),(\omega_a^j,\gamma_a^j))\right),
\end{aligned}
$$

where $\mathbf{M}_a$ and $\mathbf{F}_a$ are respectively the exterior moment and force applied in node a.

This choice is motivated by remarks in Marsden and West [90, page 427], where the authors describe how to choose discrete forces. Other choices of fiber preserving forces are possible.

We give the discrete Lagrange d'Alembert equations (6.2.13), for a given map $\tau:\mathfrak{se}(3)\to SE(3)$ which is the exponential map or the Cayley transform.

Discrete Lagrange-d'Alembert equations for the left boundary a_0 of the beam.

$$\begin{cases} -\Delta t \begin{pmatrix} U_{a_0} \\ V_{a_0} \end{pmatrix} - \mu^j_{a_0} + \mathrm{Ad}^*_{\tau(\Delta t\, \xi^{j-1}_{a_0})} \mu^{j-1}_{a_0} + \Delta t \begin{pmatrix} (\Lambda^j_{a_0})^{-1} (\mathbf{M}^j_{a_0})^- \\ (\Lambda^j_{a_0})^{-1} (\mathbf{F}^j_{a_0})^- \end{pmatrix} = 0, \\ \mu^j_{a_0} = \frac{l_K}{2} \left(\left(\mathrm{d}^R \tau(\Delta t\, \xi^j_{a_0}) \right)^{-1} \right)^* \begin{pmatrix} J\omega^j_{a_0} \\ M\gamma^j_{a_0} \end{pmatrix}, \\ g^{j+1}_{a_0} = g^j_{a_0} \tau(\Delta t\, \xi^j_{a_0}), \end{cases}$$

where U_{a_0}, and V_{a_0} are defined in (6.2.18).

Discrete Lagrange-d'Alembert equations for the right boundary a_N of the beam.

$$\begin{cases} -\Delta t \begin{pmatrix} U_{a_N} \\ V_{a_N} \end{pmatrix} - \mu^j_{a_N} + \mathrm{Ad}^*_{\tau(\Delta t\, \xi^{j-1}_{a_N})} \mu^{j-1}_{a_N} + \Delta t \begin{pmatrix} (\Lambda^j_{a_N})^{-1} (\mathbf{M}^j_{a_N})^- \\ (\Lambda^j_{a_N})^{-1} (\mathbf{F}^j_{a_N})^- \end{pmatrix} = 0, \\ \mu^j_{a_N} = \frac{l_K}{2} \left(\left(\mathrm{d}^R \tau(\Delta t\, \xi^j_{a_N}) \right)^{-1} \right)^* \begin{pmatrix} J\omega^j_{a_{a_N}} \\ M\gamma^j_{a_{a_N}} \end{pmatrix}, \\ g^{j+1}_{a_N} = g^j_{a_N} \tau(\Delta t\, \xi^j_{a_N}), \end{cases}$$

where U_{a_N}, and V_{a_N} are defined in (6.2.19).

Discrete Lagrange-d'Alembert equations for any node $a \notin \{a_0, a_N\}$.

$$\begin{cases} -\Delta t \begin{pmatrix} U_a \\ V_a \end{pmatrix} - \mu^j_a + \mathrm{Ad}^*_{\tau(\Delta t\, \xi^{j-1}_a)} \mu^{j-1}_a + \Delta t \begin{pmatrix} (\Lambda^j_a)^{-1} (\mathbf{M}_a)^- \\ (\Lambda^j_a)^{-1} (\mathbf{F}_a)^- \end{pmatrix} = 0, \\ \mu^j_a = l_K \left(\left(\mathrm{d}^R \tau(\Delta t\, \xi^j_a) \right)^{-1} \right)^* \begin{pmatrix} J\omega^j_a \\ M\gamma^j_a \end{pmatrix}, \\ g^{j+1}_a = g^j_a \tau(\Delta t\, \xi^j_a), \end{cases}$$

where U_a, and V_a are defined in (6.2.20).

6.3 Numerical simulations

Initial conditions. The initial condition at time t^0 are $(\Lambda^0, x^0) \in SE(3)$, and $(\omega^0, \gamma^0) \in \mathfrak{se}(3)$.

Stresses

Let the given stored energy $\Psi_K(\Omega_K, \Gamma_K)$ in terms of strain, for an element K, be :

$$\Psi_K(\Omega_K, \Gamma_K) = \frac{l_K}{4} \Big\{ (\Gamma_a - \mathbf{E}_3)^T \mathbf{C}_1 (\Gamma_a - \mathbf{E}_3) + (\Gamma_{a+1} - \mathbf{E}_3)^T \mathbf{C}_1 (\Gamma_{a+1} - \mathbf{E}_3) + (\Omega_a)^T \mathbf{C}_2 \Omega_a + (\Omega_{a+1})^T \mathbf{C}_2 \Omega_{a+1} \Big\}.$$

The expressions of the stress (see, for example, [83]) are $\frac{\partial \Psi_K}{\partial \Omega}$ and $\frac{\partial \Psi_K}{\partial \Gamma}$, that is, for each node a and $a+1$

$$\mathbf{M}_a = \frac{l_K}{2}\mathbf{C}_2\Omega_a, \qquad \mathbf{M}_{a+1} = \frac{l_K}{2}\mathbf{C}_2\Omega_a,$$
$$\mathbf{N}_a = \frac{l_K}{2}\mathbf{C}_1\left(\Gamma_a - \mathbf{E}_3\right), \qquad \mathbf{N}_{a+1} = \frac{l_K}{2}\mathbf{C}_1\left(\Gamma_{a+1} - \mathbf{E}_3\right).$$

6.4 Example

Parameters of the beam : length $L = 0.5$, mass density $\varrho = 10^3$, square cross-section with edge length $a = 0.05$, Poisson ratio $\nu = 0.35$, and Young's modulus $E = 10^4$, for the following test:

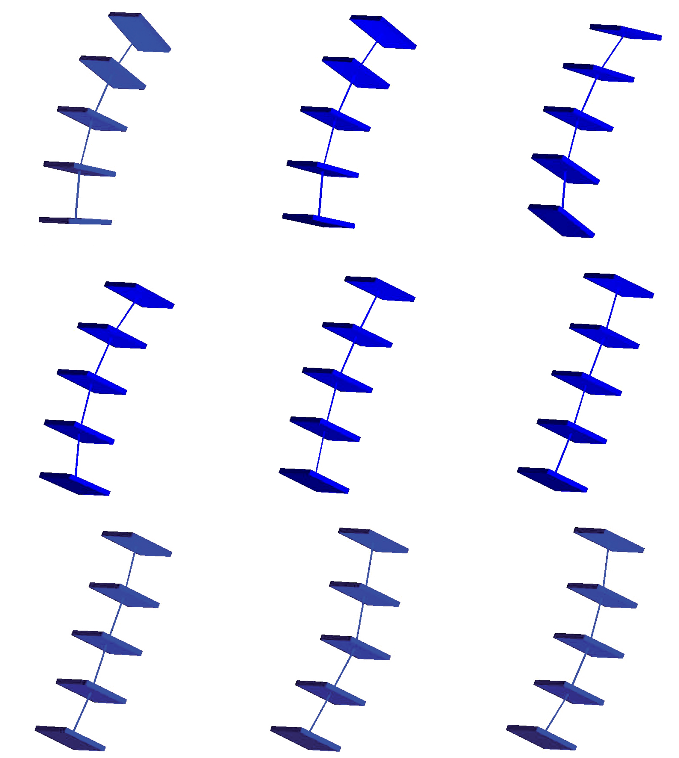


Figure 6.4.1: Free motion of the cross-sections of the beam relative to each other, when initial bending. Snapshots of the motion and deformation.

6.4.1 Remark The implementation, in progress, is performed by M. Kobilarov (University Johns Hopkins).

Chapter 7

Lie algebra variational integrator of geometrically exact plate dynamics

Introduction

In this chapter we consider the geometrically exact model of plate as defined in Simo, Marsden, and Krishnaprasad [110] and Simo, and Fox [108]. The space of configuration of this plate is very similar to that of the exact model of beam defined in Simo [107]. Indeed, for the beam, the space of configuration is $\mathcal{C}^\infty([0,L], SE(3))$, whereas for the plate it is $\mathbf{N} = \mathcal{C}^\infty(\mathcal{A}, S^2_{\mathbf{E}} \times \mathbb{R}^3)$ which is a subset of $Q = \mathcal{C}^\infty(\mathcal{A}, SE(3))$, where $\mathcal{A} \subset \mathbb{R}^2$ is an open set with smooth boundary, and compact closure, and $S^2_{\mathbf{E}}$ is the set of rotations whose rotation axis is normal to the vertical direction $\mathbf{E}$. We note that $S^2_{\mathbf{E}}$ is a sub-set of $SO(3)$ and not a sub-group.

In order to maintain the matrix of rotation in $S^2_{\mathbf{E}}$ we introduce a holonomic constraint $\Phi : Q \to \mathbb{R}^d$, such that $\mathbf{N} = \Phi^{-1}(0) \subset Q$. Thus the solutions of the Euler-Lagrange equations stay in $\mathbf{N}$.

This plate model is different from the one introduced in the classical paper of Ericksen and Truesdell [29]. The difference is that Simo considered a constrained-frame point of view in order to stay in $S^2_{\mathbf{E}} \times \mathbb{R}^3$. Thus the equations of motion and the Poisson bracket are very close to those of the geometrically exact beam model considered in chapter 4. (See Simo, Marsden, and Krishnaprasad [110]).

During the past decade, the Kirchhoff theory of thin plates and the Kirchhoff-Love theory of thin shells were often chosen in association with the finite element method to study finite membrane stretching, as well as large deflections. The energy functional of this model depends on curvature; consequently, the equations contain second-order derivatives of displacement. The use of subdivision surfaces ensures the testing of deformed geometries to be of Sobolev H^2 class (see Cirak, Ortiz, and Schröder [23]). However, the strain measures deduced from the deformation of the middle surface are generally obtained after linearization of the kinematics, which is not the case for the Simo model that uses Lie groups. We develop a Lie algebra variational integrator, as in Chapter 6, to take full advantage of its numerically efficiency and properties.

7.1 Lagrangian dynamics of a plate in $\mathbb{R}^3$

7.1.1 Basic kinematics of a plate

We review from Simo, Marsden, and Krishnaprasad [110] and from Simo, and Fox [108], the kinematic description of a plate in the ambient space $\mathbb{R}^3$.

We denote by $\mathfrak{so}(3)$ the Lie algebra of $SO(3)$ consisting of skew symmetric matrices endowed with the Lie bracket $[\xi,\eta]=\xi\eta-\eta\xi$. The adjoint representation of $SO(3)$ on its Lie algebra is denoted by $\mathrm{Ad}_\Lambda\,\xi=\Lambda\xi\Lambda^{-1}$, where $\xi\in\mathfrak{so}(3)$ and $\Lambda\in SO(3)$.

We denote by $S^2=\{\mathbf{t}\in\mathbb{R}^3\mid\|\mathbf{t}\|=1\}$, the unit sphere and by

$$T_{\mathbf{t}}S^2=\left\{\mathbf{w}\in\mathbb{R}^3\mid\mathbf{w}^T\mathbf{t}=0\right\}$$

its tangent space at $\mathbf{t}$. Let $\mathbf{E}\in\mathbb{R}^3$ and define $S_{\mathbf{E}}$ to be the set of rotations $\Lambda\in SO(3)$ whose rotation axis is normal to $\mathbf{E}$, that is

$$S^2_{\mathbf{E}}=\{\Lambda\in SO(3)\mid\text{there is }\psi\in\mathbb{R}^3,\ \text{satisfying }\psi\neq 0,\ \Lambda\psi=\psi,\ \text{and }\psi^T\mathbf{E}=0\},$$

which is a subset of $SO(3)$ and not a subgroup. The tangent space at the identity $\mathbf{I}\in S^2_{\mathbf{E}}$ is

$$T_{\mathrm{I}}S^2_{\mathbf{E}}=\left\{\widehat{\eta}\in\mathfrak{so}(3)\mid\eta^T\mathbf{E}=0\right\}\tag{7.1.1}$$

and the tangent space at an arbitrary element Λ reads

$$T_\Lambda S^2_{\mathbf{E}}=\left\{\Lambda\widehat{\eta}\mid\widehat{\eta}\in T_I S^2_{\mathbf{E}}\right\}.$$

We now recall from Simo, and Fox [108] a fundamental result concerning a relation between S^2 and $S^2_{\mathbf{E}}$.

7.1.1 Proposition *Given any two vectors $\mathbf{E},\mathbf{t}\in S^2$ with $\mathbf{t}\neq-\mathbf{E}$ there exists a unique $\Lambda\in S^2_{\mathbf{E}}$ such that*

$$\mathbf{t}=\Lambda\mathbf{E},$$

where

$$\Lambda:=(\mathbf{t}^T\mathbf{E})\mathbf{I}+\widehat{(\mathbf{E}\times\mathbf{t})}+\frac{1}{1+\mathbf{t}^T\mathbf{E}}(\mathbf{E}\times\mathbf{t})\otimes(\mathbf{E}\times\mathbf{t})$$

7.1.2 Deformation expressed relative to the inertial frame

Given a fixed orthonormal basis $\{\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}\}$ of $\mathbb{R}^3$ referred to as the *material frame*, the configuration of a plate is defined by specifying the position of its mid-surface by means of a map

$$\phi:\mathcal{A}\subset\mathrm{span}(\mathbf{E}_1,\mathbf{E}_2)\to\mathbb{R}^3,\quad\mathbf{u}=(u_1,u_2)\mapsto\phi(\mathbf{u}),$$

and the orientation of the deformation director $\mathbf{t}(\mathbf{u})$ attached to $\phi(\mathbf{u})$, where $\mathcal{A}$ is a compact subset with piecewise smooth boundary. The orientation of the director is obtained from $\mathbf{E}$ through the orthogonal transformation $\Lambda : \mathcal{A} \to S^2_{\mathbf{E}}$, such that

$$\mathbf{t}(\mathbf{u}) = \Lambda(\mathbf{u})\mathbf{E}, \quad \text{as long as} \quad \mathbf{t} \neq -\mathbf{E}.$$

The configuration of the plate is thus completely determined by the maps ϕ and Λ in the configuration space

$$Q := C^\infty \left(\mathcal{A},\ S^2_{\mathbf{E}} \times \mathbb{R}^3\right) \ni \Phi = (\Lambda, \phi).$$

If boundary conditions are imposed, then they need to be included in this configuration space. For example if $\mathcal{A} = [0, L_1] \times [0, L_2]$ is a rectangle, we can consider the following boundary conditions: $\phi(0, u_2) = (0, u_2)$ which means that the boundary $\{0\} \times [0, L_2]$ is fixed; $\Lambda(0, u_2) = Id$, which means that the director $\mathbf{t}$ is parallel to $\mathbf{E}$ along $\{0\} \times [0, L_2]$. One can add the condition $\frac{\partial \phi}{\partial u_1}(0, u_2) = \lambda \mathbf{E}_1$ for $\lambda > 0$, which means that the curve $u_1 \mapsto \phi(u_1, u_2)$ is orthogonal to the boundary $\{0\} \times [0, u_2]$ at $u_1 = 0$. Similarly we can impose boundaries conditions at the other edges of the rectangle.

Suppose that the plate is in the configuration determined by $(\Lambda, \phi) \in Q$ and that its thickness is given by a compact subset $[h^-, h^+] \subset \mathbb{R}$, then the set occupied by the plate is

$$\mathcal{B} = \left\{ X \in \mathbb{R}^3 \mid X = \phi(\mathbf{u}) + \xi \Lambda(\mathbf{u})\mathbf{E}, \ \text{with} \ \ (\mathbf{u}, \xi) \in \mathcal{A} \times [h^-, h^+] \right\},$$

where ϕ maps the mid-plane $\mathcal{A}$ to the mid-surface $\phi(\mathcal{A}) \subset \mathbb{R}^3$, and $\Lambda(\mathbf{u})\mathbf{E} = \mathbf{t}(\mathbf{u})$ is the unit vector attached to the point $\phi(\mathbf{u}) \in \mathbb{R}^3$ not necessarily normal to the mid-surface $\phi(\mathcal{A})$.

The time evolution of the plate is described by a curve $(\Lambda(t), \phi(t)) \in Q$ in the configuration space. The *material velocity* V_Φ is defined by

$$V_\Phi(\mathbf{u}, t) := \frac{d}{dt} \left(\Lambda(\mathbf{u}, t), \phi(\mathbf{u}, t)\right) = \left(\dot{\Lambda}(\mathbf{u}, t), \dot{\phi}(\mathbf{u}, t)\right),$$

and thus belongs to the tangent space $T_{(\Lambda, \phi)}Q$ of Q at (Λ, ϕ).

The *convective angular velocity* and *convective linear velocity* are the maps $\widehat{\omega}, \gamma : \mathcal{A} \to T_{\mathbf{I}} S^2_{\mathbf{E}} \times \mathbb{R}^3$ defined by

$$\widehat{\omega} := \Lambda^T \dot{\Lambda}, \quad \gamma := \Lambda^T \dot{\phi} \tag{7.1.2}$$

Note that this definition can be rewritten, using the group structure of $SE(3)$, as

$$(\widehat{\omega}, \gamma) = (\Lambda, \phi)^{-1} (\dot{\Lambda}, \dot{\phi}), \quad \text{where } (\Lambda, \phi) \in Q.$$

7.1.3 Kinetic energy

We present below the Lagrangian function of the plate.

The kinetic energy is found by integrating the kinetic energy of the material points over the whole body. Denoting $\mathcal{D} := \mathcal{A} \times [h^-, h^+]$, we have

$$\begin{aligned}
T(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) &= \frac{1}{2} \int_{\mathcal{D}} \left\| \dot{\phi}(\mathbf{u}) + \xi \dot{\mathbf{t}}(\mathbf{u}) \right\|^2 \rho(\mathbf{u}, \xi) d\mathcal{A} d\xi \\
&= \frac{1}{2} \int_{\mathcal{D}} \left\| \dot{\phi}(\mathbf{u}) + \xi \Lambda(\mathbf{u}) \widehat{\omega}(\mathbf{u}) \mathbf{E} \right\|^2 \rho(\mathbf{u}, \xi) d\mathcal{A} d\xi \\
&= \frac{1}{2} \int_{\mathcal{D}} \left[\left\| \dot{\phi}(\mathbf{u}) \right\|^2 + \| \xi \widehat{\omega}(\mathbf{u}) \mathbf{E} \|^2 \right] \rho(\mathbf{u}, \xi) d\mathcal{A} d\xi,
\end{aligned}$$

where $\rho(\mathbf{u}, \xi)$ is the mass density and where we used the fact that the mid-surface ϕ passes through the center of mass, i.e.

$$\int_{h^-}^{h^+} \xi \mathbf{E} \rho(\mathrm{u}, \xi) d\xi = 0, \quad \text{for all} \quad \mathbf{u} \in \mathcal{A}.$$

For simplicity, we assume that $\rho(\mathrm{u}, \xi) = \rho_0$ is a constant, so that we necessarily have $-h^- = h^+ =: h/2$. Using the relation $\widehat{A} B = -\widehat{B} A$ for $A, B \in \mathbb{R}^3$ we get

$$T(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) = \frac{1}{2} \int_{\mathcal{A}} \left(M \left\| \dot{\phi}(\mathbf{u}) \right\|^2 + \omega(\mathbf{u})^T J \omega(\mathbf{u}) \right) d\mathcal{A},$$

where we defined the inertia tensor

$$J := -\int_{-h/2}^{h/2} \rho_0 \, \xi^2 \left(\widehat{\mathbf{E}} \right)^2 d\xi,$$

and the distributed loads per unit surface $M := \rho_0 h$.

Since J is a positive definite matrix, it can be diagonalized, and provide eigenvalues $\{J_1, J_2, J_3\}$. The associated eigenvectors are principal axis.

7.1.2 Remark Note that using the equalities

$$\| \widehat{\omega} \mathbf{E} \|^2 = \mathrm{Tr} \left(\mathbf{E}^T \widehat{\omega}^T \widehat{\omega} \mathbf{E} \right) = \mathrm{Tr} \left(\widehat{\omega} \mathbf{E} \mathbf{E}^T \widehat{\omega}^T \right),$$

we can rewrite the kinetic energy as

$$T(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) = \frac{1}{2} \int_{\mathcal{A}} M \left\| \dot{\phi} \right\|^2 + \mathrm{Tr} \left[\widehat{\omega} J_d \widehat{\omega}^T \right] d\mathcal{A},$$

where

$$J_d := \int_{-h/2}^{h/2} \rho_0 \xi^2 \mathbf{E} \mathbf{E}^T d\xi.$$

7.1.4 Potential energy

The potential energy is given by the sum of interior potential energy (bending energy) and exterior potential energy (gravitational energy and energy created by external force and torque).

Bending energy

Given a configuration $(\Lambda, \phi) \in G$, the *deformation gradient* is defined as

$$F(\mathbf{u}) = (\mathbf{d}\Lambda(\mathbf{u}), \mathbf{d}\phi(\mathbf{u}))\,,$$

where $\mathbf{d}$ denotes the derivative with respect to $\mathbf{u}$. As in Simo, Marsden, and Krishnaprasad [110], we will use the convected deformation gradients defined by

$$\widehat{\Omega} := \Lambda^{-1}\mathbf{d}\Lambda \in \Omega^1\left(\mathcal{A}, T_I S^2_{\mathbf{E}}\right), \quad \Gamma := \Lambda^{-1}\mathbf{d}\phi \in \Omega^1(\mathcal{A}, \mathbb{R}^3). \tag{7.1.3}$$

In coordinates, $\mathbf{d}\Lambda$, $\mathbf{d}\phi$, Ω, Γ are denoted by

$$\partial_\alpha\Lambda, \;\; \partial_\alpha\phi, \;\; \Omega_\alpha = \Lambda^{-1}\partial_\alpha\Lambda, \;\; \Gamma_\alpha = \Lambda^{-1}\partial_\alpha\phi, \quad \alpha \in \{1, 2\}.$$

The bending energy is assumed to depend on the deformation gradient only through the quantities Ω_α and Γ_α that is, we have

$$\Pi_{int}(\Lambda, \phi) = \int_{\mathcal{D}} \Psi_{int}(\Omega_\alpha, \Gamma_\alpha) d\mathcal{A} d\xi,$$

where $\Psi_{int}(\Omega_\alpha, \Gamma_\alpha)$ is the stored energy function.

We assume that the unstressed state is unstretched and unsheared. That is to say that we have $\phi_{,1}(\mathbf{u}, t = 0) = \mathbf{E}_1$, $\phi_{,2}(\mathbf{u}, t = 0) = \mathbf{E}_2$ and $\Lambda(\mathbf{u}, t = 0) = Id$, for all $\mathbf{u} \in \mathcal{A}$. Also by considering that the thickness is small compared to its length, and that the material is homogeneous, we can interpret the stored energy by the following quadratic model

$$\Pi_{int}(\Lambda, \phi) = \frac{1}{2} \int_{\mathcal{A}} \sum_{\alpha,\beta} \Big\{ (\Gamma_\alpha - \mathbf{E}_\alpha)^T \mathbb{C}_{\alpha\beta} (\Gamma_\beta - \mathbf{E}_\beta) + \Omega_\alpha^T \mathbb{D}_{\alpha,\beta} \Omega_\beta \Big\} d\mathcal{A}\;,$$

for $\alpha \in \{1, 2\}$, where $\mathbb{C}_{11}, \mathbb{C}_{22}, \mathbb{C}_{12} = \mathbb{C}_{21}, \mathbb{D}_{11}, \mathbb{D}_{22}$, and $\mathbb{D}_{12} = \mathbb{D}_{21}$ are symmetric matrices, whose values depends on the thickness and on the material of which is composed the plate.

We note that the internal energy is invariant under the left action of elements of $SO(3)$, i.e. $\Pi_{int}(R\Lambda, R\phi) = \Pi_{int}(\Lambda, \phi)$, for all $R \in SO(3)$.

Exterior potential energy

We consider the potential energy created by exterior forces

$$\Pi_{ext}(\phi) = \int_{\mathcal{A}} (\langle \mathbf{q}, \phi \rangle) d\mathcal{A},$$

where $\mathbf{q} = q\mathbf{E}$ is the distributed loads per unit surface. In this form $\Pi_{ext}(\phi)$ is not invariant under the left action of elements of $SO(3)$.

7.1.5 Equation of motions and constraints for the plate

In order to implement our numerical method, we will work on the bigger configuration space $Q = C^\infty(\mathcal{A}, SE(3)) \ni (\Lambda, \phi)$ and impose the condition $\Lambda \in S^2_{\mathbf{E}}$ via the holonomic constraint. The equation of the motion under constraint is obtained by the following theorem (see Marsden and West in [90])

7.1.3 Theorem *Given a Lagrangian* $L : TQ \to \mathbb{R}$ *with holonomic constraint* $\Phi : Q \to \mathbb{R}^d$, *set* $\mathbf{N} = \Phi^{-1}(0) \subset Q$ *and* $L^{\mathbf{N}} = L\big|_{T\mathbf{N}}$. *Then the following are equivalent :*

(i) $q \in \mathcal{C}(\mathbf{N})$ *extremizes* $\mathfrak{S}^{\mathbf{N}}$ *and hence solves the Euler-Lagrange equations for* $L^{\mathbf{N}}$;

(ii) $q \in \mathcal{C}(Q)$ *and* $\lambda \in C(\mathbb{R}^d)$ *satisfy the constrained Euler-Lagrange equations*

$$\frac{\partial L}{\partial q^i}(q(t), \dot{q}(t)) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t))\right) = \left\langle \lambda(t), \frac{\partial \Phi}{\partial q^i}(q(t)) \right\rangle,$$
$$\Phi(q(t)) = 0;$$

(iii) $(q, \lambda) \in \mathcal{C}(Q \times \mathbb{R}^d)$ *extremizes* $\overline{\mathfrak{S}}(q, \lambda) = \mathfrak{S}(q) - \langle \lambda, \Phi(q) \rangle$, *and hence solves the Euler-Lagrange equations for the augmented Lagrangian* $\overline{L} : T(Q \times \mathbb{R}^d) \to \mathbb{R}$, *as*

$$\overline{L}(q, \lambda, \dot{q}, \dot{\lambda}) = L(q, \dot{q}) - \langle \lambda, \Phi(q) \rangle .$$

The constraint that we are dealing with for the plate is as follows :

$$\Phi : Q \to \mathbb{R}, \quad \Phi(\Lambda, \phi) = \frac{1}{2}\operatorname{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda)\right), \tag{7.1.4}$$

which constrains the dynamic to the submanifold $\mathbf{N} = S^2_{\mathbf{E}} \times \mathbb{R}^3 = \Phi^{-1}(0)$, in such a way that is an embedding $i : \mathbf{N} \to Q$.

Constrained Euler-Lagrange equations for the plate

The associated augmented Lagrangian $\overline{L} : T(\mathbf{Q} \times \mathbb{R}) \to \mathbb{R}$ reads

$$\begin{aligned}\overline{L}(\Lambda, \phi, \lambda, \dot{\Lambda}, \dot{\phi}, \dot{\lambda}) = & \frac{1}{2}\int_{\mathcal{A}} \left[M \left\| \dot{\phi} \right\|^2 + \omega^T J \omega \right] d\mathcal{A} \\ & - \frac{1}{2}\int_{\mathcal{A}} \sum_{\alpha,\beta} \Big((\Gamma_\alpha - \mathbf{E}_\alpha)^T \mathbb{C}_{\alpha\beta} (\Gamma_\beta - \mathbf{E}_\beta) + \Omega_\alpha^T \mathbb{D}_{\alpha,\beta} \Omega_\beta \Big) d\mathcal{A} \\ & - \int_{\mathcal{A}} (\langle \mathbf{q}, \phi \rangle + \langle \mathbf{N}, \phi \rangle) d\mathcal{A} - \int_{\mathcal{A}} \lambda \Phi(\Lambda, \phi) d\mathcal{A},\end{aligned}$$

where $\lambda \in \mathbb{R}$. The Euler-Lagrange equations are obtained by applying Hamilton's principle to the action

$$\mathfrak{S}(\Lambda, \phi, \lambda) = \int_{t_0}^{t_1} \overline{L}\left(\Lambda(t), \phi(t), \lambda(t), \dot{\Lambda}(t), \dot{\phi}(t), \dot{\lambda}(t)\right) dt.$$

Consider variations

$$\varepsilon \mapsto (\Lambda_\varepsilon, \phi_\varepsilon, \lambda_\varepsilon)$$

of the curves (Λ, ϕ, λ), with fixed endpoints. The infinitesimal variations are

$$\delta\Lambda = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Lambda_\varepsilon, \quad \delta\phi = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \phi_\varepsilon, \quad \delta\lambda = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \lambda_\varepsilon$$

and vanish at the endpoints. Since the Lagrangian is expressed in terms of the auxiliary variables ω, Ω_α and Γ_α, it is useful to compute the variations $\delta\omega$, $\delta\Omega_\alpha$ and $\delta\Gamma_\alpha$ induced by the variations $\delta\Lambda$ and $\delta\phi$. Writing $\delta\Lambda = \Lambda\widehat{\eta}$, a direct computation shows that we have

$$\begin{aligned}
\delta\omega &= \dot{\eta} + \omega \times \eta = \dot{\eta} + \widehat{\omega}\eta \\
\delta\Omega_\alpha &= \partial_\alpha\eta + \Omega_\alpha \times \eta = \partial_\alpha\eta + \widehat{\Omega}_\alpha\eta \\
\delta\Gamma_\alpha &= \Lambda^T \delta\partial_\alpha\phi + \Gamma_\alpha \times \eta.
\end{aligned}$$

We will denote by $\overline{d^R \exp_\xi}$ and $\overline{d^R \exp_\xi^{-1}}$ the maps $d^R \exp_\xi$ and $d^R \exp_\xi^{-1}$, as described in (6.2.4) and (6.2.5), when seen as linear maps on $\mathbb{R}^3$. Using this notation, the derivative of the constraint reads

$$\begin{aligned}
\mathbf{D}\Phi(\Lambda, \phi) \cdot \delta\Lambda &= \frac{1}{2}\operatorname{Tr}\left(\widehat{\mathbf{E}}^T d^R\exp_\xi^{-1}\left(\delta\Lambda\Lambda^{-1}\right)\right) = \frac{1}{2}\operatorname{Tr}\left(\widehat{\mathbf{E}}^T d^R\exp_\xi^{-1}\left(\mathrm{Ad}_\Lambda\widehat{\eta}\right)\right) \\
&= \mathbf{E} \cdot \overline{d^R\exp_\xi^{-1}}(\Lambda\eta)
\end{aligned}$$

Applying Hamilton's principle we get

$$\begin{aligned}
\delta\mathfrak{S} = \int_{t_0}^{t_1} \Bigg[&\int_{\mathcal{A}} \left(M\dot{\phi}^T \left(\delta\dot{\phi}\right) + \omega^T J \delta\omega \right) d\mathcal{A} \\
&- \int_{\mathcal{A}} \sum_{\alpha,\beta} \left((\Gamma_\alpha - \mathbf{E}_\alpha)^T \mathbb{C}_{\alpha\beta}\ \delta\Gamma_\beta + \Omega_\alpha^T \mathbb{D}_{\alpha,\beta}\ \delta\Omega_\beta \right) d\mathcal{A} \\
&- \int_{\mathcal{A}} \left(\mathbf{q}^T \delta\phi\right) d\mathcal{A} - \int_{\mathcal{A}} \left(\Phi(\Lambda)\delta\lambda + \lambda\delta\Phi(\Lambda)\right) d\mathcal{A} \Bigg] dt.
\end{aligned}$$

Taking into account of the formulas for the variations $\delta\omega$, $\delta\Omega_\beta$, $\delta\Gamma_\beta, \delta\Phi$, this integral is

$$\begin{aligned}
\int_{t_0}^{t_1} \Bigg[&\int_{\mathcal{A}} \left(M\dot{\phi}^T \left(\delta\dot{\phi}\right) + \omega^T J(\dot{\eta} + \widehat{\omega}\eta) \right) d\mathcal{A} \\
&- \int_{\mathcal{A}} \sum_{\alpha,\beta} \left((\Gamma_\alpha - \mathbf{E}_\alpha)^T \mathbb{C}_{\alpha\beta}\ \left(\Lambda^T \delta\partial_\beta\phi + \Gamma_\beta \times \eta\right) + \Omega_\alpha^T \mathbb{D}_{\alpha,\beta}\ \left(\partial_\beta\eta + \widehat{\Omega}_\beta\eta\right) \right) d\mathcal{A} \\
&- \int_{\mathcal{A}} \left(\mathbf{q}^T \delta\phi\right) d\mathcal{A} - \int_{\mathcal{A}} \left(\Phi(\Lambda, \phi)\delta\lambda + \lambda\mathbf{E} \cdot \overline{d^R\exp_\xi^{-1}}(\Lambda\eta)\right) d\mathcal{A} \Bigg] dt.
\end{aligned}$$

Now we isolate the quantities η, $\delta\phi$ by integrating by parts and obtain

$$\int_{t_0}^{t_1}\Bigg[\int_{\mathcal{A}}\Big(-M\ddot{\phi}^T\delta\phi+\left(-\dot{\omega}^TJ+\omega^TJ\widehat{\omega}\right)\eta\Big)\,d\mathcal{A}$$
$$+\sum_{\alpha,\beta}\Bigg\{\int_{\mathcal{A}}\partial_\beta\left((\Gamma_\alpha-\mathbf{E}_\alpha)^T\mathbb{C}_{\alpha,\beta}\Lambda^T\right)\delta\phi\,d\mathcal{A}-\int_0^{L_\alpha}\left[(\Gamma_\alpha-\mathbf{E}_\alpha)^T\mathbb{C}_{\alpha,\beta}\Lambda^T\delta\phi\right]_0^{L_\beta}du_\alpha$$
$$-\int_{\mathcal{A}}\Big((\mathbb{C}_{\alpha,\beta}(\Gamma_\alpha-\mathbf{E}_\alpha)\times\Gamma_\beta)^T+\Omega_\alpha^T\mathbb{D}_{\alpha,\beta}\widehat{\Omega}_\beta-\partial_\beta(\Omega_\alpha^T\mathbb{D}_{\alpha,\beta})\Big)\eta d\mathcal{A}$$
$$-\int_0^{L_\alpha}\left[\Omega_\alpha^T\mathbb{D}_{\alpha,\beta}\eta\right]_0^{L_\beta}du_\alpha\Bigg\}-\int_{\mathcal{A}}\Big(\left(\mathbf{q}^T\right)\delta\phi+\Phi(\Lambda,\phi)\delta\lambda+\lambda\mathbf{E}^T\overline{d^R\mathrm{exp}_\xi^{-1}}\Lambda\eta\Big)\,d\mathcal{A}\Bigg]\,dt.$$

We thus obtain the Euler-Lagrange equations

$$\begin{cases} J\dot{\omega}+\omega\times J\omega+\displaystyle\sum_{\alpha,\beta}\Big(\mathbb{C}_{\alpha,\beta}(\Gamma_\alpha-\mathbf{E}_\alpha)\times\Gamma_\beta \\ \qquad\qquad -\Omega_\beta\times\mathbb{D}_{\alpha,\beta}\Omega_\alpha-\partial_\beta(\mathbb{D}_{\alpha,\beta}\Omega_\alpha)\Big)-\lambda\Lambda^T\overline{d^R\mathrm{exp}_\xi^{-1}}^T\mathbf{E}=0 \\ M\ddot{\phi}-\displaystyle\sum_{\alpha,\beta}\Big(\partial_\beta\left(\Lambda\mathbb{C}_{\alpha,\beta}(\Gamma_\alpha-\mathbf{E}_\alpha)\right)\Big)+\mathbf{q}=0 \\ \Phi(\Lambda,\phi)=0 \end{cases} \tag{7.1.5}$$

with boundary conditions

$$\begin{cases} (\Gamma_\alpha-\mathbf{E}_\alpha)|_{u_\beta=0}=0 \\ (\Gamma_\alpha-\mathbf{E}_\alpha)|_{u_\beta=L_\beta}=0 \\ \Omega_\alpha(0)=\Omega_\alpha(L_\beta)=0. \end{cases} \tag{7.1.6}$$

7.2 Lie algebra variational integrator for the plate

In this section we develop a Lie algebra variational integrator for constrained systems, see e.g. Marsden, and West §3.4, §3.5 in [90].

7.2.1 Lie group structure

As we have seen, in the Lagrangian representation the motion of the plate is described by variables $\phi(\mathbf{u},t)\in\mathbb{R}^3$, which is the position of the mid-surface, and $\Lambda(\mathbf{u},t)\in S^2_{\mathbf{E}}$ denotes the rotation of the director $\mathbf{t}$ in relation with $\mathbf{E}$. In this section we will use the fact that $\mathbf{N}\subset C^\infty(\mathcal{A},SE(3))$, where $SE(3)$ is the special Euclidean group with group multiplication and inversion given by

$$(\Lambda_1,\phi_1)(\Lambda_2,\phi_2)=(\Lambda_1\Lambda_2,\phi_1+\Lambda_1\phi_2)\,,\quad(\Lambda,\phi)^{-1}=\left(\Lambda^{-1},-\Lambda^{-1}\phi\right).$$

The angular and linear convected velocities and the angular and linear convected strain, defined respectively in (7.1.2) and (7.1.3), are

$$\begin{aligned}\widehat{\omega} &= \Lambda^{-1}\dot{\Lambda} \in \mathcal{F}(\mathcal{A}, \mathfrak{so}(3)),\\ \gamma &= \Lambda^{-1}\dot{\phi} \in \mathcal{F}(\mathcal{A}, \mathbb{R}^3),\\ \widehat{\Omega}_\alpha &= \Lambda^{-1}\frac{\partial \Lambda}{\partial u_\alpha} \in \Omega^1(\mathcal{A}, \mathfrak{so}(3)),\\ \Gamma_\alpha &= \Lambda^{-1}\frac{\partial \phi}{\partial u_\alpha} \in \Omega^1(\mathcal{A}, \mathbb{R}^3).\end{aligned}$$

Recall also that, from (7.1.1), we have

$$(\omega)^T\mathbf{E} = 0, \quad (\Omega_\alpha)^T\mathbf{E} = 0.$$

Trivialized Euler-Lagrange equations on Lie groups. We now quickly recall the expression of the trivialized Euler-Lagrange equations on the tangent bundle of a Lie group G.

Let $L : TG \to \mathbb{R}$ be a Lagrangian defined on the tangent bundle TG to a Lie group G. Using the left trivialization $TG \simeq G \times \mathfrak{g}$ of the tangent bundle (see Bobenko, and Suris [11]), we get the function $\mathcal{L} : G \times \mathfrak{g} \to \mathbb{R}$ defined by

$$\mathcal{L}(g, \xi) := L(g, \dot{g}), \quad \dot{g} := g\xi.$$

By applying Hamilton principle we obtain the Euler-Lagrange equation in terms of $\mathcal{L}$. For an interval of time $[t_0, t_1]$ and given the boundaries points $(g(t_0), \xi(t_0))$ and $(g(t_1), \xi(t_1))$ held fixed, we let

$$\delta \int_{t_0}^{t_1} \mathcal{L}(g, \xi)dt = \int_{t_0}^{t_1} \left(\left\langle \frac{\partial \mathcal{L}}{\partial g}, \delta g \right\rangle + \left\langle \frac{\partial \mathcal{L}}{\partial \xi}, \delta \xi \right\rangle \right) dt = 0,$$

where the variation δg vanishes at the endpoints and the variation of ξ is given by $\delta\xi = \dot{\eta} + \mathrm{ad}_\xi\, \eta$, with $\eta = g^{-1}\delta g$, (see Marsden, and Ratiu [89] p.438). Using integration by parts we get

$$\delta \int_{t_0}^{t_1} \mathcal{L}(g, \xi)dt = \int_{t_0}^{t_1} \left(\left\langle g^{-1}\frac{\partial \mathcal{L}}{\partial g} + \mathrm{ad}^*_\xi \frac{\partial \mathcal{L}}{\partial \xi}, \eta \right\rangle - \left\langle \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \xi}\right), \eta \right\rangle \right) dt = 0,$$

for all $\eta \in \mathfrak{g}$. Hence the Euler-Lagrange equations in terms of $\mathcal{L}$ read

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \xi}\right) - \mathrm{ad}^*_\xi \frac{\partial \mathcal{L}}{\partial \xi} = g^{-1}\frac{\partial \mathcal{L}}{\partial g}.$$

The trivialized augmented Lagrangian for the plate. We trivialize the tangent bundle TQ by using the Lie group structure of $SE(3)$. More precisely, we use the diffeomorphism $(\Lambda, \phi, \dot{\Lambda}, \dot{\phi}) \mapsto (\Lambda, \phi, \Lambda^{-1}\dot{\Lambda}, \Lambda^{-1}\dot{\phi})$. Thus the trivialized augmented Lagrangian $\overline{\mathcal{L}} : C^\infty(\mathcal{A}, SE(3) \times \mathbb{R}) \times C^\infty(\mathcal{A}, \mathfrak{se}(3) \times T\mathbb{R})) \to \mathbb{R}$

for the plate to be

$$\overline{\mathcal{L}}\left(\Lambda,\phi,\lambda,\widehat{\omega},\gamma,\dot{\lambda}\right)=\frac{1}{2}\int_{\mathcal{A}}M\left\|\gamma\right\|^{2}d\mathcal{A}+\frac{1}{2}\int_{\mathcal{A}}\omega^{T}J\omega\;d\mathcal{A}$$
$$-\Pi_{int}(\Lambda,\phi)-\Pi_{ext}(\phi)-\int_{\mathcal{A}}\langle\lambda,\Phi(\Lambda,\phi)\rangle\,d\mathcal{A}. \tag{7.2.1}$$

7.2.2 Constrained Lie algebra variational integrator

In this subsection we present a variational integrator for mechanics on Lie groups based on the paper of Iserles, Munthe-Kaas, Nørsett, and Zanna [50] that uses the right trivialized derivative of the exponential map, also known as the right logarithmic derivative as defined in (6.2.3). We will later apply this variational integrator to the plate.

This right logarithmic derivative of the smooth map $\tau : M \to G$, which can be the exponential or the Cayley maps, was used also in Bou-Rabee, and Marsden [15] to develop a variety of integrators of variational partitioned Runge-Kutta type for Lie groups. In control theory, Kobilarov and Marsden [60] developed a structure preserving variational integrator to actuate a system, based on the rigid body model, to move from its current state to a desired state with minimum control effort or time.

The integrator we will present is developed to treat numerically mechanical systems on finite dimensional Lie groups.

Review on discrete variational mechanics with constraints

Suppose that a time step Δt has been fixed, denote by $\{t^j = j\Delta t \mid j = 0, ..., N\}$ the sequence of time, and by $\mathcal{C}_d(Q) = \{q_d : \{t^j\}_{j=0}^N \to Q, \quad q_d(t^j) = q^j\}$ the discrete path space. Let $L_d : Q \times Q \to \mathbb{R}$, $L_d = L_d(q^j, q^{j+1})$ be a discrete Lagrangian which we think of as approximating the action integral of L along the curve segment between q^j and q^{j+1}, that is, we have

$$L_d(q^j, q^{j+1}) \approx \int_{t^j}^{t^{j+1}} L(q(t), \dot{q}(t))dt,$$

where $q(t^j) = q^j$ and $q(t^{j+1}) = q^{j+1}$.

Let the holonomic constraint $\Phi : Q \to \mathbb{R}^d$. Then we constrain the dynamics to the submanifold $\mathbf{N} = \Phi^{-1}(0) \subset Q$. Given that 0 is a regular point of Φ, $\mathbf{N}$ is a submanifold of Q, and we can define an embedding $i^{\mathbf{N}\times\mathbf{N}} : \mathbf{N} \times \mathbf{N} \to Q \times Q$.

We denote by $C_d(\mathbb{R}^d) = C_d(\{\Delta t, ..., (N-1)\Delta t\}, \mathbb{R}^d)$ the set of the maps $\lambda_d : \{\Delta t, ..., (N-1)\Delta t\} \to \mathbb{R}^d$ with no boundary conditions.

Thus we recall the theorem 3.4.1 as given in Marsden and West [90].

7.2.1 Theorem *Given a discrete Lagrangian system $L_d : Q \times Q \to \mathbb{R}$ with holonomic constraint $\Phi : Q \to \mathbb{R}^d$, set $\mathbf{N} = \Phi^{-1}(0) \subset Q$ and $L_d^{\mathbf{N}} = L_d\big|_{\mathbf{N}\times\mathbf{N}}$. Then the following statements are equivalent:*

(i) $q_d = \{q^j\}_{j=0}^N \in C_d(\mathbf{N})$ *extremize* $\mathfrak{S}_d^{\mathbf{N}} = \mathfrak{S}_d\big|_{\mathbf{N}\times\mathbf{N}}$ *and hence solve the discrete Euler-Lagrange equations for* $L_d^{\mathbf{N}}$*;*

(ii) $q_d = \{q^j\}_{j=0}^N \in \mathcal{C}_d(Q)$, *and* $\lambda_d = \{\lambda^j\}_{j=1}^{N-1} \in \mathcal{C}_d(\mathbb{R}^d)$ *satisfy the constrained discrete Euler-Lagrange equations*

$$\begin{aligned} D_2 L_d(q^{j-1}, q^j) + D_1 L_d(q^j, q^{j+1}) &= \left\langle \lambda^j, \nabla\Phi(q^j)\right\rangle, \\ \Phi(q^j) &= 0; \end{aligned}$$

(iii) $(q_d, \lambda_d) = \{(q^j, \lambda^j)\}_{j=0}^N \in \mathcal{C}_d(Q \times \mathbb{R}^d)$ *extremize* $\widetilde{\mathfrak{S}}_d(q_d, \lambda_d) = \mathfrak{S}_d(q_d) - \langle \lambda_d, \Phi_d(q_d)\rangle_{l_2}$ *and hence solve the discrete Euler-Lagrange equations for either of the augmented discrete Lagrangians* $L_d^+, L_d^- : (Q \times \mathbb{R}^d) \times (Q \times \mathbb{R}^d) \to \mathbb{R}$ *defined by*

$$\begin{aligned} \overline{L}_d^+(q^j, \lambda^j, q^{j+1}, \lambda^{j+1}) &= L_d(q^j, q^{j+1}) - \left\langle \lambda^{j+1}, \Phi(q^{j+1})\right\rangle, \\ \overline{L}_d^-(q^j, \lambda^j, q^{j+1}, \lambda^{j+1}) &= L_d(q^j, q^{j+1}) - \left\langle \lambda^{j}, \Phi(q^{j})\right\rangle. \end{aligned}$$

Constrained integrator on Lie group

Let $\mathcal{L} : G \times \mathfrak{g} \to \mathbb{R}$ be a trivialized Lagrangian defined on the trivialized tangent bundle TG of a Lie group G and let $\tau : \mathfrak{g} \to G$ be a map with $\tau(0) = e$. We assume that τ is a C^2-diffeomorphism in a neighborhood of the origin. The discrete Lagrangian $\mathscr{L}_d : G \times \mathfrak{g} \to \mathbb{R}$ is defined as an approximation of the action functional over one time step, namely, we have

$$\mathscr{L}_d(g^j, \xi^j) \approx \int_{t^j}^{t^{j+1}} L(g(t), \dot{g}(t))dt,$$

where $g(t)$ is the unique solution of the Euler-Lagrange equations such that $g(t^j) = g^j$ and $g(t^{j+1}) = g^{j+1}$ and where

$$\tau(\Delta t\, \xi^j) = (g^j)^{-1} g^{j+1} =: f^j. \tag{7.2.2}$$

We assume that the time step is small enough so that $(g^j)^{-1}g^{j+1}$ is in a neighborhood of the identity element of G where the map τ is a diffeomorphism.

We constrain $g^{j+1} \in S^2_{\mathbf{E}} \times \mathbb{R}^3$ via the holonomic constraint $\Phi : G \to \mathbb{R}$, with set $\mathbf{N} = \Phi^{-1}(0) \subset G$, and the discrete Lagrangian $\mathscr{L}_d^N = \mathscr{L}_d\big|_{N\times N}$.

In our applications, the Lagrangian is always of the classical form kinetic minus potential energy, where the kinetic energy is G-invariant. So we can write the augmented discrete Lagrangian as

$$\begin{aligned} \overline{\mathscr{L}}_d^+(g^j, \lambda^j, \xi^j, \lambda^{j+1}) &= \mathscr{L}(g^j, \xi^j) - \left\langle \lambda^{j+1}, \Phi\left(g^j \tau(\Delta t\, \xi^j)\right)\right\rangle, \\ \overline{\mathscr{L}}_d^-(g^j, \lambda^j, \xi^j, \lambda^{j+1}) &= \mathscr{L}(g^j, \xi^j) - \left\langle \lambda^{j}, \Phi\left(g^j\right)\right\rangle. \end{aligned} \tag{7.2.3}$$

We now compute the variation $\delta\xi^j$ induced by variations of g^j. Defining $\eta^j := (g^j)^{-1}\delta g^j$ and $f^j := (g^j)^{-1}g^{j+1}$, we have

$$\begin{aligned} \Delta t\, \delta\xi^j &\overset{(6.2.6)}{=} T_{f^j}\tau^{-1}\left(\delta f^j\right) \overset{(6.2.5)}{=} \left(\mathrm{d}^R\tau(\Delta t\, \xi^j)\right)^{-1}\left(\left(\delta f^j \tau(\Delta t\, \xi^j)^{-1}\right)\right) \\ &= \left(\mathrm{d}^R\tau(\Delta t\, \xi^j)\right)^{-1}\left[\left(-(g^j)^{-1}\delta g^j (g^j)^{-1} g^{j+1} + (g^j)^{-1}\delta g^{j+1}\right)(f^j)^{-1}\right] \\ &= \left(\mathrm{d}^R\tau(\Delta t\, \xi^j)\right)^{-1}\left(-\eta^j + \mathrm{Ad}_{\tau(\Delta t\, \xi^j)}\, \eta^{j+1}\right). \end{aligned} \tag{7.2.4}$$

Constrained discrete Lagrangian $\overline{\mathscr{L}_d^-}$. The variation of the constraint $\Phi^j := \Phi\left(g^j\right)$ is written

$$\delta\Phi^j = D_{g^j}\Phi^j \cdot \delta g^j = (g^j)^{-1} D_{g^j}\Phi^j \cdot \eta^j$$

The constrained discrete Euler-Lagrange equations are obtained by applying the discrete Hamilton's principle to $\overline{\mathscr{L}_d^-}(g^j, \lambda^j, \xi^j, \lambda^{j+1})$. Taking into account that $\eta^0 = \eta^N = 0$, we get

$$\begin{aligned}
\delta\mathfrak{S}_d &= \sum_{j=0}^{N-1} D_{g^j}\mathscr{L}_d^j \cdot \delta g^j + D_{\xi^j}\mathscr{L}_d^j \cdot \delta\xi^j - \Phi^j \cdot \delta\lambda^j - \left\langle \lambda^j, D_{g^j}\Phi^j \cdot \delta g^j \right\rangle \\
&= \sum_{j=0}^{N-1} D_{g^j}\mathscr{L}_d^j \cdot g^j\eta^j + D_{\xi^j}\mathscr{L}_d^j \cdot \frac{1}{\Delta t}\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\left(-\eta^j + \mathrm{Ad}_{\tau(\Delta t\,\xi^j)}\,\eta^{j+1}\right) \\
&\quad - \Phi^j \cdot \delta\lambda^j - (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right)\cdot\eta^j \\
&= \sum_{j=1}^{N-1}\Bigg\{(g^j)^{-1}\left(D_{g^j}\mathscr{L}_d^j\right) - \frac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^j}\mathscr{L}_d^j\right) - (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right) \\
&\quad + \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^{j-1})\right)^{-1}\right)^*\left(D_{\xi^{j-1}}\mathscr{L}_d^{j-1}\right)\Bigg\}\cdot\eta^j - \sum_{j=0}^{N-1}\Phi^j\cdot\delta\lambda^j.
\end{aligned}$$

Where we denote $\mathscr{L}_d^j := \mathscr{L}_d(g^j, \xi^j)$ and $\Phi^{j+1} = \Phi\left(g^j\tau(\Delta t\,\xi^j)\right)$. The discrete trajectories $\{(g^j, \xi^j)\}$ and $\{\lambda^j\}$ satisfy the constrained discrete Euler-Lagrange equations

$$\begin{cases}
(g^j)^{-1}\left(D_{g^j}\mathscr{L}_d^j\right) - \dfrac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^j}\mathscr{L}_d^j\right) - (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right) \\
\qquad + \dfrac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^{j-1})\right)^{-1}\right)^*\left(D_{\xi^{j-1}}\mathscr{L}_d^{j-1}\right) = 0, \\
\Phi(g^j\tau(\Delta t\,\xi^j)) = 0, \quad \text{with} \;\; (g^j)^{-1}g^{j+1} = \tau(\Delta t\,\xi^j).
\end{cases} \tag{7.2.5}$$

Thus for given $(g^{j-1}, \xi^{j-1}, \lambda^{j-1})$, by (7.2.2), we obtain $g^j = g^{j-1}\tau(\Delta t\,\xi^{j-1})$, and we solve the first equation with the constrain $\Phi(g^j\tau(\Delta t\,\xi^j)) = 0$ in order to get ξ^j and λ^j.

This yields a discrete-time flow map $(g^{j-1}, \xi^{j-1}, \lambda^{j-1}) \to (g^j, \xi^j, \lambda^j)$, and this process is repeated. The discrete Euler-Lagrange equations may be written as follows

$$\begin{cases}
\dfrac{1}{\Delta t}\left(\mu^j - \mathrm{Ad}^*_{\tau(\Delta t\,\xi^{j-1})}\mu^{j-1}\right) = (g^j)^{-1}\left(D_{g^j}\mathscr{L}_d^j\right) - (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right), \\
\mu^j = \left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^j}\mathscr{L}_d^j\right) \\
\Phi(g^j\tau(\Delta t\,\xi^j)) = 0, \quad \text{with} \;\; (g^j)^{-1}g^{j+1} = \tau(\Delta t\,\xi^j).
\end{cases} \tag{7.2.6}$$

In the context of the Lie algebra variational integrators, we define the discrete Legendre transforms $\mathbb{F}^{\pm}\overline{\mathscr{L}}_d^- : G \times \mathbb{R}^d \times \mathfrak{g} \times \mathbb{R}^d \to G \times \mathbb{R}^d \times \mathfrak{g}^*$, given by

$$\begin{aligned}
\mathbb{F}^{+}\overline{\mathscr{L}}_d^{-,j} &= \left(g^{j+1},\ \lambda^{j+1},\ \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^j)}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^j}\mathscr{L}_d^j\right)\right)\\
&= \left(g^{j+1},\ \lambda^{j+1},\ \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\,\xi^j)}\,\mu^j\right),\\
\mathbb{F}^{-}\overline{\mathscr{L}}_d^{-,j} &= \left(g^j, \lambda^j,\ \frac{1}{\Delta t}\left(\left(\mathrm{d}^R\tau(\Delta t\,\xi^j)\right)^{-1}\right)^*\left(D_{\xi^j}\mathscr{L}_d^j\right) - (g^j)^{-1}\left(D_{g^j}\mathscr{L}_d^j\right)\right.\\
&\qquad\left. + (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right)\right)\\
&= \left(g^j, \lambda^j,\ \frac{1}{\Delta t}\mu^j - (g^j)^{-1}\left(D_{g^j}\mathscr{L}_d^j\right) + (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right)\right).
\end{aligned}$$

We note that equation (7.2.5) can be written in terms of the Legendre transform as

$$\mathbb{F}^{+}\overline{\mathscr{L}}_d^{-,j-1} = \mathbb{F}^{-}\overline{\mathscr{L}}_d^{-,j}, \quad \text{with} \;\; \Phi(g^j\tau(\Delta t\,\xi^j)) = 0.$$

The infinitesimal generator of left multiplication on G for $\zeta \in \mathfrak{g}$ has the expression $\zeta_G(g^j) = \zeta g^j$. Using the expression (1.2.8), one computes that the discrete momentum maps $\mathbf{J}^{\pm}_{\overline{\mathscr{L}}_d^j} : G\times\mathbb{R}^d\times\mathfrak{g}\times\mathbb{R}^d \to \mathfrak{g}^*$ associated to the discrete Lagrangian $\overline{\mathscr{L}}_d^{-,j}$ are given by

$$\begin{aligned}
\mathbf{J}^{+}_{\overline{\mathscr{L}}_d^j} &= \frac{1}{\Delta t}\,\mathrm{Ad}^*_{(g^j)^{-1}}\,\mu^j,\\
\mathbf{J}^{-}_{\overline{\mathscr{L}}_d^j} &= \mathrm{Ad}^*_{(g^j)^{-1}}\left(\frac{1}{\Delta t}\mu^j - (g^j)^{-1}\left(D_{g^j}\mathscr{L}_d^j\right) + (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right)\right).
\end{aligned}$$

Note that constrained discrete Euler-Lagrange equations (7.2.5) can also be written in terms of the spatial discrete momentum maps $\mathbf{J}^{\pm}_{\overline{\mathscr{L}}_d}$ as

$$\mathrm{Ad}^*_{g^j}\mathbf{J}^{+}_{\overline{\mathscr{L}}_d^{j-1}} = \mathrm{Ad}^*_{g^j}\mathbf{J}^{-}_{\overline{\mathscr{L}}_d^j}, \quad \text{and} \;\; \Phi(g^j\tau(\Delta t\,\xi^j)) = 0.$$

7.2.2 Remark Suppose that a G group action let $\mathbf{N} = \Phi^{-1}(0)$ invariant, then the discrete Noether theorem holds on the constrained system, and the momentum map is preserved (see Marsden, and West [90]).

7.2.3 Forced constrained discrete Euler-Lagrange equations

Given the discrete augmented trivialized Lagrangian $\overline{\mathscr{L}}_d^-$, defined in (7.2.3), discrete external Lagrangian forces can be incorporated in the dynamics by replacing the discrete Hamilton principle with the discrete Lagrange-d'Alembert principle. For Lie algebra integrators, the discrete Lagrange-d'Alembert principle reads

$$\delta \sum_{j=0}^{N-1} \overline{\mathscr{L}}_d^-(g^j, \lambda^j, \xi^j, \lambda^{j+1}) + \sum_{j=0}^{N-1} \left[\mathcal{F}_d^-(g^j, \xi^j) \cdot \delta g^j + \mathcal{F}_d^+(g^{j+1}, \xi^j) \cdot \delta g^{j+1} \right] = 0,$$

for all variations δg^j with $\delta g^0 = \delta g^N = 0$, where $\mathcal{F}^-(g^j, \xi^j) \in T^*_{g^j} G$ and $\mathcal{F}^+(g^{j+1}, \xi^j) \in T^*_{g^{j+1}} G$ are the discrete external Lagrangian forces. [1] Using the notation $\eta^j = (g^j)^{-1} \delta g^j$, we have

$$\begin{aligned} \delta g^{j+1} &= \delta g^j \tau(\Delta t\, \xi^j) + g^j \delta(\tau(\Delta t\, \xi^j)) \\ &= g^j \eta^j \tau(\Delta t\, \xi^j) + g^j \left(\mathrm{d}^R \tau(\Delta t\, \xi^j) \cdot \Delta t\, \delta \xi^j \right) \tau(\Delta t\, \xi^j) \\ &\overset{(7.2.4)}{=} g^j \left(\mathrm{Ad}_{\tau(\Delta t\, \xi^j)} \eta^{j+1} \right) \tau(\Delta t\, \xi^j). \end{aligned}$$

$$\begin{aligned} \delta \mathfrak{S}_d = & \sum_{j=0}^{N-1} D_{g^j} \overline{\mathscr{L}}_d^-(g^j, \lambda^j, \xi^j, \lambda^{j+1}) \cdot \delta g^j + D_{\xi^j} \overline{\mathscr{L}}_d^-(g^j, \lambda^j, \xi^j, \lambda^{j+1}) \cdot \delta \xi^j \\ & + D_{\lambda^j} \overline{\mathscr{L}}_d^-(g^j, \lambda^j, \xi^j, \lambda^{j+1}) \cdot \delta \lambda^j + \mathcal{F}_d^-(g^j, \xi^j) \cdot \delta g^j + \mathcal{F}_d^+(g^{j+1}, \xi^j) \cdot \delta g^{j+1} \\ = & \sum_{j=0}^{N-1} D_{g^j} \mathscr{L}_d^j \cdot g^j \eta^j + D_{\xi^j} \mathscr{L}_d^j \cdot \frac{1}{\Delta t} \left(\mathrm{d}^R \tau(\Delta t\, \xi^j) \right)^{-1} \left(-\eta^j + \mathrm{Ad}_{\tau(\Delta t\, \xi^j)} \eta^{j+1} \right) \\ & - \Phi^j \cdot \delta \lambda^j - (\lambda^j)^T \left((g^j)^{-1} D_{g^j} \Phi^j \right) \cdot \eta^j \\ & + \mathcal{F}_d^-(g^j, \xi^j) \cdot g^j \eta^j + \mathcal{F}_d^+(g^{j+1}, \xi^j) \cdot \left(g^j \left(\mathrm{Ad}_{\tau(\Delta t\, \xi^j)} \eta^{j+1} \right) \tau(\Delta t\, \xi^j) \right) \\ = & \sum_{j=1}^{N-1} \Bigg\{ (g^j)^{-1} \left(D_{g^j} \mathscr{L}_d^j \right) - \frac{1}{\Delta t} \mu^j + \frac{1}{\Delta t} \mathrm{Ad}^*_{\tau(\Delta t\, \xi^{j-1})} \mu^{j-1} \\ & - (\lambda^j)^T \left((g^j)^{-1} D_{g^j} \Phi^j \right) + (g^j)^{-1} \left(\mathcal{F}_d^-(g^j, \xi^j) \right) \\ & + (g^j)^{-1} \left(\mathcal{F}_d^+(g^j, \xi^{j-1}) \right) \Bigg\} \cdot \eta^j - \sum_{j=0}^{N-1} \Phi^j \cdot \delta \lambda^j. \end{aligned}$$

Thus, the discrete constrained Lagrange d'Alembert equations are

$$\begin{cases} (g^j)^{-1} \left(D_{g^j} \mathscr{L}_d^j \right) - \dfrac{1}{\Delta t} \mu^j + \dfrac{1}{\Delta t} \mathrm{Ad}^*_{\tau(\Delta t\, \xi^{j-1})} \mu^{j-1} - (\lambda^j)^T \left((g^j)^{-1} D_{g^j} \Phi^j \right) \\ \qquad + (g^j)^{-1} \left(\mathcal{F}_d^-(g^j, \xi^j) \right) + (g^j)^{-1} \left(\mathcal{F}_d^+(g^j, \xi^{j-1}) \right) = 0, \\ \mu^j = \left(\left(\mathrm{d}^R \tau(\Delta t\, \xi^j) \right)^{-1} \right)^* \left(D_{\xi^j} \mathscr{L}_d^j \right), \\ \Phi(g^j \tau(\Delta t\, \xi^j)) = 0, \quad \text{with} \;\; (g^j)^{-1} g^{j+1} = \tau(\Delta t\, \xi^j). \end{cases} \tag{7.2.7}$$

[1] These discrete forces are chosen in such a way that the second term in the variational principle is an approximation of the virtual work done by the force field in the continuous case.

When forces are present, one has to incorporate them in the discrete Legendre transforms and the discrete momentum maps, as explained in Marsden, and West [90]. In the context of Lie algebra variational integrators, the forced discrete Legendre transforms $\mathbb{F}^{f\pm}\overline{\mathscr{L}}_d^- : G \times \mathbb{R}^d \times \mathfrak{g} \times \mathbb{R}^d \to G \times \mathbb{R}^d \times \mathfrak{g}^*$ are

$$\begin{aligned}
&\mathbb{F}^{f+}\overline{\mathscr{L}}_d^-(g^j, \lambda^j, \xi^j, \lambda^{j+1}) \\
&\quad = \left(g^{j+1}, \lambda^{j+1},\ \frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\, \xi^j)}\mu^j + (g^{j+1})^{-1}\left(\mathcal{F}_d^+(g^{j+1}, \xi^j)\right) \right), \\
&\mathbb{F}^{f-}\overline{\mathscr{L}}_d^-(g^j, \lambda^j, \xi^j, \lambda^{j+1}) \\
&\quad = \left(g^j, \lambda^j,\ -(g^j)^{-1}\left(D_{g^j}\mathscr{L}_d^j\right) + \frac{1}{\Delta t}\mu^j + (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right) \right. \\
&\qquad\qquad\qquad \left. - (g^j)^{-1}\left(\mathcal{F}_d^-(g^j, \xi^j)\right) \right).
\end{aligned}$$

And the forced constrained discrete Lagrangian momentum maps $\mathbf{J}^{f\pm}_{\mathscr{L}_d}$: $G \times \mathfrak{g} \to \mathfrak{g}^*$, for $g^j \in G$ and $\xi^j \in \mathfrak{g}$, are

$$\begin{aligned}
&\mathbf{J}^{f+}_{\overline{\mathscr{L}}_d^-}(g^j, \lambda^j, \xi^j, \lambda^{j+1}) \\
&\quad = \mathrm{Ad}^*_{(g^{j+1})^{-1}}\left(\frac{1}{\Delta t}\mathrm{Ad}^*_{\tau(\Delta t\, \xi^j)}\mu^j + (g^{j+1})^{-1}\left(\mathcal{F}_d^+(g^{j+1}, \xi^j)\right) \right), \\
&\mathbf{J}^{f-}_{\overline{\mathscr{L}}_d^-}(g^j, \lambda^j, \xi^j, \lambda^{j+1}) \\
&\quad = \mathrm{Ad}^*_{(g^j)^{-1}}\left(-(g^j)^{-1}\left(D_{g^j}\mathscr{L}_d^j\right) + \frac{1}{\Delta t}\mu^j + (\lambda^j)^T\left((g^j)^{-1}D_{g^j}\Phi^j\right) \right. \\
&\qquad\qquad\qquad \left. - (g^j)^{-1}\left(\mathcal{F}_d^-(g^j, \xi^j)\right) \right).
\end{aligned}$$

7.3 Spatial and temporal discretization

7.3.1 Spatial discretization

We return to the plate whose equations of motion are (7.1.5), with boundary conditions (7.1.6), for the augmented Lagrangian (4.1.9). We recall that the constraint submanifold is $\mathbf{N} = S^2_{\mathbf{E}} \times \mathbb{R}^3 = \Phi^{-1}(0)$ with the constraint $\Phi : Q \to \mathbb{R}$ as defined in (7.1.4).

Spatial discretization of the variables. We suppose that $\mathcal{A}$ is a rectangle and we decompose it by $N_1 \times N_2$ rectangles K of size $l_1 \times l_2$ whose nodes are denoted by $a,\ a+1,\ a+2$, and $a+3$.

$$\begin{array}{ccc}
a+2 & \text{———} & a+3 \\
| & & | \\
a & \text{———} & a+1
\end{array} \tag{7.3.1}$$

Given the configurations $(\Lambda_a, \mathbf{x}_a)$, $(\Lambda_{a+1}, \mathbf{x}_{a+1})$, $(\Lambda_{a+2}, \mathbf{x}_{a+2})$, and $(\Lambda_{a+3}, \mathbf{x}_{a+3})$ at nodes a, $a+1$, $a+2$, $a+3$, we extend the spatial discretization defined by Crisfield, and Jelenic [24] in order to get the frame indifference [2]. So we consider the following interpolations over the subinterval K:

$$\begin{aligned}\Lambda_h(u_1,u_2) &:= \Lambda_a \exp\left(\frac{u_1}{l_1}\widehat{\psi}_1\right)\exp\left(\frac{u_2}{l_2}\;\widehat{\theta}_1(u_1)\right),\\ &= \Lambda_a \exp\left(\frac{u_2}{l_2}\widehat{\psi}_3\right)\exp\left(\frac{u_1}{l_1}\;\widehat{\theta}_2(u_2)\right)\end{aligned} \tag{7.3.2}$$

where $u_1 \in [0, l_1]$, $u_2 \in [0, l_2]$, and

$$\exp(\widehat{\theta}_1(u_1)) = \exp\left(\frac{u_1}{l_1}\widehat{\psi}_1\right)^T \Lambda_a^T \Lambda_{a+2}\; \exp\left(\frac{u_1}{l_1}\widehat{\psi}_2\right),$$

$$\exp(\widehat{\theta}_2(u_2)) = \exp\left(\frac{u_2}{l_2}\widehat{\psi}_3\right)^T \Lambda_a^T \Lambda_{a+1}\; \exp\left(\frac{u_2}{l_2}\widehat{\psi}_4\right),$$

$$\begin{aligned}&\exp(\widehat{\theta}_2(0)) = \exp(\widehat{\psi_1}) = \Lambda_a^T\Lambda_{a+1}, \quad \exp(\widehat{\theta}_2(l_2)) = \exp(\widehat{\psi_2}) = \Lambda_{a+2}^T\Lambda_{a+3},\\ &\exp(\widehat{\theta}_1(0)) = \exp(\widehat{\psi_3}) = \Lambda_a^T\Lambda_{a+2}, \quad \exp(\widehat{\theta}_1(l_1)) = \exp(\widehat{\psi_4}) = \Lambda_{a+1}^T\Lambda_{a+3},\end{aligned} \tag{7.3.3}$$

and

$$\phi_h(u_1,u_2) := \left[\mathbf{x}_a + \frac{u_1}{l_1}\Delta\mathbf{x}_{a,a+1}\right]\left(\frac{l_2-u_2}{l_2}\right) + \left[\mathbf{x}_{a+2} + \frac{u_1}{l_1}\Delta\mathbf{x}_{a+2,a+3}\right]\left(\frac{u_2}{l_2}\right), \tag{7.3.4}$$

with

$$\begin{aligned}\Delta\mathbf{x}_{a,a+1} &= \mathbf{x}_{a+1} - \mathbf{x}_a, \qquad & \Delta\mathbf{x}_{a+2,a+3} &= \mathbf{x}_{a+3} - \mathbf{x}_{a+2},\\ \Delta\mathbf{x}_{a,a+2} &= \mathbf{x}_{a+2} - \mathbf{x}_a, \qquad & \Delta\mathbf{x}_{a+1,a+3} &= \mathbf{x}_{a+3} - \mathbf{x}_{a+1}.\end{aligned} \tag{7.3.5}$$

As expected we get

$$\begin{aligned}&\Lambda_h(0,0) = \Lambda_a, \quad \Lambda_h(l_1,0) = \Lambda_{a+1}, \quad \Lambda_h(0,l_2) = \Lambda_{a+2}, \quad \Lambda_h(l_1,l_2) = \Lambda_{a+3},\\ &\phi_h(0,0) = \mathbf{x}_a, \quad \phi_h(l_1,0) = \mathbf{x}_{a+1}, \quad \phi_h(0,l_2) = \mathbf{x}_{a+2}, \quad \phi_h(l_1,l_2) = \mathbf{x}_{a+3}.\end{aligned}$$

From now on, we will use the notations

$$\Lambda_K = (\Lambda_a, \Lambda_{a+1}, \Lambda_{a+2}, \Lambda_{a+3})^T \quad \text{and} \quad \mathbf{x}_K = (\mathbf{x}_a, \mathbf{x}_{a+1}, \mathbf{x}_{a+2}, \mathbf{x}_{a+3})^T.$$

In order to get the interpolated convected variables $\widehat{\omega}_h(u_1,u_2)$, $\widehat{\gamma}_h(u_1,u_2)$, $\Omega_{h,\alpha}(u_1,u_2)$, and $\Gamma_{h,\alpha}(u_1,u_2)$, we replace the original variables $\Lambda(u_1,u_2)$ and $\phi(u_1,u_2)$ by the approximations $\Lambda_h(u_1,u_2)$ and $\phi_h(u_1,u_2)$ as defined in (7.3.2),

[2]Consider a rigid motion of (Λ, ϕ) given by the transformation $(\widetilde{\Lambda}, \widetilde{\phi}) = (R\Lambda, \mathbf{v} + R\phi)$, where $R \in SO(3)$, and $\mathbf{v} \in \mathbb{R}^3$. Then, since $(\widetilde{\Lambda}_a)^T\widetilde{\Lambda}_{a+1} = \Lambda_a^T\Lambda_{a+1}$ and $\widetilde{\Lambda}_a^T\Delta\widetilde{\mathbf{x}}_a = \Lambda_a^T\Delta\mathbf{x}_a$, the strain measures are unchanged by this transformation.

and (7.3.4). Thus we get

$$
\begin{aligned}
\widehat{\omega}_h(u_1,u_2) &= \left(\Lambda_h^T \dot{\Lambda}_h\right)(u_1,u_2),\\
\gamma_h(u_1,u_2) &= \Lambda_h^T \Bigg(\left[\dot{\mathbf{x}}_a + \frac{u_1}{l_1}\Delta\dot{\mathbf{x}}_{a,a+1}\right]\left(\frac{l_2-u_2}{l_2}\right)\\
&\qquad\qquad + \left[\dot{\mathbf{x}}_{a+2} + \frac{u_1}{l_1}\Delta\dot{\mathbf{x}}_{a+2,a+3}\right]\left(\frac{u_2}{l_2}\right)\Bigg),\\
\widehat{\Omega}_h^1(u_1,u_2) &= \frac{\widehat{\theta}_2(u_2)}{l_1},\\
\widehat{\Omega}_h^2(u_1,u_2) &= \frac{\widehat{\theta}_1(u_1)}{l_2},\\
\Gamma_h^1(u_1,u_2) &= \Lambda_h^T\left\{\left(\frac{l_2-u_2}{l_1 l_2}\right)\Delta\mathbf{x}_{a,a+1} + \frac{u_2}{l_1 l_2}\Delta\mathbf{x}_{a+2,a+3}\right\},\\
\Gamma_h^2(u_1,u_2) &= \Lambda_h^T\left\{\left(\frac{l_1-u_1}{l_1 l_2}\right)\Delta\mathbf{x}_{a,a+2} + \frac{u_1}{l_1 l_2}\Delta\mathbf{x}_{a+1,a+3}\right\}.
\end{aligned}
\tag{7.3.6}
$$

The spatial discretization of the augmented Lagrangian (7.2.1)**.** The spatially discretized Lagrangian is obtained by inserting the variables considered in (7.3.6) in the continuous Lagrangian (7.2.1) and by considering the following approximations on rectangle K.

(i) For the kinetic energy, we make the following approximations :

$$
\frac{1}{2}\int_K M\|\gamma_h(u^1,u^2)\|^2 du^1 du^2 \approx \sum_{a\in K}\frac{l_1 l_2}{8} M\|\gamma_a\|^2,
$$
$$
\frac{1}{2}\int_K \left(\omega_h(u^1,u^2)^T J \omega_h(u^1,u^2)\right) du^1 du^2 \approx \sum_{a\in K}\frac{l_1 l_2}{8}\omega_a^T J\omega_a.
$$

(ii) Concerning the potential energy, the expression obtained by using Λ_h and ϕ_h instead of Λ and ϕ is denoted by

$$
V_K(\Lambda_K,\mathbf{x}_K) := \int_K f\,(u_1,u_2)\,du_1 du_2, \tag{7.3.7}
$$

where $f(u_1,u_2) := \Psi_{int}(\Lambda_h(u_1,u_2),\phi_h(u_1,u_2)) + \Psi_{ex}(\Lambda_h(u_1,u_2),\phi_h(u_1,u_2))$, see (4.1.5) and (4.1.6). We get

$$
\begin{aligned}
f(u_1,u_2) =& \frac{1}{2}\sum_{\alpha,\beta}\left\{(\Gamma_h^\alpha - \mathbf{E}_\alpha)^T \mathbb{C}_{\alpha\beta}(\Gamma_h^\beta - \mathbf{E}_\beta) + (\Omega_h^\alpha)^T \mathbb{D}_{\alpha,\beta}\,\Omega_h^\beta\right\}\\
&+ \langle \mathbf{q}, \phi_h(u_1,u_2)\rangle\,.
\end{aligned}
$$

We approximate the expression (7.3.7) by

$$\begin{aligned} V_K(\Lambda_K, \mathbf{x}_K) &\approx \frac{l_1 l_2}{4}\left(f(0,0) + f(0,l_2) + f(l_1,0) + f(l_1,l_2)\right) \\ &= \sum_{a\in K}\sum_{\alpha,\beta} \frac{l_1 l_2}{8}\Big\{(\Gamma_a^\alpha - \mathbf{E}_\alpha)^T \mathbb{C}_{\alpha\beta}(\Gamma_a^\beta - \mathbf{E}_\beta) + (\Omega_a^\alpha)^T \mathbb{D}_{\alpha,\beta}\,\Omega_a^\beta\Big\} \\ &\quad + \sum_{a\in K}\frac{l_1 l_2}{4}\langle \mathbf{q}, \mathbf{x}_a\rangle := \mathbb{V}_K(\Lambda_K, \mathbf{x}_K), \end{aligned} \tag{7.3.8}$$

where Γ_a and Ω_a are the strain mesures in a. [3]

(iii) Concerning the scalar product $\langle \lambda, \Phi(\Lambda,\phi)\rangle$. By replacing Λ and ϕ by Λ_h and ϕ_h, and by denoting $\lambda_K = (\lambda_a, \lambda_{a+1}, \lambda_{a+2}, \lambda_{a+3})^T$, we obtain

$$\int_{\mathcal{A}} \frac{1}{2}\left\langle \lambda_K, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_h)\right)\right\rangle d\mathcal{A} \approx \sum_{a\in K}\frac{l_1 l_2}{8}\left\langle \lambda_a, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a)\right)\right\rangle.$$

As a consequence, the trivialized form $\overline{\mathcal{L}}_K : [(SE(3)\times\mathbb{R})\times(\mathfrak{se}(3)\times\mathbb{R})]^4 \to \mathbb{R}$ of the augmented Lagrangian $\overline{L}_K : T(SE(3)\times\mathbb{R})^4 \to \mathbb{R}$, over a subinterval

[3] Based on group actions of $(\exp(\widehat{\psi}), \Delta\mathbf{x}) \in SE(3)$ on $(\Lambda, \mathbf{x})$ defined in (7.3.3), and (7.3.5), we observe that $a+1$ is on the right of a and on the left of $a+3$, otherwise $a+2$ is on the right of a and on the left of $a+3$, see (7.3.1). That is, the strains (7.3.6) are

$$\begin{aligned} \Gamma_h^1(0,0) &= \frac{\Lambda_a^T}{l_1}\Delta\mathbf{x}_{a,a+1} =: \Gamma_a^1, & \widehat{\Omega}_h^1(0,0) &= \frac{1}{l_1}\exp^{-1}(\Lambda_a^T\Lambda_{a+1}) =: \widehat{\Omega}_a^1, \\ \Gamma_h^1(0,l_2) &= \frac{\Lambda_a^T}{l_1}\Delta\mathbf{x}_{a+2,a+3} =: \Gamma_{a+2}^1, & \widehat{\Omega}_h^1(0,l_2) &= \frac{1}{l_1}\exp^{-1}(\Lambda_{a+2}^T\Lambda_{a+3}) =: \widehat{\Omega}_{a+2}^1, \end{aligned}$$

$$\begin{aligned} \Gamma_h^2(0,0) &= \frac{\Lambda_a^T}{l_2}\Delta\mathbf{x}_{a,a+2} =: \Gamma_a^2, & \widehat{\Omega}_h^2(0,0) &= \frac{1}{l_2}\exp^{-1}(\Lambda_a^T\Lambda_{a+2}) =: \widehat{\Omega}_a^2, \\ \Gamma_h^2(l_1,0) &= \frac{\Lambda_a^T}{l_2}\Delta\mathbf{x}_{a+1,a+3} =: \Gamma_{a+1}^2, & \widehat{\Omega}_h^2(l_1,0) &= \frac{1}{l_2}\exp^{-1}(\Lambda_{a+1}^T\Lambda_{a+3}) =: \widehat{\Omega}_{a+1}^2, \end{aligned}$$

$$\begin{aligned} \Gamma_h^1(l_1,0) &= \frac{\Lambda_{a+1}^T}{l_1}\Delta\mathbf{x}_{a,a+1} =: \Gamma_{a+1}^1, & \widehat{\Omega}_{a+1}^1 &= \widehat{\Omega}_a^1, \\ \Gamma_h^1(l_1,l_2) &= \frac{\Lambda_{a+3}^T}{l_1}\Delta\mathbf{x}_{a+2,a+3} =: \Gamma_{a+3}^1, & \widehat{\Omega}_{a+3}^1 &=: \widehat{\Omega}_{a+2}^1, \end{aligned}$$

$$\begin{aligned} \Gamma_h^2(0,l_2) &= \frac{\Lambda_{a+2}^T}{l_2}\Delta\mathbf{x}_{a,a+2} =: \Gamma_{a+2}^2, & \widehat{\Omega}_{a+2}^2 &= \widehat{\Omega}_a^2, \\ \Gamma_h^2(l_1,l_2) &= \frac{\Lambda_{a+3}^T}{l_2}\Delta\mathbf{x}_{a+1,a+3} =: \Gamma_{a+3}^2, & \widehat{\Omega}_{a+3}^2 &=: \widehat{\Omega}_{a+1}^2, \end{aligned}$$

K, is given by

$$\begin{aligned}\overline{\mathcal{L}}_K(\Lambda_K, \mathbf{x}_K, \lambda_K, \widehat{\omega}_K, \gamma_K, \dot{\lambda}_K) = &\sum_{a\in K} \frac{l_1 l_2}{8}\Big(M\|\gamma_a\|^2 + \omega_a^T J\omega_a\Big)\\ &- \mathbb{V}_K(\Lambda_K, \mathbf{x}_K)\\ &- \sum_{a\in K}\frac{l_1 l_2}{8}\Big\langle \lambda_a, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a)\right)\Big\rangle, \quad (7.3.9)\end{aligned}$$

where we use the notations

$$\widehat{\omega}_K = (\widehat{\omega}_a, \widehat{\omega}_{a+1}, \widehat{\omega}_{a+2}, \widehat{\omega}_{a+3})^T, \quad \text{and} \quad \gamma_K = (\gamma_a, \gamma_{a+1}, \gamma_{a+2}, \gamma_{a+3})^T.$$

Note that the spatial discrete form of the trivialized Lagrangian of the total system is $\overline{\mathcal{L}}_{\mathcal{T}} = \sum_{K\in\mathcal{T}} \overline{\mathcal{L}}_K$, read

$$\begin{aligned}&\overline{\mathcal{L}}_{\mathcal{T}}\Big((\Lambda_a, \mathbf{x}_a, \lambda_a, \omega_a, \gamma_a, \dot{\lambda}_a)_{a\in\mathcal{N}}\Big)\\ &= \sum_{a\in int(\mathcal{N})}\left(\frac{l_1 l_2}{2}M\|\gamma_a\|^2 + \frac{l_1 l_2}{2}\omega_a^T J\omega_a - \frac{l_1 l_2}{2}\Big\langle \lambda_a, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a)\right)\Big\rangle\right)\\ &+ \sum_{a\in\partial\mathcal{N}\setminus Corners}\left(\frac{l_1 l_2}{4}M\|\gamma_a\|^2 + \frac{l_1 l_2}{4}\omega_a^T J\omega_a - \frac{l_1 l_2}{4}\Big\langle \lambda_a, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a)\right)\Big\rangle\right)\\ &+ \sum_{a\in Corners}\left(\frac{l_1 l_2}{8}M\|\gamma_a\|^2 + \frac{l_1 l_2}{8}\omega_a^T J\omega_a - \frac{l_1 l_2}{8}\Big\langle \lambda_a, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a)\right)\Big\rangle\right)\\ &- \sum_{K\in\mathcal{T}} \mathbb{V}_K(\mathbf{x}_K, \Lambda_K).\end{aligned}$$

7.3.1 Remark For convenience we take in count external corners and not the internal corners.

7.3.2 Temporal discretization of the Lagrangian

Temporal discretization. Given a node a, the discrete time evolution of this node is given by the discrete curve $(\Lambda_a^j, \mathbf{x}_a^j, \lambda_a^j) \in SE(3)\times\mathbb{R}$. The discrete variables g^j and $f^j = (g^j)^{-1}g^{j+1}$ associated to this node are $(\Lambda_a^j, \mathbf{x}_a^j)$ and

$$\begin{aligned}\left((\Lambda_a^j)^T\Lambda_a^{j+1}, (\Lambda_a^j)^T(\mathbf{x}_a^{j+1} - \mathbf{x}_a^j)\right) &= (\Lambda_a^j, \mathbf{x}_a^j)^{-1}(\Lambda_a^{j+1}, \mathbf{x}_a^{j+1})\\ &= \tau\left(\Delta t\,(\omega_a^j, \gamma_a^j)\right)\end{aligned}$$

The discrete Lagrangian $\overline{\mathscr{L}}_K^j$ approximating the action of the Lagrangian $\overline{\mathcal{L}}_K$, defined in (7.3.9), during the interval $[t^j, t^{j+1}]$, and over an element K of size $l_1 \times l_2$, is therefore

$$\begin{aligned}\overline{\mathscr{L}}_K^j =&\Delta t\,\frac{l_1 l_2}{8}\sum_{a\in K}\left(M\left\|\gamma_a^j\right\|^2 + (\omega_a^j)^T J\omega_a^j - \Big\langle \lambda_a^j, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a^j)\right)\Big\rangle\right)\\ &-\ \Delta t\ \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right),\end{aligned}$$

where

$$\begin{aligned}\Delta t\, \xi_a^j := \Delta t\, (\omega_a^j, \gamma_a^j) &= \tau^{-1}\left((g_a^j)^{-1} g_a^{j+1}\right)\\ &= \tau^{-1}\left((\Lambda_a^j)^{-1}\Lambda_a^{j+1}, (\Lambda_a^j)^{-1}\Delta\mathbf{x}_a^j\right)\end{aligned}$$

Then the discrete action, for a plate where the size of the elements K are equal, with 4 external corners, and 4 edges, is as follows

$$\begin{aligned}\mathfrak{S}_d = &\Delta t\, \frac{l_1 l_2}{8} \sum_{K\in\mathcal{T}} \sum_{j=1}^{N-1} \sum_{a\in K} \left(\left\langle \mathbb{J}\xi_a^j, \xi_a^j\right\rangle - \left\langle \lambda_a^j, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a^j)\right)\right\rangle\right)\\ &- \Delta t \sum_{K\in\mathcal{T}} \sum_{j=1}^{N-1} \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right)\\ = &\Delta t\, \frac{l_1 l_2}{2} \sum_{a\in int(\mathcal{T})} \sum_{j=1}^{N-1} \left(\left\langle \mathbb{J}\xi_a^j, \xi_a^j\right\rangle - \left\langle \lambda_a^j, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a^j)\right)\right\rangle\right)\\ &+ \Delta t\, \frac{l_1 l_2}{4} \sum_{a\in edge\backslash cor.} \sum_{j=1}^{N-1} \left(\left\langle \mathbb{J}\xi_a^j, \xi_a^j\right\rangle - \left\langle \lambda_a^j, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a^j)\right)\right\rangle\right)\\ &+ \Delta t\, \frac{l_1 l_2}{8} \sum_{a\in corner} \sum_{j=1}^{N-1} \left(\left\langle \mathbb{J}\xi_a^j, \xi_a^j\right\rangle - \left\langle \lambda_a^j, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a^j)\right)\right\rangle\right)\\ &- \Delta t \sum_{K\in\mathcal{T}} \sum_{j=1}^{N-1} \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right),\end{aligned}$$

where $\mathbb{J} : \mathfrak{se}(3) \to \mathfrak{se}(3)^*$ is the linear operator which has the matrix

$$\mathbb{J} = \begin{pmatrix} J & 0 \\ 0 & M\mathbf{I}_3 \end{pmatrix}.$$

Thus, we can associate with each node a discrete Lagrangian $\mathscr{L}_a^j$, depending on its position. That is

(i) $a \in int(\mathcal{T})$

$$\begin{aligned}\mathscr{L}_a^j = &\Delta t\, \frac{l_1 l_2}{2} \left(\left\langle \mathbb{J}\xi_a^j, \xi_a^j\right\rangle - \left\langle \lambda_a^j, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a^j)\right)\right\rangle\right)\\ &- \Delta t \sum_{K\ni a} \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right),\end{aligned}$$

(ii) $a \in edge\backslash corner$

$$\begin{aligned}\mathscr{L}_a^j = &\Delta t\, \frac{l_1 l_2}{4} \left(\left\langle \mathbb{J}\xi_a^j, \xi_a^j\right\rangle - \left\langle \lambda_a^j, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a^j)\right)\right\rangle\right)\\ &- \Delta t \sum_{K\ni a} \mathbb{V}_K\left(\Lambda_K^j, \mathbf{x}_K^j\right),\end{aligned}$$

(iii) $a \in corner$

$$\mathscr{L}_a^j = \Delta t \, \frac{l_1 l_2}{8} \left(\left\langle \mathbb{J}\xi_a^j, \xi_a^j \right\rangle - \left\langle \lambda_a^j, \mathrm{Tr}\left(\widehat{\mathbf{E}}^T \exp^{-1}(\Lambda_a^j) \right) \right\rangle \right) - \Delta t \sum_{K \ni a} \mathbb{V}_K \left(\Lambda_K^j, \mathbf{x}_K^j \right).$$

7.3.3 Lie algebra variational integrator

In order to obtain a Lie algebra variational integrator as defined in (7.2.6) we have to calculate $D_{\xi_a^j} \mathscr{L}_a^j$, and $D_{g_a^j} \mathscr{L}_a^j$, that is

Derivative $D_{(\Psi_a^j, H_a^j)} \mathscr{L}_a^j$.

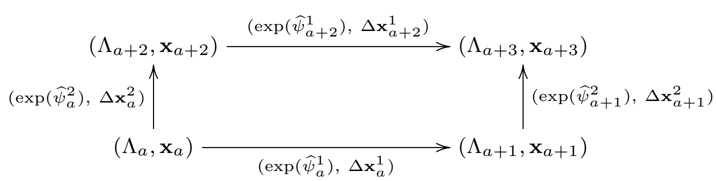
$$a \in int(\mathcal{T}), \quad \left\{ D_{\xi_a^j} \mathscr{L}_a^j = \Delta t \, l_1 l_2 \begin{pmatrix} J\omega_a^j \\ M\gamma_a^j \end{pmatrix}, \right.$$

$$a \in edge \backslash corner, \quad \left\{ D_{\xi_a^j} \mathscr{L}_a^j = \Delta t \, \frac{l_1 l_2}{2} \begin{pmatrix} J\omega_a^j \\ M\gamma_a^j \end{pmatrix}, \right.$$

$$a \in corner, \quad \left\{ D_{\xi_a^j} \mathscr{L}_a^j = \Delta t \, \frac{l_1 l_2}{4} \begin{pmatrix} J\omega_a^j \\ M\gamma_a^j \end{pmatrix}. \right.$$

Derivative $D_{\Lambda_a} \mathbb{V}_K$: In order to facilitate the calculations, we denote

$\widehat{\psi}_{a+1}^1 = \widehat{\psi}_a^1 := \widehat{\psi}_1 = \exp^{-1}(\Lambda_a^T \Lambda_{a+1})$, $\quad \widehat{\psi}_{a+3}^1 = \widehat{\psi}_{a+2}^1 := \widehat{\psi}_2 = \exp^{-1}(\Lambda_{a+2}^T \Lambda_{a+3})$,
$\widehat{\psi}_{a+2}^2 = \widehat{\psi}_a^2 := \widehat{\psi}_3 = \exp^{-1}(\Lambda_a^T \Lambda_{a+2})$, $\quad \widehat{\psi}_{a+3}^2 = \widehat{\psi}_{a+1}^2 := \widehat{\psi}_4 = \exp^{-1}(\Lambda_{a+1}^T \Lambda_{a+3})$,
$\Delta\mathbf{x}_{a+1}^1 = \Delta\mathbf{x}_a^1 := \Delta\mathbf{x}_{a,a+1} = \mathbf{x}_{a+1} - \mathbf{x}_a$, $\quad \Delta\mathbf{x}_{a+3}^1 = \Delta\mathbf{x}_{a+2}^1 := \Delta\mathbf{x}_{a+2,a+3} = \mathbf{x}_{a+3} - \mathbf{x}_{a+2}$,
$\Delta\mathbf{x}_{a+2}^2 = \Delta\mathbf{x}_a^2 := \Delta\mathbf{x}_{a,a+2} = \mathbf{x}_{a+2} - \mathbf{x}_a$, $\quad \Delta\mathbf{x}_{a+3}^2 = \Delta\mathbf{x}_{a+1}^2 := \Delta\mathbf{x}_{a+1,a+3} = \mathbf{x}_{a+3} - \mathbf{x}_{a+1}$.



For the node a, there are two possibilities: a can be on the left or on the right, with respect to $a+1$ or $a+2$. And the position is given by the Lie group actions (7.3.3), and (7.3.5). On the right when it is the image, on the left when it is the origin. (The same for the other nodes $a+1, a+2$, and $a+3$.)

(i) At a on the left of $a+1$ and $a+2$. As calculated in the appendix §(7.4) we get :

$$\begin{aligned}(\Lambda_a^T D_{\Lambda_a} \mathbb{V}_K)^\vee &= \frac{l_1 l_2}{4l_1}\mathbb{C}_{1,1}\left(\Gamma_a^1 - \mathbf{E}_1\right)\times\Lambda_a^T\Delta\mathbf{x}_a^1 + \frac{l_1 l_2}{4l_2}\mathbb{C}_{2,2}\left(\Gamma_a^2 - \mathbf{E}_2\right)\times\Lambda_a^T\Delta\mathbf{x}_a^2\\
&+ 2\frac{l_1 l_2}{2l_1}\left(\left(\left(I + \mathrm{cay}(\widehat{\psi}_a^1)^T\right)^{-1}\widehat{\mathbb{D}_{1,1}\Omega_a^1}\left(\widehat{\psi}_a^1 - 2I\right)\right)^{(A)}\right)^\vee\\
&+ 2\frac{l_1 l_2}{2l_2}\left(\left(\left(I + (\mathrm{cay}(\widehat{\psi}_a^2))^T\right)^{-1}\widehat{\mathbb{D}_{2,2}\Omega_a^2}\left(\widehat{\psi}_a^2 - 2I\right)\right)^{(A)}\right)^\vee\\
&+ \frac{l_1 l_2}{4l_2}\mathbb{C}_{1,2}\left(\Gamma_a^1 - \mathbf{E}_1\right)\times\Lambda_a^T\Delta\mathbf{x}_a^2 + \frac{l_1 l_2}{4l_1}\mathbb{C}_{2,1}\left(\Gamma_a^2 - \mathbf{E}_2\right)\times\Lambda_a^T\Delta\mathbf{x}_a^1\\
&+ \frac{l_1 l_2}{2l_2}\left(\left(\left(I + (\mathrm{cay}(\widehat{\psi}_a^2))^T\right)^{-1}\widehat{\mathbb{D}_{1,2}\Omega_a^1}\left(\widehat{\psi}_a^2 - 2I\right)\right)^{(A)}\right)^\vee\\
&+ \frac{l_1 l_2}{2l_1}\left(\left(\left(I + \mathrm{cay}(\widehat{\psi}_a^1)^T\right)^{-1}\widehat{\mathbb{D}_{2,1}\Omega_a^2}\left(\widehat{\psi}_a^1 - 2I\right)\right)^{(A)}\right)^\vee\\
&+ \frac{l_1 l_2}{2l_2}\left(\left(\left(I + (\mathrm{cay}(\widehat{\psi}_{a+2}^2))^T\right)^{-1}\widehat{\mathbb{D}_{1,2}\Omega_{a+2}^1}\left(\widehat{\psi}_{a+2}^2 - 2I\right)\right)^{(A)}\right)^\vee\\
&+ \frac{l_1 l_2}{2l_1}\left(\left(\left(I + \mathrm{cay}(\widehat{\psi}_{a+1}^1)^T\right)^{-1}\widehat{\mathbb{D}_{2,1}\Omega_{a+1}^2}\left(\widehat{\psi}_{a+1}^1 - 2I\right)\right)^{(A)}\right)^\vee.\end{aligned}$$

(ii) At $a+3$ on the right of $a+1$ and $a+2$. As calculated in the appendix §(7.4) we get :

$$\begin{aligned}&(\Lambda_{a+3}^T D_{\Lambda_{a+3}} \mathbb{V}_K)^\vee\\
&= \frac{l_1 l_2}{4l_1}\mathbb{C}_{1,1}\left(\Gamma_{a+3}^1 - \mathbf{E}_1\right)\times\Lambda_{a+3}^T\Delta\mathbf{x}_{a+3}^1 + \frac{l_1 l_2}{4l_2}\mathbb{C}_{2,2}\left(\Gamma_{a+3}^2 - \mathbf{E}_2\right)\times\Lambda_{a+3}^T\Delta\mathbf{x}_{a+3}^2\\
&+ 2\frac{l_1 l_2}{2l_1}\left(\left((\mathrm{cay}(\widehat{\psi}_{a+3}^1) + I)^{-1}\widehat{\mathbb{D}_{1,1}\Omega_{a+3}^1}(2I - \widehat{\psi}_{a+3}^1)\mathrm{cay}(\widehat{\psi}_{a+3}^1)\right)^{(A)}\right)^\vee\\
&+ 2\frac{l_1 l_2}{2l_2}\left(\left((\mathrm{cay}(\widehat{\psi}_{a+3}^2) + I)^{-1}\widehat{\mathbb{D}_{2,2}\Omega_{a+3}^2}(2I - \widehat{\psi}_{a+3}^2)\mathrm{cay}(\widehat{\psi}_{a+3}^2)\right)^{(A)}\right)^\vee\\
&+ \frac{l_1 l_2}{4l_2}\mathbb{C}_{1,2}\left(\Gamma_{a+3}^1 - \mathbf{E}_1\right)\times\Lambda_{a+3}^T\Delta\mathbf{x}_{a+3}^2 + \frac{l_1 l_2}{4l_1}\mathbb{C}_{1,2}\left(\Gamma_{a+3}^2 - \mathbf{E}_2\right)\times\Lambda_{a+3}^T\Delta\mathbf{x}_{a+3}^1\\
&+ \frac{l_1 l_2}{2l_2}\left(\left((\mathrm{cay}(\widehat{\psi}_{a+3}^2) + I)^{-1}\widehat{\mathbb{D}_{1,2}\Omega_{a+3}^1}(2I - \widehat{\psi}_{a+3}^2)\mathrm{cay}(\widehat{\psi}_{a+3}^2)\right)^{(A)}\right)^\vee\\
&+ \frac{l_1 l_2}{2l_1}\left(\left((\mathrm{cay}(\widehat{\psi}_{a+3}^1) + I)^{-1}\widehat{\mathbb{D}_{2,1}\Omega_{a+3}^2}(2I - \widehat{\psi}_{a+3}^1)\mathrm{cay}(\widehat{\psi}_{a+3}^1)\right)^{(A)}\right)^\vee\\
&+ \frac{l_1 l_2}{2l_2}\left(\left((\mathrm{cay}(\widehat{\psi}_{a+1}^2) + I)^{-1}\widehat{\mathbb{D}_{1,2}\Omega_{a+1}^1}(2I - \widehat{\psi}_{a+1}^2)\mathrm{cay}(\widehat{\psi}_{a+1}^2)\right)^{(A)}\right)^\vee\\
&+ \frac{l_1 l_2}{2l_1}\left(\left((\mathrm{cay}(\widehat{\psi}_{a+2}^1) + I)^{-1}\widehat{\mathbb{D}_{2,1}\Omega_{a+2}^2}(2I - \widehat{\psi}_{a+2}^1)\mathrm{cay}(\widehat{\psi}_{a+2}^1)\right)^{(A)}\right)^\vee\end{aligned}$$

(iii) At $a+1$ on the right of a and on the left of $a+3$, we get :

$$
\begin{aligned}
&(\Lambda_{a+1}^T D_{\Lambda_{a+1}} \mathbb{V}_K)^\vee \\
&= \frac{l_1 l_2}{4 l_1} \mathbb{C}_{1,1} \left(\Gamma_{a+1}^1 - \mathbf{E}_1\right) \times \Lambda_{a+1}^T \Delta \mathbf{x}_{a+1}^1 + \frac{l_1 l_2}{4 l_2} \mathbb{C}_{2,2} \left(\Gamma_{a+1}^2 - \mathbf{E}_2\right) \times \Lambda_{a+1}^T \Delta \mathbf{x}_{a+1}^2 \\
&\quad + \frac{l_1 l_2}{4 l_2} \mathbb{C}_{1,2} \left(\Gamma_{a+1}^1 - \mathbf{E}_1\right) \times \Lambda_{a+1}^T \Delta \mathbf{x}_{a+1}^2 + \frac{l_1 l_2}{4 l_1} \mathbb{C}_{1,2} \left(\Gamma_{a+1}^2 - \mathbf{E}_2\right) \times \Lambda_{a+1}^T \Delta \mathbf{x}_{a+1}^1 \\
&\quad + 2 \frac{l_1 l_2}{2 l_1} \left(\left((\mathrm{cay}(\widehat{\psi}_{a+1}^1) + I)^{-1} \widehat{\mathbb{D}_{1,1} \Omega_{a+1}^1} (2I - \widehat{\psi}_{a+1}^1) \mathrm{cay}(\widehat{\psi}_{a+1}^1) \right)^{(A)} \right)^\vee \\
&\quad + 2 \frac{l_1 l_2}{2 l_2} \left(\left(\left(I + (\mathrm{cay}(\widehat{\psi}_{a+1}^2))^T \right)^{-1} \widehat{\mathbb{D}_{2,2} \Omega_{a+1}^2} \left(\widehat{\psi}_{a+1}^2 - 2I \right) \right)^{(A)} \right)^\vee \\
&\quad + \frac{l_1 l_2}{2 l_2} \left(\left(\left(I + (\mathrm{cay}(\widehat{\psi}_{a+1}^2))^T \right)^{-1} \widehat{\mathbb{D}_{1,2} \Omega_{a+1}^1} \left(\widehat{\psi}_{a+1}^2 - 2I \right) \right)^{(A)} \right)^\vee \\
&\quad + \frac{l_1 l_2}{2 l_1} \left(\left((\mathrm{cay}(\widehat{\psi}_{a+1}^1) + I)^{-1} \widehat{\mathbb{D}_{2,1} \Omega_{a+1}^2} (2I - \widehat{\psi}_{a+1}^1) \mathrm{cay}(\widehat{\psi}_{a+1}^1) \right)^{(A)} \right)^\vee \\
&\quad + \frac{l_1 l_2}{2 l_2} \left(\left(\left(I + (\mathrm{cay}(\widehat{\psi}_{a+3}^2))^T \right)^{-1} \widehat{\mathbb{D}_{1,2} \Omega_{a+3}^1} \left(\widehat{\psi}_{a+3}^2 - 2I \right) \right)^{(A)} \right)^\vee \\
&\quad + \frac{l_1 l_2}{2 l_1} \left(\left((\mathrm{cay}(\widehat{\psi}_{a}^1) + I)^{-1} \widehat{\mathbb{D}_{2,1} \Omega_{a}^2} (2I - \widehat{\psi}_{a}^1) \mathrm{cay}(\widehat{\psi}_{a}^1) \right)^{(A)} \right)^\vee .
\end{aligned}
$$

(iv) At $a+2$ on the right of a and on the left of $a+3$,

$$
\begin{aligned}
&(\Lambda_{a+2}^T D_{\Lambda_{a+2}} \mathbb{V}_K)^\vee \\
&= \frac{l_1 l_2}{4 l_1} \mathbb{C}_{1,1} \left(\Gamma_{a+2}^1 - \mathbf{E}_1\right) \times \Lambda_{a+2}^T \Delta \mathbf{x}_{a+2}^1 + \frac{l_1 l_2}{4 l_2} \mathbb{C}_{2,2} \left(\Gamma_{a+2}^2 - \mathbf{E}_2\right) \times \Lambda_{a+2}^T \Delta \mathbf{x}_{a+2}^2 \\
&\quad + \frac{l_1 l_2}{4 l_2} \mathbb{C}_{1,2} \left(\Gamma_{a+2}^1 - \mathbf{E}_1\right) \times \Lambda_{a+2}^T \Delta \mathbf{x}_{a+2}^2 + \frac{l_1 l_2}{4 l_1} \mathbb{C}_{1,2} \left(\Gamma_{a+2}^2 - \mathbf{E}_2\right) \times \Lambda_{a+2}^T \Delta \mathbf{x}_{a+2}^1 \\
&\quad + 2 \frac{l_1 l_2}{2 l_1} \left(\left(\left(I + \mathrm{cay}(\widehat{\psi}_{a+2}^1)^T \right)^{-1} \widehat{\mathbb{D}_{1,1} \Omega_{a+2}^1} \left(\widehat{\psi}_{a+2}^1 - 2I \right) \right)^{(A)} \right)^\vee \\
&\quad + 2 \frac{l_1 l_2}{2 l_2} \left(\left((\mathrm{cay}(\widehat{\psi}_{a+2}^2) + I)^{-1} \widehat{\mathbb{D}_{2,2} \Omega_{a+2}^2} (2I - \widehat{\psi}_{a+2}^2) \mathrm{cay}(\widehat{\psi}_{a+2}^2) \right)^{(A)} \right)^\vee \\
&\quad + \frac{l_1 l_2}{2 l_2} \left(\left((\mathrm{cay}(\widehat{\psi}_{a+2}^2) + I)^{-1} \widehat{\mathbb{D}_{1,2} \Omega_{a+2}^1} (2I - \widehat{\psi}_{a+2}^2) \mathrm{cay}(\widehat{\psi}_{a+2}^2) \right)^{(A)} \right)^\vee \\
&\quad + \frac{l_1 l_2}{2 l_1} \left(\left(\left(I + \mathrm{cay}(\widehat{\psi}_{a+2}^1)^T \right)^{-1} \widehat{\mathbb{D}_{2,1} \Omega_{a+2}^2} \left(\widehat{\psi}_{a+2}^1 - 2I \right) \right)^{(A)} \right)^\vee \\
&\quad + \frac{l_1 l_2}{2 l_2} \left(\left((\mathrm{cay}(\widehat{\psi}_{a}^2) + I)^{-1} \widehat{\mathbb{D}_{1,2} \Omega_{a}^1} (2I - \widehat{\psi}_{a}^2) \mathrm{cay}(\widehat{\psi}_{a}^2) \right)^{(A)} \right)^\vee \\
&\quad + \frac{l_1 l_2}{2 l_1} \left(\left(\left(I + \mathrm{cay}(\widehat{\psi}_{a+3}^1)^T \right)^{-1} \widehat{\mathbb{D}_{2,1} \Omega_{a+3}^2} \left(\widehat{\psi}_{a+3}^1 - 2I \right) \right)^{(A)} \right)^\vee .
\end{aligned}
$$

The derivatives $D_{\mathbf{x}_a} \mathbb{V}_K$ are :

(i) At a on the left of $a+1$ and $a+2$,

$$\begin{aligned} D_{\mathbf{x}_a}\mathbb{V}_K =& \frac{l_1 l_2}{4l_1}(-\Lambda_a)\mathbb{C}_{1,1}\left(\Gamma_a^1 - \mathbf{E}_1\right) + \frac{l_1 l_2}{4l_1}(-\Lambda_{a+1})\mathbb{C}_{1,1}\left(\Gamma_{a+1}^1 - \mathbf{E}_1\right) \\ &+ \frac{l_1 l_2}{4l_2}(-\Lambda_a)\mathbb{C}_{2,2}\left(\Gamma_a^2 - \mathbf{E}_2\right) + \frac{l_1 l_2}{4l_2}(-\Lambda_{a+2})\mathbb{C}_{2,2}\left(\Gamma_{a+2}^2 - \mathbf{E}_2\right) \\ &+ \frac{l_1 l_2}{4l_1}(-\Lambda_a)\mathbb{C}_{1,2}\left(\Gamma_a^2 - \mathbf{E}_2\right) + \frac{l_1 l_2}{4l_1}(-\Lambda_{a+1})\mathbb{C}_{1,2}\left(\Gamma_{a+1}^2 - \mathbf{E}_2\right) \\ &+ \frac{l_1 l_2}{4l_2}(-\Lambda_a)\mathbb{C}_{2,1}\left(\Gamma_a^1 - \mathbf{E}_1\right) + \frac{l_1 l_2}{4l_2}(-\Lambda_{a+2})\mathbb{C}_{2,1}\left(\Gamma_{a+2}^1 - \mathbf{E}_1\right) \\ &+ \frac{l_1 l_2}{4}\mathbf{q}_a. \end{aligned}$$

(ii) At $a+3$ on the right of $a+1$ and $a+2$,

$$\begin{aligned} D_{\mathbf{x}_{a+3}}\mathbb{V}_K =& \frac{l_1 l_2}{4l_1}(\Lambda_{a+2})\mathbb{C}_{1,1}\left(\Gamma_{a+2}^1 - \mathbf{E}_1\right) + \frac{l_1 l_2}{4l_1}(\Lambda_{a+3})\mathbb{C}_{1,1}\left(\Gamma_{a+3}^1 - \mathbf{E}_1\right) \\ &+ \frac{l_1 l_2}{4l_2}(\Lambda_{a+1})\mathbb{C}_{2,2}\left(\Gamma_{a+1}^2 - \mathbf{E}_2\right) + \frac{l_1 l_2}{4l_2}(\Lambda_{a+3})\mathbb{C}_{2,2}\left(\Gamma_{a+3}^2 - \mathbf{E}_2\right) \\ &+ \frac{l_1 l_2}{4l_1}(\Lambda_{a+3})\mathbb{C}_{1,2}\left(\Gamma_{a+3}^2 - \mathbf{E}_2\right) + \frac{l_1 l_2}{4l_1}(\Lambda_{a+2})\mathbb{C}_{1,2}\left(\Gamma_{a+2}^2 - \mathbf{E}_2\right) \\ &+ \frac{l_1 l_2}{4l_2}(\Lambda_{a+3})\mathbb{C}_{2,1}\left(\Gamma_{a+3}^1 - \mathbf{E}_1\right) + \frac{l_1 l_2}{4l_2}(\Lambda_{a+1})\mathbb{C}_{2,1}\left(\Gamma_{a+1}^1 - \mathbf{E}_1\right) \\ &+ \frac{l_1 l_2}{4}\mathbf{q}_{a+3}. \end{aligned}$$

(iii) At $a+1$ on the left of $a+3$ and on the right of a,

$$\begin{aligned} D_{\mathbf{x}_{a+2}}\mathbb{V}_K =& \frac{l_1 l_2}{4l_1}(-\Lambda_{a+2})\mathbb{C}_{1,1}\left(\Gamma_{a+2}^1 - \mathbf{E}_1\right) + \frac{l_1 l_2}{4l_1}(-\Lambda_{a+3})\mathbb{C}_{1,1}\left(\Gamma_{a+3}^1 - \mathbf{E}_1\right) \\ &+ \frac{l_1 l_2}{4l_2}(\Lambda_{a+2})\mathbb{C}_{2,2}\left(\Gamma_{a+2}^2 - \mathbf{E}_2\right) + \frac{l_1 l_2}{4l_2}(\Lambda_a)\mathbb{C}_{2,2}\left(\Gamma_a^2 - \mathbf{E}_2\right) \\ &+ \frac{l_1 l_2}{4l_1}(-\Lambda_{a+2})\mathbb{C}_{1,2}\left(\Gamma_{a+2}^2 - \mathbf{E}_2\right) + \frac{l_1 l_2}{4l_1}(-\Lambda_{a+3})\mathbb{C}_{1,2}\left(\Gamma_{a+3}^2 - \mathbf{E}_2\right) \\ &+ \frac{l_1 l_2}{4l_2}(\Lambda_{a+2})\mathbb{C}_{2,1}\left(\Gamma_{a+2}^1 - \mathbf{E}_1\right) + \frac{l_1 l_2}{4l_2}(\Lambda_a)\mathbb{C}_{2,1}\left(\Gamma_a^1 - \mathbf{E}_1\right) \\ &+ \frac{l_1 l_2}{4}\mathbf{q}_{a+2}. \end{aligned}$$

(iv) At $a+2$ on the left of $a+3$ and on the right of a,

$$\begin{aligned} D_{\mathbf{x}_{a+1}}\mathbb{V}_K =& \frac{l_1 l_2}{4l_1}(\Lambda_{a+1})\mathbb{C}_{1,1}\left(\Gamma_{a+1}^1 - \mathbf{E}_1\right) + \frac{l_1 l_2}{4l_1}(\Lambda_a)\mathbb{C}_{1,1}\left(\Gamma_a^1 - \mathbf{E}_1\right) \\ &+ \frac{l_1 l_2}{4l_2}(-\Lambda_{a+1})\mathbb{C}_{2,2}\left(\Gamma_{a+1}^2 - \mathbf{E}_2\right) + \frac{l_1 l_2}{4l_2}(-\Lambda_{a+3})\mathbb{C}_{2,2}\left(\Gamma_{a+3}^2 - \mathbf{E}_2\right) \\ &+ \frac{l_1 l_2}{4l_1}(\Lambda_{a+1})\mathbb{C}_{1,2}\left(\Gamma_{a+1}^2 - \mathbf{E}_2\right) + \frac{l_1 l_2}{4l_1}(\Lambda_a)\mathbb{C}_{1,2}\left(\Gamma_a^2 - \mathbf{E}_2\right) \\ &+ \frac{l_1 l_2}{4l_2}\left(\Gamma_{a+1}^1 - \mathbf{E}_1\right)^T \mathbb{C}_{2,1}(-\Lambda_{a+1}^T) + \frac{l_1 l_2}{4l_2}(-\Lambda_{a+3})\mathbb{C}_{2,1}\left(\Gamma_{a+3}^1 - \mathbf{E}_1\right) \\ &+ \frac{l_1 l_2}{4}\mathbf{q}_{a+1}. \end{aligned}$$

From the preceding results, we obtain that the forced discrete Euler-Lagrange equations read

Discrete Euler-Lagrange equation (7.2.5) **for a corner of the plate.**

$$\begin{cases} \mu_a^j + \begin{pmatrix} \lambda_a^j \Lambda_a^T \overline{d^R \exp_{\zeta_a^j}^{-1}}^T \mathbf{E} \\ 0 \end{pmatrix} = -\Delta t \begin{pmatrix} U_a \\ V_a \end{pmatrix} + \mathrm{Ad}^*_{\tau(\xi_a^{j-1})} \mu_a^{j-1}, \\ \mu_a^j = \Delta t \, \dfrac{l_1 l_2}{4} \left(\left(\mathrm{d}^R \tau(\xi_a^j) \right)^{-1} \right)^* \begin{pmatrix} J\omega_a^j \\ M\gamma_a^j \end{pmatrix}, \\ \mathrm{Tr} \left(\widehat{\mathbf{E}}^T \exp^{-1} \left(\Lambda_a^j \exp(\Psi_a^j) \right) \right) = 0, \quad \text{with} \;\; g_a^{j+1} = g_a^j \tau(\xi_a^j), \end{cases}$$

$$\text{with} \quad U_a = \left(\Lambda_a^T D_{\Lambda_a} \mathbb{V}_K \right)^\vee \Big|_{t=t^j}, \qquad V_a = \Lambda_a^T D_{\mathbf{x}_a} \mathbb{V}_K \Big|_{t=t^j}.$$

Discrete Euler-Lagrange equation for an edge of the plate.

$$\begin{cases} \mu_a^j + \begin{pmatrix} \lambda_a^j \Lambda_a^T \overline{d^R \exp_{\zeta_a^j}^{-1}}^T \mathbf{E} \\ 0 \end{pmatrix} = -\Delta t \begin{pmatrix} U_a \\ V_a \end{pmatrix} + \mathrm{Ad}^*_{\tau(\xi_a^{j-1})} \mu_a^{j-1}, \\ \mu_a^j = \Delta t \, \dfrac{l_1 l_2}{2} \left(\left(\mathrm{d}^R \tau(\xi_a^j) \right)^{-1} \right)^* \begin{pmatrix} J\omega_a^j \\ M\gamma_a^j \end{pmatrix}, \\ \mathrm{Tr} \left(\widehat{\mathbf{E}}^T \exp^{-1} \left(\Lambda_a^j \exp(\Psi_a^j) \right) \right) = 0, \quad \text{with} \;\; g_a^{j+1} = g_a^j \tau(\xi_a^j), \end{cases}$$

$$\text{with} \qquad U_a = \left(\Lambda_a^T \sum_{K \ni a} D_{\Lambda_a} \mathbb{V}_K \right)^\vee \Bigg|_{t=t^j}, \qquad V_a = \Lambda_a^T \sum_{K \ni a} D_{\mathbf{x}_a} \mathbb{V}_K \Bigg|_{t=t^j}.$$

Discrete Euler-Lagrange equation for the interior of the plate.

$$\begin{cases} \mu_a^j + \begin{pmatrix} \lambda_a^j \Lambda_a^T \overline{d^R \exp_{\zeta_a^j}^{-1}}^T \mathbf{E} \\ 0 \end{pmatrix} = -\Delta t \begin{pmatrix} U_a \\ V_a \end{pmatrix} + \mathrm{Ad}^*_{\tau(\xi_a^{j-1})} \mu_a^{j-1}, \\ \mu_a^j = \Delta t \, l_1 l_2 \left(\left(\mathrm{d}^R \tau(\xi_a^j) \right)^{-1} \right)^* \begin{pmatrix} J\omega_a^j \\ M\gamma_a^j \end{pmatrix}, \\ \mathrm{Tr} \left(\widehat{\mathbf{E}}^T \exp^{-1} \left(\Lambda_a^j \exp(\Psi_a^j) \right) \right) = 0, \quad \text{with} \;\; g_a^{j+1} = g_a^j \tau(\xi_a^j), \end{cases}$$

$$\text{with} \qquad U_a = \left(\Lambda_a^T \sum_{K \ni a} D_{\Lambda_a} \mathbb{V}_K \right)^\vee \Bigg|_{t=t^j}, \qquad V_a = \Lambda_a^T \sum_{K \ni a} D_{\mathbf{x}_a} \mathbb{V}_K \Bigg|_{t=t^j}.$$

7.3.2 Remark For the algorithms presented below we need to decide whether $\tau : \mathfrak{se}(3) \to SE(3)$ is the exponential map or the Cayley transform.

7.4 Appendix : intermediate calculation

The potential energy (7.3.8) may be written as follows

$$\begin{aligned}\mathbb{V}(\Lambda_K, x_K) = &\sum_{a\in K} \frac{l_1 l_2}{8}\Bigg\{ \begin{pmatrix} \Gamma_a^1 - \mathbf{E}_1 \\ \Gamma_a^2 - \mathbf{E}_2 \end{pmatrix}^T \begin{pmatrix} \mathbb{C}_{11} & \mathbb{C}_{12} \\ \mathbb{C}_{21} & \mathbb{C}_{22} \end{pmatrix} \begin{pmatrix} \Gamma_a^1 - \mathbf{E}_1 \\ \Gamma_a^2 - \mathbf{E}_2 \end{pmatrix} \\ &+ \begin{pmatrix} \Omega_a^1 \\ \Omega_a^2 \end{pmatrix}^T \begin{pmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{pmatrix} \begin{pmatrix} \Omega_a^1 \\ \Omega_a^2 \end{pmatrix} \Bigg\} + \sum_{a\in K} \frac{l_1 l_2}{4} \langle \mathbf{q}, \mathbf{x}_a \rangle \\ = &\sum_{a\in K} \frac{l_1 l_2}{8}\Bigg\{ (\Gamma_a^1 - \mathbf{E}_1)^T \mathbb{C}_{11} (\Gamma_a^1 - \mathbf{E}_1) + (\Gamma_a^2 - \mathbf{E}_2)^T \mathbb{C}_{22} (\Gamma_a^2 - \mathbf{E}_2) + 2(\Gamma_a^1 - \mathbf{E}_1)^T \mathbb{C}_{12} (\Gamma_a^2 - \mathbf{E}_2) \\ &+ (\Omega_a^1)^T \mathbb{C}_{11} \Omega_a^1 + (\Omega_a^2)^T \mathbb{C}_{22} \Omega_a^2 + 2(\Omega_a^1)^T \mathbb{C}_{12} \Omega_a^2 \Bigg\} + \sum_{a\in K} \frac{l_1 l_2}{4} \langle \mathbf{q}, \mathbf{x}_a \rangle .\end{aligned}$$

We denote

$$\mathbb{U}_K^a := \frac{l_1 l_2}{8}\Big((\Gamma_a^\alpha - \mathbf{E}_\alpha)^T \mathbb{C}_{\alpha\beta} (\Gamma_a^\beta - \mathbf{E}_\beta) + (\Omega_a^\alpha)^T \mathbb{D}_{\alpha,\beta}\ \Omega_a^\beta \Big).$$

We compute $\widehat{\psi}_a$ using the Cayley transform, i.e., $\widehat{\psi}_a = \operatorname{cay}^{-1}(\Lambda_a^T \Lambda_{a+1})$, over element K of size $l_1 \times l_2$. If $\delta\Lambda_a = \Lambda_a \widehat{\xi} \in T_{\Lambda_a} SO(3)$, we have :

$$\begin{aligned} D_{\Lambda_a} \widehat{\psi}_a \cdot \delta\Lambda_a &= 2(\delta\Lambda_a)^T \Lambda_{a+1} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ &\quad - 2(\Lambda_a^T \Lambda_{a+1} - I)(\Lambda_a^T \Lambda_{a+1} + I)^{-1} \delta\Lambda_a^T \Lambda_{a+1} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ &= -2\widehat{\xi} \Lambda_a^T \Lambda_{a+1} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ &\quad + 2(\Lambda_a^T \Lambda_{a+1} - I)(\Lambda_a^T \Lambda_{a+1} + I)^{-1} \widehat{\xi} \Lambda_a^T \Lambda_{a+1} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ &= \left(\widehat{\psi}_a - 2I\right) \widehat{\xi} \left(I + \Lambda_{a+1}^T \Lambda_a \right)^{-1} = \left(\widehat{\psi}_a - 2I\right) \widehat{\xi} \left(I + \operatorname{cay}(\widehat{\psi}_a)^T \right)^{-1} . \end{aligned}$$

$$\begin{aligned} D_{\Lambda_{a+1}} \widehat{\psi}_a \cdot \delta\Lambda_{a+1} =& 2\Lambda_a^T \delta\Lambda_{a+1} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ & - 2(\Lambda_a^T \Lambda_{a+1} - I)(\Lambda_a^T \Lambda_{a+1} + I)^{-1} \Lambda_a^T \delta\Lambda_{a+1} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ =& 2\Lambda_a^T \Lambda_{a+1} \widehat{\xi} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ & - 2(\Lambda_a^T \Lambda_{a+1} - I)(\Lambda_a^T \Lambda_{a+1} + I)^{-1} \Lambda_a^T \Lambda_{a+1} \widehat{\xi} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ =& (2I - \widehat{\psi}_a) \Lambda_a^T \Lambda_{a+1} \widehat{\xi} (\Lambda_a^T \Lambda_{a+1} + I)^{-1} \\ =& (2I - \widehat{\psi}_a) \operatorname{cay}(\widehat{\psi}_a) \widehat{\xi} (\operatorname{cay}(\widehat{\psi}_a) + I)^{-1}. \end{aligned}$$

Derivative $D_{\Lambda_a}\mathbb{U}_K^a$ in a on the left of $a+1$ and $a+2$:

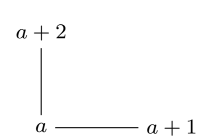
$$
\begin{aligned}
&D_{\Lambda_a}\mathbb{U}_K^a\cdot\delta\Lambda_a\\
&= \frac{l_1l_2}{8l_\beta}(\Gamma_a^\alpha-\mathbf{E}_\alpha)^T\,\mathbb{C}_{\alpha,\beta}\;\delta\Lambda_a^T\Delta\mathbf{x}_a^\beta + \frac{l_1l_2}{8l_\beta}(\Omega_a^\alpha)^T\mathbb{D}_{\alpha,\beta}D_{\Lambda_a}\psi_a^\beta\cdot\delta\Lambda_a\\
&\quad+ \frac{l_1l_2}{8l_\alpha}\left(\Gamma_a^\beta-\mathbf{E}_\beta\right)^T\mathbb{C}_{\alpha,\beta}\;\delta\Lambda_a^T\Delta\mathbf{x}_a^\alpha + \frac{l_1l_2}{8l_\alpha}(\Omega_a^\beta)^T\mathbb{D}_{\alpha,\beta}D_{\Lambda_a}\psi_a^\alpha\cdot\delta\Lambda_a\\
&= -\frac{l_1l_2}{8l_\beta}\,\mathrm{Tr}\left((\Gamma_a^\alpha-\mathbf{E}_\alpha)^T\,\mathbb{C}_{\alpha,\beta}\widehat{\xi}\Lambda_a^T\Delta\mathbf{x}_a^\beta\right) - \frac{l_1l_2}{8l_\alpha}\,\mathrm{Tr}\left(\left(\Gamma_a^\beta-\mathbf{E}_\beta\right)^T\mathbb{C}_{\alpha,\beta}\widehat{\xi}\Lambda_a^T\Delta\mathbf{x}_a^\alpha\right)\\
&\quad- \frac{l_1l_2}{8l_\beta}\,\mathrm{Tr}\left(\widehat{\mathbb{D}_{\alpha,\beta}\Omega_a^\alpha}(D_{\Lambda_a}\widehat{\psi}_a^\beta\cdot\delta\Lambda_a)\right) - \frac{l_1l_2}{8l_\alpha}\,\mathrm{Tr}\left(\widehat{\mathbb{D}_{\alpha,\beta}\Omega_a^\beta}(D_{\Lambda_a}\widehat{\psi}_a^\alpha\cdot\delta\Lambda_a)\right)\\
&= -\frac{l_1l_2}{8l_\beta}\,\mathrm{Tr}\left(\left(\Lambda_a^T\Delta\mathbf{x}_a^\beta\,(\Gamma_a^\alpha-\mathbf{E}_\alpha)^T\,\mathbb{C}_{\alpha,\beta}\right)^{(A)}\widehat{\xi}\right)\\
&\quad- \frac{l_1l_2}{8l_\beta}\,\mathrm{Tr}\left(\left(\left(I+\mathrm{cay}(\widehat{\psi}_a^\beta)^T\right)^{-1}\widehat{\mathbb{D}_{\alpha,\beta}\Omega_a^\alpha}\left(\widehat{\psi}_a^\beta-2I\right)\right)^{(A)}\widehat{\xi}\right)\\
&\quad- \frac{l_1l_2}{8l_\alpha}\,\mathrm{Tr}\left(\left(\Lambda_a^T\Delta\mathbf{x}_a^\alpha\left(\Gamma_a^\beta-\mathbf{E}_\beta\right)^T\mathbb{C}_{\alpha,\beta}\right)^{(A)}\widehat{\xi}\right)\\
&\quad- \frac{l_1l_2}{8l_\alpha}\,\mathrm{Tr}\left(\left(\left(I+(\mathrm{cay}(\widehat{\psi}_a^\alpha))^T\right)^{-1}\widehat{\mathbb{D}_{\alpha,\beta}\Omega_a^\beta}\left(\widehat{\psi}_a^\alpha-2I\right)\right)^{(A)}\widehat{\xi}\right)\\
&= \frac{l_1l_2}{4l_\beta}\left(\left(\Lambda_a^T\Delta\mathbf{x}_a^\beta\,(\Gamma_a^\alpha-\mathbf{E}_\alpha)^T\,\mathbb{C}_{\alpha,\beta}\right)^{(A)}\right)^\vee\cdot\xi\\
&\quad+ \frac{l_1l_2}{4l_\beta}\left(\left(\left(I+(\mathrm{cay}(\widehat{\psi}_a^\beta))^T\right)^{-1}\widehat{\mathbb{D}_{\alpha,\beta}\Omega_a^\alpha}\left(\widehat{\psi}_a^\beta-2I\right)\right)^{(A)}\right)^\vee\cdot\xi\\
&\quad+ \frac{l_1l_2}{4l_\alpha}\left(\left(\Lambda_a^T\Delta\mathbf{x}_a^\alpha\left(\Gamma_a^\beta-\mathbf{E}_\beta\right)^T\mathbb{C}_{\alpha,\beta}\right)^{(A)}\right)^\vee\cdot\xi\\
&\quad+ \frac{l_1l_2}{4l_\alpha}\left(\left(\left(I+\mathrm{cay}(\widehat{\psi}_a^\alpha)^T\right)^{-1}\widehat{\mathbb{D}_{\alpha,\beta}\Omega_a^\beta}\left(\widehat{\psi}_a^\alpha-2I\right)\right)^{(A)}\right)^\vee\cdot\xi,
\end{aligned}
$$

then, using the formula $\left((\mathbf{v}\mathbf{w}^T)^{(A)}\right)^\vee = \frac{1}{2}\mathbf{w}\times\mathbf{v}$, we get :

$$
\begin{aligned}
(\Lambda_a^T D_{\Lambda_a}\mathbb{U}_K^a)^\vee &= \frac{l_1l_2}{8l_\beta}\mathbb{C}_{\alpha,\beta}\,(\Gamma_a^\alpha-\mathbf{E}_\alpha)\times\Lambda_a^T\Delta\mathbf{x}_a^\beta + \frac{l_1l_2}{8l_\alpha}\mathbb{C}_{\alpha,\beta}\left(\Gamma_a^\beta-\mathbf{E}_\beta\right)\times\Lambda_a^T\Delta\mathbf{x}_a^\alpha\\
&\quad+ \frac{l_1l_2}{4l_\beta}\left(\left(\left(I+(\mathrm{cay}(\widehat{\psi}_a^\beta))^T\right)^{-1}\widehat{\mathbb{D}_{\alpha,\beta}\Omega_a^\alpha}\left(\widehat{\psi}_a^\beta-2I\right)\right)^{(A)}\right)^\vee\\
&\quad+ \frac{l_1l_2}{4l_\alpha}\left(\left(\left(I+\mathrm{cay}(\widehat{\psi}_a^\alpha)^T\right)^{-1}\widehat{\mathbb{D}_{\alpha,\beta}\Omega_a^\beta}\left(\widehat{\psi}_a^\alpha-2I\right)\right)^{(A)}\right)^\vee.
\end{aligned}
$$

Derivative $D_{\Lambda_a}\mathbb{V}_K$ is obtained by the following calculation :

$$D_{\Lambda_a}\mathbb{V}_K = D_{\Lambda_a}\mathbb{U}_K^a + D_{\Lambda_a}\mathbb{U}_K^{a+1} + D_{\Lambda_a}\mathbb{U}_K^{a+2}.$$



Derivative $D_{\Lambda_{a+3}}\mathbb{U}_K^{a+3}$ in $a+3$ on the right of node $a+1$ and $a+2$:

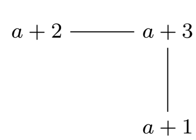
$$
\begin{aligned}
&D_{\Lambda_{a+3}}\mathbb{U}_K^{a+3}\cdot\delta\Lambda_{a+3}\\
&=\frac{l_1l_2}{8l_\beta}\left(\Gamma^\alpha_{a+3}-\mathbf{E}_\alpha\right)^T\mathbb{C}_{\alpha,\beta}\ \delta\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\beta_{a+3}+\frac{l_1l_2}{8l_\beta}(\Omega^\alpha_{a+3})^T\mathbb{D}_{\alpha,\beta}D_{\Lambda_{a+3}}\psi^\beta_{a+3}\cdot\delta\Lambda_{a+3}\\
&\quad+\frac{l_1l_2}{8l_\alpha}\left(\Gamma^\beta_{a+3}-\mathbf{E}_\beta\right)^T\mathbb{C}_{\alpha,\beta}\ \delta\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\alpha_{a+3}+\frac{l_1l_2}{8l_\alpha}(\Omega^\beta_{a+3})^T\mathbb{D}_{\alpha,\beta}D_{\Lambda_{a+3}}\psi^\alpha_{a+3}\cdot\delta\Lambda_{a+3}\\
&=-\frac{l_1l_2}{8l_\beta}\operatorname{Tr}\left(\left(\Gamma^\alpha_{a+3}-\mathbf{E}_\alpha\right)^T\mathbb{C}_{\alpha,\beta}\widehat{\xi}\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\beta_{a+3}\right)-\frac{l_1l_2}{8l_\alpha}\operatorname{Tr}\left(\left(\Gamma^\beta_{a+3}-\mathbf{E}_\beta\right)^T\mathbb{C}_{\alpha,\beta}\widehat{\xi}\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\alpha_{a+3}\right)\\
&\quad-\frac{l_1l_2}{8l_\alpha l_\beta}\operatorname{Tr}\left(\widehat{\mathbb{D}_{\alpha,\beta}\psi^\alpha_{a+3}}(D_{\Lambda_{a+3}}\widehat{\psi}^\beta_{a+3}\cdot\delta\Lambda_{a+3})\right)-\frac{l_1l_2}{8l_\alpha l_\beta}\operatorname{Tr}\left(\widehat{\mathbb{D}_{\alpha,\beta}\psi^\beta_{a+3}}(D_{\Lambda_{a+3}}\widehat{\psi}^\alpha_{a+3}\cdot\delta\Lambda_{a+3})\right)\\
&=-\frac{l_1l_2}{8l_\beta}\operatorname{Tr}\left(\left(\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\beta_{a+3}\left(\Gamma^\alpha_{a+3}-\mathbf{E}_\alpha\right)^T\mathbb{C}_{\alpha,\beta}\right)^{(A)}\widehat{\xi}\right)\\
&\quad-\frac{l_1l_2}{8l_\alpha}\operatorname{Tr}\left(\left(\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\alpha_{a+3}\left(\Gamma^\beta_{a+3}-\mathbf{E}_\beta\right)^T\mathbb{C}_{\alpha,\beta}\right)^{(A)}\widehat{\xi}\right)\\
&\quad-\frac{l_1l_2}{8l_\alpha l_\beta}\operatorname{Tr}\left(\left((\operatorname{cay}(\widehat{\psi}^\beta_{a+3})+I)^{-1}\widehat{\mathbf{D}_{\alpha,\beta}\psi^\alpha_{a+3}}(2I-\widehat{\psi}^\beta_{a+3})\operatorname{cay}(\widehat{\psi}^\beta_{a+3})\right)^{(A)}\widehat{\xi}\right)\\
&\quad-\frac{l_1l_2}{8l_\alpha l_\beta}\operatorname{Tr}\left(\left((\operatorname{cay}(\widehat{\psi}^\alpha_{a+3})+I)^{-1}\widehat{\mathbf{D}_{\alpha,\beta}\psi^\beta_{a+3}}(2I-\widehat{\psi}^\alpha_{a+3})\operatorname{cay}(\widehat{\psi}^\alpha_{a+3})\right)^{(A)}\widehat{\xi}\right)\\
&=\frac{l_1l_2}{4l_\beta}\left(\left(\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\beta_{a+3}\left(\Gamma^\alpha_{a+3}-\mathbf{E}_\alpha\right)^T\mathbb{C}_{\alpha,\beta}\right)^{(A)}\right)^\vee\cdot\xi+\frac{l_1l_2}{4l_\alpha}\left(\left(\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\alpha_{a+3}\left(\Gamma^\beta_{a+3}-\mathbf{E}_\beta\right)^T\mathbb{C}_{\alpha,\beta}\right)^{(A)}\right)^\vee\cdot\xi\\
&\quad+\frac{l_1l_2}{4l_\alpha l_\beta}\left(\left((\operatorname{cay}(\widehat{\psi}^\beta_{a+3})+I)^{-1}\widehat{\mathbb{D}_{\alpha,\beta}\psi^\alpha_{a+3}}(2I-\widehat{\psi}^\beta_{a+3})\operatorname{cay}(\widehat{\psi}^\beta_{a+3})\right)^{(A)}\right)^\vee\cdot\xi\\
&\quad+\frac{l_1l_2}{4l_\alpha l_\beta}\left(\left((\operatorname{cay}(\widehat{\psi}^\alpha_{a+3})+I)^{-1}\widehat{\mathbb{D}_{\alpha,\beta}\psi^\beta_{a+3}}(2I-\widehat{\psi}^\alpha_{a+3})\operatorname{cay}(\widehat{\psi}^\alpha_{a+3})\right)^{(A)}\right)^\vee\cdot\xi.
\end{aligned}
$$

Thus, using (4.2.29), we get :

$$
\begin{aligned}
(\Lambda^T_{a+3}D_{\Lambda_{a+3}}\mathbb{U}_K^{a+3})^\vee&=\frac{l_1l_2}{8l_\beta}\mathbb{C}_{\alpha,\beta}\left(\Gamma^\alpha_{a+3}-\mathbf{E}_\alpha\right)\times\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\beta_{a+3}+\frac{l_1l_2}{8l_\alpha}\mathbb{C}_{\alpha,\beta}\left(\Gamma^\beta_{a+3}-\mathbf{E}_\beta\right)\times\Lambda^T_{a+3}\boldsymbol{\Delta}\mathbf{x}^\alpha_{a+3}\\
&\quad+\frac{l_1l_2}{4l_\beta}\left(\left((\operatorname{cay}(\widehat{\psi}^\beta_{a+3})+I)^{-1}(\widehat{\mathbb{D}_{\alpha,\beta}\Omega^\alpha_{a+3}})(2I-\widehat{\psi}^\beta_{a+3})\operatorname{cay}(\widehat{\psi}^\beta_{a+3})\right)^{(A)}\right)^\vee\\
&\quad+\frac{l_1l_2}{4l_\alpha}\left(\left((\operatorname{cay}(\widehat{\psi}^\alpha_{a+3})+I)^{-1}(\widehat{\mathbb{D}_{\alpha,\beta}\Omega^\beta_{a+3}})(2I-\widehat{\psi}^\alpha_{a+3})\operatorname{cay}(\widehat{\psi}^\alpha_{a+3})\right)^{(A)}\right)^\vee.
\end{aligned}
$$

Derivative $D_{\Lambda_{a+3}}\mathbb{V}_K$ is obtained by the following calculation :

$$
D_{\Lambda_{a+3}}\mathbb{V}_K=D_{\Lambda_{a+3}}\mathbb{U}_K^{a+1}+D_{\Lambda_{a+3}}\mathbb{U}_K^{a+2}+D_{\Lambda_{a+3}}\mathbb{U}_K^{a+3}.
$$



Derivative $D_{\mathbf{x}_a}\mathbb{V}_K$ in a on the left of $a+1$ and $a+2$:

$$\begin{aligned}
&D_{\mathbf{x}_a}\mathbb{V}_K \cdot \delta\mathbf{x}_a \\
&= \frac{l_1 l_2}{4 l_1}\left(\Gamma^1_a - \mathbf{E}_1\right)^T \mathbb{C}_{1,1}(-\Lambda^T_a \delta\mathbf{x}_a) + \frac{l_1 l_2}{4 l_1}\left(\Gamma^1_{a+1} - \mathbf{E}_1\right)^T \mathbb{C}_{1,1}(-\Lambda^T_{a+1} \delta\mathbf{x}_a) \\
&\quad + \frac{l_1 l_2}{4 l_2}\left(\Gamma^2_a - \mathbf{E}_2\right)^T \mathbb{C}_{2,2}(-\Lambda^T_a \delta\mathbf{x}_a) + \frac{l_1 l_2}{4 l_2}\left(\Gamma^2_{a+2} - \mathbf{E}_2\right)^T \mathbb{C}_{2,2}(-\Lambda^T_{a+2} \delta\mathbf{x}_a) \\
&\quad + \frac{l_1 l_2}{4 l_1}\left(\Gamma^2_a - \mathbf{E}_2\right)^T \mathbb{C}_{2,1}(-\Lambda^T_a \delta\mathbf{x}_a) + \frac{l_1 l_2}{4 l_1}\left(\Gamma^2_{a+1} - \mathbf{E}_2\right)^T \mathbb{C}_{2,1}(-\Lambda^T_{a+1} \delta\mathbf{x}_a) \\
&\quad + \frac{l_1 l_2}{4 l_2}\left(\Gamma^1_a - \mathbf{E}_1\right)^T \mathbb{C}_{1,2}(-\Lambda^T_a \delta\mathbf{x}_a) + \frac{l_1 l_2}{4 l_2}\left(\Gamma^1_{a+2} - \mathbf{E}_1\right)^T \mathbb{C}_{1,2}(-\Lambda^T_{a+2} \delta\mathbf{x}_a) \\
&\quad + \frac{l_1 l_2}{4}\left\langle \mathbf{q}_a, \delta\mathbf{x}_a \right\rangle .
\end{aligned}$$

Derivative $D_{\mathbf{x}_{a+3}}\mathbb{V}_K$ in $a+3$ on the right of $a+1$ and $a+2$:

$$\begin{aligned}
&D_{\mathbf{x}_{a+3}}\mathbb{V}_K \cdot \delta\mathbf{x}_{a+3} \\
&= \frac{l_1 l_2}{4 l_1}\left(\Gamma^1_{a+2} - \mathbf{E}_1\right)^T \mathbb{C}_{1,1}(\Lambda^T_{a+2} \delta\mathbf{x}_{a+3}) + \frac{l_1 l_2}{4 l_1}\left(\Gamma^1_{a+3} - \mathbf{E}_1\right)^T \mathbb{C}_{1,1}(\Lambda^T_{a+3} \delta\mathbf{x}_{a+3}) \\
&\quad + \frac{l_1 l_2}{4 l_2}\left(\Gamma^2_{a+1} - \mathbf{E}_2\right)^T \mathbb{C}_{2,2}(\Lambda^T_{a+1} \delta\mathbf{x}_{a+3}) + \frac{l_1 l_2}{4 l_2}\left(\Gamma^2_{a+3} - \mathbf{E}_2\right)^T \mathbb{C}_{2,2}(\Lambda^T_{a+3} \delta\mathbf{x}_{a+3}) \\
&\quad + \frac{l_1 l_2}{4 l_1}\left(\Gamma^2_{a+3} - \mathbf{E}_2\right)^T \mathbb{C}_{2,1}(\Lambda^T_{a+3} \delta\mathbf{x}_{a+3}) + \frac{l_1 l_2}{4 l_1}\left(\Gamma^2_{a+2} - \mathbf{E}_2\right)^T \mathbb{C}_{2,1}(\Lambda^T_{a+2} \delta\mathbf{x}_{a+3}) \\
&\quad + \frac{l_1 l_2}{4 l_2}\left(\Gamma^1_{a+3} - \mathbf{E}_1\right)^T \mathbb{C}_{1,2}(\Lambda^T_{a+3} \delta\mathbf{x}_{a+3}) + \frac{l_1 l_2}{4 l_2}\left(\Gamma^1_{a+1} - \mathbf{E}_1\right)^T \mathbb{C}_{1,2}(\Lambda^T_{a+1} \delta\mathbf{x}_{a+3}) \\
&\quad + \frac{l_1 l_2}{4}\left\langle \mathbf{q}_{a+3}, \delta\mathbf{x}_{a+3} \right\rangle .
\end{aligned}$$

Derivative $D_{\mathbf{x}_{a+1}}\mathbb{V}_K$ in $a+1$ on the left of $a+3$ and on the right of a :

$$\begin{aligned}
&D_{\mathbf{x}_{a+2}}\mathbb{V}_K \cdot \delta\mathbf{x}_{a+2} \\
&= \frac{l_1 l_2}{4 l_1}\left(\Gamma^1_{a+2} - \mathbf{E}_1\right)^T \mathbb{C}_{1,1}(-\Lambda^T_{a+2} \delta\mathbf{x}_{a+2}) + \frac{l_1 l_2}{4 l_1}\left(\Gamma^1_{a+3} - \mathbf{E}_1\right)^T \mathbb{C}_{1,1}(-\Lambda^T_{a+3} \delta\mathbf{x}_{a+2}) \\
&\quad + \frac{l_1 l_2}{4 l_2}\left(\Gamma^2_{a+2} - \mathbf{E}_2\right)^T \mathbb{C}_{2,2}(\Lambda^T_{a+2} \delta\mathbf{x}_{a+2}) + \frac{l_1 l_2}{4 l_2}\left(\Gamma^2_a - \mathbf{E}_2\right)^T \mathbb{C}_{2,2}(\Lambda^T_a \delta\mathbf{x}_{a+2}) \\
&\quad + \frac{l_1 l_2}{4 l_1}\left(\Gamma^2_{a+2} - \mathbf{E}_2\right)^T \mathbb{C}_{2,1}(-\Lambda^T_{a+2} \delta\mathbf{x}_{a+2}) + \frac{l_1 l_2}{4 l_1}\left(\Gamma^2_{a+3} - \mathbf{E}_2\right)^T \mathbb{C}_{2,1}(-\Lambda^T_{a+3} \delta\mathbf{x}_{a+2}) \\
&\quad + \frac{l_1 l_2}{4 l_2}\left(\Gamma^1_{a+2} - \mathbf{E}_1\right)^T \mathbb{C}_{1,2}(\Lambda^T_{a+2} \delta\mathbf{x}_{a+2}) + \frac{l_1 l_2}{4 l_2}\left(\Gamma^1_a - \mathbf{E}_1\right)^T \mathbb{C}_{1,2}(\Lambda^T_a \delta\mathbf{x}_{a+2}) \\
&\quad + \frac{l_1 l_2}{4}\left\langle \mathbf{q}_{a+2}, \delta\mathbf{x}_{a+2} \right\rangle .
\end{aligned}$$

Derivative $D_{\mathbf{x}_{a+2}}\mathbb{V}_K$ in $a+2$ on the left of $a+3$ and on the right of a :

$$
\begin{aligned}
&D_{\mathbf{x}_{a+1}}\mathbb{V}_K \cdot \delta\mathbf{x}_{a+1}\\
&= \frac{l_1 l_2}{4l_1}\left(\Gamma^1_{a+1} - \mathbf{E}_1\right)^T \mathbb{C}_{1,1}(\Lambda^T_{a+1}\delta\mathbf{x}_{a+1}) + \frac{l_1 l_2}{4l_1}\left(\Gamma^1_{a} - \mathbf{E}_1\right)^T \mathbb{C}_{1,1}(\Lambda^T_{a}\delta\mathbf{x}_{a+1})\\
&\quad + \frac{l_1 l_2}{4l_2}\left(\Gamma^2_{a+1} - \mathbf{E}_2\right)^T \mathbb{C}_{2,2}(-\Lambda^T_{a+1}\delta\mathbf{x}_{a+1}) + \frac{l_1 l_2}{4l_2}\left(\Gamma^2_{a+3} - \mathbf{E}_2\right)^T \mathbb{C}_{2,2}(-\Lambda^T_{a+3}\delta\mathbf{x}_{a+1})\\
&\quad + \frac{l_1 l_2}{4l_1}\left(\Gamma^2_{a+1} - \mathbf{E}_2\right)^T \mathbb{C}_{2,1}(\Lambda^T_{a+1}\delta\mathbf{x}_{a+1}) + \frac{l_1 l_2}{4l_1}\left(\Gamma^2_{a} - \mathbf{E}_2\right)^T \mathbb{C}_{2,1}(\Lambda^T_{a}\delta\mathbf{x}_{a+1})\\
&\quad + \frac{l_1 l_2}{4l_2}\left(\Gamma^1_{a+1} - \mathbf{E}_1\right)^T \mathbb{C}_{1,2}(-\Lambda^T_{a+1}\delta\mathbf{x}_{a+1}) + \frac{l_1 l_2}{4l_2}\left(\Gamma^1_{a+3} - \mathbf{E}_1\right)^T \mathbb{C}_{1,2}(-\Lambda^T_{a+3}\delta\mathbf{x}_{a+1})\\
&\quad + \frac{l_1 l_2}{4}\left\langle \mathbf{q}_{a+1}, \delta\mathbf{x}_{a+1}\right\rangle .
\end{aligned}
$$

7.5 Conclusions

We obtained a constrained Lie algebra variational integrator for the geometrically exact model of plates. Moreover, another integrator for plates is in progress; it will be obtained with the gained experience from the study of variational integrators for beams. Then we will start to implement and compare the different integrators for plates.

Chapter 8

Dissipation and discrete affine Euler-Poincaré

Introduction

This chapter studies the dissipation added to the reduced discrete affine Euler-Poincaré system with symmetry.

Phenomenons of dissipation and instability for Euler-Poincaré systems on the Lie algebra, or equivalently for Lie-Poisson systems on the duals of the Lie algebras, were studied in Bloch, Krishnaprasad, Marsden, and Ratiu [10]. This paper is a reference for our work. Thus a dissipative force is construct which dissipates the energy, but angular momentum is conserved, or equivalently symmetries are conserved. In the context of the Lie-Poisson systems, this means that the coadjoint orbits remain invariant.

In view of the objective that we have set to find the equilibrium position of a structure, it is essential to preserve symmetries when applying dissipation.

Dampers in satellites act this way. That is, once a structure is deployed in space, it is subject to vibrations due to guidance systems, space debris. Then the damping mechanisms must remove the vibrations without modify the angular momentum maps.

Energy-Dissipative Momentum-Conserving or EDMC algorithms, verifying the laws in the non-linear range, were recently developed by several authors (see, e.g., Armero and Romero [4]), where conditions are imposed on the algorithm in order to conserve the momentum (such as the mid-point scheme).

Our point of view is different as there are no conditions on the discrete Lagrangian. Then we take into account the discrete theory, established by the use of the laws of mechanics.

8.1 Review of continuous Euler-Poincaré systems with forces

8.1.1 Forced Euler-Lagrange equations and momentum map conservation

A Lagrangian force is a fiber preserving map $F : TQ \to T^*Q$ over the identity. Given such a force, it is standard to modify Hamilton's principle, seeking stationary points of the action, to the Lagrange-d'Alembert principle, which seeks curve $q(t)$ satisfying

$$\delta \int_0^T L(q(t), \dot{q}(t))dt + \int_0^T F(q(t), \dot{q}(t)) \cdot \delta q(t)dt = 0. \tag{8.1.1}$$

This is equivalent to the forced Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F(q, \dot{q}).$$

Given the action of a Lie group G on configuration manifold Q by $G \times Q \to Q$, we define the Lagrangian momentum map $\mathbf{J}_L : TQ \to \mathfrak{g}^*$ to be

$$\langle \mathbf{J}_L(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle \,,$$

where ξ_Q is the infinitesimal generator and $\mathbb{F}L : TQ \to T^*Q$ is the fiber derivative.

Let L be a G-invariant Lagrangian. Evaluating (8.1.1) for a variation of the form $\delta q = \xi_Q(q)$, gives

$$\begin{aligned}
&\int_0^T dL \cdot \xi_{TQ} dt + \int_0^T F \cdot \xi_Q dt \\
&= \int_0^T F(q, \dot{q}) \cdot \xi_Q(q) dt \\
&= \int_0^T \left[\frac{\partial L}{\partial q}(q, \dot{q}) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q, \dot{q}) + F(q, \dot{q})(q, \dot{q}) \right] + \Theta_L \cdot \xi_{TQ}\Big|_0^T \\
&= \left[(\mathbf{J}_L \circ F_L^T)(q(0), \dot{q}(0)) - \mathbf{J}_L(q(0), \dot{q}(0)) \right] \cdot \xi,
\end{aligned}$$

where $F_L^T : TQ \to TQ$ is the Lagrangian flow at the frozen time T, and where we took into account the fact that L is invariant along ξ_Q, and the definition of the Lagrangian momentum $\mathbf{J}_L$.

Thus if we consider the effects of forcing on the evolution of momentum maps, we observe that when the forcing is orthogonal to the group action the momentum map is conserved. This is the content of the next theorem.

8.1.1 Theorem (Forced Noether's theorem) *Consider an action of the Lie group G on the manifold Q. Consider a Lagrangian system $L : TQ \to \mathbb{R}$ with* *forcing $F : TQ \to T^*Q$ such that L is G-invariant. Let $q(t)$ be a solution of the Euler-Lagrange equations with force. Then for all $\xi \in \mathfrak{g}$*

$$\frac{d}{dt}\left\langle \mathbf{J}_L(q(t), \dot{q}(t)), \xi \right\rangle = \left\langle F(q(t), \dot{q}(t)), \xi_Q(q(t)) \right\rangle . \tag{8.1.2}$$

In particular, if $\langle F(v_q), \xi_Q(q)\rangle = 0$ for all $v_q \in TQ$ and all $\xi \in \mathfrak{g}$. Then the Lagrangian momentum map $\mathbf{J}_L : TQ \to \mathfrak{g}^*$ will be preserved by the flow F_L^t of the Euler-Lagrange equations with force F, so that $\mathbf{J}_L \circ F_L^t = \mathbf{J}_L$ for all t.

As seen in (1.4.3) we recall an equivalent result for discrete mechanics, where the condition to conserve discrete momentum map $J_{L_d} : Q \times Q \to \mathfrak{g}^*$ is that the discrete force $F_d : Q \times Q \to T^*Q$ and the discrete infinitesimal generator $\xi_{Q\times Q} : Q \times Q \to T(Q \times Q)$ verify $\langle F_d, \xi_{Q\times Q}\rangle = 0$.

Note that the forcing F is not required to be G-equivariant. However, in the following theorem from Bloch, Krishnaprasad, Marsden, and Ratiu [10] we recall an important particular class of equivariant force fields.

8.1.2 Theorem *Consider a G-invariant Lagrangian L and a G-equivariant force field F. The solutions of the Euler-Lagrange equation with force preserve the inverse images of the coadjoint orbits in $\mathfrak{g}^*$ by the Lagrangian momentum map $\mathbf{J}_L$ if and only if for each $v_q \in TQ$, there is some $\eta(v_q) \in \mathfrak{g}$ such that*

$$\langle F(v_q), \xi_Q(q)\rangle = \langle \mathbf{J}_L(v_q), [\eta(v_q), \xi]\rangle \tag{8.1.3}$$

for all $\xi \in \mathfrak{g}$.

This result is the consequence of a number of properties, namely : the coadjoint orbits have a symplectic structure. Moreover, the Lie-Poisson bracket and the coadjoint orbit symplectic structure are consistent. Thus, for a given coadjoint orbit $\mathcal{O}$, if $v(t) \in \mathbf{J}_L^{-1}(\mathcal{O})$ or equivalently if $\mathbf{J}(v(t)) \in \mathcal{O}$, then $\mathbf{J}(v(t))$ verify Lie Poisson equations

$$\frac{d\mathbf{J}(v(t))}{dt} = \mathrm{ad}^*_{\eta(t)}\mathbf{J}(v(t)).$$

Given the relation (8.1.2) in the Forced Noether's theorem, we obtain the condition (8.1.3).

8.1.2 Euler-Poincaré reduction with forces

The Euler-Poincaré equation may be found in many papers, and books as in [89]. We note that Lie-Poisson and Euler-Poincaré equations are equivalent if the fiber derivative of Lagrangian L is a diffeomorphism from TG to T^*G. Now we recall from Bloch, Krishnaprasad, Marsden, and Ratiu [10] the Euler-Poincaré reduction with forcing.

8.1.3 Theorem *Let G be a Lie group. $L : TG \to \mathbb{R}$ a left invariant Lagrangian, and let $F : TG \to T^*G$ be an equivariant Lagrangian force relative to the canonical left actions of G on TG and T^*G, respectively. Let $\ell : \mathfrak{g} \to \mathbb{R}$ and $\mathbf{f} : \mathfrak{g} \to \mathfrak{g}^*$ be the restriction of L and F to $T_eG = \mathfrak{g}$. For a curve $g(t) \in G$, consider the curve $\xi(t) = T_{g(t)}L_{g(t)^{-1}}\dot{g}(t) \in \mathfrak{g}$. Then the following statements are equivalent :*

(i) *$g(t)$ satisfies the Euler-Lagrange equations with forcing for L on G.*

(ii) *The integral Lagrange-d'Alembert principle*

$$\delta \int_a^b L(g(t), \dot{g}(t))dt + \int_a^b F(g(t), \dot{g}(t)) \cdot \delta g(t) dt = 0$$

holds for all variations $\delta g(t)$ with fixed endpoints.

(iii) *The Euler-Poincaré equations with forcing hold*

$$\frac{d}{dt}\frac{\delta \ell}{\delta \xi} - \mathrm{ad}_\xi^* \frac{\delta \ell}{\delta \xi} = \mathbf{f}(\xi). \tag{8.1.4}$$

(iv) *The variational principle*

$$\delta \int_a^b \ell(\xi(t))dt + \int_a^b \mathbf{f}(\xi(t)) \cdot \eta(t) dt = 0$$

holds on $\mathfrak{g}$, using variations of the form

$$\delta \xi = \dot{\eta} + [\xi, \eta],$$

where η vanishes at the endpoints.

8.1.4 Remark From Theorem 8.1.1, it follows that in the particular case where $Q = G$, with forcing $F : TG \to T^*G$, the momentum map is preserved by the flow of the forced Euler-Legrange equations if and only if $F = 0$.

In the particular case where $Q = G$, with a G-equivariant force field $F : TG \to T^*G$, and $\mathbf{f} : \mathfrak{g} \to \mathfrak{g}^*$ its restriction to $T_eG = \mathfrak{g}$. Then Theorem 8.1.2 yields the following condition in order to preserve the momentum.

8.1.5 Corollary *The solutions of the Euler-Poincaré equations with forcing* (8.1.4) *preserve the coadjoint orbits of $\mathfrak{g}^*$, provided the force field $\mathbf{f}$ is given by*

$$\mathbf{f}(\zeta) = \mathrm{ad}_{\eta(\zeta)}^* \frac{\delta \ell}{\delta \zeta}. \tag{8.1.5}$$

for some smooth map $\eta : \mathfrak{g} \to \mathfrak{g}$.

Indeed, the condition that the integral curves preserves the coadjoint orbits of $\mathfrak{g}^*$ is given by (8.1.3). Since the infinitesimal generator for a left Lie group action is $\xi_G(g) = \xi g$, and, knowing that the Legendre transform $\mathbb{F}L$ of a G-invariant Lagrangian L is G-equivariant, we get the condition (8.1.5).

8.1.3 Euler-Poincaré reduction for semi-direct products with equivariant forces

The main difference between the left invariant Lagrangian considered in the theorem above and the ones we shall work with below is that functions L and ℓ depend on another parameter $a \in V^*$, where V is a representation space for the Lie group G and L has an invariance property relative to both arguments. Note that L is not a Lagrangian function as it is not defined on a tangent bundle.

Recall that the semi-direct product $S = G \circledS V$, as previously defined, is associated to a representation ρ of the Lie group G on the vector space V, and that we consider the left action of G on $TG \times V^*$ given by $g \cdot (v_h, a) = (T_h L_g v_h, \rho_g^*(a)) = (gv_h, ga)$.

We assume that the function $L : TG \times V^* \to \mathbb{R}$ is left G-invariant, then we define a reduced function $\ell : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$\ell(g^{-1}v_g, g^{-1}a) = L(v_g, a).$$

For a particular $a_0 \in V^*$ we define the Lagrangian $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) = L(v_g, a_0)$, then L_{a_0} is left invariant under the lift to TG of the left action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.

For a curve $g(t) \in G$, let $\xi(t) := g(t)^{-1}\dot{g}(t)$ and define the curve $a(t)$ as the unique solution of the following linear differential equation with time-dependent coefficients $\dot{a}(t) = -a(t)\xi(t)$, with initial condition $a(0) = a_0$. The solution can be written as $a(t) = \rho_{g(t)}^*(a_0) = g(t)^{-1}a_0$.

Given an equivariant Lagrangian force $F : TG \to T^*G$, we consider its restriction $\mathbf{f} : \mathfrak{g} \to \mathfrak{g}^*$ to $\mathfrak{g}$. Then Euler-Poincaré reduction theorem for semidirect products generalizes to the case with forcing as follows.

8.1.6 Theorem *With the preceding notations the following are equivalent :*

(i) *With a_0 held fixed, $g(t)$ satisfies the Euler-Lagrange equations with forcing for L_{a_0} on G.*

(ii) *The integral Lagrange-d'Alembert principle*

$$\delta \int_a^b L_{a_0}(g(t), \dot{g}(t))dt + \int_a^b F(g(t), \dot{g}(t)) \cdot \delta g(t)dt = 0$$

holds for all variations $\delta g(t)$ with fixed endpoints.

(iii) *The Euler-Poincaré equations with forcing hold on $\mathfrak{g} \times V^*$*

$$\frac{d}{dt}\frac{\delta \ell}{\delta \xi} - \mathrm{ad}_\xi^* \frac{\delta \ell}{\delta \xi} - \frac{\delta \ell}{\delta a} \diamond a = \mathbf{f}. \tag{8.1.6}$$

(iv) *The variational principle*

$$\delta \int_a^b \ell(\xi(t), a(t))dt + \int_a^b \mathbf{f}(\xi(t)) \cdot \eta(t)dt = 0$$

holds on $\mathfrak{g} \times V^*$*, using variations of* ξ *and* a *of the form*

$$\delta\xi = \dot{\eta} + [\xi, \eta], \quad \delta a = -\eta a,$$

where $\eta \in \mathfrak{g}$ *vanishes at the endpoints.*

Proof. The equivalence of (**i**) and (**ii**) holds according to the previous one theorem.

Next we show the equivalence of (**iii**) and (**iv**).

$$\begin{aligned}
-\int_a^b \mathbf{f}(\xi(t)) \cdot \eta(t) dt &= \delta \int_a^b \ell(\xi(t), a(t)) dt \\
&= \int_a^b \left(\left\langle \frac{\delta \ell}{\delta \xi}, \delta \xi \right\rangle + \left\langle \frac{\delta \ell}{\delta a}, \delta a \right\rangle \right) dt \\
&= \int_a^b \left(\left\langle \frac{\delta \ell}{\delta \xi}, \dot{\eta} + \mathrm{ad}_\xi \eta \right\rangle - \left\langle \frac{\delta \ell}{\delta a}, \eta a \right\rangle \right) dt \\
&= \int_a^b \left\langle -\frac{d}{dt}\frac{\delta \ell}{\delta \xi} + \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} + \frac{\delta \ell}{\delta a} \diamond a, \eta \right\rangle dt
\end{aligned}$$

And we show that (**ii**) and (**iv**) are equivalent. Not that $L_{a_0}(g(t), \dot{g}(t))$ and $\ell(\xi(t), a(t))$ are equals, as $L : TG \times V^* \to \mathbb{R}$ is G-invariant, and $a(t) = g(t)^{-1} a_0$. All variation δg induced and are induced by variations $\delta\xi$, and the variation between δg and η is given by $\eta = g^{-1}\delta g$ vanishing at endpoints. Moreover force fields are equivariant relative to the canonical left actions so

$$\begin{aligned}
\int_a^b F\left(g(t), \dot{g}(t)\right) \cdot \delta g(t) dt &= \int_a^b g(t) \cdot F\left(g(t)^{-1} g(t), g(t)^{-1} \dot{g}(t)\right) \cdot \delta g(t) dt \\
&= \int_a^b F\left(g(t)^{-1} g(t), g(t)^{-1} \dot{g}(t)\right) \cdot \eta(t) dt \\
&= \int_a^b \mathbf{f}\left(\xi(t)\right) \cdot \eta(t) dt
\end{aligned}$$

where $\eta(t) = g(t)^{-1}\delta g(t)$. As a consequence we get the equivalence. ■

8.2 Affine Euler-Poincaré reduction with forces

8.2.1 Remark The preceding theorem easily generalizes to the case of affine actions. If we assume that the function $L : TG \times V^* \to \mathbb{R}$ is left G-invariant under the affine action as described in (5.1.6). Given an equivariant fiber preserving map $F : TG \times V^* \to T^*G$ over the identity, such that

$$F(hv_g, ha_{ref}) = hF(v_g, a_{ref}).$$

We get the same result as previously, except that one has to replace the Euler-Poincaré equations with forcing (8.1.6) by the equations

$$\frac{d}{dt}\frac{\delta \ell}{\delta \xi} - \mathrm{ad}^*_\xi \frac{\delta \ell}{\delta \xi} - \frac{\delta \ell}{\delta a} \diamond a + \mathbf{d}c^T \left(\frac{\delta \ell}{\delta a} \right) = \mathbf{f}, \tag{8.2.1}$$

where $\mathbf{f} : \mathfrak{g} \times V^* \to \mathfrak{g}^*$ is the restriction of F to $\mathfrak{g} \times V^*$, and that the variations of a are now given by $\delta a = -\eta a - \mathbf{d}c(\eta)$. Indeed, as the map F is equivariant, we see that

$$\begin{aligned}\int_a^b F\left(v_{g(t)}(t), a_{ref}\right) \cdot \delta g(t) dt &= \int_a^b F\left(g(t)^{-1} v_{g(t)}(t), \theta_{g(t)^{-1}}(a_{ref})\right) \cdot \eta(t) dt \\ &= \int_a^b \mathbf{f}\left(\xi(t), a(t)\right) \cdot \eta(t) dt.\end{aligned}$$

With a G-equivariant force field $F : TG \times V^* \to T^*G$, and $\mathbf{f} : \mathfrak{g} \times V^* \to \mathfrak{g}^*$ its restriction to $T_eG = \mathfrak{g}$. Given the isotropy group $G^c_{a_0}$ of a_0 as defined in (5.1.4) and its Lie algebra $\mathfrak{g}^c_{a_0}$, the Theorem 8.1.2 yields the following condition in order to preserve the momentum.

8.2.2 Corollary *With $a_0 \in V^*$ held fixed, the solutions of the Affine Euler-Poincaré equations with forcing* (8.2.1) *preserve the coadjoint orbits of* $(\mathfrak{g}^c_{a_0})^*$, *provided the force field $\mathbf{f} : \mathfrak{g}^c_{a_0} \times V^* \to (\mathfrak{g}^c_{a_0})^*$ is given by*

$$\mathbf{f}(\zeta) = \mathrm{ad}^*_{\eta(\zeta)} \frac{\delta \ell}{\delta \zeta}. \tag{8.2.2}$$

for some smooth map $\eta : \mathfrak{g}^c_{a_0} \to \mathfrak{g}^c_{a_0}$, and $\ell : \mathfrak{g}^c_{a_0} \times V^* \to \mathbb{R}$.

Proof. With $a_0 \in V^*$ held fixed, consider a $G^c_{a_0}$-invariant Lagrangian L_{a_0} and a G-equivariant force field $F_{a_0}(v_g) = F(v_g, a_0)$. Thus by theorem (8.1.2) the solution of the Euler-Lagrange equation with force preserve the inverse image of the coadjoint orbits in $(\mathfrak{g}^c_{a_0})^*$, if and only if for each $v_g \in TG$, there is a map $\eta(v_g) \in \mathfrak{g}^c_{a_0}$ such that

$$\langle F_{a_0}(v_g), \xi_G(g) \rangle = \left\langle \mathbf{J}_{L_{a_0}}(v_g), [\eta(v_g), \xi] \right\rangle,$$

for all $\xi \in \mathfrak{g}^c_{a_0}$. Since $\xi_G(g) = \xi g$ and $\mathbf{J}_{L_{a_0}}(v_g) = \mathbb{F}L_{a_0}(v_g) g^{-1}$ we get

$$\begin{aligned}\langle F_{a_0}(v_g), \xi g \rangle &= \left\langle F_{a_0}(v_g) g^{-1}, \xi \right\rangle, \qquad \text{and} \\ \left\langle \mathbf{J}_{L_{a_0}}(v_g), [\eta(v_g), \xi] \right\rangle &= \left\langle \mathrm{ad}^*_{\eta(v_g)} \mathbb{F}L_{a_0}(v_g) g^{-1}, \xi \right\rangle.\end{aligned}$$

By G-equivariance of F_{a_0} we obtain

$$\left\langle F_{a_0}(v_g) g^{-1}, \xi \right\rangle = \left\langle \mathrm{Ad}^*_{g^{-1}} F(g^{-1} v_g), \xi \right\rangle. \tag{1}$$

As L_{a_0} is only $G^c_{a_0}$-invariant, then $\mathbb{F}L_{a_0}$ is $G^c_{a_0}$-equivariant. As a consequence, from now on, we choose $g \in G^c_{a_0}$, and we get

$$\mathrm{ad}^*_{\eta(v_g)} \circ \mathbb{F}L_{a_0}(v_g) \cdot g^{-1} = \mathrm{ad}^*_{\eta(v_g)} \circ \mathrm{Ad}^*_{g^{-1}} \circ \mathbb{F}L_{a_0}(g^{-1} v_g).$$

Moreover, as $\mathrm{Ad}_{g^{-1}} \circ \mathrm{ad}_{\eta(v_g)} = \mathrm{ad}_{\mathrm{Ad}_{g^{-1}} \eta(v_g)} \circ \mathrm{Ad}_{g^{-1}}$, we obtain

$$\left\langle \mathbf{J}_{L_{a_0}}(v_g), [\eta(v_g), \xi] \right\rangle = \left\langle \mathrm{Ad}^*_{g^{-1}} \circ \mathrm{ad}_{\mathrm{Ad}_{g^{-1}} \eta(v_g)} \circ \mathbb{F}L_{a_0}(g^{-1} v_g), \xi \right\rangle. \tag{2}$$

The equation (8.1.3) is equivalent to (1) = (2). Thus we obtain

$$\mathbf{f}_{a_0}(\zeta) = \mathrm{ad}^*_{\mathrm{Ad}_{g^{-1}}\eta(v_g)}\mathbb{F}L_{a_0}(\zeta),$$

with $\zeta = g^{-1}v_g \in \mathfrak{g}^c_{a_0}$, $\eta(v_g)$, $\xi \in \mathfrak{g}^c_{a_0}$, and $g \in G^c_{a_0}$. By taking $g = e$ we get (8.2.2). ■

8.3 Reduction of discrete forced Lagrangian systems with symmetries

8.3.1 Discrete affine Euler-Poincaré reduction with forces

In this subsection we generalize the discrete affine Euler-Poincaré reduction for semidirect products (§5.3.2) to the case with forcing, with $Q = G$. We will assume that the discrete forces $F_d^{\pm} : G \times G \times V^* \to T^*G$ are G-equivariant, that is, for a fixed $a_{ref} \in V^*$, they read

$$F_d^{\pm}(gg^j, gg^{j+1}, \theta_g(a_{ref})) = gF_d^{\pm}(g^j, g^{j+1}, a_{ref}), \quad \text{for all } g \in G.$$

This allows us to define the reduced discrete forces $\mathbf{f}_d^{\pm} : G \times V^* \to \mathfrak{g}^*$ by

$$\begin{aligned} \mathbf{f}_d^-(f^j, a^j) &:= F_d^-\left(e, (g^j)^{-1}g^{j+1}, \theta_{(g^j)^{-1}}(a_{ref})\right) \\ \mathbf{f}_d^+((f^j)^{-1}, a^{j+1}) &:= F_d^-\left((g^{j+1})^{-1}g^j, e, \theta_{(g^{j+1})^{-1}}(a_{ref})\right). \end{aligned} \tag{8.3.1}$$

The geometric setting is the same as in §5.3.2, namely, we suppose that the discrete function $L_d : G \times G \times V^* \to \mathbb{R}$ is G-invariant under the action

$$h \cdot (g^j, g^{j+1}, a) = (hg^j, hg^{j+1}, \theta_h(a)), \quad h, g^j, g^{j+1} \in G,\ a \in V^*,$$

and, for a particular $a_{ref} \in V^*$, we define the discrete reduced function $\ell_d : G \times V^* \to \mathbb{R}$ by

$$\ell_d\left(f^j, a^j\right) = \ell_d\left((g^j)^{-1}g^{j+1}, \theta_{(g^j)^{-1}}(a_{ref})\right) = L_d(g^j, g^{j+1}, a_{ref}).$$

And we define $L_{d,a_{ref}} : G \times G \to \mathbb{R}$, by $L_{d,a_{ref}}(g^j, g^{j+1}) = L_d(g^j, g^{j+1}, a_{ref})$. Then $L_{d,a_{ref}}$ is invariant under the left action of isotropy subgroup $G^c_{a_{ref}}$ on G. As well $F_{d,a_{ref}} : G \times G \to T^*G$, by $F_{d,a_{ref}}(g^j, g^{j+1}) = F_d(g^j, g^{j+1}, a_{ref})$.

With the same notations as in Theorem 5.3.1, its generalization to the case with forces is the following.

8.3.1 Theorem (Discrete affine Euler-Poincaré reduction with force) *The following are equivalent :*

(i) *With $a_{ref} \in V^*$ held fixed, the discrete Lagrange d'Alembert principle*

$$\delta \sum_{j=0}^{N-1} L_{d,a_{ref}}(g^j, g^{j+1}) + \sum_{j=0}^{N-1}\left[F^+_{d,a_{ref}}(g^j, g^{j+1}) \cdot \delta g^{j+1} + F^-_{d,a_{ref}}(g^j, g^{j+1}) \cdot \delta g^j\right] = 0, \tag{8.3.2}$$

holds, for variations δg^j with $\delta g^0 = \delta g^N = 0$.

(ii) *The discrete path $\{g^j\}_{j=0}^N$ satisfies the discrete Euler-Lagrange equations with forcing*

$$\begin{aligned} &D_2 L_{d,a_{ref}}(g^{j-1}, g^j) + D_1 L_{d,a_{ref}}(g^j, g^{j+1}) \\ &\quad + F^+_{d,a_{ref}}(g^{j-1}, g^j) + F^-_{d,a_{ref}}(g^j, g^{j+1}) = 0, \qquad \textit{for all } j = 1, ..., N-1. \end{aligned}$$

(iii) *The constrained discrete variational principle*

$$\delta \sum_{j=0}^{N-1} \ell_d(f^j, a^j) + \sum_{j=0}^{N-1} \left[\mathbf{f}^+_d((f^j)^{-1}, a^{j+1}) \cdot \eta^{j+1} + \mathbf{f}^-_d(f^j, a^j) \cdot \eta^j \right] = 0, \tag{8.3.3}$$

holds on $G \times V^*$, using variations of f^j and a^j of the form

$$\delta f^j = T_e L_{f^j} \left(-\operatorname{Ad}_{(f^j)^{-1}}(\eta^j) + \eta^{j+1} \right), \qquad \delta a^j = -\eta^j a^j - \mathbf{d}c(\eta^j),$$

where $\{\eta^j\}_{j=0}^N$ is a sequence in $\mathfrak{g}$ satisfying $\eta^0 = \eta^N = 0$.

(iv) *The discrete affine Euler-Poincaré with forcing are valid*

$$\begin{aligned} &-\operatorname{Ad}^*_{(f^j)^{-1}} T^*_e L_{f^j} D_{f^j} \ell^j_d + T^*_e L_{f^{j-1}} D_{f^{j-1}} \ell^{j-1}_d + D_{a^j} \ell^j_d \diamond a^j \\ &\quad - \mathbf{d}c^T \left(D_{a^j} \ell^j_d \right) + \mathbf{f}^+_d \left((f^{j-1})^{-1}, a^j \right) + \mathbf{f}^-_d (f^j, a^j) = 0. \end{aligned} \tag{8.3.4}$$

Proof. The equivalence of (**i**) and (**ii**) holds according to Theorem 1.4.1.

Now, using the result of Theorem 5.3.1 the equivalence of (**iii**) and (**iv**) is easy to show. We already know that

$$\begin{aligned} \delta \sum_{j=0}^{N-1} \ell_d(f^j, a^j) = \sum_{0}^{N-1} &\left\langle -\operatorname{Ad}^*_{(f^j)^{-1}} T^*_e L_{f^j} \left(D_{f^j} \ell^j_d \right), \eta^j \right\rangle + \left\langle T^*_e L_{f^j} \left(D_{f^j} \ell^j_d \right), \eta^{j+1} \right\rangle \\ &+ \left\langle D_{a^j} \ell^j_d \diamond a^j \ , \eta^j \right\rangle - \left\langle \mathbf{d}c^T \left(D_{a^j} \ell^j_d \right), \eta^j \right\rangle, \end{aligned}$$

for all variations $\{\eta^j\}_{j=0}^N$ vanishing at endpoints. So the constrained variational principle (8.3.3) is clearly equivalent to the forced discrete affine Euler-Poincaré equations

$$\begin{aligned} &-\operatorname{Ad}^*_{(f^j)^{-1}} T^*_e L_{f^j} \left(D_{f^j} \ell^j_d \right) + T^*_e L_{f^{j-1}} \left(D_{f^{j-1}} \ell^{j-1}_d \right) \\ &\quad + D_{a^j} \ell^j_d \diamond a^j - \mathbf{d}c^T \left(D_{a^j} \ell^j_d \right) + \mathbf{f}^+_d \left((f^{j-1})^{-1}, a^j \right) + \mathbf{f}^-_d (f^j, a^j) = 0. \end{aligned}$$

Now we consider the equivalence between (**i**) and (**iii**). By G-invariance the actions associated to $L_{d,a_{ref}}(g^j, g^{j+1})$ and $\ell_d(f^j, a^j)$ are equal. We already know that the variations δg^j induce and are induced by the constrained variations δf^j. It remains to be shown that the right hand side of (8.3.2) and (8.3.3) are equal. By equivariance of $F^\pm_d$ and (8.3.1), we have

$$\begin{aligned} F^-_d(g^j, g^{j+1}, a_{ref}) \cdot \delta g^j &= g^j \cdot F^-_d \left(e, f^j, \theta_{(g^j)^{-1}} a_{ref} \right) \cdot \delta g^j \\ &= \mathbf{f}^-_d \left(f^j, a^j \right) \cdot \eta^j \end{aligned}$$

and

$$\begin{aligned} F_d^+(g^j, g^{j+1}, a^j) \cdot \delta g^{j+1} &= g^{j+1} \cdot F_d^+\left((f^j)^{-1}, e, \theta_{(g^{j+1})^{-1}} a_{ref}\right) \cdot \delta g^{j+1} \\ &= \mathbf{f}_d^+\left((f^j)^{-1}, a^{j+1}\right) \cdot \eta^{j+1}. \end{aligned}$$

Then we get the result. ■

The discrete affine Euler-Poincaré equations with force (8.3.4) implicitly define the *forced discrete Lagrangian map*

$$F_{\ell_d} : G \times V^* \to G \times V^*, \quad (f^{j-1}, a^{j-1}) \mapsto (f^j, a^j).$$

Chapter 9

Discrete mechanical connection

Introduction

The mechanical connection originates from the works of Smale [113], Abraham and Marsden [1]. Then after, it was explicitly described by Kummer [62], Guichardet [37], Shapere and Wilczeck [105], Simo, Lewis, and Marsden [109], and Montgomery [92].

Principal connections, and in particular mechanical connections, are an important tool, which allows one to split the trajectories into a horizontal and a vertical part. The vertical equation gives the trajectories along the orbit associated to the action of a Lie group G, and the horizontal is perpendicular to that orbit. The first one is associated with the Euler-Poincaré equation, and the last one with the Euler-Lagrange equation. (See Cendra, Marsden, and Ratiu [22].)

Moreover mechanical connections allow to study the stability and bifurcation of relative equilibria. Where the relative equilibria are the dynamic orbits generated by the symmetry group, which correspond to equilibrium points in the quotient space. When stability of a relative equilibrium is lost, one can get bifurcation, instability and chaos. (See Hernández-Garduno, and Marsden [43].)

Furthermore, mechanical connections play an important role in the energy-momentum method. (See Lewis, and Simo [72].)

In discrete mechanics, a nice theory of discrete connections was established by Leok, Marsden, and Weinstein [67] through the pair groupoid composition. Unfortunately, it seems that this theory does not provide expressions that can be directly applied to concrete problems.

In this chapter, we obtain definitions and expressions in coordinates of the discrete mechanical connection, as well as of the discrete vertical and horizontal trajectories which are reminiscent of the continuous expressions.

9.1 Discrete Euler-Lagrange equations

9.1.1 Euler-Lagrange variational operator

We consider a smooth manifold Q which describes the configuration of the system under study. Let $\Omega(Q; q^1, q^2)$ be the set of all smooth curves $q : I \to Q$ satisfying $q(t^1) = q^1$ and $q(t^2) = q^2$ for given $q^1, q^2 \in Q$. The ***state*** or ***velocity phase space*** is the tangent bundle TQ of the manifold Q; we shall use interchangeably the notations v_q and $(q, \dot{q})$ for the tangent vectors at $q \in Q$. The ***phase space*** is its dual T^*Q, the cotangent bundle of Q, whose elements are denoted by α_q or (q, p).

Two smooth curves $q_1, q_2 : (-\varepsilon, \varepsilon) \to Q$ are said to be ***second order equivalent*** at $q \in Q$, if $q_1(0) = q_2(0) = q$, $\left.\frac{d}{dt}\right|_{t=0} q_1(t) = \left.\frac{d}{dt}\right|_{t=0} q_2(t)$, and in a chart (U, φ) with $q \in U$ we have, in addition, $\left.\frac{d^2}{dt^2}\right|_{t=0} (\varphi \circ q_1)(t) = \left.\frac{d^2}{dt^2}\right|_{t=0} (\varphi \circ q_2)(t)$. The set of these equivalence classes, denoted by $[q(t)]^{(2)}$ or by triples $(q, \dot{q}, \ddot{q})$, form a locally trivial fiber bundle $T^{(2)}Q = \ddot{Q} \to Q$, called the ***second order tangent bundle*** of Q. In classical notation of jet spaces, $T^{(2)}Q = J^2_0(\mathbb{R}, Q)$, i.e., the second order jets of maps from $\mathbb{R}$ to Q whose source is fixed at the origin $0 \in \mathbb{R}$. Note that $T^{(2)}Q$ is a submanifold of the double tangent bundle TTQ.

Given another manifold R and smooth map $f : Q \to R$, there is a naturally induced smooth locally trivial fiber bundle map $T^{(2)}f : T^{(2)}Q \to T^{(2)}R$ given by $T^{(2)}f(q, \dot{q}, \ddot{q}) := (r, \dot{r}, \ddot{r})$, where $(r, \dot{r}, \ddot{r})$ is the equivalence class of the curve $f(q(t))$ in R if the equivalence class of the curve $q(t)$ in Q is $(q, \dot{q}, \ddot{q})$. In the alternative notation this is easier to write, namely, $T^{(2)}f\left([q]^{(2)}\right) := [(f \circ q)(t)]^{(2)}$.

We recall the classical result linking the Euler-Lagrange equations to the calculus of variations.

9.1.1 Theorem *Let $L : TQ \to \mathbb{R}$ be a smooth function, called from now on, the* ***Lagrangian****. The* ***action*** *of L is defined by*

$$\mathfrak{S}(L)(q(\cdot)) = \int_{t^1}^{t^2} L(q(t), \dot{q}(t))dt$$

for all curves $q(\cdot) \in \Omega(Q; q^1, q^2)$. Let $q(t, \lambda)$ be a deformation of a curve $q(t)$ fixing the endpoints, i.e., $q(t^1, \lambda) = q(t^1)$ and $q(t^2, \lambda) = q(t^2)$ for all $\lambda \in (-\varepsilon, \varepsilon)$ and $q(t, 0) = q(t)$. Let

$$\delta q(t) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} q(t, \lambda) \in T_{q(t)}Q$$

be the corresponding variation. Since the endpoints are fixed, we have $\delta q(t^1) = 0$, $\delta q(t^2) = 0$.

There is a unique bundle map

$$\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$$

such that, for any deformation $q(t, \lambda)$, keeping the endpoints fixed, we have

$$\mathbf{d}\mathfrak{S}(L)(q(\cdot)) \cdot \delta q(\cdot) = \int_{t^1}^{t^2} \mathcal{EL}(L)(q(t), \dot{q}(t), \ddot{q}(t)) \cdot \delta q(t) dt,$$

where

$$\mathbf{d}\mathfrak{S}(L)(q(\cdot)) \cdot \delta q(\cdot) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \mathfrak{S}(L)(q(\cdot, \lambda)) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \int_{t^1}^{t^2} L(q(t,\lambda), \dot{q}(t,\lambda)) dt.$$

The 1-form bundle value map $\mathcal{EL}(L)$ is called the ***Euler-Lagrange operator*** *and has the following expression in standard local charts*

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q})_i dq^i = \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) dq^i,$$

where, on the right hand side, it is understood that one formally takes the time derivative and then replaces $\frac{d}{dt} q$ by $\dot{q}$ and $\frac{d}{dt}\dot{q}$ by $\ddot{q}$.

9.1.2 Discrete Euler-Lagrange operator

With this background, the following result, the discrete analogue of Theorem 9.1.1, is obvious.

9.1.2 Theorem *Given a manifold Q, let $L_d : Q \times Q \to \mathbb{R}$ be a given discrete Lagrangian, and*

$$\mathfrak{S}(L_d)(q_d) = \sum_{k=0}^{N-1} L_d \left(q^j, q^{j+1}\right)$$

the associated discrete action map.

Let q^j_ε be a deformation of a position q^j in $\mathcal{C}_d(Q) := \left\{ q_d : \{t^j\}_{j=0}^N \to Q \right\}$, the ***space of discrete curves****, such that $q^0_\varepsilon = q^0$ and $q^N_\varepsilon = q^N$ for any ε in an open interval containing $0 \in \mathbb{R}$ and $q^j_0 = q^j$ for all $j = 0, 1, \ldots, N$. Let*

$$\delta q^j := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q^j_\varepsilon \in T_{q^j} Q$$

be the corresponding variation. Since the endpoints are fixed, we have $\delta q^0 = \delta q^N = 0$.

Then there is a unique bundle map

$$\mathcal{DEL}(L_d) : T^{(2)} Q_d \to T^* Q,$$

where the ***discrete second-order submanifold***

$$T^{(2)} Q_d = \ddot{Q}_d := \{((a,b),(b,c)) \mid a, b, c \in Q\} \tag{9.1.1}$$

is naturally embedded in $(Q \times Q) \times (Q \times Q)$, such that, for any deformation q^j_ϵ, keeping the endpoints fixed, we have

$$\mathbf{d}\mathfrak{S}(L_d)(q_d) \cdot \delta q_d = \sum_{k=1}^{N-1} \mathcal{DEL}(L_d)\left((q^{j-1}, q^j), (q^j, q^{j+1})\right) \cdot \delta q^j,$$

where

$$\mathbf{d}\mathfrak{S}(L_d)(q_d)\cdot\delta q_d = \left.\frac{d}{d\epsilon}\right|_{\varepsilon=0}\sum_{k=0}^{N-1} L_d\left(q_\varepsilon^j, q_\varepsilon^{j+1}\right)$$

The discrete 1*-form bundle-valued map* $\mathcal{DEL}(L_d)$ *is called the* ***discrete Euler-Lagrange operator*** *and has the following local expression*

$$\mathcal{DEL}(L_d)\left((q^{j-1},q^j),(q^j,q^{j+1})\right)dq^j = \left(D_2L_d(q^{j-1},q^j)+D_1L_d(q^j,q^{j+1})\right)dq^j \tag{9.1.2}$$

for $j=1,\ldots,N-1$*. The* ***discrete Euler-Lagrange equations*** *are given by the system*

$$\mathcal{DEL}(L_d)\left((q^{j-1},q^j),(q^j,q^{j+1})\right) = \left(D_2L_d(q^{j-1},q^j)+D_1L_d(q^j,q^{j+1})\right)=0 \tag{9.1.3}$$

for all $j=1,\ldots,N-1$*.*

9.2 Discrete Euler-Poincaré equations

9.2.1 Euler-Poincaré variational operator

We begin by recalling from Cendra, Marsden, and Ratiu [22] a modern formulation of Poincaré's theorem (Poincaré [95]). Let G be a lie group with Lie algebra $\mathfrak{g}$. For every $u\in\mathfrak{g}$, $\mathrm{ad}_u:\mathfrak{g}\ni v\mapsto[u,v]\in\mathfrak{g}$ denotes the adjoint representation of $\mathfrak{g}$ on itself. In what follows $\mathrm{ad}_u^*:\mathfrak{g}^*\to\mathfrak{g}^*$ denotes the dual of the linear map ad_u relative to the canonical duality pairing $\langle\cdot,\cdot\rangle:\mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$.

9.2.1 Theorem *Let* G *be a Lie group,* $L:TG\to\mathbb{R}$ *a left* G*-invariant Lagrangian, and*

$$\mathfrak{S}(L)(g(\cdot)) = \int_{t^1}^{t^2} L(g(t),\dot g(t))dt$$

the action functional of L *defined on* $\Omega(G;g^1,g^2)$*. Let* $\ell := L|_{\mathfrak{g}}$ *and*

$$\mathfrak{S}(\ell)(v(\cdot)) = \int_{t^1}^{t^2}\ell(v(t))dt \tag{9.2.1}$$

the ***reduced action*** *functional defined on* $\Omega(\mathfrak{g})$*, the space of curves in* $\mathfrak{g}$ *with no conditions imposed at* t^1 *and* t^2*. Then the following are equivalent :*

(i) *the curve* $g(t)$ *satisfies the Euler-Lagrange equations* $\mathcal{EL}(L)(g,\dot g,\ddot g)=0$ *on* G*;*

(ii) *the curve* $g(t)$ *is a critical point of the action functional* $\mathfrak{S}(L)$ *for variations* δg *vanishing at the endpoints;*

(iii) *the curve* $v(t)$ *solves the Euler-Poincaré equations*

$$\frac{d}{dt}\frac{\partial\ell}{\partial v} = \mathrm{ad}_v^*\frac{\partial\ell}{\partial v}\,;$$

(iv) *the curve $v(t)$ is a critical point of the reduced action functional* (9.2.1) *for variations of the form*

$$\delta v = \dot{\eta} + [v, \eta],$$

where $\eta(t) \in \mathfrak{g}$ is an arbitrary curve that vanishes at the endpoints. These variations δv are exactly the variations induced by left translation of arbitrary deformations $g(t, \lambda)$ of the curve $g(t) = g(t, \lambda)$ such that $\delta g(t^1) = \delta g(t^2) = 0$, i.e., $\delta v = g^{-1}\delta g$.

In addition, there is a unique map, the ***Euler-Poincaré operator****,*

$$\mathcal{EP}(\ell) : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}^*$$

such that, for any deformation $v(t, \lambda) = g(t, \lambda)^{-1}\dot{g}(t, \lambda) \in \mathfrak{g}$ induced on $\mathfrak{g}$ by a deformation $g(t, \lambda) \in G$ of $g(t) \in \Omega(G; g_0, g_1)$ keeping the endpoints fixed, and thus $\delta g(t_i) = 0$, for $i = 0, 1$, we have

$$\mathbf{d}\mathfrak{S}(\ell)(v(\cdot)) \cdot \delta v(\cdot) = \int_{t_1}^{t_2} \mathcal{EP}(\ell)(v(t), \dot{v}(t)) \cdot \eta(t) dt,$$

where

$$\mathbf{d}\mathfrak{S}(\ell)(v(\cdot)) \cdot \delta v(\cdot) = \left. \frac{d}{d\lambda} \right|_{\lambda=0} \mathfrak{S}(\ell)(v(\cdot, \lambda))$$

and

$$\delta v(t) = \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} \left(g(t, \lambda)^{-1} \dot{g}(t, \lambda) \right) = \dot{\eta}(t) + [v(t), \eta(t)], \quad \textit{with} \quad \eta = g^{-1}\delta g.$$

The Euler-Poincaré operator has the expression

$$\mathcal{EP}(\ell)(v, \dot{v}) = \mathrm{ad}_v^* \frac{\partial \ell}{\partial v} - \frac{d}{dt} \frac{\partial \ell}{\partial v},$$

where on the right hand side one takes formally the time derivative and then replaces $\frac{d}{dt}v$ by $\dot{v}$.

9.2.2 Discrete Euler-Poincaré operator; Lie group version

There are two ways of discretizing the Euler-Poincaré equations while keeping the geometric structure of the smooth case in mind. The first one yields a discrete Lagrangian defined on the group, whereas the second is based on a discrete Lagrangian on the Lie algebra. In this subsection we develop the group version.

The following theorem, the discrete analogue of Theorem 9.2.1, is established by using the diffeomorphism $(G \times G)/G \ni [g_1, g_2] \stackrel{\sim}{\longmapsto} g_1^{-1} g_2 \in G$ as in Bobenko, and Suris [11] and Marsden, Pekarsky, and Shkoller [86] (the Lie group G acts on $G \times G$ by left translation on each factor).

9.2.2 Theorem *Given a Lie group G, let $L_d : G \times G \to \mathbb{R}$ be a given G-invariant discrete Lagrangian and*

$$\mathfrak{S}(L_d)(g_d) = \sum_{k=0}^{N-1} L_d\left(g^j, g^{j+1}\right) \tag{9.2.2}$$

the associated discrete action sum defined on $\mathcal{C}_d(G)$, the space of all discrete paths in G. Let $\ell_d : G \to \mathbb{R}$ be the the ***Lie group reduced discrete Lagrangian*** *defined by $\ell_d(g_1^{-1}g_2) := L_d(g_1, g_2)$, and*

$$\mathfrak{S}(\ell_d)(f_d) = \sum_{k=0}^{N-1} \ell_d(f^j)$$

the associated ***Lie group discrete reduced action*** *sum defined on the space of discrete paths $\mathcal{C}_d(G) = \left\{f_d : \{t^j\}_{j=0}^N \to G\right\}$, with $f^j = (g^j)^{-1}g^{j+1}$. Then the following are equivalent :*

(i) *the discrete path g_d satisfies the discrete Euler-Lagrange equations on G*

$$\mathcal{DEL}(L_d)((g^{j-1}, g^j), (g^j, g^{j+1})) = 0;$$

(ii) *the discrete path g_d consists of a finite sequence of critical points of the discrete action sum $\mathfrak{S}(L_d)$ for variations vanishing at the endpoints, i.e., $\mathbf{d}\mathfrak{S}(L_d)(g_d) \cdot \delta g_d = 0$ for all variations δg_d satisfying $\delta g^0 = \delta g^N = 0$;*

(iii) *the discrete path f_d satisfies the* ***Lie group discrete Euler-Poincaré equations***

$$-T^* R_{f^j} \mathbf{D}\ell_d(f^j) + \mathrm{Ad}^*_{f^{j-1}} T^* R_{f^{j-1}} \mathbf{D}\ell_d(f^{j-1}) = 0, \qquad j = 1, \ldots, N-1,$$

where $\mathbf{D}$ denotes usual differentiation in a vector space;

(iv) ***Lie group discrete Euler-Poincaré variational principle****: the discrete path f_d consists of a finite sequence of critical points of the discrete action sum $\mathfrak{S}(\ell_d)$ for variations of the form*

$$\delta f^j = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} f^j_\varepsilon = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \left(g^j_\varepsilon\right)^{-1} g^{j+1}_\varepsilon = T_e R_{f^j}\left(-\eta^j + \mathrm{Ad}_{f^j}\eta^{j+1}\right),$$

where $\eta^j = (g^j)^{-1}\delta g^j \in \mathfrak{g}$ is a sequence of Lie algebra elements such that $\eta^0 = 0$ and $\eta^N = 0$.

Moreover, there is a unique smooth map

$$\mathcal{DEP}(\ell_d) : G \times G \to \mathfrak{g}^*$$

such that, for any deformation $f^j_\varepsilon = (g^j_\varepsilon)^{-1}g^{j+1}_\varepsilon \in G$, keeping the endpoints fixed, i.e., $\delta g^0 = 0$ and $\delta g^N = 0$, we have

$$\mathbf{d}\mathfrak{S}(\ell_d)(f_d) \cdot \delta f_d = \sum_{j=1}^{N-1} \mathcal{DEP}(\ell_d)(f^{j-1}, f^j) \cdot \eta^j,$$

where

$$\mathbf{d}\mathfrak{S}(\ell_d)(f_d)\cdot\delta f_d = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\sum_{k=0}^{N-1}\ell_d(f_\varepsilon^j).$$

The discrete 1*-form bundle-valued map* $\mathcal{DEP}(\ell_d)$ *is called the* ***Lie group discrete Euler-Poincaré operator****, and has the following local expression:*

$$\mathcal{DEP}(\ell_d)\left(f^{j-1},f^j\right) = -T_e^*R_{f^j}\mathbf{D}\ell_d(f^j) + \mathrm{Ad}^*_{f^{j-1}}T_e^*R_{f^{j-1}}\mathbf{D}\ell(f^{j-1}).$$

The ***Lie group discrete Euler-Poincaré equations*** *are the system*

$$\mathcal{DEP}(\ell_d)\left(f^{j-1},f^j\right) = -T_e^*R_{f^j}\mathbf{D}\ell_d(f^j)+\mathrm{Ad}^*_{f^{j-1}}T_e^*R_{f^{j-1}}\mathbf{D}\ell(f^{j-1}) = 0 \quad (9.2.3)$$

for all $j=1,\ldots,N-1$.

Proof. The following calculation yields for any sequence $\eta^j\in T_{f^j}G$ satisfying $\eta^0=0$ and $\eta^N=0$

$$\begin{aligned}\left.\frac{d}{d\lambda}\right|_{\lambda=0}\mathfrak{S}(\ell_d)(f_d(\lambda)) &= \sum_{j=0}^{N-1}\mathbf{D}\ell_d(f^j)\cdot\left(T_eR_{f^j}\left(-\eta^j+\mathrm{Ad}_{f^j}\eta^{j+1}\right)\right)\\ &= \sum_{j=1}^{N-1}\left[-\mathbf{D}\ell_d(f^j)\cdot T_eR_{f^j}\eta^j+\mathbf{D}\ell_d(f^{j-1})\cdot T_eR_{f^{j-1}}\mathrm{Ad}_{f^{j-1}}\eta^j\right]\\ &= \sum_{j=1}^{N-1}\left[-T_e^*R_{f^j}\mathbf{D}\ell_d(f^j)+\mathrm{Ad}^*_{f^{j-1}}T_e^*R_{f^{j-1}}\mathbf{D}\ell_d(f^{j-1})\right]\cdot\eta^j,\end{aligned}$$

so we get the desired result. □

9.2.3 Discrete Euler-Poincaré operator; Lie algebra version

In this subsection, we present a discrete version of the Euler-Poincaré operator that has the discrete Lagrangian defined on the Lie algebra, as opposed to the Lie group, that was discussed in §9.2.2.

In this approach we will need the logarithmic derivative of a map with values in a Lie group. Let $G\times M\to M$ be a smooth left action and $\tau : M\to G$ is a smooth map. The *right logarithmic derivative* of τ at $m\in M$ is the linear map defined by

$$\mathrm{d}^R\tau(m) := T_{\tau(m)}R_{\tau(m)^{-1}}\circ T_m\tau : T_mM\to\mathfrak{g}.$$

Thus, if $t\mapsto m(t)$ is a smooth curve in M, we have

$$\frac{d}{dt}\tau\left(m(t)\right) = \left(\mathrm{d}^R\tau(m(t))\cdot\dot m(t)\right)\tau(m(t)). \quad (9.2.4)$$

For example, we apply these formulas to the exponential map $\exp:\mathfrak{g}\to G$ which has the advantage that the right logarithmic derivative is known explicitly, namely, if $\xi\in\mathfrak{g}$, we have

$$\mathrm{d}^R\exp(\xi) = T_{\exp\xi}R_{\exp(-\xi)}\circ T_\xi\exp = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\mathrm{ad}_\xi^n = \frac{e^{\mathrm{ad}_\xi}-I}{\mathrm{ad}_\xi} : \mathfrak{g}\to\mathfrak{g},$$

a linear map from $\mathfrak{g}$ to itself. Therefore, if $t \mapsto \xi(t)$ is a smooth curve in $\mathfrak{g}$, we have

$$\frac{d}{dt}\exp(\xi(t)) = \left(\mathrm{d}^R \exp(\xi(t)) \cdot \xi'(t)\right) \exp(\xi(t)) \in T_{\exp \xi(t)} G. \tag{9.2.5}$$

There are similar considerations for the left logarithmic derivative.

As proved in Bobenko, and Suris [11], the variational problem for the functional (9.2.2) is equivalent to finding extremals of the functional $\mathfrak{S}(\mathcal{L}_d) = \sum_{j=0}^{N-1} \mathcal{L}_d\left(g^j, (g^j)^{-1} g^{j+1}\right)$, where $\mathcal{L}_d : G \times G \to \mathbb{R}$, $\mathcal{L}_d\left(g^j, (g^j)^{-1} g^{j+1}\right) := L_d\left(g^j, g^{j+1}\right)$, is the left trivialized discrete Lagrangian. Then, for a given C^2-diffeormorphism $\tau : \mathfrak{g} \to G$ defined on an open neighbourhood of the origin with $\tau(0) = e$, this trivialized Lagrangian $\mathcal{L}_d$ may be transformed to the Lie algebraic left trivialized Lagrangian $\mathscr{L}_d : G \times \mathfrak{g} \to \mathbb{R}$, where $\mathscr{L}_d(g^j, \xi^j) := \mathcal{L}_d\left(g^j, (g^j)^{-1} g^{+1}\right)$ for $\xi^j := \tau^{-1}((g^j)^{-1} g^{j+1})$. The following result is also a discrete analogue of Theorem 9.2.1.

9.2.3 Theorem *Given G a Lie group, $L_d : G \times G \to \mathbb{R}$ be a left G-invariant Lagrangian, and*

$$\mathfrak{S}(L_d)(g_d) = \sum_{k=0}^{N-1} L_d\left(g^j, g^{j+1}\right)$$

the discrete action sum defined on $\mathcal{C}_d(G)$. Let $\mathfrak{l}_d : \mathfrak{g} \to \mathbb{R}$ be defined by $\mathfrak{l}(\xi^j) := \mathscr{L}_d(e, \xi^j)$, and

$$\mathfrak{S}(\mathfrak{l}_d)(\xi_d) = \sum_{k=0}^{N-1} \mathfrak{l}_d(\xi^j)$$

*the **Lie algebra discrete reduced action** sum defined on*

$$\mathcal{C}_d(\mathfrak{g}) = \left\{\xi_d : \{t^j\}_{j=0}^N \to \mathfrak{g}\right\},$$

with $\xi^j = \tau^{-1}((g^j)^{-1} g^{j+1})$ and no conditions imposed at t^0 and t^N. Then the following are equivalent :

(i) *the discrete path g_d satisfies the discrete Euler-Lagrange equations on G*

$$\mathcal{DEL}(L_d)((g^{j-1}, g^j), (g^j, g^{j+1})) = 0;$$

(ii) *the discrete path g_d consists of a finite sequence of critical points of the discrete action sum $\mathfrak{S}(L_d)$ for variations vanishing at the endpoints, i.e., $\mathbf{d}\mathfrak{S}(L_d)(g_d) \cdot \delta g_d = 0$ for all variations δg_d satisfying $\delta g^0 = \delta g^N = 0$;*

(iii) *the discrete trajectory $\xi_d \subset \mathfrak{g}$ satisfies the **Lie algebra discrete Euler-Poincaré equations***

$$\left(\left(\mathrm{d}^R \tau(\xi^j)\right)^{-1}\right)^* \left(\mathbf{D}\mathfrak{l}_d(\xi^j)\right) = \mathrm{Ad}_{\tau(\xi^{j-1})} \left(\left(\mathrm{d}^R \tau(\xi^{j-1})\right)^{-1}\right)^* \left(\mathbf{D}\mathfrak{l}_d(\xi^{j-1})\right),$$

where $\mathbf{D}$ denotes usual differentiation in the vector space $\mathfrak{g}$.

(iv) ***Lie algebra discrete Euler-Poincaré variational principle**: the discrete path ξ_d consists of a finite sequence of critical points of the discrete action sum $\mathfrak{S}(\mathfrak{l}_d)$ for variations of the form*

$$\delta\xi^j = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \xi^j_\varepsilon = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \tau^{-1}\left(\left(g^j_\varepsilon\right)^{-1} g^{j+1}_\varepsilon\right) = \left(\mathrm{d}^R\tau(\xi^j)\right)^{-1}\left(-\eta^j + \mathrm{Ad}_{f^j}\eta^{j+1}\right),$$

where $\eta^j = (g^j)^{-1}\delta g^j \in \mathfrak{g}$ is a sequence of Lie algebra elements such that $\eta^0 = 0$ and $\eta^N = 0$.

*Define the **discrete first order submanifold** $\dot{\mathfrak{g}}_d$ of $\mathfrak{g}\times\mathfrak{g}$ by*

$$\dot{\mathfrak{g}}_d := \{(a,b)\in\mathfrak{g}\times\mathfrak{g} \mid \tau(a) = A^{-1}B,\ \tau(b) = B^{-1}C, \quad \text{for some} \quad A,B,C\in G\}.$$

There is a unique smooth map

$$\mathcal{DEP}(\mathfrak{l}_d) : \dot{\mathfrak{g}}_d \to \mathfrak{g}^*,$$

such that for any deformation $\xi^j_\varepsilon \in \mathfrak{g}$, induced on $\mathfrak{g}$ by a deformation f^j_ε of $f^j = (g^j)^{-1}g^{j+1}$ keeping the endpoints fixed, we have

$$\mathbf{d}\mathfrak{S}_d(\mathfrak{l}_d)\cdot\delta\xi_d = \sum_{j=1}^{N-1}\mathcal{DEP}(\mathfrak{l}_d)(\xi^{j-1},\xi^j)\cdot\eta^j,$$

where

$$\mathbf{d}\mathfrak{S}(\mathfrak{l}_d)(\xi_d)\cdot\delta\xi_d = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\sum_{k=0}^{N-1}\mathfrak{l}_d(\xi^j_\varepsilon).$$

*The map $\mathcal{DEP}(\mathfrak{l}_d)$ is called the **Lie algebra discrete Euler-Poincaré operator** and its expression is given by*

$$\mathcal{DEP}(\mathfrak{l}_d) = \mathrm{Ad}^*_{\tau(\xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\xi^{j-1})\right)^{-1}\right)^*\left(\mathbf{D}\mathfrak{l}_d(\xi^{j-1})\right) - \left(\left(\mathrm{d}^R\tau(\xi^j)\right)^{-1}\right)^*\left(\mathbf{D}\mathfrak{l}_d(\xi^j)\right).$$

*The **Lie algebra discrete Euler-Poincaré equations** are the system*

$$\mathcal{DEP}(\mathfrak{l}_d)\left(\xi^{j-1},\xi^j\right) = \mathrm{Ad}^*_{\tau(\xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\xi^{j-1})\right)^{-1}\right)^*\left(\mathbf{D}\mathfrak{l}_d(\xi^{j-1})\right) - \left(\left(\mathrm{d}^R\tau(\xi^j)\right)^{-1}\right)^*\left(\mathbf{D}\mathfrak{l}_d(\xi^j)\right) = 0 \tag{9.2.6}$$

for all $j = 1,\ldots,N-1$.

Proof. By calculation of the variational principle $\delta\mathfrak{S}_d(\mathfrak{l}_d) = 0$, we obtain

$$\begin{aligned}
\delta\mathfrak{S}_d(\mathfrak{l}_d)(\xi_d) &= \sum_{j=0}^{N-1} \mathbf{D}\mathfrak{l}_d(\xi^j)\cdot\delta\xi^j \\
&= \sum_{j=0}^{N-1} \mathbf{D}\mathfrak{l}_d(\xi^j)\cdot\left(\mathrm{d}^R\tau(\xi^j)\right)^{-1}\left(-\eta^j + \mathrm{Ad}_{\tau(\xi^j)}\,\eta^{j+1}\right) \\
&= \sum_{j=1}^{N-1} \Big[\mathrm{Ad}^*_{\tau(\xi^{j-1})}\left(\left(\mathrm{d}^R\tau(\xi^{j-1})\right)^{-1}\right)^*\left(\mathbf{D}\mathfrak{l}_d(\xi^{j-1})\right) \\
&\qquad - \left(\left(\mathrm{d}^R\tau(\xi^j)\right)^{-1}\right)^*\left(\mathbf{D}\mathfrak{l}_d(\xi^j)\right)\Big]\cdot\eta^j,
\end{aligned}$$

which yields (9.2.6). □

9.3 Discrete mechanical connection

In this section we recall some basic facts about mechanical connections (see e.g. Marsden, Misiołek, Ortega, Permultter, and Ratiu [84]). Then we define a discrete mechanical connection and study its properties.

9.3.1 Principal connections

Let $\Phi : G \times Q \to Q$ be a free and proper action of a Lie group G on a manifold Q. Let $\mathfrak{g}$ denote the Lie algebra of G. If $\xi \in \mathfrak{g}$, its associated ***infinitesimal generator*** is the vector field $\xi_Q \in \mathfrak{X}(Q)$ whose value at $q \in Q$ is defined by

$$\xi_Q(q) := \left.\frac{d}{dt}\right|_{t=0} \Phi(\exp t\xi, q).$$

Thus, the flow of ξ_Q is $F_t(q) = \Phi(\exp t\xi, q)$. Let $\pi : Q \to Q/G$ denote the canonical projection; it is a surjective submersion and Q/G carries the quotient manifold structure. We shall denote elements of Q/G by $x := [q]_G := \pi(q)$.

The ***vertical subbundle*** is the vector subbundle $VQ \subset TQ$ whose fiber at $q \in Q$ is $\ker T_q\pi = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}$. We have $T_q\Phi_g(V_qQ) = V_{gq}Q$ for all $q \in Q$ and $g \in G$.

A ***principal connection*** one-form on the (left) principal bundle $\pi : Q \to Q/G$ is a $\mathfrak{g}$-valued one-form $\mathcal{A} \in \Omega^1(Q, \mathfrak{g})$ satisfying

$$\mathcal{A}\left(T_q\Phi_g(v_q)\right) = \mathrm{Ad}_g\left(\mathcal{A}(v_q)\right) \quad \text{and} \quad \mathcal{A}(\xi_Q(q)) = \xi, \tag{9.3.1}$$

for all $g \in G$, $v_q \in TQ$, and $\xi \in \mathfrak{g}$. The ***horizontal bundle*** is the vector subbundle $HQ \subset TQ$ whose fiber at $q \in Q$ is $H_qQ := \ker\mathcal{A}(q)$. As before, we have $T_q\Phi_g(H_qQ) = H_{gq}Q$ for all $q \in Q$ and $g \in G$. In addition $TQ = VQ \oplus HQ$. Conversely, a vector subbundle $HQ \subset TQ$ satisfying these two properties uniquely determines a connection one-form $\mathcal{A}$. The associated projections onto VQ and HQ, respectively, are denoted by

$$v_q = \mathrm{ver}(v_q) + \mathrm{hor}(v_q) = \mathcal{A}(v_q)_Q(q) + \mathrm{hor}(v_q).$$

Since $T\pi : TQ \to T(Q/G)$ restricted to H_qQ is a vector space isomorphism with $T_x(Q/G)$, where $x := \pi(q)$, given $q \in Q$ and $X_x \in T_x(Q/G)$, the vector $X_q^h := (T_q\pi)^{-1}(X_x)$ is the ***horizontal lift*** at q of X_x. Note that if $v_q \in T_qQ$, then $\mathrm{hor}(v_q) = (T_q\pi(v_q))_q^h$. Given a curve $x(t) \in Q/G$, $x(0) = x_0$, the ***horizontal lift*** of $x(t)$ passing at $t = 0$ through q_0 is the curve $x_{q_0}^h(t)$ in Q such that $x_{q_0}^h(0) = q_0$ and $\frac{d}{dt}x_{q_0}^h(t)$ is a horizontal vector for all t. If $X \in \mathfrak{X}(Q/G)$, its ***horizontal lift*** $X^h \in \mathfrak{X}(Q)$ is defined by $X^h(q) := (T_q\pi)^{-1}(X(\pi(q)) \in H_qQ$.

The curvature two-form $\mathcal{B} \in \Omega^2(Q, \mathfrak{g})$ is given by

$$\mathcal{B}(u_q, v_q) = \mathbf{d}\mathcal{A}(\mathrm{hor}(u_q), \mathrm{hor}(v_q)) = \mathbf{d}\mathcal{A}(u_q, v_q) - [\mathcal{A}(u_q), \mathcal{A}(v_q)], \qquad (9.3.2)$$

where $\mathbf{d}\mathcal{A}$ is the exterior derivative of the one-form $\mathcal{A}$. The following formula links the curvature 2-form $\mathcal{B}$ to the horizontal lift operation on vector the fields $X_1, X_2 \in \mathfrak{X}(Q/G)$:

$$[X_1, X_2]^h(q) - \left[X_1^h, X_2^h\right](q) = \left(\mathcal{B}\left(X_1^h(q), X_2^h(q)\right)\right)_Q(q). \qquad (9.3.3)$$

9.3.2 Mechanical connection in geometric mechanics

In geometric mechanics there is a natural principal connection that we now describe. Consider a G-invariant Lagrangian $L : TQ \to \mathbb{R}$. We assume that there is a *G-invariant Riemannian metric* γ on the configuration space Q and that the Lagrangian is of the form

$$L(v_q) = \frac{1}{2}\gamma(v_q, v_q) - V(q), \qquad (9.3.4)$$

for a G-invariant potential $V : Q \to \mathbb{R}$. The associated Legendre transform $\mathbb{F}L : TQ \to T^*Q$ becomes in this case

$$\mathbb{F}L(v_q) \cdot w_q = \gamma(v_q, w_q).$$

We denote by $\mathbf{J}_L : TQ \to \mathfrak{g}^*$ the ***(Lagrangian) momentum map*** defined by

$$\langle \mathbf{J}_L(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle = \gamma(v_q, \xi_Q(q)). \qquad (9.3.5)$$

By choosing local coordinates on Q and a basis $\{\mathbf{e}_a\}_{a=1}^m$ of $\mathfrak{g}$, we write the infinitesimal generator and the Lagrangian momentum map

$$[\xi_Q(q)]^i = K_a^i(q)\xi^a \quad \text{and} \quad (\mathbf{J}_L(v_q))_a = v^j\gamma_{ij}K_a^i(q).$$

For each $q \in Q$, we define the ***locked inertia tensor*** $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ by

$$\langle \mathbb{I}(q)\eta, \xi \rangle = \gamma\left(\eta_Q(q), \xi_Q(q)\right), \qquad \forall\ \eta, \xi \in \mathfrak{g}. \qquad (9.3.6)$$

Since the action is free, $\mathbb{I}(q)$ is invertible, so (9.3.6) defines an inner product on $\mathfrak{g}$. In coordinates, the locked inertia tensor reads

$$\mathbb{I}(q)_{ab} = g_{ij}K_a^i(q)K_b^j(q).$$

We recall that the locked inertia tensor is G-equivariant, i.e.

$$\mathbb{I}(gq)\mathrm{Ad}_g\xi = \mathrm{Ad}^*_{g^{-1}}\mathbb{I}(q)\xi. \tag{9.3.7}$$

If Q is endowed with a G-invariant Riemannian metric γ, then there is a natural associated connection defined by the condition that the horizontal subspace is the orthogonal to the vertical subspace $HQ := (VQ)^\perp$. It is called the ***mechanical connection*** and the associated connection one-form reads

$$\mathcal{A}(v_q) = \mathbb{I}(q)^{-1}\mathbf{J}_L(v_q), \tag{9.3.8}$$

where $\mathbb{I}$ and $\mathbf{J}_L$ are associated to the action Φ and the Riemannian metric γ. The horizontal subspace is hence given by

$$H_qQ = \ker(\mathcal{A}_q) = \{v_q \in TQ \mid \mathbf{J}_L(v_q) = 0\}.$$

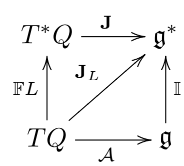


Figure 9.3.1: Diagram defining the mechanical connection

9.3.1 Remark The physical interpretation of the mechanical connection in concrete examples, where G is equal to or a subgroup of $SO(3)$, is angular velocity. For example in the case of the spherical pendulum, we have $Q = S^2_R$, the sphere of radius R in $\mathbb{R}^3$ centered at the origin, $G = S^1$ acts on the sphere S^2_R by rotations about the vertical axis, the Riemannian metric on S^2_R is the pull back of the standard metric on $\mathbb{R}^3$ given by the inner product of vectors, and V is the gravitational potential for a mass m and is hence given by $V(\theta, \varphi) = -mgR\cos\theta$, where g is the magnitude of the gravitational acceleration and $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi]$ are the spherical coordinates with the convention that θ is measured from the negative Oz-axis to the positive one and φ measures the angle in the horizontal (x, y)-plane starting at the positive Ox-axis. The mechanical connection $\mathcal{A} \in \Omega^1(S^2, \mathbb{R})$ has in this case the expression $\mathcal{A}(\theta, \varphi, \dot\theta, \dot\varphi) = \dot\varphi$, which is the angular velocity of the rotation about the Oz-axis (see, e.g., [82, §3.5] for this computation).

In the case of the free rigid body, $Q = SO(3)$, the left invariant Riemannian metric has the expression at the origin equal to $\mathbb{I}\mathbf{u} \cdot \mathbf{v}$, where $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ is the diagonalized moment of inertia tensor of the body, $I_1 > 0$, $I_2 > 0$, $I_3 > 0$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and there is no potential energy. In this case, the mechanical connection has the expression $\mathcal{A}(A, \dot A) = \dot A A^{-1}$, the spatial angular velocity, were $A \in SO(3)$, $\dot A \in T_A SO(3)$.

9.3.3 Discrete momentum maps

Let $\Phi : G \times Q \to Q$ be a left Lie group action. There is a natural induced action on $Q \times Q$ given by

$$\Phi_g^{Q\times Q}(q^j, q^{j+1}) := \left(\Phi_g(q^j), \Phi_g(q^{j+1})\right), \tag{9.3.9}$$

and with infinitesimal generator denoted by $\xi_{Q\times Q}(q^j, q^{j+1})$. Given a discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$ (not necessarily G-invariant), the ***discrete Lagrangian momentum maps*** $\mathbf{J}^+_{L_d}, \mathbf{J}^-_{L_d} : Q \times Q \to \mathfrak{g}^*$ are defined by

$$\begin{aligned}
\left\langle \mathbf{J}^+_{L_d}(q^j, q^{j+1}), \xi \right\rangle &= \left\langle \Theta^+_{L_d}(q^j, q^{j+1}), \xi_{Q\times Q}(q^j, q^{j+1}) \right\rangle \\
&= \left\langle D_2 L_d(q^j, q^{j+1}), \xi_Q(q^{j+1}) \right\rangle \\
\left\langle \mathbf{J}^-_{L_d}(q^j, q^{j+1}), \xi \right\rangle &= \left\langle \Theta^-_{L_d}(q^j, q^{j+1}), \xi_{Q\times Q}(q^j, q^{j+1}) \right\rangle \\
&= \left\langle -D_1 L_d(q^j, q^{j+1}), \xi_Q(q^j) \right\rangle .
\end{aligned}$$

(see (1.2.5) for the definition of $\Theta^\pm_d$). Note that we have

$$\mathbf{J}^\pm_{L_d} = \left(\mathbb{F}^\pm L_d\right)^* \mathbf{J},$$

where $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ is the *cotangent lift momentum map* given by $\langle \mathbf{J}(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle$.

It is important to note that if the discrete curve $\{q^j\}_{j=0}^N$ verifies the discrete Euler-Lagrange equations, then we have the equality already mentioned in (1.2.9)

$$\mathbf{J}^+_{L_d}(q^{j-1}, q^j) = \mathbf{J}^-_{L_d}(q^j, q^{j+1}), \quad \text{for all } j = 1, ..., N-1.$$

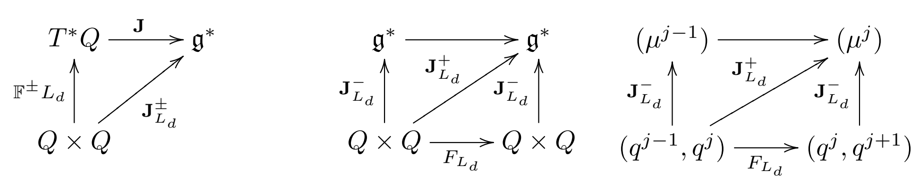


Figure 9.3.2: On the left: the definition of the discrete momentum maps. Two diagrams on the right: illustration of the equality (1.2.9).

When G acts on $Q \times Q$ by special discrete symplectic maps, that is, if $(\Phi_g^{Q\times Q})^* \Theta^\pm_{L_d} = \Theta^\pm_{L_d}$, then the discrete Lagrangian momentum maps are G-equivariant, that is,

$$\begin{aligned}
\mathbf{J}^+_{L_d} \circ \Phi_g^{Q\times Q} &= \mathrm{Ad}^*_{g^{-1}} \mathbf{J}^+_{L_d}, \\
\mathbf{J}^-_{L_d} \circ \Phi_g^{Q\times Q} &= \mathrm{Ad}^*_{g^{-1}} \mathbf{J}^-_{L_d}.
\end{aligned} \tag{9.3.10}$$

This happens, for example, if the discrete Lagrangian L_d is G-invariant, i.e., $L_d \circ \Phi_g^{Q\times Q} = L_d$ for any $g \in G$, since in this case $\Phi_g^{Q\times Q}$ is a special discrete symplectic map. Moreover, in this case the two momentum maps coincide: $\mathbf{J}^+_{L_d} = \mathbf{J}^-_{L_d}$, and therefore, from (1.2.9) we obtain the discrete Noether Theorem.

9.3.2 Theorem (Discrete Noether Theorem) *Let $L_d : Q \times Q \to \mathbb{R}$ be a given discrete Lagrangian invariant under the lift* (9.3.9) *of the left action $\Phi : G \times Q \to Q$. Then the corresponding discrete Lagrangian momentum map $\mathbf{J}_{L_d} : Q \times Q \to \mathfrak{g}^*$ is a conserved quantity of the discrete Lagrangian map $F_{L_d} : Q \times Q \to Q \times Q$, that is, $\mathbf{J}_{L_d} \circ F_{L_d} = \mathbf{J}_{L_d}$.*

To introduce a future development of this work and a better understanding of the discrete horizontal lift we will prove the following lemma.

9.3.3 Lemma *Let the discrete state space $Q \times Q$ and let $J^{\pm}_{L_d} : Q \times Q \to \mathfrak{g}^*$ be a discrete G-equivariant Lagrangian momentum map of a Lie group action of G on $Q \times Q$. Let $G \cdot \mu^j$, and $G \cdot \mu^{j+1}$ be the coadjoint orbits through μ^j and $\mu^{j+1} \in \mathfrak{g}^*$. Let $G_\mu = \{g \in G \mid g \cdot \mu = \mu\}$ be the coadjoint isotropy group. Then we have*

(i) $$(\mathbf{J}^-_{L_d})^{-1}(G \cdot \mu^j) = G \cdot (\mathbf{J}^-_{L_d})^{-1}(\mu^j) = \left\{\Phi^{Q\times Q}_g(q^j, q^{j+1}) \mid g \in G \;\; \textit{and} \;\; \mathbf{J}^-_{L_d}(q^j, q^{j+1}) = \mu^j\right\};$$

(ii) $$(\mathbf{J}^+_{L_d})^{-1}(G \cdot \mu^{j+1}) = G \cdot (\mathbf{J}^+_{L_d})^{-1}(\mu^{j+1}) = \left\{\Phi^{Q\times Q}_g(q^j, q^{j+1}) \mid g \in G \;\; \textit{and} \;\; \mathbf{J}^+_{L_d}(q^j, q^{j+1}) = \mu^{j+1}\right\};$$

(iii) $G_{\mu^j} \cdot (q^j, q^{j+1}) = \big(G \cdot (q^j, q^{j+1})\big) \cap (\mathbf{J}^-_{L_d})^{-1}(\mu^j)$;

(iv) $G_{\mu^{j+1}} \cdot (q^j, q^{j+1}) = \big(G \cdot (q^j, q^{j+1})\big) \cap (\mathbf{J}^+_{L_d})^{-1}(\mu^{j+1})$.

Proof. (i) We have

$$\begin{aligned}(q^j, q^{j+1}) \in (\mathbf{J}^-_{L_d})^{-1}(G \cdot \mu^j) &\iff \mathbf{J}^-_{L_d}(q^j, q^{j+1}) = \mathrm{Ad}^*_{g^{-1}}\mu^j, \quad \text{for some } g \in G \\ &\iff \mu^j = \mathrm{Ad}^*_g \mathbf{J}^-_{L_d}(q^j, q^{j+1}) = \mathbf{J}^-_{L_d}(g^{-1}q^j, g^{-1}q^{j+1}) \\ &\iff (g^{-1}q^j, g^{-1}q^{j+1}) \in (\mathbf{J}^-_{L_d})^{-1}(\mu^j) \\ &\iff (q^j, q^{j+1}) = \Phi^{Q\times Q}_g(g^{-1}q^j, g^{-1}q^{j+1}) \in G \cdot (\mathbf{J}^-_{L_d})^{-1}(\mu).\end{aligned}$$

(ii) Same proof as (i) with μ^j replaced by μ^{j+1}.

(iii) For any $g \in G$, we have

$$\begin{aligned}(gq^j, gq^{j+1}) \in (\mathbf{J}^-_{L_d})^{-1}(\mu^j) &\iff \mu^j = \mathbf{J}^-_{L_d}(gq^j, gq^{j+1}) = \mathrm{Ad}^*_{g^{-1}}\mathbf{J}^-_{L_d}(q^j, q^{j+1}) = \mathrm{Ad}^*_{g^{-1}}(\mu^j) \\ &\iff \mu^j \in G_{\mu^j}.\end{aligned}$$

(iv) Same proof as (iii) with μ^j replaced by μ^{j+1}. ■

Since we considered G-equivariant discrete momentum maps $\mathbf{J}^{\pm}_{L_d}$, it follows that G_{μ^j} leaves $(\mathbf{J}^-_{L_d})^{-1}(\mu^j)$ invariant and $G_{\mu^{j+1}}$ leaves $(\mathbf{J}^+_{L_d})^{-1}(\mu^{j+1})$ invariant. Thus, the orbit space $(\mathbf{J}^{\pm}_{L_d})^{-1}(\mu^j)/G_{\mu^j}$ is well defined.

9.3.4 Discrete mechanical connection and discrete variational mechanics

This subsection is devoted to the definition of a discrete mechanical connection, which is related to the definition in the continuous case, and is compatible with the general framework of discrete Lagrangian mechanics given in [90].

From now on we assume that $L_d : Q \times Q \to \mathbb{R}$ is the discrete Lagrangian associated to a simple classical G-invariant mechanical system, that is, $L : TQ \to \mathbb{R}$ is of the form

$$L(v_q) = \frac{1}{2}\gamma(v_q, v_q) - V(q),$$

where γ is a G-invariant Riemannian metric and $V : Q \to \mathbb{R}$ is a G-invariant potential. The discrete Lagrangian is of the form

$$L_d(q^j, q^{j+1}) = K(q^j, q^{j+1}) - V(q^j) \tag{9.3.11}$$

and is supposed to inherit the symmetry of L, that is, we suppose that L_d is G-invariant under the diagonal action $\Phi^{Q\times Q}$ (see (9.3.9)). Note that in this case, there is only one discrete momentum map since $\mathbf{J}^+_{L_d}(q^j, q^{j+1}) = \mathbf{J}^-_{L_d}(q^j, q^{j+1})$.

Given a discrete Lagrangian of the form (9.3.11) and the locked inertia tensor $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ (see (9.3.6)) associated to γ, we define the *discrete mechanical connections* $\mathcal{A}^{\pm}_d$ as follows.

9.3.4 Definition (Discrete mechanical connection) *The discrete mechanical connections are the maps* $\mathcal{A}^{\pm}_{L_d} : Q \times Q \to \mathfrak{g}$ *defined by*

$$\begin{aligned} \mathcal{A}^+_{L_d}(q^j, q^{j+1}) &= \mathbb{I}(q^{j+1})^{-1}\mathbf{J}^+_{L_d}(q^j, q^{j+1}), \\ \mathcal{A}^-_{L_d}(q^j, q^{j+1}) &= \mathbb{I}(q^j)^{-1}\mathbf{J}^-_{L_d}(q^j, q^{j+1}). \end{aligned} \tag{9.3.12}$$

Note that the expressions in coordinates are given by

$$\begin{aligned} \left(\mathcal{A}^+_{L_d}(q^j, q^{j+1})\right)^a &= \left(\mathbb{I}(q^{j+1})^{-1}\right)^{ab}\left(D_2L_d(q^j, q^{j+1})\right)_l\left(K(q^{j+1})\right)^l_b, \\ \left(\mathcal{A}^-_{L_d}(q^j, q^{j+1})\right)^a &= \left(\mathbb{I}(q^j)^{-1}\right)^{ab}\left(-D_1L_d(q^j, q^{j+1})\right)_l\left(K(q^j)\right)^l_b. \end{aligned}$$

9.3.5 Remark The maps $\mathcal{A}^{\pm}_{L_d}$ assign to each discrete path $(q^j, q^{j+1}) \in Q \times Q$ the corresponding angular velocities at times t^j and t^{j+1}, respectively. If the system under consideration is formed by a chain of rigid bodies with ball-socket joints or is formed by an elastic material, the locked inertia tensor is the inertia tensor of the rigid body obtained by freezing all the joints or by rigidifying a given configuration of the elastic material. For example, in the case of the double spherical pendulum, $\mathbb{I}(r_1, r_2) = m_1r_1^2 + m_2r_2^2$, where r_1 and r_2 are the distance of the masses m_1 and m_2 to the z-axis. The mechanical connection in this case gives the angular velocities of the two bodies about the Oz-axis (see, e.g. Marsden [82, §3.5]).

9.3.6 Proposition (**i**) *The maps* $\mathcal{A}^{\pm}_{L_d} : Q \times Q \to \mathfrak{g}$ *are G-equivariant, i.e. for all* $q^j, q^{j+1} \in Q$ *and* $g \in G$, *we have*

$$\mathcal{A}^{\pm}_{L_d}\left(\Phi^{Q\times Q}_g(q^j, q^{j+1})\right) = \mathrm{Ad}_g\mathcal{A}^{\pm}_{L_d}(q^j, q^{j+1}). \tag{9.3.13}$$

(**ii**) *The discrete and continuous mechanical connections are related via the identities*

$$\begin{aligned} \mathcal{A}^-_{L_d}(q^j, q^{j+1}) &= \mathcal{A}(q^j)\left(\mathbb{F}^-L_d(q^j, q^{j+1})^\sharp\right) \\ \mathcal{A}^+_{L_d}(q^j, q^{j+1}) &= \mathcal{A}(q^{j+1})\left(\mathbb{F}^+L_d(q^j, q^{j+1})^\sharp\right), \end{aligned} \tag{9.3.14}$$

where $\sharp : T^*Q \to TQ$ *is the index raising operator associated with the metric* γ*, and hence* $\mathbb{F}L_d^-(q^j, q^{j+1})^\sharp \in T_{q^j}Q$ *and* $\mathbb{F}L_d^+(q^j, q^{j+1})^\sharp \in T_{q^{j+1}}Q$*.*
(**iii**) *If the discrete curve* $\{q^j\}_{j=0}^N$ *verifies the discrete Euler-Lagrange equations, then*

$$\mathcal{A}^+_{L_d}(q^{j-1}, q^j) = \mathcal{A}^-_{L_d}(q^j, q^{j+1}), \quad \textit{for all } j = 1, ..., N-1. \tag{9.3.15}$$

Proof. Using the equivariance of the locked inertia tensor and of the discrete Lagrangian momentum maps, we get

$$\begin{aligned}
\mathcal{A}^+_{L_d}\left(\Phi_g^{Q\times Q}(q^j, q^{j+1})\right) &= \mathbb{I}(gq^{j+1})^{-1}\mathbf{J}_d^+\left(\Phi_g^{Q\times Q}(q^j, q^{j+1})\right)\\
&\overset{(9.3.10)}{=} \mathbb{I}(gq^{j+1})^{-1}\mathrm{Ad}^*_{g^{-1}}\mathbf{J}_d^+(q^j, q^{j+1})\\
&\overset{(9.3.7)}{=} \mathrm{Ad}_g\mathbb{I}(q^{j+1})^{-1}\mathbf{J}_d^+(q^j, q^{j+1})\\
&= \mathrm{Ad}_g\mathcal{A}^\pm_{L_d}(q^j, q^{j+1}),
\end{aligned}$$

similarly for $\mathcal{A}^-_{L_d}$. The equalities (9.3.14) follow from the definitions. Equality (9.3.15) follows easily from the relation (1.2.9) that holds when the discrete Euler-Lagrange equations are verified. ■

Note that (9.3.13) implies the relation

$$\left.\frac{d}{dt}\right|_{t=0}\mathcal{A}^\pm_{L_d}\left(\Phi^{Q\times Q}_{\exp(t\xi)}(q^j, q^{j+1})\right) = \left[\xi, \mathcal{A}^\pm_{L_d}(q^j, q^{j+1})\right], \quad \text{for all } \xi \in \mathfrak{g}.$$

The relations between the various maps considered so far are illustrated in the diagram below.

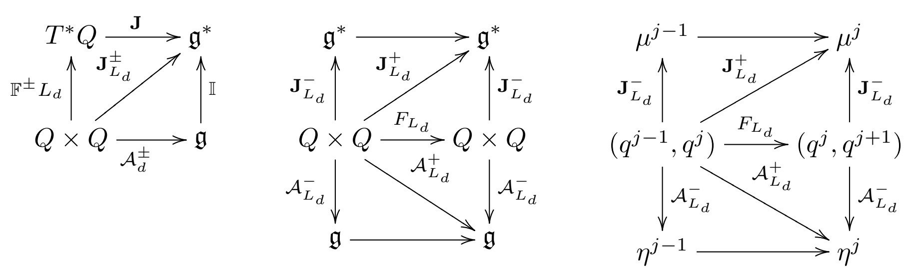


Figure 9.3.3: On the left: definition of the discrete mechanical connection. Two diagrams on the right: illustration of the equalities (1.2.9) and (9.3.15).

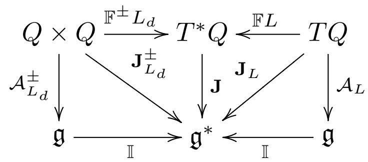


Figure 9.3.4: Illustration of the equalities (9.3.14).

9.3.5 Discrete horizontal space

In this subsection we show that it is possible to define a splitting of the elements $(q^j, q^{j+1}) \in Q \times Q$ in horizontal and vertical components, in a compatible way with the discrete mechanical connection we just defined.

We begin by noting that (9.3.14) and the usual vertical plus horizontal decomposition of a vector relative to the smooth connection $\mathcal{A}$, implies that

$$\mathrm{ver}_{q^j}\left(\mathbb{F}^- L_d(q^j, q^{j+1})^\sharp\right) = \left[\mathcal{A}^-_{L_d}(q^j, q^{j+1})\right]_Q (q^j) \tag{9.3.16}$$

$$\mathrm{ver}_{q^{j+1}}\left(\mathbb{F}^+ L_d(q^j, q^{j+1})^\sharp\right) = \left[\mathcal{A}^+_{L_d}(q^j, q^{j+1})\right]_Q (q^{j+1}). \tag{9.3.17}$$

Given a regular discrete Lagrangian L_d, we get a discrete vertical space.

9.3.7 Definition *The* $(-)$ ***discrete vertical space*** *at* q^j *is*

$$V^-_{q^j} Q = \left\{(q^j, q^{j+1}) \mid \left(\mathbb{F}^- L_d(q^j, q^{j+1})\right)^\sharp \in V_{q^j} Q\right\}, \tag{9.3.18}$$

and the $(+)$ ***discrete vertical space*** *at* q^{j+1} *is*

$$V^+_{q^{j+1}} Q = \left\{(q^j, q^{j+1}) \mid \left(\mathbb{F}^+ L_d(q^j, q^{j+1})\right)^\sharp \in V_{q^{j+1}} Q\right\}. \tag{9.3.19}$$

To define the discrete horizontal space we will shift the problem to the cotangent bundle T^*Q. We recall that, given the decomposition $TQ = VQ \oplus HQ$, we have $T^*Q = (VQ)^\circ \oplus (HQ)^\circ$, where $(VQ)^\circ$ and $(HQ)^\circ$ are the annihilators of VQ and HQ, that is,

$$\begin{aligned}(V_q Q)^\circ &= \left\{\alpha_q \in T^*_q Q \mid \langle \alpha_q, v_q\rangle = 0, \ \ \forall v_q \in V_q Q\right\},\\ (H_q Q)^\circ &= \left\{\beta_q \in T^*_q Q \mid \langle \beta_q, w_q\rangle = 0, \ \ \forall w_q \in H_q Q\right\}.\end{aligned}$$

Since the Legendre transform of the smooth Lagrangian L coincides with the flat (index lowering) operator induced by the metric γ, we have the relations

$$\mathbb{F}L(V_q Q) = (H_q Q)^\circ, \qquad \mathbb{F}L(H_q Q) = (V_q Q)^\circ,$$

and the equivalence $v_q \in H_q Q \iff \mathbb{F}L(v_q) \in (V_q Q)^\circ$.

In the definition below, we adapt this relation to the discrete case, in order to obtain the notion of discrete horizontal space.

9.3.8 Definition *Given the discrete mechanical connections* $\mathcal{A}^{\pm}_{L_d} : Q \times Q \to \mathfrak{g}$, *the* $(-)$ ***discrete horizontal space*** *at* q^j *is*

$$H^-_{q^j} Q = \left\{ (q^j, q^{j+1}) \mid \mathbb{F}^- L_d(q^j, q^{j+1}) \in (V_{q^j} Q)^\circ \right\}, \tag{9.3.20}$$

and the $(+)$ ***discrete horizontal space*** *at* q^{j+1} *is*

$$H^+_{q^{j+1}} Q = \left\{ (q^j, q^{j+1}) \mid \mathbb{F}^+ L_d(q^j, q^{j+1}) \in (V_{q^{j+1}} Q)^\circ \right\}. \tag{9.3.21}$$

9.3.9 Proposition *We have the following equivalences*

$$\begin{aligned}(q^j, q^{j+1}) \in H^-_{q^j} Q &\Longleftrightarrow \mathcal{A}^-_{L_d}(q^j, q^{j+1}) = 0 \Longleftrightarrow \mathbf{J}^-_{L_d}(q^j, q^{j+1}) = 0 \\ &\Longleftrightarrow \mathbb{F}^- L_d(q^j, q^{j+1})^\sharp \in H_{q^j} Q\end{aligned}$$

$$\begin{aligned}(q^j, q^{j+1}) \in H^+_{q^{j+1}} Q &\Longleftrightarrow \mathcal{A}^+_{L_d}(q^j, q^{j+1}) = 0 \Longleftrightarrow \mathbf{J}^+_{L_d}(q^j, q^{j+1}) = 0 \\ &\Longleftrightarrow \mathbb{F}^+ L_d(q^j, q^{j+1})^\sharp \in H_{q^{j+1}} Q\end{aligned}$$

Moreover, if the discrete Lagrangian is G*-invariant we have*

$$(q^j, q^{j+1}) \in H^-_{q^j} Q \Longleftrightarrow (q^j, q^{j+1}) \in H^+_{q^{j+1}} Q.$$

Proof. We have the following equivalences

$$\begin{aligned}(q^j, q^{j+1}) \in H^-_{q^j} Q &\overset{(9.3.20)}{\Longleftrightarrow} \left\langle \mathbb{F}^- L_d(q^j, q^{j+1}), \xi_Q(q^j) \right\rangle = 0, \ \ \forall\, \xi \in \mathfrak{g} \\ &\overset{(9.3.5)}{\Longleftrightarrow} \left\langle \mathbf{J}\left(\mathbb{F}^- L_d(q^j, q^{j+1})\right), \xi \right\rangle = 0, \ \ \forall\, \xi \in \mathfrak{g} \\ &\overset{(9.3.12)}{\Longleftrightarrow} \left\langle \mathbb{I}(q^j)\left(\mathcal{A}^-_{L_d}(q^j, q^{j+1})\right), \xi \right\rangle = 0, \ \ \forall\, \xi \in \mathfrak{g} \\ &\overset{(9.3.6)}{\Longleftrightarrow} \gamma\left(\left[\mathcal{A}^-_{L_d}(q^j, q^{j+1})\right]_Q (q^j), \xi_Q(q^j) \right) = 0, \ \ \forall\, \xi \in \mathfrak{g} \\ &\Longleftrightarrow \left[\mathcal{A}^-_{L_d}(q^j, q^{j+1})\right]_Q (q^j) = 0 \\ &\overset{(9.3.16)}{\Longleftrightarrow} \operatorname{ver}_{q^j}\left(\mathbb{F}^- L_d(q^j, q^{j+1})^\sharp\right) = 0,\end{aligned}$$

where γ is the Riemannian metric on Q. The other equivalences follow from the fact that the action is free and $\mathbb{I}(q)$ is an isomorphism.

If the discrete Lagrangian is G-invariant the additional equivalence follows from the equality $\mathbf{J}^-_{L_d}(q^j, q^{j+1}) = \mathbf{J}^+_{L_d}(q^j, q^{j+1})$. ■

Horizontal trajectories. Recall that if the discrete Lagrangian is G-invariant, then the momentum maps coincide, i.e., $\mathbf{J}^+_{L_d} = \mathbf{J}^-_{L_d}$, and are the same conserved quantity. So, if $(q^0, q^1) \in H^-_{q^0} Q$ (or, equivalently, $(q^0, q^1) \in H^+_{q^1} Q$), then the solution $\{q^j\}_{j=0}^N$ of the discrete Euler-Lagrange equations is necessarily horizontal, that is, $(q^j, q^{j+1}) \in H^-_{q^j} Q$ (or, equivalently $(q^j, q^{j+1}) \in H^+_{q^j} Q$), for all $j = 0, ..., N-1$.

9.3.10 Definition *The **discrete horizontal projections** at q^j and q^{j+1} are the maps*

$$\begin{aligned} \operatorname{hor}^-_{q^j} &: \{q^j\} \times Q \to H^-_{q^j} Q \subset \{q^j\} \times Q \\ \operatorname{hor}^+_{q^{j+1}} &: Q \times \{q^{j+1}\} \to H^+_{q^{j+1}} Q \subset Q \times \{q^{j+1}\} \end{aligned}$$

defined by

$$\begin{aligned} \operatorname{hor}^-_{q^j}(q^j, q^{j+1}) &:= \left(\mathbb{F}^- L_d\right)^{-1} \left(\operatorname{hor}_{q^j}\left(\mathbb{F}^- L_d(q^j, q^{j+1})^\sharp\right)\right)^\flat, \\ \operatorname{hor}^+_{q^{j+1}}(q^j, q^{j+1}) &:= \left(\mathbb{F}^+ L_d\right)^{-1} \left(\operatorname{hor}_{q^{j+1}}\left(\mathbb{F}^+ L_d(q^j, q^{j+1})^\sharp\right)\right)^\flat. \end{aligned} \tag{9.3.22}$$

Note that we have the equivalences

$$\begin{aligned} (q^j, q^{j+1}) \in H^-_{q^j} Q &\Longleftrightarrow \operatorname{hor}^-_{q^j}(q^j, q^{j+1}) = (q^j, q^{j+1}) \\ (q^j, q^{j+1}) \in H^+_{q^{j+1}} Q &\Longleftrightarrow \operatorname{hor}^+_{q^j}(q^j, q^{j+1}) = (q^j, q^{j+1}) \end{aligned}$$

and, in view of (9.3.16) and (9.3.17), the relations

$$\begin{aligned} \Big(\mathbb{F}L^-_d(\operatorname{hor}^-_{q^j}(q^j, q^{j+1}))\Big)^\sharp + \operatorname{ver}_{q^j}\left(\mathbb{F}L^-_d(q^j, q^{j+1})^\sharp\right) &= \left(\mathbb{F}L^-_d(q^j, q^{j+1})\right)^\sharp \\ \Big(\mathbb{F}L^+_d(\operatorname{hor}^+_{q^{j+1}}(q^j, q^{j+1}))\Big)^\sharp + \operatorname{ver}_{q^{j+1}}\left(\mathbb{F}L^+_d(q^j, q^{j+1})^\sharp\right) &= \left(\mathbb{F}L^+_d(q^j, q^{j+1})\right)^\sharp. \end{aligned} \tag{9.3.23}$$

These identities suggest the definition of the two discrete vertical projections.

9.3.11 Definition *The **discrete vertical projections** at q^j and q^{j+1}, expressed in $\{q^j\} \times Q$ and $Q \times \{q^{j+1}\}$, are the maps*

$$\begin{aligned} \operatorname{ver}^-_{q^j} &: \{q^j\} \times Q \to V^-_{q^j} Q \subset \{q^j\} \times Q \\ \operatorname{ver}^+_{q^{j+1}} &: Q \times \{q^{j+1}\} \to V^+_{q^{j+1}} Q \subset Q \times \{q^{j+1}\} \end{aligned}$$

defined by

$$\begin{aligned} \operatorname{ver}^-_{q^j}(q^j, q^{j+1}) &= (\mathbb{F}L^-_d)^{-1}\left(\left[\mathcal{A}^-_{L_d}(q^j, q^{j+1})\right]_Q (q^j)\right)^\flat, \\ \operatorname{ver}^+_{q^{j+1}}(q^j, q^{j+1}) &= (\mathbb{F}L^+_d)^{-1}\left(\left[\mathcal{A}^+_{L_d}(q^j, q^{j+1})\right]_Q (q^{j+1})\right)^\flat. \end{aligned} \tag{9.3.24}$$

The definition and (9.3.23) imply the analogue of the horizontal plus vertical decomposition in the discrete setting, namely,

$$\begin{aligned} \mathbb{F}L^-_d(\operatorname{hor}^-_{q^j}(q^j, q^{j+1}))^\sharp + \mathbb{F}L^-_d(\operatorname{ver}^-_{q^j}(q^j, q^{j+1}))^\sharp &= \mathbb{F}L^-_d(q^j, q^{j+1})^\sharp, \\ \mathbb{F}L^+_d(\operatorname{hor}^+_{q^{j+1}}(q^j, q^{j+1}))^\sharp + \mathbb{F}L^+_d(\operatorname{ver}^+_{q^{j+1}}(q^j, q^{j+1}))^\sharp &= \mathbb{F}L^+_d(q^j, q^{j+1})^\sharp. \end{aligned} \tag{9.3.25}$$

We summarize the previous discussion in the following commutative diagram:

$$\begin{array}{ccccc} Q \times Q & \xrightarrow{\ \mathbb{F}L^\pm_d\ } & T^*Q & \xrightarrow{\ \sharp\ } & TQ \\ \Big\downarrow{\scriptstyle \operatorname{ver}^\pm \oplus \operatorname{hor}^\pm} & & \Big\| & & \Big\downarrow{\scriptstyle \operatorname{ver} \oplus \operatorname{hor}} \\ VQ^\pm \times_Q HQ^\pm & \xleftarrow{\ (\mathbb{F}L^\pm_d)^{-1}\ } & (VQ)^* \oplus (HQ)^* & \xleftarrow{\ \flat\ } & VQ \oplus HQ. \end{array}$$

We shall need later on the behavior of the discrete Legendre transformation of a discrete G-invariant classical Lagrangian under the group action. Thus, the smooth Lagrangian is equal to the kinetic energy of a G-invariant metric γ on Q minus a G-invariant potential $V : Q \to \mathbb{R}$. Assume that the discrete Lagrangian L_d is also G-invariant. We shall establish the following formulas

$$\begin{aligned} T_{q^j}\Phi_g\left(\mathbb{F}^- L_d(q^j, q^{j+1})^\sharp\right) &= \mathbb{F}^- L_d(gq^j, gq^{j+1})^\sharp \\ T_{q^{j+1}}\Phi_g\left(\mathbb{F}^+ L_d(q^j, q^{j+1})^\sharp\right) &= \mathbb{F}^+ L_d(gq^j, gq^{j+1})^\sharp. \end{aligned} \tag{9.3.26}$$

To do this, we begin by proving that for any $v \in T_qQ$ and $g \in G$ we have

$$T^*_{g^{-1}q}\Phi_g\left(v^\flat\right) = \left(T_q\Phi_{g^{-1}}(v)\right)^\flat \tag{9.3.27}$$

or, equivalently, for any $\alpha \in T^*_qQ$ and $g \in G$

$$\left(T^*_{g^{-1}q}\Phi_g(\alpha)\right)^\sharp = T_q\Phi_{g^{-1}}\left(\alpha^\sharp\right), \tag{9.3.28}$$

where $\flat : T_qQ \ni v \mapsto \gamma(v, \cdot) \in T^*_qQ$ is the index lowering operator defined by the Riemannian metric γ and $\sharp : T^*Q \to TQ$ is its inverse. Indeed, for any $u \in T_{g^{-1}q}Q$, by G-invariance of γ, we have

$$\left\langle T^*_{g^{-1}q}\Phi_g\left(v^\flat\right), u\right\rangle = \gamma\left(v, T_{g^{-1}q}\Phi_g u\right) = \gamma\left(T_q\Phi_{g^{-1}}(v), u\right) = \left\langle \left(T_q\Phi_{g^{-1}}(v)\right)^\flat, u\right\rangle$$

which proves (9.3.27). Identity (9.3.28) is obtained by letting $\alpha = v^\flat$.

Now we prove the first identity in (9.3.26). Taking the derivative with respect to q^j in the identity $L_d(gq^j, gq^{j+1}) = L_d(q^j, q^{j+1})$, we obtain

$$D_1L_d\left(q^j, q^{j+1}\right) = D_1L_d\left(gq^j, gq^{j+1}\right) \circ T_{q^j}\Phi_g$$

and hence

$$\begin{aligned} \mathbb{F}^- L_d(q^j, q^{j+1})^\sharp &\overset{(1.2.3)}{=} -D_1L_d\left(q^j, q^{j+1}\right)^\sharp = -\left(D_1L_d\left(gq^j, gq^{j+1}\right) \circ T_{q^j}\Phi_g\right)^\sharp \\ &= -\left(T^*_{q^j}\Phi_g\left(D_1L_d\left(gq^j, gq^{j+1}\right)\right)\right)^\sharp \\ &\overset{(9.3.28)}{=} -T_{gq^j}\Phi_{g^{-1}}\left(D_1L_d\left(gq^j, gq^{j+1}\right)\right)^\sharp. \end{aligned}$$

Thus, applying $T_{q^j}\Phi_g$ to this relation and using the definition of $\mathbb{F}^-L_d$ yields the first identity in (9.3.26).

9.3.12 Lemma *For all $g \in G$, we have*

$$\begin{aligned} \Phi_g^{Q\times Q}\left(H^-_{q^j}Q\right) &= H^-_{\Phi_g(q^j)}Q \\ \Phi_g^{Q\times Q}\left(H^+_{q^{j+1}}Q\right) &= H^+_{\Phi_g(q^{j+1})}Q \\ \Phi_g^{Q\times Q}(\mathrm{hor}^-_{q^j}(q^j, q^{j+1})) &= \mathrm{hor}^-_{\Phi_g(q^j)}(\Phi_g(q^j), \Phi_g(q^{j+1})) \\ \Phi_g^{Q\times Q}(\mathrm{hor}^+_{q^{j+1}}(q^j, q^{j+1})) &= \mathrm{hor}^+_{\Phi_g(q^{j+1})}(\Phi_g(q^j), \Phi_g(q^{j+1})) \end{aligned}$$

Proof. We have the following equivalences : $(q^j, q^{j+1}) \in H^-_{q^j} Q \Longleftrightarrow \mathcal{A}^-_{L_d}(q^j, q^{j+1}) = 0 \Longleftrightarrow \mathrm{Ad}_g \mathcal{A}^-_{L_d}(q^j, q^{j+1}) = 0 \overset{(9.3.13)}{\Longleftrightarrow} \Phi^{Q\times Q}_g(q^j, q^{j+1}) \in H^-_{\Phi_g(q^j)} Q$, which proves the first equality. The second is shown in an identical manner.

For the next, using the relation $(\Phi^{Q\times Q}_g)^* \mathbb{F}L^\pm_d = \mathbb{F}L^\pm_d$, where $\mathbb{F}L^\pm_d$ are locally isomorphisms, and the definition $\mathrm{hor}(v_q) = v_q - (\mathcal{A}(v_q))_Q(q)$, we obtain

$$\begin{aligned}
\mathrm{hor}^-_{\Phi_g(q^j)}(\Phi^{Q\times Q}_g(q^j, q^{j+1})) &\overset{(9.3.22)}{=} \left(\mathbb{F}L^-_d\right)^{-1} \left(\mathrm{hor}_{\Phi_g(q^j)} \left(\mathbb{F}L^-_d(\Phi^{Q\times Q}_g(q^j, q^{j+1}))^\sharp\right)\right)^\flat \\
&\overset{(9.3.26)}{=} \left(\mathbb{F}L^-_d\right)^{-1} \left(\mathrm{hor}_{\Phi_g(q^j)} \left(T\Phi_g(\mathbb{F}L^-_d(q^j, q^{j+1}))^\sharp\right)\right)^\flat \\
&= \left(\mathbb{F}L^-_d\right)^{-1} \left(T\Phi_g \, \mathrm{hor}_{q^j} \left((\mathbb{F}L^-_d(q^j, q^{j+1}))^\sharp\right)\right)^\flat \\
&\overset{(9.3.27)}{=} \left(\mathbb{F}L^-_d\right)^{-1} T^*\Phi_{g^{-1}} \left(\mathrm{hor}_{q^j} \left((\mathbb{F}L^-_d(q^j, q^{j+1}))^\sharp\right)\right)^\flat \\
&\overset{(9.3.26)}{=} \Phi^{Q\times Q}_g \left(\mathbb{F}L^-_d\right)^{-1} \left(\mathrm{hor}_{q^j} \left((\mathbb{F}L^-_d(q^j, q^{j+1}))^\sharp\right)\right)^\flat .
\end{aligned}$$

The last equality is verified in the same way. ■

Concluding Remarks

We have described a series of Lie group and Lie algebra variational integrators for flexible beams and plates which are synchronous or asynchronous, for finite-element nonlinear dynamics, in order to provide tools to study complex structures, composed of beams and thin plates.

At the same time, we developed a discrete theory, like the discrete affine Euler-Poincaré reduction. Other parts of this theory can be applied to a general configuration space which may or may not be a Lie group. For example, we introduced discrete mechanical connections.

The results we obtained by implementations and benchmarks have always verified the theory. It seems that the algorithms we get are faster than energy-momentum preserving algorithms. However, we know that much work remains to be done, because the development of these integrators is associated with the development of the theory.

In this thesis, we have not studied the order of approximation of the results as it was done in Marsden and West [90]. This aspect of the theory is an exciting direction of research for the future.

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Submitted papers

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