Lie group and Lie algebra variational integrators for flexible beam and plate in $\mathbb{R}^3$

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1 Introduction

The purpose of this research project is to develop variational integrators synchronous or asynchronous, which can be used as tools to study complex structures composed of plates and beams subjected to large deformations and stress.

We consider the geometrically exact models of beam and plate, whose configuration spaces are Lie groups. These models are suitable for modeling objects subjected to large deformations, where the stored energy chosen is adapted for the types of materials used in our field (isotropic or composite).

The work of J. E. Marsden, and of his doctoral and post-doctoral students, were the basis for the development of variational integrators which are symplectic and perfectly preserve symmetries. Furthermore, discrete mechanical systems with symmetry can be reduced. In addition, by a "good discretization", the strain measures are unchanged by superposed rigid motion.

The idea behind this work is to take advantage of the properties of these integrators to define the equilibrium position of structures, which are generally unknown, as well as to determine the constraints, while preserving the invariants of the structure.

Along with solving this problem, we continue the approach of J.E. Marsden which consists to lay the foundations of discrete mechanics, with its theorems, its axioms, its definitions, which have the same value as the laws of continuous mechanics but for a discrete domain. That is, the discrete trajectories of a motion obtained by variational integrators satisfy these discrete laws.

2 State of research done in the chosen field during the past year

2.1 Study of the dynamic of the spring pendulum

We consider the spring pendulum where the mass is attached to the point pivot by a spring. The potential energy of the spring pendulum is not only given by the exterior gravitational field, giving rise to the exterior gravitational potential W_{ext}, but also of an internal potential $W_{spring} = \frac{1}{2}k\Delta\mathbf{x}^2$, where $\Delta\mathbf{x}$ is the elongation of the spring, and k is the spring constant. This internal energy is similar to the energy due to the elongation of the beam $W_{elongation} = \frac{1}{2}EA\left(\frac{\partial\phi_x}{\partial x}\right)^2$, where $\partial\phi_x/\partial x$ is the longitudinal strain of the beam in a point x of the line of centroids, E is the Young modulus, and A is the cross-sectional area.

For physicists, the behavior of the elastic spherical pendulum is quite similar to the molecular oscillations of CO_2 and with 2nd harmonic generation in nonlinear laser optics, as indicated by Holms [2008].

The space of configuration is a Lie group, thus we develop a Lie group variational integrator as so to study the motion. By this simple example we highlight the properties of variational integrators.

(Ref 1 : Demoures F. Lie group variational integrator and spring pendulum, 2012.)

2.2 Study of dynamic and static of beams under large overall motions

The goal of this study is to derive a structure preserving integrator for geometrically exact beam dynamics. We use the Lagrangian variational formulation of the continuous problem to obtain a Lie group variational integrator that preserves the symmetries and symplectic structure at the discrete level. In addition, the algorithm exhibits almost-perfect energy conservation. The geometrically exact theory of beam dynamics was developed in Simo [1985], Simo, Marsden, and Krishnaprasad [1988]. In the paper we submitted, we take advantage of this geometric approach to deduce a discrete variational principle in convective representation, thereby obtaining a structure preserving integrator. We derive a numerical scheme for the geometrically exact theory of beams by using variational integrators.

The configuration space is $Q = \{\phi = (\phi_0, \Lambda) \mid C^\infty\left([0, L], \mathbb{R}^3 \times SO(3)\right)\}$, where ϕ_0 is a mid-line in $C^\infty\left([0, L], \mathbb{R}^3\right)$, and $\Lambda \in SO(3)$. Given $\{\mathbf{E}_I\}$, the standard basis in $\mathbb{R}^3$, the beam is explicitly described as$x = \phi(X^1, X^2, X^3, t) := \phi_0(S, t) + \sum_{\alpha=1}^{2} X^\alpha \Lambda \mathbf{E}_\alpha(X^3, t)$.

We considered two temporal discretizations, associated to synchronized and asynchronized time variational integrators. As well we took in count the conservation of discrete momentum map which is conserved for a given symmetry. The strain of the corresponding discrete model remains objective (frame-indifferent). This is a fundamental property of three-dimensional elasticity which can be violated by certain interpolations of rotations. The inherent property to preserve the symmetries allows us to properly define the equilibrium position.

This study was carried out with F. Gay-Balmaz (Cnrs - Ecole Normale Sup Paris), and the implementation of the synchronous Lie group variational integrator was performed by S. Leyendecker (University of Erlangen-Nuremberg), and S. Ober-Blöbaum (University of Paderborn) while the implementation in progress of the asynchronous Lie group variational integrator is performed by T. Leitz (University of Erlangen-Nuremberg).

(Ref 2 : Demoures F., Gay-Balmaz F., Leyendecker S., Ober-Blobaum S., Ratiu T.S., and Weinand Y. *Discrete variational Lie group formulation of geometrically exact beam dynamics*, 2012. (Submitted paper.))

2.3 Study of affine Euler-Poincaré reduction applied to a beam model

Reduction is an important tool to investigate many aspects of mechanical systems with symmetry. Indeed, apart from the computational simplification afforded by reduction, reduction also is an interesting way to identify invariant subsystems.

The affine Euler-Poincaré reduction is concerned with some important themes. Namely, the semi-direct product of a group G with a vector space V, where the construction of a semi-direct product involves a Lie group representation, secondly the one-cocycle $c \in \mathcal{F}(G, V^*)$ and the associated affine representation $\theta : G \to GL(V^*)$, and finally the reduction which may be Euler-Poincaré or Lie-Poisson.

The theory of affine Euler-Poincaré that brings together these three themes was developed in Gay-Balmaz, and Ratiu [2009] for fluid mechanics, and in Ellis, Gay-Balmaz, Holm, Putkaradze, and Ratiu [2010] for charged molecular strands.

For the beam the gravity breaks the $SO(3)$ symmetry. The potential energy is only invariant under rotations S^1 about vertical axis $\mathbf{E}$. In this case it is more interesting to consider the Lagrangian $L : TG \times V^* \to \mathbb{R}$ defined on $TG \times V^*$, where V^* is the space of linearly advected quantities such as strain (Ω, Γ) or the direction $\chi = \Lambda^{-1}\mathbf{E}$. Then the Lagrangian $L : TG \times V^* \to \mathbb{R}$ is left G-invariant under the affine action $(v_h, a) \mapsto (g\, v_h, \theta_g\, a) = (g\, v_h, g\, a + c(g))$ where $g, h \in G$, $v_h \in TG$, $a \in V^*$, and $c \in \mathcal{F}(G, V^*)$ is a one-cocycle, then we can consider the affine Euler-Poincaré theory.

We develop the discrete affine Euler Poincaré theory, in order to obtain a Lie group G-invariant discrete Lagrangian, and a discrete reduction.

This study was carried out with F. Gay-Balmaz (Cnrs - Ecole Normale Sup Paris).

2.4 Study of Lie algebra variational integrator of geometrically exact beam dynamics

The issue is to obtain a variational integrator based on a Lie algebra point of view, which analyzes the deformations of the geometrically exact model of a beam introduced and described in (2.2).

Take the configuration space of the beam to be $Q := C^\infty([0, \ell], SE(3))$, the space of smooth curves defined on the closed interval $[0, \ell]$ with values in the special Euclidean Lie group $SE(3)$. For the given Lie group $G = SE(3)$, the Lagrangian $L : TQ \to \mathbb{R}$ of the beam studied in this paper, has the form $L(g, \dot{g}) = \frac{1}{2}\gamma(\dot{g}, \dot{g}) - V(g)$, where γ

is a *G-invariant Riemannian metric* on the configuration space Q, and $V : Q \to \mathbb{R}$ is the G-invariant potential energy. Then, pushing forward L by left trivialization $(g, \dot{g}) \mapsto (g, g^{-1}\dot{g})$ gives the trivialized Lagrangian $\mathcal{L}(g, \xi) := L(g, \dot{g}), \quad \dot{g} := g\xi$, which is consistent with the convected representation.

In this study we discretize spatially the interval $[0, \ell]$ by a set $\mathcal{T}$ of N simplexes K with nodes a, while the objectivity strain measure is preserved (frame-indifference). Next, we discretize temporally, and approximate the convected velocities $\xi_a = g_a^{-1}\dot{g}_a$, at each node by elements in the Lie algebra $\mathfrak{g} = \mathfrak{se}(3)$. We obtain the discrete Lagrangian $\mathscr{L}_K^j : G \times \mathfrak{g} \to \mathbb{R}$ approximating the action of the trivialized Lagrangian $\mathcal{L}_K : G \times \mathfrak{g} \to \mathbb{R}$ over the interval $[t^j, t^{j+1}]$, for elements K of length l_K. By applying the discrete Hamilton variational principle, we get the discrete Euler-Lagrange equations. The associated discrete evolution operator is $F : G \times \mathfrak{g} \to G \times \mathfrak{g}, \quad (g^{j-1}, \xi^{j-1}) \mapsto (g^j, \xi^j)$, where $\tau : \mathfrak{g} \to G$ with $\tau(0) = e$ is a smooth map such that $\tau(\xi^j) = (g^j)^{-1}g^{j+1}$; for example, τ may be exponential map or the Cayley transform. Thus we obtain an integrator which has the properties of all the variational integrators and is numerically efficient.

This study was carried out with F. Gay-Balmaz (Cnrs - Ecole Normale Sup Paris), and the implementation, in progress, is performed Marin Kobilarov (Johns Hopkins University).

2.5 Study of dynamic plates under large overall motions

We consider the geometrically exact model of plate as defined in Simo, Marsden, and Krishnaprasad [1988] and Simo, and Fox [1989]. The space of configuration of this plate is very similar to that of the exact model of beam defined in Simo [1985]. Indeed, for the beam, the space of configuration is $\mathcal{C}^\infty([0, L], SE(3))$, whereas for the plate it is $\mathbf{N} = \mathcal{C}^\infty(\mathcal{A}, S^2_{\mathbf{E}} \times \mathbb{R}^3)$ which is a subset of $Q = \mathcal{C}^\infty(\mathcal{A}, SE(3))$, where $\mathcal{A} \subset \mathbb{R}^2$ is an open set with smooth boundary, and compact closure, and $S^2_{\mathbf{E}}$ is the set of rotations whose rotation axis is normal to the vertical direction $\mathbf{E}$. We note that $S^2_{\mathbf{E}}$ is a sub-set of $SO(3)$ and not a sub-group. In order to maintain the matrix of rotation in $S^2_{\mathbf{E}}$ we introduce an holonomic constraint $\Phi : Q \to \mathbb{R}^d$, such that $\mathbf{N} = \Phi^{-1}(0) \subset Q$. Thus the solutions of the Euler-Lagrange equations stay in $\mathbf{N}$.

And we developed a Lie algebra variational integrator to take full advantage of its numerically efficiency, and properties. Moreover, another integrator for plates is in progress; it will be obtained with the gained experience from the study of variational integrators for beams. Then we will start to implement and compare the different integrators for plates.

This study was carried out with F. Gay-Balmaz (Cnrs - Ecole Normale Sup Paris).

2.6 Study of dissipation added to the reduced discrete affine Euler-Poincaré system

The purpose of this study is the dissipation added to the reduced discrete affine Euler-Poincaré system with symmetry.

Phenomenons of dissipation and instability for Euler-Poincaré systems on the Lie algebra, or equivalently for Lie-Poisson systems on the duals of the Lie algebras, were studied in Bloch, Krishnaprasad, Marsden, and Ratiu [1996]. This paper is a reference for our work. Thus a dissipative force is construct which dissipates the energy, but angular momentum is conserved, or equivalently symmetries are conserved.

In view of the objective that we have set to find the equilibrium position of a structure, it is essential to preserve symmetries when applying dissipation.

Our point of view is different from that of Energy-Dissipative Momentum-Conserving algorithms as there are no conditions on the discrete Lagrangian. Then after, we take into account the discrete theory that we establish from the law of the mechanics.

This study was carried out with F. Gay-Balmaz (Cnrs - Ecole Normale Sup Paris).

2.7 Study of Discrete mechanical connection

Principal connections, and in particular mechanical connections, are an important tool, which allows one to split the trajectories into a horizontal and a vertical part. The vertical equation gives the trajectories along the orbit associated to the action of a Lie group G, and the horizontal is perpendicular to that orbit. The first one is associated with the Euler-Poincaré equation, and the last one with the Euler-Lagrange equation. (See Cendra, Marsden, and Ratiu [2001].)

Moreover mechanical connections allow to study the stability and bifurcation of relative equilibria. Where the relative equilibria are the dynamic orbits generated by the symmetry group, which correspond to equilibrium points in the quotient space. When stability of a relative equilibrium is lost, one can get bifurcation, instability and chaos. (See Hernández-Garduno, and Marsden [2004].) Furthermore, mechanical connections play an important role in the energy-momentum method. (See Lewis, and Simo [1990].)

We obtain definitions and expressions in coordinates of the discrete mechanical connection, as well as of the discrete vertical and horizontal trajectories which are reminiscent of the continuous expressions.

This study was carried out with Francois Gay-Balmaz (Cnrs - Ecole Normale Sup Paris).

2.8 Importants news

2.8.1 Thesis exam.

François Demoures thesis exam took place on Wednesday 10th October 2012.

2.8.2 Cooperations

- **Francois Gay-Balmaz** (CNRS-Normale Sup Paris), theory
- **Sigrid Leyendecker** (Erlangen), implementation, advices, and achievement of benchmarks

- **Sina Ober-Blöbaum** (Paderborn), implementation, and advices
- **Marin Kobilarov** (Johns Hopkins), implementation, and advices
- **Mathieu Desbrun** (Caltech), general advices

3 Output

3.1 Conference activite

Coorganizers (F. Demoures, T.S. Ratiu and Y. Weinand) of the Workshop *Discrete Mechanics and Integrators*, Bernoulli Center, EPFL, October 6, 2011.

http://bernoulli.epfl.ch/PublicProgramsLstOut.php

3.2 Submitted papers

- Demoures F., Gay-Balmaz F., Leyendecker S., Ober-Blobaum S., Ratiu T.S., and Weinand Y. *Discrete variational Lie group formulation of geometrically exact beam dynamics*, 2012.
- Demoures F. *Lie group variational integrator and spring pendulum*, 2012.

3.3 Public defense of F. Demoures thesis

On Friday 16th November 2012.

3.4 Papers in progress

- F. Demoures, F. Gay-Balmaz, S. Leyendecker, S. Ober-Blöbaum, and T. S. Ratiu. *Structure preserving discretization of geometrically exact beams.*
- F. Demoures, F. Gay-Balmaz, M. Kobilarov, and T. S. Ratiu. *Multisymplectic Lie algebra variational integrator for a geometrically exact beam in* $\mathbb{R}^3$.