

Weak second order explicit stabilized methods for stiff stochastic differential equations

Assyr Abdulle¹, Gilles Vilmart², and Konstantinos C. Zygalakis³

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Abstract

We introduce a new family of explicit integrators for stiff Itô stochastic differential equations (SDEs) of weak order two. These numerical methods belong to the class of one-step stabilized methods with extended stability domains and do not suffer from the stepsize reduction faced by standard explicit methods. The family is based on the standard second order orthogonal Runge-Kutta Chebyshev methods (ROCK2) for deterministic problems. The convergence, and the mean-square and asymptotic stability properties of the methods are analyzed. Numerical experiments, including applications to nonlinear SDEs and parabolic stochastic partial differential equations are presented and confirm the theoretical results.

Keywords: Stiff SDEs, explicit stochastic methods, stabilized methods, orthogonal Runge-Kutta Chebyshev, S-ROCK.

AMS subject classification (2010): 65C30, 60H35, 65L20

1 Introduction

We consider stiff systems of Itô stochastic differential equations for which standard explicit integrators – e.g., the well-known Euler-Maruyama method – face a severe step size restriction [15, 13, 17]. Such problems are usually solved numerically by (semi)-implicit methods, which can be expensive for large systems and difficult to implement for complex problems. Recently, a new class of explicit stabilized methods called S-ROCK has been introduced for stiff problems [2, 4]. On the one hand, these methods (fully explicit) are as easy to implement as the Euler-Maruyama method. On the other hand, their extended mean-square stability regions [15] (for suitable test problems) make them much more efficient than classical explicit methods for stiff problems.

In this paper, we introduce a weak second order family of explicit stabilized integrators based on the second order ROCK2 integrators for deterministic stiff problems. The main feature of the algorithm is that it has an arbitrarily large mean-square stability region that grows quadratically with respect to the number of drift function evaluations. For an efficient implementation, the integrators are derivative free, and the number of diffusion function evaluations is independent of the number of Wiener processes involved, similarly to the methods introduced in [28]. As an additional feature, the proposed methods have a large asymptotic stability region close to the origin.

Up to now, with the exception of [19], only weak first order explicit stabilized methods have been proposed for stiff stochastic problems. In [19] an attempt to generalize the S-ROCK methods

¹Mathematics Section, École Polytechnique Fédérale de Lausanne, Station 8, 1015 Lausanne, Switzerland, Assyr.Abdulle@epfl.ch

²École Normale Supérieure de Cachan, Antenne de Bretagne, INRIA Rennes, IRMAR, CNRS, UEB, av. Robert Schuman, F-35170 Bruz, France, Gilles.Vilmart@bretagne.ens-cachan.fr

³School of Mathematics, University of Southampton, Southampton SO17 1BJ, UK, k.zygalakis@soton.ac.uk

to weak second order has been proposed. However, this generalization involves the solution of a large number of order conditions and the resulting methods have less favorable mean-square stability properties than the methods proposed in this paper (see e.g., [2, 4]).

This paper is organized as follows. We recall in Section 2 the concept of stabilized methods for stiff SDEs. In Section 3, we introduce our new weak second order explicit stabilized integrators and analyze their weak order of convergence and stability properties. Finally, we present in Section 4 various numerical experiments, both for linear and non-linear stiff SDEs and for parabolic stochastic partial differential equations (SPDEs), that illustrate the efficiency of the proposed methods.

2 Stabilized stochastic methods

We briefly recall the notions of weak convergence, mean-square and asymptotic stabilities.

2.1 Weak stochastic integrators

We consider the Itô stochastic system of differential equations

$$dX(t) = f(X(t))dt + \sum_{r=1}^m g^r(X(t))dW_r(t), \quad X(0) = X_0, \quad (1)$$

where $X(t)$ is a random variable with values in \mathbb{R}^N , $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is the drift term, $g^r : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $r = 1, \dots, m$ are the diffusion terms, and $W_r(t)$, $r = 1, \dots, m$ are independent one-dimensional Wiener processes. The drift and diffusion functions are assumed smooth enough, Lipschitz continuous and to satisfy a growth bound in order to ensure a unique (mean-square bounded) solution of (1) [6, 17]. For the numerical approximation of (1) we consider the discrete map

$$X_{n+1} = \Psi(X_n, h, \xi_n), \quad (2)$$

where $\Psi(\cdot, h, \xi_n) : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $X_n \in \mathbb{R}^N$ for $n \geq 0$, h denotes the timestep size, and ξ_n denotes a random vector. The numerical approximation (2), starting from the exact initial condition X_0 of (1) is said to have weak order τ if for all functions¹ $\phi : \mathbb{R}^N \rightarrow \mathbb{R} \in C_P^{2(\tau+1)}(\mathbb{R}^N, \mathbb{R})$,

$$|\mathbb{E}(\phi(X_n)) - \mathbb{E}(\phi(X(t_n)))| \leq Ch^\tau, \quad (3)$$

and to have strong order τ if

$$\mathbb{E}(|X_n - X(t_n)|) \leq Ch^\tau, \quad (4)$$

for any $t_n = nh \in [0, T]$ with $T > 0$ fixed, for h small enough with constants C independent of h .

Remark 2.1. *A well-known theorem of Milstein [23] allows to infer the global weak order from the error after one step. Assuming that $f, g^r \in C_P^{2(\tau+1)}(\mathbb{R}^N, \mathbb{R}^N)$ are Lipschitz continuous, that the moments of the exact solution of (1) exist and are bounded (up to a sufficiently high order), and that the local error bound for all $\phi \in C_P^{2(\tau+1)}(\mathbb{R}^N, \mathbb{R})$ and all initial values $X(0) = X_0$ satisfies*

$$|\mathbb{E}(\phi(X_1)) - \mathbb{E}(\phi(X(t_1)))| \leq Ch^{\tau+1}, \quad (5)$$

for all h sufficiently small, then the global error bound (3) holds. Here the constant C is again independent of h . For the strong convergence we have the following result [24]. If

$$\mathbb{E}|X_1 - X(t_1)| \leq Ch^{\tau+1/2} \quad \text{and} \quad |\mathbb{E}(X_1) - \mathbb{E}(X(t_1))| \leq Ch^{\tau+1}, \quad (6)$$

for all initial values $X(0) = X_0$, then the global error bound (4) holds.

¹Here and in what follows, $C_P^\ell(\mathbb{R}^N, \mathbb{R})$ denotes the space of ℓ times continuously differentiable functions $\mathbb{R}^N \rightarrow \mathbb{R}$ with all partial derivatives with polynomial growth.

The simplest method to approximate solutions to (1) is the so-called Euler-Maruyama method

$$X_{n+1} = X_n + hf(X_n) + \sum_{r=1}^m g^r(X_n)\Delta W_{n,r}, \quad (7)$$

where $\Delta W_{n,r} \sim \mathcal{N}(0, h)$, $r = 1, \dots, m$ are independent Wiener increments. This method has strong order 1/2 and weak order 1 for general a system of Itô SDEs [21]. Various higher order weak methods have been considered in the literature [17, 25]. For example, weak second order methods were proposed by Milstein [22, 23], Platen [26], Talay [30] and Tocino and Vigo-Aguiar [33], and more recently Runge-Kutta type methods of Rössler [27]. We mention also the extrapolation methods of Talay and Tubaro [31] and of [18] that combines methods with different stepsizes to achieve higher weak order convergence. We recall here the weak second order Taylor method known as the Milstein-Talay method [30] that will be useful in what follows.

$$\begin{aligned} X_1 &= X_0 + hf(X_0) + \sum_{r=1}^m g^r(X_0)\Delta W_r + \sum_{q,r=1}^m (g^r(X_0))'g^q(X_0)I_{q,r} \\ &+ \frac{h^2}{2} \left(f'(X_0)f(X_0) + \frac{1}{2} \sum_{r=1}^m f''(X_0)(g^r(X_0), g^r(X_0)) \right) + \sum_{r=1}^m (f(X_0))'g^r(X_0)I_{r,0} \\ &+ \sum_{r=1}^m \left((g^r(X_0))'f(X_0) + \sum_{q=1}^m \frac{1}{2} (g^r(X_0))''(g^q(X_0), g^q(X_0)) \right) I_{0,r}, \end{aligned} \quad (8)$$

where $I_{r,0}, I_{0,r}, I_{q,r}$ denote the stochastic integrals defined by

$$I_{r,0} = \int_{t_0}^{t_1} \int_{t_0}^t dW_r(s)dt, \quad I_{0,r} = \int_{t_0}^{t_1} \int_{t_0}^t ds dW_r(t), \quad I_{q,r} = \int_{t_0}^{t_1} \int_{t_0}^t dW_q(s)dW_r(t). \quad (9)$$

Notice that for notational brevity, we shall always write X_1 and X_0 in place of X_{n+1} and X_n when introducing an integrator. As given above, the method (8) is not practical for implementation: it contains derivatives which are expensive in general, and the stochastic integrals that are difficult to simulate. We will discuss these issues in Section 3.1 where we derive a stabilized explicit version of the Milstein-Talay method suitable for stiff problems.

2.2 Stability concepts for SDEs

In practice it is not only the order of convergence that guarantees an efficient approximation of an SDE, but also the long-time behavior of the solution. Stability properties of the exact and the numerical solutions are important to understand this behavior. Widely used characterizations of stability for SDEs are the mean-square and the asymptotic stability (in the large) [6, 14]. The former measures the stability of moments, the latter measures the overall behavior of sample paths. In particular, we have the following definitions. The steady solution $X \equiv 0$ of (1) with $f(0) = g^r(0) = 0$, $r = 1, \dots, m$ is called stochastically asymptotically stable in the large if

$$\lim_{t \rightarrow \infty} |X(t)| = 0, \quad \text{with probability 1 for all } |X_0| < \delta, \quad (10)$$

mean-square stable if there exists $\delta > 0$, such that

$$\lim_{t \rightarrow \infty} \mathbb{E}(|X(t)|^2) = 0 \quad \text{for all } |X_0| < \delta. \quad (11)$$

2.2.1 The stochastic scalar test equation with multiplicative noise

To gain insight on the stability behavior of a numerical method, we consider a class of linear scalar test problems widely used in the literature [29, 15, 9, 32] (see also [7, 8] where other test equations have been considered recently),

$$dX = \lambda X dt + \mu X dW(t), \quad X(0) = 1, \quad (12)$$

in dimensions $N = m = 1$, with fixed complex scalar parameters λ, μ . The exact solution of (12), given by $X(t) = \exp((\lambda + \frac{1}{2}\mu^2)t + \mu W(t))$, is stochastically asymptotically stable if and only if

$$\lim_{t \rightarrow \infty} |X(t)| = 0, \text{ with probability } 1 \iff \mathcal{S}_{\text{SDE}}^{\text{AS}} := \left\{ (\lambda, \mu) \in \mathbb{C}^2 ; \Re(\lambda - \frac{1}{2}\mu^2) < 0 \right\}, \quad (13)$$

and mean-square stable if and only if

$$\lim_{t \rightarrow \infty} \mathbb{E}(|X(t)|^2) = 0 \iff (\lambda, \mu) \in \mathcal{S}_{\text{SDE}}^{\text{MS}} := \left\{ (\lambda, \mu) \in \mathbb{C}^2 ; \Re(\lambda) + \frac{1}{2}|\mu|^2 < 0 \right\}. \quad (14)$$

We name the domains $\mathcal{S}_{\text{SDE}}^{\text{MS}} \subset \mathcal{S}_{\text{SDE}}^{\text{AS}}$ the mean-square and asymptotic stability domains of the test equation (12), respectively.

2.2.2 Stability of numerical integrators for SDEs

We now look for conditions such that a numerical method (2) applied to the linear test problem (12) yields numerically stable solutions. Similarly to the continuous case we say that the numerical method (2) applied to (12) is said to be

- numerically asymptotically stable if $\lim_{n \rightarrow \infty} |X_n| = 0$ with probability 1;
- numerically mean-square stable if $\lim_{n \rightarrow \infty} \mathbb{E}(|X_n|^2) = 0$.

Applying a numerical method to the test SDE (12) usually yields [15] the following one step difference equation

$$X_{n+1} = R(p, q, \xi_n) X_n, \quad (15)$$

where $p = \lambda h, q = \mu \sqrt{h}$, and ξ_n is a random variable (e.g. a Gaussian $\xi_n \sim \mathcal{N}(0, 1)$ or a discrete random variable). Once this difference equation is formulated, it is not difficult to define the domains of mean-square and asymptotic stability of the numerical method applied to the test SDE (12). In particular, for the numerical mean-square stability, we have [15]

$$\lim_{n \rightarrow \infty} \mathbb{E}(|X_n|^2) = 0 \iff \mathcal{S}_{\text{num}}^{\text{MS}} := \left\{ (p, q) \in \mathbb{C}^2 ; \mathbb{E}|R(p, q, \xi)|^2 < 1 \right\}, \quad (16)$$

and for the numerical asymptotic stability, assuming $R(p, q, \xi) \neq 0$ with² probability 1 and $\mathbb{E}((\log |R(p, q, \xi)|)^2) < \infty$, it is shown in [15, Lemma 5.1] the equivalence

$$\lim_{n \rightarrow \infty} |X_n| = 0, \text{ with probability } 1 \iff \mathcal{S}_{\text{num}}^{\text{AS}} := \left\{ (p, q) \in \mathbb{C}^2 ; \mathbb{E}(\log |R(p, q, \xi)|) < 0 \right\}. \quad (17)$$

A-stability We denote $\mathcal{S}_{\text{num}}^{\text{AS}}, \mathcal{S}_{\text{num}}^{\text{MS}}$, respectively, the above domains of asymptotic and mean-square stability. A numerical integrator is called asymptotically A-stable if $\mathcal{S}_{\text{SDE}}^{\text{AS}} \subseteq \mathcal{S}_{\text{num}}^{\text{AS}}$ and mean-square A-stable if $\mathcal{S}_{\text{SDE}}^{\text{MS}} \subseteq \mathcal{S}_{\text{num}}^{\text{MS}}$. In addition if we restrict $(p, q) \in \mathbb{R}^2$ then the domains of stability $\mathcal{S}_{\text{num}}^{\text{AS}} \cap \mathbb{R}^2, \mathcal{S}_{\text{num}}^{\text{MS}} \cap \mathbb{R}^2$ will be called regions of stability.

²Notice that if $R(p, q, \xi) = 0$ with a non-zero probability, then (15) is clearly numerically asymptotically stable.

3 Weak second order explicit stabilized methods

In this section we introduce the S-ROCK2 algorithm for systems of SDEs (1) in the most general setting for non-commutative noise of arbitrary dimension m . The method is obtained by a combination of an efficient weak second order Milstein-Talay scheme with a stabilization procedure based on the second order deterministic stabilized Runge-Kutta method ROCK2.

3.1 Efficient derivative free explicit Milstein-Talay method

We briefly discuss the Milstein-Talay method (8) and explain an efficient implementation of (8) that will be helpful to understand our new S-ROCK2 methods. First it is well-known that one can replace the stochastic integrals $I_{r,0}, I_{0,r}, I_{q,r}$ by discrete random increments without altering the weak order. Consider independent discrete random variables $\chi_l, \xi_l, l = 1 \dots m$ satisfying

$$\mathbb{P}(\chi_l = \pm 1) = 1/2, \quad \mathbb{P}(\xi_l = \pm\sqrt{3}) = 1/6, \quad \mathbb{P}(\xi_l = 0) = 2/3, \quad (18)$$

then both $I_{r,0}$ and $I_{0,r}$ can be replaced by $h^{3/2}\xi_r$ and $I_{q,r}$ can be replaced by

$$J_{q,r} = \begin{cases} h(\xi_r\xi_r - 1)/2 & \text{if } q = r \\ h(\xi_q\xi_r - \chi_q)/2 & \text{if } r < q \\ h(\xi_q\xi_r + \chi_r)/2 & \text{if } r > q. \end{cases} \quad (19)$$

Thus the method (8) with discrete random increments reads

$$\begin{aligned} \hat{X}_1 &= X_0 + hf(X_0) + \sqrt{h} \sum_{r=1}^m g^r(X_0)\xi_r + \sum_{q,r=1}^m (g^r(X_0))'g^q(X_0)J_{q,r} \\ &+ \frac{h^2}{2} \left(f'(X_0)f(X_0) + \frac{1}{2} \sum_{r=1}^m f''(X_0)(g^r(X_0), g^r(X_0)) \right) \\ &+ \sum_{r=1}^m \left((g^r(X_0))'f(X_0) + \frac{1}{2} \sum_{q=1}^m (g^r(X_0))''(g^q(X_0), g^q(X_0)) + (f(X_0))'g^r(X_0) \right) \frac{h^{3/2}\xi_r}{2}. \end{aligned} \quad (20)$$

We next briefly discuss derivative free methods. First (see [17]), using additional Runge-Kutta stages allows to remove $f'f, (g^r)'f, f'g^r, f''(g^r, g^r)$ without altering the weak order two of (20),

$$\begin{aligned} K_1 &= X_0 + hf(X_0), \quad K_2 = K_1 + \sqrt{h} \sum_{r=1}^m g^r(X_0)\xi_{n,r}, \\ \tilde{X}_1 &= X_0 + \frac{h}{2} \left(f(K_2) + f(X_0) \right) + \sqrt{h} \sum_{r=1}^m g^r \left(\frac{X_0 + K_1}{2} \right) \xi_r \\ &+ \sum_{q,r=1}^m (g^r(X_0))'g^q(X_0)J_{q,r} + \frac{h^{3/2}}{4} \sum_{q,r=1}^m (g^r(X_0))''(g^q(X_0), g^q(X_0))\xi_r. \end{aligned} \quad (21)$$

Next, we use the approximation

$$\sum_{q,r=1}^m (g^r(X_0))'g^q(X_0)J_{q,r} = \frac{1}{2} \sum_{r=1}^m \left[g^r \left(X_0 + \sum_{q=1}^m g^q(X_0)J_{q,r} \right) - g^r \left(X_0 - \sum_{q=1}^m g^q(X_0)J_{q,r} \right) \right] + \mathcal{O}(h^3), \quad (22)$$

that again does not alter the weak second order of the method and requires only 3 evaluations of each function g^r (see [28]). Notice that in contrast a naive finite difference approximation

e.g., $\frac{1}{2h} \sum_{q,r=1}^m \left[g^r \left(x + hg^q(x) \right) - g^r \left(x - hg^q(x) \right) \right] J_{q,r}$, would require $2m+1$ evaluations of each function g^r at the points $x, x \pm hg^q(x)$. Finally, replacing the last term in (21) by

$$\frac{\sqrt{h}}{4} \sum_{r=1}^m \left[g^r \left(X_0 + \sqrt{h} \sum_{q=1}^m g^q(X_0) \chi_q \right) \xi_q - 2\sqrt{h} g^r(X_0) \xi_q + g \left(X_0 - \sqrt{h} \sum_{q=1}^m g^q(X_0) \chi_q \right) \xi_q \right], \quad (23)$$

we obtain the scheme

$$\begin{aligned} K_1 &= X_0 + hf(X_0), \quad K_2 = K_1 + \sqrt{h} \sum_{r=1}^m g^r(X_0) \xi_r, \\ \bar{X}_1 &= X_0 + \frac{h}{2} \left(f(X_0) + f(K_2) \right) + \sum_{r=1}^m g^r \left(\frac{X_0 + K_1}{2} \right) \sqrt{h} \xi_r \\ &+ \frac{1}{2} \sum_{r=1}^m \left(g^r \left(X_0 + \sum_{q=1}^m g^q(X_0) J_{q,r} \right) - g^r \left(X_0 - \sum_{q=1}^m g^q(X_0) J_{q,r} \right) \right) \\ &+ \frac{1}{4} \sqrt{h} \sum_{r=1}^m \left(g^r \left(X_0 + \sqrt{h} \sum_{q=1}^m g^q(X_0) \chi_q \right) - 2g^r(X_0) + g \left(X_0 - \sqrt{h} \sum_{q=1}^m g^q(X_0) \chi_q \right) \right) \xi_r. \end{aligned} \quad (24)$$

Each step of the above scheme necessitates only five evaluations of the diffusion functions g^r , $r = 1, \dots, m$, independently of the dimension m . Efficient derivative free second order methods have been introduced by Roessler and Komori [28]. The method (24) – a modification of the second order method in [17, eq.(2.70) Chap.14] – seems not to have appeared in the literature, in particular the finite difference discretisation of the term $(g^r)''(g^q, g^q)$ by (23) seems new. We give here a direct proof of its weak order two that will be useful in what follows.

Lemma 3.1. *Consider the system of SDEs (1) with $f, g^r \in C_P^6(\mathbb{R}^N, \mathbb{R}^N)$, Lipschitz continuous. Then the derivative free Milstein-Talay method (24) for the approximation of (1) satisfies*

$$|\mathbb{E}(\phi(X(nh))) - \mathbb{E}(\phi(\bar{X}_n))| \leq Ch^2, \quad 0 \leq nh \leq T$$

for all $\phi \in C_P^6(\mathbb{R}^N, \mathbb{R})$, where C is independent of n, h .

Proof. We show that $|\mathbb{E}(\phi(\bar{X}_1)) - \mathbb{E}(\phi(X(t_1)))| \leq Ch^3$ and conclude by Remark 2.1. Since, we already know [30] that $|\mathbb{E}(\phi(\hat{X}_1)) - \mathbb{E}(\phi(X(t_1)))| \leq Ch^3$, where \hat{X}_1 is the weak Milstein-Talay method (20), it remains to show that

$$|\mathbb{E}(\phi(\bar{X}_1)) - \mathbb{E}(\phi(\hat{X}_1))| \leq Ch^3. \quad (25)$$

We first observe that

$$\begin{aligned} \frac{h}{2} (f(K_2) + f(X_0)) &= hf(X_0) + \frac{h^2}{2} \left(f'(X_0) f(X_0) + \frac{1}{2} \sum_{r=1}^m f''(X_0) (g^r(X_0), g^r(X_0)) \right) \\ &+ \sum_{r=1}^m (f(X_0))' (g^r(X_0)) \frac{h^{3/2} \xi_r}{2} + h^2 R_1 + h^{5/2} R_2 + \mathcal{O}(h^3), \end{aligned} \quad (26)$$

where $R_1 = \frac{1}{4} \sum_{r,s=1}^m f''(X_0) (g^r(X_0), g^r(X_0)) (\xi_r \xi_s - \delta_{r,s})$ and $\delta_{r,s}$ is the Kronecker delta function. As $\mathbb{E}(\xi_r \xi_s) = \delta_{r,s}$ we have $\mathbb{E}(R_1) = 0$. Observe that $\mathbb{E}(R_2) = 0$, as R_2 contains random variables of the form $\xi_q \xi_r \xi_s$ whose expectation is zero for all $q, r, s = 1, \dots, m$. Second, we see that

$$\sum_{r=1}^m g^r \left(\frac{X_0 + K_1}{2} \right) \sqrt{h} \xi_r = \sum_{r=1}^m \left(\sqrt{h} g^r(X_0) + \frac{1}{2} h^{3/2} (g^r(X_0))' f(X_0) \right) \xi_r + h^{5/2} R_3 + \mathcal{O}(h^3), \quad (27)$$

where $\mathbb{E}(R_3) = 0$. Third, we have

$$\begin{aligned} & \frac{\sqrt{h}}{4} \sum_{r=1}^m \left(g^r(X_0 + \sqrt{h} \sum_{q=1}^m g^q(X_0) \chi_q) - 2g^r(X_0) + g(X_0 - \sqrt{h} \sum_{q=1}^m g^q(X_0) \chi_q) \right) \xi_r \\ &= \frac{h^{3/2}}{4} \sum_{q,r=1}^m (g^r(X_0))''(g^q(X_0), g^q(X_0)) \xi_r + h^{3/2} R_4 + h^{5/2} R_5 + \mathcal{O}(h^3), \end{aligned} \quad (28)$$

where $R_4 = \frac{1}{4} \sum_{q,p=1}^m (g^r(X_0))''(g^q(X_0), g^p(X_0)) (\chi_q \chi_p - \delta_{q,p}) \xi_r$ and $\mathbb{E}(R_5) = 0$ (this last random variable contains again products of random variable of the form $\xi_q \xi_r \xi_s$). As $\mathbb{E}(\chi_q \chi_p - \delta_{q,p}) = 0$ we have also that $\mathbb{E}(R_4) = 0$. Combining the estimates (22),(26),(27),(28) we obtain

$$\bar{X}_1 - \hat{X}_1 = h^{3/2} R_4 + h^2 R_1 + h^{5/2} (R_2 + R_3 + R_5) + \mathcal{O}(h^3). \quad (29)$$

Setting $G = \sum_{r=1}^m g^r(X_0) \xi_r$ and expanding $\phi(\bar{X}_1) = \phi(\hat{X}_1)'(\bar{X}_1 - \hat{X}_1) + \mathcal{O}(h^3)$, we deduce

$$\begin{aligned} \phi(\bar{X}_1) - \phi(\hat{X}_1) &= \phi'(X_0 + \sqrt{h}G + hf(X_0) + \mathcal{O}(h^{3/2}))(\bar{X}_1 - \hat{X}_1) + \mathcal{O}(h^3) \\ &= \phi'(X_0)(\bar{X}_1 - \hat{X}_1) + h^2 \phi''(X_0)(G, R_4) + h^{5/2} \phi''(X_0)(G, R_1) \\ &+ h^{5/2} \phi''(X_0)(f(X_0), R_4) + \frac{h^{5/2}}{2} \phi'''(X_0)(G, G, R_4) + \mathcal{O}(h^3). \end{aligned}$$

We notice that each of the above terms have expectancy zero (for the terms involving both G and R_4 , we use the independence of the random variables χ_q, ξ_r). This shows (25) and the proof is complete. \square

3.2 Stabilization procedure

All the second order methods considered in this paper (drift implicit or explicit) applied to the linear test problem (12) have a stability function (15) of the form

$$R(p, q, \xi) = A(p) + B(p)q\xi + C(p)\frac{q^2}{2}(\xi^2 - 1), \quad (30)$$

where ξ is either a Gaussian random variable $\mathcal{N}(0, 1)$ or a three points discrete random variable (18). The numerical mean-square stability domain (16) for methods having the above stability function can be characterized by [15]

$$S_{num}^{MS} = \left\{ (p, q) \in \mathbb{C}^2 ; |A(p)|^2 + |B(p)q|^2 + \frac{1}{2}|C(p)q^2|^2 < 1 \right\}, \quad (31)$$

while the numerical asymptotic stability domain, assuming here a three points discrete random variable is used³can be characterized by [15]

$$S_{num}^{AS} = \left\{ (p, q) \in \mathbb{C}^2 ; \left| (A(p) + C(p)q^2)^2 - 3B^2(p)q^2 \right| \left| A(p) - \frac{1}{2}C(p)q^2 \right|^4 < 1 \right\}. \quad (32)$$

For example, for the weak Milstein-Talay method (20) and its derivative free version (24), we have

$$A(p) = 1 + p + \frac{1}{2}p^2, \quad B(p) = 1 + p, \quad C(p) = 1. \quad (33)$$

Since visualizing the domain of stability for $(p, q) \in \mathbb{C}^2$ is difficult we restrict ourselves to study the case where $(p, q) \in \mathbb{R}^2$. It can be seen in Figure 1 that the weak Milstein-Talay method (20)

³Notice that for $\xi \sim \mathcal{N}(0, 1)$ numerical asymptotic stability domains S_{num}^{AS} are more difficult to characterize [15].

Milstein-Talay method, see (33)

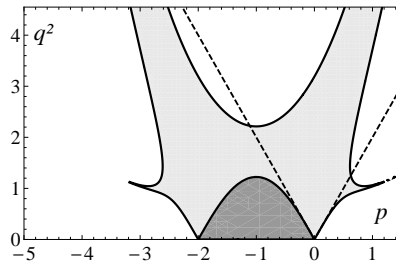


Figure 1: Mean-square stability region (dark gray) and asymptotic stability region (dark and light grays) of the explicit second order Milstein-Talay method with stability function (30).

has restricted mean-square and asymptotic stability regions. This is expected for classical explicit methods and our goal is to introduce a stabilization procedure that permits to enlarge them significantly. We first define for $a > 0$ the following “portion of the true mean-square stability region”

$$\mathcal{S}_a^{MS} = \{(p, q) \in (-a, 0) \times \mathbb{R} ; p + \frac{1}{2}|q|^2 < 0\}, \quad (34)$$

and define for a given method

$$\ell = \sup\{a > 0 ; \mathcal{S}_a^{MS} \subset \mathcal{S}_{num}^{MS}\}, \quad d = \sup\{a > 0 ; (-a, 0) \times \{0\} \subset \mathcal{S}_{num}^{MS}\}, \quad (35)$$

where d is the size of the stability domain along the deterministic p -axis (observe that $d \geq \ell$). For the Milstein-Talay method we have $\ell = 0, d = 2$. In contrast the S-ROCK2 methods (see (47), Section 3.3) have values ℓ, d increasing quadratically with the stage parameter s . In turn, the ratio of stability versus work increases linearly, while for classical explicit methods it is bounded.

Weak order one S-ROCK methods [4] For ordinary differential equations (ODEs),

$$\frac{dX(t)}{dt} = f(X(t)), \quad X(0) = X_0, \quad (36)$$

a well-know stabilization procedure for the Euler method has been proposed in [34]. Its construction is based on the classical Chebyshev polynomials $T_s(\cos x) = \cos(sx)$. Given an integer $s \geq 1$, the number of stages, and a damping parameter $\eta \geq 0$, we define the following Runge-Kutta method (first order Chebyshev method) with step size h by the following explicit recursion where

$$\begin{aligned} K_0 &= X_0, \quad K_1 = X_0 + h \frac{\omega_1}{\omega_0} f(K_0), \\ K_j &= 2h \frac{T_{j-1}(\omega_0)}{T_j(\omega_0)} f(K_{j-1}) + 2\omega_0 \frac{T_{j-1}(\omega_0)}{T_j(\omega_0)} K_{j-1} - \frac{T_{j-2}(\omega_0)}{T_j(\omega_0)} K_{j-2}, \quad j = 2, \dots, s \\ X_1 &= K_s, \end{aligned} \quad (37)$$

$\omega_0 = 1 + \frac{\eta}{s^2}$, $\omega_1 = \frac{T_s(\omega_0)}{T_s'(\omega_0)}$. Applied to the linear test problem $dX(t)/dt = \lambda X(t)$ the method (37) gives $X_1 = R_s(p)X_0$, where $p = \lambda h$ and where $R_s(p)$, called the stability function (polynomial) of the method, is given by $R_s(p) = T_s(\omega_0 + \omega_1 p)/T_s(\omega_0)$. We emphasize that (37) denotes in fact a family of methods indexed by the stage number s . A crucial property of the methods (37) is

$$|R_s(p)| \leq 1 \quad \text{for all } p \in (-d_s, 0), \quad (38)$$

with $d_s \simeq C \cdot s^2$, for s large enough, where C depends on the damping parameter η (for $\eta = 0$, $C = 2$). Thus the length d_s of the stability domain

$$\mathcal{S} := \{p \in \mathbb{C}; |R(z)| \leq 1\} \quad (39)$$

of the methods increases quadratically with s on the negative real axis. This quadratic growth of the stability domain is the key feature of such methods compared to standard explicit integrators.

The idea for stabilizing the Euler-Maruyama (7) is now simply to damp its stability function $R(p, q, \xi) = 1 + p + q\xi$, obtained by applying (7) to (12) using $R_s(p)$ (with a value of the damping η optimized for each s , see [4]). The corresponding Runge-Kutta type scheme reads [4]

$$\begin{aligned} K_0 &= X_0, & K_1 &= X_0 + h \frac{\omega_1}{\omega_0} f(K_0), \\ K_j &= 2h \frac{T_{j-1}(\omega_0)}{T_j(\omega_0)} f(K_{j-1}) + 2\omega_0 \frac{T_{j-1}(\omega_0)}{T_j(\omega_0)} K_{j-1} - \frac{T_{j-2}(\omega_0)}{T_j(\omega_0)} K_{j-2}, & j &= 2, \dots, s \\ X_1 &= K_s + \sum_{r=1}^m g^r(K_s) \Delta W_r. \end{aligned} \tag{40}$$

The method (40) will be denoted by S-ROCK(1/2,1). Another method of strong order 1 and weak order 1 has been considered in [4]. A derivative free version, denoted S-ROCK(1,1), can be obtained by replacing the last line in (40) by

$$X_1 = K_s + \sum_{r=1}^m g^r(K_s) \Delta W_r + \frac{1}{2} \sum_{r=1}^m \left(g^r(K_s + \sum_{q=1}^m g^q(K_s) I_{q,r}) - g^r(K_s - \sum_{q=1}^m g^q(K_s) I_{q,r}) \right),$$

where $I_{q,r}$ are defined in (9) and by considering a larger damping η . It turns out that S-ROCK(1/2,1) and S-ROCK(1,1) include a portion of the true mean-square stability region that scales like $\ell_s \simeq 0.33 \cdot s^2$ and $\ell_s \simeq 0.19 \cdot s^2$, respectively.

Second order stabilization Similarly as for the weak order one S-ROCK method, the idea is to stabilize the weak second order method (24). We start with a deterministic stabilized second order Chebyshev method. Recall that the derivation of optimal stability functions suitable for the stabilization of second order (deterministic method) is a non trivial task and various strategies have been proposed [20, 34, 5, 1]. We choose here the second order orthogonal Runge-Kutta Chebyshev methods (ROCK2) introduced in [5]. The idea is to search for a stability polynomial

$$R_s(p) = w_2(p) P_{s-2}(p), \tag{41}$$

where $P_{s-2}(p)$ is a member family of polynomials $\{P_j(z)\}_{j \geq 0}$ orthogonal with respect to the weight function $\frac{w_2(x)^2}{\sqrt{1-x^2}}$. The polynomial P_{s-2} has degree $s-2$, while w_2 is a positive polynomial of degree two (depending on s). One constructs the polynomials w_2 such that R_s satisfies [5]

$$R_s(p) = 1 + p + \frac{p^2}{2} + \mathcal{O}(p^3), \tag{42}$$

together with a large stability interval (38), increasing as $\simeq 0.81 \cdot s^2$ along the negative real axis. Thanks to the recurrence relation of the orthogonal polynomials $\{P_j(z)\}_{j \geq 0}$, a method of order two for (36) based on a recurrence formula can be constructed ⁴

$$\begin{aligned} K_0 &= X_0, & K_1 &= K_0 + \mu_1 h f(K_0), \\ K_j &= \mu_j h f(K_{j-1}) - \nu_j K_{j-1} - \kappa_j K_{j-2}, & j &= 2, \dots, s-2, \\ K_{s-1} &= K_{s-2} + 2\tau h f(K_{s-2}), \\ X_1 &= K_{s-2} + \left(2\sigma - \frac{1}{2} \right) h f(K_{s-2}) + \frac{1}{2} h f(K_{s-1}). \end{aligned} \tag{43}$$

⁴The two last stages of the method are written in a slightly different way as in the ROCK2 method [5, Equ. (26-27)] as (43) is more convenient for an extension to stochastic integrators. We emphasize that it has the same order and stability properties as the ROCK2 method [5, Equ. (26-27)].

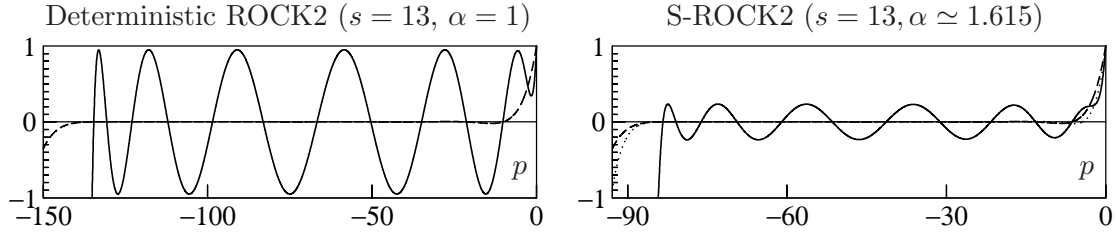


Figure 2: Comparison of polynomials involved in ROCK2 and S-ROCK2 for $s = 13$. Polynomials $R_{s,\alpha}$ (solid lines), $P_{s-2}(\alpha p)$ (dashed lines). Right picture: $P_s(\alpha p)$ (dotted lines).

The parameters μ_j, k_j (depending on s) are obtained from the three-term recurrence relation [5, eq. (24)-(25)] of the orthogonal polynomials $\{P_j(z)\}_{j \geq 0}$, while σ, τ (that also depend on s) satisfy $w_2(p) = 1 + 2\sigma p + \tau p^2$ and are chosen such that (42) holds.

In preparation for the extension of the ROCK2 methods to stochastic problems, we now explain a novel strategy to introduce damping in the scheme (43). The idea is to consider the following scheme for a fixed scalar parameter α .

$$\begin{aligned}
K_0 &= X_0, & K_1 &= K_0 + \alpha \mu_1 h f(K_0), \\
K_j &= \alpha \mu_j h f(K_{j-1}) - \nu_j K_{j-1} - \kappa_j K_{j-2}, & j &= 2, \dots, s-2, \\
K_{s-1} &= K_{s-2} + 2\tau_\alpha h f(K_{s-2}) \\
X_1 &= K_{s-2} + \left(2\sigma_\alpha - \frac{1}{2}\right) h f(K_{s-2}) + \frac{1}{2} h f(K_{s-1}).
\end{aligned} \tag{44}$$

Notice that for $\alpha = 1$, we recover the original ROCK2 method (43). Applied to the linear test problem $dX/dt = \lambda X, X(0) = X_0$ this method yields (setting $p = h\lambda$ and $X_0 = 1$)

$$X_1 = (1 + 2\sigma_\alpha p + \tau_\alpha p^2) P_{s-2}(\alpha p) =: R_{s,\alpha}(p). \tag{45}$$

Lemma 3.2. *The method (44) has second order for the system of ODEs (36) for any α provided*

$$\sigma_\alpha = \frac{1-\alpha}{2} + \alpha\sigma, \quad \tau_\alpha = \frac{(\alpha-1)^2}{2} + 2\alpha(1-\alpha)\sigma + \alpha^2\tau. \tag{46}$$

Proof. Recall that for second order deterministic methods, standard Runge-Kutta order conditions for linear and nonlinear problems are identical. From (42) we deduce the relation $P'_{s-2}(0) = 1 - 2\sigma$ and $\frac{1}{2}P''_{s-2}(0) = \frac{1}{2} - \tau - 2\sigma(1 - 2\sigma)$. Imposing the second order condition (45) yields

$$(1 + 2\sigma_\alpha p + \tau_\alpha p^2) \left(1 + P'_{s-2}(0)\alpha p + \frac{1}{2}P''_{s-2}(0)(\alpha p)^2 + \mathcal{O}(p^3)\right) = 1 + p + \frac{p^2}{2} + \mathcal{O}(p^3),$$

which is equivalent to the relations (46). □

In Figure 2, we plot, for $s = 13$, the polynomials $P_{s-2}(\alpha p)$ and $R_{s,\alpha}(p)$ (defined in (45)) involved in the standard ROCK2 ($\alpha = 1$, left picture) method and the S-ROCK2 method ($\alpha \simeq 1.615$ right picture) introduced in the next section. It can be seen that increasing α reduces the amplitude of the oscillations of $R_{s,\alpha}(p)$. The appropriate choice of α is discussed below.

3.3 The S-ROCK2 methods

We introduce here our new explicit stabilized integrator, obtained by stabilizing the Milstein-Talay method (24) with (44).

S-ROCK-2 integrator of weak order two Given X_0 , computed X_1 as follows.

$$\begin{aligned}
K_0 &= X_0, \quad K_1 = K_0 + \alpha\mu_1 hf(K_0), \\
K_j &= \mu_j \alpha hf(K_{j-1}) - \nu_j K_{j-1} - \kappa_j K_{j-2}, \quad j = 2, \dots, s, \\
K_{s-1}^* &= K_{s-2} + 2\tau_\alpha hf(K_{s-2}) + \sqrt{h} \sum_{r=1}^m g^r(K_s) \xi_r, \\
X_1 &= K_{s-2} + \left(2\sigma_\alpha - \frac{1}{2}\right) hf(K_{s-2}) + \frac{1}{2} hf(K_{s-1}^*) + \sqrt{h} \sum_{r=1}^m g^r(K_{s-1}) \xi_r, \\
&+ \frac{1}{2} \sum_{r=1}^m \left(g^r \left(K_s + \sum_{q=1}^m g^q(K_s) J_{q,r} \right) - g^r \left(K_s - \sum_{q=1}^m g^q(K_s) J_{q,r} \right) \right) \\
&+ \frac{1}{4} \sqrt{h} \sum_{r=1}^m \left(g^r \left(K_s + \sqrt{h} \sum_{q=1}^m g^q(K_s) \chi_q \right) - 2g^r(K_s) + g \left(K_s - \sqrt{h} \sum_{q=1}^m g^q(K_s) \chi_q \right) \right) \xi_r, \quad (47)
\end{aligned}$$

where $\alpha = 1/(2P'_{s-1}(0))$ and $\sigma_\alpha, \tau_\alpha$ are given by (46). Here, the constants $\mu_j, \nu_j, \kappa_j, \sigma, \tau$ depend on s and are the same as for the standard deterministic ROCK2 integrator (43).

integrator	work			stability	
	#f	#g ^r	#random	d _s	ℓ _s
s steps of Milstein-Talay (24)	2s	5s	2ms	2s	0
one step of S-ROCK2 (47)	s + 2	6	2m	≈ 0.42(s + 2) ²	≈ 0.42(s + 2) ²

Table 1: Computational complexity for an SDE in dimensions N (drift) and m (diffusion).

Numerical computations show that the S-ROCK2 method includes a portion of the true mean-square stability region \mathcal{S}_ℓ^{MS} that grows with the stage number as $\ell_{\text{S-ROCK2}} \simeq 0.42(s+2)^2$. The computational complexity of one step of the S-ROCK2 method with stepsize h is reported in Table 1 and compared to s steps with stepsize h/s of the weak second order Milstein-Talay method (24). As observed, one step of the S-ROCK2 method (47) requires at each step $\#f = s+2$ evaluations of the drift function, $\#g^r = 6$ evaluations of the diffusion functions g^r , $r = 1, \dots, m$, and $\#random = 2m$ simulations of independent discrete random variables, independently of the dimensions N, m of the considered SDE. The main feature of our S-ROCK2 integrators is that the mean-square stability region sizes ℓ_s, d_s grow quadratically with respect to the computational work $\#f + \#g^r$, while $\ell_s = 0$ and d_s grows only linearly for the standard explicit methods.

Remark 3.3. (*Diagonal noise*) When $N = m$ and $(g^1(x), \dots, g^N(x)) = \text{diag}(g_1(x^1), \dots, g_N(x^N))$ is a diagonal matrix where $g_k(x)$ depends only on x^k , one can replace the two last lines of (47) by $G(K_s) = (G_1(K_{s,1}), \dots, G_N(K_{s,N}))^T$ where $G_k(y) = \frac{1}{2} \left(g_k(y + g_k(y) J_{k,k}) - g_k(y - g_k(y) J_{k,k}) \right) + \frac{1}{4} \left(\sqrt{h} g_k(y + \sqrt{h} g_k(y)) - 2\sqrt{h} g_k(y) + \sqrt{h} g_k(K_s - \sqrt{h} g_k(y)) \right) \xi$.

We next prove that the method (47) has indeed weak second order.

Theorem 3.4. Consider the SDE (1) with $f, g^r \in C_P^6(\mathbb{R}^N, \mathbb{R}^N)$, Lipschitz continuous. Then the S-ROCK2 method (47) for the approximation of (1) satisfies

$$|\mathbb{E}(\phi(X(nh))) - \mathbb{E}(\phi(X_n))| \leq Ch^2, \quad 0 \leq nh \leq T$$

for all $\phi \in C_P^6(\mathbb{R}^N, \mathbb{R})$, where C is independent of n, h .

Proof. Noticing that $K_j = X_0 + c_j hf(X_0) + \mathcal{O}(h^2)$ and using Lemma 3.2 yields

$$K_{s-2} + \left(2\sigma_\alpha - \frac{1}{2}\right)hf(K_{s-2}) + \frac{1}{2}hf(K_{s-2} + 2\tau_\alpha hf(K_{s-2})) = X_n + \left(hf + \frac{h^2}{2}f'f\right)(X_n) + \mathcal{O}(h^3).$$

The choice $\alpha = 1/P'_{s-1}(0)$ implies $P_{s-1}(\alpha p) = 1 + \frac{p}{2} + \mathcal{O}(p^2)$ which yields $K_{s-1} = X_0 + \frac{h}{2}f(X_n) + \mathcal{O}(h^2)$. Using in addition that $K_s = X_0 + \mathcal{O}(h)$, we deduce $X_1 - \bar{X}_1 = h^2 R_1 + h^{5/2} R_2 + \mathcal{O}(h^3)$, where \bar{X}_1 is defined by the derivative free Milstein-Talay method (24) and $\mathbb{E}(R_1) = \mathbb{E}(R_2) = \mathbb{E}(R_1 \xi_r) = 0$ for all r . We conclude using Remark 2.1, similarly to the end of the proof of Lemma 3.1. \square

Remark 3.5. *Considering independent Gaussian variables $\xi_r \sim \mathcal{N}(0, 1)$ the strong order of the integrator (47) becomes one in the case of a commutative noise. In the general case of non-commutative noise, if one replaces the discrete random variables $J_{l,k}$ by the usual multiple stochastic integrals (9), then (47) becomes of strong order one.*

Stability analysis We now focus on the mean-square and asymptotic stability properties of our new explicit weak second order integrator.

Proposition 3.6. *Consider the S-ROCK2 method (47) applied to the linear test equation (12). The numerical asymptotic stability domain \mathcal{S}_{num}^{AS} defined in (17) is given by (32) and the numerical mean-square stability domain \mathcal{S}_{num}^{MS} defined in (16) is given by (31) where the polynomial functions $A(p), B(p), C(p)$ are defined by*

$$A(p) = P_{s-2}(\alpha p)(1 + 2\sigma_\alpha p + \tau_\alpha p^2), \quad B(p) = P_{s-1}(\alpha p) + \frac{p}{2}P_s(\alpha p), \quad C(p) = P_s(\alpha p). \quad (48)$$

Proof. Follows from noticing that applied to the linear test problem (12), the method (47) yields (30) with $A(p), B(p), C(p)$ given by (48). \square

In Figure 4 we plot asymptotic (light gray) and mean-square (dark gray) stability regions for different values of the stage number s . In all cases, we observe the inclusions $\mathcal{S}_{num}^{MS} \cap \mathbb{R}^2 \subset \mathcal{S}_{num}^{AS} \cap \mathbb{R}^2$. The dotted lines where $p < 0$ correspond to the boundary of the true mean-square stability region, while the dotted lines where $p > 0$ (see bottom pictures) correspond to the boundary of the true asymptotic stability region (given by $p + |q|^2/2 = 0$ and $p - q^2/2 = 0$, respectively). In the bottom pictures we zoom close to the origin and observe that the asymptotic stability region contains the mean square stability region. Furthermore, a large portion of the true asymptotic stability region is included in the numerical one close the the origin, $\{(p, q) \in \mathbb{R}^2 ; |p| \leq 1, |p| < |q| < 2.7\} \subset \mathcal{S}_{num}^{AS}$.

We briefly comment the ideas behind the definition of the scheme (47) that allow favorable mean-square stability properties. Recall that the deterministic stability functions $A(p)$ oscillates around the negative real axis (see top left picture in Figure 2). If the absolute value of the local maxima and minima of $A(p)$ are close to one, then in view of (31) we will have gaps in the mean-square stability domains (in the regions where $A(p)$ is bounded). We therefore have to damp this term by introducing and tuning the value of α as explained before in Section 3.2. Next it can be observed that the polynomials $P_{s-2}(p), P_{s-1}(p), P_s(p)$ oscillates around zero with a small amplitude of size $\mathcal{O}(p^{-2})$ along the real axis. These polynomials are thus good candidates to damp the Milstein term which corresponds to the q^2 term in the stability function (30). However, simply working with the polynomials $P_{s-2}(p)$ leaves a gap in the mean-square stability regions close to the origin near to the extrema of $P_{s-2}(p)$. For this reason, we consider instead the

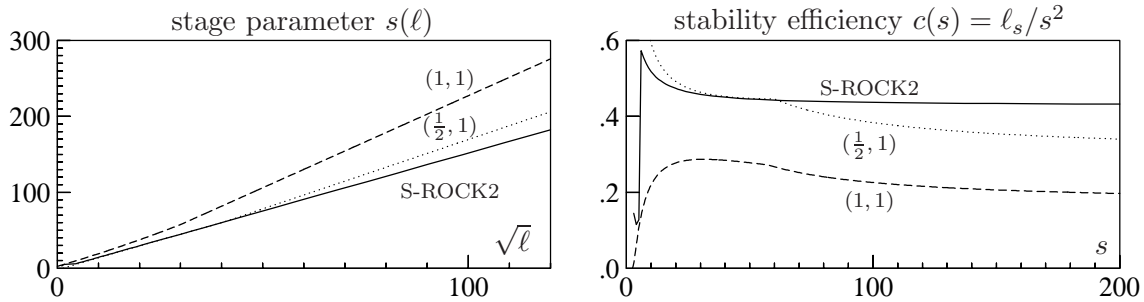


Figure 3: Comparison of S-ROCK2 (solid lines) and the weak order one S-ROCK methods (1, 1) (dashed lines), $(\frac{1}{2}, 1)$ (dotted lines). Left picture: optimal stage parameter s as a function of $\sqrt{\ell}$, where ℓ is given by (35). Right picture: stability efficiency $c(s) = \ell_s/s^2$.

polynomial $P_s(p)$ to damp the Milstein term which has the advantage that $P'_s(0) > P'_{s-2}(0)$ (faster decay near the origin). The additional advantage of considering simultaneously the three polynomials P_{s-2}, P_{s-1} , and P_s is that their extrema close to the origin do not coincide. This permits to avoid a gap in the mean-square stability domain close to the origin. The polynomials P_{s-1}, P_s (corresponding to the internal stages K_{s-1}, K_s in (47)) are obtained by computing two additional stages compared to the deterministic ROCK2 (44) method.

3.4 Comparison with other stiff integrators

Comparison with S-ROCK method In Figure 3 we plot the length ℓ defined in (35) of the portion of the true mean-square stability region \mathcal{S}_ℓ^{MS} as a function of the number of stages used. As we can see the behaviour of the S-ROCK2 method is $\ell \simeq Cs^2$ similarly to the S-ROCK methods of weak order 1 discussed in Section 3.2. Furthermore, once can also see that the S-ROCK2 method is actually more efficient from a stability point of view, since the stability efficiency factor $c(s) = \ell_s/s^2$ converges numerically to about 0.42 for large s , which is larger than the S-ROCK(1/2,1/2) and S-ROCK(1,1) values of 0.33 and 0.19, respectively.

Comparison with a class of diagonally drift-implicit Runge-Kutta methods [11] We have already discussed the asymptotic and mean-square stability properties of the Milstein-Talay method (24), which has restricted mean-square and asymptotic stability regions (see Figure 1). We next consider the weak second-order diagonally implicit Runge-Kutta method (DDIRDI5) derived in [11] with the following aims: reducing the computational cost of a fully drift-implicit method such as the modified θ -Milstein method [3], and improving the stability domains compared to explicit methods. For these method we have (see Fig. 5 for an example with $c_1 = c_2 = 1$)

$$A(p) = 1 + p \frac{1 + (1/2 - c_1 - c_2)p}{(1 - c_1p)(1 - c_2p)}, \quad B(p) = 1 + p \frac{1 - (c_1 + c_2)p/2}{(1 - c_1p)(1 - c_2p)}, \quad C(p) = 1. \quad (49)$$

Even though the mean-square stability domain has been improved close to the deterministic p axis, it can be seen that this method does not cover a portion of the true mean square stability region. Here as for the explicit Milstein-Talay method, we have $\ell = 0$. This can be seen in Figure 5, where $\mathcal{S}_{num}^{MS} \cap \mathbb{R}^2$ (dark gray) is strictly contained below the curve $p + \frac{1}{2}|q|^2 = 0$ (see dashed lines), which is the boundary of the true mean-square stability region. In fact, the proposition below shows that $\ell = 0$ for the the whole class of integrators satisfying (49).

Proposition 3.7. *Consider a weak order two integrator with a stability function of the form (30) when applied to the linear test equation (12). A necessary condition for $\ell \neq 0$ in (35) is*

$$1 + 2B''(0) - 4C'(0) \leq 0. \quad (50)$$

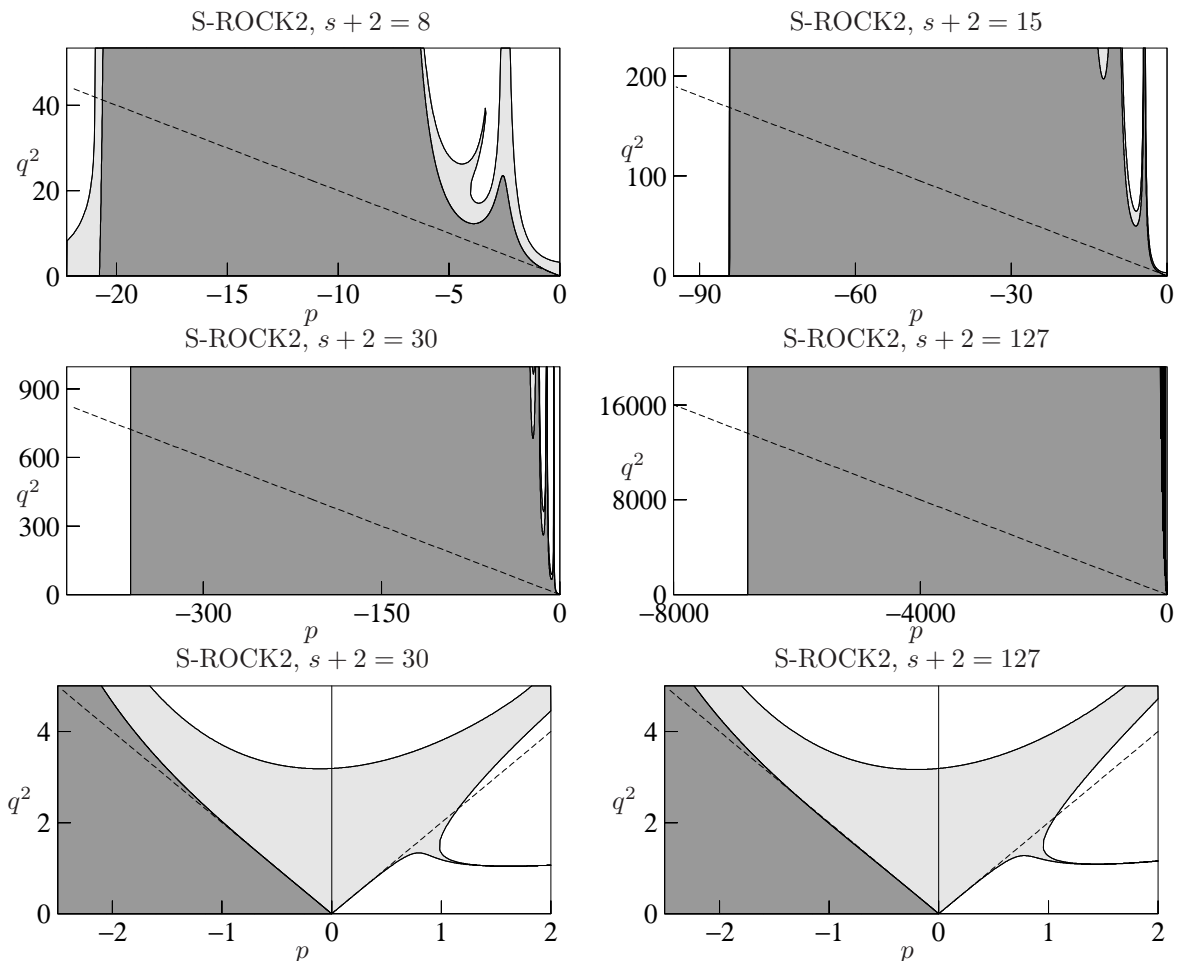


Figure 4: Mean-square stability regions (dark gray) and asymptotic stability regions (dark and light grays) of S-ROCK2 for $s + 2 = 8, 15, 30,$ and 127 stages, respectively.

In particular, for the class of integrators with stability function given by (49), we have $\ell = 0$ for all $c_1, c_2 \geq 0$.

Proof. Using that $\mathbb{E}|R(p, q, \xi)|^2$ can be written as an increasing function of q^2 (see (31)), we have

$$\sup_{(p,q) \in \mathcal{S}_\ell^{MS}} \mathbb{E}(|R(p, q, \xi)|^2) = \sup_{p \in [-\ell, 0)} \mathbb{E}(|R(p, \sqrt{-2p}, \xi)|^2).$$

Using the weak order two assumption, we have $A(p) = 1 + p + p^2/2 + \mathcal{O}(p^3)$ and $B(p) = 1 + p + \mathcal{O}(p^2)$, a Taylor expansion yields $\mathbb{E}(|R(p, \sqrt{-2p}, \xi)|^2) = 1 + (4C'(0) - 2B''(0) - 1)p^3 + \mathcal{O}(p^4)$ which is smaller than 1 for small $-p > 0$ only if (50) holds. For the class of integrators (49), we have $1 + 2B''(0) - 4C'(0) = 1 + 2(c_1 + c_2) > 0$, so the condition (49) is violated. \square

Remark 3.8. Notice that if $C(p) = 1$, then the numerical mean-square stability domain is bounded in the q direction by $|q|^4/2 < 1$. This can be observed in Figure 5 where the mean-square stability region $\mathcal{S}_{num}^{MS} \cap \mathbb{R}^2$ (dark gray) of the DDIRDI5 method is bounded vertically by $|q|^2 < \sqrt{2}$.

Comparison with a family of weak second order explicit integrators [19] A family of explicit stabilized integrators for stiff Stratonovitch systems of SDEs has been introduced recently in [19]. Although the stability domain of these methods grows quadratically along the p -axis, with

DDIRDI5 method, $c_1 = c_2 = 1$ in (49)

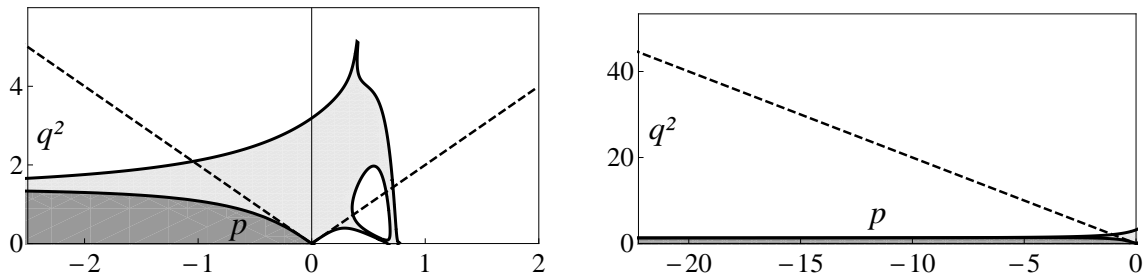


Figure 5: Mean-square stability region (dark gray) and asymptotic stability region (dark and light grays) of the implicit DDIRDI5 with stability function (30) given by (49) (compare with Fig. 4).

$d_s \simeq 0.54s^2$ (according to the left pictures in [19, Fig. 3]), it can be observed that the mean-square stability domain contains uncontrolled gaps for $|q|^2 > \sqrt{2}$ and we have $\ell_s \leq \sqrt{2}$.

4 Numerical experiments

We now present various different numerical experiments with our newly constructed methods. In Section 4.1 we confirm the order of weak convergence of the S-ROCK2 method on two different non-linear problems. Then, in Section 4.2, we test the performance of the S-ROCK2 method for a non-linear stiff problem and compare it to other weak second order methods, while in Section 4.3 we present numerical results for a stochastic PDE arising from Neurosciences.

4.1 Weak convergence rates

We start our numerical investigations by considering two non-stiff non-linear SDEs. The first one is

$$dX(t) = \left(\frac{1}{4}X(t) + \frac{1}{2}\sqrt{X(t)^2 + 1} \right) dt + \sqrt{\frac{X(t)^2 + 1}{2}} dW(t), \quad X(0) = 0. \quad (51)$$

The exact solution is $X(t) = \sinh(t/2 + W(t)/\sqrt{2})$, it satisfies $\mathbb{E}((\operatorname{arcsinh} X(t))^2) = t^2/4 + t/2$.

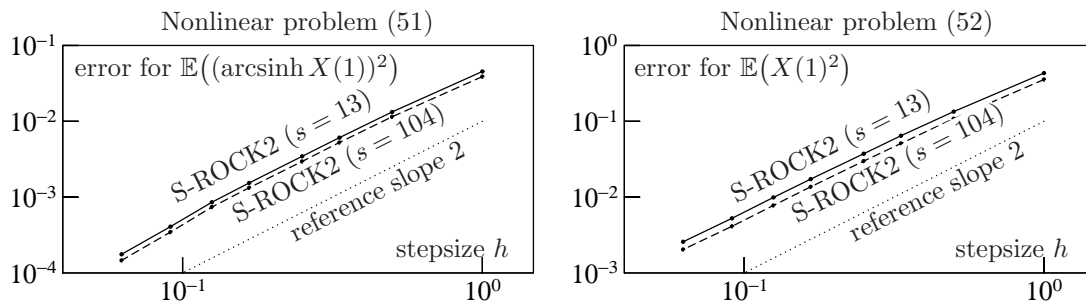


Figure 6: S-ROCK2 method with $s = 13$ stages (solid lines) and $s = 104$ (dashed lines). Weak error at final time $T = 1$ versus the stepsize h for problems (51), (52), where $1/h = 1, 2, 3, 4, 6, 8, 11, 16$.

The second test problem is another nonlinear SDE [11] with 10 independent driving Wiener

processes,

$$dX(t) = X(t)dt + \sum_{j=1}^{10} a_j^{-1} \sqrt{X(t) + b_j^{-1}} dW_j(t), \quad X(0) = 1,$$

with non-commutative noise, where the values of the constants $a_j, j = 1, \dots, 10$ are respectively 10, 15, 20, 25, 40, 25, 20, 15, 20, 25, and the values of $b_j, j = 1, \dots, 10$ are respectively 2, 4, 5, 10, 20, 2, 4, 5, 10, 20. For this problem by applying Itô's formula to $\phi(x) = x^2$, taking expectations and using the fact that $\mathbb{E}(X(t)) = e^t$, one calculates

$$\mathbb{E}(X^2(t)) = (-68013 - 458120e^t + 14926133e^{2t})/14400000. \quad (52)$$

We apply the S-ROCK2 method to both problems (51) and (52) and approximate respectively $\mathbb{E}((\operatorname{arcsinh} X(T))^2)$ and $\mathbb{E}(X^2(T))$ up to the final time $T = 1$ using 10^9 realisations and different step sizes h . We plot the results in Figure 6 and observe that the S-ROCK2 method converges with weak second order for both problems, which confirms the statement of Theorem 3.4. We plot the results for $s = 13$ (solid lines) and $s = 104$ stages (dashed lines) using the same set of generated random numbers. The fact that the obtained curves are nearly identical illustrates that the error constants of the S-ROCK2 methods are nearly independent of the stage number of the integrator, similarly to the standard deterministic ROCK2 integrator (43).

4.2 A nonlinear stiff problem

		parameters: $\varepsilon = 10^{-3}, \lambda = -4$		
method	stepsize	work	$\mathbb{E}(X(1)^2) - 1$	$\mathbb{E}(X(1)) - 1$
Milstein-Talay	$h = 1/25$	$\#f = 500, \#g = 1250$	∞ (unstable)	∞ (unstable)
Milstein-Talay	$h = 1/30$	$\#f = 600, \#g = 1500$	-8.5e-3 (stable)	-8.5e-2 (stable)
SROCK2($s = 4$)	$h = 1$	$\#f = 60, \#g = 60$	-5.1e-10 (stable)	-2.7e-10 (stable)
		parameters: $\varepsilon = 1, \lambda = -10^{-2}$		
method	stepsize	work	$\mathbb{E}(X(1)^2) - 1$	$\mathbb{E}(X(1)) - 1$
Milstein-Talay	$h = 1/280$	$\#f = 5600, \#g = 14000$	∞ (unstable)	∞ (unstable)
Milstein-Talay	$h = 1/300$	$\#f = 6000, \#g = 15000$	0.e0 (stable)	0.e0 (stable)
SROCK2($s = 7$)	$h = 1/4$	$\#f = 360, \#g = 240$	-1.1e-15 (stable)	-5.6e-16 (stable)
		parameters: $\varepsilon = 1, \lambda = -10^{-3}$		
method	stepsize	work	$\mathbb{E}(X(1)^2) - 1$	$\mathbb{E}(X(1)) - 1$
Milstein-Talay	$h = 1/2800$	$\#f = 56000, \#g = 140000$	∞ (unstable)	∞ (unstable)
Milstein-Talay	$h = 1/3000$	$\#f = 60000, \#g = 150000$	0.e0 (stable)	0.e0 (stable)
SROCK2($s = 23$)	$h = 1/4$	$\#f = 1000, \#g = 240$	-4.0e-12 (stable)	-2.0e-12 (stable)

Table 2: Stability efficiency. Work versus stiffness for the second order Milstein-Talay method (24) and the S-ROCK2 methods applied to the nonlinear SDE (53). Averages over 10^6 samples.

To illustrate the advantage of including a whole “portion” of the mean-square stability region of the linear test problem (12) we consider the following nonlinear scalar problem with a one-dimensional noise,

$$dX(t) = -\lambda X(t)(1 - X(t))dt - \mu X(t)(1 - X(t))dW(t), \quad X(0) = 0.95, \quad (53)$$

on time interval $(0, T)$ of length $T = 10$, which is derived from a population dynamics model [12, Chap. 6.2] (see also [4, Example 5.2]). Notice that if one linearizes this problem close to the steady solution $X(t) = 1$, it yields the linear test problem (12). Here, the initial condition $X(0) = 0.95$ is chosen close to this stationary solution. We consider the parameters $\lambda < 0, \mu = \sqrt{-\lambda(2 - \varepsilon)}$ for which the linear test problem (12) is asymptotic and mean-square stable because

$\lambda + |\mu|^2/2 = -\varepsilon/2 < 0$. Recall that for this linear test problem, the S-ROCK2 method is mean-square stable provided $|\lambda|h < \ell_s$, as studied in Section 3.3, while the Milstein-Talay method (20) or (24) is mean-square stable only if h is small enough. We consider two cases.

1. **Case $\varepsilon \ll 1$** We consider first the case where $\lambda < 0$ is a constant of moderate size, e.g. $\lambda = -4$, while $\varepsilon > 0$ is small. Considering the linear test equation (12), the S-ROCK2 method is stable provided $h|\lambda| < \ell_s$, and the stepsize $h = 1$ can be used for $s \geq 4$ independently of the smallness of ε . In contrast, for the weak second order Milstein-Talay method, a calculation shows that the mean-square stability condition becomes $h|\lambda| \leq \varepsilon/4 + \mathcal{O}(\varepsilon^2)$, and the number of steps needed for a stable integration grows like $\mathcal{O}(4\lambda\varepsilon^{-1})$ as $\varepsilon \rightarrow 0$.
2. **Case $|\lambda| \gg 1$** We next consider the case where $\varepsilon = 1$ and $|\lambda| \gg 1$. The S-ROCK method is stable provided $h|\lambda| < \ell_s \simeq Cs^2$ and the minimal number of steps needed for a stable integration grows like $\mathcal{O}(\sqrt{|\lambda|})$ as $\lambda \rightarrow \infty$. In contrast, for the weak second order Milstein-Talay method, a calculation yields the mean-square stability condition $h|\lambda| \leq 2$, and the number of steps of the method for a stable integration grows like $\mathcal{O}(|\lambda|)$ as $\lambda \rightarrow -\infty$.

We observe in Table 2 the spectacular improvement of the efficiency when switching from the weak second order Milstein-Talay method (24) to the S-ROCK2 methods applied to the nonlinear SDE (53). Indeed, considering three different sets of parameters λ, ε , we observe in all cases that the S-ROCK2 allows one to use a much larger timestep than for the Milstein-Talay method. This significantly reduces the computational cost of each sample, measured in terms of the number of evaluations of the drift and the noise (see $\#f, \#g^r$). We highlight that for the considered parameters ε, λ a severe step size restrictions similar to the explicit Milstein-Talay method would be obtained for the drift-implicit integrators DDIRDI5 of [11] as discussed in Section 3.4, because their mean-square stability domain is bounded on the q axis by $|q|^2 \leq \sqrt{2}$ (see Rem. 3.8 and Fig. 5).

4.3 Electric potential in a neuron

We consider here the problem of the propagation of an electric potential $V(x, t)$ in a neuron [35]. This potential is governed by a system of non-linear PDEs called the Hodgkin-Huxley equations [16], but in certain ranges of values of V , this system of PDEs can be well approximated by the cable equation [35]. In particular, if the neuron is subject to a uniform input current density over the dendrites and if certain geometric constraints are satisfied, then the electric potential satisfies the following linear cable equation with uniform input current density.

$$\begin{aligned} \frac{\partial V}{\partial t}(x, t) &= \nu \frac{\partial^2 V}{\partial x^2}(x, t) - \beta V(x, t) + \sigma(V(x, t) + V_0)\dot{W}(x, t), \quad 0 \leq x, t \leq 1, \\ \frac{\partial V}{\partial x}(0, t) &= \frac{\partial V}{\partial x}(1, t) = 0, \quad t > 0, \quad V(x, 0) = V_0(x), \quad 0 \leq x \leq 1, \end{aligned} \quad (54)$$

where $\dot{W}(x, t) = \frac{\partial^2}{\partial x \partial t} w(x, t)$ is a space-time white noise meant in the Stratonovich sense. Here we have assumed that the distance between the origin (or soma) to the dendritic terminals is 1, and that the soma is located at $x = 0$. Furthermore, the white noise term is describing the effect of the arrival of random impulses and the multiplicative noise structure depicts the fact that the response of the neuron to a current impulse may depend on a local potential [35]. The quantity of interest is the threshold time

$$\tau = \inf\{t > 0; V(t, 0) > \lambda\}, \quad (55)$$

since when the potential at the soma (somatic depolarization) exceeds the threshold λ the neuron fires an action potential. The SPDE (54) yields, after space discretization with finite differences

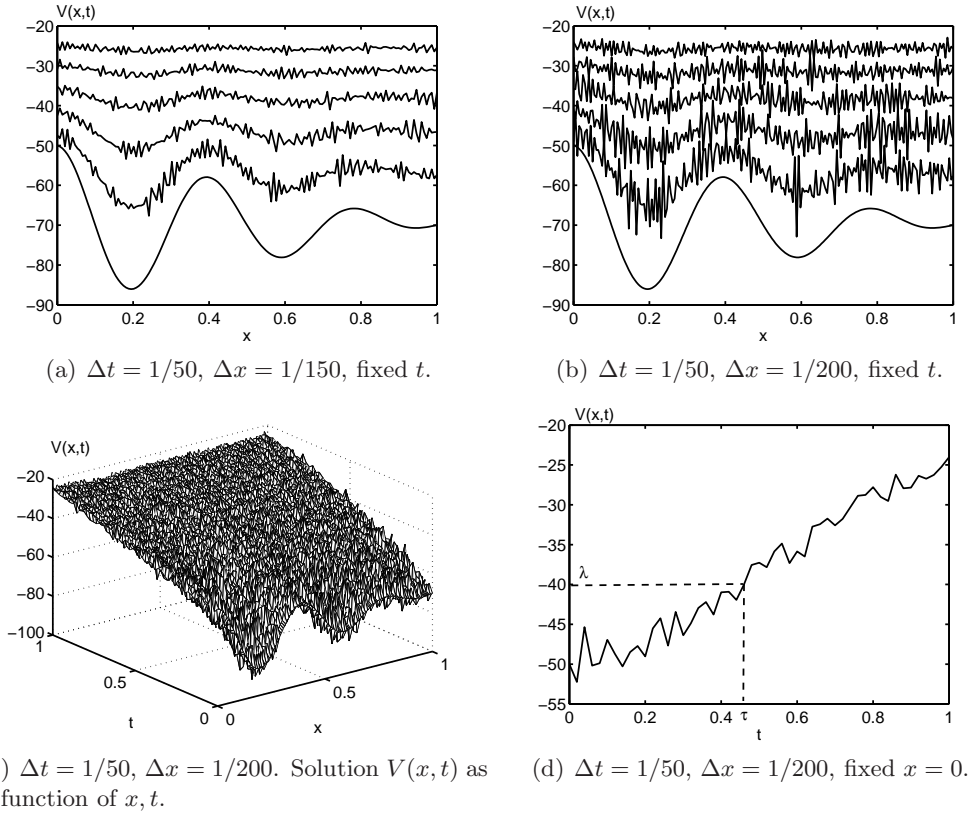


Figure 7: Samples of realisations of the SPDE (54) (discretized in space) using S-ROCK2 with $s + 2 = 8$ stages (resp. 11) for $\Delta x = 1/150$ (resp. $\Delta x = 1/200$). Figures (a),(b): solutions as functions of x at fixed times $t = 0, 0.2, 0.4, \dots, 1.0$ (increasing with time, from bottom to top). Figure (d): solution as a function of t for $x = 0$.

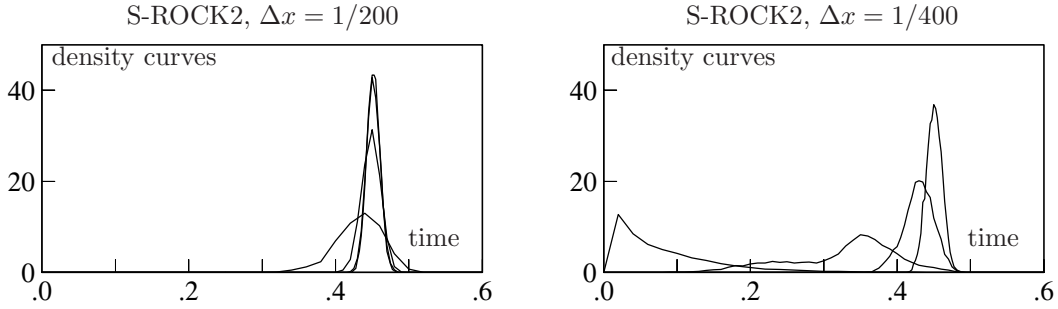


Figure 8: Density plots of the threshold time (55) in the SPDE (54) for various space mesh sizes $\Delta x = 1/100, 1/200, 1/400$. The four curves in each plot correspond respectively to $\Delta t = 1/50, 1/100, 1/200, 1/400$ (the variances decrease when Δt decreases).

[10] the following stiff system of SDE where $V(x_i, t) \approx u_i$, with $x_i = i\Delta x$, $\Delta x = 1/N$,

$$du_i = \nu \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} dt - \beta u_i dt + \sigma \frac{u_i + V_0}{\sqrt{\Delta x}} \circ dw_i, \quad i = 0, \dots, N, \quad (56)$$

where the Neumann condition imposes $u_{-1} = u_0$ and $u_{N+1} = u_N$. Here w_0, \dots, w_N are independent standard Wiener processes, and $\circ dw_i$ indicates Stratonovich noise. Converting this equation

into an equivalent system of Itô SDEs, we obtain

$$du_i = \nu \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} dt + \left(\frac{\sigma^2}{\Delta x} (u_i + V_0) - \beta u_i \right) dt + \sigma \frac{u_i + V_0}{\sqrt{\Delta x}} dw_i, \quad i = 0, \dots, N. \quad (57)$$

We consider the initial condition $V_0(x) = -70 + 20 \cos(5\pi x)(1 - x)$ and the constants $\nu = 10^{-2}$, $\sigma = 4 \cdot 10^{-3}$, $\beta = 1$, $V_0 = 10$, $\lambda = -40$. We consider the time interval $(0, T)$ with $T = 1$. Notice that the noise in (57) is in diagonal form, so we can apply Remark 3.3. We now plot the empirical histograms for the threshold time τ calculated over 10^7 realisations of (57).

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