FINITE ELEMENT HETEROGENEOUS MULTISCALE METHOD
FOR TRANSIENT WAVE PROPAGATION

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Talk Abstract

A finite element heterogeneous multiscale method (FE-HMM) is proposed for the time dependent wave equation with highly oscillatory, albeit not necessarily periodic, co-efficients. It is based on a finite element discretization of an effective wave equation at the macro scale, whose a priori unknown effective coefficients are computed "on the fly" on sampling domains within each macro finite element at the micro scale $\varepsilon > 0$. Since the sampling domains scale in size with ε, which corresponds to the finest scales in the possibly highly heterogeneous medium, the computational work is independent of ε. In [1], we proved optimal error estimates in the energy norm and the L^2 norm with respect to the micro and macro scale mesh parameters, h and H, and also convergence to the homogenized solution as $\varepsilon \to 0$.

Introduction

Transient wave phenomena from a wide range of applications, such as seismic inversion, medical imaging and therapy, or the design of multiphase composite materials are often modeled by the wave equation with highly oscillatory coefficients. Yet when waves propagate across inhomogeneous media with microscopic heterogeneities, assumed to occur at a scale larger than atomistic, standard numerical methods, such as finite element methods (FEM) or finite difference (FD) methods, become inefficient. As classical numerical schemes require grid resolution of the medium down to its finest scales, they typically lead to extremely large problem sizes although the scales of interest, such as the wavelength, often occur at a macroscopic level. In contrast, classical homogenization theory provides the analytical framework for deriving a properly averaged (homogenized) field a^0 that captures the essential macroscopic features of the highly oscillatory velocity field a^ε, as $\epsilon \to 0$. Unfortunately, explicit analytical formulas for a^0 are only available in simple situations, such as periodic or random stationary fields. To circumvent these difficulties, we propose a FE-HMM which computes the unknown effective coefficients "on the fly", that is without a priori knowledge of a^0.

Model Problem

We consider the wave equation

$$\partial_{tt} u_\varepsilon - \nabla \cdot (a^\varepsilon \nabla u_\varepsilon) = F \ \text{ in } \Omega \times]0, T[$$
$$u_\varepsilon = 0 \ \text{ on } \partial\Omega \times]0, T[\tag{1}$$
$$u_\varepsilon(x, 0) = f(x), \ \ \partial_t u_\varepsilon(x, 0) = g(x) \ \text{ in } \Omega,$$

where $a^\varepsilon(x) \in (L^\infty(\Omega))^{d \times d}$ is symmetric, uniformly elliptic, and bounded. Here $\varepsilon > 0$ represents a small scale in the problem, which characterizes the multiscale nature of the tensor $a^\varepsilon(x)$.

Homogenization

For every $\varepsilon > 0$, (1) has a unique (weak) solution u_ε under suitable additional assumptions on the source term, F, and the initial conditions f, g. Regarding the issue of convergence of u_ε, as $\varepsilon \to 0$, classical homogenization theory yields:

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly* in } L^\infty(0, T; H_0^1(\Omega)),$$
$$u_\varepsilon \to u_0 \qquad \text{strongly in } L^2(0, T; L^2(\Omega)),$$

where u_0 solves the "homogenized" wave equation:

$$\partial_{tt} u_0 - \nabla \cdot (a^0 \nabla u_0) = F \ \text{ in } \Omega \times]0, T[$$
$$u_0 = 0 \ \text{ on } \partial\Omega \times]0, T[\tag{2}$$
$$u_0(x, 0) = f(x), \ \ \partial_t u_0(x, 0) = g(x) \ \text{ in } \Omega,$$

Unfortunately, the homogenized tensor (or squared velocity field) $a^0(x)$ of this homogenized equation is seldom explicitly known.

FE-HMM

If $a^0(x)$ is explicitly known, the variational formulation of (2) immediately yields for $0 < t < T$:

$$\partial_{tt}(u_0(t, .), v) + B_0(u_0(t, .), v) = (F(t, .), v),$$

for all $v \in H_0^1(\Omega)$, where

$$B_0(v, w) = \int_\Omega a^0(x) \nabla v \cdot \nabla w \, dx.$$

Then, we can apply a standard Galerkin FE method for the FE-space $S_0^\ell(\Omega, \mathcal{T}_H)$ of continuous, piecewise polynomial functions of degree ℓ, defined over a macro triangulation $\mathcal{T}_H$; in particular, $H \gg \varepsilon$ is allowed, since a^0 is no longer highly oscillatory.

Inside each macro element $K \in \mathcal{T}_H$, we now choose a quadrature formula, given by its integration points $x_{j,K}$ and quadrature weights $\omega_{j,K}$ ($j = 1, \ldots, J$) to evaluate the integrals required for the mass and stiffness matrices. This leads to the semi-discrete FE formulation: find $u_0^H \in S_0^\ell(\Omega, \mathcal{T}_H)$ such that

$$\partial_{tt}(u_0^H(.,t), v^H) + B_{0,H}(u_0^H(.,t), v^H) = (F(.,t), v^H),$$

for all $v^H \in S_0^\ell(\Omega, \mathcal{T}_H)$, where

$$B_{0,H}(v^H, w^H) =$$
$$\sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \omega_{j,K} a^0(x_{j,K}) \nabla v_H(x_{j,K}) \cdot \nabla w_H(x_{j,K}).$$

However, as $a^0(x)$ is generally unknown, we shall approximate the bilinear form $B_{0,H}$ with the FE-HMM bilinear form B_H:

$$B_H(v^H, w^H) =$$
$$\sum_{K \in \mathcal{T}_H} \sum_{j=1}^{J} \frac{\omega_{j,K}}{|K_\delta|} \int_{K_\delta} a^\varepsilon(x) \nabla v_{K_\delta}^h \cdot \nabla w_{K_\delta}^h \, dx, \quad (3)$$

where $v_{K_\delta}^h$ (resp. $w_{K_\delta}^h$) are the solution of the micro problem 4 on each sampling domain $K_\delta = K_\delta(x_{j,K})$ around each quadrature point: find $v_{K_\delta}^h$ such that $(v_{K_\delta}^h - v_{lin,K_\delta}^H) \in S^q(K_\delta, \mathcal{T}_h)$ and

$$\int_{K_\delta} a^\varepsilon(x) \nabla v_{K_\delta}^h \cdot \nabla z^h \, dx = 0, \quad (4)$$

for all $z^h \in S^q(K_\delta, \mathcal{T}_h)$, where $S^q(K_\delta, \mathcal{T}_h)$ is the micro FE space defined above and

$$u_{lin,K_\delta}^H = u^H(x_{j,K}) + (x - x_{j,K}) \cdot \nabla u^H(x_{j,K}) \quad (5)$$

is a piecewise linear approximation of $u^H \in S_0^\ell(\Omega, \mathcal{T}_H)$ about the integration point $x_{j,K}$ (see [2],[3] for details). The multi-scale FE discretization is illustrated in Fig. 1.

The following optimal L^2-error estimate was proved for the FE-HMM solution $u^H(x,t)$ for locally periodic coefficients in [1]:

$$\|u_0 - u_H\|_{L^\infty(0,T;L^2(\Omega))} \le C \left(H^{\ell+1} + \varepsilon + (h/\varepsilon)^{2q} \right),$$
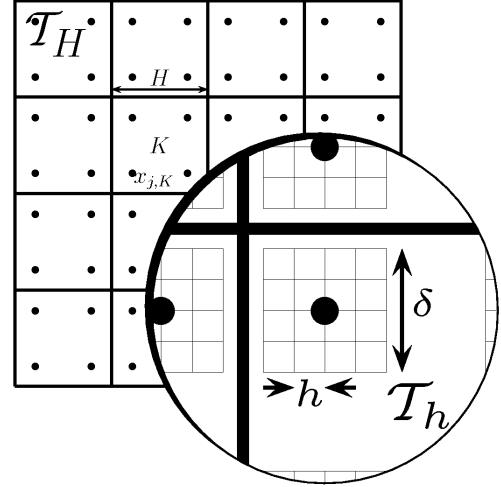

Figure 1: The FE-HMM spatial discretization.

where ℓ is the order of the macro solver, q the order of the micro solver and C is a constant independent of ε, H and h. The proof relies on a new Strang-type lemma for the wave equation, which allows to re-use former results from the FE-HMM for elliptic equations [4].

Numerical examples

To illustrate the usefulness of our FE-HMM, we shall now apply it to two simple problems. First, we consider a one-dimensional heterogeneous medium, where we can compute a reference solution u_ε by fully resolving the medium down to the finest scale ε. Then, we consider a two-dimensional example, where the homogenized tensor, a^0, is explicitly known. In both cases, we use the standard second-order leap-frog method for time discretization. The time-step restriction for stability is independent of small-scale parameters, such as h and ε.

One-dimensional example

We first consider the one-dimensional wave equation in $\Omega =]-3, 5[$ with $F = 0$ until the final time $T = 3$, where the initial conditions correspond to a right-moving Gaussian pulse. The squared velocity field $a^\varepsilon(x)$, shown in Fig. 2, is not periodic but nonetheless highly oscillatory for increasingly smaller values of ε. It is given by

$$a^\varepsilon(x) = \begin{cases} 2 + \sqrt{2} + \sin\left(\frac{2\pi x}{\varepsilon}\right) & \text{for } x \in \Omega_+, \\ \sqrt{2} + \sin\left(\frac{2\pi x}{\varepsilon}\right) & \text{elsewhere,} \end{cases} \quad (6)$$

where $\Omega_+ = \bigcup_{n=1}^{5}(n - 0.5, n)$.

We now set $\varepsilon = 10^{-3}$ and compute the FE-HMM solution u^H on a coarse mesh with $H = 10^{-2}$ and time step $\Delta t = 10^{-3}$. The sampling domains are of size
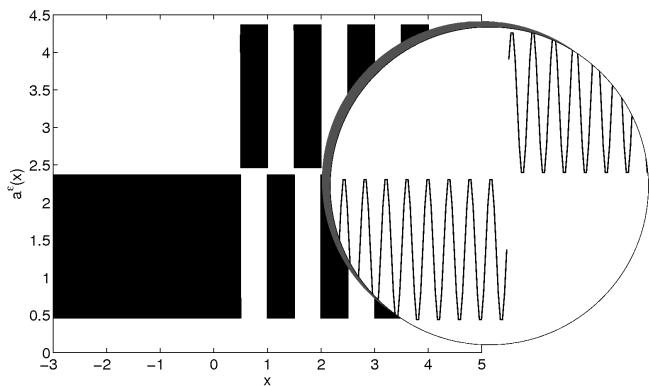
Figure 2: The squared velocity $a^\varepsilon(x)$ as defined in (6) is shown for $\varepsilon = 10^{-3}$.

$\delta = 10^{-3}$, each partitioned in equidistant sub-intervals of size $h = 10^{-4}$. Both the macro and the micro FE spaces consist of continuous piecewise linear functions with $\ell = q = 1$. For reference, we also compute u_ε by a fully resolved standard FE solution on a highly refined mesh with $H = 10^{-4}$ and $\Delta t = 10^{-5}$ to avoid numerical artifacts. Due to the small size of ε we expect u_ε to be very close to u_0 (in the L^2 norm).
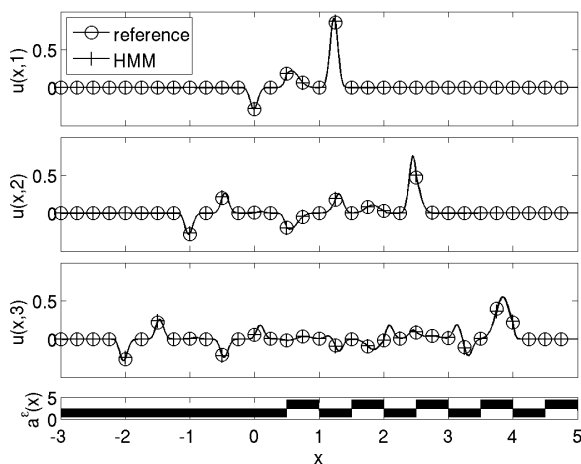
Figure 3: Snapshots of u^H and u_ε at times $t = 1, 2, 3$.

Both u^H and u_ε are shown in Fig. 3 and we observe that they essentially coincide (at this scale) until the final time. Hence the FE-HMM is able to capture the main features of the underlying heterogeneous medium at a much smaller computational cost, and in fact independently of ε. Note, however, that the FE-HMM is unable to capture long-time dispersive effects, which classical homogenization cannot reproduce either.

In Table 1, the computational cost of a standard fully

Table 1: The computational cost of a fully resolved FE solution of (1) with $\varepsilon = 10^{-3}$ is compared to the cost of the FE-HMM.

	fully resolved FEM	FE-HMM
number of elements	80000	800
computing the mass matrix	13.5 s	0.0074 s
computing the stiffness matrix	18.1 s	3.7 s
number of time steps	300000	3000
time for time-stepping	9306.8 s	0.408 s

resolved FE solution of the wave equation (1) with $\varepsilon = 10^{-3}$ from Fig. 3 is compared to the cost of the FE-HMM. Since $H/h = 100$ and the problem is one-dimensional, the fully resolved FE solution requires about a hundred times more elements. The resulting hundredfold more stringent CFL condition, however, also forces the fully resolved FE method to use a one hundred times smaller time step. Hence the time iteration of the fully resolved FE method is about 10,000 times more expensive than that of the FE-HMM here. Regarding the total execution time, that huge gain in computing time is slightly mitigated by the higher relative cost in computing the stiffness matrix. Indeed in the case of the FE-HMM, the computation of the stiffness matrix requires in each macro-element the solution of an additional micro-cell problem of size δ and mesh-size h. As the stiffness matrix is computed only once upfront, its computational cost hardly matters in comparison to that of the subsequent time integration. Nonetheless the computation of the stiffness matrix for the FE-HMM is in fact about five times cheaper than that of the fully resolved FEM, because the number of coarse elements still is so much smaller.

As the cost of the FE-HMM is independent of ε, the relative cost of the FE-HMM becomes arbitrarily small at even smaller values of ε; the fully resolved FE method, however, may simply no longer be computable. In three space dimensions, for instance, the fully resolved FE solution would require one million times more memory and one hundred million times more computer time for the same values of h, H, and ε.

Two-dimensional example

Next, we let $\Omega =]0, 3[\times]0, 1[\subset \mathbb{R}^2$, $T = 3$ and divide Ω into two distinct sub-regions – see Fig. 4:

$$\begin{aligned} \Omega_1 &= \{x = (x_1, x_2) \in \Omega : x_2 \geq x_1 - 1\} \\ \Omega_2 &= \{x = (x_1, x_2) \in \Omega : x_2 < x_1 - 1\}. \end{aligned} \quad (7)$$

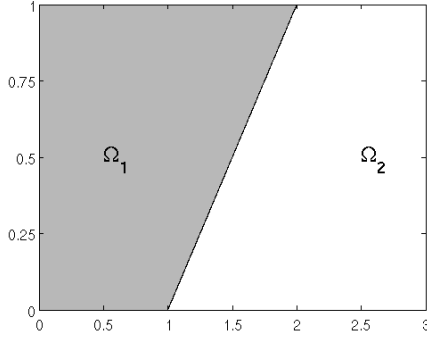We consider the wave equation (1) in two space dimensions with $F = 0$ and zero initial condition, $f = g \equiv 0$.

Figure 4: The computational domain Ω for the two-dimensional example.

The squared velocity field is given by

$$a^{\varepsilon}(x) = \begin{cases} 1.1I & \text{for } x \in \Omega_1, \\ (1.1 + \sin(2\pi\frac{x_1}{\varepsilon}))I & \text{for } x \in \Omega_2, \end{cases} \quad (8)$$

where I is the 2×2 identity matrix.

On the upper and lower boundary, we impose homogeneous Neumann boundary conditions (BC) and on the right boundary we impose a homogeneous Dirichlet BC. On the left boundary we set the time-dependent Dirichlet condition,

$$u(x,t) = \sin(4\pi t), \quad (9)$$

which corresponds to a plane wave incoming from the left. The discretization parameters H, ε, δ and h are chosen as before, but we choose quadrilateral biquadratic finite elements.
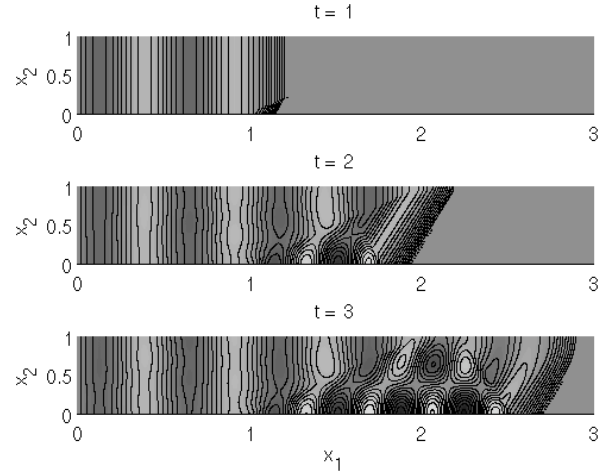
For reference, we compute the FE solution u_0^H of the wave equation with that analytically computed homogenized tensor In Ω_1, no homogenization is needed whereas in Ω_2 there is a closed formula for the homogenized tensor a^0, due to the simple layered structure of a^{ε}:

$$a^0(x) = \begin{cases} 1.1I & \text{for } x \in \Omega_1, \\ \begin{pmatrix} \sqrt{0.21} & 0 \\ 0 & 1.1 \end{pmatrix} & \text{for } x \in \Omega_2. \end{cases} \quad (10)$$
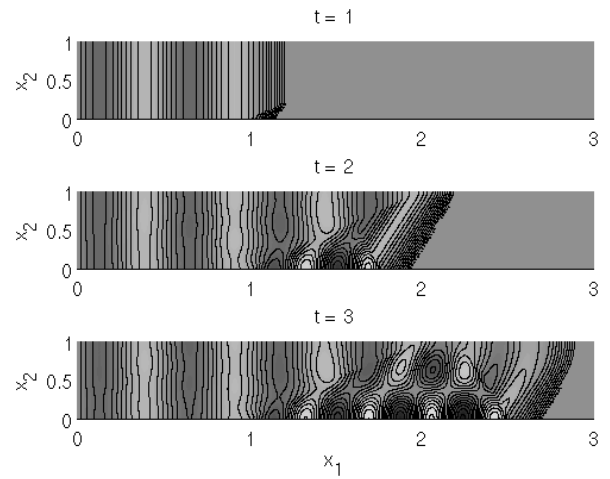
Snapshots of the FE-HMM solution u^H and the homogenized solution u_0 are shown in Fig. 5 at different times; both coincide at this scale.

(a)

(b)

Figure 5: Two-dimensional example. Snapshots are shown at times $t = 1, 2, 3$: (a) u^H; (b) u_0

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