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On the efficiency of numerical homogenization methods

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We study finite element (FE) discretizations of second-order elliptic problems of the form

$$-\nabla \cdot (a^\varepsilon(x)\nabla u^\varepsilon(x)) = f(x) \text{ in } \Omega, \qquad u^\varepsilon(x) = 0 \text{ on } \partial\Omega, \tag{1}$$

where Ω is a bounded convex polyhedron in $\mathbb{R}^d$, with $d \leq 3$. Zero Dirichlet boundary conditions are taken here for simplicity and we emphasize that other boundary conditions can be treated with the multiscale method described below. We also note that while we discuss linear elliptic problems, the multiscale method (and the a-priori analysis) can be derived for time-dependent problems (parabolic or hyperbolic) or nonlinear problems [7],[8],[9].

The $d \times d$ tensor $a^\varepsilon(x)$, assumed to be uniformly elliptic and bounded, is allowed to vary on a very small spatial scale denoted by ε. This behavior of a^ε makes a standard numerical approximation very costly if not impossible. Numerical homogenization methods are multiscale methods, inspired by homogenization theory [11],[14], aiming at computing an effective solution, $u^0(x)$, of the equation (1). This effective solution solves a homogenized problem, whose data are usually unknown in explicit form.

The FE-HMM method for computing a numerical approximation of the effective (homogenized) solution $u^0(x)$ is based on a macro FE space made of piecewise polynomial functions of degree ℓ defined on a macro partition $\mathcal{T}_H$ of Ω, and micro FEMs for the solution of so-called cell problems (involving the oscillating tensor of equation (1)) defined on sampling domains located at quadrature points within the macro partition. A proper averaging of the micro solutions allows then to define the effective bilinear form whose solution gives an approximation of the homogenized solution u^0. Supplemented with appropriate numerical correctors, the FE-HMM solution can also capture, in certain situation, the fine scale solution $u^\varepsilon(x)$ (see [1],[2],[3],[12],[13]).

A fully discrete a priori convergence analysis shows that the complexity of the FE-HMM (as any numerical homogenization method) is superlinear with respect to the macroscopic degrees of freedom [1],[2],[3]. In particular, macro and micro meshes have to be refined simultaneously to obtain optimal convergence rates with a minimal computational cost. For high dimensional problems or high order macro methods (for which a lot of sampling domains have to be used) the FE-HMM can become costly.

Three ways of reducing the complexity. We discuss three (non exclusive)

ways to reduce the complexity of numerical homogenization methods as the FE-HMM. To simplify the presentation we consider a tensor of the form $a^\varepsilon = a(x, x/\varepsilon)$.

Case of very regular micro oscillations of $a(x, \cdot)$. In this situation, the macro solver of the FE-HMM can be coupled with pseudo-spectral methods on the sampling domains. Taking advantage of the fast convergence of the micro solvers, this strategy can reduce significantly the computational cost provided enough regularity of the oscillating tensor at the micro scale [4].

Case of a regular dependence on the macro variable of $a(\cdot, x/\varepsilon)$. In such a situation, following the reduced basis framework, the idea is to compute in an offline stage a low dimensional subspace of the micro solutions used to define the effective bilinear form in the FE-HMM. These parametrized micro solutions are selected by a Greedy algorithm. Let $\mathcal{M}$ be the space of all cell problem solutions indexed by $\nu \in \Omega$, the barycenter of the sampling domains and $\eta = 1, \ldots, d$, the d solutions of the cell problem in each sampling domain. The goal is to find an $N-$dimensional subspace $\mathcal{M}_N$ of $\mathcal{M}$ that "minimizes" the distance $\sup_{\xi \in \mathcal{M}} \operatorname{dist}(\xi, \mathcal{M}_N)$ [10]. This precomputed set of selected solutions of cell problems (computed with high accuracy) can then be used in an online stage to obtain cheap micro solutions.

Case of low regularity of u^0. In this situation, explicit localized error indicators for robust and reliable adaptive mesh refinement can be derived. A (non-uniformly) refinement of the macromesh can be coupled to a refinement of the micromesh covering the sampling domains [5],[6]. As new microsolutions need not to be re-computed in non refined elements, adaptive mesh refinement allows for an important computational saving compared to uniform refinement strategies.

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Data assimilation for dynamical systems

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(joint work with Georg Gottwald (University of Sydney), Kay Bergemann (Universität Potsdam), Eugenia Kalnay, Javier Amezcua, Kayo Ide (University of Maryland))

The basic task of data assimilation (nonlinear filtering) can be explained for second-order Langevin dynamics

$$dq = v dt, \tag{1}$$

$$dv = -V'(q)dt - \gamma v dt + \sqrt{\sigma} dw(t) \tag{2}$$

where $w(t)$ denotes standard Brownian motion. The parameters $\gamma > 0$ and $\sigma > 0$ are assumed to be known as well as the initial conditions $q(0)$ and $v(0)$. In addition it is assumed that one has "measurements" $Q(t)$ of the "true" positions $q_T(t)$ which satisfy the stochastic differential equation

$$dQ = v_T(t)dt + \sqrt{r} du(t) \tag{3}$$

where $u(t)$ is again Brownian motion, $(q_T(t), v_T(t))$, $t \geq 0$, is an unknown solution (the "truth"), and $r > 0$ is known.

Both naive approaches of either solving (1)-(2) with the given initial conditions or integrating (3) to obtain $Q(t)$ are able to track the reference solution $(q_T(t), v_T(t))$ over long periods of time. Instead one has to resort to filtering or smoothing techniques to combine (1)-(3).

In recent year, the ensemble Kalman filter [1] has emerged as a powerful nonlinear filter for intermittent data assimilation. We have extended this technique to continuous data assimilation problems as outlined above. In particular, the ensemble Kalman-Bucy filter [5, 7] leads to the following augmented system of stochastic differential equations. We first rewrite (1)-(3) in more abstract form as

$$dx = f(x,t)dt + \Sigma^{1/2} dw(t), \tag{4}$$

$$dy = Hx dt + R^{1/2} du(t) \tag{5}$$