Distributed Synthesis and Control of Constrained Linear Systems

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***Abstract*— This work presents an approach for both distributed synthesis and control for a network of discrete-time constrained linear systems without central coordinator. Every system in the network is dynamically coupled to a number of neighboring systems and it is assumed that communication among neighbors is possible. A model predictive controller based on distributed optimization is introduced, by which every system in the network can compute feasible and stabilizing control inputs online. Stability of the closed-loop network of systems is guaranteed by introducing local terminal cost functions and sets, which together satisfy invariance conditions in a distributed way. This includes in particular that the local terminal sets are not static but evolve over time. It is shown that synthesis of both quadratic terminal cost functions and corresponding terminal sets can be done by distributed optimization. Finally, closed-loop performance of the proposed controller is demonstrated on a coupled array of inverted pendulums.**

I. INTRODUCTION

This paper considers both distributed synthesis and control for a network of dynamically coupled linear systems. None of these systems is aware or the global network's state or model. Instead, each system has a set of local information which it obtains by measuring its own state and by sharing knowledge with a number of neighboring systems. The fact that each system takes all decisions based on local information implies a strong need for both distributed synthesis and distributed control.

Many researchers have considered the synthesis of distributed controllers for unconstrained linear systems. For some classes of systems, the control synthesis problem has been proven to be intractable, although other tractable classes have been identified. In [1] for instance, it is shown that the synthesis problem is convex if the communication structure of a system is quadratically invariant to its coupling structure. Since the resulting problem is infinite dimensional however, no universal synthesis method exists, although in some cases practical methods are available. Centralized methods to obtain distributed linear state feedback controllers include the solution of a linear matrix inequality (LMI) [2], [3] or the construction of a vector Lyapunov function [4]. Distributed methods include LMI based $\mathcal{H}_\infty$ controller synthesis [5], [6] or gradient based optimization of an infinite horizon quadratic cost function [7].

For constrained linear systems, model predictive control (MPC) has proven to be a powerful centralized control method. At every point in time, a finite horizon optimal control problem, i.e. the MPC problem, is solved and the control input is defined as the first element of the optimal input sequence. Stability is usually achieved by forcing the last element of the optimal state trajectory to lie in a terminal invariant set, in which a Lyapunov function for the unconstrained system under a nominal control law is known. If this Lyapunov function is used as a terminal cost in the finite horizon problem, stability of the closed loop system under the MPC control law is guaranteed [8]. This paper provides sufficient, and practical conditions that extend the standard theory to the distributed case.

In order to maintain constraint satisfaction and stability, distributed MPC approaches are often more conservative than centralized ones. In some approaches, a global MPC problem is solved by distributed optimization, but conservative terminal sets consisting of a single point are used to guarantee closed loop stability [9], [10]. In other approaches, restrictions are imposed such that stabilizing control inputs can be found by decoupled local optimization problems. These include solving a local min-max problem, taking the influence of neighbors into account as disturbance [11] or staying close to previously computed feasible local trajectories [12].

In this paper, a novel distributed MPC controller for discrete-time constrained linear systems is proposed. Compared with existing approaches, conservatism is reduced by combining distributed optimization with nonconservative decentralized terminal sets. Stability and invariance are guaranteed by updating the size of these terminal sets dynamically, utilizing recent research on decentralized Lyapunov functions in [13]. This scheme of decentralized invariance follows similar ideas as recently presented in [14]. However, the approach taken here focuses not only on invariance, but also on stability and on simultaneous distributed synthesis of terminal control laws, terminal costs and terminal invariant sets. The proposed distributed MPC controller results in a larger region of attraction compared to a controller based on a terminal set consisting of a single point. As an additional, and important, novelty, a synthesis method is proposed, by which the computation of all terminal costs and sets can be carried out in a completely distributed way.

The structure of the paper is as follows: In Section II, the distributed problem is introduced. In Section III, stability conditions for distributed MPC are derived by adapting an

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idea on structured control Lyapunov functions from [13]. Subsequently, in Section IV, a distributed synthesis method for an MPC controller which fulfills these stability conditions is presented and in Section V, performance of the proposed distributed MPC controller is demonstrated by a numerical example.

II. PRELIMINARIES AND PROBLEM SETUP

A. Preliminaries and Notation

For a vector $x \in \mathbb{R}^n$, $||x||$ denotes its Euclidean norm. By $\mathrm{diag}(S_1, S_2, \ldots, S_n)$, we denote a block-diagonal matrix with matrices S_1 to S_n on the main diagonal and zeros everywhere else. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and if $f(0) = 0$. A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}_\infty$ if $f \in \mathcal{K}$ and if $\lim_{s\to\infty} f(s) = \infty$.

B. Decentralized Constrained Linear Systems

Throughout the paper, we deal with an ordered set of M dynamically coupled linear systems. The state of system i is denoted as $x_i \in \mathbb{R}^{m_i}$, whereas its input is denoted as $u_i \in \mathbb{R}^{p_i}$. Coupling is caused by the states while inputs only affect the systems locally. Consequently, the dynamics of system i can be written as

$$x_i^+ = \left(\sum_{j=1}^{M} A_{ij} x_j \right) + B_i u_i \ , \tag{1}$$

where $A_{ij} \in \mathbb{R}^{m_i \times m_j}$ and $B_i \in \mathbb{R}^{m_i \times p_i}$. The dynamics of the overall network of systems are obtained as

$$x^+ = Ax + Bu \ , \tag{2}$$

where $x = [x_1^T, \ldots, x_M^T]^T \in \mathbb{R}^m$ is its state and $u = [u_1^T, \ldots, u_M^T]^T \in \mathbb{R}^p$ is its input. $A \in \mathbb{R}^{m\times m}$ consists of the blocks A_{ij} and $B = \mathrm{diag}(B_1, \ldots, B_M) \in \mathbb{R}^{m\times p}$ is block-diagonal, where it is assumed that the pair (A, B) is controllable. Throughout the paper, we will refer to a system described by (1) as a subsystem and to the system described by (2) as the global system.

Using the subsystem dynamics (1), we define the notion of neighboring subsystems and specify the communication structure in the overall network.

Definition II.1 (Neighboring subsystems) *Subsystem j is a neighbor of subsystem i if $A_{ij} \neq 0$. $\mathcal{N}_i$ denotes the ordered set of neighbors of subsystem i including i and $x_{\mathcal{N}_i} \in \mathbb{R}^{m_{\mathcal{N}_i}}$ contains the states of all subsystems in $\mathcal{N}_i$.*

Assumption II.2 (Communication) *Two subsystems i and j can communicate with each other only if either $j \in \mathcal{N}_i$ or $i \in \mathcal{N}_j$.*

The states and inputs of every subsystem i are subject to local constraints

$$x_i \in \mathcal{X}_i, \ \ u_i \in \mathcal{U}_i \ , \tag{3}$$

where $\mathcal{X}_i = \{x_i \in \mathbb{R}^{m_i} | G_x^i x_i \leq f_x^i\}$ and $\mathcal{U}_i = \{u_i \in \mathbb{R}^{p_i} | G_u^i u_i \leq f_u^i\}$ are polytopes containing the origin in their interior. By taking the cartesian product of all local constraint sets, we denote the global state and input constraint sets as

$$\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_M, \ \ \mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_M \ . \tag{4}$$

Remark II.3 *The particular class of constrained distributed systems introduced in this section has been chosen for the sake of simplicity in the notation. Nevertheless, the ideas presented in the following sections can be extended to more general problem formulations, e.g. combined constraints.*

C. Distributed MPC Problem Formulation

First, consider the global MPC problem

$$V^*(x) = \min_{\mathbf{u}} \ V_f(x(N)) + \sum_{k=0}^{N-1} l(x(k), u(k)) \tag{5a}$$

$$\text{s.t. } x(0) = x \tag{5b}$$

$$x(k+1) = Ax(k) + Bu(k), \ \forall k \in \{0, \ldots, N-1\} \tag{5c}$$

$$(x(k), u(k)) \in \mathcal{X} \times \mathcal{U}, \ \forall k \in \{0, \ldots, N-1\} \tag{5d}$$

$$x(N) \in \mathcal{X}_f \ . \tag{5e}$$

We define $\mathbb{X}_N$ as the set of x, for which (5) is feasible. Both the stage cost $l(x, u)$ and the terminal cost $V_f(x)$ are convex functions and the terminal set $\mathcal{X}_f \subseteq \mathbb{R}^m$ is convex compact and contains the origin in its interior. Furthermore, $\mathbf{u} = \{u(0), \ldots, u(N-1)\}$ denotes an input trajectory over the finite horizon N and $\mathbf{u}^*(x)$ denotes an optimal choice of $\mathbf{u}$ in the sense of (5). The first element of $\mathbf{u}^*(x)$ is denoted $u^*(x)$ and defines a state feedback control law, which results in the nonlinear global closed loop system dynamics

$$x^+ = Ax + Bu^*(x) \ . \tag{6}$$

Sufficient conditions for stability of system (6) on $\mathbb{X}_N$ are fulfilled if a terminal control law $u_f(\cdot) : \mathcal{X}_f \to \mathbb{R}^p$ and $\mathcal{K}_\infty$ class functions α_1, α_2 and α_3 exist such that

$$\forall x \in \mathcal{X}_f\colon x \in \mathcal{X}, \ u_f(x) \in \mathcal{U}, \ Ax + Bu_f(x) \in \mathcal{X}_f \tag{7a}$$

$$\alpha_1(||x||) \leq V_f(x) \leq \alpha_2(||x||) \tag{7b}$$

$$V_f(Ax + Bu_f(x)) - V_f(x) \leq -\alpha_3(||x||) \ . \tag{7c}$$

Condition (7a) implies that $\mathcal{X}_f$ is a positive invariant set and conditions (7b) and (7c) imply that on $\mathcal{X}_f$, $V_f(x)$ is a Lyapunov function for the closed loop system

$$x^+ = Ax + Bu_f(x) \ . \tag{8}$$

In a distributed system, central computation of $u^*(x)$ is not possible due to the lack of global system information at subsystem level. However, an appropriately structured global MPC problem can be solved by well-established distributed optimization methods [15]. In these methods, a global convex optimization problem is decomposed into a number of smaller subproblems, which are coupled through shared variables. By iterative negotiation on these shared variables, the subproblems' variable sets converge to the global optimal solution, while in every iteration only communication between coupled subproblems is required.

To comply with the communication constraints from Assumption II.2, we would like to decompose problem (5) into one subproblem per subsystem i, each of which only contains the variables u_i and $x_{\mathcal{N}_i}$. The only elements in (5), which do not inherently allow such a decomposition are the stage cost $l(x,u)$, the terminal cost $V_f(x)$ and the terminal set $\mathcal{X}_f$. The main challenge in the formulation of the distributed MPC problem is thus the choice of these elements, in particular the choice of the stability-critical terminal cost and terminal set. These are generally dense, but for the sake of decomposability we impose the structural constraints

$$l(x,u) = \sum_{i=1}^{M} l_i(x_{\mathcal{N}_i}, u_i) \tag{9a}$$

$$V_f(x) = \sum_{i=1}^{M} V_f^i(x_i) \tag{9b}$$

$$\mathcal{X}_f = \mathcal{X}_f^1 \times \ldots \times \mathcal{X}_f^M \ , \tag{9c}$$

where $\mathcal{X}_f^i \subseteq \mathbb{R}^{m_i}$. A distributed synthesis approach for a terminal cost and a terminal set, structured as in (9b) and (9c), is presented in the next two sections.

III. DISTRIBUTED INVARIANCE

A. Decentralized Terminal Cost

One sufficient condition for stability of an MPC controller is fulfilled if the terminal cost function is a Lyapunov function for the unconstrained system under a terminal control law. A Lyapunov function structured as (9b) strongly resembles the vector Lyapunov function methodology presented in [4]. In this methodology, a Lyapunov function for a network of coupled subsystems is constructed as a weighted sum of Lyapunov functions for the uncoupled subsystems. However, this methodology does not directly include the synthesis of a nominal control law, existence of which is another sufficient stability condition for MPC. Therefore, we will exploit another approach, synthesizing a control law and a Lyapunov function simultaneously.

A naïve way to choose a terminal cost function $V_f^i(x_i)$ for subsystem i would be to demand it to decrease in every time step, even under full coupling. Such an approach, however, would be very conservative. Consider for instance a subsystem i, whose state x_i rests at the origin where $V_f^i(x_i) = 0$. If the state x_j of a neighboring subsystem $j \in \mathcal{N}_i$ is nonzero, x_i will necessarily be driven away from the origin, causing $V_f^i(x_i)$ to increase. Therefore, as proposed in [13], it is desirable to allow a local terminal cost to increase, as long as at the same time the global terminal cost decreases.

Theorem III.1 (Implied by Theorem III.4 in [13]) *If there exists a positive invariant set $\mathcal{X}_f \subseteq \mathbb{R}^m$, and there exist functions $V_f^i(x_i)$, $\gamma_i(x_{\mathcal{N}_i})$, $u_f^i(x_{\mathcal{N}_i})$ and $l_i(x_{\mathcal{N}_i}, u_f^i(x_{\mathcal{N}_i}))$ as well as functions α_1^i, α_2^i and $\alpha_3^i \in \mathcal{K}_\infty$, such that $\forall x \in \mathcal{X}_f$ and $\forall i \in \{1,\ldots,M\}$*

$$\alpha_1^i(||x_i||) \le V_f^i(x_i) \le \alpha_2^i(||x_i||) \tag{10}$$

$$\alpha_3^i(||x_{\mathcal{N}_i}||) \le l_i(x_{\mathcal{N}_i}, u_f^i(x_{\mathcal{N}_i})) \tag{11}$$

$$V_f^i(x_i^+) - V_f^i(x_i) \le -l_i(x_{\mathcal{N}_i}, u_f^i(x_{\mathcal{N}_i})) + \gamma_i(x_{\mathcal{N}_i}) \tag{12}$$

$$\sum_{i=1}^{M} \gamma_i(x_{\mathcal{N}_i}) \le 0 \ , \tag{13}$$

then the function $V_f(x) = \sum_{i=1}^{M} V_f^i(x_i)$ is a Lyapunov function for system (8) under control law $u_f(x) = [u_f^{1,T}(x_{\mathcal{N}_1}),\ldots,u_f^{M,T}(x_{\mathcal{N}_M})]^T$ and (8) is asymptotically stable on $\mathcal{X}_f$.

B. Decentralized Invariant Terminal Set

Given a Lyapunov function $V_f(x)$ for the closed loop system (8), any feasible level set $\mathcal{X}_f = \{x \in \mathbb{R}^m | V_f(x) \le \alpha\} \subseteq \mathcal{X}$ thereof is invariant and would therefore be a potential terminal set for a stabilizing centralized MPC controller. However, for distributed optimization we want the terminal set to be decomposable as in (9c), which is nontrivial and will be discussed in the following.

Consider for every subsystem a level set of the local terminal cost function $V_f^i(x_i)$ as

$$\mathcal{X}_f^i = \{x_i \in \mathbb{R}^{m_i} | V_f^i(x_i) \le \alpha_i\} \ , \tag{14}$$

where $\sum_{i=1}^{M} \alpha_i \le \alpha$. We then have

$$(x_1,\ldots,x_M) \in \mathcal{X}_f^1 \times \ldots \times \mathcal{X}_f^M \Rightarrow x^+ \in \mathcal{X}_f \ . \tag{15}$$

However, the converse of (15) does not hold. Hence, as time evolves, the state of the global system remains in $\mathcal{X}_f$ but the state of any subsystem i might leave $\mathcal{X}_f^i$, which compromises invariance. A remedy for this issue is to update the size of the local terminal sets according to the dynamics

$$\alpha_i^+ = \alpha_i + \gamma_i(x_{\mathcal{N}_i}) \ , \tag{16}$$

which leads to a dynamic terminal set $\mathcal{X}_f^i(\alpha_i)$ for every subsystem. The following results will show invariance of the dynamic set $\mathcal{X}_f^1(\alpha_1) \times \ldots \times \mathcal{X}_f^M(\alpha_M)$.

Lemma III.2 $\forall i \in \{1,\ldots,M\}$: *Given a local terminal set $\mathcal{X}_f^i(\alpha_i) = \{x_i \in \mathbb{R}^{m_i} | V_f^i(x_i) \le \alpha_i\}$, it holds that*

$$x_i \in \mathcal{X}_f^i(\alpha_i) \Rightarrow x_i^+ \in \mathcal{X}_f^i(\alpha_i^+) \ . \tag{17}$$

Proof: $\forall i \in \{1,\ldots,M\}$: Consider $x_i \in \mathcal{X}_f^i(\alpha_i)$, which implies that $V_f^i(x_i) \le \alpha_i$. By Theorem III.1 we have

$$V_f^i(x_i^+) \le V_f^i(x_i) - l_i(x_{\mathcal{N}_i}, u_f^i(x_{\mathcal{N}_i})) + \gamma_i(x_{\mathcal{N}_i}) \tag{18}$$

$$\le V_f^i(x_i) + \gamma_i(x_{\mathcal{N}_i}) \le \alpha_i + \gamma_i(x_{\mathcal{N}_i}) = \alpha_i^+ \tag{19}$$

and thus $x_i^+ \in \mathcal{X}_f^i(\alpha_i^+)$. ■

Theorem III.3 *Given a feasible set $\mathcal{X}_f = \{x \in \mathbb{R}^m | V_f(x) \le \alpha\} \subseteq \mathcal{X}$ and $\forall i \in \{1,\ldots,M\}$ local terminal sets $\mathcal{X}_f^i(\alpha_i) = \{x_i \in \mathbb{R}^{m_i} | V_f^i(x_i) \le \alpha_i\}$, it holds that*

$$\mathcal{X}_f^1(\alpha_1) \times \ldots \times \mathcal{X}_f^M(\alpha_M) \subseteq \mathcal{X}_f \Rightarrow \mathcal{X}_f^1(\alpha_1^+) \times \ldots \times \mathcal{X}_f^M(\alpha_M^+) \subseteq \mathcal{X}_f \ . \tag{20}$$

Proof: $\mathcal{X}_f^1(\alpha_1) \times \ldots \times \mathcal{X}_f^M(\alpha_M) \subseteq \mathcal{X}_f$ implies $\sum_{i=1}^M \alpha_i \leq \alpha$. Using the update rule (16) and Theorem III.1, we see that

$$\sum_{i=1}^M \alpha_i^+ = \sum_{i=1}^M \alpha_i + \sum_{i=1}^M \gamma_i(x_{\mathcal{N}_i}) \leq \sum_{i=1}^M \alpha_i \leq \alpha \ , \quad (21)$$

which implies (20). ∎

Remark III.4 *Another method considering dynamic invariant sets for distributed systems was recently proposed in [14]. However, while the dynamics in [14] are state-independent, the dynamics presented in this paper depend on the current state of the system and can therefore adapt to specific scenarios.*

IV. DISTRIBUTED SYNTHESIS

For the remainder of this paper, consider local linear state feedback control laws $u_f^i(x_{\mathcal{N}_i}) = K_{\mathcal{N}_i} x_{\mathcal{N}_i}$ and quadratic local functions $l_i(x_{\mathcal{N}_i}, u_f^i) = x_{\mathcal{N}_i}^T Q_{\mathcal{N}_i} x_{\mathcal{N}_i} + u_f^{i,T} R_i u_f^i$, $V_f^i(x_i) = x_i^T P_i x_i$ and $\gamma_i(x_{\mathcal{N}_i}) = x_{\mathcal{N}_i}^T \Gamma_{\mathcal{N}_i} x_{\mathcal{N}_i}$, where $Q_{\mathcal{N}_i}$, R_i and P_i are positive definite matrices. The first part of this section is concerned with distributed synthesis of P_i, $K_{\mathcal{N}_i}$ and $\Gamma_{\mathcal{N}_i}$, such that Theorem III.1 is satisfied. The second part is then concerned with distributed computation of local terminal sets $\mathcal{X}_f^i(\alpha_i)$ under polytopic state and input constraints.

Remark IV.1 *Note that the matrices $\Gamma_{\mathcal{N}_i}$ are not required to be negative semi-definite. Quite the opposite, positive- or indefiniteness allows a local cost increase, given that the global cost still decreases. This reduces conservatism in the choice of P_i and $K_{\mathcal{N}_i}$.*

Since both local and global considerations need to be taken into account in the computations, some additional notation is introduced for convenience. Let for every subsystem $U_i \in \{0,1\}^{m_i \times m}$ and $W_i \in \{0,1\}^{m_{\mathcal{N}_i} \times m}$ be matrices whose rows are linearly independent unit vectors, such that for all $x \in \mathcal{X}$

$$x_i = U_i x, \ \ x_{\mathcal{N}_i} = W_i x \ . \quad (22)$$

A. Decentralized Synthesis of Terminal Cost

By defining $A_{\mathcal{N}_i} = U_i A W_i^T$, the dynamics of subsystem i can be written as $x_i^+ = (A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i}) x_{\mathcal{N}_i}$, and conditions (12) and (13) from Theorem III.1 can be written as the set of nonlinear matrix inequalities

$$\begin{aligned} &(A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i})^T P_i (A_{\mathcal{N}_i} + B_i K_{\mathcal{N}_i}) - \bar{P}_i \\ &\quad \leq -(Q_{\mathcal{N}_i} + K_{\mathcal{N}_i}^T R_i K_{\mathcal{N}_i}) + \Gamma_{\mathcal{N}_i}, \ \ \forall i \in \{1, \ldots, M\} \quad (23) \\ &\sum_{i=1}^M W_i^T \Gamma_{\mathcal{N}_i} W_i \leq 0 \ , \quad (24) \end{aligned}$$

where $\bar{P}_i = W_i U_i^T P_i U_i W_i^T$. The following discussion will demonstrate that conditions (23) and (24) can equivalently be written as a set of LMIs. This is beneficial since it poses a distributed convex feasibility problem, which can be solved by efficient numerical tools. Consider now for every subsystem i the substitution $E_i := P_i^{-1}$. Consequently, $E = diag(E_1, \ldots, E_M) = P^{-1}$ and $\bar{E}_i = W_i U_i^T P_i^{-1} U_i W_i^T$. We additionally introduce $E_{\mathcal{N}_i} = W_i E W_i^T$, which leads to the substitions $H_{\mathcal{N}_i} := E_{\mathcal{N}_i} \Gamma_{\mathcal{N}_i} E_{\mathcal{N}_i}$ and $Y_{\mathcal{N}_i} := K_{\mathcal{N}_i} E_{\mathcal{N}_i}$. Furthermore, consider the following Lemma.

Lemma IV.2 (Proposition 8.1.2 in [16]) *Given the symmetric matrices $S_1 \in \mathbb{R}^{n \times n}$, $S_2 \in \mathbb{R}^{n \times n}$ and the matrix $T \in \mathbb{R}^{q \times n}$. If rank$(T) = n$, then $S_1 \leq S_2$ if and only if $T S_1 T^T \leq T S_2 T^T$.*

Theorem IV.3 *Condition (23) is equivalent to the LMIs*

$$\begin{bmatrix} \bar{E}_i + H_{\mathcal{N}_i} & E_{\mathcal{N}_i} A_{\mathcal{N}_i}^T + Y_{\mathcal{N}_i}^T B_i^T & E_{\mathcal{N}_i} Q_{\mathcal{N}_i}^{1/2} & Y_{\mathcal{N}_i}^T R_i^{1/2} \\ A_{\mathcal{N}_i} E_{\mathcal{N}_i} + B_i Y_{\mathcal{N}_i} & E_i & 0 & 0 \\ Q_{\mathcal{N}_i}^{1/2} E_{\mathcal{N}_i} & 0 & I & 0 \\ R_i^{1/2} Y_{\mathcal{N}_i} & 0 & 0 & I \end{bmatrix} \geq 0, \ \forall i \in \{1, \ldots, M\} \quad (25)$$

and condition (24) is equivalent to the LMI

$$\sum_{i=1}^M W_i^T H_{\mathcal{N}_i} W_i \leq 0 \ , \quad (26)$$

Proof: First, we prove that (25) is equivalent to (23). Multiplying inequality (23) by $E_{\mathcal{N}_i}$ from both sides yields the nonlinear matrix inequality

$$\begin{aligned} &\bar{E}_i + E_{\mathcal{N}_i} \Gamma_{\mathcal{N}_i} E_{\mathcal{N}_i} - (E_{\mathcal{N}_i} Q_{\mathcal{N}_i} E_{\mathcal{N}_i} + Y_{\mathcal{N}_i}^T R_i Y_{\mathcal{N}_i}) \\ &- (E_{\mathcal{N}_i} A_{\mathcal{N}_i}^T + Y_{\mathcal{N}_i}^T B_i^T) P_i (A_{\mathcal{N}_i} E_{\mathcal{N}_i} + B_i Y_{\mathcal{N}_i}) \geq 0 \ . \quad (27) \end{aligned}$$

Applying the Schur complement, we obtain

$$\begin{aligned} &\begin{bmatrix} \bar{E}_i + E_{\mathcal{N}_i} \Gamma_{\mathcal{N}_i} E_{\mathcal{N}_i} & (E_{\mathcal{N}_i} A_{\mathcal{N}_i}^T + Y_{\mathcal{N}_i}^T B_i^T) \\ (A_{\mathcal{N}_i} E_{\mathcal{N}_i} + B_i Y_{\mathcal{N}_i}) & E_i \end{bmatrix} \\ &- \begin{bmatrix} E_{\mathcal{N}_i} Q_{\mathcal{N}_i}^{1/2} & Y_{\mathcal{N}_i}^T R_i^{1/2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{\mathcal{N}_i}^{1/2} E_{\mathcal{N}_i} & 0 \\ R_i^{1/2} Y_{\mathcal{N}_i} & 0 \end{bmatrix} \geq 0 \ . \quad (28) \end{aligned}$$

Applying the Schur complement a second time we obtain (25). To prove that (26) is equivalent to (24), we proceed as

$$\sum_{i=1}^M W_i^T \Gamma_{\mathcal{N}_i} W_i \leq 0 \quad (29)$$

$$\Leftrightarrow E \left(\sum_{i=1}^M W_i^T \Gamma_{\mathcal{N}_i} W_i \right) E = \sum_{i=1}^M E W_i^T \Gamma_{\mathcal{N}_i} W_i E \leq 0 \quad (30)$$

$$\Leftrightarrow \sum_{i=1}^M W_i^T E_{\mathcal{N}_i} \Gamma_{\mathcal{N}_i} E_{\mathcal{N}_i} W_i = \sum_{i=1}^M W_i^T H_{\mathcal{N}_i} W_i \leq 0 \ , \quad (31)$$

where (30) follows from Lemma IV.2 and (31) follows from the fact that E is block-diagonal. ∎

Remark IV.4 *The problem of finding $E_{\mathcal{N}_i}$, $Y_{\mathcal{N}_i}$ and $H_{\mathcal{N}_i}$ that satisfy (25) and (26) can be posed as a distributed LMI. (25) directly decomposes into one LMI per subsystem i, which is coupled to its neighbors by $E_{\mathcal{N}_i}$. For decomposition of (26), the structure in the sum on the left hand side can be exploited. In particular, the nonzero block of a matrix $W_i^T H_{\mathcal{N}_i} W_i$ overlaps with the nonzero block of a matrix $W_j^T H_{\mathcal{N}_j} W_j$ only if either $j \in \mathcal{N}_i$ or $i \in \mathcal{N}_j$.*

B. Decentralized Synthesis of Terminal Set

The problem of finding the largest feasible level set of a given convex quadratic function under polytopic state and input constraints is represented by the linear program (LP)

$$\alpha_{max} = \operatorname{argmin}_{\alpha} - \alpha \tag{32a}$$
$$\text{s.t. } ||P_i^{1/2}(G_x^{i,j})^T||_2^2\alpha \le (f_x^{i,j})^2 \tag{32b}$$
$$\forall j \in \{1,\dots,l_x^i\},\ \forall i \in \{1,\dots,M\}$$
$$||P_{\mathcal{N}_i}^{1/2}K_{\mathcal{N}_i}^T(G_u^{i,j})^T||_2^2\alpha \le (f_u^{i,j})^2 \tag{32c}$$
$$\forall j \in \{1,\dots,l_u^i\},\ \forall i \in \{1,\dots,M\}\ ,$$

where $P_{\mathcal{N}_i} = E_{\mathcal{N}_i}^{-1}$ and l_x^i, l_u^i are the numbers of halfspaces defining the constraint polytopes $\mathcal{X}_i$ and $\mathcal{U}_i$. This problem obviously decomposes into one subproblem per subsystem, whereas the subproblem of each subsystem is coupled to the subproblems of all its neighbors by the variable α. Thus, problem (32) can be solved by distributed optimization and the size α_{max} of the largest feasible level set of $V_f(x)$ can be obtained at every subsystem. Since initial feasible level sets $\mathcal{X}_f^i(\alpha_i)$ need to satisfy $\sum_{i=1}^M \alpha_i \le \alpha_{max}$, they could be obtained simply by dividing α_{max} by M. This requires local knowledge of M, which is a rather mild assumption.

Remark IV.5 *An initial terminal set for every subsystem i can also be found by solving the distributed MPC problem for a given initial condition, treating the local α_i as optimization variables: Consider a spanning tree over the network of subsystems. Each subproblem in the tree, additionally to its local α_i, optimizes over a running sum variable, which equals its own share α_i plus the sum of all α_j down the tree. The root subproblem can then enforce $\sum_{i=1}^M \alpha_i \le \alpha_{max}$.*

A summary of the offline distributed synthesis process leading to a stabilizing distributed MPC is given in Algorithm 1, online distributed MPC is described in Algorithm 2.

Algorithm 1 Offline distributed MPC synthesis

1: Solve system of LMIs (25) and (26) by distributed optimization to find local terminal costs.
2: Solve LP (32) by distributed optimization to locally find the size α_{max} of the largest feasible level set of the global terminal cost.
3: Find initial sizes α_i for the local terminal sets such that $\sum_{i=1}^M \alpha_i \le \alpha_{max}$ (for instance by the approach layed out in Remark IV.5).

Algorithm 2 Online Distributed MPC, executed at every subsystem

1: Measure local state x_i.
2: Solve local MPC problem by distributed optimization.
3: Apply the input obtained in step 2.
4: Update α_i: $\alpha_i^+ = \alpha_i + x_{\mathcal{N}_i}^T(N)\Gamma_i x_{\mathcal{N}_i}(N)$
5: Go to step 1

V. NUMERICAL EXAMPLE

The unstable system presented in this section consists of a one-dimensional array of inverted pendulums with masses of 1kg and rod lengths of 1m. Each pendulum represents a subsystem which is coupled to its adjacent pendulums by a spring of 3N/m and a damper of 3Ns/m. Furthermore, every pendulum can apply a torque input $u_i = T_i$ in its pivot. The state of each pendulum consists of its angle and its angular velocity, hence $x_i = [\phi_i, \dot{\phi}_i]^T$, the discrete-time model is obtained by the Euler method with sample time 0.1s. Polytopic local state and input constraints of the form $||x_i||_\infty \le 10$ and $||u_i||_\infty \le 10$ are imposed and the local stage cost matrices $Q_i = I$ and $R_i = I$.

Both a terminal cost (LMI) and a terminal set (LP) were computed by distributed optimization. In particular, the alternating direction method of multipliers [15] was used as a solver method. The number of iterations required for satisfactory convergence is illustrated in Fig. 1 for different array lengths. As the length is increased, the required number of iterations stagnates for the LMI, while it keeps increasing for the LP. This is due to the fact that the variable α couples all subproblems in the LP (32), while in the LMI all coupling is local.

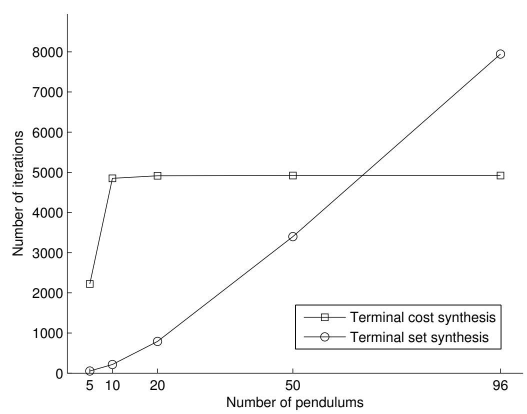

Fig. 1: Number of iterations required to synthesize local terminal costs and sets by distributed optimization.

For an array of 5 pendulums, an initial condition of $x = [1.3, 0, \dots, 0]^T$ and a prediction horizon of 10 steps, the performance of three controllers was compared: (i) centralized MPC with dense LQR terminal cost and set, (ii) the distributed MPC proposed in this paper and (iii) distributed MPC with terminal set $\mathcal{X}_f = \{0\}$. The simulated closed-loop trajectories for the input torques and angles of the first two pendulums are illustrated in Fig. 2. It is remarkable that the accumulated cost of the proposed distributed MPC controller (ii) is only 3.8% higher than the one of centralized MPC (i), while the accumulated cost of controller (iii) is 49% higher. For controller (ii) and a prediction horizon of 40 steps, the evolution of the local terminal sets is illustrated in Fig. 3. Note that as time evolves, the sizes of all terminal sets are shrinking except for the one of pendulum 3, which is continuously expanding.

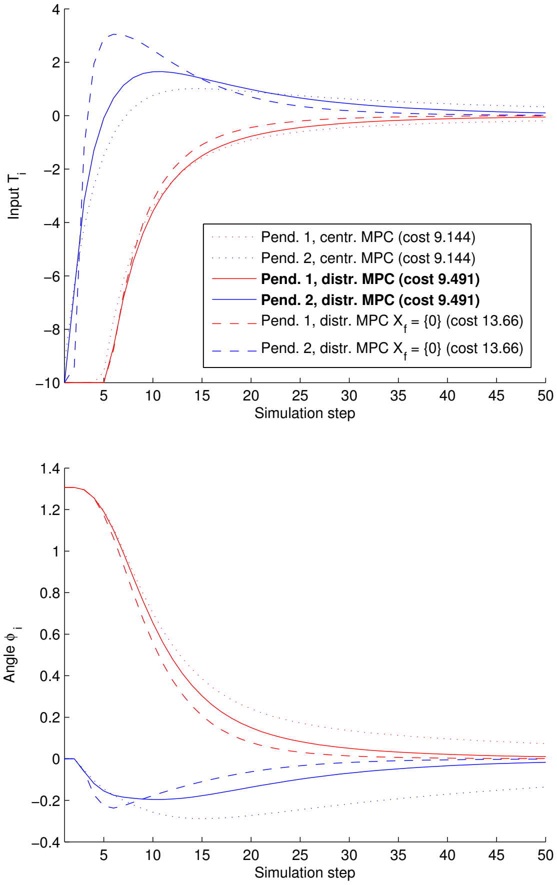

Fig. 2: Torque and angle trajectories for pendulum 1 (red) and 2 (blue) for different controllers.

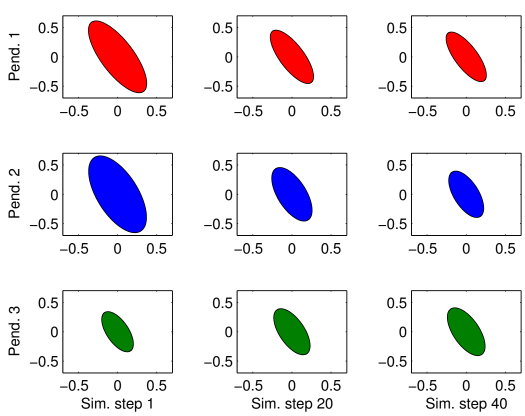

Fig. 3: Evolution of the shapes of the local terminal sets under update rule (16) and a prediction horizon of 40 steps.

For the 5 pendulum array, the regions of attraction under the three MPC configurations were investigated by deflecting ϕ_1 to the border of infeasibility, while keeping all other states at the origin. As illustrated in Fig. 4, for short prediction horizons, the proposed MPC approach results in a region of attraction which is similar to the one for centralized MPC and significantly larger than the one for MPC with $\mathcal{X}_f = \{0\}$. For longer prediction horizons, the regions of attraction converge, as expected, to the same maximum stabilizable set.

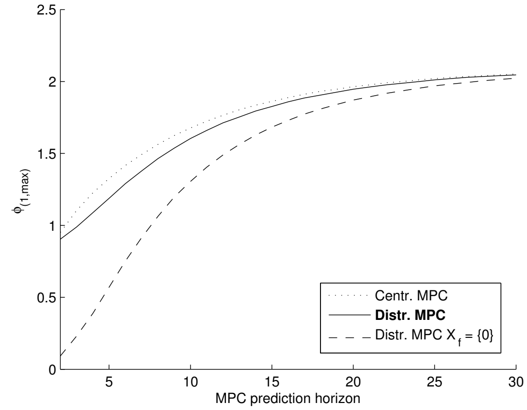

Fig. 4: Comparison of the region of attraction for different MPC configurations.

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