An algorithmic decomposition of claw-free graphs leading to an $O(n^3)$-algorithm for the weighted stable set problem

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Abstract

We propose an algorithm for solving the maximum weighted stable set problem on claw-free graphs that runs in $O(n^3)-$time, drastically improving the previous best known complexity bound. This algorithm is based on a novel decomposition theorem for claw-free graphs, which is also introduced in the present paper. Despite being weaker than the well-known structure result for claw-free graphs given by Chudnovsky and Seymour [5], our decomposition theorem is, on the other hand, algorithmic, i.e. it is coupled with an $O(n^3)-$time procedure that actually produces the decomposition. We also believe that our algorithmic decomposition result is interesting on its own and might be also useful to solve other kind of problems on claw-free graphs.

1 Introduction

Given a graph $G(V, E)$, a *matching* is a set of non incident edges of E and a *stable set* is a set of pairwise non adjacent vertices of V. Edmonds [6] proved that the weighted matching problem can be solved in polytime ($O(|V|^4)$ for any graph. This worst case complexity was later improved by other authors, the best bound currently being $O(|V|(|V|log|V| + |E|)$ [9].

Given a (multi-)graph G, one can define the *line-graph* H of G as the intersection graph of the edges of G. G is called a *root-graph* of H. A graph is then said to be *line* if it is the line-graph of some graph G. There is a one-to-one correspondence between matchings in G and stable sets in H. Therefore, since G can be computed efficiently (this can be done in $O(max\{|E|, |V|\})$-time for line-graphs of simple graphs [18] and line graphs of multi-graphs [10, pp 67-68]), the stable set problem in H is equivalent to a matching problem and can thus be solved in time $O(|V|^2log(|V|))$ (observe that the root graphs will have $|V|$ edges and $O(|V|)$ vertices). Line graphs have the property that the neighborhood of any vertex can be covered by two cliques, and the graphs with this latter property are called *quasi-line graphs*. A graph is *claw-free* if no vertex has a stable set of size three in its neighborhood. Claw-free graphs thus generalize in turn quasi-line graphs. Interestingly, while the stable set problem is $\mathcal{NP}$-hard in general, it was proven it can be solved in polynomial time for claw-free graphs: Sbihi [19] and later Lovász and Plummer [12] gave algorithms for the cardinality case, while Minty [13] solved the weighted version. The Minty algorithm was revised by Nakamura and Tamura [14] and later simplified by Schrijver[20] and can be implemented to run in time $O(|V|^6)$ (Recently Nobili and Sassano [15] reported to us that they could build upon the main ideas of Minty's algorithm and solve the problem in time $O(|V|^4log(|V|)$, a significant improvement).

A deep decomposition theorem for claw-free graphs was recently introduced by Chudnovsky and Seymour [5]. Moreover, in a recent paper Oriolo, Pietropaoli and Stauffer [16] proposed a new approach to solve the maximum weighted stable set (MWSS) problem on graphs that admit a suitable decomposition. However, the previous two results cannot be combined together as to get an algorithm to solve the MWSS problem on claw-free graphs, since no polynomial time algorithm is known to get the decomposition in [5], and finding it seems to be quite a challenging open question [10]. Nevertheless, by means of some graph reductions, and an algorithmic decomposition theorem for a subclass of quasi-line graphs, Oriolo, Pietropaoli and Stauffer [16] developed a $O(|V|^6)$-time algorithm to solve the problem.

In this paper, we provide a new decomposition theorem for claw-free graphs and a $O(n^3)$ algorithm to actually obtain the decomposition. Namely, we prove that in $O(n^3)$-time we either recognize that a

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claw-free graph G is net-free and quasi-line, or that it has small stability number and a long odd anti-wheel, or return a suitable decomposition of G. Our theorem is inspired by ideas and tools developed by Chudnovsky and Seymour (as well as by the weaker decomposition theorem in [16]), but it is a stand-alone result that, even if less detailed than their theorem, is particularly useful when dealing with the MWSS problem. In fact, building upon a few algorithmic results from the literature, and following the approach in [16] for finding a MWSS on graphs that admit a suitable decomposition, we show that we can solve the MWSS problem in claw-free graphs in $O(|V|^3)$ time. This is to the best of our knowledge the fastest algorithm to solve the problem and this improves drastically upon previous known algorithms. Moreover it almost closes the algorithmic gap between stable set in claw-free graphs and matching (in fact, as observed earlier, the MWSS problem in line-graphs can be solved in $O(|V|^2 log(|V|))$ time).

The paper is organized as follows: in Section 2 we settle notations and give general definitions. In Section 3, we provide a number of preliminary definitions and related basic results. Section 4 and Section 5 are devoted to provide decomposition theorems and algorithms for quasi-line and claw-free graphs respectively. In Section 6 we show how to exploit these decomposition results to obtain an algorithm for the maximum weighted stable set problem on claw-free graphs.

2 Notations and definitions

For a non-negative integer k, we let $[k]$ denote the set $\{1, 2, \ldots, k\}$.

Let $G(V, E)$ be a simple graph. The complement of G is denoted by $\overline{G}$, while $G[S]$ denotes the subgraph induced by a set $S \subseteq V$. If $S \subseteq V$, we let $G \setminus S := G[V \setminus S]$. A *stable set* is a set of pairwise non-adjacent vertices, while a *clique* is a set of pairwise adjacent vertices. We denote by $\alpha(G)$ ($\alpha_w(G)$) the maximum size (resp. weighted with respect to $w : V \mapsto \mathbb{R}$) stable set in G, and, for a set $S \subseteq V$, we let $\alpha(S) = \alpha(G[S])$ (resp. $\alpha_w(S) = \alpha_w(G[S])$). We denote by $N(v)$ the *open neighborhood* of a vertex $v \in V$, i.e. the set of vertices that are adjacent to v; we let $N[v] = N(v) \cup \{v\}$ be the *closed neighborhood.* For a set $S \subseteq V$ we let $N(S) := \bigcup_{v \in S} N(v) \setminus S$ and $N[S] := \bigcup_{v \in S} N[v]$. A vertex u is *universal* to v if $N[v] \subseteq N[u]$ and we let $U(v)$ be the set of vertices that are universal to v.

We also denote by $N_j(v)$ the set of vertices that are at distance j (in terms of number of edges) from v (therefore $N_1(v) = N(v)$). A vertex $v \in V$ is *simplicial* if $N(v)$ is a clique, and we denote by $S(G)$ the set of simplicial vertices of G.

DEFINITION 2.1. *We say that a clique K of a connected graph G is* distance simplicial *if, for every j, $\alpha(N_j(K)) \leq 1$. In this case, we also say that G is distance simplicial with respect to K.*

A $k-$*hole* is a chordless cycle with k vertices, and it is *odd* if k is odd. A k-*anti-hole* is the complement of a $k-$hole, and it is *odd* if k is odd. A $k-$*wheel* is a graph with vertex set $\{v\} \cup C$, where C induces a k-hole and v is complete to C: v is the *center* of the wheel. Analogously, a $k-$*anti-wheel* is a graph with vertex set $\{v\} \cup C$, where C induces a k-anti-hole, and v is complete to C: v is the *center* of the anti-wheel. Note that a 5-wheel and a 5-anti-wheel define indeed isomorphic graphs.

We say that G is *claw-free* if none of its vertices has a stable set of size three in its neighborhood. We say that G is *quasi-line* if the neighborhood of each $v \in V$ can be partitioned into two cliques, that is, $\overline{G[N(v)]}$ is bipartite. Note that G is claw-free if it has no 3-anti-wheels, while it is quasi-line if it has no odd anti-wheels. We say that a k-anti-wheel is *long* if $k > 5$. The line graph $H = \mathcal{L}(G)$ of a multi-graph $G = (V, E)$ (i.e. loops, parallel edges are allowed) is defined as follows: we associate a vertex of H to every edge of G. Then two vertices are adjacent in H if and only if the corresponding edges were incident in G. We call G a *root graph* of H. Krausz [11] proved the following characterization of line graphs.

LEMMA 2.1. *[11] A graph $G(V, E)$ is line if and only if there exists a family of cliques $\mathcal{F}$ such that every edge in E is covered by a clique from the family, and moreover every vertex in V is covered by at most two cliques from the family.*

3 Preliminaries

In this section we present a rather long list of preliminary results and definitions.

3.1 Strips Chudnovsky and Seymour [4] introduced a composition operation in order to define their decomposition result for claw-free graphs. This composition procedure is general and applies to non-claw-free graphs as well. We borrow (but slightly change) some definitions from their work.

Definition 3.1. *A* strip $(G, \mathcal{A})$ *is a graph G (not necessarily connected) with a multi-family $\mathcal{A}$ of either one or two designated non-empty cliques of G.*

If $\mathcal{A}$ is made of a single clique, then $(G, \mathcal{A})$ is a 1-strip*, if $\mathcal{A}$ is made of two cliques A_1, A_2, then $(G, \mathcal{A})$ is a* 2-strip *(in this case, possibly $A_1 = A_2$ since $\mathcal{A}$ is a multi-family). The cliques in $\mathcal{A}$ are called the* extremities *of the strip, while the* core *of the strip is made of the vertices that do not belong to the extremities.*

Let $\mathcal{G} = (G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$ be a family of k vertex disjoint strips; we can *compose* the strips in $\mathcal{G}$, according to the operation we define below. Note that we denote by $\bigcup_{j\in[k]} \mathcal{A}^j$ the *multi-set* whose elements are the *extremities* from each $\mathcal{A}^j$: again, it is a multi-set, as the two extremities of a same strip need not to be different.

Definition 3.2. *Let $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$ be k vertex disjoint strips and let $\mathcal{P} := \{P_1, ..., P_m\}$ be a partition of the multi-set of the extremities $\bigcup_{j\in[k]} \mathcal{A}^j$. The* composition of $\{(G^j, \mathcal{A}^j), j \in [k]\}$ w.r.t. the partition $\mathcal{P}$ *is the graph G such that:*

- *$V(G) = \bigcup_{j=1}^{k} V(G^j)$;*
- *two vertices $u, v \in V(G)$ are adjacent if and only if either $u, v \in V(G^j)$ and $\{u, v\} \in E(G^j)$, for some $j \in [k]$, or there exist $A \in \mathcal{A}^i$ and $A' \in \mathcal{A}^j$, for some $1 \le i \le j \le k$, such that $u \in A$, $v \in A'$, and A and A' are in the same class of $\mathcal{P}$.*

In this case, we say that $(\mathcal{F}, \mathcal{P})$ with $\mathcal{F} = \{(G^j, \mathcal{A}^j), j \in [k]\}$, defines a strip decomposition *of G. Note also that, for each class $P \in \mathcal{P}$, the set of vertices $\bigcup_{A\in P} A$ is a clique of G, that is called a* partition-clique.

In the following, when we say that G is the composition of some set of strips with respect to some partition $\mathcal{P}$, we mean that the strips are vertex disjoint and that $\mathcal{P}$ gives a partition of the multi-set of the extremities of these strips. We skip the straightforward proof of the next lemma:

Lemma 3.1. *Let G be the composition of some strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$, with respect to some partition $\mathcal{P}$. Then the following statements hold:*

- *for each $j \in [k]$, the core $C(G^j)$ of the strip $(G^j, \mathcal{A}^j)$ is anti-complete to $V(G) \setminus V(G^j)$ and $G[C(G^j)] = G^j[C(G^j)]$;*
- *for each $j \in [k]$, $G[V(G^j)] = G^j[V(G^j)]$ if either G^j is a 1-strip, or it is a 2-strip and its extremities belong to different classes of $\mathcal{P}$; else $G[V(G^j)]$ is obtained from $G^j[V(G^j)]$ making its extremities complete to each other.*
- *each edge between different strips G^i and G^j is an edge between their extremities and is induced by some partition-clique.*

One can easily build a graph G that is the composition of strips $\{(G^j, \mathcal{A}^j), j \in [k]\}$ such that each G^j is claw-free/quasi-line/line but G itself is not claw-free/quasi-line/line. However, this is not possible, as soon as we require that, for each strip, the property we are interested in (claw-freeness/quasi-lineness/lineness) holds on an *auxiliary* graph that we associate to the strip. This leads to the following:

Definition 3.3. *We say that a strip $(G, \mathcal{A})$ is claw-free/quasi-line/line if the graph G_+ that is obtained from G as follows:*

- *if G is a 2-strip, with $\mathcal{A} = \{A_1, A_2\}$, add two additional vertices a_1, a_2 such that $N(a_i) = A_i$, for $i = 1, 2$;*
- *if G is a 1-strip, with $\mathcal{A} = \{A_1\}$, add one additional vertex a_1 such that $N(a_1) = A_1$,*

is claw-free/quasi-line/line.

We skip the proof of the following simple lemma.

Lemma 3.2. *The composition of claw-free/quasi-line strips is a claw-free/quasi-line graph.*

Lemma 3.3. *Let G be the composition of k line strips $\{(G^j, \mathcal{A}^j), j \in [k]\}$ with respect to a partition $\mathcal{P}$. Then G is a line graph.*

Proof. (*For the sake of completeness we give the proof, which is however similar to the proof of Claim 7 in [16].*) From Lemma 2.1, we know that the strips G^j are line if and only if, for all $j = 1, ..., k$, there exists a set of cliques $\mathcal{F}^j$ of G^j_+ such that: every edge from G^j_+ is covered by a clique of $\mathcal{F}^j$; each vertex in G^j_+ is covered by at most two cliques of $\mathcal{F}^j$. In fact, we may assume without loss of generality that the set $\mathcal{F}^j$ is also such that the vertices from $V(G^j_+) \setminus V(G^j)$ are covered by exactly one clique of $\mathcal{F}^j$ (if a vertex v of $V(G^j_+) \setminus V(G^j)$ is covered by two cliques F_1, F_2, then we can slightly change $\mathcal{F}^j$ into $\mathcal{F}^j \setminus (F_1 \cup F_2) \cup \{N[v]\}$). We denote by F^j the set of cliques of $\mathcal{F}^j$ covering the vertices from $V(G^j_+) \setminus V(G^j)$ (F^j is of cardinality one if $(G^j, \mathcal{A}^j)$ is a 1-strip and two if $(G^j, \mathcal{A}^j)$ is a 2-strip). Let $\widetilde{\mathcal{F}}^j := \mathcal{F}^j \setminus F^j$. Consider the family of cliques $\widetilde{\mathcal{F}}$ of G made of the union of $\widetilde{\mathcal{F}}^j$ for all j and the partition-cliques defined by $\mathcal{P}$. By definition of composition, $\widetilde{\mathcal{F}}$ covers all edges in G and moreover every vertex in G is covered by at most two cliques. The result follows then again from Lemma 2.1.

(We point out that one might impose properties on the strips $\{(G^j, \mathcal{A}^j), j \in [k]\}$ in order to avoid using the artifact of additional vertices, and still get an analogous of Lemma 3.2 and Lemma 3.3, see [5]: for our purpose, this unnecessarily complicates the exposition.)

In the next section we show an algorithm to decompose quasi-line graphs. *Articulation cliques* are the main tool for getting this decomposition; however it is convenient to introduce them in the more general framework of claw-free graphs as we will also make use of this concept in claw-free graphs.

3.2 Articulation cliques Let $G(V, E)$ be a claw-free graph. A vertex $v \in V$ such that $N[v]$ can be covered by two maximal cliques K_1 and K_2 (not necessarily different) is called *regular*, and it is called *strongly regular* when this covering is unique: in this case, we also say that K_1 and K_2 are *crucial* for v. A vertex such that $N[v]$ cannot be covered by two maximal cliques K_1 and K_2 is called *irregular*.

Note that each irregular vertex of G is the center of an odd k-anti-wheel with $k \geq 5$, and that G is quasi-line if every vertex is regular. Moreover, if G is a line graph, then every vertex is strongly regular (however, this condition is not sufficient to define line graphs).

Definition 3.4. *A maximal clique K of a claw-free graph G is an* articulation clique *if, for each $v \in K$, K is crucial for v.*

The following lemmas show two first families of articulation cliques. We omit the easy proof of the first one.

Lemma 3.4. *Let $G(V, E)$ be a claw-free graph. Let v be a simplicial vertex of G (i.e. $v \in S(G)$). Then $N[v]$ is an articulation clique.*

Lemma 3.5. *Let G be a claw-free graph and let K be a maximal clique of G and $v \in K$. If $N(v) \setminus K \neq \emptyset$ and*

(1) either $N(v) \setminus K$ is anti-complete to some vertex in K,

(2) or v is regular, $K \setminus U[v]$ can be partitioned into two non empty cliques X_1 and X_2 and $N(v) \setminus K$ can be partitioned into two non empty cliques Y_1 and Y_2, such that X_1 is anti-complete to Y_2, X_2 is anticomplete to Y_1, and there is a missing edge between X_1 and Y_1,

then K is crucial for v. In particular, if K has no simplicial vertex and for each $v \in K$ condition (1) or (2) is satisfied, then K is an articulation clique.

Proof. For each $v \in K$, we show that if (1) or (2) holds, then v is strongly regular and K is crucial for v. Let us first assume that v satisfies condition (1), i.e. there exists a vertex $w \in K$ such that $N(v) \setminus K$ is anticomplete to w. This implies that $N(v) \setminus K$ is a clique, otherwise any stable set of size two in $N(v) \setminus K$, say $\{t, z\}$, would cause the claw $(v; w, t, z)$. Thus, v is regular. Then, in each covering of $N[v]$ with two maximal cliques K_1, K_2, we can assume w.l.o.g that $w \in K_1$ and $N(v) \setminus K \subseteq K_2$. We have $U[v] \subseteq K$ since $\forall u \in N(v) \setminus K$, $uw \notin E$. Moreover $U[v] = K_1 \cap K_2$. Now since $\forall u \in K \setminus U[v]$, there exists $z_u \in N(v) \setminus K : uz_u \notin E$, it follows that $K \setminus U[v] \subseteq K_1$. Therefore $K \subseteq K_1$ and by maximality $K = K_1$.

Now suppose $v \in K$ satisfies condition (2); in particular v is regular. Let K_1, K_2 be again a covering of $N[v]$ with two maximal cliques. Then we can assume w.l.o.g. that $X_1 \subseteq K_1$ and $Y_2 \subseteq K_2$. By hypothesis, a vertex $y \in Y_1$ is non-complete to X_1: then, $y \in K_2$, which implies $X_2 \subseteq K_1$. Last, as Y_1 is anticomplete to X_2, $Y_1 \subseteq K_2$. Summing up, we showed that $X_1 \cup X_2 \subseteq K_1$ $Y_1 \cup Y_2 \subseteq K_2$. As $K = U[v] \cup X_1 \cup X_2$ and $U[v] = K_1 \cap K_2$, it follows that $K \subseteq K_1$ and by maximality, $K = K_1$. We conclude that v is strongly regular and K is crucial for v.

The next lemma shows that we may easily characterize the vertices of a claw-free graph, as well as find the set of all articulation cliques.

LEMMA 3.6. *Let $G(V, E)$ be a claw-free graph. (i) For each vertex $\in V$, we may check in time $O(|V|^2)$ if v is either regular, or strongly regular (and find its crucial cliques), or irregular (and find an odd k-anti-wheel centered in v, $k \geq 5$). (ii) We can list all articulation cliques of G in time $O(|V|^3)$.*

Proof. (*For the sake of completeness we give the proof, which is however similar to the proof of Lemma 20 in [16].*) Let $n = |V|$. (i) For each $v \in V$, consider the graph $H = \overline{G[N(v) \setminus U(v)]}$. Then v is regular if H is bipartite and is irregular otherwise. If H is bipartite, then v is strongly regular if and only H is connected: in this case, $S_1 \cup U(v) \cup \{v\}$ and $S_2 \cup U(v) \cup \{v\}$ are the crucial cliques for v, where S_1 and S_2 are the classes of the unique coloring of H. If H is not bipartite, then the vertices of any chordless cycle of H together with v induce an odd k-anti-wheel on G, with $k \geq 5$, since H has no triangles. The statement trivially follows.

(ii) It follows from (i) that we may build in time $O(n^3)$ the set R of strongly regular vertices and, for each vertex $v \in R$, its crucial cliques. Altogether, the family $\mathcal{K}$, that is made of cliques that are crucial for some strongly regular vertex, has size at most $2n$, as no vertex either has two crucial cliques, or it has none. Now, an articulation clique of G is a clique K such that: $K \in \mathcal{K}$; every $v \in K$ is strongly regular and K is crucial for v. The statement follows.

We will show later that every claw-free graph G, that has some articulation clique, admits a strip decomposition where each partition-clique is indeed an articulation clique of G. In order to produce this decomposition, we need to“reverse” the composition operation defined earlier. We start with G quasi-line.

4 An algorithm to decompose quasi-line graphs

4.1 Ungluing articulation cliques in quasi-line graphs An interesting class of articulation cliques in quasi-line graphs are those generated by nets. A *net* $\{v_1, v_2, v_3; s_1, s_2, s_3\}$ is the graph with vertices $v_1, v_2, v_3, s_1, s_2, s_3$ and edges v_1v_2, v_1v_3, v_2v_3, and v_is_i for $i = 1, 2, 3$. We say that G is *net-free* if no induced subgraph of G is isomorphic to a net and call *net clique* every maximal clique of G that contains v_1, v_2, v_3.

LEMMA 4.1. *In a quasi-line graph every net clique is an articulation clique.*

Proof. (*For the sake of completeness we give the proof, which is however similar to the proof of Lemma 22 in [16].*) Let K be a net clique: we must show that K is crucial for every vertex $v \in K$. For $i = 1, 2, 3$, let K_i be the set of vertices from K that are adjacent to s_i, and $K_4 := K \setminus (K_1 \cup K_2 \cup K_3)$. Note that $\{K_1, K_2, K_3, K_4\}$ is a partition of K, since a vertex $v \in K$ that is adjacent to two vertices from s_1, s_2, s_3, say s_1 and s_2, implies the claw $(v; s_1, s_2, v_3)$.

First, suppose $v \in K_1$. Let (Q_1, Q_2) be a pair of maximal cliques such that $N[v] = Q_1 \cup Q_2$ (such a pair exists, since the graph is quasi-line). Assume w.l.o.g. that $s_1 \in Q_1$, it follows that $K \setminus K_1 \subseteq Q_2$. We now show that every vertex $z \in N(v) \setminus K$ is not complete to $K \setminus K_1$. Suppose the contrary, i.e. there exists $z \in N(v) \setminus K$ that is complete to $K \setminus K_1$. Since K is maximal, there exists $w \in K_1$, $w \neq v$, such that $wz \notin E$. Since z is adjacent to v, it cannot be adjacent to both s_2 and s_3 (otherwise there would be the claw $(z; s_2, s_3, v)$). Assume w.l.o.g. z is not linked to s_3. Let z_3 be a vertex in K_3. Then by construction $z_3z \in E$, $ws_3 \notin E$, and $(z_3; s_3, w, z)$ is a claw, a contradiction. Therefore, every vertex in $z \in N(v) \setminus K$ is not complete to $K \setminus K_1$ and so it must belong to Q_1. It follows that $Q_1 = (N(v) \setminus K) \cup \{v\} \cup U(v)$ and $Q_2 = K$, that is, (Q_1, K) is the unique covering of $N(v)$ into two maximal cliques, thus K is crucial for v. The same holds for any vertex v in K_2 or K_3.

Now suppose that $v \in K_4$. If v is a simplicial vertex, then the statement is trivial. Now suppose that there exists $w \notin K$ such that $wv \in E$. Observe that w is adjacent to at most one vertex of $\{s_1, s_2, s_3\}$: if the contrary, assume w.l.o.g. $s_1, s_2 \in N(w)$, there would be the claw $(w; v, s_1, s_2)$. Hence there exists a stable set of size three in $\{w, s_1, s_2, s_3\}$ containing w and we are back to the previous case.

Let K be an articulation clique of a quasi-line graph G. The ungluing of K requires a partition of the vertices of K into suitable classes. These classes are the equivalence classes defined by an equivalence relation $\mathcal{R}$ on the vertices of K. Call *bound* a vertex of K that belongs to two articulation cliques of G (note that no vertex belongs to *more* than two articulation cliques). Then, for $u, v \in K$, $u\mathcal{R}v$ if and only if:

(i) either $u = v$;

(ii) or both u and v are bound and they belong to the same articulation cliques;

(iii) or u and v are neither simplicial nor bound and $(N(v) \setminus K) \cup (N(u) \setminus K)$ is a clique.

We claim that $\mathcal{R}$ define an equivalence relation on the vertices of K. In fact, while symmetry and reflexivity of $\mathcal{R}$ are by definition, transitivity follows either from definition or from the next lemma.

LEMMA 4.2. *Let $G(V, E)$ be a quasi-line graph, K an articulation clique with three distinct non-simplicial vertices $u, v, z \in K$. If $(N(u) \setminus K) \cup (N(z) \setminus K)$ and $(N(v) \setminus K) \cup (N(z) \setminus K)$ are cliques, then also $(N(u) \setminus K) \cup (N(v) \setminus K)$ is a clique.*

Proof. In the following, for $y \in K$, we let $\widetilde{N}(y) = N(y) \setminus K$. Since u, v, z are non simplicial, it follows that $\widetilde{N}(u)$, $\widetilde{N}(v)$ and $\widetilde{N}(z)$ are non-empty. Now suppose the statement is false; therefore there exist w_1 and $w_2 \in \widetilde{N}(u) \cup \widetilde{N}(v)$ that are non-adjacent. Since $\widetilde{N}(u)$ and $\widetilde{N}(v)$ are cliques, it follows that w.l.o.g. $w_1 \in \widetilde{N}(u) \setminus \widetilde{N}(v)$ and $w_2 \in \widetilde{N}(v) \setminus \widetilde{N}(u)$. Furthermore, note that $w_1, w_2 \notin \widetilde{N}(z)$, since this would contradict the hypothesis. Then pick any vertex t from $\widetilde{N}(z)$: tz, tw_1, $tw_2 \in E$, and w_1z, $w_2z \notin E$ hold; thus, $(t; z, w_1, w_2)$ is a claw, a contradiction.

The above discussion leads to the following:

DEFINITION 4.1. *Let G be quasi-line and K an articulation clique. We denote by $\mathcal{Q}(K)$ the family of the equivalence classes defined by $\mathcal{R}$ and call each class of $\mathcal{Q}(K)$ a* spike *of K.*

We are now ready to introduce the operation of ungluing for the articulation cliques of a quasi-line graph.

DEFINITION 4.2. *Let G be a quasi-line graph and $\mathcal{K}(G)$ the family of all articulation cliques of G. The* ungluing *of the cliques in $\mathcal{K}(G)$ consists of removing, for each clique $K \in \mathcal{K}(G)$, the edges between different spikes of $\mathcal{Q}(K)$. We denote the resulting graph by $G_{|\mathcal{K}(G)}$.*

In the following, we denote by $\mathcal{K}(G)$ the family of all articulation cliques of G and let $\mathcal{Q}(\mathcal{K}(G)) = \bigcup_{K \in \mathcal{K}(G)} \mathcal{Q}(K)$, i.e. the family of all spikes of cliques in $\mathcal{K}(G)$.

LEMMA 4.3. *Let $G(V, E)$ be a quasi-line graph. We can build the graph $G_{|\mathcal{K}(G)}$ and the family $\mathcal{Q}(\mathcal{K}(G))$ in time $O(|V|^3)$.*

Proof. As we proved in Lemma 3.6, we can list all cliques in $\mathcal{K}(G)$ in time $|V|^3$ and $|\mathcal{K}(G)| \leq 2n$. We are going to show that given an articulation clique $K \in \mathcal{K}(G)$, $\mathcal{Q}(K)$ can be computed in time $O(|K|n^2)$. As a result, and since each vertex of G is in at most two articulation cliques, the time required to compute $\mathcal{Q}(\mathcal{K}(G))$ is then $\sum_{K \in \mathcal{K}(G)} O(|K|n^2) = O(n^3)$. The algorithm consists of a preprocessing phase plus three steps. In the preprocessing we compute and remove from K all the simplicial and bound vertices, while at the same time recording them in the corresponding spikes of $\mathcal{Q}(K)$; note that this does not affect the remaining spikes from $\mathcal{Q}(K)$. Suppose then that K is an articulation clique of G without bound and simplicial vertices: this implies that two vertices $u, v \in K$ are in the same spike if and only if $\widetilde{N}(u) \cup \widetilde{N}(v)$ is a clique, where we used again the notation $\widetilde{N}(u) := N(u) \setminus K$. In the first step, we construct a bipartite graph $G'(K \cup N(K), E')$ such that $u \in K$ and $v \in N(K)$ are adjacent if and only if either $v \in \widetilde{N}(u)$ or v is complete to $\widetilde{N}(u)$. In the second step, we construct, building on G', a graph $G''(K, E'')$ where $u, v \in K$ are adjacent if and only if $\widetilde{N}(u) \cup \widetilde{N}(v)$ is a clique. By construction, two vertices from K belong to the same spike of $\mathcal{Q}(K)$ if and only if they are in the same component of G''. Thus, in the last phase we compute the components of G'', and we are done.

We start with the preprocessing phase. For each $u \in K$, we can build $\widetilde{N}(u)$, and check whether u is simplicial or it belongs to a second articulation clique, in time $O(n)$ (thanks to the list $\mathcal{K}(G)$), inserting those at the same time in the corresponding spikes. We then discard those vertices from K and move to the first step of the algorithm. Since the set $\widetilde{N}(u)$ is available for each $u \in K$, the graph G' can be now built in time $O(|K|n^2)$. Now, in order to build the graph G'' that is defined in the second step, it is enough to consider each pair of vertices $u, v \in K$ and simply check whether $ut \in E'$ for each $t \in \widetilde{N}(v)$. Hence, the graph G'' can be built in time $O(|K|^2 n)$. Finally, in the third step, we can compute the components of G'' in time $O(n^2)$.

Thus, $\mathcal{Q}(\mathcal{K}(G))$ can be computed in $O(n^3)-$time. Last, observe that $O(n^2)-$time suffices in order to build the graph $G_{|\mathcal{K}(G)}$ from G.

Lemma 4.4. *Let G be a connected quasi-line graph such that $\mathcal{K}(G)$ is non-empty. Let $\mathcal{C}$ be the connected components of $G|_{\mathcal{K}(G)}$. Then:*

(i) *$G|_{\mathcal{K}(G)}$ is quasi-line.*

(ii) *If $Q \in \mathcal{Q}(\mathcal{K}(G))$, then $Q \subseteq V(C)$ for some $C \in \mathcal{C}$, and C is distance simplicial w.r.t. Q.*

(iii) *For each $C \in \mathcal{C}$, there are either one or two spikes from $\mathcal{Q}(\mathcal{K}(G))$ that belong to C. Therefore, if we let $\mathcal{A}(C)$ be the family of these spikes (possibly a multi-set), then $(C, \mathcal{A}(C))$ is a strip.*

(iv) *G is the composition of the strips $\{(C, \mathcal{A}(C)) : C \in \mathcal{C}\}$ with respect to the partition $\mathcal{P}$ that puts two extremities in the same class if and only if they are spikes from a same articulation clique.*

The long proof of this lemma is postponed to the appendix. Meanwhile, we observe the following:

Observation 1. *Observe that, by definition, every simplicial vertex defines a spike of $N[v]$ on its own, i.e. $\{v\}$ is a spike of $N[v]$ (recall that $N[v]$ is an articulation clique, by Lemma 3.4). Therefore, every simplicial vertex v in $S(G)$ will appear in a 1-strip on its own i.e. there will be a strip $(\{v\}, \{\{v\}\})$ in the strip decomposition of G provided by Lemma 4.4.*

4.2 Quasi-line graphs without articulation cliques We are now ready to give our main decomposition result on quasi-line graphs.

Theorem 4.1. *Let $G(V, E)$ be a connected quasi-line graph with n vertices. In time $O(|V|^3)$ Algorithm 1:*

(i) *either recognizes that G is net-free;*

(ii) *or provides a decomposition into $k \leq n$ strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$, such that each graph G^j is distance simplicial with respect to each clique $A \in \mathcal{A}^j$. Moreover, the partition-cliques are the articulation cliques of G.*

Algorithm 1

Require: A connected quasi-line graph G.

Ensure: The algorithm either recognizes that G is net-free, or returns a strip decomposition of G as to satisfy (ii).

1: Find the family $\mathcal{K}(G)$ of all articulation cliques of G. If $\mathcal{K}$ is empty, then G has no net cliques and then it is net-free, **stop**.
2: Unglue the articulation cliques in $\mathcal{K}(G)$ as to build the graph $G_{|\mathcal{K}(G)}$.
3: Let $\mathcal{C}$ be the components of $G_{|\mathcal{K}(G)}$. For each component $C \in \mathcal{C}$, let $\mathcal{A}(C)$ be the (multi-)set of spikes in C.
4: **Return** the family of strips $\{(C, \mathcal{A}(C)) : C \in \mathcal{C}\}$ and the partition $\mathcal{P}$ of $\bigcup_{C \in \mathcal{C}} \mathcal{A}(C)$ that puts two extremities in the same class if and only if they are spikes from a same articulation clique.

Proof of Theorem 4.1. We first show that the algorithm is correct. If $\mathcal{K}(G) = \emptyset$, it follows from Lemma 4.1 that G is net-free. Otherwise, by part (iv) of Lemma 4.4, we know that G is the composition of strips $\{(C, \mathcal{A}(C)) : C \in \mathcal{C}\}$ w.r.t. the partition of $\bigcup_{C \in \mathcal{C}} \mathcal{A}(C)$ that puts two extremities in the same class if and only if they are spikes from a same articulation clique. As edges between different spikes of any articulation cliques of G are removed in $G|_{\mathcal{K}(G)}$, this implies that the partition cliques are the articulation cliques of G. Moreover, as $\cup\{V(C) : C \in \mathcal{C}\}$ partitions V, those strips are at most n. Last, observe that each $C \in \mathcal{C}$ is distance simplicial with respect to each $A \in \mathcal{A}(C)$ by part (ii) of Lemma 4.4.

We now move to complexity issues. We can find all articulation cliques of G in time $O(n^3)$, thanks to Lemma 3.6. Moreover, we can build the graph $G_{|\mathcal{K}(G)}$ and the family $\mathcal{Q}(\mathcal{K}(G))$ in time $O(n^3)$, thanks to Lemma 4.3; this also immediately gives the partition $\mathcal{P}$. In order to compute $\mathcal{C}$ and the sets $\mathcal{A}(C)$ for each $C \in \mathcal{C}$, a breadth-first algorithm suffices. □

5 An algorithm to decompose claw-free graphs

The main result of this section is the following theorem (Algorithm 2 is presented later):

THEOREM 5.1. *Let $G(V, E)$ be a connected claw-free graph with n vertices. In time $O(|V|^3)$ Algorithm 2:*

(i) either recognizes that G is quasi-line and net-free;

(ii) or recognizes that G has an odd anti-wheel and stability number at most 3;

(iii) or provides a decomposition into $k+t \le n$ strips $(F^1, \mathcal{A}^1), \ldots, (F^k, \mathcal{A}^k), (H^1, \mathcal{B}^1), \ldots, (H^t, \mathcal{B}^t)$, with respect to a partition $\mathcal{P}$, such that each graph F^j is distance simplicial with respect to each clique $A \in \mathcal{A}^j$ and each H^j has an induced 5-wheel and stability number at most 3. Moreover, the set of partition-cliques is a subset of the set of articulation cliques of G.

Even if Theorem 5.1 resembles Theorem 4.1 (note however that for Theorem 5.1 the partition-cliques are a *subset* of the articulation cliques of G), the way we build the strip decomposition for claw-free graphs is slightly different than for quasi-line graphs. In fact, in Algorithm 1 the strips are produced all together (in a way, simultaneously) from the ungluing of the articulation cliques of G. Algorithm 2 will instead first find and "remove" a family $\mathcal{H}$ of suitable non-quasi-line strips of G, that we call *hyper-line* strips, as to produce a quasi-line graph $G|_{\mathcal{H}}$. Then it will proceed as Algorithm 1 and build a strip decomposition $(\mathcal{F}, \mathcal{P})$ of $G|_{\mathcal{H}}$. Finally, it will suitably "combine" the strips in $\mathcal{F} \cup \mathcal{H}$ and the partition $\mathcal{P}$ to derive a strip decomposition of G.

We start therefore with the crucial definition of hyper-line strips. In the following, we say that a strip $(H, \mathcal{A})$ has *disjoint extremities* if $\mathcal{A}$ is either made of a single clique, or is made of two vertex disjoint cliques.

DEFINITION 5.1. *Let $G(V, E)$ be a claw-free graph and $(H, \mathcal{A})$ a strip with disjoint extremities. We say that H is an hyper-line strip of G if:*

- *H is an induced subgraph of G, i.e. $H = G[V(H)]$;*
- *the core $C(H, \mathcal{A})$ of the strip $(H, \mathcal{A})$ is anti-complete to $V \setminus V(H)$;*
- *for each $A \in \mathcal{A}$, $A \cup K(A)$ is an articulation clique of G, where $K(A) := N(A) \setminus V(H)$.*

Observe that, if $(H, \mathcal{A})$ is an hyper-line strip of G, then G is the composition of the strips $(H, \mathcal{A})$ and $(G \setminus V(H), \mathcal{K}(\mathcal{A}))$, with respect to the partition $\{\{A, K(A)\}, A \in \mathcal{A}\}$, where we let $\mathcal{K}(\mathcal{A}) := \{K(A), A \in \mathcal{A}\}$. Observe also that by definition, no vertex of A_2 (resp. A_1) is complete to $A_1 \cup K(A_1)$ (resp. $A_2 \cup K(A_2)$).

As we discussed above, in order to get a strip decomposition of a claw-free graph G, we will suitably "remove" hyper-line strips from G as to end up with a quasi-line graph. This leads to the following definition.

DEFINITION 5.2. *Let $G(V, E)$ be a claw-free graph and $\mathcal{H}$ a family of vertex disjoint hyper-line strips of G. We denote by $G|_{\mathcal{H}}$ the graph obtained from G by deleting the vertices in the core of the strips and the edges between the extremities of the 2-strips, that is:*

- *$V(G|_{\mathcal{H}}) = V(G) \setminus \bigcup_{(H,\mathcal{A}) \in \mathcal{H}} C(H, \mathcal{A})$;*
- *$E(G|_{\mathcal{H}}) = \{uv \in E : u, v \in V(G|_{\mathcal{H}})\} \setminus \{uv : u \in A_1, v \in A_2, A_1 \neq A_2 \in \mathcal{A}, (H, \mathcal{A}) \in \mathcal{H}\}$.*

The next lemma shows some properties of the graph $G|_{\mathcal{H}}$ that will be very useful for combining the strips in $\mathcal{H}$ with the strips coming from the decomposition of $G|_{\mathcal{H}}$.

Lemma 5.1. *Let G be a claw-free graph, $\mathcal{H}$ a family of vertex disjoint hyper-line strips of G and A an extremity of a strip $(H, \mathcal{A}) \in \mathcal{H}$. In the graph $G|_{\mathcal{H}}$, each vertex $v \in A$ is simplicial and $N_{G|_{\mathcal{H}}}[v] = A \cup K(A)$; moreover, each articulation clique of $G|_{\mathcal{H}}$ is an articulation clique of G.*

The proof of Lemma 5.1 is a bit technical, and we prefer to postpone it to the Appendix. The same applies with the proof of the following theorem, that is the crucial tool for the decomposition of a claw-free graph.

Theorem 5.2. *Let G^0 be a connected claw-free but not quasi-line graph with n vertices. In time $O(n^3)$ we may:*

(i) either recognize that G^0 has stability number at most 3;

(ii) or build a family $\mathcal{H}$ of vertex disjoint hyper-line strips of G such that:

- *each strip in $\mathcal{H}$ contains a 5-wheel of G and has stability number at most 3;*
- *$G|_{\mathcal{H}}$ is quasi-line and non-empty, i.e. $V(G|_{\mathcal{H}}) \neq \emptyset$.*

So suppose that G is claw-free and that by Lemma 5.1 we have found a family $\mathcal{H}$ of vertex disjoint hyper-line strips of G such that $G|_{\mathcal{H}}$ is quasi-line. Following Algorithm 1, we decompose $G|_{\mathcal{H}}$ and get a strip decomposition $(\mathcal{F}, \mathcal{P})$. We now show how to "combine" the strips in $\mathcal{F} \cup \mathcal{H}$ and the partition $\mathcal{P}$ as to derive a strip decomposition of G.

Let A be an extremity of a strip $(H, \mathcal{A}) \in \mathcal{H}$. It follows from Lemma 5.1 and Lemma 3.4 that $A \cup K(A)$ is an articulation clique of $G|_{\mathcal{H}}$, and therefore (see Theorem 4.1) a partition-clique of $G|_{\mathcal{H}}$. In particular, each vertex $v \in A$ will determine a 1-strip $(\{v\}, \{\{v\}\})$ of $\mathcal{F}$ (see Observation 1), and, therefore, the class $P(A) \in \mathcal{P}$ corresponding to the partition-clique $A \cup K(A)$ is such that $\{\{v\}, v \in A\} \subseteq P(A)$.

Now we build upon $(\mathcal{F}, \mathcal{P})$ and get a strip decomposition for G: we simply remove from $\mathcal{F}$ all 1-strips of the form $(\{v\}, \{\{v\}\})$, with $v \in A$, A being an extremity of a strip $(H, \mathcal{A}) \in \mathcal{H}$, and "replace" them with the strips in $\mathcal{H}$. Analogously, for each A being an extremity of a strip $(H, \mathcal{A}) \in \mathcal{H}$, we replace in the class $P(A)$ the set of extremities $\{\{v\}, v \in A\} \subseteq P(A)$, with A.

We summarize the procedure outlined above in the following algorithm:

Algorithm 2

Require: A connected claw-free but not quasi-line graph G.

Ensure: The algorithm either recognizes that G has stability number at most 3, or returns a strip decomposition of G as to satisfy statement *(iii)* of Theorem 5.1.

1: By Theorem 5.2, either conclude that G has stability number at most 3: **stop**, or build a family $\mathcal{H}$ of vertex disjoint hyper-line strips of G such that $G|_{\mathcal{H}}$ is quasi-line, and each strip contains a 5-wheel of G and has stability number at most 3.

2: Use Algorithm 1 to find a strip decomposition $(\mathcal{F}, \mathcal{P})$ of $G|_{\mathcal{H}}$. Let $\mathcal{F}' = \mathcal{F}$ and $\mathcal{P}' = \mathcal{P}$.

3: For each A being an extremity of a strip $(H, \mathcal{A}) \in \mathcal{H}$ do:
 remove from $\mathcal{F}'$ all 1-strips made of vertices from A, i.e. $\mathcal{F}' = \mathcal{F}' \setminus \{(\{v\}, \{\{v\}\}),\ v \in A\}$;
 replace the class $P(A) \in \mathcal{P}'$ with the class $P'(A)$, i.e. $\mathcal{P}' = (\mathcal{P}' \cup P'(A)) \setminus P(A)$, where:
  $P(A)$ is the class of $\mathcal{P}$ that contains the set of extremities $\{\{v\}, v \in A\}$;
  $P'(A) = (P(A) \cup A) \setminus \{\{v\}, v \in A\}$

4: **Return** the family of strips $\mathcal{F}' \cup \mathcal{H}$ and the partition $\mathcal{P}'$.

Proof of Theorem 5.1. Correctness of Algorithm 2 easily follows from the above discussion. Note also that the algorithm runs in $O(n^3)$-time, as its crucial steps 1 and 2 can be performed in $O(n^3)$-time. The statement then immediately follows, as soon as we use Theorem 4.1 for characterizing the quasi-line strip of $\mathcal{F}'$, and Lemma 5.1 to conclude that each articulation clique of $G|_{\mathcal{H}}$ is an articulation clique of G. □

6 The maximum weighted stable set problem in claw-free graphs

In this section, we are going to use our algorithmic decomposition result to derive a simple algorithm for the maximum weighted stable set problem in a claw-free graph G. We start by defining a simple graph reduction for composition of strips that only "shift" the weighted stability number $\alpha_w(G)$.

6.1 The maximum weighted stable set problem in composition of strips Let $G(V,E)$ be the composition of $k \geq 1$ strips $H_1 = (G^1, \mathcal{A}^1), \dots, H_k = (G^k, \mathcal{A}^k)$, with respect to a partition $\mathcal{P}$ and let $w : V(G) \mapsto \mathbb{R}$. We now show that we can always substitute H_1 with a simple *gadget strip* H_1' and reduce the problem of finding a maximum weighted stable set (MWSS) on G to the same problem on a graph G' that is the composition of $H_1', H_2, \dots, H_k$ with respect to a partition $\mathcal{P}'$.

The main idea is the following. Observe that the only possible obstruction to combine a stable set T of $G \setminus V(G^1)$ and a stable set U of G^1 into a stable set of G are the adjacencies in the partition-cliques involving the extremities of H_1. Because those extremities are cliques, there are four ways U may intersect the extremities of H_1 (by now assume that H_1 is a 2-strip): U contains a vertex in both extremities; U contains a vertex in one or the other extremity; U does not contain any vertex in the extremities. When one is interested in the MWSS for G then, given a stable set for $G \setminus V(G^1)$, one obviously wants to pick the MWSS among those from configurations that are compatible with respect to T. Hence, we can replace H_1 with another strip H_1' as long as H_1 and H_1' agree on the values of a few crucial stable sets.

Because the composition (and thus the adjacencies between $G \setminus V(G^1)$ and G^1) is slightly different if (i) H_1 is a 1-strip (in this case $\mathcal{A}^1 = \{A_1\}$, and there exists $P \in \mathcal{P} : A_1 \in P$); (ii) H_1 is a 2-strip (i.e. $\mathcal{A}^1 = \{A_1, A_2\}$) and its extremities are in the same class of the partition $\mathcal{P}$ (i.e. there exists $P \in \mathcal{P} : A_1, A_2 \subseteq P$) or (iii) H_1 is a 2-strip and its extremities are in different classes of the partition $\mathcal{P}$ (i.e. there exist $P_1 \neq P_2 \in \mathcal{P} : A_i \in P_i\ i = 1, 2$), we need to distinguish those cases.

Let us define $w'(v) = w(v)$ for all $v \notin V(G^1)$ and let us denote by C_k the complete graph on $k \geq 1$ vertices labeled $c_1, ..., c_k$: C_1 is the trivial graph with a single vertex while C_3 is a triangle.

Moreover:

- In case (i), we define $H_1' = (C_1, \{c_1\})$; $\delta_1 = \alpha_w(G^1 \setminus A_1)$; $w'(c_1) = \alpha_w(G^1) - \delta_1$; $\mathcal{P}' := (\mathcal{P} \cup (P \cup \{c_1\} \setminus A_1)) \setminus P$.
- In case (ii), we define $H_1' = (C_1, \{c_1\})$; $\delta_1 = \alpha_w(G^1 \setminus (A_1 \cup A_2))$; $w'(c_1) = \max\{\alpha_w(G^1 \setminus A_1), \alpha_w(G^1 \setminus A_2), \alpha_w(G^1 \setminus A_1 \Delta A_2)\} - \delta_1$; $\mathcal{P}' := (\mathcal{P} \cup (P \cup \{c_1\} \setminus \{A_1, A_2\})) \setminus P$.
- In case (iii): we define $H_1' = (C_3, \{\{c_1, c_3\}, \{c_2, c_3\}\})$; $\delta_1 = \alpha_w(G^1 \setminus (A_1 \cup A_2))$; $w'(c_1) = \alpha_w(G^1 \setminus A_2) - \delta_1$, $w'(c_2) = \alpha_w(G^1 \setminus A_1) - \delta_1$ and $w'(c_3) = \alpha_w(G^1) - \delta_1$; $\mathcal{P}' := (\mathcal{P} \setminus (P_1 \cup P_2)) \cup ((P_1 \setminus A_1) \cup \{c_1, c_3\}) \cup ((P_2 \setminus A_2) \cup \{c_2, c_3\})$.

The next lemma follows easily from the above discussion.

LEMMA 6.1. *Let G' be the composition of $H_1', H_2, \dots, H_k$ with respect to the partition $\mathcal{P}'$, with $w' : V(G') \mapsto \mathbb{R}$ and δ_1 as defined above. Then $\alpha_w(G) - \delta_1 = \alpha_{w'}(G')$. Moreover any* MWSS *of G' (with respect to w') can be converted into a* MWSS *of G (with respect to w) if the following stable sets are known: a* MWSS *of G^1; a* MWSS *of G^1 not intersecting A, for each $A \in \mathcal{A}^1$; a* MWSS *of G^1 not intersecting $A_1 \Delta A_2$ (only if $\mathcal{A}^1 = \{A_1, A_2\}$ and A_1, A_2 are in the same class of $\mathcal{P}$); a* MWSS *of G^1 not intersecting $\bigcup_{A \in \mathcal{A}^1} A$.*

Proof. (*For the sake of completeness we give the proof, which is however similar to the proof of Lemma 5 in [16].*) We first show that $\alpha_w(G) - \delta_1 = \alpha_{w'}(G')$. (*i*) Let S be a maximum weighted stable set of G. First suppose that S picks a vertex in A_1. Then $S \cap V(G^1)$ is a maximum weighted stable set in G^1 (otherwise we would swap with a better one in G^1). Also S is not picking any vertex belonging to an extremity in P other than A_1, and therefore $S' = (S \setminus V(G^1)) \cup \{c_1\}$ is a stable set of G'. Moreover $\alpha_w(G) = w(S) = w'(S') - w'(c_1) + w(S \cap V(G^1)) = w'(S') - w'(c_1) + \alpha_w(G^1) = w'(S') + \delta_1 \leq \alpha_{w'}(G') + \delta_1$. Suppose now that S does not pick any vertex from A_1. Then $S \cap V(G^1)$ is a maximum weighted stable set in $G^1 \setminus A_1$, while $S \setminus V(G^1)$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w(S \cap V(G^1)) + w(S \setminus V(G^1)) \leq \delta_1 + \alpha_{w'}(G')$.

Conversely, let S' be a maximum weighted stable set of G'. First suppose that S' picks c_1. In this case, for any stable set S of G^1, $(S' \setminus c_1) \cup S$ is a stable set of G. Therefore, if in particular we choose S as a maximum weighted stable set of G^1, $\alpha_w(G) \geq w((S' \setminus c_1) \cup S) = w'(S') - w'(c_1) + \alpha_w(G^1) = \alpha_{w'}(G') + \delta_1$. Now suppose that S' does not pick c_1. In this case, for any stable set S of $G^1 \setminus A_1$, $S' \cup S$ is a stable set of G. Therefore, if in particular we choose S as a maximum weighted stable set of $G^1 \setminus A_1$, $\alpha_w(G) \geq w(S' \cup S) = w'(S') + \alpha_w(G^1 \setminus A_1) = \alpha_{w'}(G') + \delta_1$.

(ii). This case easily reduces to the previous one. In fact, let $\overline{G}^1$ be the graph obtained from G^1 making A_1 complete to A_2, $\overline{H_1}$ be the 1-strip $(\overline{G}^1, A_1 \cup A_2)$, and finally $\overline{\mathcal{P}}$ be the partition obtained from $\mathcal{P}$ by replacing P with $P \cup \{A_1 \cup A_2\} \setminus \{A_1, A_2\}$. Then G is the composition of $\overline{H}_1, H_2, \ldots, H_k$ with respect to the partition $\overline{\mathcal{P}}$. Now the statement follows from the previous case, as soon as we observe that $\alpha_w(\overline{G}^1) = \max\{\alpha_w(G^1 \setminus A_1), \alpha_w(G^1 \setminus A_2), \alpha_w(G^1 \setminus A_1 \Delta A_2)\}$ and $\alpha_w(\overline{G}^1 \setminus A_1 \cup A_2) = \alpha_w(G^1 \setminus A_1 \cup A_2)$.

(iii). Let S be a maximum weighted stable set of G. First suppose that S intersects both A_1 and A_2. Then $S \cap V(G^1)$ is a maximum weighted stable set in G^1. Also $S' = (S \setminus V(G^1)) \cup \{c_3\}$ is a stable set of G'. Moreover $\alpha_w(G) = w(S) = w'(S') - w'(c_3) + w(S \cap V(G^1)) = w'(S') - w'(c_3) + \alpha_w(G^1) = w'(S') + \delta_1 \leq \alpha_{w'}(G') + \delta_1$. Suppose now that S picks a vertex in A_1 but no vertex in A_2. Then $S \cap V(G^1)$ is a maximum weighted stable set in $G^1 \setminus A_2$. Also $S' = (S \setminus V(G^1)) \cup \{c_1\}$ is a stable set of G'. Moreover $\alpha_w(G) = w(S) = w'(S') - w'(c_1) + w(S \cap V(G^1)) = w'(S') - w'(c_1) + \alpha_w(G^1 \setminus A_2) = w'(S') + \delta_1 \leq \alpha_{w'}(G') + \delta_1$. The case where S picks a vertex in A_2 but no vertex in A_1 goes along the same lines. Finally suppose now that S does not pick any vertex from $A_1 \cup A_2$. Then $S \cap V(G^1)$ is a maximum weighted stable set in $G^1 \setminus (A_1 \cup A_2)$, while $S \setminus V(G^1)$ is a stable set of G'. Therefore, $\alpha_w(G) = w(S) = w(S \cap V(G^1)) + w(S \setminus V(G^1)) \leq \delta_1 + \alpha_{w'}(G')$.

Conversely, let S' be a maximum weighted stable set of G'. First suppose that S' picks c_3. In this case, for any stable set S of G^1, $(S' \setminus c_3) \cup S$ is a stable set of G. Therefore, if in particular we choose S as a maximum weighted stable set of G^1, $\alpha_w(G) \geq w((S' \setminus c_3) \cup S) = w'(S') - w'(c_3) + \alpha_w(G^1) = \alpha_{w'}(G') + \delta_1$. Now suppose that S' picks c_1. In this case, for any stable set S of $G^1 \setminus A_2$, $(S' \setminus c_1) \cup S$ is a stable set of G. Therefore, if in particular we choose S as a maximum weighted stable set of $G^1 \setminus A_2$, $\alpha_w(G) \geq w((S' \setminus c_1) \cup S) = w'(S') - w'(c_1) + \alpha_w(G^1 \setminus A_2) = \alpha_{w'}(G') + \delta_1$. The case where S' picks c_2 goes along the same lines. Finally suppose that S' does not pick any vertex from C_3. In this case, for any stable set S of $G^1 \setminus (A_1 \cup A_2)$, $S' \cup S$ is a stable set of G. Therefore, if in particular we choose S as a maximum weighted stable set of $G^1 \setminus (A_1 \cup A_2)$, $\alpha_w(G) \geq w(S' \cup S) = w'(S') + \alpha_w(G^1 \setminus (A_1 \cup A_2)) = \alpha_{w'}(G') + \delta_1$.

Finally observe that it follows from above that we may immediately derive from S' a maximum weighted stable set of G if we are given: a MWSS of G^1; a MWSS of G^1 not intersecting A, for each $A \in \mathcal{A}^1$; a MWSS of G^1 not intersecting $A_1 \Delta A_2$ (only if $\mathcal{A}^1 = \{A_1, A_2\}$ and A_1, A_2 are in the same class of $\mathcal{P}$); a MWSS of G^1 not intersecting $\bigcup_{A \in \mathcal{A}^1} A$.

Trivially, we can apply the above procedure iteratively to each strip H_i. The problem of finding a MWSS on G reduces therefore to the same problem on the graph G' that is the composition of $H'_1, H'_2, \ldots, H'_k$ with respect to a partition $\mathcal{P}'$. The following lemma shows some key properties of G'.

Corollary 6.1. *G' is a line graph and in time $O(k)$ we can built a root graph $\tilde{G}$ with $O(k)$ vertices and edges.*

Proof. The strips H'_j, $j = 1, ..., k$ are line strips and therefore it follows from Lemma 3.3 that G' is a line graph. (Note also that by construction G' has at most $3k$ vertices). Moreover, the proof of the same lemma suggests how to build a root graph for G' with $O(k)$ vertices and edges in $O(k)$-time: we skip the details.

Since the number k of strips is bounded by $O(|V(G)|)$, it follows that we have reduced, provided we can efficiently compute the weights w' for the vertices of each strip H'_i, the maximum weighted stable set problem on G to a weighted matching problem on the graph $\tilde{G}$, that has $O(|V(G)|)$ vertices and edges. This latter problem can be solved in time $O(|V(G)|^2 \log |V(G)|)$ by [9]. Also note that the computation of the weights w' for the vertices of some strip H'_i requires the solution of some MWSS problems on induced subgraphs of G^i. Thus, we have proved the following:

Theorem 6.1. *The maximum weighted stable set problem on a graph G, that is the composition of some set of strips $(G^1, \mathcal{A}^1), \ldots, (G^k, \mathcal{A}^k)$, can be solved in $O(|V(G)|^2 \log |V(G)| + \sum_{i=1,\ldots,k} p_i(|V(G^i)|))$-time, if each G^i belongs to some class of graphs, where the same problem can be solved in time $O(p_i(|V(G^i)|))$.*

6.2 The maximum weighted stable set problem in quasi-line net free graphs and distance-simplicial graphs It follows from Theorem 5.1 and Theorem 6.1 that, in order to get an algorithm for the MWSS problem on a claw-free graph G, we are left with showing: (i) how to find a MWSS in a graph

G that is quasi-line and net-free; (ii) how to find a MWSS in a graph G that is distance simplicial with respect to some clique (assuming that we will use enumeration for finding a MWSS in a graph that has stability number at most 3).

The construction and the algorithm given by Pulleyblank and Shepherd in [17] for solving the MWSS in *distance claw-free* graphs can be used to solve (i) in time $O(|V(G)|^4)$ and (ii) in time $O(|V(G)|^2)$. In the following, we build upon their technique and get an $O(|V(G)|^3)$-time algorithm for (i).

First, we need a definition. Recall (see Definition 2.1) that a vertex v of a connected graph G is *distance simplicial* if, for every j, $\alpha(N_j(v)) \leq 1$.

Definition 6.1. *A vertex v of a connected graph G is* almost distance-simplicial *if $\alpha(N_j(v)) \leq 1$ for every $j \geq 2$, and $\alpha(N(v) \cup N_2(v)) \leq 2$. A graph is* almost distance-simplicial *if there exists v that is distance simplicial.*

The next two lemmas directly follow from the results in [17]. For the sake of completeness, we show a proof for the first lemma; the proof of the second lemma goes along the same lines and so we skip it.

Lemma 6.2. *Let $G(V,E)$ be a connected graph and z an almost distance-simplicial vertex of G. The maximum weighted stable set problem in G can be solved in time $O(|V|^2)$.*

Proof. We are given a weight function $w : V(G) \mapsto \mathbb{R}$. We build upon a construction and an algorithm from Pulleyblank and Shepherd [17] for distance-claw-free graphs. Let $p \in \mathbb{N}$ minimum such that $N_i(z) = \emptyset$. We denote $N_0(z) = \{z\}$ and $\mathcal{S}_i, i = 0, ..., p$ the set of all stables set in $G[N_i(z)]$ (including the empty set).

Let us now define an auxiliary directed graph $D(G)$. The vertices of $D(G)$ consist of $\{v^i_S : S \in \mathcal{S}_i, i = 0, ..., p\}$ together with two special nodes u^*, v^*. The arc set A of $D(G)$ is defined as follows. For each $S \in S_p$, $(v^p_S, v^*) \in A$; $(u^*, v^0_\emptyset), (u^*, v^0_{\{z\}}) \in A$ and for all $i = 0, ..., p-1$ and each stable set S of $G[N_i(z) \cup N_{i+1}(z)]$, $(v^i_{S \cap N_i(z)}, v^i_{S \cap N_{i+1}(z)}) \in A$. We assign weights w' to the arcs of $D(G)$ as follows: for each arc $a = (x, v^*)$, $w'_a = 0$ and for each arc $a = (x, v^i_S)$, $w'_a = \sum_{y \in S} w_y$. The maximum weighted stable set problem in G is equivalent to the longest directed (u^*, v^*)-path in the acyclic graph $D(G)$ and can thus be solved in time $O(|E(D(G))|)$ (see e.g. [1]). By hypothesis, $\sum_{j=2}^{p} |\mathcal{S}_i| = O(V(G))$ and $|\mathcal{S}_1 \cup \mathcal{S}_2| = O(|V(G)|^2)$. Therefore, from the handshaking lemma, $|E(D(G))| = O(|V(G)|^2)$ and thus the results follows.

Lemma 6.3. *Let $G(V,E)$ be a connected graph and K a clique of G such that G is distance simplicial with respect to K. The maximum weighted stable set problem in G can be solved in time $O(|V|^2)$.*

It follows from Lemma 6.3 and Theorem 6.1 that the MWSS problem in a graph G that is the composition of a set of (given) distance simplicial strips can be solved in time $O(|V(G)|^2 \log |V(G)|)$. The $O(|V(G)|^3)$-time algorithm for a quasi-line and net-free graph G requires more results from the literature.

Definition 6.2. *A triple $\{x, y, z\}$ of vertices of a graph G is an asteroidal triple (AT) if for every two of these vertices there is a path between them avoiding the closed neighborhood of the third. A graph G is called asteroidal triple-free (AT-free) if it has no asteroidal triple.*

Brandstadt and Dragan [2] proved the following.

Lemma 6.4. *For every vertex v in a $\{$claw,net$\}$-free graph $G(V,E)$, $G(V \setminus N[v])$ is $\{$claw,AT$\}$-free.*

Now using the celebrated 2LexBFS algorithm, Hempel and Krastch [8] proved the additional following result (Lemma 6 in [8]).

Lemma 6.5. *Given a $\{$claw,AT$\}$-free graph $G(V,E)$, one can find in $O(|E|)$ an almost distance simplicial vertex in G.*

Corollary 6.2. *The maximum weighted stable set problem in a $\{$claw,net$\}$-free graph can be solved in time $O(|V|^3)$.*

Proof. For each $v \in V$ we compute the maximum weighted stable set picking v by solving the maximum weighted stable problem on $G(V \setminus N[v])$. This can be done in $O(|V|^2)$ time because of Lemma 6.4, Lemma 6.5 and Lemma 6.2. We choose the best stable set over those $|V|$ choices (and the empty set in case where $w : V \mapsto \mathbb{R}^-$). The results follows.

6.3 The algorithm for the maximum weighted stable set problem in claw-free graphs We are now ready to put all the bricks together and present our $O(|V(G)|^3)$ time algorithm for the weighted stable set problem in claw-free graphs.

Algorithm 3

Require: A connected claw-free graph G and a function $w : V(G) \mapsto \mathbb{R}$.
Ensure: The algorithm find a maximum weighted stable set in G with respect to w.

1: Use Algorithm 2 to detect in $O(|V|^3)$ time if $\alpha(G) \leq 3$, G is {claw,net}-free or provide a decomposition of G that obeys Theorem 5.1.
2: If $\alpha(G) \leq 3$, solve the problem by enumeration in $O(|V|^3)$ time;
3: if G is {claw,net}-free, then use Corollary 6.2 to solve the problem in $O(|V|^3)$ time;
4: if G is the composition of k strip $H_1, ..., H_k$, then apply Theorem 6.1 to solve the problem. Observe that in this case: if H_i is a distance simplicial strip, then the maximum weighted stable set problems can be solved in time $O(|V(H_i)|^2)$ (use Lemma 6.3); if H_i is a 5-wheel strip , then the maximum weighted stable set problems can be solved in time $O(|V(H_i)|^3)$ (by enumeration). Then a maximum weighted stable set in G can be computed in time $O(|V(G)|^3)$ from Theorem 6.1.

References

[1] R. Ahuja, T. L. Magnanti, and J. B. Orlin. Network Flows: Theory, Algorithms, and Applications. Prentice-Hall, Englewood Cliffs, NJ, 1993.
[2] A. Brandstädt and F. F. Dragan. On linear and circular structure of (claw, net)-free graphs. *Discrete Applied Mathematics* 129, pages 285–303, 2003.
[3] A. Brandstädt and C. Hoáng. On clique separators, nearly chordal graphs, and the Maximum Weight Stable Set problem. *Theoretical Computer Science* 389, pages 295–306, 2007.
[4] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. *Surveys in Combinatorics 2005, London Math. Soc. Lecture Note Series*, 327, pages 153–171, 2005.
[5] M. Chudnovsky and P. Seymour. Claw free Graphs IV. Global structure. *Journal of Combinatorial Theory. Ser B*, 98, pages 1373–1410, 2008.
[6] J. Edmonds. Maximum matching and a polyhedron with 0,1-vertices. *Journal of Research of the National Bureau of Standards*, 69, pages 125–130, 1965.
[7] J. Fouquet. A strengthening of Ben Rebeas lemma. *Journal of Combinatorial Theory* 59, pages 35-40, 1993.
[8] H. Hemper, D. Kratsch. On claw-free asteroidal triple-free graphs. *Discrete Applied Mathematics* 121, pages 155–180, 2002.
[9] H.N. Gabow. Data structures for weighted matching and nearest common ancestors with linking. Proceedings of the 1st Annual ACM-SIAM Symposium on Discrete Algorithms ,1990.
[10] A. King Claw-free graphs and two conjectures on omega, Delta, and chi. Ph.D. Thesis, McGill University, 2009.
[11] J. Krausz. Demonstration nouvelle d'un theoreme de Whitney sur le reseaux. *Mat. Fix. Lapok*, 50, pages 75-85, 1943.
[12] L. Lovász, and M.D. Plummer, *Matching theory.* North Holland, Amsterdam, 1986.
[13] G. J. Minty. On maximal independent sets of vertices in claw-free graphs. *Journal on Combinatorial Theory*, 28, pages 284–304, 1980.
[14] D. Nakamura and A. Tamura. A revision of Minty's algorithm for finding a maximum weighted stable set of a claw-free graph. *Journal of the Operations Research Society of Japan*, 44(2), pages 194–2004, 2001.
[15] P. Nobili and A. Sassano. Personal Communication, 2010.
[16] G. Oriolo, U. Pietropaoli and G. Stauffer. A new algorithm for the maximum weighted stable set problem in claw-free graphs. In A. Lodi, A. Panconesi and G. Rinaldi, editors, *Proceedings Thirteenth IPCO Conference*, pages 77–96, 2008.
[17] W. R. Pulleyblank and F. B. Shepherd. Formulations for the stable set polytope of a claw-free graph. In G. Rinaldi and L. Wolsey, editors, *Proceedings Third IPCO Conference*, pages 267–279, 1993.
[18] N. D. Roussopoulos, (1973). A max m,n algorithm for determining the graph H from its line graph G. *Information Processing Letters* 2 (4), pages 108–112, 1973.
[19] N. Sbihi. Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans étoile. *Discrete Mathematics* **29**, pages 53–76, 1980.
[20] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency (3 volumes).* Algorithms and Combinatorics 24. Berlin: Springer., 2003.

A Proof of Lemma 4.4

Before starting with the proof of Lemma 4.4 itself, we give an intermediate structural result.

LEMMA A.1. *Let $G(V, E)$ be a connected quasi-line net-free graph, and let K be a non-empty clique of G such that $N(K)$ is a clique, but $K \cup N(K)$ is not a clique. Then G is* distance simplicial *with respect to K.*

Proof. Suppose by contradiction that there exists $j \geq 2$ such that $\alpha(N_j(K)) \geq 2$: we choose j to be minimal, i.e. for all $h < j$, $\alpha(N_h(K)) = 1$. Let $\{s_1, s_2\}$ be a stable set of size 2 in $N_j(K)$. For $i = 1, 2$, define the non-empty sets $S_i = N(s_i) \cap N_{j-1}(K)$, and note that $S_1 \cap S_2 = \emptyset$. In fact, suppose to the contrary there exists $v \in S_1 \cap S_2$; then, $(v; u, s_1, s_2)$ is a claw, for each u in $N(v) \cap N_{j-2}(K)$. This implies that (S, S_1, S_2) is a partition on $N_{j-1}(K)$, where we defined $S = N_{j-1}(K) \setminus (S_1 \cup S_2)$.

Claim 1. *For $i = 1, 2$, if $v \in S_i$ and $u \notin S_i$, then $N(v) \cap N_{j-2}(K) \subseteq N(u) \cap N_{j-2}(K)$.*

Proof. Suppose there exists a vertex w in $N_{j-2}(K)$ adjacent to v but not u, then $(v; w, u, s_i)$ is a claw, a contradiction.

Claim 2. $\cup_{i=1,2}(S_i \cup (N(S_i) \cap N_{j-2}(K)))$ *is a clique.*

Proof. As $N_{j-1}(K)$ and $N_{j-2}(K)$ are cliques by construction, it suffices to show that for any pair $u, v \in S_1 \cup S_2$, $N(u) \cap N_{j-2}(K) = N(v) \cap N_{j-2}(K)$. This immediately follows from Claim 1 for $u \in S_1$, $v \in S_2$. Thus assume w.l.o.g. that $u, v \in S_1$, and let $w \in N(u) \cap N_{j-2}(K)$, and pick $x \in S_2$. By applying twice Claim 1, it first follows that $xw \in E$, and then that $vw \in E$, concluding the proof.

Claim 3. $j = 2$.

Proof. Trivially $j \geq 2$, as $N(K)$ is a clique. Now suppose $j \geq 3$, and let $v \in S_1$, $u \in S_2$, $w \in N(v) \cap N_{j-2}(K)$ and $s_3 \in N(w) \cap N_{j-3}(K)$. We already argued that $vs_2, us_1 \notin E$: moreover, by Claim 2, $w \in N(u)$ and by construction s_3 is non-adjacent to u, v, s_1, s_2, while w is non-adjacent to s_1, s_2. Thus, $\{v, u, w; s_1, s_2, s_3\}$ is a net, contradicting the hypothesis.

We conclude the proof by showing that, if $j = 2$, all vertices from K are simplicial (i.e. $K \cup N(K)$ is a clique), contradicting the hypothesis. Pick $u, v \in K$, and suppose there exists $w \in N[u] \setminus N[v]$. Recall that $N(K) = S_1 \cup S_2 \cup S$. Suppose first that $w \in S_1 \cup S_2$. Then, by Claim 2, v is anticomplete to $S_1 \cup S_2$, while u is complete to $S_1 \cup S_2$. This implies that $(u, v_1, v_2; s_1, s_2, v)$ is a net, for some vertices $v_1 \in S_1$ and $v_2 \in S_2$, i.e. a contradiction. Now let $w \in S$. Then, by Claim 1, v is also anticomplete to $S_1 \cup S_2$. Recall that, by Claim 2, u is either complete or anticomplete to $S_1 \cup S_2$. If the former holds, then we can construct a net as done for the previous case. If conversely the latter holds, $(w, v_1, v_2; u, s_1, s_2)$ is a net. In both cases, we derive a contradiction. This shows that $N[u] = N[v]$ for arbitrary $u, v \in K$. As $N(K) = S_1 \cup S_2 \cup S$ is a clique, we conclude that $K \cup N(K)$ is a clique.

We now move to the proof of Lemma 4.4.

Claim 4. *If $Q_1 \cap Q_2 \neq \emptyset$ for some spikes $Q_1, Q_2 \in \mathcal{Q}(G)$, then $Q_1 = Q_2 = K_1 \cap K_2$ for some articulation cliques $K_1, K_2 \in \mathcal{K}(G)$.*

Proof. Let v be a vertex that is contained in two spikes $Q_1, Q_2 \in \mathcal{Q}(G)$. Since $\mathcal{Q}(K)$ partitions $V(K)$ for each articulation clique K, Q_1, Q_2 are spikes of different articulation cliques K_1, K_2. This implies that v is contained into two articulation cliques and thus, by definition, $Q_1 = K_1 \cap K_2 = Q_2$, concluding the proof.

It is convenient to define the following three families of spikes from $\mathcal{Q}(G)$.

(*i*) those formed by simplicial vertices of G;

(*ii*) those formed by vertices that belong to two articulation cliques of G, i.e. those formed by *bound* vertices;

(*iii*) those formed by vertices that belong to exactly one articulation clique, say K, and have a neighbor that is not in K.

One immediately checks that the three classes above provide a partition of $\mathcal{Q}(G)$ and thus, using Claim 4, induce a partition of vertices from $V(\cup_{K \in \mathcal{K}(G)} K)$.

Similarly to what it was done in the proof of Lemma 4.2, for an articulation clique K, a vertex $v \in K$, and a set $Q \in \mathcal{Q}(K)$, we use the notation $\widetilde{N}_K(v) := N(v) \setminus K$ and $\widetilde{N}_K(Q) := N(Q) \setminus K$. We omit the subscript K when it is clear from the context. Note that with $N()$, $\mathcal{Q}()$,... we denote those sets in the graph G, while we add the subscript $G|_{\mathcal{K}(G)}$ when we refer to the corresponding sets in the graph $G|_{\mathcal{K}(G)}$. We start with some basic facts on articulation cliques.

Claim 5. *Let Q_1 be a spike of an articulation clique $K \in \mathcal{K}(G)$.*

(*i*) *if Q_1 is a spike of type (ii), then all vertices from Q_1 are copies, and $N[Q_1] = K \cup K'$, $K' \in \mathcal{K}(G)$ being the other articulation clique Q_1 is contained into;*

(*ii*) *if Q_1 is a spike of type (iii), then $\widetilde{N}_K(Q_1)$ is a clique;*

(*iii*) *if Q_1 is a spike of type (iii) and Q_2 is a different spike of $K \in \mathcal{K}(G)$ such that $\widetilde{N}_K(Q_1) \subseteq \widetilde{N}_K(Q_2)$, then Q_2 is of type (ii), and there exits $K' \in \mathcal{K}(G)$ distinct from K such that $N[Q_2] = K \cup K'$ and $\widetilde{N}_K(Q_1) \subseteq K'$.*

(*iv*) *if Q_1 is a spike of type (iii) and Q_2 is a different spike of $K \in \mathcal{K}(G)$ also of type (iii), then for each $q_1 \in Q_1$ and $q_2 \in Q_2$, there exists a vertex $v \in \widetilde{N}_K(q_1) \setminus N_K(q_2)$, $w \in \widetilde{N}_K(q_2) \setminus \widetilde{N}_K(q_1)$ such that $vw \notin E$. Moreover, $v \in \widetilde{N}_K(Q_1) \setminus \widetilde{N}_K(Q_2)$, $w \in \widetilde{N}_K(Q_2) \setminus \widetilde{N}_K(Q_1)$.*

(*v*) *if Q_1 is a spike of type (ii) or (iii) for each $v \in Q_1$ the unique pair of cliques covering $N(v)$ is $K, \widetilde{N}_K(v) \cup U[v]$. In case Q_1 is a spike of type (ii), $U[v] = Q_1$.*

Proof. (*i*) Pick $q \in Q_1$, and recall that, by definition, both K and K' are crucial for v. Thus, K, K' is the unique pair of maximal cliques covering $N[q]$, and consequently $N[q] = K \cup K'$. Moreover, all vertices from Q_1 are copies, since they have the same closed neighborhood.
(*ii*) Suppose not: then u, v are not adjacent, for some $u, v \in \widetilde{N}_K(Q_1)$. Since $\widetilde{N}_K(t)$ is a clique for each $t \in Q_1$, this implies that $N(u) \cap Q_1$ is distinct from $N(v) \cap Q_1$. Let $q \in N(u) \cap Q_1$, $q' \in N(v) \cap Q_1$. Then, $\widetilde{N}_K(q)$ is not complete to $\widetilde{N}_K(q')$, contradicting the definition of spike.
(*iii*) Q_2 is not of type (*i*), since $\widetilde{N}_K(Q_2)$ is non-empty. Suppose now Q_2 is of type (*iii*), and pick $q_1 \in Q_1$, $q_2 \in Q_2$. By hypothesis $\widetilde{N}_K(q_2) \subseteq \widetilde{N}_K(Q_2)$ and $\widetilde{N}_K(q_1) \subseteq \widetilde{N}_K(Q_1) \subseteq \widetilde{N}_K(Q_2)$ are complete to each other, since they contained in the $\widetilde{N}_K(Q_2)$, which is a clique by part (*ii*). This implies that q_1, q_2 belong to the same spike of K, a contradiction. Thus, Q_2 is a spike of type (*ii*), and consequently $N[Q_2] = K \cup K'$ for some $K' \in \mathcal{K}(G)$ from part (*i*). Moreover, since $\widetilde{N}_K(Q_1) \cap K = \emptyset$, $\widetilde{N}_K(Q_1) \subseteq K'$.
(*iv*). $\widetilde{N}_K(q_1) \cup \widetilde{N}_K(q_2)$ is not a clique, since this would contradict the fact that q_1, q_2 belong to the different spikes. Thus, there exists a missing edge vw. By definition of articulation clique, $\widetilde{N}_K(q_1)$ and $\widetilde{N}_K(q_2)$ are cliques, thus we can assume w.l.o.g. that $v \in \widetilde{N}_K(q_1)$, $w \in \widetilde{N}_K(q_2)$. Moreover, since $\widetilde{N}(Q_1)$ is a clique, $w \notin \widetilde{N}(Q_1)$. Similarly $v \notin \widetilde{N}(Q_2)$.
(*v*). Since K is crucial for v, there exists a unique pair of maximal cliques K, H that cover $N[v]$. Recall that $N[v] = K \cup \widetilde{N}_K(v)$, with $\widetilde{N}_K(v) \cap K = \emptyset$. Thus, $\widetilde{N}_K(v) \subseteq H$. Moreover, a vertex $v \in K$ belongs to H if and only if it is complete to $\widetilde{N}_K(v)$, and thus universal to v. Suppose now that Q_1 is of type (*ii*). H is then an articulation clique and by definition, $Q_1 = K \cap H$ and thus by maximality of K and H, $Q_1 = U[v]$.

Claim 6. *Let $Q \in \mathcal{Q}(K) \setminus \mathcal{Q}(K')$ for some $K, K' \in \mathcal{K}(G)$. Then $\widetilde{N}_K(Q) \cap K' \subseteq Q'$ for some $Q' \in \mathcal{Q}(K')$.*

Proof. By Claim 4, $Q \cap K' = \emptyset$. Now suppose, by contradiction, that $\widetilde{N}_K(Q) \cap Q', \widetilde{N}_K(Q) \cap Q'' \neq \emptyset$, for some distinct $Q', Q'' \in \mathcal{Q}(K')$. We first argue that Q', Q'' are spikes of type (*iii*). They are not of type (*i*), since they are adjacent to Q, which we argued lie outside K'. Suppose now that Q'' is of type (*ii*), i.e. it is also contained in some articulation clique K''. By Claim 5, $N[Q''] = K' \cup K''$. Since $Q \cap K' = \emptyset$, it follows $Q \cap N(Q'') \subseteq K''$, and thus $Q \subset K''$, since in this case Q belongs to two articulation cliques, and all its vertices are copies. Consequently $N[Q] = K \cup K''$. By construction, Q' does not intersect K, K'', contradicting the fact that Q is adjacent to Q'. Thus, both Q' and Q'' are spikes of type (*iii*). By Claim 5, there exists $u \in \widetilde{N}_{K'}(Q') \setminus \widetilde{N}_{K'}(Q'')$ and $v \in \widetilde{N}_{K'}(Q'') \setminus \widetilde{N}_{K'}(Q')$ such that $uv \notin E$. At least one between u and v does not belong to K, say w.l.o.g. u. As $Q \cap \widetilde{N}_{K'}(Q') \neq \emptyset$ by hypothesis, and $\widetilde{N}_{K'}(Q')$ is a clique, then u is a vertex outside K that is adjacent to some vertex of Q, i.e. $u \in \widetilde{N}_K(Q)$. This leads to contradiction, since $\widetilde{N}_K(Q)$ is a clique by Claim 5 and u is anticomplete to Q''.

Claim 7. *For each $K \in \mathcal{K}(G)$, for each $Q \in \mathcal{Q}(K)$ of type (iii), we have $N_{G_{|\mathcal{K}(G)}}(q) \setminus Q = \widetilde{N}_K(q)$ for all $q \in Q$. In particular $N_{G_{|\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q)$, and moreover $N_{G_{|\mathcal{K}(G)}}(Q)$ is a (non empty) clique in $G|_{\mathcal{K}(G)}$.*

Proof. $N_{G_{|\mathcal{K}(G)}}(q) \backslash Q = \widetilde{N}_K(q)$ follows from the fact that vertices of Q are in exactly one articulation clique and thus adjacencies are removed only with vertices in K. Moreover, $\widetilde{N}_K(Q)$ is a clique in G. Suppose $N_{G_{|\mathcal{K}(G)}}(Q)$ is not a clique in $G_{\mathcal{K}(G)}$: this implies that there exists an articulation clique $K' \in \mathcal{K}(G)$ and two distinct sets $Q', Q'' \in \mathcal{Q}(K')$ such that $Q' \cap \widetilde{N}_K(Q)$, $Q'' \cap \widetilde{N}_K(Q) \neq \emptyset$. Moreover, $K' \neq K$ since $\widetilde{N}_K(Q) \cap K = \emptyset$ by definition, and consequently $Q \notin \mathcal{Q}(K')$, since Q is a spike of type (iii). But the we contradict Claim 6.

Claim 8. *Let $K \in \mathcal{K}(G)$, Q_1, Q_2 different sets of $\mathcal{Q}(K)$, and $v \in \widetilde{N}(Q_1) \cap \widetilde{N}(Q_2)$. Then for each covering of $N[v]$ with two maximal cliques H_1, H_2, we can assume w.l.o.g. $Q_1 \cap N(v) \subseteq H_1$ and $Q_2 \cap N(v) \subseteq H_2$. Thus, in particular, v is adjacent to at most two spikes from $\mathcal{Q}(K)$. In case where both Q_1 and Q_2 are of type (iii) we can assume w.l.o.g. that, $Q_1 \cap N(v) \subseteq H_1 \setminus H_2$ and $Q_2 \cap N(v) \subseteq H_2 \setminus H_1$.*

Proof. Let $u_i \in Q_i \cap N(v)$, for $i = 1, 2$. Note that u_1 (resp. u_2) is not simplicial, since $\widetilde{N}(u_1) \neq \emptyset$ (resp. $\widetilde{N}(u_2) \neq \emptyset$). First suppose that both spikes Q_1 and Q_2 are of type (iii): by Claim 5, there exist $z_1 \in \widetilde{N}(u_1) \setminus \widetilde{N}(u_2)$ and $z_2 \in \widetilde{N}(u_2) \setminus \widetilde{N}(u_1)$ such that $z_1 z_2 \notin E$. Note that $z_1, z_2 \in N(v)$. It follows that any covering of $N[v]$ with two maximal cliques H_1, H_2 is such that w.l.o.g. $u_1, z_1 \in H_1$ and $u_2, z_2 \in H_2$. But since z_1 is anticomplete to Q_2 (else $\{z_1, z_2\} \subseteq \widetilde{N}(Q_2)$ is not a clique contradicting Claim 5 (ii)) and similarly z_2 is anticomplete to Q_1, the results follows.

Now suppose that at least one of Q_1, Q_2 is also a spike of a second articulation clique K', say w.l.o.g. Q_1. As $N[Q_1] = \Gamma[Q_1] = K \cup K'$, we have $N(v) \cap Q_1 = Q_1$ and since $v \notin K$, then $v \in K'$. Moreover, by definition of spikes, for all $w \in N(v) \cap Q_2$, $w \notin K'$. Thus the unique pair of maximal cliques that cover $N[v]$ is K', H, with $Q_1 \subseteq K'$, $N(v) \cap Q_2 \subseteq H$, the result follows.

Claim 9. *Let $v \in V \setminus V(\mathcal{K}(G))$, then v is regular in $G|_{\mathcal{K}(G)}$. Moreover, if v is strongly regular in $G|_{\mathcal{K}(G)}$, it is also strongly regular in G and if H_1', H_2' is the unique pair of maximal cliques in $G|_{\mathcal{K}(G)}$ that cover $N_{G_{|\mathcal{K}(G)}}(v)$, H_1',H_2' is the unique pair of maximal cliques in G that cover $N(v)$.*

Proof. Let v be a vertex of $V \setminus V(\mathcal{K}(G))$ and H_1, H_2 a pair of maximal cliques that cover $N(v)$ (G is quasi-line so every vertex is regular). By Claim 8, for each articulation clique $K \in \mathcal{K}(G)$, v is adjacent to at most two spikes from K. Let $K_1, ..., K_m$ (possibly $m = 0$) be the articulation cliques from $\mathcal{K}(G)$ such that $N(v)$ intersects each of $K_1, ..., K_m$ in exactly two spikes, and denote those spikes by Q_i^1, Q_i^2. Note that all these spikes are of type (iii) (recall that v belongs to no articulation clique of G), thus they are pairwise non-intersecting by Claim 4. By Claim 8, we can assume w.l.o.g. that $Q_1^1, \ldots, Q_m^1 \subseteq H_1 \setminus H_2$, $Q_1^2, \ldots, Q_m^2 \subseteq H_2 \setminus H_1$. Observe that since $v \in V \setminus V(\mathcal{K}(G))$, we have $N_{G_{|\mathcal{K}(G)}}[v] = N[v] = H_1 \cup H_2$. But H_1 and H_2 are still maximal cliques of $G|_{\mathcal{K}(G)}$ since adjacencies are removed only between Q_i^1 and Q_i^2 during ungluing and $Q_i^1 \subseteq H_1 \setminus H_2$ and $Q_i^2 \subseteq H_2 \setminus H_1$. Thus v is regular in $G|_{\mathcal{K}(G)}$.

This shows that if H_1, H_2, H_1', H_2' are two different pairs of maximal cliques that cover $N(v)$, H_1, H_2, H_1', H_2' are also two different pairs of maximal cliques that cover $N_{|G_{\mathcal{K}(G)}}(v)$ and thus strongly regularity of v in $G_{|\mathcal{K}(G)}$ implies strong regularity of v in G. Moreover the unique pair of cliques covering the neighborhood of v in G and $G_{\mathcal{K}(G)}$ are the same.

Claim 10. *$G|_{\mathcal{K}(G)}$ is quasi-line.*

Proof. We show that each vertex is regular in $G|_{\mathcal{K}(G)}$. Let $vinV(\mathcal{K}(G))$, this follows from Claim 7 for vertices of type (iii) and it is immediate for vertices of type (i) and (ii). For vertices in $V \setminus V(\mathcal{K}(G))$, this follows from Claim 9.

Claim 11. *For each $K \in \mathcal{K}(G)$, for each $Q \in \mathcal{Q}(K)$ of type (iii), if $Q \cup \widetilde{N}_K(Q)$ is a clique in G, then $Q \cup \widetilde{N}_K(Q) = Q \cup N_{G_{|\mathcal{K}(G)}}(Q)$ induces a (clique) connected component of $G|_{\mathcal{K}(G)}$.*

Proof. Let $K \in \mathcal{K}(G)$, $Q \in \mathcal{Q}(K)$ of type (iii) such that $Q \cup \widetilde{N}_K(Q)$ is a clique. From Claim 7 it follows that $N_{G_{|\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q)$ is a clique, and thus $Q \cup N_{G_{|\mathcal{K}(G)}}(Q)$ is also a clique in $G|_\mathcal{K}(G)$ with $N_{G_{|\mathcal{K}(G)}}(Q)$ non-empty (no edge between Q and $N_{G_{|\mathcal{K}(G)}}(Q)$ is removed in the ungluing since it would mean that a vertex of Q is in a second articulation clique, a contradiction).

Suppose first that there exists a vertex $w \in K \backslash Q$ that is complete to $\widetilde{N}_K(Q)$. By Claim 5, $w \in K \cap K_1$ for some $K_1 \in \mathcal{K}(G)$ with $K_1 \neq K$, $N[w] = K \cup K_1$, and $\widetilde{N}_K(Q) \subseteq K_1$. First, note that all vertices from $\widetilde{N}_K(Q)$ must be in the same spike Q_1 from $\mathcal{Q}(K_1)$, otherwise $N_{G_{|\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q)$ would not be a clique in $G|_{\mathcal{K}(G)}$, contradicting Claim 7. Observe that Q_1 does not intersect K, otherwise $Q_1 = K \cap K_1$. Moreover Q_1 is of type (iii). Indeed, the vertices in Q are neither in K_1, nor in any other clique other than K (otherwise they would be bounds) and thus $Q \subseteq \widetilde{N}_{K_1}(Q_1)$. Since $Q \neq \emptyset$ it follows that the vertices of Q_1 are not simplicial and moreover they are not bounds, since in this case, this would imply that the vertices from Q belong to two different articulation cliques and are thus also bounds, a contradiction. We now argue that actually $\widetilde{N}_{K_1}(Q_1) = Q$. Using again Claim 7, $N_{G|_{\mathcal{K}(G)}}(Q_1) = \widetilde{N}_{K_1}(Q_1)$ and it is a clique in $G|_{\mathcal{K}(G)}$. Since $Q \subseteq \widetilde{N}_{K_1}(Q_1)$, all vertices from $\widetilde{N}_{K_1}(Q_1)$ are complete to Q in $G|_{\mathcal{K}(G)}$. But $N_{G|_{\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q) \subseteq Q_1$, thus it follows that $\widetilde{N}_{K_1}(Q_1) \setminus K = \emptyset$. But $\widetilde{N}_{K_1}(Q_1) \cap K \subseteq Q'$ with $Q' \in \mathcal{Q}(K)$, else $N_{G|_{\mathcal{K}(G)}}(Q_1) = \widetilde{N}_{K_1}(Q_1)$ would not be a clique in $G|_{\mathcal{K}(G)}(Q)$ contradicting Claim 7. It follows that $Q = Q'$ and thus $\widetilde{N}_{K_1}(Q_1) = Q$. Moreover, for each $t \in Q_1$, $\widetilde{N}_{K_1}(t)$ is non-empty and contained in Q, which implies $t \in \widetilde{N}_K(Q)$. Thus, $Q_1 = \widetilde{N}_K(Q)$. Summing up, we have that $N_{G|_{\mathcal{K}(G)}}(Q) = \widetilde{N}_K(Q)$, and $N_{G|_{\mathcal{K}(G)}}(\widetilde{N}_K(Q)) = N_{G|_{\mathcal{K}(G)}}(Q_1) = \widetilde{N}_{K_1}(Q_1) = Q$. The results follows.

Thus, we can suppose there exists no $w \in K \setminus Q$ that is complete to $\widetilde{N}_K(Q)$. We now prove that this case leads to contradiction, by showing that $Q \cup \widetilde{N}_K(Q)$ is an articulation clique in G, and thus Q is not a spike of type (iii). Note first that, in this case, $Q \cup \widetilde{N}_K(Q)$ is a maximal clique in G, and it is crucial for all vertices in Q by Claim 5 (v). We now show it is also crucial for all vertices from $\widetilde{N}_K(Q)$. For each $v \in \widetilde{N}_K(Q)$, let $T_v := N(v) \setminus (Q \cup \widetilde{N}_K(Q))$. If $T_v = \emptyset$, then v is simplicial and $Q \cup \widetilde{N}_K(Q)$ is crucial for v.

Now suppose that $T_v \cap K = \emptyset$, and note that this implies that T_v is anticomplete to Q, since $N[Q] = K \cup \widetilde{N}_K(Q)$. Applying Lemma 3.5 (1), it follows that $Q \cup \widetilde{N}_K(Q)$ is crucial for v.

Last, we suppose $T_v \cap K \neq \emptyset$. Let $S := (N(v) \cap K) \setminus Q$ (note $S \neq \emptyset$ by hypothesis). By Claim 8, $S \subseteq Q'$ for some $Q' \in \mathcal{Q}(K)$ distinct from Q, and for each bipartition H_1, H_2 of $N[v]$ into two maximal cliques, w.l.o.g. $Q \subseteq H_1, S \subseteq H_2$. Now, since we are in the case where no $w \in K \setminus Q$ is complete to $\widetilde{N}_K(Q)$, in particular no vertex in S is complete to $\widetilde{N}_K(Q)$, thus $S \cap H_1 = \emptyset$. Note that the unique maximal clique that contains Q in $N[v] \setminus S$ is $Q \cup \widetilde{N}_K(Q)$. Thus, also in this case $Q \cup \widetilde{N}_K(Q)$ is crucial for v.

Since $Q \cup \widetilde{N}_K(Q)$ is a maximal clique in G and it is crucial for all its vertices, it is an articulation clique of G, and this concludes the proof.

Claim 12. *$G|_{\mathcal{K}(G)}$ contains no net cliques.*

Proof. Let K be an articulation clique of $G|_{\mathcal{K}(G)}$. We will show in the following that K cannot be a net clique. Since $E(G|_{\mathcal{K}(G)}) \subseteq E$, K is a clique in G. By construction, K is not an articulation clique of G, unless K is a made of a unique spike, but then, the statement is trivial for spikes of type (i) or (ii) and it follows from Claim 7 for spike of type (iii), thus we can suppose that this does not hold.

Suppose first K is not maximal in G: then there exists a vertex w that is complete to K in G, while it is not complete to K in $G|_{\mathcal{K}(G)}$. This implies that there exist spikes $Q_1' \neq Q_2'$ of some articulation clique $K' \in \mathcal{K}(G)$ such that $w \in Q_2'$, $Q_1' \cap K \neq \emptyset$. This also implies that w is anticomplete to Q_1' in $G|_{\mathcal{K}(G)}$ and that $Q_2' \cap K = \emptyset$, otherwise K is not a clique in $G|_{\mathcal{K}(G)}$. Now suppose that K is a net clique in $G|_{\mathcal{K}(G)}$, this implies that there exists vertices $a_1, a_2, a_3 \in K$, $v_1, v_2, v_3 \notin K$ such that $a_i v_j \in E_{\mathcal{K}(G)}$ if and only if $i = j$, and $v_i v_j \notin E_{\mathcal{K}(G)}$ for each $i \neq j$. Recall (cfr. the proof of Lemma 4.1) that (S_1, S_2, S_3, S) is a partition of the vertices of K, for $S_i = N_{G|_{\mathcal{K}(G)}}(v_i) \cap K$, and $S = K \setminus (S_1 \cup S_2 \cup S_3)$. Since Q_1' is a clique in $G|_{\mathcal{K}(G)}$ and $N_{G|_{\mathcal{K}(G)}}(Q_1')$ is a clique in $G|_{\mathcal{K}(G)}$ (trivial if Q_1' is of type (i) or (ii) and by Claim 7 if of type (iii)), Q_1' intersects at most one set among S_1, S_2, S_3. Thus, we can suppose $Q_1' \cap (S_1 \cup S_2) = \emptyset$, which implies that $Q_1' \cap K$ is anticomplete to $\{v_1, v_2\}$ in $G|_{\mathcal{K}(G)}$. Now we show that w is complete to S_1 and S_2 in $G|_{\mathcal{K}(G)}$. Suppose to the contrary that w.l.o.g. it is not complete to S_1. Then $S_1' = S_1 \setminus N_{G|_{\mathcal{K}(G)}}(w) \neq \emptyset$ and there exist spikes $Q_1'' \neq Q_2''$ of some articulation clique $K'' \in \mathcal{K}(G)$ such that $w \in Q_2''$, $S_1' \subseteq Q_1''$ (S_1' is not in different spikes of K'', since else K would not be a clique of $G|_{\mathcal{K}(G)}$). Also $K'' \neq K'$ since otherwise some edges between S_1' and $Q_1' \cap K$ would be missing in $G|_{\mathcal{K}(G)}$. It follows that $w \in K'' \cap K'$ is a vertex of type (ii) and $N[w] = K'' \cup K'$. But again since Q_1'' and $N_{G|_{\mathcal{K}(G)}}(Q_1'')$ are cliques in $G|_{\mathcal{K}(G)}$, Q_1'' intersects at most one set among S_1, S_2, S_3 and thus this set is S_1. Therefore w is anticomplete to S_2 in G since no other spikes of K', K'' intersect K (again otherwise K would not be a clique in $G|_{\mathcal{K}(G)}$), but this is a contradiction. We can conclude in particular that $w \neq v_1, v_2$. Note now that w is adjacent to v_1, v_2 in $G|_{\mathcal{K}(G)}$ otherwise, for instance, $(a_1; v_1, w, q)$ is a claw in $G|_{\mathcal{K}(G)}$ for any $q \in Q_1 \cap K$, contradicting Claim 10. Thus $wv_1, wv_2 \in E(G|_{\mathcal{K}(G)})$, while $v_1 v_2 \notin E(G|_{\mathcal{K}(G)})$ by hypothesis. Recall that w belongs to some spike Q_2' from K', K' being an articulation clique of $\mathcal{K}(G)$. Again, both Q_2' and $N_{G|_{\mathcal{K}(G)}}(Q_2')$ are cliques in $G|_{\mathcal{K}(G)}$. So either v_1 or $v_2 \in Q_2'$, but not both: say w.l.o.g. $v_2 \in Q_2'$. Now as we already observed,

$Q_2' \cap K = \emptyset$ and thus $a_2 \notin Q_2'$. Therefore a_2, v_1 is a stable set of size 2 in $N_{G_{|\mathcal{K}(G)}}(Q_2')$, contradicting the fact that $N_{G_{|\mathcal{K}(G)}}(Q_2')$ is a clique.

Thus, K is maximal in G. This implies that, for each $u \in K$, $U[v] \subseteq K$. We show now that $K \in \mathcal{K}(G)$. Indeed suppose not: by maximality of K this means that K is non-crucial in G for some vertex $v \in K$. Note that it must be that $v \in V(\mathcal{K}(G))$. Indeed for each $u \notin V(\mathcal{K}(G))$, u is strongly regular in $G|_{\mathcal{K}(G)}$ (K is an articulation clique in $G|_{\mathcal{K}(G)}$) and thus K is crucial in G by Claim 9. Thus, $v \in K_1 \cap Q_1$ for some $K_1 \in \mathcal{K}(G)$, $Q_1 \in \mathcal{Q}(K_1)$. We have $K_1 \neq K$, otherwise we already have $K = K_1 \in \mathcal{K}(G)$, and we also have $K_1 \cap K \subseteq Q_1$, else K is not a clique in $G|_{\mathcal{K}(G)}$. Moreover Q_1 is of type (iii) since else $K \subseteq N_{G|_{\mathcal{K}(G)}}[v] = Q_1$, contradicting the fact that K is not contained in a spike. Let K_1, H be the unique pair of maximal cliques that cover $N[v]$ in G. Note that $\widetilde{N}_{K_1}(v) \supseteq K \setminus Q_1$. Suppose first that $\widetilde{N}_{K_1}(v) = K \setminus Q_1$ (in particular $\widetilde{N}_{K_1}(v) \neq \emptyset$ else $K = Q_1$). Then by Claim 5 (v), $(K \setminus Q_1) \cup U[v] = K \cup U[v], K_1$ is the unique pair of maximal cliques covering $N[v]$. As $U[v] \subseteq K$, this implies that K is crucial for v in G, a contradiction. Thus $\widetilde{N}_{K_1}(v) \setminus K \neq \emptyset$. Since $\widetilde{N}_{K_1}(v) = N_{G|_{\mathcal{K}(G)}}(v) \setminus Q_1$ is a clique in $G_{\mathcal{K}(G)}$ by Claim 7, $\widetilde{N}_{K_1}(v), Q_1$ is a pair of cliques that cover $N_{G|_{\mathcal{K}(G)}}(v)$. Expand them respectively to K', K'', as to be maximal in $G|_{\mathcal{K}(G)}$. Observe that since K is crucial for v in $G|_{\mathcal{K}(G)}$, one of K' or K'' is equal to K. But since we assumed $\widetilde{N}_{K_1}(v) \setminus K \neq \emptyset$ and K is maximal in G, this implies that $K = K''$ and thus $Q_1 \subseteq K$. Because we already observed $K_1 \cap K \subseteq Q_1$, it follows $Q_1 = K_1 \cap K$.

Now suppose that K is a net clique in $G_{|\mathcal{K}(G)}$. This implies again that there exist vertices $a_1, a_2, a_3 \in K$, $v_1, v_2, v_3 \notin K$ such that $a_i v_j \in E_{\mathcal{K}(G)}$ if and only if $i = j$, and $v_i v_j \notin E_{\mathcal{K}(G)}$ for each $i \neq j$ and that (S_1, S_2, S_3, S) is a partition of the vertices of K, for $S_i = N_{G|_{\mathcal{K}(G)}}(v_i) \cap K$, and $S = K \setminus (S_1 \cup S_2 \cup S_3)$. Since Q_1, $N_{G|_{\mathcal{K}(G)}}(Q_1)$ are cliques in $G|_{\mathcal{K}(G)}$, Q_1 again intersects at most one of the sets S_1, S_2, S_3 and thus w.l.o.g. $Q_1 \cap (S_1 \cup S_2) = \emptyset$. But since $Q_1 \subseteq K$, no vertex of S_3 belongs to Q_1, else v_3, a_1 is a stable set of size 2 in $N_{G_{|\mathcal{K}(G)}}(Q_1)$. Thus, $Q_1 \subseteq S$ is anticomplete to $\{v_1, v_2, v_3\}$. Let $w \in \widetilde{N}_{K_1}(v) \setminus K = N_{G|_{\mathcal{K}(G)}}(v) \setminus K$, which we assumed non-empty. Note that $w \neq v_1, v_2, v_3$ and w is non-complete to K in G, since otherwise we contradict the maximality of K. Since w is complete to $K \setminus Q_1$ in $G|_{\mathcal{K}(G)}$ by Claim 7), it follows that $wq \notin E_{\mathcal{K}(G)}$ for some $q \in Q_1$, and $wa_1, wa_2, wa_3 \in E_{\mathcal{K}(G)}$. Moreover, $wv_1, wv_2, wv_3 \in E$ since otherwise, for instance, $(a_1; v_1, q, w)$ is a claw in $G_{|\mathcal{K}(G)}$, contradicting Claim 10. But then $(w; v_1, v_2, v_3)$ is also a claw in $G_{|\mathcal{K}(G)}$, again a contradiction. This shows that K is maximal and crucial for each of its vertices in G, and thus $K \in \mathcal{K}(G)$. This is a contradiction, and concludes the proof.

Claim 10 implies part (i) of the statement.

We now show part (ii). Let $Q \in \mathcal{Q}(K)$ for some $K \in \mathcal{K}(G)$. By definition of ungluing, $Q \subset V(C)$ for some component C of $G|_{\mathcal{K}(G)}$. If the component coincides with Q, the statement is trivial, thus suppose that Q has non-empty neighborhood in $G|_{\mathcal{K}(G)}$. This implies that Q is of type (iii). If $Q \cup N_{G|_{\mathcal{K}(G)}}(Q)$ is a clique, Claim 11 implies that it is a clique-component of $G|_{\mathcal{K}(G)}$, and again the statement is trivial. So suppose it is not; as C is quasi-line, it has no net clique (by Claim 12) and thus no nets, and $N_{G|_{\mathcal{K}(G)}}(Q)$ is a clique (by Claim 7), we use Lemma A.1 and conclude that part (ii) holds true.

We now show part (iii). Let $C \in \mathcal{C}$. If C contains no spikes, then it is a connected component of G, contradicting the fact that G is connected. Thus C contains at least one spike. Now suppose it has at least three, say A_1, A_2,..., A_l for $l \geq 3$. Recall that, by Claim 4 and by definition of ungluing, they are disjoint. Moreover, again by definition of ungluing, we can assume they all are of type (iii).

Claim 13. *C is not a clique.*

Proof. Suppose the contrary. By construction, $A_1, A_2, ..., A_l$ are spikes from l different articulation cliques $K_1, K_2, ..., K_l \in \mathcal{K}(G)$, else C would not be a clique. Let $A_{l+1} := V(C) \setminus (A_1 \cup ... \cup A_l)$. Since $N_{G|_{\mathcal{K}(G)}}(A_i) = \widetilde{N}_{K_i}(A_i)$ for $i = 1, ..., l$, this implies that for $i = 1, ..., l$, $\widetilde{N}_{K_i}(A_i) = \cup_{j=1,...,l+1; j \neq i} A_j$. Now suppose there exists some vertex $w \in K_1 \setminus A_1$ that is complete to $\widetilde{N}_{K_i}(A_1) = A_2 \cup ... \cup A_{l+1}$. $w \in Q_1'$ for some $Q_1' \in \mathcal{Q}(K_1)$. Then by Claim 5 (iii), Q_1' is a spike of type (ii), i.e. $Q_1' = K_1 \cap K'$ for some $K' \in \mathcal{K}(G) \neq K_1$. In particular, A_2, A_3 are in the intersection of two different articulation cliques ($K' \neq K_2, K_3$ since $A_2 \cup A_3 \subseteq K'$ but $A_2 \cup A_3 \not\subseteq K_2, K_3$), contradicting the fact that they are spikes of type (iii). A similar argument works for $w \in K_j \setminus A_j$, for all $j = 1, ..., l$. Thus, by Claim 5 (v), for $i = 1, ..., l$, for each $v \in A_i$, the unique bipartition of $N[v]$ into two maximal cliques is given by $(K_i, \cup_{i=1,...,l+1} A_i)$ and consequently $\cup_{i=1,...,l+1} A_i$ is crucial for v in G. We now argue that it is also crucial in G for vertices of A_{l+1}. Pick $v \in A_{l+1}$, by definition, it does not belong to any articulation clique from $\mathcal{K}(G)$: then $N[v] = N_{G|_{\mathcal{K}(G)}}[v] = \cup_{i=1,\dots,l+1} A_i$, thus v is simplicial and consequently $\cup_{i=1,\dots,l+1} A_i$ is crucial for v in G. This shows that $\cup_{i=1,\dots,l+1} A_i$ is an articulation clique in G, a contradiction.

We now conclude the proof of part *(iii)* by showing that one among A_1, A_2, A_3 is such that $N_{G|_{\mathcal{K}(G)}}[A_i] = A_i \cup N_{G|_{\mathcal{K}(G)}}(A_i)$ is a clique and thus by Claim 11, C is a clique, contradicting Claim 13. Recall that, by definition, A_1, A_2, A_3 are disjoint, and by part *(ii)*, C is distance simplicial w.r.t. A_1, A_2 and A_3. For $i = 2, 3$, let j_i be the maximum integer such that $N_{j_i}(A_1) \cap A_i \neq \emptyset$, where $N_j(A_1)$ is the $j-$th neighborhood of A_1 in $G|_{\mathcal{K}(G)}$. Observe that $A_i \cap N_{j_i-k}(A_1) = \emptyset$ for all $i = 2, 3$ and $j_i \geq k \geq 2$.

Claim 14. $j_2 \neq j_3$.

Proof. Suppose $j_2 = j_3$: then $A_2 \subseteq N_{j_2}(A_1)$; this is trivial (by disjointness) if $j_2 = 1$, and if $j_2 > 1$, then the neighborhood of A_2 in C would contain a vertex from $N_{j_2-2}(A_1)$ and a vertex from $A_3 \cap N_{j_2}(A_1)$, contradicting the fact that $N_{G|_{\mathcal{K}(G)}}(A_2)$ is a clique. Similarly, $A_3 \subseteq N_{j_2}(A_1)$. Thus, $A_2 \cup A_3$ is a clique. As $N_{G|_{\mathcal{K}(G)}}(A_2)$, $N_{G|_{\mathcal{K}(G)}}(A_3)$ are cliques, $N_{G|_{\mathcal{K}(G)}}(A_2) \setminus A_3$ is complete to A_3 and $N_{G|_{\mathcal{K}(G)}}(A_3) \setminus A_2$ is complete to A_2. In particular this implies that $N(A_2) \cap N_{j_2+1}(A_1) = \emptyset$ and $N(A_3) \cap N_{j_2+1}(A_1) = \emptyset$, that $N_{G|_{\mathcal{K}(G)}}[A_2] = N_{G|_{\mathcal{K}(G)}}[A_3]$, thus $N_{G|_{\mathcal{K}(G)}}[A_2]$ is a clique, a contradiction.

From the previous claim, we can assume w.l.o.g. $j_2 \leq j_3 - 1$. As $N_{G|_{\mathcal{K}(G)}}(A_2)$ is a clique, A_2 is anticomplete to $N_{j_2+1}(A_1)$; as $N_{j_2+1}(A_1)$ is non-empty, also $N_{j_2}(A_1) \setminus A_2$ is non-empty, and it is complete to A_2. But because $N_{j_2}(A_1) \setminus A_2$ is non-empty, $A_2 \cap N_{j_2-1}(A_1) = \emptyset$, else $N_{G|_{\mathcal{K}(G)}}(A_2)$ is not a clique (it has 2 non adjacent neighbors in $N_{j_2-2}(A_1)$ and in $N_{j_2}(A_1) \setminus A_2$). Using again the fact that $N_{G|_{\mathcal{K}(G)}}(A_2)$ is a clique, we conclude that $N_{j_2-1}(A_1) \cap N_{G|_{\mathcal{K}(G)}}(A_2)$ is complete to $N_{j_2}(A_1) \setminus A_2$.

Moreover, $N_{j_2-1}(A_1) \subseteq N(A_2)$, otherwise $N_2(A_2)$ picks two non adjacent vertices in $N_{j_2-1}(A_1)$ and $N_{j_2+1}(A_1)$, contradicting the fact that A_2 is distance simplicial. Thus, $N_{j_2}(A_1) \setminus A_2$ is complete to $N_{j_2-1}(A_1)$.

We now show that $N_{G|_{\mathcal{K}(G)}}[A_2]$ is a clique, thus concluding the proof of part *(iii)*. Suppose it is not: from what argued above, the only possibility is that there exists $u \in A_2$, $w \in N_{j_2-1}(A_1) \subseteq N_{G|_{\mathcal{K}(G)}}(A_2)$ with $uw \notin E(C)$. Now pick $z \in N_{j_2+1}(A_1)$ and $t \in N_{j_2}(A_1) \cap N_{G|_{\mathcal{K}(G)}}(z)$. By construction, $wz, uz \notin E(C)$, while $tu, tw, tz \in E(C)$, thus $(t; u, w, z)$ is a claw, a contradiction.

We are left with part *(iv)*. First observe that the set of strips $\{(C, \mathcal{A}(C)), C \in \mathcal{C}\}$ is well-defined, since by part *(iii)* for each $C \in \mathcal{C}$, $\mathcal{A}(C)$ is a multi-set with one or two cliques. Let G' be the graph obtained by composing $\{(C, \mathcal{A}(C)), C \in \mathcal{C}(G|_{\mathcal{K}(G)})\}$ with respect to the partition $\mathcal{P}$ that puts two extremities in the same class if and only if they are spikes from a same articulation clique. By definition of ungluing, $\cup\{V(C) : C \in \mathcal{C}\}$ partitions V, thus $V(G) = V(G')$. By definition of composition, two vertices u, v of G' are adjacent if and only if $uv \in E(C)$ for some C, or $u \in A_1, v \in A_2$ and A_1, A_2 belong to the same set of the partition $\mathcal{P}$. By the definition of $\mathcal{P}$, this implies that $uv \in E(G')$ if and only if $uv \in E(G)$. Thus $G' = G$ and we conclude the proof.

B Proofs of Section 5

(*We point out that Theorem 5.2 builds in a non-trivial way upon Lemma 12 in [16].*) Both the proof of Lemma 5.1 and Theorem 5.2 require a few preliminary lemmas about the iterative removal of hyper-line strips from a claw-free graph G.

So let G be a claw-free graph and $(H, \mathcal{A})$ an hyper-line strip of G. Following Definition 5.2 (and with a little abuse of notation, as we should write $G|_{\{(H,\mathcal{A})\}}$), we let $G|_{(H,\mathcal{A})}$ be the graph such that:

- $V(G|_{(H,\mathcal{A})}) = V(G) \setminus C(H, \mathcal{A})$;
- $E(G|_{(H,\mathcal{A})}) = \{uv \in E : u, v \in V(G|_{(H,\mathcal{A})})\} \setminus \{uv \in E : u \in A_1, v \in A_2, A_1 \neq A_2 \in \mathcal{A}\}$.

The following lemma summarizes a few properties of the graph $G|_{(H,\mathcal{A})}$.

Lemma B.1. *Let G be a graph and $(H, \mathcal{A})$ an hyper-line strip of G. Then:*

(i) *$G|_{(H,\mathcal{A})}$ is claw-free.*

(ii) *A vertex of G that is regular and belongs to $G|_{(H,\mathcal{A})}$ remains regular.*

(iii) *A vertex of G that is irregular and belongs to $G|_{(H,\mathcal{A})}$ remains irregular. In particular, if W is a 5-wheel of G centered in $a \notin C(H, \mathcal{A})$, then W is also a 5-wheel of $G|_{(H,\mathcal{A})}$.*

(iv) *The set of simplicial vertices of* $G|_{(H,\mathcal{A})}$ *is given by* $S(G) \cup \{v \in A, A \in \mathcal{A}\}$*. Moreover, if* v *is a simplicial vertex of* G*, then its neighborhood is the same in* G *and* $G|_{(H,\mathcal{A})}$*.*

(v) *If* K *is an articulation clique of* G *that does not take vertices from* $C(H,\mathcal{A})$*, then* K *is an articulation clique of* $G|_{(H,\mathcal{A})}$*.*

(vi) *If* K *is an articulation clique of* $G|_{(H,\mathcal{A})}$*, then it is also an articulation clique of* G*.*

(vii) *If* $(\overline{H},\overline{\mathcal{A}})$ *is an hyper-line strip of* $G|_{(H,\mathcal{A})}$ *that is vertex disjoint from* $(H,\mathcal{A})$*, then it is also an an hyper-line strip of* G*.*

(viii) *If* $(\overline{H},\overline{\mathcal{A}})$ *is an hyper-line strip of* G *that is vertex disjoint from* $(H,\mathcal{A})$*, then it is also an an hyper-line strip of* $G|_{(H,\mathcal{A})}$*.*

Proof. The statements easily follow from a few remarks. First of all, the vertices that belong to an extremity $A \in \mathcal{A}$ are simplicial vertices of $G|_{(H,\mathcal{A})}$. As for a vertex v of $G|_{(H,\mathcal{A})}$ that does not belong to any extremity $A \in \mathcal{A}$, note that it has the same neighborhood in G and $G|_{(H,\mathcal{A})}$. In particular, if $(H,\mathcal{A})$ is a 1-strip, also the adjacencies between vertices in $N(v)$ are unchanged; if $(H,\mathcal{A})$ is a 2-strip with disjoint extremities, then the adjacencies between vertices in $N(v)$ change only if $v \in K(A_1) \cap K(A_2)$. In this case, v is a strongly regular (non-simplicial) vertex of G, and it is also a strongly regular (non-simplicial) vertex of $G|_{(H,\mathcal{A})}$, as it is still adjacent to vertices of A_1 and A_2, that are simplicial and therefore define articulation cliques of $G|_{(H,\mathcal{A})}$. Statements $(i) - (iv)$ easily follow.

Consider now an articulation clique K of G. As we already pointed out, if $K \in \{A \cup K(A), A \in \mathcal{A}\}$, then K is still an articulation cliques of $G|_{(H,\mathcal{A})}$. Now suppose that $K \notin \{A \cup K(A), A \in \mathcal{A}\}$; as we discussed above, the adjacencies between vertices in $N(v)$ change only if $v \in K(A_1) \cap K(A_2)$. Clearly, no such a vertex belong to K, as in this case it will belongs to three articulation cliques of G. Therefore K is also an articulation cliques of $G|_{(H,\mathcal{A})}$. So statement (v) follows, and by reversing this argument also statement (vi) holds. Finally, let $(\overline{H},\overline{\mathcal{A}})$ be an hyper-line strip of $G|_{(H,\mathcal{A})}$ that is vertex disjoint from $(H,\mathcal{A})$. Observe that the vertices of $\overline{H}$ induce the same subgraph in G and $G|_{(H,\mathcal{A})}$, and therefore $\overline{H}$ is an induced subgraph of G. Then, the fact that $(\overline{H},\overline{\mathcal{A}})$ is also an an hyper-line strip of G, i.e. statement (vii), easily follows from statement (vi). Analogously, statement $(viii)$ easily follows from statement (v).

Observation 2. *Let* $\mathcal{H} = \{(H^1,\mathcal{A}^1),\dots,(H^t,\mathcal{A}^t)\}$ *be a family of vertex disjoint hyper-line strips of a claw-free graph* G^0*. For* $i = 1,\dots t,$*, let* $G^i = G^{i-1}|_{(H^i,\mathcal{A}^i)}$ *(note that each* $(H^i,\mathcal{A}^i)$ *is an hyper-line strip of* G^{i-1} *because of the last statement in Lemma B.1). Define* $G|_{\mathcal{H}}$ *according to Definition 5.2. It is easy to check that then the graph* $G^t \equiv G|_{\mathcal{H}}$*.*

In other words, either we *simultaneously* remove a set of strip, or we remove them *sequentially*, we get the same graph.

Proof of Lemma 5.1. Given the above observation, the proof of the lemma easily follows by induction from statements (iv) and (vi) of Lemma B.1. □

We now move to the proof of Theorem 5.2. We start with a couple of classical lemmas.

Lemma B.2. [7] *Let* G *be a connected claw-free graph* G *with* $\alpha(G) \geq 4$*. For each vertex* $v \in V(G)$*, each odd-hole in* $\overline{G[N(v)]}$ *has length five. In particular, if* G *does not contain a* 5*-wheel, it is quasi-line.*

Lemma B.3. [12] *Let* $G(V,E)$ *be a claw-free graph with an induced* 5*-wheel centered in* $a \in V$*. Then* $\alpha(a \cup N(a) \cup N_2(a)) \leq 3$*.*

The next lemma then easily follows.

Lemma B.4. *Let* G *be a connected claw-free graph but not quasi-line graph with* n *vertices. In time* $O(n^3)$ *we may either recognize that* $\alpha(G) \leq 3$*, or build, for each irregular vertex, a* 5*-wheel* $W(a)$ *centered in* a*.*

Proof. We know from Lemma 3.6 that in time $O(n^3)$ we may find, for each irregular vertex v, an odd k-anti-wheel $W(a)$ centered in v, $k \geq 5$. If there exists a vertex a such that $W(a)$ is a long odd anti-wheel, then $\alpha(G) \leq 3$ by Lemma B.2.

The core of the proof is the following fundamental lemma, that investigates the structure of a claw-free graph with a 5−wheel.

LEMMA B.5. *Let $G(V,E)$ be a connected and claw-free graph and $a \in V$ the center of a 5-wheel W of G. Then*

(i) *either G has no simplicial vertex and $\alpha(G) \leq 3$;*

(ii) *or there exists an hyper-line strip $(H,\mathcal{A})$ such that $a \in V(H)$, no vertex of H is simplicial and $\alpha(H) \leq 3$.*

Moreover, if we are given the set $S(G)$, we can decide whether G satisfies (i) or find the hyper-line strip $(H,\mathcal{A})$ from point (ii) in time $O(n^2)$.

Proof. We postpone the complexity issues to the end of the proof, and start by showing that each graph G that fulfills the hypothesis, satisfies conditions (i) or (ii) of the statement. In order to do that, we have to gather some more information on the structure of G. In the following, we denote the 5-wheel centered in a by $W = (a; u_1, u_2, u_3, u_4, u_5)$. Also, for $i \in [5]$, we denote by S_i the set of vertices in $N_2(a)$ whose adjacent vertices in W are exactly u_i, u_{i+1} (where we identify u_6 with u_1) and such that they either have a neighbor in $N_3(a)$, or they are simplicial. We also let $\widetilde{N_2}(a)$ be the set of vertices in $N_2(a) \setminus \bigcup_{i=1..5} S_i$.

We now investigate some properties of the graph G in the first three neighborhoods of the irregular vertex a.

CLAIM 15. *Let v be a vertex of $N_2(a)$. The following statements hold:*

(i) *The vertices of $\{u_1, u_2, u_3, u_4, u_5\}$ that are adjacent to v are at least two and they have consecutive indices.*

(ii) *If v has a neighbor in $N_3(a)$ or is simplicial, then v has exactly two neighbors in $\{u_1, u_2, u_3, u_4, u_5\}$, and they have consecutive indices.*

Proof. We first prove that v has at least one neighbor in $\{u_1, u_2, u_3, u_4, u_5\}$. By contradiction, suppose there exists $v \in N_2(a)$ that is anticomplete to $\{u_1, u_2, u_3, u_4, u_5\}$. Since $v \in N_2(a)$, there exists $u \notin W$ such that $au \in E$ and $uv \in E$. Such a u must be adjacent to at least three consecutive vertices in $\{u_1, u_2, u_3, u_4, u_5\}$, otherwise there would exist a claw centered in a and picking u and two non-adjacent vertices. Thus w.l.o.g. let $uu_1 \in E$, $uu_2 \in E$, $uu_3 \in E$. But then there is a claw: $(u; u_1, v, u_3)$.

Now observe that if v is adjacent to some vertex in $\{u_1, u_2, u_3, u_4, u_5\}$, say u_1, then it is adjacent to u_5 or u_2 too, otherwise there would exist a claw: $(u_1; v, u_2, u_5)$. Statement (i) easily follows.

Now suppose that v has a neighbor $x \in N_3(a)$. Observe that v cannot be adjacent to two non-adjacent vertices in $\{u_1, u_2, u_3, u_4, u_5\}$, say u_1 and u_3, otherwise there would exist a claw: $(v; x, u_1, u_3)$. It follows that v has exactly two neighbors in $\{u_1, u_2, u_3, u_4, u_5\}$, and they have consecutive indices. Similarly if v is simplicial it cannot be adjacent to two non-adjacent vertices in $\{u_1, u_2, u_3, u_4, u_5\}$ and thus it follows that v has exactly two neighbors in $\{u_1, u_2, u_3, u_4, u_5\}$, and they have consecutive indices

From Claim 15, it follows that the only vertices from $N_2(a)$ with an adjacent in $N_3(a)$ are those from $\bigcup_{i=1..5} S_i$.

CLAIM 16. *If v is a simplicial vertex in a claw-free graph G, and G has an induced 5−wheel W with center a, then $v \notin \{a\} \cup N(a) \cup \widetilde{N_2}(a)$.*

Proof. First observe that all vertices of a 5−wheel are non-simplicial. Now let $u \in N(a) \setminus W$. In order to prevent claws, u is adjacent to two non-consecutive vertices in the 5−wheel, and thus it is not simplicial. Last, take a simplicial vertex $v \in N_2(a)$; since it has to be adjacent to at least two vertices from $u_1, \dots, u_5$ by Claim 15, in order to be simplicial it must be adjacent to exactly two consecutive vertices from $u_1, \dots, u_5$, say u_1, u_2. Then, by definition, $v \in S_1$, which implies $v \notin \widetilde{N_2}(a)$.

CLAIM 17. *If $\bigcup_{i=1..5} S_i = \emptyset$, we are in case (i) of the statement.*

Proof. In this case, $V = \{a\} \cup N(a) \cup \widetilde{N_2}(a)$. By Claim 16, G has no simplicial vertices; By Lemma B.3, G has stability number at most 3.

Thus, in the following, we can suppose that $\bigcup_{i=1,\dots,5} S_i \neq \emptyset$.

Claim 18. *For $i = 1, 2, \dots, 5$, the set $S_i \cup S_{i+1}$ is a clique.*

Proof. Suppose the contrary, that is, there exist $x, y \in S_i \cup S_{i+1}$ that are not adjacent. Then, there would be the claw: $(u_{i+1}; a, x, y)$.

Claim 19. *For $i = 1, \dots, 5$, the set $S_i \cup (N(S_i) \cap (N(a) \cup \widetilde{N_2}(a)))$ is a clique.*

Proof. W.l.o.g. we prove this claim for S_1 (we can assume $S_1 \neq \emptyset$ otherwise it is trivial). For sake of shortness, let $Q = N(S_1) \cap (N(a) \cup \widetilde{N_2}(a))$. We know from the above claim that S_1 is a clique. We now show that every vertex in S_1 is complete to Q. Suppose the contrary: then there exist $x \in N(a) \cup \widetilde{N_2}(a)$, $x \neq u_1, u_2$, and $y, z \in S_1$ such that $xy \in E$ and $xz \notin E$. As y is non-simplicial ($z, x \in n(y)$ and $zx \notin E$), it has a neighbor in $N_3(a)$, say w. Observe that $wx, wu_1, wu_2 \notin E$, therefore x must be adjacent to u_1 and u_2: else, say $xu_1 \notin E$, there would be the claw $(y; x, u_1, w)$. Moreover, in order to avoid the claws $(u_1; u_5, x, z)$ and $(u_2; u_3, x, z)$, it follows that u_5x and $u_3x \in E$. But then $(x; u_3, u_5, y)$ is a claw.

Finally we show that Q is a clique. Suppose the contrary. There exists $v, x \in Q$ that are not adjacent. We have just shown that S_1 is complete to Q, thus let $y \in S_1$, and we have that xy and $vy \in E$. As y is non-simplicial, there exists a vertex w of $N_3(a)$ that is adjacent to y, then there is the claw $(y; x, v, w)$.

Claim 20. *Let $s \in S_i$ for some $i \in \{1, .., 5\}$. $N_3(a) \cap N(s)$ is a clique.*

Proof. Suppose there exists $x, y \in N_3(a) \cap N(s)$ with $xy \notin E$. Let $z \in N(s) \cap N(a)$, then $(s; x, y, z)$ is a claw.

Claim 21. *Let $s \in S_i$ and $t \in S_j$ for some $i \neq j \in \{1, .., 5\}$. If $st \in E$, then $N_3(a) \cap N(s) = N_3(a) \cap N(t)$.*

Proof. Suppose that there exists $x \in N_3(a) \cap N(s)$ and $xt \notin E$. Since $i \neq j$, there exists $y \in \{u_1, ..., u_5\}$ such that $y \in N(s) \setminus N(t)$. But then $(s; x, y, t)$ is a claw.

Claim 22. *Let $\mathcal{S}$ be the union of at least two non-empty subsets S_i. If $\mathcal{S}$ is a clique, then $\mathcal{S} \cup (N_3(a) \cap N(\mathcal{S}))$ is a clique.*

Proof. Suppose that $S_i \cup S_j \subseteq \mathcal{S}$, $i \neq j$. For all $s \in S_i$, $t \in S_j$, $i \neq j$, $N_3(a) \cap N(s) = N_3(a) \cap N(t)$ by Claim 21. If we iterate this argument, we can conclude that each vertex $s \in \mathcal{S}$ has the same neighbors in $N_3(a)$. Finally, by Claim 20, $N_3(a) \cap N(s)$ is a clique and therefore $\mathcal{S} \cup (N_3(a) \cap N(\mathcal{S}))$ is a clique.

We now switch back to the statement we want to prove. By hypothesis and because of the properties of sets S_i shown above, we are in exactly one of the following cases.

1. There is a single set $S_1, \dots, S_5$ that is non-empty.
2. The set $\bigcup_{i=1..5} S_i$ is not a clique and the sets $S_1, \dots, S_5$ that are non-empty are two.
3. The set $\bigcup_{i=1..5} S_i$ is not a clique and the sets $S_1, \dots, S_5$ that are non-empty are three and non-consecutive.
4. The sets $\bigcup_{i=1..5} S_i$ is a clique and the sets $S_1, \dots, S_5$ that are non-empty are at least two.
5. The sets $\bigcup_{i=1..5} S_i$ is not a clique, and the sets S_i that are non-empty are consecutive, at least three.

We are now going to show that in cases $1-4$, we satisfy condition (ii) of the statement, while in case 5 we satisfy condition (i) of the statement. More precisely, we show that in cases $1-4$ there exists a strip $(H, \mathcal{A})$ with disjoint extremities such that H is an induced subgraph of G and the following properties hold:

(j) $C(H, \mathcal{A})$ is anticomplete to $V \setminus V(H)$;

(jj) For $A \in \mathcal{A}$, $A \cup K(A)$ is an articulation clique of G;

(jjj) $a \in V(H)$ and $\alpha(H) \leq 3$;

(jv) $S(G) \cap V(H) = \emptyset$;

and that for case 5, $\alpha(G) \leq 3$ and $S(G) = \emptyset$.

Let us consider case 1. Assume w.l.o.g. that $S_1 \neq \emptyset$. In this case, we set $H = G[\{a\} \cup N(a) \cup \widetilde{N}_2(a)]$, $A_1 = N(S_1) \cap (N(a) \cup \widetilde{N}_2(a))$. We then consider the strip $(H, \{A_1\})$. Then, statement (j) holds by construction, (jjj) holds by construction and by Lemma B.3, (jv) holds by Claim 16. We are left with showing (jj). Note first that $K(A_1) = S_1$. First observe that $A_1 \cup K(A_1)$ is a clique, because of Claim 19, and it is maximal by construction. If $A_1 \cup K(A_1)$ contains a simplicial vertex, then it is an articulation clique by Lemma 16, thus suppose it has none. Then each vertex of S_1 has an adjacent in $N_3(a)$, which is anticomplete to A_1 by construction, thus condition (1) from Lemma 3.5 holds for those vertices. Now fix $v \in A_1$; as v is not simplicial, it has a neighbor w not in A_1, which is by construction anticomplete to S_1. Condition (1) from Lemma 3.5 also holds for those vertices, and this concludes the proof.

Let us consider case 2. Assume w.l.o.g. that $S_1, S_3 \neq \emptyset$, and let $s_1 \in S_1$, $s_3 \in S_3$ be a pair of non-adjacent vertices. Observe that $N(a) \cap N(S_1) \cap N(S_3) = \emptyset$, since any vertex from this set, say v, would be the center of the claw $(v; s_1, s_3, a)$. Now let $Q = N(S_1) \cap N(S_3) \cap \widetilde{N}_2(a)$. Set $H = G[\{a\} \cup N(a) \cup \widetilde{N}_2(a) \setminus Q]$, $A_1 = (N(S_1) \cap (N(a) \cup \widetilde{N}_2(a))) \setminus Q$, $A_2 = (N(S_3) \cap (N(a) \cup \widetilde{N}_2(a))) \setminus Q$. As $N(a) \cap N(S_1) \cap N(S_3) = \emptyset$, then A_1, A_2 are disjoint. We consider the strip $(H, \{A_1, A_2\})$. Then, statement (jjj) holds by Lemma B.3, (jv) holds by Claim 16. If (j) does not hold, then there must be vertex v in Q that is adjacent to some vertex $w \in C(H, \mathcal{A})$, since S_1 and S_3 are, by construction, anticomplete to $C(H, \mathcal{A})$. Thus $(v; w, s_1, s_3)$ is a claw, a contradiction. We are left to show (jj). Note that $K(A_1) = S_1 \cup Q$ and $K(A_2) = S_3 \cup Q$. First observe that $A_1 \cup K(A_1)$ and $A_2 \cup K(A_2)$ are cliques, because of Claim 19, and they are maximal by construction. We are left to show that they are articulation cliques. We show the statement for $A_1 \cup K(A_1)$, since the same argument holds for $A_2 \cup K(A_2)$. The proof builds on Lemma 3.5, i.e. we show that each vertex in $A_1 \cup K(A_1)$ fits either case (1) or case (2) of the lemma.

We first need to investigate $N(Q)$. By construction, $N(Q) \cap N_3(a) = \emptyset$. Note first that Q is complete to $(N(a) \cup \widetilde{N}_2(a)) \cap N(S_1)$, since those vertices are in the clique A_1, and similarly Q is complete to $(N(a) \cup \widetilde{N}_2(a)) \cap N(S_3)$. Now we show that $N(Q) \setminus (N(S_1) \cup N(S_3)) = \emptyset$: suppose the contrary, i.e. there exists $w \in N(Q) \setminus (N(S_1) \cup N(S_3))$, then $(q; w, s_1, s_3)$ is a claw, for a vertex $q \in Q$ such that $qw \in E$. Thus, $N[Q] = ((N(a) \cup \widetilde{N}_2(a)) \cap N(S_1)) \cup ((N(a) \cup \widetilde{N}_2(a)) \cap N(S_1)) = A_1 \cup K(A_1) \cup A_2 \cup K(A_2)$. This implies that all vertices of Q are regular and copies of one another.

Now fix $v \in Q$, and set $X_1 = S_1$, $X_2 = A_1$, $Y_1 = S_3$, $Y_2 = A_2$: X_1 is non-complete to Y_1 by hypothesis; since $N(a) \cap N(S_1) \cap N(S_3)$ is empty, and $Q \cap A_1, Q \cap A_2 = \emptyset$, X_2 is anticomplete to Y_1 and Y_1 is anticomplete to X_2. As $X_1 \cup X_2 = A_1 \cup K(A_1) \setminus Q$ and $Y_1 \cup Y_2 = N(Q) \setminus (A_1 \cup K(A_1))$, and we already argued that vertices from Q are copies, v satisfies case (2) of Lemma 3.5. Next, pick $v \in S_1$. If v is simplicial, we are fine. So suppose not. Then note that $N(v) \setminus (A_1 \cup K(A_1)) \subseteq (S_3 \cup N_3(a))$, which implies that $N(v) \setminus (A_1 \cup K(A_1))$ is anticomplete to u_1; thus, v satisfies case (1) of Lemma 3.5. Now pick $v \in (A_1 \cup K(A_1)) \setminus (Q \cup S_1) \subseteq N(a) \cup \widetilde{N}_2(a)$. v is not simplicial by Claim 16. Note that thus $N(v) \setminus (A_1 \cup K(A_1))$ is non-empty and it is anticomplete to S_1 by construction. Thus, v satisfies case (1) of Lemma 3.5. Thus we conclude that $A_1 \cup K(A_1)$ is an articulation clique.

Let us consider case 3. Assume w.l.o.g. that $S_1, S_2, S_4 \neq \emptyset$. Also, recall that $S_4 \cup (N(S_4) \cap (N(a) \cup \widetilde{N}_2(a)))$ is a clique by Claim 19. As we show in the following, each vertex in S_4 is either complete or anticomplete to $S_1 \cup S_2$. Indeed, suppose that $s_4 \in S_4$ has a non-adjacent in $S_1 \cup S_2$, say w.l.o.g. $s_1 \in S_1$. It follows that s_4 is anti-complete to S_2, otherwise there exists $s_2 \in S_2 \cap N(s_4)$ and $(s_2; s_4, s_1, u_3)$ is a claw. Applying a similar reasoning, s_4 is anti-complete to S_1. Thus we can partition S_4 in $(\bar{S}_4, \bar{\bar{S}}_4)$, where the vertices in $\bar{S}_4$ are those complete to $S_1 \cup S_2$. Note that $\bar{S}_4$ may be empty, while $\bar{\bar{S}}_4$ is not by hypothesis; moreover, they are both cliques by Claim 18. Claims 20 and 21 imply that $T \cup (N(T) \cap N_3(a))$ is a clique, for $T = S_1 \cup S_2 \cup \bar{S}_4$. Finally, the vertices in $S_1 \cup S_2$ are not simplicial, and therefore have neighbors in $N_3(a)$. Let $Q = N(T) \cap N_3(a)$, and note that Q is a non-empty clique.

Now set $H = G[\{a\} \cup N(a) \cup \widetilde{N}_2(a) \cup S_1 \cup S_2]$, $A_1 = S_1 \cup S_2$, $A_2 = N(S_4) \cap (N(a) \cup \widetilde{N}_2(a)))$ and we consider the strip $(H, \{A_1, A_2\})$. First, observe that A_1, A_2 are vertex disjoint. Then, statement (j) holds by construction, statement (jjj) holds by Lemma B.3, (jv) holds by Claim 16 and because no vertex in $S_1 \cup S_2$ is simplicial. We are left to show (jj). Note that $K(A_1) = \bar{S}_4 \cup Q$ and $K(A_2) = S_4$.

Now observe that $A_1 \cup K(A_1) = T \cup (N(T) \cap N_3(a))$ and thus it is a clique. We can also conclude that $A_2 \cup K(A_2)$ is a clique using Claim 19. By construction, they are both maximal.

We now show that $A_1 \cup K(A_1)$ is an articulation clique. Again, we use Lemma 3.5. Note that we can suppose that no vertex in $A_1 \cup K(A_1)$ is simplicial, otherwise the statement immediately follows from Lemma 3.4: since $A_1 \cup K(A_1)$ is a maximal clique, this implies that each vertex in $A_1 \cup K(A_1)$ has a neighbor outside $A_1 \cup K(A_1)$. Now pick a vertex $v \in S_1$, and note that $N(v) \setminus (A_1 \cup K(A_1))$ is contained in $N(a) \cup \widetilde{N}_2(a)$, and thus it is anticomplete to $N_3(a) \cap N(S_1) \subseteq A_1 \cup K(A_1)$, which we already shown is non-empty. Thus, v satisfies case (1) of Lemma 3.5. Similarly for $v \in S_2$. Now take a vertex v in Q; as it is not simplicial, $\emptyset \neq N(v) \setminus (A_1 \cup K(A_1)) \subseteq N_4(a) \cup (N_3(a) \setminus Q) \cup \bar{\bar{S}}_4$, which implies that $N(v) \setminus (A_1 \cup K(A_1))$ is anti-complete to $S_1 \cup S_2$ and again v satisfies case (1). Last, take a vertex v in $\bar{S}_4$. As $N(\bar{S}_4) \subseteq Q \cup S_1 \cup S_2 \cup \bar{\bar{S}}_4 \cup A_2$, v is regular. First observe that all vertices of $\bar{S}_4$ are copies, since $\bar{S}_4$ is a clique and $N[\bar{S}_4] = A_1 \cup K(A_1) \cup A_2 \cup K(A_2)$. Then fix $v \in \bar{S}_4$: case (2) of Lemma 3.5 applies with $X_1 = A_1$, $X_2 = Q$, $Y_1 = N(S_4) \cap (N(a) \cup \widetilde{N}_2(a))$, $Y_2 = \bar{\bar{S}}_4$. Thus we conclude that $A_1 \cup K(A_1)$ is an articulation clique.

We now show that $A_2 \cup K(A_2)$ is an articulation clique: again, we use Lemma 3.5 for the proof. We can again suppose that each vertex of $A_2 \cup K(A_2)$ has a neighbor outside $A_2 \cup K(A_2)$. For a vertex in $\bar{S}_4$, we set $X_1 = A_2$, $X_2 = \bar{\bar{S}}_4$, $Y_1 = A_1$, $Y_2 = Q$, and conclude that case (2) applies. Now take a vertex v in A_2: each neighbor of v that is not in $A_2 \cup K(A_2)$ is anticomplete to $\bar{\bar{S}}_4$, so case (1) applies. Take a vertex v in $\bar{\bar{S}}_4$: each neighbor of v that is not in $A_2 \cup K(A_2)$ belongs to $N_3(a)$, and therefore it is anticomplete to $u_4 \in A_2$. This shows that case (1) applies also in this case, and consequently that $A_2 \cup K(A_2)$ is an articulation clique.

Let us consider case 4. Note that, in this case, each non-empty S_i is made of non-simplicial vertices, and therefore each vertex in some S_i has a neighbor in $N_3(a)$. By Claim 22, it also follows that $N_3(a)$ is a clique and it is complete to $\bigcup_{i=1..5} S_i$. In this case, we set $H = G[\{a\} \cup N(a) \cup N_2(a)]$, $A_1 = \bigcup_{i=1..5} S_i$ and we consider the strip $(H, \{A_1\})$. Statement (j) holds by construction, (jjj) by Lemma B.3, (jv) by Claim 16 and because each non-empty S_i is made of non-simplicial vertices. We are left with statement (jj): let $K(A_1) = N_3(a)$. We already argued that $A_1 \cup K(A_1)$ is a clique, it is maximal by construction, and again we can suppose it has no simplicial vertex. In particular, each vertex in $K(A_1)$ has a neighbor in $N_4(a)$, which is by definition anticomplete to A_1. As each vertex in A_1 has a neighbor in $N(a) \cup \widetilde{N_2}(a)$ that is anticomplete to $K(A_1)$, we apply case (i) of Lemma 3.5 to conclude that $A_1 \cup K(A_1)$ is an articulation clique.

Let us now consider case 5. As we already mentioned, we are going to show that $\alpha(G) \leq 3$ and $S(G) = \emptyset$. Assume w.l.o.g. that $S_1, S_2, \ldots, S_k$, $k \geq 3$, are non-empty, with either $k = 5$ or $S_{k+1} = \emptyset$. By iteratively applying Claims 18 and 22, it follows that $N_3(a)$ is complete to $S_1 \cup S_2 \cup \ldots \cup S_k$, and that $N_3(a)$ is a clique. Now observe that $N_4(a) = \emptyset$. In fact, otherwise let z be a vertex of $N_3(a)$ that has some adjacent $w \in N_4(a)$. By hypothesis, there exist $x, y \in S_1 \cup S_2 \cup \ldots \cup S_k$ that are not adjacent, then there would be the claw $(z; w, x, y)$.

Let s_1, s_2, s_3 be vertices in respectively S_1, S_2, S_3. We now show that $\widetilde{N_2}(a) = \emptyset$. Suppose the contrary and let $z \in \widetilde{N_2}(a)$. Observe that $\{u_1, u_2, u_3, u_4\} \subseteq N(S_1) \cup N(S_2) \cup N(S_3)$. On the other hand, $\{u_1, u_2, u_3, u_4\} \cap N(z)$ is non-empty from part (i) of Claim 15. It is a routine to check that then $\{u_1, u_2, u_3, u_4, s_1, s_2, s_3\} \subseteq N(z)$. In fact, suppose e.g. that $u_1 z \in E$: then $s_1 z \in E$ in order to avoid the claw $(u_1; a, s_1, z)$ and $u_2 z \in E$ in order to avoid the claw $(s_1; u_2, z, w)$, where $w \in N_3(a)$ is adjacent to s_1. If we iterate this argument, we can show that indeed $\{u_1, u_2, u_3, u_4, s_1, s_2, s_3\} \subseteq N(z)$. But this leads to a contradiction, since $(z; u_1, u_4, s_2)$ is a claw. It follows that $N_2(a) = S_1 \cup S_2 \cup \ldots \cup S_k$ and therefore $N_2(a)$ is complete to $N_3(a)$. We know from Lemma B.3 that $\alpha(G[\{a\} \cup N(a) \cup N_2(a)]) \leq 3$. If $\alpha(G) \geq 4$, then there must exist a stable set S of size 4 picking exactly one vertex in $N_3(a)$ (since $N_3(a)$ is a clique and we showed $N_4(a) = \emptyset$). It follows that $|S \cap (\{a\} \cup N(a))| = 3$, which is a contradiction.

We are left to show that no vertex in G is simplicial. Recall that in this case $V = \{a\} \cup N(a) \cup (\bigcup_{i=1..5} S_i) \cup N_3(a)$. No vertex of $\{a\} \cup N(a)$ is simplicial by Claim 16. No vertex of $\bigcup_{i=1..5} S_i$ is simplicial, since we already argued that each vertex of $\bigcup_{i=1..5} S_i$ has a neighbor in $N_3(a)$. Last, observe that no vertex in $N_3(a)$ is simplicial, since $N_3(a)$ is complete to $\bigcup_{i=1..5} S_i$, that is not a clique by hypothesis.

We now move to complexity issues. We can compute the sets $N_j(a)$ for $j = 1, 2, 3$ in time $O(m)$. Recall that, by definition, for $i \in [5]$, S_i is the subset of $N_2(a)$ formed by those vertices a) whose neighbors in W are exactly u_i and u_{i+1}, and such that $b1$) they have a neighbor in $N_3(a)$, or $b2$) they are simplicial. Given a vertex of $N_2(a)$, we can check conditions a), $b1$), $b2$) in time $O(n)$ (recall that we are given the set $S(G)$). Thus $O(n^2)$ is sufficient to build sets $S_1, \dots, S_5$ and $\widetilde{N_2}(a)$, and to check for each pair i, j, if $S_i \cup S_j$ is a clique. If $\bigcup_{i=1..5} S_i = \emptyset$, from Claim 17 we are in case (i), thus we can suppose $\bigcup_{i=1..5} S_i \neq \emptyset$. Then we can distinguish between cases $1-5$ and construct the strip $(H, \mathcal{A})$ with the required properties in time $O(n^2)$.

Proof of Theorem 5.2. Let G be a claw-free but not quasi-line graph G. Using Lemma B.4 and the trivial fact that the set $S(G)$ can be computed in $O(n^3)$-time, it suffices to show that, given

- for each irregular vertex of G, a 5-wheel $W(a)$ centered in a;
- for some irregular vertex a, an hyper-line strip $(H, \mathcal{A})$ such that $a \in V(H)$, no vertex of H is simplicial and $\alpha(H) \leq 3$;
- the set $S(G)$ of simplicial vertices of G;

we can build in time $O(n^3)$ a family $\mathcal{H}$ of vertex disjoint hyper-line strips of G such that:

- each strip in $\mathcal{H}$ contains a 5-wheel of G and has stability number at most 3;
- $G|_{\mathcal{H}}$ is quasi-line and non-empty.

So, let G^1 be the graph $G|_{(H,\mathcal{A})}$. Note that G^1 is not necessarily connected. Let C be a component of $G|_{(H,\mathcal{A})}$. By (i) of Lemma B.1 C is claw-free; by $(ii)-(iii)$ of the same lemma either C is quasi-line, or we have, for each irregular vertex a of C, a 5-wheel $W(a)$ centered in a. Suppose that C is not quasi-line, and pick an irregular vertex a. Observe that, by construction, C has some vertex from the extremities of $(H, \mathcal{A})$, and therefore a simplicial vertex. Hence, it follows from Lemma B.5 that there exists an hyper-line strip $(H_1, \mathcal{A}_1)$ such that $a \in V(H_1)$, no vertex of H_1 is simplicial and $\alpha(H_1) \leq 3$. Note in particular, that $(H, \mathcal{A})$ and $(H_1, \mathcal{A}_1)$ are vertex disjoint: this is because the vertices of the core of $(H, \mathcal{A})$ do not belong to $G|_{(H,\mathcal{A})}$, while the vertices of its extremities are simplicial.

Then we proceed by induction and define a series $G^1, \dots, G^t$ of graphs such that, for $i \in [t]$, $G^i = G^{i-1}|_{(H^i,\mathcal{A}^i)}$ (we let $G^0 := G$), where, for $i \in [t]$:

- $(H^i, \mathcal{A}^i)$ is an hyper-line strip of G^{i-1}, that is vertex disjoint from $(H^1, \mathcal{A}^1), \dots, (H^{i-1}, \mathcal{A}^{i-1})$;
- $(H^i, \mathcal{A}^i)$ contains a 5-wheel of G, has stability number at most 3 and no simplicial vertices;
- G^t is quasi-line.

Note, in particular, that the graph G^t is non-empty, since the removal of each strip produces some simplicial vertex, none of which belongs to any hyper-line strip that we remove, so they belong to G^t. Also recall that, if let $\mathcal{H} = \{(H^1, \mathcal{A}^1), \dots, (H^t, \mathcal{A}^t)\}$, then by Observation 2, $G^t \equiv G|_{\mathcal{H}}$. In order to conclude the proof, we must show that the family $\{(H^i, \mathcal{A}^i)\}_{i=1}^t$ can be produced in time $O(n^3)$. Trivially, $t \leq n$, since the strips are vertex disjoint. Also, using statement (iv) of Lemma B.1, for each graph $i = 1, \dots, t$, the set $S(G^i)$ can be easily built from the set $S(G^{i-1})$. We already observed that, by part (iii) of Lemma B.1, we are given a 5−wheel for each irregular vertex in G^i. Thus, Lemma B.5 guarantees that we can build the family $\mathcal{H}$ in $O(n^3)$-time.