On Sampling a High-Dimensional Bandlimited Field on a Union of Shifted Lattices

Jayakrishnan Unnikrishnan and Martin Vetterli
Audiovisual Communications Laboratory, School of Computer and Communication Sciences.
Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland
Email: {jay.unnikrishnan, martin.vetterli}@epfl.ch

***Abstract*—We study the problem of sampling a high-dimensional bandlimited field on a union of shifted lattices under certain assumptions motivated by some practical sampling applications. Under these assumptions, we show that simple necessary and sufficient conditions for perfect reconstruction can be identified. We also obtain an explicit scheme for reconstructing the field from its samples on the various shifted lattices. We illustrate our results using examples.**

I. Introduction

Consider the problem of sampling and reconstructing a d-dimensional field $f(r) : r \in \mathbb{R}^d$, where r represents a d-dimensional spatial location. If f is bandlimited, the classical sampling theorem of Petersen and Middleton [1] provide schemes for sampling and reconstructing the field from its values measured on a lattice in $\mathbb{R}^d$. Various results on more general sampling configurations are also known (see, e.g., [2] [3] for a summary). One example of such a sampling configuration is a union of shifted lattices. Sampling on such sets arises naturally in a number of applications in signal processing and remote sensing [4], in interlaced sampling schemes in tomography [5], in recovery of functions from their local averages [6], and also in spatial sampling using mobile sensors [7] . Such a configuration is easy to analyze [8] when the points lie on a union of shifted versions of the same lattice. Recently more general results on sampling on a union of shifted lattices have been obtained in an abstract harmonic analysis framework [9] but, to the best of our knowledge, there are no works that study the general problem in the classical signal processing context.

In this paper, we study the problem of sampling on a union of shifted lattices, under certain assumptions on the lattices. We describe practical spatial sampling scenarios where these assumptions are satisfied. Under these assumptions, we identify simple necessary and sufficient conditions for perfect reconstruction of bandlimited fields. We also provide an explicit reconstruction scheme for perfectly recovering the bandlimited field from the measurements taken at the sampling points. Our solution is easier to understand and implement than the more general result on sampling on a union of lattices in the abstract harmonic analysis literature.

A. Notation

For vectors $x, y \in \mathbb{R}^d$ we use $\langle x, y\rangle$ to denote the inner-product in Euclidean space. For any set $\Omega \subset \mathbb{R}^d$ and $r \in \mathbb{R}^d$ we use $\overset{\circ}{\Omega}$ to denote the interior of Ω in Euclidean space, $\chi_\Omega(.)$ to denote the characteristic function of Ω, and $\Omega(r)$ to denote a translated version of Ω, defined as:

$$\Omega(r) := \{x \in \mathbb{R}^d : x - r \in \Omega\}.$$

B. Paper organization

We begin by providing background on related problems in Section II. In Section III we discuss the problem of sampling on unions of shifted lattices and present our results and examples. We conclude in Section IV.

II. Background and related results

We denote a field in d-dimensional space by a complex-valued mapping $f : \mathbb{R}^d \mapsto \mathbb{C}$. For a field $f(.)$, we define its Fourier transform $F(.)$ as

$$F(\omega) = \int_{\mathbb{R}^d} f(r) \exp(-\mathrm{i}\langle \omega, r\rangle) dr, \qquad \omega \in \mathbb{R}^d$$

where i denotes the imaginary unit, and $\langle u, v\rangle$ denotes the scalar product between vectors u and v in $\mathbb{R}^d$. We use $\mathcal{B}_\Omega$ to denote the collection of fields with finite energy such that the Fourier transform F of f is supported on a set $\Omega \subset \mathbb{R}^d$, i.e.,

$$\mathcal{B}_\Omega := \{f \in L^2(\mathbb{R}^d) : F(\omega) = 0 \text{ for } \omega \notin \Omega\}. \qquad (1)$$

A bandlimited field has the advantage that it is possible to represent the field using only the samples of the field measured at a discrete collection of points. We say that a collection of points Λ forms a *sampling set* for Ω if any field $f \in \mathcal{B}_\Omega$ can be uniquely reconstructed using only its values on the points in Λ. We use $\mathcal{S}_\Omega$ to denote the collection of all sampling sets for Ω. We now discuss known results on conditions for sampling and reconstruction of bandlimited fields.

A. Sampling on a lattice

The most well-known result on sampling of bandlimited fields deals with sampling on a lattice. A lattice Λ in $\mathbb{R}^d$ is defined as a collection of points of the form

$$\Lambda = \{\sum_{i=1}^{d} m_i v_i : m \in \mathbb{Z}^d\}$$

where $\{v_i : 1 \le i \le d\}$ forms a basis for $\mathbb{R}^d$. We say that the lattice Λ is generated by the vectors $\{v_i : 1 \le i \le d\}$. The necessary and sufficient conditions on Λ for reconstructing

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a field f in $\mathcal{B}_\Omega$ using the samples of the field taken on Λ follow from the results of [1]. Let $\{u_i : 1 \le i \le d\}$ denote a reciprocal set of lattice vectors in $\mathbb{R}^d$ satisfying $\langle u_i, v_j \rangle = 2\pi\delta_{ij}$. Then the associated sampled field spectrum is composed of spectral repetitions of f and is supported on the set $\bigcup_{m\in\mathbb{Z}^d} \Omega\left(\sum_{i=1}^{d} m_i u_i\right)$. If the spectral repetitions overlap in the Fourier domain we say that the samples are *aliased* and in this case it is not possible to perfectly reconstruct the field from its samples. The repetitions do not overlap provided

$$\Omega \cap \Omega(\sum_{i=1}^{d} m_i u_i) = \emptyset \text{ for all } m \in \mathbb{Z}^d \setminus \{0\}^d.$$

This is the condition required to ensure that any field $f \in \mathcal{B}_\Omega$ can be reconstructed based on samples on the lattice Λ. For example if a field in $\mathbb{R}^2$ is bandlimited to a set Ω in the form of a circular disc of radius ρ, then it can be reconstructed exactly by sampling on a rectangular lattice with $v_1 = (X, 0)$ and $v_2 = (0, Y)$, provided $X < \frac{\pi}{\rho}$ and $Y < \frac{\pi}{\rho}$.

B. Sampling on a union of shifts of a lattice

A shift of a lattice Λ is a set of the form $\Lambda + w : \{x + w : x \in \Lambda\}$ where w is a vector representing the shift. A union of such shifted versions of the same lattice Λ gives a more general configuration of sampling points, e.g., $\bigcup_{k=1}^{N}(\Lambda + w^k)$ where $w^k \in \mathbb{R}^d$ are the shifts. Such a configuration of sampling points is also known by other names such as *periodic sampling* or *bunched sampling*. Various authors have studied such sampling configurations and identified schemes for perfect reconstruction from samples taken on such a configuration of points (see, e.g., [5] and references therein). One approach for studying such sampling problems is to view this as a special case of Papoulis' *generalized sampling* [10] in higher dimensions. Since a shift in the spatial domain can be interpreted as a filtering operation, the samples obtained from each individual lattice in the union can be viewed as samples of a filtered version of the field taken at points on the original lattice Λ. Thus this problem can be interpreted as a multichannel sampling problem and solved using Papoulis' approach [10] as outlined in [8].

A more general sampling configuration is a union of shifted versions of lattices that are not necessarily identical. In the following section, we consider this setting in detail. We describe known results and then present new results.

III. Sampling on a union of shifted lattices

Consider lattices $\Lambda_1, \Lambda_2, \ldots, \Lambda_N$ in $\mathbb{R}^d$ where

$$\Lambda_k := \{\sum_{j=1}^{d} m_j v_j^k : m \in \mathbb{Z}^d\}$$

where for each k, the vectors $\{v_j^k : 1 \le j \le d\}$ forms a basis for $\mathbb{R}^d$. For each k let $\{u_j^k : 1 \le j \le d\}$ denote a reciprocal set of lattice vectors defined by the relation

$$\langle u_j^k, v_i^k \rangle = 2\pi\delta_{ij} \text{ for all } k.$$

We now study the problem of sampling bandlimited fields $f \in \mathcal{B}_\Omega$ on the union of shifted versions of these lattices given by

$$U = \bigcup_{k=1}^{N} (\Lambda_k + w^k) \qquad (2)$$

where $w^k \in \mathbb{R}^d$ represent the shifts. This problem has been studied by various authors in one dimension [11] and higher dimensions [9], [12]. However, in higher dimensions, the conditions and algorithms for perfect reconstruction in [9] are stated for locally compact abelian groups under an abstract harmonic analysis framework. In this section, we show that under certain conditions on the shifted lattices, this problem admits a simple and explicit solution.

As we saw in Section II-A, the spectrum of the sampled field obtained by samples taken on each individual lattice Λ_k is composed of spectral repetitions in d directions. We also saw that if the repetitions in the sampled spectrum on a lattice do not overlap, then the original field can be reconstructed exactly from the samples on this lattice. The same reasoning holds for sampling on a shifted lattice since a spatial shift of the field $f \in \mathcal{B}_\Omega$ does not change the support of its Fourier transform. However, while sampling a field $f \in \mathcal{B}_\Omega$ on a union of shifted lattices U, it may be possible to perfectly reconstruct the field f based on all the samples even if the samples on each individual shifted lattice are aliased (e.g., see Example 3.2 later in the paper). In this section we present some conditions and schemes for perfect recovery. Let $\mathcal{Q} \subset \mathbb{R}^d$ be defined as

$$\mathcal{Q} := \left\{ \sum_{k=1}^{N} (-1)^{n_k} \frac{u_1^k}{2} : n_k \in \{0,1\}, 1 \le k \le N \right\}.$$

The following proposition gives a simple necessary condition on U and Ω for perfect reconstruction.

Proposition 3.1: Let $\Omega \subset \mathbb{R}^d$ be a compact set and let U denote a union of shifted lattices of the form (2). Suppose $\mathcal{Q} \subset \mathring{\Omega}(s)$ for some $s \in \mathbb{R}^d$. Then $U \notin \mathcal{S}_\Omega$.

Proof: Consider the field

$$f_\epsilon(r) = \exp(-\mathrm{i}\langle s, r\rangle) \prod_{k=1}^{N} \sin\left(\tfrac{1}{2}\langle u_1^k, r - w^k\rangle\right) \prod_{\ell=1}^{d} \frac{\sin(r_\ell \epsilon)}{r_\ell \epsilon}, \quad r \in \mathbb{R}^d.$$

It is easily verified that the field f_ϵ vanishes at all points on U. Hence f_ϵ cannot be distinguished from the function that is identically zero using the field samples on U. Moreover, for ϵ small enough, the field f_ϵ has a Fourier transform supported within a subset of Ω. Thus $U \notin \mathcal{S}_\Omega$. ■

Under certain assumptions the simple necessary conditions stated above are also sufficient to ensure that $U \in \mathcal{S}_\Omega$. We require the following assumptions on U and Ω:

(A1) Ω is a compact convex set.

(A2) For each $k \in \{1, 2, \ldots, N\}$ the vectors $\{v_j^k : j \geq 2\}$ are small enough so that the following condition holds

$$\Omega \cap \Omega(\sum_{j=1}^{d} m_j u_j^k) = \emptyset \text{ for all } m \in \tilde{\mathbb{Z}}^d$$

where $\tilde{\mathbb{Z}}^d := \{m \in \mathbb{Z}^d : m_j \neq 0 \text{ for some } j \geq 2\}$.

The assumption **(A2)** is equivalent to saying that each lattice Λ_k is finely sampled along the directions $\{v_j^k : j \geq 2\}$ so that the overlaps in the corresponding sampled spectrum, if any, arise due to repetitions in the direction u_1^k. Hence under assumption **(A2)** the spectrum from each shifted lattice $\Lambda_k + w^k$ is aliased at most in one direction - the direction of u_1^k.

Assumption **(A1)** is often satisfied in practical applications of sampling low-pass phenomena. We now discuss practical sampling scenarios where the assumption **(A2)** is satisfied. Suppose that the process of sampling the field is accomplished by measuring the field values along hyperplanes at fine resolutions. For a field in $\mathbb{R}^2$, this means that the measurement procedure involves scanning the field along straight lines at high resolutions. Such a scenario is well motivated in spatial sampling using mobile sensors [7]. A mobile sensor moving along a straight line can sample the field at high resolutions along its path without incurring any additional cost in terms of distance traveled. Now consider a measurement setup in which the measurements are taken by sensors moving along a sequence of equispaced parallel lines. We refer to such a sequence of equispaced parallel lines as a uniform set of lines, like in [7]. If the samples taken on the various lines are themselves aligned then the collection of all such samples lies on a shifted lattice in $\mathbb{R}^2$. If we have a collection of several such uniform sets, then the collection of all samples lie on a union of shifted lattices in $\mathbb{R}^2$. Furthermore, since the measurements are taken at a fine resolution along the lines assumption **(A2)** is satisfied. An example of such a sampling configuration is shown in Figure 1(a). Similarly, one could consider a scheme for measuring fields in $\mathbb{R}^3$ on several uniform sets of planes. If the samples from the planes on a uniform set are aligned, then they form a shifted lattice in $\mathbb{R}^3$. The collection of all samples from several such uniform sets lie on a union of shifted lattices in $\mathbb{R}^3$ and since the measurements are taken at a fine resolution along the planes, it follows that assumption **(A2)** is satisfied.

Motivated by the practical examples provided above, we show that, under assumptions **(A1)** and **(A2)**, the necessary conditions from Proposition 3.1 are also sufficient for perfectly reconstructing bandlimited fields in $\mathcal{B}_\Omega$ from their measurements on U. We have the following theorem.

Theorem 3.2: Let U be a union of shifted lattices in $\mathbb{R}^d$ defined in (2), and $\Omega \subset \mathbb{R}^d$ satisfy[1] Assumptions **(A1)** and **(A2)**. Suppose that the vectors $\{u_1^1, u_1^2, \ldots, u_1^N\}$ are linearly independent. Then we have

$$U \in \mathcal{S}_\Omega \text{ if } \mathcal{Q} \not\subseteq \Omega(s) \text{ for all } s \in \mathbb{R}^d, \text{ and} \tag{3}$$

$$U \notin \mathcal{S}_\Omega \text{ if } \mathcal{Q} \subset \mathring{\Omega}(s) \text{ for some } s \in \mathbb{R}^d. \tag{4}$$

Sketch of proof: The result of (4) follows from Proposition 3.1. The sufficient condition of (3) is proved by demonstrating an explicit reconstruction strategy. This follows by the same steps as used to prove [7, Thm 4.3]. The proof is long and hence omitted. We refer the reader to [7] for details. ■

Now that we have identified the conditions for perfect reconstruction, we now seek explicit reconstruction schemes for reconstructing the bandlimited fields assuming that the sufficient condition of (3) holds. The reconstruction scheme can be obtained by studying the spectra of the sampled fields. In the sequel we use the following abusive notation for $n \in \mathbb{Z}^d$. We use $\langle n, v^k \rangle$ to denote $\sum_{\ell=1}^{d} n_\ell v_\ell^k$ and $\langle n, u^k \rangle$ to denote $\sum_{\ell=1}^{d} n_\ell u_\ell^k$. Let f_s^k denote the sampled impulse stream from the k^{th} shifted lattice $\Lambda_k + w^k$ defined as

$$f_s^k(r) = \sum_{n \in \mathbb{Z}^d} f(\langle n, v^k \rangle + w^k)\delta(r - \langle n, v^k \rangle - w^k) \tag{5}$$

where $\delta(.)$ represents the Dirac-delta function in d-dimensions. It follows from [1] that the Fourier transform of this sampled impulse stream is given by

$$F_s^k(\omega) = \sum_{n \in \mathbb{Z}^d} \exp(\mathrm{i} \sum_{\ell=1}^{d} n_\ell \langle w^k, u_\ell^k \rangle) F(\omega + \langle n, u^k \rangle), \omega \in \mathbb{R}^d.$$

Under assumption **(A2)** this spectrum satisfies

$$F_s^k(\omega) = \sum_{n \in \mathbb{Z}} \exp(\mathrm{i} n \langle w^k, u_1^k \rangle) F(\omega + n u_1^k), \omega \in \Omega. \tag{6}$$

In the following proposition we provide the structure of the reconstruction scheme. To maintain continuity the proof is relegated to the appendix.

Proposition 3.3: Let $f \in \mathcal{B}_\Omega$ denote a bandlimited field. Under the conditions of Theorem 3.2, there exists a partition of Ω into mutually disjoint sets Ω_ℓ, $\ell = 1, 2, \ldots, L$, integers $1 \leq m_\ell < \infty$, and scalars $\beta_{i,k,n}$ for $i \in \{1, \ldots, L\}, j \in \{1, \ldots, m_i\}, k \in \{1, \ldots, N\}, n \in \mathbb{Z}^d$ such that

$$f(r) = \sum_{n \in \mathbb{Z}^d} \sum_{i=1}^{L} \sum_{k=1}^{N} \beta_{i,k,n} f(\langle n, v^k \rangle + w^k) \hat{\chi}_{\Omega_i}(r - \langle n, v^k \rangle - w^k), r \in \mathbb{R}^d \tag{7}$$

where $\hat{\chi}_{\Omega_i}(.)$ represents the inverse Fourier transform of the characteristic function of Ω_i. □

Thus the interpolation formula of (7) can be used to reconstruct the bandlimited field $f \in \mathcal{B}_\Omega$ using only its samples. The exact values of $\beta_{i,k,n}$ and Ω_i depends on the specific problem at hand. Thus (7) represents the interpolation formula for reconstructing the field f from its measurements on a union of shifted lattices U under assumptions **(A1)** and **(A2)**. We now discuss some examples where we can explicitly evaluate the values of $\beta_{i,k,n}$ and Ω_i.

[1] A different set of conditions for perfect reconstruction can be obtained without Assumption **(A1)** provided a stronger version of **(A2)** holds.

Example 3.1 (Rectangular lattices and isotropic field): Suppose Ω is a circular disc of radius ρ centered at the origin, corresponding to an isotropic bandlimited fields. And suppose U is a union of shifted lattices as in (2) where Λ_1 and Λ_2 are both rectangular lattices with basis vectors

$$v_1^1 = (0, \Delta), v_2^1 = (\epsilon, 0), v_1^2 = (\Delta, 0), v_2^2 = (0, \epsilon). \quad (8)$$

Assume that the lattices are not shifted so that $w^1 = w^2 = 0$. The sampling configuration U is illustrated in Figure 1(a) where Λ_1 is shown in black and Λ_2 in red. If $\epsilon < \frac{\pi}{\rho}$ then Assumption **(A2)** holds. Suppose that this condition holds. From Theorem 3.2, it follows that perfect reconstruction is possible (i.e. $U \in \mathcal{S}_\Omega$) whenever $\Delta < \Delta^* := \frac{\sqrt{2}\pi}{\rho}$. Under this condition the sampled spectra from the two lattices are as depicted in Figure 1(b). We have shown only the supports of the spectral repetitions that intersect with the main lobe. Based on the sampled spectra, we can partition Ω into five distinct regions as shown in Figure 1(c). Here Ω_1 represents the portion of Ω that is not aliased in either sampled spectrum, Ω_2 and Ω_3 represent the portions that are aliased only in the sampled spectrum from the first lattice and Ω_4 and Ω_5 are the portions aliased only in the sampled spectrum from the second lattice. Hence the sampled spectra satisfy

$$\begin{aligned} F_s^1(\omega) &= F(\omega), \qquad \omega \in \Omega_1 \cup \Omega_4 \cup \Omega_5 \\ F_s^2(\omega) &= F(\omega), \qquad \omega \in \Omega_2 \cup \Omega_3. \end{aligned}$$

Thus $F(.)$ can be recovered from the sampled spectra as

$$\begin{aligned} &F(\omega) \\ &= F_s^1(\omega)\chi_{\Omega_1 \cup \Omega_4 \cup \Omega_5}(\omega) + F_s^2(\omega)\chi_{\Omega_2 \cup \Omega_3}(\omega) \\ &= \sum_{n \in \mathbb{Z}^2} f(\langle n, v^1 \rangle) \exp(-\mathrm{i} \sum_{\ell=1}^{2} n_\ell \langle v_\ell^1, \omega \rangle) \chi_{\Omega_1 \cup \Omega_4 \cup \Omega_5}(\omega) + \\ &\qquad \sum_{n \in \mathbb{Z}^2} f(\langle n, v^2 \rangle) \exp(-\mathrm{i} \sum_{\ell=1}^{2} n_\ell \langle v_\ell^2, \omega \rangle) \chi_{\Omega_2 \cup \Omega_3}(\omega). \end{aligned}$$

Taking inverse Fourier transforms we have

$$\begin{aligned} f(r) &= \sum_{n \in \mathbb{Z}^2} f(\langle n, v^1 \rangle) \hat{\chi}_{\Omega_1 \cup \Omega_4 \cup \Omega_5}(r - \langle n, v^1 \rangle) + \\ &\qquad \sum_{n \in \mathbb{Z}^2} f(\langle n, v^2 \rangle) \hat{\chi}_{\Omega_2 \cup \Omega_3}(r - \langle n, v^2 \rangle) \end{aligned}$$

which is the interpolation formula for perfectly reconstructing the field from the samples on the two lattices. This is the equivalent of (7) for the current example. □

In the above example every part of Ω was unaliased in at least one of the sampled spectra and thus reconstruction was straightforward. However, in some cases it is possible to reconstruct the field exactly even when this condition does not hold, as we illustrate in the following example.

Example 3.2 (Rectangular lattices and non-isotropic field): Suppose $\Omega := \{\omega \in \mathbb{R}^2 : \omega_y \geq 0, |\omega_x| + |\omega_y| \leq \rho\}$ is a triangular region. As before let U be a union of two rectangular lattices Λ_1 and Λ_2 with basis vectors given by

$$v_1^1 = (\Delta, 0), v_2^1 = (0, 2\epsilon), v_1^2 = (0, 2\Delta), v_2^2 = (\epsilon, 0).$$

Suppose $\epsilon < \frac{\pi}{\rho}$ which ensures that Assumption **(A2)** holds. From Theorem 3.2, it follows that perfect reconstruction is possible (i.e. $U \in \mathcal{S}_\Omega$) whenever $\Delta < \Delta^* := \frac{2\pi}{\rho}$. Under this condition, the partitions of Ω are as shown in Figure 2. We see that unlike in the previous example now there are some portions of Ω that are aliased in both the sampled spectra. Now the sampled spectra satisfy the following relations:

$$\begin{aligned} F_s^1(\omega) &= F(\omega), \quad \omega \in \Omega_1 \cup \Omega_2 \cup \Omega_3 \\ F_s^2(\omega) &= F(\omega), \quad \omega \in \Omega_4 \cup \Omega_5 \\ F_s^2(\omega) &= F(\omega) + F(\omega + u_1^2), \quad \omega \in \Omega_6 \cup \Omega_7. \end{aligned}$$

Thus the original spectrum can be recovered as

$$\begin{aligned} F(\omega) &= F_s^1(\omega)\chi_{\Omega_1 \cup \Omega_2 \cup \Omega_3}(\omega) + F_s^2(\omega)\chi_{\Omega_4 \cup \Omega_5}(\omega) \\ &\qquad + (F_s^2(\omega) - F_s^1(\omega + u_1^2))\chi_{\Omega_6 \cup \Omega_7}(\omega) \end{aligned}$$

and the original field can be recovered by inverting the Fourier spectrum as in the previous example. □

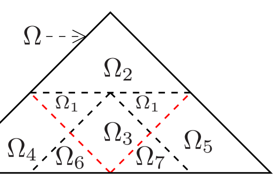

Fig. 2. Partition of Ω in Example 3.2.

IV. Conclusion

We have studied the problem of sampling and reconstructing bandlimited fields on a union of shifted lattices under assumptions **(A1)** and **(A2)**. These assumptions are well justified in some practical problems of interest such as the sampling of spatial fields using mobile sensors. Under these assumptions we provide simple conditions for perfect reconstruction that are easier to verify than those provided in known results [9] on the general problem of sampling on a union of shifted lattices. We also provide explicit reconstruction schemes that are easier to implement than the known iterative reconstruction schemes for the general problem.

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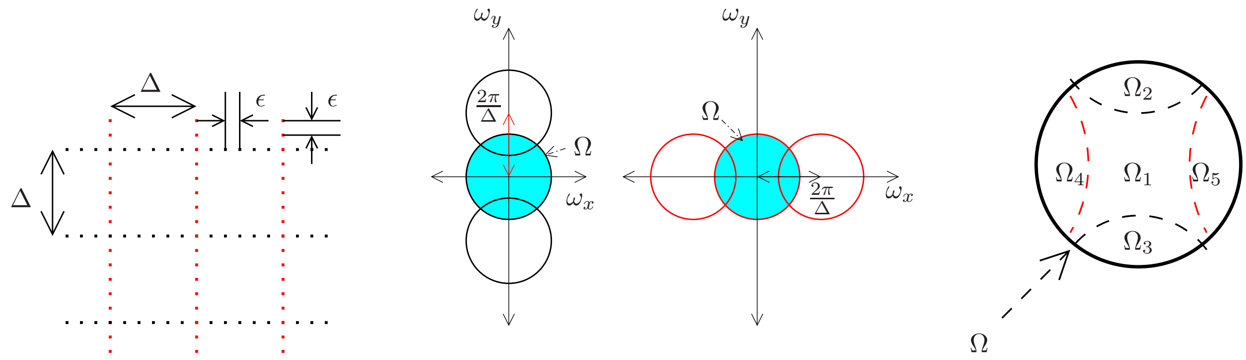
Appendix

Proof of Proposition 3.3: For $\omega \in \Omega$ let $\{x^1(\omega), x^2(\omega), \ldots, x^{m_\omega}(\omega)\}$ be all the elements $x \in \mathbb{Z}^N$ for which $\omega \in \Omega(\sum_{k=1}^N x_k u_1^k)$. Without loss of generality, we assume $x^1(\omega)$ to be 0. Let

$$I(\omega) := \{x \in \mathbb{Z}^N : \omega - \sum_{k=1}^{N} x_k u_1^k \in \Omega\}$$

or equivalently, $I(\omega) = \{x^1(\omega), x^2(\omega), \ldots, x^{m_\omega}(\omega)\}$. Also let $W(\omega) := \{\omega - \sum_{k=1}^N x_k u_1^k : x \in I(\omega)\}$. Grouping together



(a) Union of two rectangular lattices. (b) Sampled spectra from the two lattices. (c) Partition of Ω

Fig. 1. Sampled spectra from two rectangular lattices and the associated partitioning of Ω for Example 3.1.

the $\omega \in \Omega$ for which the set $I(\omega)$ coincide yields a partition of Ω into mutually disjoint subsets $\Omega_\ell, \ell = 1, 2, \ldots, L$. Stated formally the decomposition reads: $\Omega = \cup_{\ell=1}^{L} \Omega_\ell$ with

$$\forall \ell \in \{1, \ldots, L\} \quad \text{there exists} \quad M_\ell \subset \mathbb{Z}^N, \quad M_\ell = \{0 = x^{\ell 1}, \ldots, x^{\ell m_\ell}\}$$

such that

$$\forall \omega \in \Omega_\ell, x \in \mathbb{Z}^N : (\omega - \sum_{k=1}^{N} x_k u_1^k) \in \Omega \Leftrightarrow x \in M_\ell.$$

From (6) along with the definitions above, it follows that for any $\omega \in \Omega$, the sampled spectra $F_s^k(\zeta) : \zeta \in W(\omega), k \in \{1, 2, \ldots, N\}$ measured on $W(\omega)$ can be expressed as linear combinations of the original spectrum $F(\zeta) : \zeta \in W(\omega)$ measured on $W(\omega)$. Clearly, for any set Ω_i the structure of these equations are identical for all $\omega \in \Omega_i$. Furthermore, by Theorem 3.2 it follows that these equations can be solved to obtain the value of the original spectrum $F(\omega)$ at ω as a linear combination of $F_s^k(\zeta) : \zeta \in W(\omega), k \in \{1, 2, \ldots, N\}$. Thus it follows that there exists scalars $\tilde{\beta}_{i,j,k} : i, j \in \{1, 2, \ldots, L\}, k \in \{1, 2, \ldots, N\}$ such that for all $i \in \{1, 2, \ldots, L\}$ we have

$$F(\omega)\chi_{\Omega_i}(\omega) = \sum_{k=1}^{N} \sum_{j=1}^{m_i} \tilde{\beta}_{i,j,k} F_s^k(\omega - \sum_{q=1}^{N} x_q^{ij} u_1^q) \chi_{\Omega_i}(\omega).$$

Summing up over all i we get

$$F(\omega) = \sum_{i=1}^{L} \sum_{j=1}^{m_i} \sum_{k=1}^{N} \tilde{\beta}_{i,j,k} F_s^k(\omega - \sum_{q=1}^{N} x_q^{ij} u_1^q) \chi_{\Omega_i}(\omega).$$

Substituting the Fourier transform of (5) above, we obtain

$$F(\omega) = \sum_{i=1}^{L} \sum_{j=1}^{m_i} \sum_{k=1}^{N} \tilde{\beta}_{i,j,k} \chi_{\Omega_i}(\omega) \sum_{n \in \mathbb{Z}^d} f(\langle n, v^k \rangle + w^k) \exp\left(-\mathrm{i} \left\langle (\sum_{\ell=1}^{d} n_\ell v_\ell^k + w^k), (\omega - \sum_{q=1}^{N} x_q^{ij} u_1^q) \right\rangle \right).$$

Taking inverse transform we get (7) where $\hat{\chi}_{\Omega_i}(.)$ represents the inverse Fourier transform of $\chi_{\Omega_i}(.)$, and

$$\beta_{i,k,n} = \sum_{j=1}^{m_i} \tilde{\beta}_{i,j,k} \exp\left(\mathrm{i} \sum_{q=1}^{N} x_q^{ij} \left\langle \sum_{\ell=1}^{d} n_\ell v_\ell^k + w^k, u_1^q \right\rangle \right).$$

■

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