A data-driven approach to mixed-sensitivity control with application to an active suspension system

Simone Formentin and Alireza Karimi

***Abstract*—In this paper, a data-driven approach is proposed to tune fixed-order controllers for unknown stable LTI plants in a mixed-sensitivity loop-shaping framework. The method requires a single set of input-output samples and it is based on convex optimization techniques; moreover, it asymptotically guarantees the internal stability of the closed-loop system. The effectiveness of the method is illustrated with application to the control of an active suspension system.**

***Index Terms*—data-driven control, identification for control, Youla-Kučera parameterization, mixed $\mathcal{H}_2$-$\mathcal{H}_\infty$ loop-shaping, convex optimization**

I. Introduction

One of the main methods to specify the desired behaviour of a control system is to describe the frequency-domain relations between the different signals in the loop. In robust control theory, $\mathcal{H}_2$-$\mathcal{H}_\infty$ loop-shaping are design techniques that allow one to find a trade-off between different features, *e.g.* tracking and noise rejection, by means of $\mathcal{H}_2$-$\mathcal{H}_\infty$ optimization methods.

The fact that any feedback controller design must reflect a compromise between insensitivity to different disturbances and good stability margins is first identified in [1], where the mixed-sensitivity criterion is introduced as a suitable quality measure of the closed-loop behaviour. Among all different approaches for the solution of such control-design problem, the Youla-Kučera parameterization [2] represents one of the most successful. As a matter of fact, by parameterizing the feedback controller with the Youla-Kučera parameter Q, the mixed-sensitivity problem becomes convex in the unknown Q and the final controller is guaranteed to internally stabilize the closed-loop system. However, in case of fixed-order controller, the loop-shaping problem becomes much more complex, as model-reduction techniques (see *e.g.* [3], [4]) must be employed and closed-loop stability may be compromised.

In the classical model-based controller design, a model of the plant is derived from first-principle methods or identified using experimental data and then a controller is designed based on the available model. This framework can be very useful in the industrial practice, as witnessed by the huge number of contributions in different application fields, see *e.g.* [5], [6], [7]. In situations that the first-principle methods cannot be used, three optimization problems must be solved to obtain the final controller. First, the best model with the desired structure that minimizes a prediction error criterion is identified. Then, a full-order controller is computed by minimizing a mixed sensitivity criterion. Finally, a fixed-order controller that best fits the frequency response of the optimal controller is obtained by optimization or some order reduction techniques.

In this paper, a different philosophy is proposed to solve the mixed-sensitivity problem in the data-driven framework. Since the unique aim of model identification is the design of the controller, in the proposed approach this first step is skipped by directly identifying the Youla-Kučera parameter from a single set of data that minimizes the control criterion. The final reduced-order controller K is then deduced from the same data-set as the one that approximate the optimal controller. The design issue is naturally converted into a convex data-driven optimization problem, if Q and K are linearly parameterized. Furthermore, in both noiseless and noisy environments, the method is "one-shot", *i.e.* it requires only one set of input-output (I/O) samples, and it allows the designer to avoid all the reasoning about the physics of the system, by still guaranteeing the closed-loop stability.

Noniterative data-driven methodologies for fixed-order controller design already exist in the model-reference control framework, *e.g.* the Correlation-based Tuning (CbT [8]) and Virtual Reference Feedback Tuning (VRFT [9], [10]). Some recent developments of these approaches can be found, *e.g.*, in [11], [12], [13], [14]. As far as the authors are aware, this is the first time where the noniterative data-driven philosophy is applied to the mixed-sensitivity loop-shaping problem. A preliminary version of this work can be found in [15].

An iterative data-driven solution to the problem has instead been presented in [16]. However, the technique presented in this work is substantially different from the one in [16], for several reasons: it also deals with $\mathcal{H}_\infty$ loop-shaping problems, it is noniterative, it has no strict constraints for the identification experiments, it guarantees internal stability with the resulting controller for the real system and it is based on data collected during open-loop operation of the system (without using preliminary stabilizing controllers).

In this paper, the effectiveness of the method will be experimentally shown on the control of an active suspension system, where the goal of the feedback action is the rejection of some disturbance effects. Some data-driven control solutions of an analogous system are performed in [17] and in [18], using extended versions of the Virtual Reference Feedback Tuning (VRFT) and of the Correlation-based Tuning (CbT) methods, respectively. However, it should be stressed that the present contribution is different for some reasons. Firstly, VRFT and CbT do not follow a mixed sensitivity criterion with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance criterion

Simone Formentin and Alireza Karimi are with the Laboratoire d'Automatique, École Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne (Switzerland), e-mail: {simone.formentin/alireza.karimi}@epfl.ch

Manuscript received May 1, 2012; revised...?

but a "model-matching" rationale. Moreover, the stability issue was not considered in those contributions.

The paper is structured as follows. In Section II, the mixed-sensitivity loop-shaping problem is formulated in a system-theoretic analytical framework. Section III presents the data-driven method in detail, while a theoretical comparison with model-based design is given in Section IV. Section V presents the application of the method on the active suspension system. The paper is ended by some concluding remarks.

II. Preliminaries

Consider the unknown stable LTI SISO plant $G_o(q^{-1})$, where q^{-1} denotes the backward-shift operator, and three weighting functions $W_s(q^{-1})$, $W_t(q^{-1})$ and $W_u(q^{-1})$. The loop-shaping control aim considered in this paper is to design an LTI fixed-order controller $K(q^{-1},\rho)$, linear in ρ, so as to minimize

$$J(\rho) = \|W_s S(\rho)\|^2 + \|W_t T(\rho)\|^2 + \|W_u U(\rho)\|^2 \quad , \tag{1}$$

where, $S(\rho) = [1 + K(\rho)G_o]^{-1}, T(\rho) = 1 - S(\rho), U(\rho) = K(\rho)S(\rho)$, and the symbol $\|\cdot\|$ might indicate either the $\mathcal{H}_2$- or the $\mathcal{H}_\infty$-norm throughout the whole paper. Notice also that from now on, the arguments t and q^{-1} are arbitrarily dropped for space reasons. The criterion (1) is generally non-convex with respect to the parameter vector ρ and, in most cases, $J(\rho) = 0$ cannot be achieved.

Consider now the Youla-Kučera reformulation [2] of (1). The set of all stabilizing controllers for $G_o(q^{-1})$ is

$$\mathcal{C} = \left\{ C(q^{-1}) = \frac{Q(q^{-1})}{1 - Q(q^{-1})G_o(q^{-1})}, Q(q^{-1}) \in \mathcal{H}_\infty \right\}. \tag{2}$$

where $\mathcal{H}_\infty$ is the set of all stable rational transfer functions with bounded infinity norm. Then, the three sensitivity functions $S(\rho), T(\rho)$ and $U(\rho)$ can be rewritten as

$$\begin{aligned} S(q^{-1}) &= \left(1 - Q(q^{-1})G_o(q^{-1})\right) && (3)\\ T(q^{-1}) &= Q(q^{-1})G_o(q^{-1}) && (4)\\ U(q^{-1}) &= Q(q^{-1}). && (5) \end{aligned}$$

It follows that the criterion (1) is convex in $Q(q^{-1})$ or in the parameters of $Q(q^{-1})$, if it is linearly parameterized. The fixed-order controller is finally found as the reduction [4] $K(q^{-1},\rho)$ of the full-order controller

$$C(q^{-1}) = \frac{\hat{Q}(q^{-1})}{1 - \hat{Q}(q^{-1})G_o(q^{-1})}, \tag{6}$$

where $\hat{Q}(q^{-1}) \in \mathcal{H}_\infty$ is the minimizer of the loop-shaping criterion (1).

In the Youla-Kučera setting, only (6), and not the reduced-order controller, is guaranteed to internally stabilize the system. However, an additional constraint based on the following sufficient condition can be included in the controller reduction procedure to overcome this problem and make $K(q^{-1},\rho)$ internally stabilizing.

Theorem 1: Let $G_o(q^{-1})$ and $\hat{Q}(q^{-1})$ be discrete-time dynamical systems in $\mathcal{H}_\infty$. The controller $K(q^{-1},\rho)$ internally stabilizes the plant $G_o(q^{-1})$ if

1) $\Delta(\rho) = G_o\left[(1 - \hat{Q}G_o)K(\rho) - \hat{Q}\right] \in \mathcal{H}_\infty$;
2) the stability radius $\hat{\gamma}(\rho) = \|\Delta(\rho)\|_\infty$ is less than 1.

Proof: Consider the scheme in Fig. 1, where $C(q^{-1})$ is the full-order controller (6). Since $\hat{Q}(q^{-1})$ belongs to $\mathcal{H}_\infty$, $C(q^{-1})$ internally stabilizes the closed-loop system opened at z. Then, both $S(q^{-1}) = 1 - \hat{Q}(q^{-1})G_o(q^{-1})$ and $T(q^{-1}) = \hat{Q}(q^{-1})G_o(q^{-1})$ are stable.
From the Small-Gain Theorem (see [19]), a sufficient condition for the closed-loop stability of the interconnected system is that the transfer function between $u(t)$ and $z(t)$ is stable and its infinity norm is less than 1 (requirements 1 and 2) and then the thesis holds. ∎

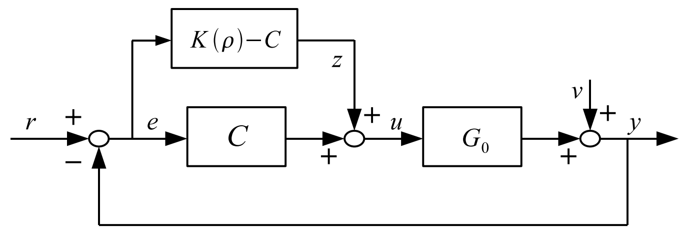

Fig. 1. Closed-loop system with controller $K(\rho)$ and explicit representation of the controller-reduction.

Remark. In Theorem 1, the second condition can be used to suitably bound the tuning of $K(q^{-1},\rho)$, provided the first requirement is satisfied. Specifically, if also stability of the fixed-order controller $K(q^{-1},\rho)$ is enforced, the first condition is always true, since $\hat{Q}(q^{-1}) \in \mathcal{H}_\infty$ by assumption and $S(q^{-1}) = 1 - \hat{Q}(q^{-1})G_o(q^{-1})$ is stable. On the other hand, if $K(q^{-1},\rho)$ contains an integrator, it is sufficient to impose that $Q(1) = 1/G_o(1)$ in the Q-design procedure. By doing this way, $S(q^{-1}) = 1 - \hat{Q}(q^{-1})G_o(q^{-1})$ has a zero at 1 and $(1 - \hat{Q}(q^{-1})G_o(q^{-1}))K(q^{-1},\rho)$ is stable for any ρ. In this paper, controllers with unstable poles will not be taken into account.

In practical situation, only an approximation $\hat{G}$ of the real system is known. Therefore, the criterion above could yield a controller that destabilizes the closed-loop system. A possible reformulation of the stability constraint for stable controllers is presented next.
Let consider only a class of controllers in $\mathcal{H}_\infty$ and suppose that a bound

$$\delta = \sup_{\hat{G}} \left\|\hat{G} - G_o\right\|_\infty$$

is known. The measure of the stability radius $\hat{\gamma}(\rho)$ for the real plant (according to its definition in Theorem 1) is such that $\hat{\gamma}(\rho) \leq \hat{\gamma}'(\rho)$, where

$$\hat{\gamma}'(\rho) = \|G_o\|_\infty \left\|\left(1 - \hat{G}\hat{Q}\right)K(\rho) - \hat{Q}\right\|_\infty + \|G_o\|_\infty \,\delta \left\|\hat{Q}K(\rho)\right\|_\infty \tag{7}$$

and $\|G_o\|_\infty$ can be either computed from data [8] or overestimated by means of $\|\hat{G}\|_\infty + \delta$. It follows that the stability bound can be transformed in the (more conservative) convex constraint

$$\hat{\gamma}'(\rho) < 1, \tag{8}$$

that only depends on $\hat{G}$ and δ. If this new formulation of the stability constraint is used in model-reduction procedure, internal stability can be guaranteed for the real system.

The standard (model-based) approach for mixed-sensitivity $\mathcal{H}_2$-$\mathcal{H}_\infty$ design is presented when a set of N input-output noisy data $\{u(t), y(t)\}$, $t = 1, .., N$ is available. These data are generated in open-loop operation according to the system dynamics, *i.e.* $y(t) = G_o(q^{-1})u(t) + v(t)$, where $v(t) = H_o(q^{-1})e(t)$, $H_o(q^{-1})$ is an unknown stable filter and $e(t)$ is a zero mean white noise. Assume also that u is a persistent exciting stationary signal [20].

MODEL-BASED ALGORITHM

1) Choose a class of SISO LTI models
$$\mathcal{G} = \left\{ G(q^{-1}, \theta)\ ,\ \theta \in \Theta \subset \mathcal{R}^{dim(\theta)} \right\}. \tag{9}$$
2) Identify a data-driven model of $G_o(q^{-1})$ as $\hat{G} = G(q^{-1}, \hat{\theta})$, where
$$\hat{\theta} = \arg\min_\theta \|G_o - G(\theta)\|^2 \tag{10}$$
3) Compute the optimal Youla-Kučera parameter $\hat{Q}$ as the rational transfer function in $\mathcal{H}_\infty$ that minimizes (1). This can be approximately done using a linearly parameterized Q [21].
4) Compute the full-order controller in $\mathcal{C}$ that guarantees the optimal sensitivity trade-off as
$$\hat{C} = \hat{Q}/(1 - \hat{Q}\hat{G}).$$
5) Compute the stable reduced-order controller via "control-oriented" model reduction [22], *e.g.* as $\hat{K} = K(q^{-1}, \hat{\rho})$, where
$$\hat{\rho} = \arg\min_\rho J_k(\rho)$$
$$J_k(\rho) = \left\| \left(1 - \hat{G}\hat{Q}\right)\hat{C} - \left(1 - \hat{G}\hat{Q}\right)K(\rho) \right\|^2 \tag{11}$$
such that (8) is satisfied.

In the following section, a suitable way to solve the data-driven mixed-sensitivity problem without identifying the plant model is proposed and analyzed.

III. Data-driven approach

Let the Youla-Kučera parameter be linearly parameterized, *i.e.* $Q(\eta) = \eta^T \beta_Q(q^{-1})$, where $\beta_Q(q^{-1})$ is a vector of orthonormal basis functions with the same dimension of η, e.g., $\beta_Q(q^{-1}) = [1 \quad q^{-1} \quad q^{-2} \ldots]$. Analogously, consider for the controller the linear parameterization

$$K(q^{-1}, \rho) = \rho^T \beta_K(q^{-1}), \tag{12}$$

where $\beta_K(q^{-1})$ is a vector of orthonormal basis functions with the same dimension of ρ.
Consider now the tuning scheme in Fig. 2. For each value of

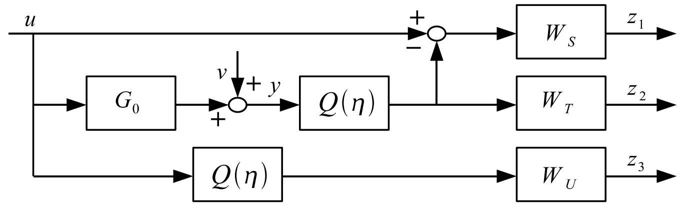

Fig. 2. Tuning scheme for the Youla-Kučera parameter.

the parameter vector, the signals $z_1(t, \eta)$, $z_2(t, \eta)$ and $z_3(t, \eta)$ can be expressed as functions of $u(t)$ and of the output $y(t)$, without including the real plant dynamics:

$$\begin{aligned} z_1(\eta) &= W_s\left(1 - G_o Q(\eta)\right)u = W_s u - W_s Q(\eta) y \\ z_2(\eta) &= W_t G_o Q(\eta) u = W_t Q(\eta) y \\ z_3(\eta) &= W_u Q(\eta) u. \end{aligned}$$

In a noiseless environment, *i.e.* when $v(t) = 0\ ,\ \forall t$, the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the generating functions $H_i(q^{-1}, \theta)$ of such signals, *i.e.* the functions such that $z_i(t, \eta) = H_i(q^{-1}, \eta)u(t)\ ,\ i = 1, 2, 3$, can be estimated from data. In detail, concerning the $\mathcal{H}_2$-norm, it holds that, for N that tends to infinity,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \left[L(q^{-1}) z_i(t, \eta)\right]^2 = \|H_i(\eta)\|_2^2\ ,\ i = 1, 2, 3,$$

where the prefilter $L(q^{-1})$ is such that $\left|L(e^{j\omega})\right|^2 = 1/U(\omega)$ and $U(\omega)$ is an estimate of the spectrum of u. An estimate of the $\mathcal{H}_\infty$-norm can be instead derived via spectral estimates as suggested in [8]. Formally, for N that tends to infinity, it holds that

$$\max_{\omega_k} \left| \frac{\Phi_i(\omega_k, \eta)}{U(\omega_k)} \right| \to \|H_i(\eta)\|_\infty\ ,\ i = 1, 2, 3,$$

where $\omega_k = 2\pi k/(2l + 1)$, $k = 1, \ldots, l + 1$ and $\Phi_i(\omega_k, \eta)$ is an estimate of the cross-spectrum between u and z_i. In detail, such spectrum may be computed as

$$\Phi_i(\omega_k, \eta) = \sum_{\tau=-l}^{l} R_i(\tau, \eta) e^{-j\tau\omega_k}$$

where $R_i(\tau, \eta)$ is an estimate of the cross-correlation function between u and z_i

$$R_i(\tau, \eta) = \frac{1}{N} \sum_{t=1}^{N} u(t - \tau) z_i(t, \eta).$$

The estimation of the $\mathcal{H}_\infty$-norm is consistent, when $l \to \infty$ and $l/N \to 0$ (see [8]). Notice that, since all signals are linear functions of η with the parameterization of Q selected above, both the $\mathcal{H}_2$ squared norm and the $\mathcal{H}_\infty$-norm are convex in the parameter vector.
Therefore, in such noiseless setting, the problem of finding $\hat{\eta}$ minimizing (1) is converted in a convex optimization problem, where the addends in the cost function, *i.e* the

weighted sensitivities that generate the z_i-s, are directly computed from data. In this way, points 1-2-3 of the standard model-based algorithm (see again Section II) are *reduced to a single identification step*.
If $v(t)$ is a generic zero-mean stochastic signal, the problem of minimizing (1) turns out to be a standard errors-in-variables (EIV) problem, where a model has to be identified starting from noisy input data. In this case, different solutions are available in the literature to make the procedure insensitive to noise (see [23] for an overview).

The same data-driven rationale can be used to condensate points 4-5 of the model-based algorithm in another data-based step, without identifying $G_o(q^{-1})$. In fact, the linearly parameterized reduced-order controller $K(q^{-1},\rho)$ can be directly identified from the same data-set used for the computation of the Youla-Kučera parameter. Moreover, it will be shown that the bias due to presence of noise can be easily handled if the problem is formulated using the correlation approach.
Consider Fig. 3, where the tracking error $\varepsilon_k(\rho)$ is defined as

$$\varepsilon_k(\rho) = (1 - G_oQ(\hat{\eta}))\,K(\rho)u - Q(\hat{\eta})u.$$

Introduce now the instrumental variable vector $\zeta(t)$

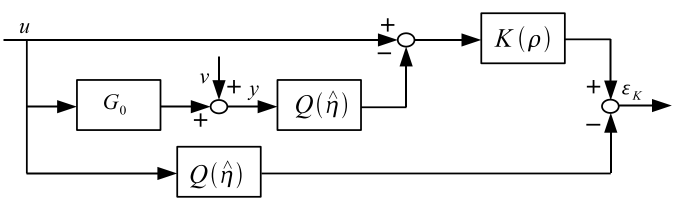

Fig. 3. Tuning scheme for the reduced-order controller.

$$\zeta(t) = [u(t + l_k), \ldots, u(t), \ldots, u(t - l_k)]^T$$

and the decorrelation criterion as

$$V_k(\rho) = \left[\frac{1}{N}\sum_{t=1}^{N}\zeta(t)\varepsilon_k(t,\rho)\right]^T \frac{1}{N}\sum_{t=1}^{N}\zeta(t)\varepsilon_k(t,\rho) \quad (13)$$

The following result holds for the formulation above in both noiseless and noisy settings.

Theorem 2: Consider the decorrelation criterion (13), where $\varepsilon_k(t,\rho)$ is generated by the linearly-parameterized controller (12) and filtered with $L_k(q^{-1})$ such that

$$\left|L_k(e^{j\omega})\right| = 1/U(\omega). \quad (14)$$

Then, as $N, l_k \to \infty$ and $l_k/N \to 0$, the minimizer $\hat{\rho}$ of $V_k(\rho)$ is with probability 1 a minimizer of (11), where $\hat{C} = Q(\hat{\eta})/(1 - Q(\hat{\eta})G_o)$ and $\hat{\eta}$ is the minimizer of (1).

Proof: Following the same procedure adopted for model-reference criterion in [8], the criterion can be proved to statistically converge to a continuous function of the cross-correlation indicators $R_k(\tau,\rho) = \mathbb{E}[u(t-\tau)\varepsilon_k(t,\rho)]$, *i.e.*

$$\lim_{N\to\infty} V_k(\rho) = \sum_{\tau=-l_k}^{\tau=l_k} R_k(\tau,\rho)^2.$$

Notice then that if $K(q^{-1},\rho)$ is stable, $(1 - Q(\hat{\eta})G_o)\,K(\rho) - Q(\hat{\eta})$ is stable and that the same holds if $K(q^{-1},\rho)$ contains an integrator and Q has been constrained such that $Q(1) = 1/G_o(1)$. As a consequence, the squared sum $\sum_{\tau=-l_k}^{l_k} R_k(\tau,\rho)^2$ and its limit $\sum_{\tau=-\infty}^{\infty} R_k(\tau,\rho)^2$ are bounded on the parameter set. Thus, as $N, l_k \to \infty$ and $l_k/N \to 0$, $V_k(\rho)$ converges uniformly to $\sum_{\tau=-\infty}^{\infty} R_k(\tau,\rho)^2$ [24]. In frequency-domain, the asymptotical value of $V_k(\rho)$ can be rewritten by means of the Parseval theorem as

$$\sum_{\tau=-\infty}^{\infty} R_k(\tau,\rho)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi^2(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}|L_k|^2\,|(1 - G_oQ(\hat{\eta}))\,K(\rho) - Q(\hat{\eta})|^2\,U^2(\omega)d\omega$$

where Ψ is defined as

$$\Psi(\omega) = L_k\left[(1 - G_oQ(\hat{\eta}))\,K(\rho) - Q(\hat{\eta})\right]U(\omega).$$

If the data-prefilter is selected according to (14), then (13) asymptotically tends to (11) with $\hat{G} = G_o$ and, since the convergence is uniform, the minimizers of the two criteria coincide. ■

The stability constraint can be included in the design problem in two different ways that explained in the following subsections.

A. Double-experiment procedure

a data-driven version of the constraint in Theorem 1 can be formulated with a second open-loop experiment, by feeding the plant with $\{y(t)\}_{t=1\ldots N}$ and collecting the output $\{y'(t)\}_{t=1\ldots N}$.

Let $\Delta(\rho)$ be the transfer function between u and a signal z_Δ, *i.e.*

$$z_\Delta(\rho) = (1 - G_oQ(\hat{\eta})))\,K(\rho)G_ou - Q(\hat{\eta})G_ou.$$

It follows that, in a noiseless environment,

$$\begin{aligned} z_\Delta(\rho) &= (1 - G_oQ(\hat{\eta})))\,K(\rho)y - Q(\hat{\eta})y \\ &= K(\rho)y - Q(\hat{\eta})K(\rho)y' - Q(\hat{\eta})y, \end{aligned}$$

that is $z_\Delta(\rho)$ can be computed as a function of known data for each value of ρ. The $\mathcal{H}_\infty$-norm of $\Delta(\rho)$ can then be asymptotically derived as suggested in [8]. It should be mentioned that if $K(\rho)$ contains an integrator, the equality constraint $Q(1) = 1/G_o(1)$ requires an additional information on the static gain of the process, as an estimate of the plant model is no more available.

B. Single-experiment procedure

If a stabilizing minimum-phase controller C_s is available, it is possible to avoid ad-hoc experiments. Consider again Fig.1, by replacing C with C_s. A different stability condition depending on C_s can be straightforwardly derived by following the same rationale in Theorem 1 and requiring that

$$\Delta_s(\rho) = \frac{G_o\,(K(\rho) - C_s)}{1 + G_oC_s} \in \mathcal{H}_\infty$$

and $\gamma_s(\rho) = \|\Delta_s(\rho)\|_\infty < 1$. In such case, $\Delta_s(\rho)$ can be seen as the transfer function between a fictitious reference $r_f(t)$ and z_{Δ_s} (see again the closed-loop scheme in Fig. 1), *i.e.*

$$z_{\Delta_s}(\rho) = \frac{G_o\left(K(\rho) - C_s\right)}{1 + G_o C_s} r_f,$$

where r_f is given by $r_f(t) = C_s^{-1}(q^{-1})u(t) + y(t)$. The expression of $z_{\Delta_s}(\rho)$ may be rewritten as

$$\begin{aligned} z_{\Delta_s}(\rho) &= \frac{G_o K(\rho)}{1 + G_o C_s} r_f - \frac{G_o C_s}{1 + G_o C_s} r_f \\ &= C_s^{-1} K(\rho) \frac{G_o C_s}{1 + G_o C_s} r_f - y \\ &= \left(C_s^{-1} K(\rho) - 1\right) y \end{aligned}$$

Therefore, $z_{\Delta_s}(\rho)$ is completely known from data and the $\mathcal{H}_\infty$-norm of $\Delta(\rho)$ can again be asymptotically derived as suggested in [8].

Remark.

- The above rationale is derived in a noiseless setting. Several techniques for dealing with noisy data in spectral estimation are available in the literature [8].
- If $K(\rho)$ contains an integrator, also C_s must have it; analogously, C_s must be stable if $K(\rho)$ is stable. However, this is easy to achieve in practical situations.
- The "double experiment" formulation of the stability-constraint is the same as that for the model-based case, with the difference that the identified model $\hat{G}$ is replaced by the "true" model G_o. This fact makes the proposed method less conservative than the standard model-based one, where knowledge and use of the additional variable δ (see again Section II) is required to guarantee internal stability for the real closed-loop system.

The data-driven algorithm can be summarized in the following three points.

DATA-DRIVEN ALGORITHM

1) Choose a class of SISO LTI models

$$\mathcal{Q} = \left\{ Q(q^{-1}, \eta) \ , \ \eta \in \Pi \subset \mathcal{R}^{dim(\eta)} \right\}. \quad (15)$$

2) Compute the optimal parameters of Q by the following optimization problem:

$$\hat{\eta} = \arg\min_{\eta} J_N(\eta) \quad (16)$$

where $J_N(\eta)$ can be any user-defined composition of sample-based estimates of $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of the weighted sensitivity functions.

3) Identify the data-based reduced-order controller as $\hat{K} = K(q^{-1}, \hat{\rho})$, where $\hat{\rho} = \arg\min_\rho V_k(\rho)$ and one of the two proposed stability constraints is satisfied.

IV. Comparison

In this section, the new approach and the standard model-based algorithm will be discussed and compared from different points of view, in order to highlight advantages and disadvantages of the two methods.

A. Asymptotic results in case of correct parameterization

Define the optimal value for the Youla-Kučera parameter Q_o and the optimal full-order controller C_o respectively as

$$Q_o = \arg\min_{Q \in \mathcal{RH}_\infty} J(G_o, Q), \quad (17)$$

$$C_o = \frac{Q_o}{1 - G_o Q_o}. \quad (18)$$

Consider then the F.I.R. extension of the selected classes of models, *i.e.* write

$$\mathcal{G} = \left\{ G(\theta, q^{-i}) = \sum_{i=0}^{n_G} \theta_i q^{-i} \right\},$$

$$\mathcal{Q} = \left\{ Q(\theta, q^{-i}) = \sum_{i=0}^{n_Q} \eta_i q^{-i} \right\}.$$

The following asymptotic result holds.

Proposition 1: Assume that $G_o \in \mathcal{G}$ in model-based procedure and $Q_o \in \mathcal{Q}$ in the data-driven case. Then:

1) the reduced-order controllers guarantee the same asymptotical loop-shaping performance in model-based and in the data-driven framework.
2) if C_o belongs to the class of considered fixed-order controllers, the minimum of $J(\rho)$ in model-based case and in data-driven algorithm coincide, as $N \to \infty$.

Proof: Let consider the model-based approach first. The FIR estimate of G_o is $G(\hat{\theta})$, where $\hat{\theta}$ is given by

$$\hat{\theta} = \left[\frac{1}{N} \sum_{t=1}^{N} \psi(t)\psi(t)^T \right]^{-1} \frac{1}{N} \sum_{t=1}^{N} \psi(t) y(t) \quad (19)$$

and $\psi(t) = [u(t), u(t-1), \ldots, u(t-n_G+1)]$. It is well known [20] that (19) can be written as the sum of three different terms: the real value θ_o, a term due to undermodeling and a third addend depending on noise variance. Since prediction error techniques are used and $G_o \in \mathcal{G}$, it holds that, asymptotically, $\hat{\theta} \to \theta_o$. Consequently, $Q \to Q_o$ and $J = J(G_o, Q_o)$. For what concerns the data-driven algorithm, the same reasoning can be applied. In few words, since $Q_o \in \mathcal{Q}$ by hyphoteses, $Q \to Q_o$ as the number of data grows, because the proposed method is consistent. This means that the Youla-Kučera parameter minimizing (1) is the same for both the approaches if large data-sets are used and, subsequently, $J = J(G_o, Q_o)$. Starting from the same expression for Q, the unique difference between (11) and (13) is the fact that (11) can be computed by means of noiseless simulated data, obtained by feeding $\hat{G}$ with $u(t)$, whereas (13) must be minimized using the set of I/O noisy data. However, the result shown in Theorem 2 assures that, as $N, l_k \to \infty$ and $l_k/N \to 0$, the minima of two cost functions coincide (thesis 1). The same result straightforwardly holds if no order-reduction is required (thesis 2). ■

B. Discussion about undermodeling

Theorem 1 states that both the approaches are consistent if the right model-order is selected for G and Q. This is not the case in many real-world applications. As obvious, a complete theoretical analysis of the differences between the two approaches would be very complex in this setting, since optimization results are strictly related to the dynamic structure of the plant. Furthermore, in model-based approach, undermodeling of G weighs on Q and then the final value of (1) also depends on how the Youla-Kučera parameter is calculated. However, a ticklish aspect concerning undermodeling of the controller can be highlighted, that is the conservatism of the stability constraint. As explained in Section II, in order to ensure the internal stability of the real closed-loop system, the modeling error must be taken into account. Moreover, the constraint $\gamma'(\rho) < 1$ may affect closed-loop performance, if the optimal solution is close to the boundary defined by the real constraint. A simple situation where this fact may happen has been presented in [15].

It must be mentioned that the formulation (7) is only one of possible solutions and that different results could be achieved in different situations. In any case, if internal stability for the real system has to be (asymptotically) guaranteed, a fair comparison between the methods must take into account a bound on the modeling error, by introducing more conservatism in the model-based approach.

V. Experimental example

The goal of this experimental example is to implement a controller for an active suspension system aimed to reject some external disturbances. In this example, the control action will be designed in order to reject the low-frequency disturbances. For more details on the active suspension applications and some comparison with other technologies, the reader is referred to [25].

The experimental setup is illustrated in Fig. 4. The system contains three mobile plates, that can move in the vertical direction independently from each other. The bottom plate represents the perturbation of the system. The user can impose a vertical position of this plate using a brushed servo motor which is linked by a belt and a ball screw to the bottom plate. The middle plate is linked by two springs to the bottom plate. There is damping due to friction in the linear bearings and between the springs and their guide. The top plate is the one representing the chassis of the device to control and it is linked by another motor to the middle plate. This second motor is the actuator of the control system, secured in parallel with a spring and shock absorber. The aim of the control problem is to minimize the energy of the acceleration of the first plate, by regulating the force provided by the second actuator, once any disturbance is given. A position sensor installed on the top plate enables the feedback control of acceleration by double derivation of the signal. The sampling time is $T_s = 20\ ms$.

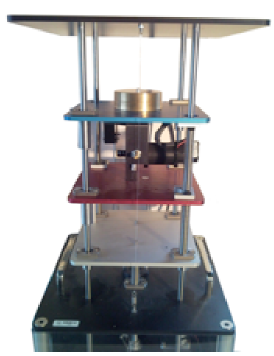

Fig. 4. The active suspension system by Quanser®.

Since the above issue is formulated as a noise rejection problem, a suitable mixed-sensitivity loop-shaping cost could be the $\mathcal{H}_2$ cost

$$\min_{\rho} J(\rho) = \min_{\rho} \|W_s S(\rho)\|_2^2 + \|W_u U(\rho)\|_2^2 \qquad (20)$$

where the weighting functions are

$$W_s = 10\frac{(1-\alpha)^2}{(1-\alpha q^{-1})^2}\ ,\ \alpha = e^{-2\pi T_s 5}\ ,\ W_u = 0.1. \qquad (21)$$

The $\mathcal{H}_2$-norm is the best choice in this case because, as already said, the objective is the energy of the acceleration. By minimizing the weighted sensitivity function together with the input sensitivity, the resulting controller will reject the noise in the frequency range indicated by W_s (that is, up to $5\ Hz$), but it will also keep the control action not too large. A weighting on the complementary sensitivity function is instead not needed as no explicit tracking performance are required; specifically, the reference acceleration will be always set to zero.

Since, in this paper, controllers are supposed to be fixed-order, the additional requirement that only up to 5 parameters can be used will be taken into account. Moreover, since the system contains two derivative actions (the actual acceleration is the second derivative of the position), the integral action will not be employed. No constraints are instead given on Q, that in the present method is seen only as a tool to compute the optimal K. Then, while for K the dimension of ρ is compulsorily 5, η can be as large as desired. However, too large Q may yield numerical problems and a smart selection of the order is advisable.

In this case, since the optimal Q theoretically contains the poles of the final complementary sensitivity function and the bandwidth of S is set to $5\ Hz$, a reasonable choice for Q is an FIR with dimension of η equal to the length of the impulse response of W_s, that is 15 (it can be easily verified that the response takes almost $0.3\ s$ to go to zero, with the given sampling time T_s). Concerning the length of the instrumental variables, the same reasoning can be done, as they are used to approximate the correlation matrix of the closed-loop output and the size of that is related to the speed of the system. However, some tests (not reported here for matter of space) showed that this dimension can be increased up to 35 without substantial changes.

In order to design the optimal controller, the open-loop data

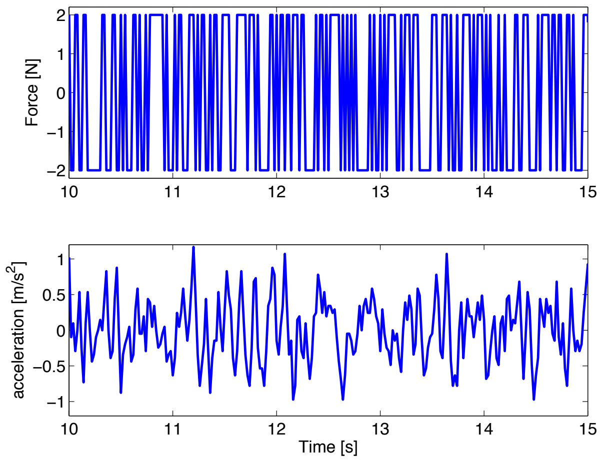

Fig. 5. Open-loop data collection for controller identification.

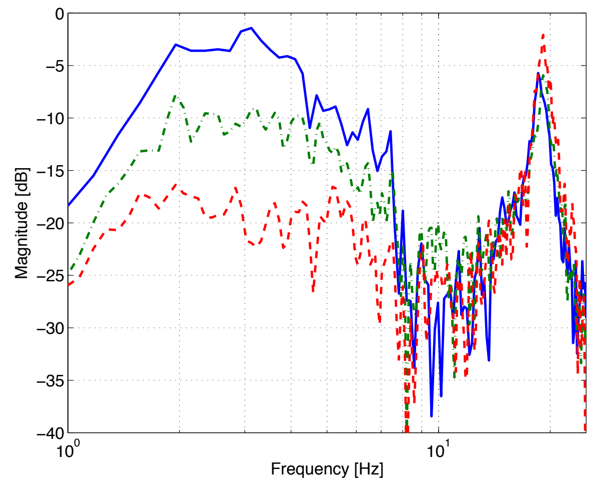

Fig. 6. Magnitude of the frequency response of the transfer function between the disturbance and the body acceleration: open-loop (solid), closed-loop with model-based controller (dash-dotted) and closed-loop with the data-driven controller (dashed).

in Fig. 5 are employed. The input force sequence has been selected as a 9^{th}-order Maximum Length Sequence (MLS) and the acceleration output has been collected. The MLS is basically a pseudo-random sequence of pulses and can be easily obtained by using a shift-register with different feedback taps. The advantage of such a choice with respect to a white noise excitation is that its spectrum is almost white but the amplitude of the time-history of the force can be kept constant at a desired level (in this case, $2\ N$). The 5-parameter controller given by the method is

$$\begin{aligned} K(\rho) &= \rho_o + \rho_1 q^{-1} + \rho_2 q^{-2} + \rho_3 q^{-3} + \rho_4 q^{-4} \\ &= 3.66 + 0.05867 q^{-1} + 2.113 q^{-2} + 0.3015 q^{-3} \\ &\quad +1.165 q^{-4} \end{aligned}$$

The improvement of the closed-loop system with the resulting controller with respect to the open-loop response can be evaluated in Fig.6, where the magnitude of the experimental frequency response of the transfer function between the disturbance and the chassis acceleration is illustrated. As expected, in the frequency range where W_s is higher, the sensitivity function of the closed-loop system gets lower and the disturbance is better filtered. It should be here stressed that also the illustrated frequency responses (for performance assessment) have been computed directly from data, specifically using experimental MLS excitation of the disturbance and Welch method [26], whereas no analytical model has been identified.

In Fig. 6, the performance given by the optimal 5-parameter model-based controller is also shown. For computing such a controller, it was not possible to employ FIR identification, as the large number of parameters required (to have a good matching) yielded too poor closed-loop performance; therefore, output error (OE) identification was used (notice that this approach is based on non-convex techniques). Specifically, the OE(6,6,1) model (22) was identified using the *oe* command of the System Identification Toolbox in Matlab [27] and validation was obtained via the *resid* command. The *h2syn* command [28] was then used to compute Q and the full-order controller C_o (18). The parameter vector of the reduced-order controller was then found by the (convex) control-oriented controller reduction procedure at the 5^{th} step of the model-based design algorithm in Section II. Specifically, in this case, the $\mathcal{H}_2$-norm was selected.

Notice that, despite in model-based approach the choice of Q is completely free, the constraint in the controller structure make the final performance worse than the data-driven approach. In particular, it can be noticed that at the frequency corresponding to the maximum amplification factor, *i.e.* $3\ Hz$, the data-driven approach leads to an additional disturbance reduction of almost $10\ dB$. At the same time, the settling time with the model-based controller is 1.7 seconds against 1.4 seconds of the closed-loop system with the data-driven controller. This is coherent with the theory presented in Section IV, as in this practical case, the dataset is not infinite and C_o might not belong to the set of 5-parameter controllers. The advantage of the data-driven approach is due to the fact that, in every step, the final control criterion is directly taken into account.

The performance of the system can be quantitatively assessed by computing the root mean square (RMS) of the estimated acceleration $\hat{a}$, that is

$$RMS = \frac{1}{r}\sqrt{\sum_{t=1}^{r} \hat{a}^2}, \tag{23}$$

over a set of r samples, provided a MLS excitation is given. The RMS value, normalized with respect to the open-loop case, is 0.76 for the model-based controller and 0.41 for the data-driven controller when the spectrum of the excitation of the disturbance is limited up to $10\ Hz$ and $r = 1000$.

The result in Fig. 6 can then be summarized by stating that the proposed approach guarantees - for the given application - an additional 35% in the reduction of the acceleration energy with respect to model-based loop-shaping.

$$\hat{G}(q^{-1}) = \frac{0.09835q^{-1} - 0.2304q^{-2} + 0.09333q^{-3} + 0.2106q^{-4} - 0.2731q^{-5} + 0.1006q^{-6}}{1 - 3.029q^{-1} + 3.971q^{-2} - 2.67q^{-3} + 0.8097q^{-4} - 0.0365q^{-5} + 0.01685q^{-6}}, \quad (22)$$

VI. Concluding remarks

A data-driven approach for controller design in mixed-sensitivity $\mathcal{H}_2$-$\mathcal{H}_\infty$ loop-shaping framework has been proposed. The method is based on convex optimization techniques and it is limited to stable plants. The main idea is to derive the Youla-Kučera parameter *directly* from a set of I/O data and to perform a second identification step to identify a fixed-order controller from the same data-set. In both the cases, the criterion to optimize is related to the final control performance - *i.e.* it is "control-oriented" - and not to the matching of the real plant. Internal stability of the closed-loop system with the resulting controller is asymptotically achieved by means of a convex $\mathcal{H}_\infty$-constraint. Furthermore, the stability constraints for the proposed technique are generally less conservative than for the standard model-based approach, with which the data-driven approach shares the same asymptotical results. In this paper, the proposed method has also been applied on an active suspension system, where experimental results have shown very good performance when compared to a standard model-based approach.

References

[1] M. Safonov, A. Laub, and G. Hartmann, "Feedback properties of multivariable systems- The role and use of the return difference matrix," *IEEE Transactions on Automatic Control*, vol. 26, pp. 47–65, 1981.

[2] J. Doyle, B. Francis, and A. Tannenbaum, *Feedback control theory*. Macmillan New York, 1992, vol. 134.

[3] G. Obinata, B. Anderson, and F. Lutze, "Model reduction for control system design," *Applied Mechanics Reviews*, vol. 54, 2001.

[4] B. Anderson and Y. Liu, "Controller reduction: concepts and approaches," *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 802–812, 1989.

[5] Z. Gao, X. Dai, T. Breikin, and H. Wang, "Novel parameter identification by using a high-gain observer with application to a gas turbine engine," *IEEE Transactions on Industrial Informatics*, vol. 4, no. 4, pp. 271–279, 2008.

[6] W. Lin, T. Su, and R. Wu, "Parameter identification of induction machine with a starting no-load low voltage test," *IEEE Transactions on Industrial Electronics*, no. 99, pp. 1–1, 2012.

[7] G. Fontes, C. Turpin, and S. Astier, "A large-signal and dynamic circuit model of a H2/O2 pem fuel cell: description, parameter identification, and exploitation," *IEEE Transactions on Industrial Electronics*, vol. 57, no. 6, pp. 1874–1881, 2010.

[8] K. van Heusden, A. Karimi, and D. Bonvin, "Data-driven model reference control with asymptotically guaranteed stability," *International Journal of Adaptive Control and Signal Processing*, 2011.

[9] M. Campi, A. Lecchini, and S. Savaresi, "Virtual reference feedback tuning: a direct method for the design of feedback controllers," *Automatica*, vol. 38, no. 8, pp. 1337–1346, 2002.

[10] S. Formentin, S. Savaresi, and L. Del Re, "Noniterative direct data-driven controller tuning for multivariable systems: theory and application," *IET Control Theory & Applications*, vol. 6, no. 9, pp. 1250–1257, 2012.

[11] L. Campestrini, D. Eckhard, M. Gevers, and A. Bazanella, "Virtual reference feedback tuning for non-minimum phase plants," *Automatica*, 2011.

[12] S. Formentin, M. Corno, S. Savaresi, and L. Del Re, "Direct data-driven control of linear time-delay systems," *Asian Journal of Control*, vol. 14, no. 3, pp. 652–663, 2012.

[13] S. Formentin, A. Karimi, and S. Savaresi, "On input design for direct data-driven controller tuning," in *IFAC Symposium on System Identification*, vol. 16, no. 1, 2012, pp. 1466–1471.

[14] J. Rojas and R. Vilanova, "Internal model controller tuning using the virtual reference approach with robust stability," in *IFAC World Congress*, vol. 18, no. 1, 2011, pp. 10 237–10 242.

[15] S. Formentin, A. Karimi, and S. Savaresi, "Direct data-driven H2-Hinf loop-shaping," in *IFAC World Congress*, vol. 18, no. 1, 2011.

[16] H. Procházka, M. Gevers, B. Anderson, and C. Ferrera, "Iterative feedback tuning for robust controller design and optimization," in *44th IEEE Conference on Decision and Control*, 2005, pp. 3602–3607.

[17] M. Campi, A. Lecchini, and S. Savaresi, "An application of the virtual reference feedback tuning method to a benchmark problem," *European Journal of Control*, vol. 9, no. 1, pp. 66–76, 2003.

[18] L. Miškovic, A. Karimi, and D. Bonvin, "Correlation-based tuning of a restricted-complexity controller for an active suspension system," *European journal of control*, vol. 9, no. 1, pp. 77–83, 2003.

[19] K. Zhou, J. Doyle, and K. Glover, *Robust and optimal control*. Prentice-Hall Englewood Cliffs, NJ, 1996.

[20] L. Ljung, *System identification: theory for the user*. Prentice-Hall Englewood Cliffs, NJ, 1999.

[21] S. Boyd and C. Barratt, "Linear controller design: limits of performance," 1991.

[22] M. Volker, C. Sonntag, and S. Engell, "Control of integrated processes: a case study on reactive distillation in a medium-scale pilot plant," *Control engineering practice*, vol. 15, no. 7, pp. 863–881, 2007.

[23] T. Söderström, "Errors-in-variables methods in system identification," *Automatica*, vol. 43, no. 6, pp. 939–958, 2007.

[24] R. Rockafeller, "Convex Analysis," *Princeton Press, NJ*, 1970.

[25] C. Alexandru and P. Alexandru, "A comparative analysis between the vehicles passive and active suspensions," *International Journal of Mechanics*, vol. 4, no. 5, 2011.

[26] P. Welch, "The use of fast fourier transform for the estimation of power spectra: a method based on time averaging over short, modified periodograms," *IEEE Transactions on Audio and Electroacoustics*, vol. 15, no. 2, pp. 70–73, 1967.

[27] L. Ljung, "System identification toolbox," *The Matlab users guide*, 1988.

[28] G. Balas, R. Chiang, A. Packard, and M. Safonov, *Robust Control Toolbox*. Natick, MA: The MathWorks, Inc, 2005.

Simone Formentin was born in Legnano, Italy, in 1984. He received his B.Sc. and M.Sc. degree summa cum laude in Control System Engineering from Politecnico di Milano (Italy) in 2006 and 2008, respectively. From 2009 to 2011, he was a Ph.D. student in Automatic Control within a joint program between Politecnico di Milano and Johannes Kepler University Linz (Austria). From December 2011 to August 2012, he was a post-doctoral fellow at Swiss Federal Institute of Technology in Lausanne (EPFL), Switzerland. He is currently a post-doctoral fellow at Università degli studi di Bergamo (Italy). His research interests include data-driven controller tuning approaches and automotive control.

Alireza Karimi received his B.Sc. and M.Sc. degrees in Electrical Engineering in 1987 and 1990 from Amir Kabir University (Tehran Polytechnic). After 3 years of industrial experience he joined Institut National Polytechnique de Grenoble (INPG) in France and received his DEA and Ph.D. degrees both on Automatic Control in 1994 and 1997, respectively. He was Assistant Professor at Electrical Engineering Department of Sharif University of Technology in Teheran from 1998 to 2000. He is currently Senior Scientist at the Automatic Control Laboratory of Swiss Federal Institute of Technology in Lausanne (EPFL), Switzerland. He is Associate Editor of European Journal of Control since 2004. His research interests include closed-loop identification, data-driven controller tuning approaches and robust control.