

GAUSSIAN UPPER DENSITY ESTIMATES FOR SPATIALLY HOMOGENEOUS SPDES

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ABSTRACT. We consider a general class of SPDEs in \mathbb{R}^d driven by a Gaussian spatially homogeneous noise which is white in time. We provide sufficient conditions on the coefficients and the spectral measure associated to the noise ensuring that the density of the corresponding mild solution admits an upper estimate of Gaussian type. The proof is based on the formula for the density arising from the integration-by-parts formula of the Malliavin calculus. Our result applies to the stochastic heat equation with any space dimension and the stochastic wave equation with $d \in \{1, 2, 3\}$. In these particular cases, the condition on the spectral measure turns out to be optimal.

1. INTRODUCTION

We are interested in establishing Gaussian type upper estimates for the density of the *mild* solution of the following class of SPDEs:

$$\mathcal{L}u(t, x) = b(u(t, x)) + \sigma(u(t, x))\dot{W}(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad (1.1)$$

where $T > 0$ is some fixed time horizon and \mathcal{L} denotes a general second order partial differential operator with constant coefficients, with appropriate initial conditions. The coefficients σ and b are real-valued functions and $\dot{W}(t, x)$ is the formal notation for a Gaussian random perturbation which is white in time and has some spatially homogeneous correlation (see Section 2.1 for a precise definition of this noise). The typical examples of operator \mathcal{L} to which our result applies are the heat operator for any spatial dimension $d \geq 1$ and the wave operator with $d \in \{1, 2, 3\}$.

If \mathcal{L} is first order in time, such as the heat operator $\mathcal{L} = \frac{\partial}{\partial t} - \Delta$, where Δ denotes the Laplacian operator on \mathbb{R}^d , then we impose initial conditions of the form

$$u(0, x) = u_0(x) \quad x \in \mathbb{R}^d, \quad (1.2)$$

for some Borel function $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$. If \mathcal{L} is second order in time, such as the wave operator $\mathcal{L} = \frac{\partial^2}{\partial t^2} - \Delta$, then we have to impose two initial conditions:

$$u(0, x) = u_0(x), \quad \frac{\partial u}{\partial t}(0, x) = v_0(x), \quad x \in \mathbb{R}^d, \quad (1.3)$$

for some Borel functions $u_0, v_0 : \mathbb{R}^d \rightarrow \mathbb{R}$.

The above class of SPDEs has been widely studied in the last two decades. Precisely, results on existence and uniqueness of solution in such a general setting have been established in [29, 7, 10, 6], while the particular cases of heat and wave equations have been studied using several frameworks in [35, 2, 8, 20, 15, 30, 28, 4].

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A fruitful line of research developed in some of the above-cited references has been to apply techniques of Malliavin calculus in order to deduce some interesting properties of the probability law of the solution at any $(t, x) \in (0, T] \times \mathbb{R}^d$. In fact, there is a whole bunch of results on existence and smoothness of the density for the stochastic heat and wave equations, for which we refer to [2, 27, 1, 20, 19, 31, 32, 33, 18]. Moreover, in the paper [23], existence and smoothness of density for the class of SPDEs (1.1) have been analyzed, unifying and improving some of the results cited so far. It is also worth mentioning that other kind of methods beyond Malliavin calculus can be used to prove the absolute continuity of the law of the solution in some particular SPDEs (see e.g. [11]).

Once the existence (and possibly smoothness) of the density of the solution to (1.1) is established, one usually gathers at some *nice* estimates for this density, such as lower and upper Gaussian type bounds. Exploiting again techniques of Malliavin calculus, this problem has been recently addressed by several authors. Precisely, as far as SPDEs with additive noise is concerned, using an explicit formula for the density proved in [21], the main result in [24] says the following (see Thm. 7 therein, and also [25] for related results). Let μ be spectral measure associated with the spatial correlation of \dot{W} , and consider the following assumption, which is necessary and sufficient for the existence and uniqueness of mild solution to (1.1) (see e.g. [7]).

Hypothesis 1.1. Let Γ be the fundamental solution associated to the operator \mathcal{L} . For all $t > 0$, $\Gamma(t)$ defines a non-negative distribution with rapid decrease such that

$$\Phi(T) := \int_0^T \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(t)(\xi)|^2 \mu(d\xi) dt < +\infty.$$

Moreover, Γ is a non-negative measure of the form $\Gamma(t, dx)dt$ such that, for all $T > 0$,

$$\sup_{0 \leq t \leq T} \Gamma(t, \mathbb{R}^d) < +\infty.$$

Then, under the above hypothesis, with vanishing initial data, $\sigma \equiv 1$ and assuming that $b \in \mathcal{C}^1$ has a bounded derivative, [24, Thm. 7] states that, for small enough t and any $x \in \mathbb{R}^d$, the density $p_{t,x}$ of $u(t, x)$ satisfies, for almost every $z \in \mathbb{R}$,

$$\frac{E |u(t, x) - m|}{C_2 \Phi(t)} \exp \left\{ -\frac{(z - m)^2}{C_1 \Phi(t)} \right\} \leq p_{t,x}(z) \leq \frac{E |u(t, x) - m|}{C_1 \Phi(t)} \exp \left\{ -\frac{(z - m)^2}{C_2 \Phi(t)} \right\}, \quad (1.4)$$

where $m = E(u(t, x))$, for some positive constants C_1, C_2 . Note that the term $\Phi(t)$ is precisely the variance of the stochastic convolution in the mild form of (1.1) when $\sigma \equiv 1$. As a consequence, this result applies to the stochastic heat equation for any $d \geq 1$ and the stochastic wave equation in the case $d \in \{1, 2, 3\}$ provided that (see [7, Sec. 3])

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < +\infty. \quad (1.5)$$

On the other hand, in the multiplicative noise setting, such kind of density estimates, particularly the lower one, become more difficult to obtain and Nourdin-Viens' density formula cannot be applied. This has been already illustrated by Kohatsu-Higa in [16] where, by means of Malliavin calculus techniques, a new method to obtain Gaussian lower bounds for general functionals of the Wiener sheet has been obtained. In the same

paper, the author has applied this result to a stochastic heat equation in $[0, 1]$ and driven by the space-time white noise.

In order to deal with SPDEs beyond the one-dimensional setting, in [26] Kohatsu-Higa's general method has been extended to the Gaussian space associated to the underlying spatially homogeneous noise \dot{W} . This allowed us to end up with the following density estimates for the stochastic heat equation in any space dimension $d \geq 1$. Assume that $b, \sigma \in \mathcal{C}^\infty$ are bounded together with all their derivatives, $|\sigma(z)| \geq c > 0$ for all $z \in \mathbb{R}$, and for some $\eta \in (0, 1)$, it holds

$$\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^\eta} \mu(d\xi) < +\infty. \quad (1.6)$$

Then, for all $(t, x) \in (0, T] \times \mathbb{R}^d$, the density $p_{t,x}$ of $u(t, x)$ verifies, for all $z \in \mathbb{R}$,

$$\begin{aligned} C_1 \Phi(t)^{-1/2} \exp \left\{ -\frac{|z - I_0(t, x)|^2}{C_2 \Phi(t)} \right\} &\leq p_{t,x}(z) \\ &\leq c_1 \Phi(t)^{-1/2} \exp \left\{ -\frac{(|z - I_0(t, x)| - c_3 T)^2}{c_2 \Phi(t)} \right\}, \end{aligned} \quad (1.7)$$

where $I_0(t, x) = (\Gamma(t) * u_0)(x)$, u_0 being the initial data. In the case $b \equiv 0$, the constant c_3 would vanish. Note that here, in comparison to (1.4), the estimates are valid for any $T > 0$.

In fact, let us point out that the upper bound in (1.7) is much easier to obtain than the lower one, and the former comes from the expression for the density popping up from the integration-by-parts formula in the Malliavin calculus framework.

Extending the lower estimate in (1.7) to the general class of SPDEs (1.1) seems to be an open problem, for the success in the application of the general strategy of [26] is closely tied to the parabolic structure of the heat equation. However, a much more humble objective, which is the one we plan to gather in the present paper, is to tackle the upper bound. In fact, we are going to seek the minimal conditions on either the coefficients b and σ and the spectral measure μ implying that the upper estimate in (1.7) remains valid for the general class of SPDEs (1.1). In particular, we will only need b and σ to be of class \mathcal{C}^2 (and bounded with bounded derivatives) and, for the particular case of the heat (resp. wave) equation with any $d \geq 1$ (resp. $d \in \{1, 2, 3\}$), the condition on μ will be simply (1.5) rather than (1.6). More precisely, the main result of the paper is the following. We use the notation $I_0(t, x)$ to denote the contribution of the initial data (see (3.2) and (3.3) for its explicit expression in the case of heat and wave equations) and suppose that the forthcoming Hypothesis 3.2 is satisfied.

Theorem 1.2. *Assume that Hypothesis 1.1 is satisfied, and that $b, \sigma \in \mathcal{C}^2$ are bounded, have bounded derivatives, and $|\sigma(z)| \geq c > 0$ for all $z \in \mathbb{R}$. Moreover, suppose that, for some $\gamma > 0$, it holds*

$$C \tau^\gamma \leq \Phi(\tau) = \int_0^\tau \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds, \quad \tau \in (0, 1]. \quad (1.8)$$

Then, for any $(t, x) \in (0, T] \times \mathbb{R}^d$, the solution $u(t, x)$ of (1.1) has a density $p_{t,x}$ which is a continuous function and satisfies, for all $z \in \mathbb{R}$,

$$p_{t,x}(z) \leq c_1 \Phi(t)^{-1/2} \exp \left\{ -\frac{(|z - I_0(t, x)| - c_3 T)^2}{c_2 \Phi(t)} \right\}, \quad (1.9)$$

where the constant c_3 vanishes whenever b does.

We remark that, though the above bound does not look exactly Gaussian, it does in an *asymptotic* point of view, namely whenever T is small or z is large. On the other hand, we note that, under (1.5), condition (1.8) is satisfied for the heat and wave equations with $\gamma = 1$ and $\gamma = 3$, respectively (see e.g. [19, Lem. 3.1] and [31, App. A]). Similarly, one can also check that the above theorem applies to the stochastic damped wave equation with any space dimension (see Example 7 in [7, Sec. 3]), where condition (1.8) is fulfilled with $\gamma = 3$.

We also point out that our result is not applicable to the case $\sigma(z) = z$ (this would be related, e.g., to the parabolic Anderson problem [3]). In fact, in such a case there are even very few results on absolute continuity of the law of solutions to SPDEs (see [27]). Nevertheless, in the recent paper [13], the authors prove existence and smoothness of the density for a stochastic heat equation with a nonlinear multiplicative noise which is white in time and with some spatial correlation (much more regular than the one considered in the present paper), and with a non-degeneracy condition on the diffusion coefficient of the form $\sigma(u_0(x_0)) \neq 0$ for some $x \in \mathbb{R}^d$. Their proof is based on a Feynman-Kac formula for the solution of the underlying equation. This technique has also been applied in [14] to study the density for a stochastic heat equation with a linear multiplicative fractional Brownian sheet.

As mentioned before, the proof of Theorem 1.2 will be based on the expression for the density arising from the application of the integration-by-parts formula (see [22, Prop. 2.1.1]). We point out that this is a well-known method that has been used in other contexts (see e.g. [12, 9]). As far as the technical obstacles is concerned, the main two ingredients needed in the proof of Theorem 1.2 are the following:

- (i) A suitable estimate, in terms of $\Phi(t)$, of the norm of the iterated Malliavin derivative in a small time interval (see Lemma 4.1 for details). This will be a consequence of a kind of analogous result for the case of the stochastic heat equation (see [26, Lem. 3.4]) and a mollifying procedure thanks to an approximation of the identity which will let us smooth the fundamental solution $\Gamma(t)$.
- (ii) A precise control of the negative moments of the norm of the Malliavin derivative of the solution, again in terms of $\Phi(t)$ (see Proposition 4.2). For this, we will adapt the proof of [23, Thm. 6.2] to our setting, where the latter allowed the authors of that paper to establish that the underlying density is a smooth function (under much more regularity on the coefficients though).

The content of the paper is organized as follows. In Section 2, we rigorously describe the Gaussian spatially homogeneous noise \dot{W} considered in equation (1.1), we introduce the corresponding Gaussian setting associated to it, together with the main notations of the Malliavin calculus machinery. Section 3 will be devoted to recall the definition of mild solution to our SPDE (1.1) and summarize the main results on existence and uniqueness of solution, Malliavin differentiability and existence and smoothness of the density. The steps (i) and (ii) detailed above will be tackled in Section 4. Finally, we will prove Theorem 1.2 in Section 5.

With a slight (but harmless) abuse of notation, as already done in this Introduction, the notation $|\cdot|$ shall denote either the modulus and norm in \mathbb{R}^d . Unless otherwise

stated, any constant c or C appearing in our computations below is understood as a generic constant which might change from line to line without further mention.

2. PRELIMINARIES

2.1. Spatially homogeneous noise. Let us explicitly describe here our spatially homogeneous noise (see e.g. [7]). Precisely, on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, this is given by a family $W = \{W(\varphi), \varphi \in \mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)\}$ of zero mean Gaussian random variables, where $\mathcal{C}_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ denotes the space of smooth functions with compact support, with the following covariance structure:

$$\mathbb{E}(W(\varphi)W(\psi)) = \int_0^\infty \int_{\mathbb{R}^d} (\varphi(t, \star) * \tilde{\psi}(t, \star))(x) \Lambda(dx) dt. \quad (2.1)$$

In this expression, Λ denotes a non-negative and non-negative definite tempered measure on \mathbb{R}^d , $*$ stands for the convolution product, the symbol \star denotes the spatial variable and $\tilde{\psi}(t, x) := \psi(t, -x)$.

In the above setting, a well-known result of harmonic analysis (see [34, Chap. VII, Théorème XVII]) implies that Λ has to be the Fourier transform of a non-negative tempered measure μ on \mathbb{R}^d , where the latter is usually called the *spectral measure* of the noise W . We recall that, in particular, for some integer $m \geq 1$ it holds

$$\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^m} \mu(d\xi) < +\infty$$

and, by definition of the Fourier transform in the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions, $\Lambda = \mathcal{F}\mu$ means that, for all ϕ belonging to the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing \mathcal{C}^∞ functions,

$$\int_{\mathbb{R}^d} \phi(x) \Lambda(dx) = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi) \mu(d\xi).$$

Therefore, we have

$$\mathbb{E}(W(\varphi)^2) = \int_0^\infty \int_{\mathbb{R}^d} |\mathcal{F}\varphi(t)(\xi)|^2 \mu(d\xi) dt.$$

A typical example of space correlation is given by $\Lambda(dx) = f(x)dx$, where f is a non-negative function which is assumed to be integrable around the origin. In this case, the covariance functional (2.1) reads

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(t, x) f(x - y) \psi(t, y) dy dx dt.$$

The space-time white noise would correspond to the case where f is the Dirac delta at the origin.

2.2. Gaussian setting and Malliavin calculus. We are going to describe the Gaussian framework which can be naturally associated to our noise W and introduce the notations involved in the Malliavin calculus techniques.

To start with, let us denote by \mathcal{H} be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ endowed with the semi-inner product

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{H}} := \int_{\mathbb{R}^d} (\phi_1 * \tilde{\phi}_2)(x) \Lambda(dx) = \int_{\mathbb{R}^d} \mathcal{F}\phi_1(\xi) \overline{\mathcal{F}\phi_2(\xi)} \mu(d\xi), \quad \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d).$$

As proved in [7, Example 6], we remind that the Hilbert space \mathcal{H} may contain distributions.

Let $T > 0$ be a fixed real number and define $\mathcal{H}_T := L^2([0, T]; \mathcal{H})$. Using an approximation argument, our noise W can be extended to a family of mean zero Gaussian random variables indexed by \mathcal{H}_T (see e.g. [6, Lemma 2.4]). With an innocuous abuse of notation, this family will be still denoted by $W = \{W(g), g \in \mathcal{H}_T\}$. Moreover, it holds $\mathbb{E}(W(g_1)W(g_2)) = \langle g_1, g_2 \rangle_{\mathcal{H}_T}$, for all $g_1, g_2 \in \mathcal{H}_T$. Thus, this family defines an *isonormal Gaussian process* on the Hilbert space \mathcal{H}_T and we shall use the differential Malliavin calculus based on it (see e.g. [22, 33]).

As usual, we denote the Malliavin derivative operator by D . Recall that it is a closed and unbounded operator defined in $L^2(\Omega)$ and taking values in $L^2(\Omega; \mathcal{H}_T)$, whose domain is denoted by $\mathbb{D}^{1,2}$. More general, for any integer $m \geq 1$ and any $p \geq 2$, the domain of the iterated Malliavin derivative D^m in $L^p(\Omega)$ will be denoted by $\mathbb{D}^{m,p}$, where we remind that D^m takes values in $L^p(\Omega; \mathcal{H}_T^{\otimes m})$. We also set $\mathbb{D}^\infty = \bigcap_{p \geq 1} \bigcap_{m \in \mathbb{N}} \mathbb{D}^{m,p}$. The space $\mathbb{D}^{m,p}$ can also be seen as the completion of the set of *smooth functionals* with respect to the semi-norm

$$\|F\|_{m,p} := \left\{ \mathbb{E}(|F|^p) + \sum_{j=1}^m \mathbb{E} \left(\|D^j F\|_{\mathcal{H}_T^{\otimes j}}^p \right) \right\}^{\frac{1}{p}}.$$

For any differentiable random variable F and any $r = (r_1, \dots, r_m) \in [0, T]^m$, $D^m F(r)$ is an element of $\mathcal{H}^{\otimes m}$ which will be denoted by $D_r^m F$.

A random variable F is said to be *smooth* if it belongs to \mathbb{D}^∞ , and a smooth random variable F is said to be *non-degenerate* if $\|DF\|_{\mathcal{H}_T}^{-1} \in \bigcap_{p \geq 1} L^p(\Omega)$. Owing to [22, Theorem 2.1.4], we know that a non-degenerate random variable has a \mathcal{C}^∞ density.

For any $t \in [0, T]$, let \mathcal{F}_t be the σ -field generated by the random variables $\{W_s(h), h \in \mathcal{H}, 0 \leq s \leq t\}$ and the \mathbb{P} -null sets, where $W_t(h) := W(1_{[0,t]}h)$.

3. SPATIALLY HOMOGENEOUS SPDES

We gather here a general result on existence and uniqueness of mild solution for our SPDE (1.1) and the main results on Malliavin calculus applied to it, namely Malliavin differentiability and existence and smoothness of density. As usual, we will also focus on the main examples of application that we have in mind, which are the stochastic heat and wave equations with $d \geq 1$ and $d \in \{1, 2, 3\}$, respectively.

We recall that, by definition, a mild solution of (1.1) is an \mathcal{F}_t -adapted random field $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ such that the following stochastic integral equation is satisfied:

$$\begin{aligned} u(t, x) = & I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) \sigma(u(s, y)) W(ds, dy) \\ & + \int_0^t \int_{\mathbb{R}^d} b(u(t-s, x-y)) \Gamma(s, dy) ds, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (3.1)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$. Here, Γ denotes the fundamental solution associated to \mathcal{L} and $I_0(t, x)$ is the contribution of the initial conditions, which we define below.

The (real-valued) stochastic integral on the right-hand side of (3.1) is understood with respect to the *cylindrical Wiener process* that can be naturally associated to our spatially homogeneous noise W (see [23, 6] and also [35, 7]). In particular, we will assume that

Hypothesis 1.1 is satisfied. Concerning the last integral on the right-hand side of (3.1), we point out that we use the notation “ $\Gamma(s, dy)$ ” because we will assume that $\Gamma(s)$ is a measure on \mathbb{R}^d .

As far as the term $I_0(t, x)$ is concerned, if \mathcal{L} is a parabolic-type operator and we consider the initial condition (1.2), then

$$I_0(t, x) = (\Gamma(t) * u_0)(x) = \int_{\mathbb{R}^d} u_0(x - y) \Gamma(t, dy). \quad (3.2)$$

On the other hand, in the case where \mathcal{L} is second order in time with initial values (1.3),

$$I_0(t, x) = (\Gamma(t) * v_0)(x) + \frac{\partial}{\partial t} (\Gamma(t) * u_0)(x). \quad (3.3)$$

Example 3.1. Owing to the considerations in [7, Section 3] (see also [23, Examples 4.2 and 4.3]), in the case of the stochastic heat equation in any space dimension $d \geq 1$ and the stochastic wave equation in dimensions $d = 1, 2, 3$, the fundamental solutions are well-known and the conditions in Hypothesis 1.1 are satisfied if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < +\infty. \quad (3.4)$$

We shall consider the following assumption on the initial conditions. In the case of the stochastic heat equation in any space dimension (resp. wave equation with dimension $d = 1, 2, 3$), sufficient conditions on u_0 (resp. u_0, v_0) implying that the hypothesis below is fulfilled are provided in [6, Lemma 4.2].

Hypothesis 3.2. $(t, x) \mapsto I_0(t, x)$ is continuous and $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |I_0(t, x)| < +\infty$.

The following well-posedness result, which is a quotation of [6, Thm. 4.3], is a slight extension of the results in [7].

Theorem 3.3. *Assume that Hypotheses 1.1 and 3.2 are satisfied and that σ and b are Lipschitz functions. Then there exists a unique solution $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ of equation (3.1). Moreover, for all $p \geq 1$,*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E(|u(t, x)|^p) < +\infty.$$

Let us now deal with the Malliavin differentiability of the solution $u(t, x)$ of (3.1). For this, we consider the Gaussian context described in Section 2.2. The following proposition summarizes a series of results in [19, 32, 23]. For the statement, we will use the following notation: for any $m \in \mathbb{N}$, set $\bar{s} := (s_1, \dots, s_m) \in [0, T]^m$, $\bar{z} := (z_1, \dots, z_m) \in (\mathbb{R}^d)^m$, $\bar{s}(i) := (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_m)$ (resp. $\bar{z}(i)$), and, for any function f and variable X for which it makes sense, set

$$\Delta^m(f, X) := D^m f(X) - f'(X) D^m X.$$

Note that $\Delta^m(f, X) = 0$ for $m = 1$ and, if $m > 1$, it only involves iterated Malliavin derivatives up to order $m - 1$.

Proposition 3.4. *Assume that Hypothesis 1.1 is satisfied and, for some $m \in \mathbb{N} \cup \{\infty\}$, $\sigma, b \in \mathcal{C}^m(\mathbb{R})$ and their derivatives of order greater than or equal to one are bounded, Then, for all $(t, x) \in [0, T] \times \mathbb{R}^d$, the random variable $u(t, x)$ belongs to $\mathbb{D}^{j,p}$ for any $j = 1, \dots, m$ and $p \geq 1$. Furthermore, for any $j \in \{1, \dots, m\}$ and $p \geq 1$, the iterated Malliavin derivative $D^j u(t, x)$ satisfies the following equation in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$:*

$$\begin{aligned} D^j u(t, x) &= Z^j(t, x) \\ &+ \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) [\Delta^j(\sigma, u(s, y)) + D^j u(s, y) \sigma'(u(s, y))] W(ds, dy) \\ &+ \int_0^t \int_{\mathbb{R}^d} [\Delta^j(b, u(t-s, x-y)) + D^j u(t-s, x-y) b'(u(t-s, x-y))] \Gamma(s, dy) ds, \end{aligned} \tag{3.5}$$

where $Z^j(t, x)$ is the element of $L^p(\Omega; \mathcal{H}_T^{\otimes j})$ given by

$$Z^j(t, x)_{\bar{s}, \bar{z}} = \sum_{i=1}^j \Gamma(t-s_i, x-dz_i) D_{\bar{s}(i), \bar{z}(i)}^{j-1} \sigma(u(s_i, z_i)).$$

A detailed description of the construction of Hilbert-space-valued stochastic integrals as the one in equation (3.5) can be found in [23, Sec. 3]. Indeed, as proved in [6, Sec. 3.6], these kind of integrals turn out to be equivalent to Hilbert-space-valued stochastic integrals *à la* Da Prato and Zabczyk [5].

The above Proposition 3.4 can be used to obtain the following results on existence and smoothness of the density for the solution $u(t, x)$. They are direct consequences of Theorems 5.2 and 6.2 in [23], with the only difference that the latter consider vanishing initial conditions.

Theorem 3.5. *Assume that Hypotheses 1.1 and 3.2 are satisfied, $b, \sigma \in \mathcal{C}^1$ have a bounded derivative, and $|\sigma(z)| \geq c > 0$ for all $z \in \mathbb{R}$. Then, for all $(t, x) \in (0, T] \times \mathbb{R}^d$, the random variable $u(t, x)$ has a law which is absolutely continuous with respect to the Lebesgue measure.*

Theorem 3.6. *Assume that Hypotheses 1.1 and 3.2 are satisfied, $\sigma, b \in \mathcal{C}^\infty$ and their derivatives of order greater than or equal to one are bounded, and that $|\sigma(z)| \geq c > 0$ for all $z \in \mathbb{R}$. Moreover, suppose that, for some $\gamma > 0$,*

$$C t^\gamma \leq \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds, \quad t \in (0, 1). \tag{3.6}$$

Then, for every $(t, x) \in (0, T] \times \mathbb{R}^d$, the law of the random variable $u(t, x)$ has a \mathcal{C}^∞ density.

As commented in the Introduction, both results apply to the stochastic heat equation with $d \geq 1$ and the stochastic wave equation with $d \in \{1, 2, 3\}$ provided that (3.4) is satisfied, since condition (3.6) holds for these examples with $\gamma = 1$ and $\gamma = 3$, respectively.

4. AUXILIARY RESULTS

This section is devoted to prove the main two ingredients needed in the proof of Theorem 1.2. The first one establishes a suitable uniform bound for the norm of the

iterated Malliavin derivative of the solution $u(t, x)$ in small time intervals. The second one deals with the negative moments of the corresponding Malliavin matrix, which here simply reduces to the norm of the Malliavin derivative of $u(t, x)$.

Lemma 4.1. *Let $0 \leq a < e \leq T$ and $p \geq 1$. Assume that Hypotheses 1.1 and 3.2 are satisfied and that, for some $m \in \mathbb{N}$, the coefficients b, σ belong to \mathcal{C}^m and all their derivatives of order greater than or equal to one are bounded. Then, there exists a positive constant C , which is independent of a and e , such that, for all $\delta \in (0, e - a]$:*

$$\sup_{(\tau, y) \in [e - \delta, e] \times \mathbb{R}^d} \mathbb{E} \left(\|D^j u(\tau, y)\|_{\mathcal{H}_{e - \delta, e}^{\otimes j}}^{2p} \right) \leq C \Phi(\delta)^{jp}, \quad (4.1)$$

for all $j \in \{1, \dots, m\}$, where we remind that, for all $t \geq 0$,

$$\Phi(t) = \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds.$$

Proof. It is similar to that of [26, Lem. 3.4], where a conditioned version of this result for the stochastic heat equation has been proved. Precisely, as already pointed out in [26, Rmk. 3.5], in our general setting we need to smooth the fundamental solution Γ as follows. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ be such that $\psi \geq 0$, its support is contained in the unit ball of \mathbb{R}^d and $\int_{\mathbb{R}^d} \psi(x) dx = 1$. For $n \in \mathbb{N}$, set $\psi_n(x) := n^d \psi(nx)$ and, for all t , $\Gamma_n(t) := \psi_n * \Gamma(t)$. It is well-known that $\Gamma_n(t)$ belongs to $\mathcal{S}(\mathbb{R}^d)$.

Let us now consider $\{u_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ the unique solution of

$$\begin{aligned} u_n(t, x) &= I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Gamma_n(t - s, x - y) \sigma(u_n(s, y)) W(ds, dy) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} b(u_n(t - s, x - y)) \Gamma(s, dy) ds. \end{aligned}$$

Since $\Gamma_n(t)$ is a smooth function (such as in the case of the heat equation), we can mimic the proof of [26, Lem. 3.4], so that we end up with estimate (4.1) with u replaced by u_n . Indeed, we should remark here that the term involving the pathwise integral with respect to $\Gamma(s, dy) ds$ does not cause any problem since we only need to use that $\Gamma(t, \mathbb{R}^d)$ is uniformly bounded in t , which is part of Hypothesis 1.1.

On the other hand, a direct consequence of the proofs of [32, Thm. 1] and [23, Prop. 6.1] is that, for all $(t, x) \in [e - \delta, e] \times \mathbb{R}^d$ and $j \in \{1, \dots, m\}$,

$$D^j u(t, x) = L^2(\Omega; \mathcal{H}_{e - \delta, e}^{\otimes j}) - \lim_{n \rightarrow \infty} D^j u_n(t, x).$$

Therefore, writing down the corresponding convergence of norms and taking supremum over $[e - \delta, e] \times \mathbb{R}^d$, we conclude the proof. \square

Proposition 4.2. *Assume that Hypotheses 1.1 and 3.2 are satisfied, that b, σ are \mathcal{C}^1 functions with bounded derivatives and that $|\sigma(z)| \geq c > 0$ for all $z \in \mathbb{R}$. Moreover, suppose that, for some $\gamma > 0$,*

$$C t^\gamma \leq \int_0^t \int_{\mathbb{R}^d} |\mathcal{F}\Gamma(s)(\xi)|^2 \mu(d\xi) ds, \quad t \in (0, 1). \quad (4.2)$$

Then, for any $p > 0$, there exists a constant $C > 0$ such that, for all $(t, x) \in (0, T] \times \mathbb{R}^d$,

$$\mathbb{E} (\|Du(t, x)\|_{\mathcal{H}_T}^{-2p}) \leq C \Phi(t)^{-p}.$$

Proof. The proof's structure is analogous as that of the proofs of [23, Thm. 6.2] and [26, Prop. 4.3], so we will only sketch the main steps.

First, owing to [22, Lem. 2.3.1], it suffices to check that, for any $q > 2$, there exists $\varepsilon_0 = \varepsilon_0(q) > 0$ such that, for all $\varepsilon \leq \varepsilon_0$,

$$\mathbb{P} \left\{ \Phi(t)^{-1} \|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon \right\} \leq C\varepsilon^q. \quad (4.3)$$

Note that the Malliavin derivative $Du(t, x)$ verifies the following equation in \mathcal{H}_T (take $m = 1$ in (3.5)):

$$\begin{aligned} Du(t, x) &= \sigma(u(\cdot, \star))\Gamma(t - \cdot, x - \star) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y) \sigma'(u(s, y)) Du(s, y) W(ds, dy) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} b'(u(s, x - y)) Du(s, x - y) \Gamma(t - s, dy) ds. \end{aligned}$$

Then, for any small $\delta > 0$ one proves that

$$\begin{aligned} \mathbb{P} \left\{ \Phi(t)^{-1} \|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon \right\} &\leq \mathbb{P} \left\{ \Phi(t)^{-1} I(t, x; \delta) \geq c \Phi(t)^{-1} \Phi(\delta) - \varepsilon \right\} \\ &\leq (c \Phi(t)^{-1} \Phi(\delta) - \varepsilon)^{-p} \Phi(t)^{-p} \mathbb{E}(|I(t, x; \delta)|^p), \end{aligned} \quad (4.4)$$

where $I(t, x; \delta) := \|R_1(t, x; \delta)\|_{\mathcal{H}_{t-\delta, t}}^2 + \|R_2(t, x; \delta)\|_{\mathcal{H}_{t-\delta, t}}^2$ and

$$\begin{aligned} R_1(t, x; \delta) &:= \int_{\cdot}^t \int_{\mathbb{R}^d} \Gamma(t - s, x - y) \sigma'(u(s, y)) Du(s, y) W(ds, dy), \\ R_2(t, x; \delta) &:= \int_{\cdot}^t \int_{\mathbb{R}^d} b'(u(t - s, x - y)) Du(t - s, x - y) \Gamma(s, dy) ds. \end{aligned}$$

Using the above Lemma 4.1 and applying standard integral estimates, one checks that

$$\mathbb{E}(|I(t, x; \delta)|^p) \leq C \Phi(\delta)^p (\Phi(\delta)^p + \Psi(\delta)^p),$$

where we have set

$$\Psi(s) := \int_0^s \Gamma(r, \mathbb{R}^d) dr.$$

Thus, going back to (4.4), we obtain

$$\mathbb{P} \left\{ \Phi(t)^{-1} \|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon \right\} \leq C (c \Phi(t)^{-1} \Phi(\delta) - \varepsilon)^{-p} \Phi(t)^{-p} \Phi(\delta)^p (\Phi(\delta)^p + \Psi(\delta)^p).$$

At this point, taking a small enough ε_0 if necessary, we can choose $\delta = \delta(\varepsilon)$ such that

$$\frac{c}{2} \Phi(t)^{-1} \Phi(\delta) = \varepsilon. \quad (4.5)$$

Hence, we have

$$\mathbb{P} \left\{ \Phi(t)^{-1} \|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon \right\} \leq C (\Phi(\delta)^p + \Psi(\delta)^p).$$

Note, on the one hand, that condition (4.5) implies $\Phi(\delta) \leq C\Phi(T)\varepsilon \leq C\varepsilon$. On the other hand, by Hypothesis 1.1 we have $\Psi(\delta) \leq C\delta$. Hence, the assumption (4.2) and what we have just said above let us infer that $\Psi(\delta) \leq C\varepsilon^{\frac{1}{\gamma}}$. Therefore,

$$\mathbb{P} \left\{ \Phi(t)^{-1} \|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon \right\} \leq C \left(\varepsilon^p + \varepsilon^{\frac{p}{\gamma}} \right),$$

so taking $p = q(\gamma \vee 1)$ we conclude that (4.3) is satisfied. \square

5. PROOF OF THE MAIN RESULT

In this section, we are going to prove Theorem 1.2. On the one hand, we note that Proposition 3.4 implies that, for any $(t, x) \in (0, T] \times \mathbb{R}^d$, the random variable $u(t, x)$ belongs to $\mathbb{D}^{2,p}$ for all $p \geq 1$. Moreover, an immediate consequence of Proposition 4.2 is that the Malliavin matrix associated to $u(t, x)$ has negative moments of all orders. Thus, applying a general criterion of the Malliavin calculus (see e.g. [22, Prop. 2.1.5] or [17, Thm. 4.1]), we obtain that the law of $u(t, x)$ has a density and it is a continuous function.

On the other hand, as explained in the Introduction, the proof of (1.9) is a matter of following exactly the same arguments as in [26, Sec. 5] and invoking the results of the previous section. Let us sketch the main steps to follow.

To start with, we use the formula for the density arising from the application of the integration-by-parts formula in the Malliavin calculus context (see e.g. [22, Prop. 2.1.1]). Precisely, denoting the density of $u(t, x)$ by $p_{t,x}$, we have

$$p_{t,x}(y) = E \left(\mathbf{1}_{\{u(t,x) > y\}} \delta \left(\frac{Du(t, x)}{\|Du(t, x)\|_{\mathcal{H}_T}^2} \right) \right), \quad y \in \mathbb{R},$$

where here δ denotes the divergence operator or Skorohod integral, that is the adjoint of the Malliavin derivative operator (see [22, Ch. 1]).

Next, taking into account the equation satisfied by $u(t, x)$ (i.e. (3.1)) and applying [22, Prop. 2.1.2], we obtain

$$\begin{aligned} p_{t,x}(y) &\leq C \mathbb{P} \{ |M_t| > |y - I_0(t, x)| - c_3 T \}^{\frac{1}{q}} \\ &\times \left\{ \mathbb{E} (\|Du(t, x)\|_{\mathcal{H}_T}^{-1}) + \left(\mathbb{E} \|D^2 u(t, x)\|_{\mathcal{H}_T^{\otimes 2}}^\alpha \right)^{\frac{1}{\alpha}} \left(\mathbb{E} \|Du(t, x)\|_{\mathcal{H}_T}^{-2\beta} \right)^{\frac{1}{\beta}} \right\}, \end{aligned} \quad (5.1)$$

where α, β, q are any positive real numbers satisfying $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{q} = 1$. In the above expression, M_t denotes the *martingale part* of the solution $u(t, x)$, that is

$$M_t = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) \sigma(u(s, y)) W(ds, dy),$$

and the term $c_3 T$ comes from the fact that, due to Hypothesis 1.1 and the boundedness of b , for all $(t, x) \in (0, T] \times \mathbb{R}^d$,

$$\left| \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) b(u(s, y)) dy ds \right| \leq c_3 T, \quad \mathbb{P}\text{-a.s.}$$

In order to estimate the terms in (5.1), we first apply the exponential martingale inequality in order to get a suitable exponential bound of the probability in (5.1) (using that $\langle M \rangle_t \leq C \Phi(t)$), and then we conveniently apply Lemma 4.1 and Proposition 4.2. Thus

$$p_{t,x}(y) \leq c_1 \Phi(t)^{-1/2} \exp \left(- \frac{(|y - I_0(t, x)| - c_3 T)^2}{c_2 \Phi(t)} \right), \quad y \in \mathbb{R},$$

where the constants c_1, c_2, c_3 do not depend on (t, x) , so we conclude the proof of Theorem 1.2. \square

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