ON THE STATIONARY DISTRIBUTION OF A STOCHASTIC TRANSFORMATION

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ABSTRACT. We find the stationary distribution of a stochastic transformation. The idea of the proof is to solve a related SDE and thus apply the fixed point theorem in a "stronger" sense than in the original formulation of the problem.

This short note is inspired by the following result proved in [1].

Let $W(t),\ t \geq 0$ be linear Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and consider the stochastic delay differential equation

$$\begin{aligned} \mathrm{d}X(t) &= X(t-1)\,\mathrm{d}W(t), \\ X(t) &= \eta(t), \quad t \in [-1,0], \end{aligned} \tag{0.1}$$

where $\eta \in \mathrm{C} := \mathrm{C}([-1,0],\mathbf{R})$, and for $t \geq 0$, we define

$$X_t(s) := X(t+s),\ s \in [-1,0].$$

Equation (0.1) has a unique solution for each initial condition $X_0 = \eta \in \mathrm{C}$ and the process X_t, $t \geq 0$ is a (strong) $\mathrm{C}-$valued Markov process with continuous paths. We define the following norms on C: Let $\|.\|$ be the sup-norm, and $\|\!|.|\!\|$ the M_2-norm defined as

$$\|\!|f|\!\|^2 := (f(0))^2 + \int_{-1}^{0} (f(s))^2\,\mathrm{d}s$$

(the Hilbert space M_2 consists of all functions from $[-1,0]$ to $\mathbf{R}$ for which this norm is finite).

Theorem 1. ([1]) *There exists a number $\Lambda \in \mathbf{R}$ such that for each $\eta \in \mathrm{C}\backslash\{0\}$, the solution X of equation* (0.1) *with initial condition η satisfies*

$$\lim_{t\to\infty} \frac{1}{t} \log \|X_t(\omega)\| = \lim_{t\to\infty} \frac{1}{t} \log \|\!|X_t(\omega)|\!\| = \Lambda \qquad \textit{a.s.}$$

For each $\eta \not\equiv 0$, the process X_t starting at $X_0 = \eta$ will almost surely never become (identically) zero. Therefore, the process

$$S_t := X_t / \|\!|X_t|\!\|,\ t \geq 0$$

is well-defined. Since equation (0.1) is linear, the process $S_t,\ t \geq 0$ is a Markov process with continuous paths (with respect to both the sup-norm and the M_2-norm on C) on the unit sphere of M_2.

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It has been proved in [1] that $S_t,\ t \geq 0$ has a unique invariant probability measure μ. It would be desirable to find an "explicit" description of μ. This we cannot provide but we will present a related theorem.

What can one say about a process R (if there is one) whose quadratic variation distribution is the same as the distribution of the quadratic variation of the integral of R with respect to an independent Brownian motion? We will show that one can obtain a fairly explicit answer to this question. The main step is to transform the question into a seemingly different problem concerned with an SDE.

Theorem 2. *There exists a filtered probability space and a continuous local martingale R on that space which satisfies $R(0) = 1$ almost surely and such that the process $M(t) := \int_0^t R(s)^2 \, \mathrm{d}s,\ t \geq 0$ has the same law as the process $[R](t),\ t \geq 0$. Any process R with these properties has the same law as the stochastic exponential Z of Brownian motion defined by $Z(t) := \exp\{B(t) - \frac{t}{2}\},\ t \geq 0$, where $B(t),\ t \geq 0$ is standard Brownian motion.*

Proof. It is straightforward to check that the process Z has all properties required of R so this proves existence. Note that, in this case, the equality of the processes M and $[Z]$ is not only in law but even almost sure.

Let $\mathrm{C}_* := \mathrm{C}([0,1], \mathbf{R})$ be equipped with the supremum norm $\|.\|$ (note that C_* and C are defined on different intervals). To show uniqueness, we consider a discrete time Markov chain X^k taking values in the space C_* defined in the following way: let $B^k,\ k \in \mathbf{N}$ be independent Brownian motions and $\eta \in \mathrm{C}_*$. Let $X^0 \equiv \eta$ and

$$X^{k+1}(t) := 1 + \int_0^t X^k(s) \, \mathrm{d}B^k(s),\ k \in \mathbf{N}_0. \tag{0.2}$$

This Markov chain has an invariant probability measure μ, namely the distribution of the process Z restricted to the interval $[0,1]$. The proof will be complete once we have proved uniqueness of the invariant probability measure.

To show uniqueness, let $\eta, \phi \in \mathrm{C}_*$, denote the processes defined via (0.2) with initial conditions η and ϕ by X^k and Y^k, resp., and define $Z^k(t) := X^k(t) - Y^k(t)$. Then we have

$$\mathbf{E}Z^{k+1}(t)^2 = \int_0^t \mathbf{E}Z^k(s)^2 \, \mathrm{d}s.$$

By induction, we obtain

$$\mathbf{E}Z^k(t)^2 \leq \|\phi - \eta\|^2 \frac{t^k}{k!}.$$

Since Z^k is a martingale, we get $\mathbf{E}\|X^k - Y^k\|^2 \to 0$ implying uniqueness of an invariant probability measure.

□

The theorem implies that if X has the same distribution as Z and W is Brownian motion not necessarily independent of Z then the law of $1 + \int_0^\cdot X(s)\mathrm{d}W(s)$ is the same as that of Z.

In the proof of uniqueness, we can start with $X^0 \equiv Z$ and consider Brownian motions B^k which are not necessarily independent from each other and from X^k's (but each X^k should be adapted to some filtration with respect to which B^k is a Brownian motion to ensure that the stochastic integral is well-defined; a similar remark applies to the initial condition Z in relation to B^0). Since the proof applies without any changes, we obtain convergence of X^k's to Z in law, for any such scheme.

References

[1] Scheutzow, M. (2012). Exponential growth rate for a singular linear stochastic delay differential equation. *Discrete Contin. Dyn. Syst. Ser. S*, to appear.

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