TORSION-FREE ENDOTRIVIAL MODULES

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Dedicated to Geoff Robinson on the occasion of his 60th birthday

Abstract. Let G be a finite group and let $T(G)$ be the abelian group of equivalence classes of endotrivial kG-modules, where k is an algebraically closed field of characteristic p. We investigate the torsion-free part $TF(G)$ of the group $T(G)$ and look for generators of $TF(G)$. We describe three methods for obtaining generators. Each of them only gives partial answers to the question but we obtain more precise results in some specific cases. We also conjecture that $TF(G)$ can be generated by modules belonging to the principal block and we prove the conjecture in some cases.

1. Introduction

If G is a finite group, let $T(G)$ be the group of equivalence classes of endotrivial kG-modules, where k is a field of characteristic p (assumed algebraically closed for simplicity). The abelian group $T(G)$ is finitely generated, hence of the form $T(G) = TT(G) \oplus F$, where $TT(G)$ is the torsion subgroup and F is a free abelian group. The purpose of this paper is to investigate the torsion-free part of $T(G)$, and in particular find generators for a suitable torsion-free direct summand F of $T(G)$. The non-uniqueness of F is actually an issue and so we work instead with the canonically defined free abelian group $TF(G) = T(G)/TT(G)$.

In many cases, $TF(G) \cong \mathbb{Z}$, generated by the class of the syzygy module $\Omega(k)$. Otherwise, by [9], G has maximal elementary abelian p-subgroups of rank 2 and its Sylow p-subgroup P has a rather special structure. In particular, the centre $Z(P)$ is cyclic, hence has a unique subgroup Z of order p.

In order to find generators for $TF(G)$, there are three known constructions, one using relative syzygies, one using suitable subquotients of a syzygy module $\Omega^n(k)$, and one involving a class in group cohomology restricting non-trivially to Z. We analyze the three constructions and extend as much as possible the results about them. The first construction works well for p-groups and also for groups with a normal Sylow p-subgroup, but it does not seem possible to extend the method to arbitrary finite groups. The second construction needs the assumption that Z is normal in G and hence cannot work otherwise. The third construction, which we call the cohomological pushout method, works well in general, but only rationally, not integrally : it provides generators for $\mathbb{Q} \otimes_{\mathbb{Z}} TF(G)$, but it produces only a subgroup of finite index in $TF(G)$. We can show that this subgroup is the whole of $TF(G)$ in some cases, but we also give examples where this is not so. The problem of describing generators of $TF(G)$ in full generality remains open, but our discussion shows where the difficulties lie and allows us to state specific questions to be solved.

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We then prove that the endotrivial modules in the principal block form a subgroup $T_0(G)$ of $T(G)$ and that $T_0(G)$ has finite index. We conjecture that $T(G) = T_0(G) + TT(G)$, in other words that $TF(G)$ can be generated by modules in the principal block. We prove that the conjecture holds in some cases, in particular if Z is a normal subgroup.

Finally we discuss control of p-fusion and we conjecture that if P is a common Sylow p-subgroup of G and G' and if a group homomorphism $\phi : G \to G'$ induces an isomorphism $\mathcal{F}_P(G) \to \mathcal{F}_P(G')$ between the canonical fusion systems of G and G' on P, then ϕ induces an isomorphism $TF(G') \to TF(G)$. We prove the conjecture in a few cases.

In a final section, we have gathered a number of examples illustrating various results of this paper.

A general remark about our methods may be useful. If $N_G(P)$ denotes the normalizer of a Sylow p-subgroup P of G, many results can be proved for $N_G(P)$ but the passage from $N_G(P)$ to G seems difficult. It is known that the restriction map $\mathrm{Res}^G_{N_G(P)} : T(G) \longrightarrow T(N_G(P))$ is injective, induced by the Green correspondence, but the non-surjectivity of this map is a crucial issue and is an obstacle for solving several of our problems (see Section 8).

2. Preliminaries

Throughout this paper, we let k denote an algebraically closed field of prime characteristic p. In addition, we assume that all modules are finitely generated.

Given a finite group H, we write k for the trivial kH-module, or, whenever H needs to be clarified, we write k_H instead. Unless otherwise specified, the symbol $\otimes$ is the tensor product $\otimes_k$ of the underlying vector spaces, and in case of kH-modules, then H acts diagonally on the factors. If M is a kH-module, and $\varphi : Q \to M$ its projective cover, then we let $\Omega^1(M)$, or simply $\Omega(M)$, denote the kernel of φ (called the first syzygy of M). Likewise, if $\vartheta : M \to Q$ is the injective hull of M (recall that kH is a self-injective ring so Q is also projective), then $\Omega^{-1}(M)$ denotes the cokernel of ϑ. Inductively, we set $\Omega^n(M) = \Omega(\Omega^{n-1}(M))$ and $\Omega^{-n}(M) = \Omega^{-1}(\Omega^{-n+1}(M))$ for all integers $n > 1$.

If G is a finite group of order divisible by p, then a kG-module M is *endotrivial* if its endomorphism algebra $\mathrm{End}_k(M)$ is isomorphic (as a kG-module) to the direct sum of the trivial module k_G and a projective kG-module. In other words, a kG-module M is endotrivial if and only if $M^* \otimes M \cong k \oplus (\mathrm{proj})$, where M^* denotes the k-dual $\mathrm{Hom}_k(M, k)$ of M, and (proj) some projective module. Recall the following basic results (see Section 2 in [9]).

Lemma 2.1. *Let G be a finite group of order divisible by p.*

(1) *Let M be a kG-module. If M is endotrivial, then M splits as the direct sum $M_\diamond \oplus (\mathrm{proj})$ for an indecomposable endotrivial kG-module $M_\diamond$, which is unique up to isomorphism.*

(2) *The relation*

$$M \sim N \iff M_\diamond \cong N_\diamond$$

on the class of endotrivial kG-modules is an equivalence relation. We let $T(G)$ be the set of equivalence classes. Every equivalence class contains a unique indecomposable module up to isomorphism.

(3) *The tensor product induces an abelian group structure on the set $T(G)$ by*

$$[M] + [N] = [M \otimes N] \ .$$

The zero element of $T(G)$ is the class $[k]$ of the trivial module, consisting of all modules of the form $k \oplus (\mathrm{proj})$. The inverse of the class of a module M is the class of the dual module M^*.

The group $T(G)$ is called the *group of endotrivial kG-modules.* It is known to be a finitely generated abelian group. In particular, the torsion subgroup $TT(G)$ of $T(G)$ is finite. We define $TF(G) = T(G)/TT(G)$. The rank of the free abelian group $TF(G)$ is called the torsion-free rank of $T(G)$ and has been determined in [9], extending work of Alperin [1]. If the p-rank of G is one, then $TF(G) = \{0\}$, so we can assume that the p-rank of G is at least 2. Let $\mathcal{C}^G$ be the set of all maximal elementary abelian p-subgroups of G and write

$$\mathcal{C}^G = \mathcal{C}_1^G \cup \ldots \cup \mathcal{C}_{n_G}^G$$

where we have the following:

(i) If the p-rank of G is 2, then $\mathcal{C}_1^G, \ldots, \mathcal{C}_{n_G}^G$ are the conjugacy classes of maximal elementary abelian p-subgroups (all of which have rank 2). If a Sylow p-subgroup P of G is not dihedral, one of these subgroups must be normal in P, say E_1, and we assume that $E_1 \in \mathcal{C}_1^G$.

(ii) If the p-rank of G is at least 3, then $\mathcal{C}_1^G$ is the set of all maximal elementary abelian p-subgroups of rank at least 3, and $\mathcal{C}_2^G, \ldots, \mathcal{C}_{n_G}^G$ are the conjugacy classes of maximal elementary abelian p-subgroups of rank 2 (if any).

For an elementary abelian p-group E of rank at least 2, Dade [15] proved that $T(E) \cong \mathbb{Z}$, generated by the class $[\Omega(k)]$. This is used in the following result, which is Theorem 3.1 in [9].

Theorem 2.2. *Assume that the p-rank of G is at least 2.*

(1) *The torsion-free rank of $T(G)$ is equal to the number n_G above.*
(2) *More precisely, choose $E_i \in \mathcal{C}_i^G$ for each $i = 1, \ldots, n_G$ and consider the homomorphism*

$$\Psi = \prod_{i=1}^{n_G} \mathrm{Res}_{E_i}^G : T(G) \longrightarrow \prod_{i=1}^{n_G} T(E_i) \cong \mathbb{Z}^{n_G}\,.$$

Then $\mathrm{Ker}(\Psi) = TT(G)$, $\mathrm{Im}(\Psi) \cong TF(G)$, *and* $\mathrm{Im}(\Psi)$ *has finite index in* $\mathbb{Z}^{n_G}$.
(3) *The map Ψ is independent of the choices of the subgroups E_i.*

Note that the most common situation occurs when the p-rank of G is at least 3 and there are no maximal elementary abelian p-subgroups of rank 2, so that $n_G = 1$. If $n_G = 1$, then $TF(G) \cong \mathbb{Z}$, generated by the class $[\Omega(k)]$, so the question of finding generators for $TF(G)$ really occurs when $n_G \geq 2$. We shall usually assume that $n_G \geq 2$.

We define the *type* of an endotrivial module M to be the n_G-tuple of integers given by $\Psi([M])$. The whole point is that, for each i, there exists an endotrivial module of type $(0, \ldots, 0, t, 0, \ldots, 0)$ for some integer t appearing in i-th position. One of the main issues when looking for generators of $TF(G)$ is to find the minimal value of t for each i. Recall that $\Omega^r(k)$ has type $(r, r, \ldots, r)$.

Our next result is general and does need the assumption that $n_G \geq 2$.

Proposition 2.3. *Let G be a finite group of order divisible by p and P a Sylow p-subgroup of G.*

(1) *The restriction map $\mathrm{Res}_P^G : T(G) \to T(P)$ has a finite kernel. In other words, it induces an injective homomorphism*

$$\mathrm{Res}_P^G : TF(G) \to TF(P)\,.$$

(2) *If G has a nontrivial normal p-subgroup, then the kernel of* $\mathrm{Res}^G_P : T(G) \to T(P)$ *consists of the classes of one-dimensional modules.*

Proof. (1) follows from part (2) of Theorem 2.2, or from Lemma 2.3 in [11], while (2) follows from Lemma 2.4 in [11]. □

We end this section with some notes on support varieties of kG-modules. The theory of support varieties played a key role in the classification of endotrivial modules over p-groups and it is a necessary ingredient in two of the constructions that appear in Sections 5 and 6. The books [5] and [14] serve as references for this material.

The basic definitions rely on the facts that the cohomology ring $\mathrm{H}^*(G,k)$ is a graded-commutative noetherian ring and for any finite-dimensional kG-modules M and N, the k-space $\mathrm{Ext}^*_{kG}(M,N)$ is a graded, finitely generated module over $\mathrm{H}^*(G,k) \cong \mathrm{Ext}^*_{kG}(k,k)$. For a finitely generated kG-module M, let $J(M)$ be the annihilator in $\mathrm{H}^*(G,k)$ of the cohomology module $\mathrm{Ext}^*_{kG}(M,M)$. Then the support variety $V_G(M)$ of M is the subvariety of the maximal ideal spectrum $V_G(k)$ of $\mathrm{H}^*(G,k)$ defined by the graded ideal $J(M)$. That is, $V_G(M)$ is the subset of $V_G(k)$ formed by all maximal ideals that contain $J(M)$. The support variety has many useful properties. Here are a few that we use in the paper.

Theorem 2.4. *Suppose that L, M and N are kG-modules.*

(1) $V_G(M) = \{0\}$ *if and only if M is projective.*
(2) $V_G(M \oplus N) = V_G(M) \cup V_G(N)$.
(3) $V_G(M \otimes N) = V_G(M) \cap V_G(N)$.
(4) $V_G(M) = \bigcup_E \mathrm{res}^*_{G,E}(V_E(M{\downarrow}^G_E))$, *where the union is over a set of representatives of the maximal elementary abelian subgroups of G and* $\mathrm{res}^*_{G,E} : V_E(M) \to V_G(M)$ *is the map on spectra induced by the restriction map on cohomology.*
(5) *If $V_G(M) = V_1 \cup V_2$ where V_1 and V_2 are closed sets such that $V_1 \cap V_2 = \{0\}$, then M has submodules M_1 and M_2 such that $V_G(M_1) = V_1$, $V_G(M_2) = V_2$ and $M \cong M_1 \oplus M_2$.*

3. Values of the type function

Our purpose is to describe more precisely the possible values of the type of endotrivial modules. We do this for a p-group P such that the torsion-free rank n_P of $TF(P)$ is at least 2. We shall see in later sections that the results extend to the case of a group G with Sylow p-subgroup P such that the unique central subgroup Z of P of order p is normal in G. This happens, for example, if P is normal in G.

We assume that the torsion-free rank n_P of $T(P)$ is at least 2, so that there exist maximal abelian elementary p-subgroups of rank 2. For $2 \leq i \leq n_P$, we choose a maximal elementary abelian p-subgroup E_i of rank 2 with $E_i \in \mathcal{C}^G_i$. The group E_i has the form $E_i = Z \times S_i$ where Z is the unique central subgroup of P of order p and S_i is a non-central subgroup of order p. Moreover, $N_P(S_i)/S_i = C_P(S_i)/S_i$ must have p-rank 1, by maximality of E_i. Thus $N_P(S_i)/S_i = C_P(S_i)/S_i$ is either cyclic or quaternion (where a quaternion group means a generalized quaternion 2-group). We define the integer m_i as follows:

$$m_i = \begin{cases} 1 & \text{if } C_P(S_i)/S_i \text{ is cyclic of order 2,} \\ 2 & \text{if } C_P(S_i)/S_i \text{ is cyclic of order} \geq 3, \\ 4 & \text{if } C_P(S_i)/S_i \text{ is quaternion.} \end{cases}$$

Our next result shows that given a finite p-group P with $n_P > 1$, then $m_2 = \cdots = m_{n_P}$.

Theorem 3.1. *Let P be a finite p-group with $n_P > 1$. Then $m_i = m_j$ for all $2 \le i, j \le n_P$. Moreover, the case where one value of m_i is equal to 1 only occurs if $p = 2$ and P is a dihedral 2-group (and $n_P = 2$).*

Proof. The statement holds if p is odd, as then all the factor groups $C_P(S_i)/S_i$ are cyclic of order at least 3, and therefore $m_2 = \cdots = m_{n_P} = 2$.

We are left with the case $p = 2$. We can assume that $n_P \ge 3$, otherwise there is nothing to prove. First assume that $m_i = 1$ for some i. Then $|C_P(S_i)| = 4$, so P has maximal class by a theorem of Suzuki (see Satz III.14.23 in [17]). Therefore, P is dihedral, semi-dihedral, or quaternion (see Theorem 5.4.5 in [16]). But a quaternion group has rank 1 and a semi-dihedral group has $n_P = 1$. So P is dihedral and has two conjugacy classes of maximal Klein four-groups. Thus $n_P = 2$ and there is nothing to prove. (Actually for both classes, we have $|C_P(S_i)/S_i| = 2$.)

Assume now that $m_i = 2$ for some $i \ge 2$, that is, $C_P(S_i) = S_i \times A$ where A is cyclic of order ≥ 4 (see Lemma 2.2 in [12]). For every $j \ge 2$, we must prove that $C_P(S_j) \cong S_j \times C_{2^q}$ for some $q \ge 2$ (in other words $C_P(S_j)/S_j$ is not quaternion). Thus we must prove that, for any non-central involution w of P, either $C_P(w) \cong C_2 \times C_{2^q}$ for some $q \ge 2$, or w is contained in a subgroup in $\mathcal{C}_1^P$. Recall that if the rank of P is 2, then $\mathcal{C}_1^P = \{E_1\}$ where $E_1 \triangleleft P$, and then $C_P(E_1)$ has index 2 in P, while if the rank of P is at least 3, then the subgroups in $\mathcal{C}_1^P$ have rank ≥ 3. We let also $Z = \langle z \rangle$ be the unique central subgroup of P of order 2.

By assumption, P contains an involution t whose centralizer has the form $C_P(t) = \langle t \rangle \times A$ with $A = \langle a \rangle \cong C_{2^m}$ for some $m \ge 2$. All such 2-groups have been classified by Janko, and we proceed by inspection of the eight possible isomorphism types, which we consider in the same order as listed by Berkovitch and Janko in [6, Theorem 48.1] (see [18, Theorem 1.1] for the original result). In the first four cases the rank of P is 2 and in the last four cases it is equal to 3.

In case (a), P is a dihedral 2-group, so P does not satisfy the assumption (because $m_i = 1$ and this case has actually been treated at the beginning of our proof). Case (b) does not occur, because it is that of quasi-dihedral groups, which have a unique elementary abelian subgroup of rank 2, hence $n_P = 1$, contrary to our assumption.

In case (c), P has rank 2, the subgroup $T = C_P(t)$ is a maximal subgroup of P and $Z(P)$ is cyclic of order ≥ 4 contained in T. Since T is abelian, $E_1 = \langle t, z \rangle$ is a normal elementary abelian subgroup of P, where z is the generator of Z, and E_1 is the unique subgroup in $\mathcal{C}_1^P$. For every involution w, that is in P but not in $T = C_P(t)$, we have $|C_P(w) : C_T(w)| = 2$, hence $C_P(w) = \langle w \rangle \times C_T(w)$. If $C_T(w)$ had rank 2, then $C_P(w)$ would have rank 3, which is impossible since P has rank 2. Thus $C_T(w)$ has rank 1, hence it is cyclic (because T is abelian) and of order ≥ 4 because it contains $Z(P)$, which is cyclic of order ≥ 4. Thus $C_P(w)$ has the desired form.

For the remaining five isomorphism types of such finite 2-groups P, we follow [6] and introduce the following notation. In addition to the above, P has a subgroup $S = AL$ of index at most 2, where $L = \langle t, b \mid t^2 = b^{2^{n-1}} = 1, b^t = b^{-1} \rangle$ is dihedral of order 2^n, $L \triangleleft P$, and $A \cap L = Z(L) = Z$. In particular, $|S| = 2^{n+m-1}$ with $m \ge 2$ and $n \ge 3$. The action of a on $\langle b \rangle$ is such that the two elements of order 4, $a^{2^{m-2}}$ and $b^{2^{n-3}}$, commute. Put also $c = a^{2^{m-2}} b^{2^{n-3}}$, which is an involution.

In case (d), every involution of P is contained in the central product $\langle a^{2^{m-2}} \rangle * L$. Thus an involution $w \in P$ is non-central if and only if either $w \in L - Z$, or $w = c$, or $w = cz$. For the involutions c and cz, we have $\langle c, z \rangle \triangleleft P$ because a and b centralize c and $c^t = a^{2^{m-2}} (b^{2^{n-3}})^t = a^{2^{m-2}} b^{2^{n-3}} z = cz$. This is the only four-group which is normal in P (unless L has order 8, see below), and hence $E_1 = \langle c, z\rangle$ belongs to $\mathcal{C}_1^P$. So we only have to consider the involutions in $L - Z$. If $w \in L - Z$, then w is L-conjugate to either t or tb, because L is dihedral. In the case where $S < P$, we know that t and tb fuse in P, so $n_P = 2$ and there is nothing to prove. So we can assume that $P = S = AL$. Then a centralizes t and stabilizes the L-conjugacy class of tb. Hence $(tb)^a = (tb)^y$, for some $y \in L$, and $\tilde{a} = ay^{-1}$ centralizes tb. We obtain isomorphic centralizers $C_P(t) = \langle t\rangle \times A$ and $C_P(tb) = \langle tb\rangle \times \tilde{A}$, where $\tilde{A} = \langle \tilde{a}\rangle$, hence the result. There is a slight technical problem if L is dihedral of order 8, because $E_1 = \langle c, z\rangle$, $E_2 = \langle t, z\rangle$, and $E_3 = \langle tb, z\rangle$ are all normal in P, and any of them can be chosen to be in $\mathcal{C}_1^P$, not necessarily E_1. But in that case all three centralizers are isomorphic (as one can easily check), so the result follows. This completes the discussion of the first four cases and we are left with the groups of rank 3.

In case (e), $P = S$ with $n \geq m + 2 \geq 5$, $b^a = b^{1+2^{n-m}}$, and $A/\langle z\rangle$ acts faithfully on L. An involution outside L must be of the form $a^{2^{m-2}}x$ with $x \in L$. By direct computation, we see that the involutions of P are conjugate to either t, tb, c, or $a^{2^{m-2}}tb$. Then $\langle c, z\rangle$ is normal in P with $c^a = c$, $c^b = cz$, and $c^t = cz$, and $C_P(c) = \langle a, b^2, tb\rangle$ is a maximal subgroup of P of rank 3. Both tb and $a^{2^{m-2}}tb$ have a centralizer of rank 3 because they commute with c and z. Hence t is the only remaining involution up to conjugation and $\langle t, z\rangle$ is the only maximal elementary abelian subgroup of rank 2 up to conjugation. Therefore $n_P = 2$ and there is nothing to prove.

In the last three possible isomorphism types of groups, we have $|P : S| = 2$ and, following [6], there is an involution $s \in P - S$, hence $P = S \rtimes \langle s\rangle$. In case (f), we have $n \geq m + 1 \geq 5$, $a^s = a^{1+2^{m-2}}b^{-2^{n-m}}$, $b^s = b^{-1}$, and $t^s = tb$. That is, $\langle b, s\rangle \cong D_{2^n}$ and $\langle t, s\rangle = L \rtimes \langle s\rangle \cong D_{2^{n+1}}$. Moreover, a routine computation gives $c^s = c$. Again, $\langle c, z\rangle$ is normal in P and $C_P(c) = \langle a, b, s\rangle$ is a maximal subgroup of P of rank 3. It follows that the involutions w of P are either involutions of L or they are contained in $C_P(c)$. If $w \in C_P(c)$, then $C_P(w)$ has rank 3, because it contains $\langle w, c, z\rangle$. Otherwise, we observe that the two conjugacy classes of involutions of L fuse, because $t^s = tb$. Thus t is the only remaining involution up to conjugation and $\langle t, z\rangle$ is the only maximal elementary abelian subgroup of rank 2 up to conjugation. Therefore $n_P = 2$ and there is nothing to prove.

In case (g), $S = A * L$ (central product over Z), and we have the same action of s on L, whereas $a^s \in aZ$. This gives us the same involutions as in (f) and again $n_P = 2$.

Case (h) is very similar, but with $a^s = a^{-1}c$ and $n, m \geq 4$. In particular, $n_P = 2$ once again. $\square$

Remark 3.2. In case P has rank 2, one may wonder if Theorem 3.1 holds in the range $1 \leq i, j \leq n_P$ instead of $2 \leq i, j \leq n_P$, but the following example shows that it not so. Take $P = (Q_8 \times \langle u\rangle) \rtimes \langle t\rangle$ where both u and t have order 2. The action of t swaps two generators of Q_8 and inverts their product, and moreover $u^t = uz$, where z is the generator of $Z(Q_8) = Z(P)$. Then $E_1 = \langle z, u\rangle$ is normal in P, so $E_1 \in \mathcal{C}_1^P$, and we get $C_P(u) = \langle u\rangle \times Q_8$. On the other hand $E_2 = \langle z, t\rangle$ belongs to $\mathcal{C}_2^P$ and we have $C_P(t) = \langle t\rangle \times A$ where A is cyclic of order 4. This is an example of a group in case (d) of the Berkovitch-Janko list.

Theorem 3.1 improves the result given as Theorem 7.1 in [12] or Theorem 7.2 in [8] as follows.

Theorem 3.3. *Let P be a p-group such that the torsion-free rank n_P of $TF(P)$ is at least 2. Let $m = m_2 = \cdots = m_{n_P}$.*

1. *For $2 \leq i \leq n_P$, there exists an endotrivial kP-module N_i with type* $(0, \ldots, 0, mp, 0, \ldots, 0)$ *(where mp appears in i-th position).*
2. *$T(P)$ is free abelian with generators* $[\Omega(k)]$, $[N_2]$, $\ldots$, $[N_{n_P}]$.

There are two available proofs based on two different constructions of the modules N_i. Both constructions will be recalled in the next sections (see Theorem 4.1 and Theorem 5.2).

4. Construction via relative syzygies

The construction of generators of $TF(G)$ as relative syzygies works well if a Sylow p-subgroup is normal in G. It does not seem to extend further to other cases. In this section, we review this construction of generators of $TF(G)$ under the assumption that a Sylow p-subgroup P of G is normal in G. Our results are only slight improvements of those obtained in [19], but we need the improvement later in Section 9.

We assume that the torsion-free rank n_P of $T(P)$ is at least 2 and we keep the notation of Section 3, with $m = m_2 = \cdots = m_{n_P}$. Then we define the module

$$N_i = \Omega^m(\Omega^{-m}_{P/S_i}(k)) ,$$

where $\Omega^{-m}_{P/S_i}(k)$ denotes the relative syzygy of the trivial module, relative to the subgroup S_i (see Section 3 in [12]). This is defined as the m-th cokernel in a minimal relative injective resolution of the trivial module. Alternatively, $\Omega^{-1}_{P/S_i}(k)$ is defined as the cokernel of the canonical map $k \to k[P/S_i]$ and then $\Omega^{-m}_{P/S_i}(k)$ is the unique indecomposable summand with vertex P in the tensor product $\Omega^{-1}_{P/S_i}(k)^{\otimes m}$.

Here is now a more precise version of Theorem 3.3. It is proved in [12] (using also [13]), but the statement now incorporates the improvement obtained in Theorem 3.1.

Theorem 4.1. *Let P be a p-group such that the torsion-free rank n_P of $TF(P)$ is at least 2.*

(1) *For $2 \le i \le n_P$, the kP-module N_i is endotrivial and has type $(0, \dots, 0, mp, 0, \dots, 0)$ (where mp appears in i-th position).*
(2) *$T(P)$ is free abelian with generators $[\Omega(k)]$, $[N_2]$, $\dots$, $[N_{n_P}]$.*

Actually, the modules in [12] are the dual modules of the modules N_i defined here, i.e. $N_i^* \cong \Omega^{-m}(\Omega^m_{P/S_i}(k))$. The advantage of our definition here is that the type is positive and also that there is a map $N_i \to k$ which splits on restriction to E_j for $j \neq i$, so that N_i resembles the module M_i constructed in Section 6 by the cohomological pushout method.

We assume that P is normal in G. When passing from P to G, some of the conjugacy classes of maximal elementary abelian p-subgroups of rank 2 may fuse (so in particular $n_G \le n_P$). For simplicity of notation, we assume that $E_1, E_2, \dots, E_{n_G}$ are representatives of $\mathcal{C}_1^G, \dots, \mathcal{C}_{n_G}^G$. Because of our assumption that P is normal in G, fusing of the classes means that for $n_G + 1 \le j \le n_P$, E_j is G-conjugate to one of $E_1, \dots, E_{n_G}$. Note that E_j can only fuse to E_1 if the p-rank of G is 2.

It was proved in [19] that the restriction map induces an isomorphism $T(G) \cong T(P)^G$. For every P-conjugacy class $\mathcal{C}_i^P$ with $2 \le i \le n_G$, an endotrivial module for G is constructed by first extending N_i to the stabilizer of $\mathcal{C}_i^P$ and then tensor-inducing to G. The stabilizer of $\mathcal{C}_i^P$ is equal to $PN_G(S_i) = PN_G(E_i)$ (see Lemma 3.5 in [19]) and we write for simplicity $J_i = PN_G(S_i)$. From the argument in [19], we know that N_i extends to J_i and our improvement is to make this extension more explicit (and more elementary).

Proposition 4.2. *Let G be a group with a normal Sylow p-subgroup P. Assume that the torsion-free rank n_P of $TF(P)$ is at least 2.*

(1) *For $2 \le i \le n_G$, the kJ_i-module $\Omega^{-m}_{J_i/S_i}(k_{J_i})$ is endotrivial and $\Omega^{-m}_{J_i/S_i}(k_{J_i}){\downarrow}^{J_i}_P \cong \Omega^{-m}_{P/S_i}(k_P)$.*

(2) *For $2 \le i \le n_G$, the kJ_i-module $\tilde{N}_i = \Omega^m(\Omega^{-m}_{J_i/S_i}(k_{J_i}))$ is endotrivial and $\tilde{N}_i{\downarrow}^{J_i}_P \cong N_i$.*

(3) *$T(G) = X(G) \oplus F$, where $X(G)$ is the subgroup of $T(G)$ generated by the classes of one-dimensional modules and where F is free abelian, generated by the classes of $\Omega(k)$ and of the tensor induced modules $\mathrm{Ten}^G_{J_i}(\tilde{N}_i)$ (for $2 \le i \le n_G$).*

(4) *Each of the modules $\mathrm{Ten}^G_{J_i}(\tilde{N}_i)$ has a trivial one-dimensional quotient, given by a map $\mathrm{Ten}^G_{J_i}(\tilde{N}_i) \to k_G$ which splits on restriction to E_j for any $j \ne i$, $1 \le j \le n_G$.*

Proof. For simplicity of notation, write $S = S_i$ and $J = J_i$, and assume that $C_P(S)/S$ is cyclic of order ≥ 3, so that $m = 2$. The other values of m are treated similarly. Let

$$\mathcal{E}: \qquad 0 \longrightarrow k_P \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow \Omega^{-2}_{P/S}(k_P) \longrightarrow 0$$

be a minimal relative injective resolution of the trivial kP-module (relative to S). For instance $Q_0 = k[P/S]$. Since J stabilizes the conjugacy class of S, every module in this sequence is J-invariant and therefore $\mathcal{E}{\uparrow}^J_P{\downarrow}^J_P$ is the direct sum of $|J/P|$ copies of $\mathcal{E}$. On the other hand, we claim that $\mathcal{E}{\uparrow}^J_P$ is a relative injective resolution of the kJ-module $k_P{\uparrow}^J_P$ (relative to S). Clearly $Q_i{\uparrow}^J_P$ is projective relative to S. Moreover the sequence $\mathcal{E}{\uparrow}^J_P{\downarrow}^J_S$ splits because

$$\mathcal{E}{\uparrow}^J_P{\downarrow}^J_S = \mathcal{E}{\uparrow}^J_P{\downarrow}^J_P{\downarrow}^P_S \cong (|J/P| \cdot \mathcal{E}){\downarrow}^P_S$$

and $\mathcal{E}{\downarrow}^P_S$ splits (by definition of a relative injective resolution).

Since relative injective resolutions are additive and $k_P{\uparrow}^J_P \cong k_J \oplus L$ for some kJ-module L, it follows that $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{F}'$, where

$$\mathcal{F}: \qquad 0 \longrightarrow k_J \longrightarrow R_0 \longrightarrow R_1 \longrightarrow \Omega^{-2}_{J/S}(k_J) \longrightarrow 0$$

is a minimal relative injective resolution of the trivial kJ-module and $\mathcal{F}'$ is a relative injective resolution of L. It follows that $\Omega^{-2}_{J/S}(k_J)$ is a direct summand of $\Omega^{-2}_{P/S}(k_P){\uparrow}^J_P$ and, since $\Omega^{-2}_{P/S}(k_P){\uparrow}^J_P{\downarrow}^J_P$ is the direct sum of $|J/P|$ copies of $\Omega^{-2}_{P/S}(k_P)$, we obtain

$$\Omega^{-2}_{J/S}(k_J){\downarrow}^J_P \cong \Omega^{-2}_{P/S}(k_P) \oplus \ldots \oplus \Omega^{-2}_{P/S}(k_P) \qquad (r \text{ summands})$$

where $1 \le r \le |J/P|$. On the other hand

$$\Omega^{-2}_{J/S}(k_J){\downarrow}^J_P \cong \Omega^{-2}_{P/S}(k_P) \oplus T$$

for some S-projective module T. This forces $r = 1$ and $T = 0$. Thus $\Omega^{-2}_{J/S}(k_J){\downarrow}^J_P \cong \Omega^{-2}_{P/S}(k_P)$, and in particular $\Omega^{-2}_{J/S}(k_J)$ is endotrivial. This proves (1).

For the proof of (2), note that $\Omega^2(\Omega^{-2}_{J/S}(k_J))$ is endotrivial by (1) and it is indecomposable. Since its restriction to P must remain indecomposable (by Proposition 2.6 in [9]), we must have

$$\Omega^2(\Omega^{-2}_{J/S}(k_J)){\downarrow}^J_P \cong \Omega^2(\Omega^{-2}_{P/S}(k_P))\,.$$

Now (3) was proved as Theorem 3.10 in [19].

For the proof of (4), we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2(\Omega^{-2}_{J/S}(k_J)) \arrow[r] \arrow[d] & I_0 \arrow[r] \arrow[d] & I_1 \arrow[r] \arrow[d] & \Omega^{-2}_{J/S}(k_J) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k_J \arrow[r] & R_0 \arrow[r] & R_1 \arrow[r] & \Omega^{-2}_{J/S}(k_J) \arrow[r] & 0
\end{tikzcd}$$

where I_0 and I_1 are projective kJ-modules and where the right-hand side map is the identity. Write $\tilde{N}_i = \Omega^2(\Omega^{-2}_{J/S}(k_J))$. Fix $j \neq i$ with $1 \leq j \leq n_G$. Since E_j intersects trivially S and all its conjugates, any module induced from S is projective on restriction to E_j. Therefore the two modules $R_0{\downarrow}^J_{E_j}$ and $R_1{\downarrow}^J_{E_j}$ are projective and so $\Omega^{-2}_{J/S}(k_J){\downarrow}^J_{E_j} \cong \Omega^{-2}(k_{E_j}) \oplus (\text{proj})$. Therefore, on restriction to E_j, we have two homotopy equivalent projective resolutions of $\Omega^{-2}(k_{E_j})$ and $\tilde{N}_i{\downarrow}^J_{E_j} = k_{E_j} \oplus (\text{proj})$. It follows that the map $\tilde{N}_i{\downarrow}^J_{E_j} \to k_{E_j}$ must be the shift of the identity, so it must be the identity on k_{E_j} and zero on a suitable projective complement. In other words, the map $\tilde{N}_i \to k_J$ splits on restriction to E_j. In particular, it is non-zero and k_J is isomorphic to a quotient of $\tilde{N}_i$. Now tensor inducing from J to G, we see that $\mathrm{Ten}^G_J(\tilde{N}_i)$ has a quotient isomorphic to $\mathrm{Ten}^G_J(k_J) \cong k_G$. Again this splits on restriction to E_j. □

5. Construction via ordinary syzygies

The second construction of generators for $TF(G)$ uses suitable subquotients of some ordinary syzygy $\Omega^n(k)$ of the trivial module. It is presented in Theorem 7.2 of [8] in the case of a p-group, and in Theorem 3.4 of [9] in the case where the group G has a normal Sylow p-subgroup P. We extend slightly this result by showing that it holds more generally if Z is normal in G, where Z is the unique central subgroup of P of order p. We cannot expect to extend further the method because it uses in an essential way the quotient group G/Z, hence the normality of Z. For an arbitrary finite group G with $n_G \geq 2$, the result of this section applies to $N_G(Z)$, so we will be left to consider the restriction map $\mathrm{Res}^G_{N_G(Z)} : T(G) \longrightarrow T(N_G(Z))$, which is injective, but its image is not easy to control (see Section 8).

Let G be a group such that the torsion-free rank n_G of $TF(G)$ is at least 2. Let P be a Sylow p-subgroup of G, let Z be the unique central subgroup of P of order p, and assume that Z is normal in G. Let $\overline{G} = G/Z$, $\overline{P} = P/Z$, and similarly for other subgroups containing Z. For $2 \leq i \leq n_G$, we let $E_i \in \mathcal{C}^G_i$. Then E_i has the form $E_i = Z \times S_i$ where S_i is a non-central subgroup of order p. We let $m = m_2$ be the integer defined in Section 3 and we set $a = mp$. Recall that Theorem 3.1 says that $m = m_i$ for each $2 \leq i \leq n_P$. In [8] and [9], the integer a is defined differently when $p = 2$, but we first check that both definitions agree.

Lemma 5.1. *Assume that $p = 2$. With the notation above, there is a unique elementary abelian 2-subgroup $\overline{A}_i$ of $\overline{P}$ which has maximal order subject to the condition that $\overline{E}_i \subseteq \overline{A}_i$. In fact, the group $\overline{N_P(E_i)}$ is abelian, hence contains a unique maximal elementary abelian 2-subgroup $\overline{A}_i$. Moreover, if A_i denotes its inverse image in G, then $|A_i|/4 = |\overline{A}_i|/2 = a$.*

Proof. Let $A_i \subseteq P$ be a subgroup such that $E_i \subseteq A_i$ and $\overline{A}_i$ is a maximal elementary abelian subgroup. Without loss, by Theorem 3.1, we may take $i = 2$ and let $A = A_2$ and $E = E_2$. Because $\overline{A}$ is abelian, E is a normal subgroup of A, hence $A \subseteq N_P(E)$. By Lemma 2.2 in [12], $N_P(E)$ is a central product $N_P(E) = D_8 * L$, where D_8 is dihedral of order 8 and L is either cyclic or quaternion. Thus $\overline{N_P(E)}$ is abelian and so $\overline{A}$ is unique. Moreover $L \cong C_P(S)/S$, so that, by definition of m in Section 3, we obtain

$$a = 2m = \begin{cases} 2 & \text{if } L \text{ is cyclic of order } 2, \\ 4 & \text{if } L \text{ is cyclic of order } \geq 4, \\ 8 & \text{if } L \text{ is quaternion.} \end{cases}$$

If L is cyclic of order 2, then $N_P(E) = D_8 = A$, hence $|\overline{A}| = 4$. If L is cyclic of order ≥ 4, then $N_P(E) = D_8 * L$ contains $A = D_8 * C_4$ and $|\overline{A}| = 8$. If L is quaternion, then $N_P(E) = D_8 * L$ contains $A = D_8 * Q_8$ and $|\overline{A}| = 16$. □

Let $Z = \langle z \rangle$ be the unique central subgroup of order p in P. For any kG-module M, we let $M_0 = \{m \in M \mid (z-1)^{p-1}m = 0\}$ and $\overline{M} = M/M_0$. Notice that $\overline{M}$ is a $k\overline{G}$-module. Applying this construction to the module $M = \Omega^a(k)$, we obtain a $k\overline{G}$-module $\overline{\Omega^a(k)}$.

We now come to the second more precise version of Theorem 3.3, which extends Theorem 7.2 of [8] and Theorem 3.4 of [9].

Theorem 5.2. *Let G be a group such that the torsion-free rank n_G of $TF(G)$ is at least 2. Let P be a Sylow p-subgroup of G, let Z be the unique central subgroup of P of order p, and assume that Z is normal in G. For $1 \leq i \leq n_G$, we let $E_i \in \mathcal{C}_i^G$ and we let $a = mp$.*

1. *For $2 \leq i \leq n_G$, there is a subquotient N_i of $\Omega^a(k)$ which is endotrivial and has type $(0, \ldots, 0, a, 0, \ldots, 0)$ (where a appears in i-th position).*
2. *$TF(G)$ is generated by $[\Omega(k)]$, $[N_2]$, $\ldots$, $[N_{n_G}]$.*

Proof. We follow quite closely the proof of Theorem 3.4 of [9] and indicate the necessary modifications. First it should be noted that one of the main ingredients is Theorem 4.2 in [8], which is explicitly stated for a group in which Z is normal. The second main ingredient is Theorem 7.2 of [8], which treats the case of a p-group.

Now we fix i such that $2 \leq i \leq n_G$, we let $\mathcal{C}$ be the set of subgroups in $\mathcal{C}_i^G$ which are contained in P, and we decompose

$$\mathcal{C} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_r \,,$$

where each $\mathcal{A}_t$ is a P-conjugacy class of maximal elementary abelian p-subgroups of rank 2 in P (for $1 \leq t \leq r$). Then, as in the proof of Theorem 3.4 of [9], the variety $V_{\overline{P}}\big(\overline{\Omega^a(k_P)}\big)$ of the $k\overline{P}$-module $\overline{\Omega^a(k_P)}$ decomposes as follows:

$$V_{\overline{P}}\big(\overline{\Omega^a(k_P)}\big) = (W_1 \cup \ldots \cup W_r) \cup \hat{W} \,,$$

where $W_t = \operatorname{res}^*_{\overline{P},\overline{A_t}}(V_{\overline{A_t}}(k))$ and $A_t \in \mathcal{A}_t$ for $1 \leq t \leq r$, and such that $(W_1 \cup \ldots \cup W_r) \cap \hat{W} = \{0\}$.

Since all the subgroups A_t fuse in G and become conjugate to E_i, we obtain

$$\operatorname{res}^*_{\overline{G},\overline{P}}(W_t) = \operatorname{res}^*_{\overline{G},\overline{P}} \operatorname{res}^*_{\overline{P},\overline{A_t}}(V_{\overline{A_t}}(k)) = \operatorname{res}^*_{\overline{G},\overline{A_t}}(V_{\overline{A_t}}(k)) = \operatorname{res}^*_{\overline{G},\overline{E_i}}(V_{\overline{E_i}}(k)) \,,$$

and therefore the variety $V = \operatorname{res}^*_{\overline{G},\overline{P}}(W_t)$ is independent of t for $1 \leq t \leq r$. It follows that

$$V_{\overline{G}}\big(\overline{\Omega^a(k_G)}\big) = V \cup \hat{V} \,,$$

where $\hat{V} = \operatorname{res}^*_{\overline{G},\overline{P}}(\hat{W})$, and such that $V \cap \hat{V} = \{0\}$. Then by Theorem 2.4 (5), $\overline{\Omega^a(k_G)} \cong M_1 \oplus M_2$, where $V_G(M_1) = V$ and $V_G(M_2) = \hat{V}$. By similar means and with some additional work, as in Lemma 4.1 and Theorem 4.2 of [8], we show that $(z-1)\Omega^a(k_G) \cong L_1 \oplus L_2$, where

$(z-1)^r L_j/(z-1)^{r+1}L_j \cong M_j$ for all $r = 0, \dots, p-2$ and $j = 1, 2$. Let U_j be the kernel of the composition

$$\Omega^a(k_G) \longrightarrow \overline{\Omega^a(k_G)} \longrightarrow M_j$$

where the first map is the natural quotient and the second is the projection. Continuing the argument of Theorem 3.3 in [9] (which is a restatement of Theorem 4.2 in [8]), we obtain that the subquotient $N_i = U_2/L_2$ is an endotrivial module such that

$$N_i{\downarrow}^G_{E_i} \cong \Omega^a(k) \oplus (\mathrm{proj}) \qquad \text{and} \qquad N_i{\downarrow}^G_{E_j} \cong k \oplus (\mathrm{proj}) \quad \text{for } 1 \leq j \leq n_G\,,\ j \neq i\,,$$

so N_i is of type $(0, \dots, 0, a, 0, \dots, 0)$ with a in i-th position.

On restriction to the Sylow p-subgroup P, the types have more components (because $n_P \geq n_G$) and we see that the type of $N_i{\downarrow}^G_P$ is

$$(0, \dots, 0, a, \dots, a, 0, \dots, 0)$$

with a repeated r times, where r is, as above, the number of P-conjugacy classes of maximal elementary abelian p-subgroups of rank 2 in P which fuse with E_i in G. Clearly, the restriction to P of any endotrivial kG-module M must have a type which is constant on those r conjugacy classes (and this remark holds for each i). By subtracting a multiple $c[\Omega(k)]$ for some $c \in \mathbb{Z}$, the first component of the type of $[M'] = [M] - c[\Omega(k)]$ can be assumed to be zero. Then, we know from Theorem 4.1 that the j-th component of the type of $[M'{\downarrow}^G_P]$ is an integral multiple of $a = mp$. Therefore, since the components are constant on elementary abelian p-subgroups of P which fuse in G, the class $[M'{\downarrow}^G_P]$ in $TF(P)$ must be a $\mathbb{Z}$-linear combination of the classes of $[N_2{\downarrow}^G_P]$, $\dots$, $[N_{n_G}{\downarrow}^G_P]$. It follows that $[M{\downarrow}^G_P]$ is a $\mathbb{Z}$-linear combination of $[\Omega(k){\downarrow}^G_P]$, $[N_2{\downarrow}^G_P]$, $\dots$, $[N_{n_G}{\downarrow}^G_P]$. Since $\mathrm{Res}^G_P : TF(G) \to TF(P)$ is injective by Proposition 2.3, $[M]$ is a $\mathbb{Z}$-linear combination of $[\Omega(k)]$, $[N_2]$, $\dots$, $[N_{n_G}]$, as required. $\square$

6. Construction via a cohomological pushout

The third construction of generators for $TF(G)$ uses a cohomology class restricting non-trivially to the unique central subgroup Z of order p in a Sylow p-subgroup P. In contrast with the previous two constructions, it can be used for any finite group G. However, it does not produce integral generators of $TF(G)$, but only generators of $\mathbb{Q} \otimes_{\mathbb{Z}} TF(G)$. We recall the construction and generalize it slightly. We also give examples where it does not produce integral generators of $TF(G)$.

For the discussion of $TF(G)$, we can assume that a Sylow p-subgroup P has maximal elementary abelian subgroups of rank 2. In particular, $Z(P)$ must be cyclic and we let Z be the unique central subgroup of P of order p. As in Section 2, we let $n = n_G$ and we choose $E_i \in \mathcal{C}^G_i$ for each $i = 1, \dots, n$.

Let q be the minimal positive integer such that there exists a one-dimensional kG-module W and a cohomology class $\zeta \in \mathrm{H}^q(G, W)$ with the property that $\mathrm{res}_{G,Z}(\zeta)$ is not nilpotent in $\mathrm{H}^q(Z, k)$ (note that W is trivial on restriction to Z). From the cohomology of cyclic groups, we see that q must be even if p is odd. The condition can be rewritten as $\mathrm{res}_{G,Z}(\zeta) \neq 0$, provided we assume that q is even if p is odd.

We only need to extend some of the results from Section 4 of [8] by allowing for a one-dimensional module W instead of the trivial module. If $\zeta \in \mathrm{H}^q(G, W)$ satisfies the above property and if $\hat{\zeta} : \Omega^q(k) \to W$ represents ζ, then the variety of the module $L_\zeta = \mathrm{Ker}(\hat{\zeta})$, decomposes as

$$V_G(L_\zeta) = V_1 \cup \ldots \cup V_n$$

with $V_i \cap V_j = \{0\}$ for $i \neq j$, where $V_i = \mathrm{res}^*_{G,E_i}(V_{E_i}(L_\zeta))$. Consequently, L_ζ decomposes as

$$L_\zeta = L_1 \oplus \ldots \oplus L_n$$

where $V_G(L_i) = V_i$. Now consider the module

$$M_i = \Omega^q(k) \big/ (\oplus_{j \neq i} L_j)$$

and the map $\zeta_i : M_i \to W$ induced by $\hat{\zeta}$, which has kernel L_i. In other words, M_i appears as a pushout in the following diagram :

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \oplus_{j\neq i} L_j \arrow[r] \arrow[d] & \oplus_{j\neq i} L_j \arrow[d] & & \\
0 \arrow[r] & L_\zeta \arrow[r] \arrow[d] & \Omega^q(k) \arrow[r, "\hat{\zeta}"] \arrow[d] & W \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_i \arrow[r] \arrow[d] & M_i \arrow[r, "\zeta_i"] \arrow[d] & W \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

For this reason, this construction of M_i is called the *cohomological pushout method.*

The following was proved in [8] and [9] in case W is the trivial module, but the proof extends without change to the case of a one-dimensional module W, because its restriction to any p-group is trivial.

Theorem 6.1. *Let q, W, ζ, L_i, M_i, and ζ_i be as above, in particular, with $\mathrm{res}_{G,Z}(\zeta)$ not nilpotent.*

1. *For each i, the kG-module M_i is endotrivial.*
2. *The map ζ_i splits on restriction to E_j for $j \neq i$. In particular, ζ_i splits on restriction to Z.*
3. *M_i is of type $(0, \ldots, 0, q, 0, \ldots, 0)$ with q in i-th position.*
4. *The classes $[\Omega(k)]$, $[M_2], \ldots, [M_n]$ generate a torsion-free subgroup of $T(G)$ of finite index.*

The question now is whether the above classes generate $TF(G)$. Some positive answers to this question will be considered in Section 7. However, it often occurs that the classes above do not generate $TF(G)$, but only a subgroup of finite index. The problem is that we obtain a module M_i of type $(0, \ldots, 0, q, 0, \ldots, 0)$, but an endotrivial module may exist with type $(0, \ldots, 0, r, 0, \ldots, 0)$ and r dividing q. In other words, it is not sufficient to require the minimality of q with the property that there exists W and $\zeta \in \mathrm{H}^q(G, W)$ such that $\mathrm{res}_{G,Z}(\zeta)$ is not nilpotent.

This problem already occurs for p-groups (in which case W must be the trivial module). By Theorem 4.1, we know the minimal value $r = mp$ such that there exists an endotrivial module with type $(0, \ldots, 0, r, 0, \ldots, 0)$. When $p = 2$, there is an example of a 2-group of order 64 with $q = 8$ and $r = 4$ (see Example 8.2 in [8]). When p is odd, we present an example of a group of order 3^5 with $q = 18$ and $r = 6$ (see Example 11.2).

One may wonder whether the one-dimensional module W may be chosen to be the trivial module, but there are examples where the minimal value of q occurs only with a non-trivial one-dimensional module W (see Example 11.1).

For reference, we present one specific result on the existence of cohomology elements restricting non-trivially to the center of the Sylow p-subgroup. It is a standard result and we present a proof for completeness.

Lemma 6.2. *Assume that $p > 2$. Let G be a finite group with the property that the rank n_G of the group $TF(G)$ is at least two. Let P be a Sylow p-subgroup of G, let Z be the unique subgroup of order p in the center of P, and let $N = N_G(Z)$. If $N \neq C_G(Z)$, then there exists no element ζ in $\mathrm{H}^{2p}(G,k)$ such that $res_{G,Z}(\zeta) \neq 0$.*

Proof. The natural action of N on Z induces an injection $N/C_G(Z) \to \mathrm{Aut}(Z) \cong \mathbb{Z}/(p-1)\mathbb{Z}$. So there exists a p'-element x in N such that x and $C_G(Z)$ generate N. Let H be the subgroup generated by x and Z. Then $H = Z \rtimes \langle x \rangle$ has order pa where a is relatively prime to p. The restriction map $\mathrm{res}_{G,Z}$ factors through $\mathrm{res}_{H,Z}$. Thus we can prove the lemma by showing that $\mathrm{H}^{2p}(H,k) = \{0\}$.

Because Z is a cyclic p-group, the groups $\mathrm{H}^m(Z,k)$ have k-dimension one for all m. In particular, $\mathrm{H}^2(Z,k)$ is spanned by a single element γ and the action of x on $\mathrm{H}^2(Z,k)$ is given by $\gamma^x = \omega\gamma$ where ω is a primitive t^{th}-root of unity, t being the order of the image of x in $\mathrm{Aut}(Z)$. Therefore, $(\gamma^p)^x = \omega^p\gamma^p \neq \gamma^p$, where $\gamma^p \in \mathrm{H}^{2p}(Z,k)$ is the p-th power of γ, and it spans $\mathrm{H}^{2p}(H,k)$. Because $\mathrm{H}^{2p}(H,k)$ is the space of H-fixed points in $\mathrm{H}^{2p}(Z,k)$, we must have that $\mathrm{H}^{2p}(H,k) = \{0\}$, as asserted. ◻

7. When the cohomological pushout method is optimal

When the cohomological pushout method of Section 6 produces generators for the whole of $TF(G)$ (and not merely a subgroup of finite index), we say that the method is *optimal* for the group G. In this section, we prove that there are various cases where the method is optimal. We start with some specific p-groups.

Proposition 7.1. *Let $P = C_p \wr C_p = E_0 \rtimes C_p$ where $E_0 = (C_p)^p$ and consider the normal subgroups $E_i = [P, E_{i-1}]$, for $1 \leq i \leq p$ (forming the lower central series of P). Consider the group $Q = P/E_i$ where $2 \leq i \leq p$. Then the cohomological pushout method is optimal for the group Q. In particular, this holds for P itself and for the extraspecial p-group P/E_2 of order p^3.*

Proof. Let $Q = P/E_i$ and $F = E_0/E_i$, so that $Q = F \rtimes C_p$. We have $Z = Z(Q) = E_{i-1}/E_i$. Since F is elementary abelian, there exists $\zeta \in \mathrm{H}^m(F,k)$ whose restriction to Z is not zero, where $m = 2$ if p is odd and $m = 1$ if $p = 2$. Let $\eta = \mathcal{N}\mathrm{orm}_{F,Q}(\zeta) \in \mathrm{H}^{mp}(Q,k)$. Using the Mackey formula for the norm map, we obtain

$$\begin{aligned} \mathrm{res}_{Q,Z}(\eta) &= \prod_{g \in [Z\backslash Q/F]} \mathcal{N}\mathrm{orm}_{Z\cap {}^gF, Z}\ \mathrm{conj}_g\ \mathrm{res}_{F, Z^g \cap F}(\zeta) \\ &= \prod_{g \in [Q/F]} \mathcal{N}\mathrm{orm}_{Z,Z}\, \mathrm{res}_{F,Z}(\zeta) = \mathrm{res}_{F,Z}(\zeta)^p \neq 0\,. \end{aligned}$$

Since ζ has degree m, the cohomology class η has degree mp and, by Theorem 6.1, we obtain endotrivial kQ-modules M_i of type $(0,\ldots,0,mp,0\ldots,0)$, with mp in i-th position and $2 \leq i \leq n_Q$.

But we know from Theorem 4.1 that the values mp are the minimal possible values for generators of $TF(Q)$ (because $C_Q(S_i)/S_i$ is cyclic of order p, so $m = 1$ if $p = 2$ and $m = 2$ if p is odd). Therefore $TF(Q)$ is generated by the classes of $\Omega(k), M_2, \ldots, M_{n_Q}$ (and in fact $TF(Q) = T(Q)$ since $TT(Q) = 0$ by [13]).

The two special cases correspond to the values $i = p$ (with $E_p = \{1\}$) and $i = 2$. □

Note that if p is odd, then P/E_2 is extraspecial of exponent p, while if $p = 2$, then $P = P/E_2$ is dihedral of order 8. Note also that the module M_i must be isomorphic to the module N_i of Theorem 4.1 because both are indecomposable of same type and, for the p-group P, the restriction map from $T(P)$ to the product $\prod_E T(E)$, taken over all elementary abelian subgroups E is injective.

We now show that we can always push the method from P to the centralizer of the subgroup Z.

Proposition 7.2. *Let G be a group such that the torsion-free rank n_G of $TF(G)$ is at least 2. Let P be a Sylow p-subgroup of G and let Z be the unique central subgroup of order p in P. If the cohomological pushout method is optimal for the group P, then so it is for the group $C_G(Z)$.*

Proof. We may assume that $G = C_G(Z)$, that is, Z is central in G. We prove that the cohomological pushout method is optimal for the group G by constructing a cohomology class as follows. Let q be the minimal positive integer such that there exists $\zeta \in \mathrm{H}^q(P, k)$ with $\mathrm{res}_{P,Z}(\zeta)$ not nilpotent. Let $\eta = \mathrm{Tr}_{P,G}(\zeta)$, where $\mathrm{Tr}_{P,G}$ is the transfer map. Using the Mackey formula and the fact that any cohomology class in $\mathrm{H}^*(Z, k)$ is invariant under G-conjugation, because Z is central, we obtain

$$\begin{aligned} \mathrm{res}_{G,Z}(\eta) &= \sum_{g\in[Z\backslash G/P]} \mathrm{Tr}_{Z\cap {}^gP,Z}\ \mathrm{conj}_g\ \mathrm{res}_{P,Z^g\cap P}(\zeta) \\ &= \sum_{g\in[G/P]} \mathrm{Tr}_{Z,Z}\ \mathrm{res}_{P,Z}(\zeta) = |G : P|\, \mathrm{res}_{P,Z}(\zeta)\,. \end{aligned}$$

Since $|G : P|$ is prime to p, this is not nilpotent.

If $\hat{\eta} : \Omega^q(k_G) \to k_G$ represents η and $L_\eta = \mathrm{Ker}(\hat{\eta})$, then

$$V_G(L_\eta) = V_1 \cup \ldots \cup V_n \quad \text{and} \quad L_\eta = L_1 \oplus \ldots \oplus L_n$$

where $n = n_G$ and $V_i = \mathrm{res}^*_{G,E_i}(V_{E_i}(L_\zeta))$. For $2 \le i \le n$, we obtain an endotrivial module

$$M_i = \Omega^q(k_G)\big/(\oplus_{j\neq i} L_j)$$

of type $(0, \ldots, 0, q, 0, \ldots, 0)$ with q in i-th position. We may have several P-conjugacy classes of maximal elementary abelian subgroups which fuse into the single G-conjugacy class $\mathcal{C}_i^G$ and so the type of $M_i{\downarrow}_P^G$ may have several values q corresponding to elementary abelian subgroups ${}^gE_i$ where $g \in G$ and ${}^gE_i \le P$. In other words, M_i has the property that, for any elementary abelian p-subgroup E of rank 2 of P,

$$[M_i{\downarrow}_E^G] \cong \begin{cases} 0 & \text{if } E \text{ is not } G\text{-conjugate to } E_i, \\ [\Omega^q(k_E)] & \text{if } E = {}^gE_i \text{ for some } g \in G. \end{cases}$$

Now any endotrivial kG-module M has the property that, if $[M{\downarrow}_E^G] = [\Omega^r(k_E)]$, then also $[M{\downarrow}_{^gE}^G] = [\Omega^r(k_{^gE})]$ for any $g \in G$ with ${}^gE \le P$. Thus the type of $M{\downarrow}_P^G$ must be constant on G-conjugacy classes. Since the cohomological pushout method is optimal for the group P, the class of $M{\downarrow}_P^G$ in $TF(P)$ is generated by classes with type $(1, 1, \ldots, 1)$ and $(0, \ldots, 0, q, 0, \ldots, 0)$ with q in j-th position, for each j with $2 \le j \le n_P$. In view of the injectivity of the restriction map to elementary abelian p-subgroups (Theorem 2.2) and because the type is constant on G-conjugacy classes, it follows that the class of M in $TF(G)$ must be generated by the classes of the modules $\Omega(k)$ and M_i for $2 \leq i \leq n_G$. Thus the cohomological pushout method is optimal for the group G. □

8. Green correspondence

Let $H = N_G(P)$ be the normalizer of a Sylow p-subgroup P of G. By Proposition 2.6 in [9], we know that $\mathrm{Res}^G_H : T(G) \longrightarrow T(H)$ is injective (induced by the Green correspondence). Clearly, this induces a group homomorphism

$$\mathrm{Res}^G_H : TF(G) \longrightarrow TF(H) \,.$$

This is is still injective because if $\mathrm{Res}^G_H([M])$ is torsion, then $\mathrm{Res}^G_H(n[M]) = 0$ for some n, hence $n[M] = 0$.

By Proposition 4.2, any element of $T(H)$ is the class of a module of the form $A \otimes M$, where M is torsion-free, generated by the modules of Proposition 4.2, and A is a one-dimensional module. However, we do not know in general which kH-modules have a Green correspondent which is endotrivial.

Question 8.1. Keep the notation above.

(1) If A is one-dimensional, find conditions implying that the Green correspondent of A is endotrivial.
(2) If M is a generator of $TF(H)$ (as constructed by one of our methods), is its Green correspondent endotrivial ?

For the first question, there are examples where the Green correspondent L is not endotrivial (see Example 11.3, or more generally Theorem 6.2 in [9]) and others where L turns out to be endotrivial (e.g. when $G = S_{2p}$ and L is the Young module $Y^{(p,p)}$, see Proposition 7.1 in [10]). A general answer to this first question has been obtained recently by Balmer [3] but, although elementary, it involves some technicalities which do not seem to be easy to deal with. For the second question, we have no example where the Green correspondent is not endotrivial, but we have too little evidence to make a conjecture.

In principle, it might happen that the Green correspondents of A and M are not endotrivial while the Green correspondent of $A \otimes M$ is endotrivial. This difficulty seems, at present, hard to overcome. In other words, if an element of $TF(H) = T(H)/TT(H)$ is in the image of Res^G_H, it does not mean that all its representatives in $T(H)$ are in the image of Res^G_H. So we see that the non-uniqueness of a torsion-free complement of the torsion subgroup is an important issue.

9. Blocks

Let $B_0(G)$ denote the principal block of the finite group G. We let $T_0(G)$ be the set of all classes in $T(G)$ such that the only indecomposable module in the class (up to isomorphism) belongs to $B_0(G)$. As instances of classes in $T_0(G)$, we have of course $[\Omega^r(k)]$, but also the classes of the modules $N_i = \Omega^m(\Omega^{-m}_{P/S_i}(k))$ of Theorem 4.1, the classes of the modules $\mathrm{Ten}^H_{J_i}(\tilde{N}_i)$ of Proposition 4.2, and the classes of the modules M_i of Theorem 6.1, because in each of those three cases, there is a non-zero map from the module to the trivial module k.

We first observe the following.

Proposition 9.1. *Let G be a finite group.*

1. *$T_0(G)$ is a subgroup of $T(G)$.*
2. *$T_0(G)$ has finite index in $T(G)$.*

Proof. If M belongs to $B_0(G)$, then so does its dual M^*. So $T_0(G)$ is invariant under passage to additive inverses. Now let M be an indecomposable endotrivial kG-module belonging to $B_0(G)$. Since tensoring with M is a stable equivalence, with inverse obtained by tensoring with M^*, we have isomorphisms

$$\mathrm{Ext}^1_{kG}(S,T) \cong \mathrm{Ext}^1_{kG}(M \otimes S, M \otimes T)$$

and it follows that if S and T belong to the same block, then so do S' and T', where S' and T' are indecomposable modules such that $M \otimes S \cong S' \oplus (\mathrm{proj})$ and $M \otimes T \cong T' \oplus (\mathrm{proj})$. Applying this to $S = k$ and T in $B_0(G)$, we see that M belongs to the same block as the indecomposable module T' in the class of $M \otimes T$. (In other words, tensoring with M preserves the principal block.) This means that T' belongs to $B_0(G)$, or in other words, that $[M] + [T] = [M \otimes T]$ belongs to $T_0(G)$. This completes the proof of (1).

For the proof of (2), we let H be the normalizer of a Sylow p-subgroup and we note that, by Proposition 4.2, $T(H) = X(H) \oplus F$, where F is free abelian generated by modules belonging to $B_0(H)$. Thus $T_0(H)$ has finite index in $T(H)$. Now

$$\mathrm{Res}^G_H : T(G) \longrightarrow T(H)$$

is induced by the Green correspondence, which preserves the principal blocks. Thus $T_0(G)$ is the kernel of

$$T(G) \longrightarrow T(H) \longrightarrow T(H)/T_0(H)\,.$$

Therefore $T(G)/T_0(G)$ embeds in the finite group $T(H)/T_0(H)$, so $T_0(G)$ has finite index in $T(G)$. □

Part (2) of Proposition 9.1 suggests the following conjecture.

Conjecture 9.2. *For any finite group G, the group $TF(G)$ can be generated by classes of modules belonging to $B_0(G)$. Equivalently, $T(G) = TT(G) + T_0(G)$. Equivalently, there exists a torsion-free subgroup F of $T(G)$ such that $T(G) = TT(G) \oplus F$ and $F \subseteq T_0(G)$.*

Proposition 9.3. *Conjecture 9.2 holds in each of the following cases:*

1. *The group G has a normal Sylow p-subgroup.*
2. *The group G has the property that the cohomological pushout method is optimal, for a class in cohomology with trivial coefficients (i.e. the endotrivial modules of Theorem 6.1 generate $TF(G)$, with the additional assumption that $W = k$).*
3. *The subgroup $N_G(P)$ is strongly p-embedded in G, where P is a Sylow p-subgroup of G.*

Proof. (1) If G has a normal Sylow p-subgroup P, we apply Proposition 4.2. Each of the modules $\Omega(k)$ and $\mathrm{Ten}^G_{J_i}(\tilde{N}_i)$ belongs to the principal block, because there is a non-trivial homomorphism $\mathrm{Ten}^G_{J_i}(\tilde{N}_i) \to k$.

(2) Suppose that $TF(G)$ is generated by the classes of the modules $\Omega(k)$, M_2, ..., M_n of Theorem 6.1. Again there is a non-trivial homomorphism $\zeta_i : M_i \to k$, so M_i belongs to $B_0(G)$.

(3) When $N_G(P)$ is strongly p-embedded in G, the restriction map $\mathrm{Res}^G_{N_G(P)} : T(G) \longrightarrow T(N_G(P))$ is an isomorphism (by Remark 2.9 in [9]), induced by the Green correspondence, which preserves the principal block. □

Note that the situation for a group G with a normal Sylow p-subgroup is quite clear. By Proposition 4.2, the classes of the modules $\Omega(k)$ and $\mathrm{Ten}_{J_i}^G(\tilde{N}_i)$ (for $2 \le i \le n_G$) generate a torsion-free subgroup F such that $T(G) = X(G) \oplus F$. Tensoring the indecomposable modules in F by a one-dimensional module A, we get modules which all belong to the block containing A. If A belongs to $B_0(G)$, then we get again modules in $B_0(G)$. Otherwise, we get modules in a different block.

Remark 9.4. Let $\mathrm{stmod}(kG)$ denote the stable category of finitely generated modules modulo projectives. It should be noted that if the cohomological pushout method is optimal for constructing endotrivial modules (using $\zeta \in \mathrm{H}^q(G,k)$ for some values of q), then all of the endotrivial modules constructed by the method are in the thick subcategory $\mathrm{Thick}(k)$ of $\mathrm{stmod}(kG)$ generated by the trivial module k. This is because, in the diagram preceding Theorem 6.1, with $W = k$, the module $\Omega^q(k)$ is in $\mathrm{Thick}(k)$ and hence L_ζ and all its direct summands are in $\mathrm{Thick}(k)$ too. Because the tensor product of two modules in $\mathrm{Thick}(k)$ is again in $\mathrm{Thick}(k)$, the classes of endotrivial modules in $\mathrm{Thick}(k)$ form a subgroup $T_k(G)$ of finite index in $T(G)$. We might ask if it is always the case that $TT(G) + T_k(G) = T(G)$. This is a stronger statement than Conjecture 9.2, since $T_k(G) \subseteq T_0(G)$.

10. Control of fusion

If P is a Sylow p-subgroup of G, we let $\mathcal{F}_P(G)$ denote the fusion system on P with morphisms induced by all conjugations by elements of G. We say that a group homomorphism $\phi : G \to G'$ *controls p-fusion* if ϕ induces an isomorphism between a Sylow p-subgroup P of G and a Sylow p-subgroup P' of G', and if moreover ϕ induces an isomorphism between $\mathcal{F}_P(G)$ and $\mathcal{F}_{P'}(G)$. The two cases of interest are the following:

(1) If A is a normal subgroup of G of order prime to p (e.g. $A = O_{p'}(G)$), the canonical group homomorphism $G \to G/A$ controls p-fusion.
(2) If H is a subgroup of G containing a Sylow p-subgroup of G, the inclusion $H \to G$ controls p-fusion if and only if the subgroup H controls p-fusion in the usual sense.

The general case is just made of these two cases, because an arbitrary homomorphism $\phi : G \to G'$ which controls p-fusion involves the composition of $G \to G/\mathrm{Ker}(\phi)$ (and $\mathrm{Ker}(\phi)$ is necessarily of order prime to p) and the inclusion $\mathrm{Im}(\phi) \to G'$ (and $\mathrm{Im}(\phi)$ necessarily contains a Sylow p-subgroup P').

If $\phi : G \to G'$ is a group homomorphism which controls p-fusion, then ϕ induces a homomorphism

$$\phi^* : T(G') \longrightarrow T(G)$$

via restriction along ϕ. Indeed, it is the composition of the ordinary restriction map

$$\mathrm{Res}^{G'}_{\mathrm{Im}(\phi)} : T(G') \longrightarrow T(\mathrm{Im}(\phi))\,,$$

the isomorphism $T(\mathrm{Im}(\phi)) \longrightarrow T(G/\mathrm{Ker}(\phi))$, and the inflation map

$$T(G/\mathrm{Ker}(\phi)) \longrightarrow T(G)\,.$$

Note that the inflation map is well defined when the kernel has order prime to p, because it maps projective modules to projective modules, hence endotrivial modules to endotrivial modules. Clearly, $\phi^* : T(G') \longrightarrow T(G)$ induces in turn a group homomorphism

$$\overline{\phi^*} : TF(G') \longrightarrow TF(G)\,.$$

Conjecture 10.1. *Let $\phi : G \to G'$ be a group homomorphism which controls p-fusion. Then the induced homomorphism $\overline{\phi^*} : TF(G') \longrightarrow TF(G)$ is an isomorphism.*

The first evidence for this conjecture is the following result, which is essentially contained in Corollary 3.2 of [9]. The result has also been obtained by completely different means in Corollary 4.5 of [2].

Proposition 10.2. *Let $\phi : G \to G'$ be a group homomorphism which controls p-fusion. Then the induced homomorphism $\phi^* : T(G') \longrightarrow T(G)$ has finite kernel and its image has finite index in $T(G)$. In other words $\overline{\phi^*} : TF(G') \longrightarrow TF(G)$ is injective and has finite cokernel.*

Proof. For simplicity, we assume that G and G' have a common Sylow p-subgroup P and that ϕ is the identity on P. Since we have control of p-fusion, the G- and G'-conjugacy classes of subgroups of P are the same. It follows that $T(G)$ and $T(G')$ have the same torsion-free rank. Thus it suffices to prove that $\phi^* : T(G') \longrightarrow T(G)$ has finite kernel.

As noticed above, ϕ^* is the composition of a restriction to a subgroup containing P, an isomorphism, and an inflation map. It is clear that the inflation map is injective. Now if H is a subgroup of G' containing P, then we know that $\mathrm{Res}^G_P : T(G) \longrightarrow T(P)$ has finite kernel (Proposition 2.3), hence so does the restriction map $\mathrm{Res}^G_H : T(G) \longrightarrow T(H)$. Therefore, the composition of these three maps defining ϕ^* has finite kernel too. □

Remark 10.3. As noticed in the proof, the inflation map is injective. Moreover, the restriction map to a subgroup containing $N_G(P)$ is always injective (induced by the Green correspondence). So $\phi^* : T(G') \longrightarrow T(G)$ is injective in many cases, but not always. For instance, if $G = P \times C$ with C of order prime to p, then $\mathrm{Res}^G_P : T(G) \longrightarrow T(P)$ has kernel consisting of the classes of the one-dimensional kC-modules, and this kernel is not trivial in general. Thus $\overline{\phi^*}$ is injective but we cannot hope for the injectivity of ϕ^*.

In Conjecture 10.1, the only question is the surjectivity of the map $\overline{\phi^*}$. This is clearly related to our main theme of finding generators for $TF(G)$.

Here is one case where the conjecture holds.

Proposition 10.4. *Let $\phi : G \to G'$ be a group homomorphism which controls p-fusion. Suppose that the cohomological pushout method is optimal for G, by means of a cohomology class $\zeta \in \mathrm{H}^q(G, k)$ with values in the trivial module k. Then the induced homomorphism $\overline{\phi^*} : TF(G') \longrightarrow TF(G)$ is an isomorphism. Moreover, the cohomological pushout method is optimal for G'.*

Proof. For simplicity, we assume that G and G' have a common Sylow p-subgroup P and that ϕ is the identity on P. The homomorphism ϕ induces a map

$$\mathrm{res}_\phi : H^*(G', k) \longrightarrow H^*(G, k)$$

and we first recall the well-known fact that res_ϕ is an isomorphism. This is clear for the inflation map induced by $G \to G/\mathrm{Ker}(\phi)$ because $\mathrm{Ker}(\phi)$ has order prime to p, hence acts trivially on the principal block and the cohomology only detects the principal block. Moreover, the restriction map induced by the inclusion $\mathrm{Im}(\phi) \to G'$ is easily seen to be an isomorphism when the subgroup $\mathrm{Im}(\phi)$ controls p-fusion (see Proposition 3.8.4 in [4]).

Now by assumption, $TF(G)$ is generated by the classes of $\Omega(k)$, M_2, …, M_n, where $n = n_G$ and M_i is constructed as in Theorem 6.1, using a class $\zeta \in \mathrm{H}^q(G, k)$ with $\mathrm{res}_{G,Z}(\zeta)$ not nilpotent.

Then we have a map $\hat{\zeta} : \Omega^q(k) \to k$ representing ζ,

$$L_\zeta = \mathrm{Ker}(\hat{\zeta}) = L_1 \oplus \ldots \oplus L_n \quad \text{and} \quad V_G(L_\zeta) = V_1 \cup \ldots \cup V_n$$

with $V_G(L_i) = V_i$, and there is a map $\zeta_i : M_i \to k$ with kernel L_i.

Let $\zeta' = {\mathrm{res}_\phi}^{-1}(\zeta) \in \mathrm{H}^q(G', k)$. Then for any elementary abelian p-subgroup E of P, we have

$$\mathrm{res}_{G',E}(\zeta') = \mathrm{res}_{G,E}\, \mathrm{res}_\phi(\zeta') = \mathrm{res}_{G,E}(\zeta)$$

and it follows that

$$V_{G'}(\zeta') = V_{G'}(L_{\zeta'}) = V_1' \cup \ldots \cup V_n'$$

where $V_i' = \mathrm{res}^*_\phi(V_i)$. Now if ζ' is represented by $\hat{\zeta}' : \Omega^q(k_{G'}) \to k_{G'}$, then

$$L_{\zeta'} = \mathrm{Ker}(\hat{\zeta}') = L_1' \oplus \ldots \oplus L_n'$$

with $V_{G'}(L_i') = V_i'$. We then construct the endotrivial module

$$M_i' = \Omega^q(k_{G'}) \big/ (\oplus_{j \neq i} L_j')\,.$$

Now ζ is also represented by the map

$$\mathrm{res}_\phi(\hat{\zeta}') : \ \mathrm{Res}_\phi(\Omega^q(k_{G'})) = \Omega^q(k_G) \oplus (\mathrm{proj}) \longrightarrow k_G$$

and therefore we obtain $\mathrm{res}_\phi(L_i') \cong L_i \oplus (\mathrm{proj})$ and $\mathrm{res}_\phi(M_i') \cong M_i \oplus (\mathrm{proj})$. This proves that the generators of $TF(G)$ are in the image of $\overline{\phi^*}$. Therefore $\overline{\phi^*}$ is surjective, hence an isomorphism.

It also follows that the images of the generators of $TF(G)$ under $(\overline{\phi^*})^{-1}$, namely $[\Omega(k_{G'})]$, $[M_2']$, $\ldots$, $[M_n']$, generate $TF(G')$. □

As an interesting special case, we note the following result.

Corollary 10.5. *Suppose that the subgroup $H = N_G(P)$ controls p-fusion in G and that the cohomological pushout method is optimal for H, by means of a cohomology class $\zeta \in \mathrm{H}^q(H,k)$ with values in the trivial module k. Then the restriction map $\mathrm{Res}^G_H : TF(G) \longrightarrow TF(H)$ is an isomorphism and the cohomological pushout method is optimal for G.*

Another situation where Conjecture 10.1 holds occurs when $G = A \rtimes P$ is p-nilpotent and $\phi : G \to G/A \cong P$ is the quotient map where $A = O_{p'}(G)$. In that case, by Corollary 3.4 in [11], any endotrivial module M has the form $M \cong V \otimes L$, where V is simple and endotrivial, and L is in the image of inflation from $G/A \cong P$. Moreover V is torsion by Remark 3.5 in [11]. It follows that $[M]$ can be modified by the torsion class $[V^*]$ to obtain a class $[L]$ in the image of the inflation map. This shows that the inflation map $\overline{\phi^*} : TF(G/A) \longrightarrow TF(G)$ is surjective, hence an isomorphism.

However, in the general case of an arbitrary group G and the quotient map $\phi : G \to G/O_{p'}(G)$, it is not clear how to prove the conjecture. It is not clear either how to prove the conjecture when H is a subgroup of G containing $N_G(P)$ and controlling p-fusion, because of the difficulty mentioned in Section 8.

11. Examples

In this final section we present the results of some calculations for finding generators of $TF(G)$ for specific finite groups G.

Example 11.1. Here is an example where the cohomological pushout method is optimal, but where we need a one-dimensional module W which is not the trivial module. Let $p = 3$, let P be extraspecial of order 27 and exponent 3, and let $G = P \rtimes V$ where V is a Klein four-group with one generator inverting the centre Z of P. In that case the minimal degree q of cohomology is $q = 6$. There is a class in $\mathrm{H}^6(G, W)$ restricting non-trivially to Z, but the one-dimensional module W cannot be the trivial module in view of Lemma 6.2.

Example 11.2. Here is an example where the cohomological pushout method is not optimal. Let $R = C_9 \rtimes C_3$ be an extraspecial 3-group of order 27 and exponent 9, generated by a and b, and let $z = a^3 = [a, b]$ be the generator of $Z = Z(R)$. Let $Q = R \times \langle c\rangle$, where $\langle c\rangle$ is a cyclic group of order 3, and let $P = Q \rtimes C_3$, where a generator d of C_3 acts on Q via $[d, a] = 1, [d, b] = c, [d, c] = z$. Then P has order 3^5 and has 3 classes of maximal elementary abelian subgroups of rank 2 and also a maximal elementary abelian subgroup of rank 3, so $T(P) = TF(P)$ has rank 4. Now Theorem 4.1 tells us that there are endotrivial modules of type $(1, 1, 1, 1)$, $(0, 6, 0, 0)$, $(0, 0, 6, 0)$, and $(0, 0, 0, 6)$ which generate $T(P)$. However, $q = 18$ is the smallest value of q such that there exists $\zeta \in \mathrm{H}^q(P, k)$ whose restriction to Z is non zero.

The final three examples were all calculated using the computer algebra system MAGMA [7]. The method was very direct. In the first step, a suitable endotrivial module U was created for the Sylow p-subgroup P by taking a relative syzygy. This module was induced to $N_G(P)$. The result was decomposed into direct summands and each indecomposable direct summand was tested for being endotrivial. The endotrival summands were induced again to a subgroup containing $N_G(P)$. Again, the induced module was decomposed into indecomposable direct summands and the summands tested to see if they were endotrivial. This process was repeated until the resulting modules were induced to G. Note that, to check that a module, in any stage, is endotrivial, it is only necessary to test if its restriction to P is the direct sum of U and a projective module. Because of the fact that decomposing a module into indecomposable directs summands is computationally difficult, it was necessary to go through a chain of subgroups of maximal length in order to keep the dimensions of the induced modules as small as possible.

Example 11.3. $PSL(3, 3)$ **in characteristic 3.** Suppose that G is the projective linear group $PSL(3, 3)$ and k is a field of characteristic 3. A Sylow 3-subgroup P of G is an extraspecial group of order 27 and exponent 3. There are four maximal elementary abelian subgroups in P, and two of these fuse in G. So G has three conjugacy classes of maximal elementary abelian 3-subgroups, and we label them so that the second and third are the classes of the two unipotent radicals E_2 and E_3 of the two maximal parabolic subgroups. The normalizer $B = N_G(P)$ is a Borel subgroup, it has order 4×27 and it is also the normalizer of the center Z of P. The centralizer of Z has index 2 in B. Hence by Lemma 6.2, there is no element in $\mathrm{H}^6(G, k)$ whose restriction to Z is not zero. The first such element occurs in degree 12. It follows that the cohomological pushout method will produce endotrivial modules of type $(0, 12, 0)$ and $(0, 0, 12)$, hence only of type (a, b, c) where any two of a, b or c differ by a multiple of 12.

On the other hand with the computer we can produce an endotrivial module M of type $(2, -4, 2)$ and dimension 55. In other words, $[M{\downarrow}^G_{E_1}] = [\Omega^2(k)]$, $[M{\downarrow}^G_{E_2}] = [\Omega^{-4}(k)]$ and $[M{\downarrow}^G_{E_3}] = [\Omega^2(k)]$. Because there is an outer automorphism of G that interchanges the two parabolic subgroups, hence E_2 and E_3, we can get also an endotrivial module of type $(2, 2, -4)$. Consequently, there exist endotrivial modules of type $(0, 6, 0)$ and $(0, 0, 6)$, hence of type (a, b, c) for any triple a, b, c such that both $a - b$ and $a - c$ are multiples of 6. The module M is in the principal block and remains indecomposable on restriction to both maximal parabolic subgroups. If U is a one-dimensional kB-module, then its kG-Green correspondent is endotrivial if and only if $U \cong k$ (this is a general fact for groups of Lie type of rank ≥ 2, by Theorem 6.2 in [9]). Hence, $TT(G) = \{0\}$ and $T(G) = TF(G) = T_0(G)$.

We can also describe $T(B)$ and $T(H)$, where $H = N_G(E_2)$ is a maximal parabolic subgroup. First note that B and H both have also three conjugacy classes of maximal elementary abelian 3-subgroups. The endotrivial kG-modules of type $(0, 6, 0)$ and $(0, 0, 6)$ remain of the same type on restriction to B, so $TF(G) \cong TF(H) \cong TF(B)$ via restriction. By Proposition 4.2, we have $T(B) \cong TF(B) \oplus X(B)$ and $X(B)$ is a Klein four-group (because B/P is a Klein four-group). Only two of the one-dimensional representations of B have a Green correspondent for the group H which is endotrivial (both are actually just one-dimensional again), hence $TT(H) = X(H) \cong \mathbb{Z}/2\mathbb{Z}$. But only the trivial module for B has a Green correspondent for the whole group G which is endotrivial. Hence we obtain

$$T(B) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ , \quad T(H) \cong \mathbb{Z}^3 \oplus \mathbb{Z}/2\mathbb{Z} \ , \quad T(G) \cong \mathbb{Z}^3 \ .$$

Example 11.4. $PSL(3, 5)$ **in characteristic 5.** In the case that $G = PSL(3, 5)$ and the characteristic of k is 5, the situation is very similar to that in the previous example. A Sylow 5-subgroup P is extraspecial of exponent 5. There are three classes of maximal elementary abelian 5-subgroups, two of which are represented by unipotent radicals of parabolic subgroups. The normalizer B of P has order $2^4 \times 5^3$. The centralizer of the center Z of P has index 4 in B. Hence, there is no element in $\mathrm{H}^{10}(G, k)$ whose restriction to Z is not zero. The first such element occurs in degree 40. It follows that the cohomological pushout method will produce endotrivial modules only of type (a, b, c) where any two of a, b or c differ by a multiple of 40.

However, we can construct an endotrivial module of type $(2, -8, 2)$ having dimension 251, and by the same argument as above, there exists an endotrivial module of type (a, b, c) for any a, b, c with $a - b$ and $a - c$ multiples of 10. The constructed module is indecomposable on restriction to both maximal parabolic subgroups. Also, $T(G) = TF(G) = T_0(G)$.

Example 11.5. M_{12} **in characteristic 3.** A Sylow 3-subgroup P of $G = M_{12}$ is extraspecial of order 27 and exponent 3. There are three G-conjugacy classes of maximal elementary abelian 3-subgroups. The centralizer of the center Z of P has index 2 in the normalizer of Z. Thus the restriction to Z of every element in $\mathrm{H}^6(G, k)$ vanishes. However, there exists an endotrivial kG-module of type $(2, -4, 2)$ and dimension 82 in the principal kG-block. And again, $T(G) = TF(G) = T_0(G)$.

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