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A posteriori error estimates for the finite element approximation of the Stokes problem

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Abstract

In this paper we propose a new technique to obtain upper and lower bounds on the energy norm of the error in the velocity field, for the Stokes problem. It relies on a splitting of the velocity error in two contributions: a projection error, that quantifies the distance of the computed solution to the space of divergence free functions, and an error in satisfying the momentum equation. We will show that both terms can be sharply estimated, from above and from below, by implicit a posteriori error estimators. In particular, the proposed estimator is based on the solution of local Stokes problems both with "Neumann-type" boundary conditions, extending the ideas presented in [12, 17] for the Laplace equation, and homogeneous Dirichlet boundary conditions. The numerical results show very good effectivity indices. The underlying idea is quite general and can be applied to other saddle point problems as well, as the ones arising in mixed formulations of second order PDEs.

1 Introduction

Techniques for *a posteriori* estimation of errors in finite element approximations of partial differential equations are becoming widely employed in applications as methods to control the accuracy of the approximation and subsequently adapt the computational mesh. The literature is particularly wide for elliptic problems: we refer to the monograph [2] for a survey on the different methods proposed so far.

Whenever the error estimation is meant for verification of the numerical solution for engineering design purposes, rather than (or additionally to) application to mesh adaptivity, a desirable feature of an error estimator is to provide guaranteed upper and lower bounds on the error.

In the case of elliptic self-adjoint problems, it is known that some implicit estimators, based on the solution of local problems on subdomains where the residual of the FE equation acts as a datum, allow to obtain sharp bounds from above and from below for the error measured in the so called energy norm. We mention, in this respect, the *element residual method* with flux equilibration [11, 1] and the more recent *weighted subdomain residual method* proposed in [8] and analyzed in [12, 17].

For the Stokes problem, many estimators have been proposed as well, extending the ideas developed in the elliptic (unconstrained) case. We mention, in particular, the works of Bank and Welfert [5] and Verfürth [20] where implicit estimators, based on the solution of local Stokes problems, are proposed. Ainsworth and Oden [3] suggest, instead, an equilibrated element residual method based on the solution of local Poisson problems. In this paper, the error is measured in an less standard norm. Other approaches can be found in [9, 14, 6] and [15] where the analysis is directed toward the estimation of the error in specific quantities of interest.

Yet, to our knowledge, all the estimators proposed so far, provide upper bounds on the error only up to unknown constants that involve, among others, the constant appearing in the *inf-sup* condition. The problem of obtaining guaranteed bounds for the error, measured in a suitable norm, is still an open question.

In this paper, we show that, for the Stokes problem and, more generally, for an elliptic constrained problem, it is actually possible to bound the error in the velocity field measured in the natural energy norm. Our work moves from the idea that the Stokes problem, is an elliptic self-adjoint problem on the constrained space of divergence free functions, thus it should be possible to extend suitably the elliptic guaranteed estimators to this case.

The derivation here presented relies on a splitting of the velocity error in two contributions: a projection error that quantifies the distance of the numerical solution to the space of divergence free functions, and an error in satisfying the momentum equation. Both terms can be bounded from above and from below by extending, in a quite straightforward way, the elliptic estimators that provide for guaranteed upper and lower bounds.

The plan of the paper is the following. In Sections 2 and 3 we present the Stokes problem as well as its finite element discretization and we characterize the error in the velocity field with respect to the residuals of the finite element formulation. In Section 4 we present a general framework in which the problem of a posteriori estimation of the velocity error can be set and we outline the main idea that underlies the definition of the upper and lower bound estimators. Next, in Sections 5 and 6 we detail a particular choice of estimators that is meant for a continuous pressure discretization (though it can be applied also for discontinuous pressure spaces). They are based on the solution of local Stokes problems on patches of elements and are generalizations of the weighted subdomain residual method given in [17] and the Babuska-Rheinboldt estimator [4].

Those estimators rely on the solution of infinite dimensional local problems, therefore, they are not directly employable in applications. In Section 7 we consider their "computable" version obtained by approximating the local problems is some enriched finite element spaces. The numerical experiments presented in Section 8 show excellent bounds for the error in the velocity field.

2 Preliminaries

Let Ω be an open bounded domain in $\mathbb{R}^d$, $d = 2,3$, with Lipschitz boundary $\partial\Omega$. We consider the Stokes problem

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

with homogeneous Dirichlet boundary conditions $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$.

Let us introduce the function spaces

$$\mathbf{V} = \left[H_0^1(\Omega)\right]^d \qquad Q = L_0^2(\Omega) \equiv \{q \in L^2(\Omega), \quad \int_\Omega q = 0\}$$

and indicate with $\|\cdot\|_1$ and $\|\cdot\|_0$ the H^1 and L^2 norms, respectively. Moreover, we denote by $\mathbf{V}'$ the dual space of $\mathbf{V}$. Then, the weak formulation of problem (1) reads: *find* $\mathbf{u} \in \mathbf{V}$ *and* $p \in Q$ *such that*

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{u}, q) = 0, & \forall q \in Q \end{cases} \tag{2}$$

where $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)$ and the forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined as follows:

$$a(\cdot,\cdot): \ \mathbf{V} \times \mathbf{V} \to \mathbb{R}; \qquad a(\mathbf{v}, \mathbf{w}) = \nu \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{w} \, d\Omega,$$
$$b(\cdot,\cdot): \ \mathbf{V} \times Q \to \mathbb{R}; \qquad b(\mathbf{v}, q) = -\int_\Omega \operatorname{div} \mathbf{v} \, q \, d\Omega,$$

The form $a(\cdot,\cdot)$ is symmetric, continuous and coercive and defines an inner product on $\mathbf{V}$. The associated norm $\|\mathbf{v}\|_a = \sqrt{a(\mathbf{v},\mathbf{v})}$ (hereafter also called *energy norm*) is equivalent to the H^1-norm. Furthermore, the form $b(\cdot,\cdot)$ is bilinear and continuous and satisfies the well known *inf-sup* condition (see, for instance, [10]): *there exists* $\beta > 0$ *such that*

$$\sup_{\mathbf{v} \in \mathbf{V} \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_a} \geq \beta \|q\|_0, \qquad \forall q \in Q. \tag{3}$$

Problem (2) is known to possess a unique solution $(\mathbf{u}, p) \in \mathbf{V} \times Q$ for any $\mathbf{f} \in \mathbf{V}'$ (see, e.g., [10]).

Let us, now, introduce a regular triangulation $\mathcal{T}_h$ of the domain Ω and two finite element spaces $\mathbf{V}_h^k \in \mathbf{V}$ and $Q_h^m \in Q$ of piecewise polynomials of degree at most k (resp. m) on each element of $\mathcal{T}_h$ [1]. Let us assume, moreover that these two spaces $(\mathbf{V}_h^k, Q_h^m)$ satisfy the discrete *inf-sup* condition

$$\sup_{\mathbf{v} \in \mathbf{V}_h^k \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_a} \geq \beta_h \|q\|_0, \qquad \forall q \in Q_h^m. \tag{4}$$

with a constant β_h independent of h. Different choices of spaces $\mathbf{V}_h^k$ and Q_h^m have been proposed in the literature in order to satisfy the previous condition (see e.g. [7], [19] for

[1] In the case of quadrilateral or brick elements, it should be understood that the polynomials are of degree at most k (resp m) in each direction parallel to the edges of the reference unit d-cube $[0,1]^d$.

quadrilateral elements). Then, the finite element discretization: *find* $\mathbf{u}_h \in \mathbf{V}_h^k$ *and* $p_h \in Q_h^m$ *such that*

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}_h^k \\ b(\mathbf{u}_h, q) = 0, & \forall q \in Q_h^m \end{cases} \tag{5}$$

admits a unique solution.
For the further discussion, we will assume that the spaces $(\mathbf{V}_h^k, Q_h^m)$ satisfy also a *local* inf-sup condition

$$\sup_{\mathbf{v} \in \mathbf{V}_h^k(\omega) \cap [H_0^1(\omega)]^d} \frac{b_\omega(\mathbf{v}, q)}{\|\mathbf{v}\|_{a,\omega}} \geq \beta_h^* \|q\|_0, \qquad \forall q \in Q_h^m(\omega) \cap L_0^2(\omega) \tag{6}$$

on each patch ω of elements sharing a common vertex, with a constant β_h^* independent on h and on the patch. In (6) we have indicated with $\mathbf{V}_h^k(\omega)$, $Q_h^m(\omega)$, $b_\omega(\cdot, \cdot)$ and $\|\cdot\|_{a,\omega}$ the restriction of the corresponding function spaces and forms onto the subdomain $\omega \subset \Omega$. Many finite element spaces proposed for the Stokes problem satisfy the local inf-sup condition as well (see e.g. [10, 18, 19]).
If we denote by $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and $E = p - p_h$ the errors introduced by the finite element approximation on the velocity and pressure, respectively, then $(\mathbf{e}, E)$ turns out to be the solution of the following problem

$$\begin{cases} a(\mathbf{e}, \mathbf{v}) + b(\mathbf{v}, E) = \mathcal{R}_h^m(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{e}, q) = \mathcal{R}_h^c(q), & \forall q \in Q \end{cases} \tag{7}$$

where the two linear functionals

$$\begin{aligned} &\mathcal{R}_h^m : \mathbf{V} \to \mathbb{R}, && \mathcal{R}_h^m(\mathbf{v}) = (\mathbf{f}, \mathbf{v}) - a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) \\ &\mathcal{R}_h^c : Q \to \mathbb{R}, && \mathcal{R}_h^c(q) = -b(\mathbf{u}_h, q) \end{aligned}$$

represent the residual in the *momentum equation* and the residual in the *continuity equation*. Owing to (5), $\mathcal{R}_h^m$ and $\mathcal{R}_h^c$ vanish, respectively, on the spaces $\mathbf{V}_h^k$ and Q_h^m, i.e.

$$\mathcal{R}_h^m(\mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h^k, \qquad \mathcal{R}_h^c(q) = 0 \quad \forall q \in Q_h^m \tag{8}$$

3 Characterization of the error $(\mathbf{e}, E)$

In view of the a posteriori error estimation, we would like to relate the norm of the error to some norms of the residuals. Indeed, the latter are known quantities, once the finite element solution has been computed, and many techniques, so called *residual based methods*, have been proposed in the literature aiming at estimating their norm: we refer to [2, 21] for a review of these methods for elliptic problems.
For the Stokes problem, in order to characterize the error $(\mathbf{e}, E)$ in terms of the residuals, we split it as follows:

$$(\mathbf{e}, E) = (\mathbf{e}_0, E_0) + (\mathbf{e}_\perp, E_\perp)$$

where $(\mathbf{e}_0, E_0)$ and $(\mathbf{e}_\perp, E_\perp)$ satisfy, respectively, the two subproblems

$$\begin{cases} a(\mathbf{e}_0, \mathbf{v}) + b(\mathbf{v}, E_0) = \mathcal{R}_h^m(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{e}_0, q) = 0, & \forall q \in Q \end{cases} \tag{9}$$

$$\begin{cases} a(\mathbf{e}_\perp, \mathbf{v}) + b(\mathbf{v}, E_\perp) = 0, & \forall \mathbf{v} \in \mathbf{V} \\ b(\mathbf{e}_\perp, q) = \mathcal{R}_h^c(q), & \forall q \in Q \end{cases} \tag{10}$$

Let us define the two subspaces of $\mathbf{V}$

$$\begin{aligned} \mathbf{V}_{\mathrm{div}} &\equiv \{\mathbf{v} \in \mathbf{V}, \quad b(\mathbf{v}, q) = 0 \quad \forall q \in Q\}, \\ \mathbf{V}_{\mathrm{div}}^\perp &\equiv \{\mathbf{v} \in \mathbf{V}, \quad a(\mathbf{v}, \mathbf{w}) = 0 \quad \forall \mathbf{w} \in \mathbf{V}_{\mathrm{div}}\}. \end{aligned}$$

$\mathbf{V}_{\mathrm{div}}$ is the subspace of $\mathbf{V}$ of divergence free functions while $\mathbf{V}_{\mathrm{div}}^\perp$ is the subspace orthogonal to $\mathbf{V}_{\mathrm{div}}$ with respect to the inner product $a(\cdot, \cdot)$.
From (9) we infer that $\mathbf{e}_0 \in \mathbf{V}_{\mathrm{div}}$. On the other hand, by taking $\mathbf{v} \in \mathbf{V}_{\mathrm{div}}$ in (10), we have $a(\mathbf{e}_\perp, \mathbf{v}) = 0$, so that $\mathbf{e}_\perp \in \mathbf{V}_{\mathrm{div}}^\perp$. As a result we have

$$\|\mathbf{e}\|_a^2 = \|\mathbf{e}_0\|_a^2 + \|\mathbf{e}_\perp\|_a^2, \tag{11}$$

the equality being achieved owing to the orthogonality property $\mathbf{e}_0 \perp \mathbf{e}_\perp$.
Let us now observe that problem (9) is an elliptic equation in the constrained space $\mathbf{V}_{\mathrm{div}}$. The following result holds

Lemma 3.1 *With the above definitions:*

$$\|\mathbf{e}_0\|_a = \sup_{\mathbf{v} \in \mathbf{V}_{\mathrm{div}}} \frac{|\mathcal{R}_h^m(\mathbf{v})|}{\|\mathbf{v}\|_a} \equiv \|\mathcal{R}_h^m\|_{\mathbf{V}'_{\mathrm{div}}}. \tag{12}$$

Proof: By taking $\mathbf{v} = \mathbf{e}_0$ in (9) we have

$$\|\mathbf{e}_0\|_a^2 = \mathcal{R}_h^m(\mathbf{e}_0), \qquad \Longrightarrow \qquad \|\mathbf{e}_0\|_a = \frac{\mathcal{R}_h^m(\mathbf{e}_0)}{\|\mathbf{e}_0\|_a} \leq \|\mathcal{R}_h^m\|_{\mathbf{V}'_{\mathrm{div}}}.$$

Furthermore, for all $\mathbf{v} \in \mathbf{V}_{\mathrm{div}}$, we have $\mathcal{R}_h^m(\mathbf{v}) = a(\mathbf{e}_0, \mathbf{v}) \leq \|e_0\|_a \|\mathbf{v}\|_a$ and the assertion follows immediately. □

On the other hand, problem (10) can be seen as a minimization problem of the a-norm $\|\cdot\|_a$ under the constraint $b(\mathbf{v}, q) = \mathcal{R}_h^c(q) \ \forall q \in Q$. More precisely, if we define the constrained set of functions

$$\mathbf{V}_{\mathcal{R}_h^c} \equiv \{\mathbf{v} \in \mathbf{V}, \ \ b(\mathbf{v}, q) = \mathcal{R}_h^c(q) \ \ \forall q \in Q\},$$

the following result holds

Lemma 3.2 *The function* $\mathbf{e}_\perp$*, solution of Problem (10), is also a solution to the minimization problem*

$$\textit{find}\quad \mathbf{w} \in \mathbf{V}_{\mathcal{R}_h^c} \quad \textit{s.t.} \qquad a(\mathbf{w},\mathbf{w}) \le a(\mathbf{v},\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}_{\mathcal{R}_h^c}$$

and, consequently

$$\|\mathbf{e}_\perp\|_a = \left(\min_{\mathbf{v}\in\mathbf{V}_{\mathcal{R}_h^c}} a(\mathbf{v},\mathbf{v}) \right)^{\frac{1}{2}} \tag{13}$$

Proof: The proof is a standard argument of functional analysis for saddle-point problems. See, for instance, [7]. □

As a consequence of Lemmas 3.1 and 3.2, the energy norm of the error on the velocity field can be fully characterized by

$$\|\mathbf{e}\|_a^2 = \|R_h^m\|^2_{\mathbf{V}'_{\mathrm{div}}} + \min_{\mathbf{v}\in\mathbf{V}_{\mathcal{R}_h^c}} a(\mathbf{v},\mathbf{v}) \tag{14}$$

In the next section we will show how exact upper and lower bounds for the quantity $\|\mathbf{e}\|_a$ can be achieved. Those bounds are obtained by constructing a posteriori error estimators separately for the two terms appearing in the right hand side of (14).

We finally mention that a characterization of the error on the pressure in the L^2-norm can be obtained thanks to the *inf-sup* condition (3). We have indeed

$$\begin{aligned}
\|E\|_0 &\le \frac{1}{\beta} \sup_{\mathbf{v}\in\mathbf{V}\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v},E)}{\|\mathbf{v}\|_a} \\
&= \frac{1}{\beta} \sup_{\mathbf{v}\in\mathbf{V}_{\mathrm{div}}^\perp\setminus\{\mathbf{0}\}} \frac{1}{\|\mathbf{v}\|_a} \left\{ \mathcal{R}_h^m(\mathbf{v}) - a(\mathbf{e},\mathbf{v}) \right\} \\
&\le \frac{1}{\beta} \left\{ \sup_{\mathbf{v}\in\mathbf{V}_{\mathrm{div}}^\perp\setminus\{\mathbf{0}\}} \frac{\mathcal{R}_h^m(\mathbf{v})}{\|\mathbf{v}\|_a} + \sup_{\mathbf{v}\in\mathbf{V}_{\mathrm{div}}^\perp\setminus\{\mathbf{0}\}} \frac{a(\mathbf{e}_\perp,\mathbf{v})}{\|v\|_a} \right\},
\end{aligned}$$

thus leading to the estimate

$$\|E\|_0 \le \frac{1}{\beta} \left\{ \|\mathcal{R}_h^m\|_{\mathbf{V}'} + \|\mathbf{e}_\perp\|_a \right\}. \tag{15}$$

where we have exploited the fact that $\|\mathcal{R}_h^m\|_{(\mathbf{V}_{\mathrm{div}}^\perp)'} \le \|\mathcal{R}_h^m\|_{\mathbf{V}'}$. This inequality has already been proved in [13, Lemma 3]. Observe that, even in the favorable case where we are able to provide an upper bound for both $\|\mathcal{R}_h^m\|_{\mathbf{V}'}$ and $\|\mathbf{e}_\perp\|_a$, the presence of the constant β in (15), which is in general unknown and difficult to estimate, prevents from achieving a guaranteed upper bound for the L^2-norm of the error on the pressure.

4 Upper and lower bounds for the velocity error

We give first a general framework that yields upper and lover bounds on the energy norm of the velocity error. We refer to Sections 5 and 6 for a description of the particular a posteriori error estimator that we have analyzed and tested numerically.

The following Proposition is an immediate consequence of Lemmas 3.1 and 3.2:

Proposition 4.1 *Given any function $\boldsymbol{\psi}_0 \in \mathbf{V}_{\rm div}$ and $\boldsymbol{\psi}_\perp \in \mathbf{V}_{\mathcal{R}_h^c}$, we have*

$$\frac{|\mathcal{R}_h^m(\boldsymbol{\psi}_0)|}{\|\boldsymbol{\psi}_0\|_a} \leq \|\mathbf{e}_0\|_a, \qquad \|\boldsymbol{\psi}_\perp\|_a \geq \|\mathbf{e}_\perp\|_a.$$

Provided we are able to build two particular functions $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_\perp$, which are reasonable approximations of $\mathbf{e}_0$ and $\mathbf{e}_\perp$, respectively, Proposition 4.1 suggests the idea to define a lower bound estimator for $\|\mathbf{e}_0\|_a$ by simply taking

$$\varepsilon_0^{low} = \frac{|\mathcal{R}_h^m(\boldsymbol{\psi}_0)|}{\|\boldsymbol{\psi}_0\|_a} \tag{16}$$

and an upper bound estimator for $\|\mathbf{e}_\perp\|_a$ by

$$\varepsilon_\perp^{up} = \|\boldsymbol{\psi}_\perp\|_a. \tag{17}$$

Let us, now, suppose that we are able to build a Hilbert space $\boldsymbol{\mathcal{M}}$, which will be called hereafter *broken* space for a reason that will become clear later, endowed with an inner product $\tilde{a}(\cdot,\cdot)$ and associated norm $\|\cdot\|_{\tilde{a}}$, satisfying the following assumptions:

A1. There exists a linear application $\mathcal{I} : \mathbf{V} \to \boldsymbol{\mathcal{M}}$ that injects $\mathbf{V}$ into $\boldsymbol{\mathcal{M}}$; i.e. $\mathcal{I}(\mathbf{V}) \subset \boldsymbol{\mathcal{M}}$.

A2. The inner product $\tilde{a}(\cdot,\cdot)$ on $\boldsymbol{\mathcal{M}}$ extends $a(\cdot,\cdot)$ on $\mathbf{V}$, i.e.

$$\tilde{a}(\mathcal{I}\mathbf{v}, \mathcal{I}\mathbf{w}) = a(\mathbf{v}, \mathbf{w}), \qquad \forall \mathbf{v}, \mathbf{w} \in \mathbf{V}.$$

This implies, in particular, that $\mathcal{I}$ is an isometry.

Let, moreover, $\tilde{\mathcal{R}}_h^m \in \boldsymbol{\mathcal{M}}'$ be a continuous extension of the functional $\mathcal{R}_h^m \in \mathbf{V}'$, i.e.

$$\tilde{\mathcal{R}}_h^m(\mathcal{I}\mathbf{v}) = \mathcal{R}_h^m(\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}.$$

The Hahn-Banach theorem assures the existence of such an extension.
Finally, we introduce a subspace $\boldsymbol{\mathcal{M}}_0 \subset \boldsymbol{\mathcal{M}}$ and an affine subspace $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c} \subset \boldsymbol{\mathcal{M}}$ such that

$$\boldsymbol{\mathcal{M}}_0 \supset \mathcal{I}(\mathbf{V}_{\rm div}), \qquad \boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c} \supset \mathcal{I}(\mathbf{V}_{\mathcal{R}_h^c}). \tag{18}$$

A straightforward consequence of Lemmas 3.1 and 3.2 is:

Proposition 4.2 *Given a broken space $\mathcal{M}$ that satisfies the assumptions A1-A2 and two sets $\mathcal{M}_0$ and $\mathcal{M}_{\mathcal{R}_h^c}$ satisfying (18), we have*

$$\sup_{\mathbf{v}\in\mathcal{M}_0} \frac{\tilde{\mathcal{R}}_h^m(\mathbf{v})}{\|\mathbf{v}\|_{\tilde{a}}} \geq \|\mathbf{e}_0\|_a, \qquad \left(\inf_{\mathbf{v}\in\mathcal{M}_{\mathcal{R}_h^c}} \tilde{a}(\mathbf{v},\mathbf{v})\right)^{\frac{1}{2}} \leq \|\mathbf{e}_\perp\|_a.$$

Again, this Proposition suggests the idea to define an upper bound estimator for $\|\mathbf{e}_0\|_a$ as

$$\varepsilon_0^{up} = \sup_{\mathbf{v}\in\mathcal{M}_0} \frac{\tilde{\mathcal{R}}_h^m(\mathbf{v})}{\|\mathbf{v}\|_{\tilde{a}}} \tag{19}$$

and a lower bound estimator for $\|\mathbf{e}_\perp\|_a$ as

$$\varepsilon_\perp^{low} = \left(\inf_{\mathbf{v}\in\mathcal{M}_{\mathcal{R}_h^c}} \tilde{a}(\mathbf{v},\mathbf{v})\right)^{\frac{1}{2}}. \tag{20}$$

Remark 4.1 *The hypothesis that $\tilde{\mathcal{R}}_h^m$ is a continuous functional on $\mathcal{M}$ ensures that the estimator ε_0^{up} is bounded.*

The two expressions in (19) and (20) may seem very complicated to compute. Yet, if the space $\mathcal{M}$ has some “broken” property, that means that the elements $\mathbf{v} \in \mathcal{M}$ are defined only locally on subdomains of Ω, without any requirement that they match between one subdomain and another, the computation of (19) and (20) reduces, in general, to the solution of local problems on subdomains.

Remark 4.2 *A typical example of broken space (see [2]) is given by the space of functions that are in $H^1(K)$ for each element K of the mesh, without any continuity requirement at the interface between two adjacent elements. More precisely,*

$$\mathcal{M} \equiv \{\mathbf{v} \in [L^2(\Omega)]^d,\ \ \mathbf{v}|_K \in [H^1(K)]^d,\ \ \mathbf{v} = 0 \ \ on\ \ \partial\Omega\} \tag{21}$$

with extended inner product

$$\tilde{a}(\mathbf{v},\mathbf{w}) = \sum_K a_K(\mathbf{v}|_K, \mathbf{w}|_K) \tag{22}$$

where $a_K(\cdot,\cdot)$ is the restriction of the bilinear form $a(\cdot,\cdot)$ to the element K. This choice of broken space leads to the well known element residual method*. Observe that the inner product defined in (22) induces only a semi-norm on $\mathcal{M}$ since it vanishes for piecewise constant functions. In order for ε_0^{up} to be bounded, we need to guarantee that also the extended residual $\tilde{\mathcal{R}}_h^m$ vanishes for piecewise constant functions. This demands, in general, for some flux equilibration techniques (see [11, 2]).*

In the following section, we will consider another possible choice of broken space that leads to the solution of local problems on patches of elements. It is an extension of the weighted subdomain residual method proposed in [8] for elliptic problems and analyzed thoroughly in [12, 17]. In the present paper, we will give a reinterpretation of that estimator in terms of broken spaces.

Once the estimators ε_0^{up}, ε_0^{low}, $\varepsilon_\perp^{up}$ and $\varepsilon_\perp^{low}$ are available, they can be simply recombined to obtain upper and lower bounds for the velocity error $\|\mathbf{e}\|_a$. Precisely, we introduce the global estimators

$$\varepsilon^{up} = \sqrt{\left(\varepsilon_0^{up}\right)^2 + \left(\varepsilon_\perp^{up}\right)^2}, \qquad \text{and} \qquad \varepsilon^{low} = \sqrt{\left(\varepsilon_0^{low}\right)^2 + \left(\varepsilon_\perp^{low}\right)^2}.$$

As a consequence of (11) we have that

$$\varepsilon^{up} \geq \|\mathbf{e}\|_a \qquad \text{and} \qquad \varepsilon^{low} \leq \|\mathbf{e}\|_a.$$

5 Patch-wise broken space

The main results stated in this section are Lemmas 5.4 and 5.5, where we define ε_0^{up} and $\varepsilon_\perp^{low}$.
Let $\{x_i, \ \ i = 1, \dots, N\}$ be the set of vertices of the mesh $\mathcal{T}_h$ (including the vertices on the boundary $\partial\Omega$) and $\{\phi_i, \ \ i = 1, \dots, N\}$ the associated set of first order Lagrangian basis functions. More precisely, if we denote with $\mathbf{F}_K$ the affine mapping from the reference triangle or square $\hat{K}$ onto each element K of the mesh, we have

$$\phi_i(x_j) = \delta_{ij}, \quad \forall i, j = 1, \dots, N$$
$$\phi_{i|_K} = \hat{\phi} \circ \mathbf{F}_K^{-1}, \quad \hat{\phi} \in \begin{cases} \mathbb{P}_1(\hat{K}) & \text{for triangular meshes} \\ \mathbb{Q}_1(\hat{K}) & \text{for quadrilateral meshes} \end{cases}$$

where we have denoted by δ_{ij} the Kronecker symbol. The support of each ϕ_i is denoted by ω_i and will be referred to as the patch of elements connected to the vertex x_i of the mesh. We denote by $h_i = \max_{K \in \omega_i} h_K$ the maximum diameter of the elements in the patch. A well known property of the Lagrange basis functions states that the set $\{\phi_i\}_{i=1}^N$ forms a partition of unity, that is $\sum_{i=1}^N \phi_i = 1$. We define the following weighted spaces on each patch ω_i:

$$\mathbf{W}(\omega_i) = \{\mathbf{v} : \omega_i \to \mathbb{R}^d, \quad \int_{\omega_i} |\nabla \mathbf{v}|^2 \phi_i < +\infty\}, \tag{23}$$

$$\mathring{\mathbf{W}}(\omega_i) = \begin{cases} \{\mathbf{v} \in \mathbf{W}(\omega_i), \quad \int_{\omega_i} \mathbf{v}\phi_i = \mathbf{0}\} & \text{if } x_i \notin \partial\Omega \\ \{\mathbf{v} \in \mathbf{W}(\omega_i), \quad \mathbf{v} = \mathbf{0} \text{ on } \partial\omega_i \cap \partial\Omega\} & \text{if } x_i \in \partial\Omega \end{cases} \tag{24}$$

$$Z(\omega_i) = \{q : \omega_i \to \mathbb{R}, \quad \int_{\omega_i} q^2 \phi_i < +\infty\} \tag{25}$$

Observe that $\mathbf{W}(\omega_i) \supset [H^1(\omega_i)]^d$ and $Z(\omega_i) \supset L^2(\omega_i)$, since $0 \leq \phi_i \leq 1,\ \forall x \in \omega_i$. We introduce, moreover, the following weighted bilinear forms on each subdomain ω_i:

$$a_{\phi_i}(\mathbf{u}, \mathbf{v}) = \nu \int_{\omega_i} (\nabla \mathbf{u} : \nabla \mathbf{v})\phi_i, \qquad \forall \mathbf{u}, \mathbf{v} \in \mathring{\mathbf{W}}(\omega_i)$$

$$b_{\phi_i}(\mathbf{v}, q) = -\int_{\omega_i} (\operatorname{div} \mathbf{v}) q \phi_i, \qquad \forall \mathbf{v} \in \mathring{\mathbf{W}}(\omega_i),\ q \in Z(\omega_i)$$

The form $a_{\phi_i}(\cdot,\cdot)$ induces an inner product in the space $\mathring{\mathbf{W}}(\omega_i)$ with associated norm $\|\mathbf{v}\|_{a,\phi_i} = \sqrt{a_{\phi_i}(\mathbf{v},\mathbf{v})}$, while $b_{\phi_i}(\mathbf{v}, q)$ can be easily shown to be continuous on $\mathring{\mathbf{W}} \times Z$ with respect to the norms $\|\cdot\|_{a,\phi_i}$ and $\|\cdot\|_{0,\phi_i} = \|\cdot\, \phi_i^{1/2}\|_{L^2(\omega_i)}$.

We now define the broken space $\boldsymbol{\mathcal{M}}$ as

$$\boldsymbol{\mathcal{M}} = \prod_{i=1}^{N} \mathring{\mathbf{W}}(\omega_i) \tag{26}$$

In other words, the elements of $\boldsymbol{\mathcal{M}}$ are sets of functions $\{\mathbf{v}_i\}_{i=1}^N$, each one defined on a subdomain ω_i. The subdomains overlap; yet, on the overlapping region, the functions $\mathbf{v}_i$ are not required to match. With this respect we can say that the space $\boldsymbol{\mathcal{M}}$ defined in (26) is a broken space.

We equip the space $\boldsymbol{\mathcal{M}}$ with the natural inner product associated with a product space, i.e. for any $\mathcal{O} = \{\mathbf{v}_i\}_{i=1}^N$ and $\mathcal{Q} = \{\mathbf{w}_i\}_{i=1}^N$ in $\boldsymbol{\mathcal{M}}$

$$\tilde{a}(\mathcal{O}, \mathcal{Q}) = \sum_{i=1}^{N} a_{\phi_i}(\mathbf{v}_i, \mathbf{w}_i) \tag{27}$$

and we define the extended residual $\tilde{\mathcal{R}}_h^m \in \boldsymbol{\mathcal{M}}'$ as

$$\tilde{\mathcal{R}}_h^m(\mathcal{O}) = \sum_{i=1}^{N} \mathcal{R}_h^m(\mathbf{v}_i \phi_i). \tag{28}$$

Following the arguments given in [17, Lemma 3], it can be proved that $\tilde{\mathcal{R}}_h^m$ is a bounded, linear functional on $\boldsymbol{\mathcal{M}}$. The proof relies on the following

Weighted Poincaré inequality*: There exists a constant $C > 0$, independent of h_i, such that*

$$\|\mathbf{v}\|^2_{L^2(\omega_i)} \leq C h_i \int_{\omega_i} |\nabla \mathbf{v}|^2 \phi_i \qquad \forall \mathbf{v} \in \mathring{\mathbf{W}}(\omega_i). \tag{29}$$

This inequality has been proved in [12] for meshes of triangles or tetrahedrons and in [17] for quadrilateral meshes in 2D. The proof for 3D “brick” elements is still missing and the applicability of this estimator in 3D problems is an open question.

The following important result holds:

Lemma 5.3 *We define the linear application* $\mathcal{I} : \mathbf{V} \to \boldsymbol{\mathcal{M}}$ *as*

$$\mathcal{I}\mathbf{v} = \{\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i\}_{i=1}^N, \quad \textit{where} \quad \bar{\mathbf{v}}_i = \begin{cases} \left(\int_{\omega_i} \mathbf{v}\phi_i\right) / \left(\int_{\omega_i} \phi_i\right), & \textit{if } x_i \notin \partial\Omega \\ \mathbf{0} & \textit{if } x_i \in \partial\Omega \end{cases} \tag{30}$$

then

i) $\mathcal{I}$ *injects* $\mathbf{V}$ *into* $\boldsymbol{\mathcal{M}}$*, i.e.* $\mathcal{I}(\mathbf{V}) \subset \boldsymbol{\mathcal{M}}$.

ii) $\tilde{a}(\mathcal{I}\mathbf{v}, \mathcal{I}\mathbf{w}) = a(\mathbf{v}, \mathbf{w}), \qquad \forall \mathbf{v}, \mathbf{w} \in \mathbf{V}$.

iii) $\tilde{\mathcal{R}}_h^m(\mathcal{I}\mathbf{v}) = \mathcal{R}_h^m(\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}$.

Proof:

i) From the previous considerations it appears clear that, for any $\mathbf{v} \in \mathbf{V}$, the function $\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i \in \overset{\circ}{\mathbf{W}}(\omega_i)$. Thus, $\mathbf{V}$ is injected in $\boldsymbol{\mathcal{M}}$ through the application $\mathcal{I}$.

ii) We have

$$\begin{aligned} \tilde{a}(\mathcal{I}\mathbf{v}, \mathcal{I}\mathbf{w}) &= \sum_{i=1}^N a_{\phi_i}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i, \mathbf{w}_{|\omega_i} - \bar{\mathbf{w}}_i) = \sum_{i=1}^N a_{\phi_i}(\mathbf{v}_{|\omega_i}, \mathbf{w}_{|\omega_i}) \\ &= \sum_{i=1}^N \int_{\omega_i} (\nabla \mathbf{v}_{|\omega_i} : \nabla \mathbf{w}_{|\omega_i})\phi_i = \sum_{i=1}^N \int_\Omega (\nabla \mathbf{v} : \nabla \mathbf{w})\phi_i = a(\mathbf{v}, \mathbf{w}). \end{aligned}$$

iii) Similarly, we have

$$\tilde{\mathcal{R}}_h^m(\mathcal{I}\mathbf{v}) = \sum_{i=1}^N \mathcal{R}_h^m((\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i)\phi_i) = \sum_{i=1}^N \mathcal{R}_h^m(\mathbf{v}_{|\omega_i}\phi_i) - \sum_{i=1}^N \bar{\mathbf{v}}_i \mathcal{R}_h^m(\phi_i)$$

The last term vanishes thanks to the Galerkin orthogonality (8), and we have, finally

$$\tilde{\mathcal{R}}_h^m(\mathcal{I}\mathbf{v}) = \sum_{i=1}^N \mathcal{R}_h^m(\mathbf{v}\phi_i) = \mathcal{R}_h^m(\mathbf{v}).$$

□

The space $\boldsymbol{\mathcal{M}}$ defined above, as well as the extended inner product $\tilde{a}$ and residual $\tilde{\mathcal{R}}_h^m$, satisfy the requirements set in the previous section.

5.1 The subspace $\boldsymbol{\mathcal{M}}_0$

A first possibility to define the broken subspace $\boldsymbol{\mathcal{M}}_0$ is

$$\boldsymbol{\mathcal{M}}_{0,S} = \{\mathcal{O} = \{\mathbf{v}_i\}_{i=1}^N \in \boldsymbol{\mathcal{M}}, \;\; b_{\phi_i}(\mathbf{v}_i, q) = 0 \;\; \forall q \in Z(\omega_i)\}. \tag{31}$$

The subscript S stands for *Stokes* since this choice leads to the solution of local Stokes-like problems. If $\mathbf{v} \in \mathbf{V}_{\text{div}}$ is a divergence free function, $\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i$ is also divergence free in ω_i and

$$b_{\phi_i}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i, q) = -\int_{\omega_i} \text{div}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i)\, q\phi_i = 0, \qquad \forall q \in Z(\omega_i).$$

Thus $\mathcal{I}$ isometrically injects $\mathbf{V}_{\text{div}}$ into $\boldsymbol{\mathcal{M}}_{0,S}$, i.e. $\mathcal{I}(\mathbf{V}_{\text{div}}) \subset \boldsymbol{\mathcal{M}}_{0,S}$. The space $\boldsymbol{\mathcal{M}}_{0,S}$ satisfies all the requirements set in Proposition 4.2. It follows immediately that the error estimator

$$\varepsilon_{0,S}^{up} = \sup_{\mathcal{O} \in \boldsymbol{\mathcal{M}}_{0,S}} \frac{\tilde{\mathcal{R}}_h^m(\mathcal{O})}{\|\mathcal{O}\|_{\tilde{a}}} \tag{32}$$

provides an upper bound of the quantity $\|\mathbf{e}_0\|_a$.
We are now in the position to state the first main result of this section

Lemma 5.4 *The upper bound estimator (32), associated with the choice of the broken subspace $\boldsymbol{\mathcal{M}}_{0,S}$ defined in (31), can be computed as*

$$\varepsilon_{0,S}^{up} = \left(\sum_{i=1}^N a_{\phi_i}(\boldsymbol{\eta}_i^{up}, \boldsymbol{\eta}_i^{up}) \right)^{\frac{1}{2}} \tag{33}$$

where the functions $\boldsymbol{\eta}_i^{up}$ are the solutions of the local constrained elliptic problems defined on each patch ω_i:
find $\boldsymbol{\eta}_i^{up} \in \overset{\circ}{\mathbf{W}}(\omega_i)$, with $b_{\phi_i}(\boldsymbol{\eta}_i^{up}, q) = 0, \;\; \forall q \in Z(\omega_i)$, s.t.

$$a_{\phi_i}(\boldsymbol{\eta}_i^{up}, \mathbf{v}) = \mathcal{R}_h^m(\mathbf{v}\phi_i), \qquad \forall \mathbf{v} \in \overset{\circ}{\mathbf{W}}(\omega_i), \;\; b_{\phi_i}(\mathbf{v}, q) = 0 \;\; \forall q \in Z(\omega_i). \tag{34}$$

Proof: The space $\boldsymbol{\mathcal{M}}_{0,S}$ is a closed subspace of $\boldsymbol{\mathcal{M}}$ and $\tilde{\mathcal{R}}_h^m$ is a linear continuous functional on $\boldsymbol{\mathcal{M}}$ (and a fortiori on $\boldsymbol{\mathcal{M}}_{0,S}$). Thus, by the Riesz representation theorem, there exists a unique element $\mathcal{H} = \{\boldsymbol{\eta}_i^{up}\}_{i=1}^N \in \boldsymbol{\mathcal{M}}_{0,S}$ that satisfies the problem

$$\tilde{a}(\mathcal{H}, \mathcal{O}) = \tilde{\mathcal{R}}_h^m(\mathcal{O}), \qquad \forall \mathcal{O} \in \boldsymbol{\mathcal{M}}_{0,S}. \tag{35}$$

and is such that

$$\sup_{\mathcal{O} \in \boldsymbol{\mathcal{M}}_{0,S}} \frac{\tilde{\mathcal{R}}_h^m(\mathcal{O})}{\|\mathcal{O}\|_{\tilde{a}}} = \|\mathcal{H}\|_{\tilde{a}} = \left(\sum_{i=1}^N a_{\phi_i}(\boldsymbol{\eta}_i^{up}, \boldsymbol{\eta}_i^{up}) \right)^{\frac{1}{2}}.$$

Thus, given $\mathcal{H}$, the estimator $\varepsilon_{0,S}^{up}$ can be computed by formula (33). Problem (35) can be written equivalently as

$$\sum_{i=1}^{N} a_{\phi_i}(\boldsymbol{\eta}_i^{up}, \mathbf{v}_i) = \sum_{i=1}^{N} \mathcal{R}_h^m(\mathbf{v}_i \phi_i), \qquad \forall \{\mathbf{v}_i\}_{i=1}^N \in \boldsymbol{\mathcal{M}}_{0,S}. \tag{36}$$

Since the functions $\mathbf{v}_i$ are completely independent one to the other (we can actually chose $\mathbf{v}_i = \mathbf{0}$ for all $i \neq j$, and $\mathbf{v}_j$ different that zero), problem (36) reduces to the set of N independent local problems stated in (34). □

Remark 5.3 *If the bilinear form b_{ϕ_i} satisfies an* inf-sup *condition, we can add the constraint appearing in (34) explicitly to the equation by a Lagrange multiplier. We obtain, in this case, the set of N local weighted Stokes problems:* find $\boldsymbol{\eta}_i^{up} \in \overset{\circ}{\mathbf{W}}(\omega_i)$ and $\xi_i^{up} \in Z(\omega_i)$ such that

$$\begin{cases} a_{\phi_i}(\boldsymbol{\eta}_i^{up}, \mathbf{v}) + b_{\phi_i}(\mathbf{v}, \xi_i^{up}) = \mathcal{R}_h^m(\mathbf{v}\phi_i), & \forall \mathbf{v} \in \overset{\circ}{\mathbf{W}}(\omega_i) \\ b_{\phi_i}(\boldsymbol{\eta}_i^{up}, q) = 0 & \forall q \in Z(\omega_i) \end{cases} \tag{37}$$

This is the formulation that we will adopt in Section 7 to define the computable error estimator. The issue whether the form b_{ϕ_i} satisfies or not the inf-sup *condition is still an open question.*

Observe that in (34) we do not have imposed any boundary condition (the test functions $\mathbf{v}$ as well as the solutions $\boldsymbol{\eta}_i^{up}$ are completely free on the boundary $\partial\omega_i$). Hence, the local problems (34) (or (37)) are of “Neumann” type.

Remark 5.4 *Another possible choice for $\boldsymbol{\mathcal{M}}_0$ is $\boldsymbol{\mathcal{M}}_{0,P} = \boldsymbol{\mathcal{M}}$ (the subscript P standing for Poisson). It is clear, indeed, that this space satisfies also the requirements of Proposition 4.2. In this case, we end up with the solution of local weighted Poisson problems, on each patch ω_i, of the form: find $\boldsymbol{\eta}_{i,P}^{up} \in \overset{\circ}{\mathbf{W}}(\omega_i)$ such that*

$$a_{\phi_i}(\boldsymbol{\eta}_{i,P}^{up}, \mathbf{v}) = \mathcal{R}_h^m(\mathbf{v}\phi_i), \qquad \forall \mathbf{v} \in \overset{\circ}{\mathbf{W}}(\omega_i), \tag{38}$$

and the upper bound estimator is defined again as $\varepsilon_{0,P}^{up} = \left(\sum_{i=1}^N a_{\phi_i}(\boldsymbol{\eta}_{i,P}^{up}, \boldsymbol{\eta}_{i,P}^{up})\right)^{\frac{1}{2}}$.
Since $\boldsymbol{\mathcal{M}}_{0,P} \supset \boldsymbol{\mathcal{M}}_{0,S}$ we immediately have that $\varepsilon_{0,P}^{up} \geq \varepsilon_{0,S}^{up}$. We expect that the estimator $\varepsilon_{0,S}$ provides a sharper upper bound for the quantity $\|\mathbf{e}_0\|_a$ than $\varepsilon_{0,P}$, yet at the expense of solving local Stokes problems instead of local Poisson ones.
The idea of solving local Poisson problems to obtain an a posteriori estimate on the error of the Stokes problem has already been considered in [3] (see also [14]). Yet, in those works, no exact upper bounds are provided for the error on the velocity field.
We finally remark that, since $\mathbf{V} \subset \boldsymbol{\mathcal{M}}_{0,P}$, the estimator $\varepsilon_{0,P}^{up}$ is also an upper bound estimator for the quantity $\|\mathcal{R}_h^m\|_{\mathbf{V}'}$, i.e.

$$\varepsilon_{0,P}^{up} \geq \sup_{\mathbf{v}\in\mathbf{V}} \frac{\mathcal{R}_h^m(\mathbf{v})}{\|\mathbf{v}\|_a}.$$

Thus, this estimator may be used to build an estimator for the error on the pressure (see inequality (15)).

5.2 The subset $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$

Proceeding in a similar way as in the previous section, we define the broken set $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$ as

$$\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c} \equiv \{\mathcal{O} = \{\mathbf{v}_i\}_{i=1}^N \in \boldsymbol{\mathcal{M}},\ \ b_{\phi_i}(\mathbf{v}_i, q) = \mathcal{R}_h^c(q\phi_i)\ \ \forall q \in Z(\omega_i)\}. \tag{39}$$

It is easy to show that $\mathcal{I}(\mathbf{V}_{\mathcal{R}_h^c}) \subset \boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$. Indeed, given any function $\mathbf{v} \in \mathbf{V}_{\mathcal{R}_h^c}$ and any $q \in Z(\omega_i)$, we have

$$b_{\phi_i}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i, q) = -\int_{\omega_i} \operatorname{div}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i)\, q\phi_i = b(\mathbf{v}, \widetilde{q\phi_i}).$$

where we have denoted by $\widetilde{q\phi_i}$ the extension of $q\phi_i$ by zero outside the domain ω_i. Since the function $\mathbf{v}$ satisfies the constraint $b(\mathbf{v}, q) = \mathcal{R}_h^c(q),\ \forall q \in L^2(\Omega)$ and the function $\widetilde{q\phi_i} \in L^2(\Omega)$, we conclude that

$$b_{\phi_i}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i, q) = \mathcal{R}_h^c(q\phi_i), \qquad \forall q \in Z(\omega_i).$$

In particular, we see that the set $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$ is not empty.
The set $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$ satisfies the assumptions given in Proposition 4.2. Thus, the estimator

$$\varepsilon_{\perp}^{low} = \left(\inf_{\mathcal{O} \in \boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}} \tilde{a}(\mathcal{O}, \mathcal{O}) \right)^{\frac{1}{2}} \tag{40}$$

provides for a lower bound of the quantity $\|\mathbf{e}_\perp\|_a$.
We are now in the position to state the second main result of this section

Lemma 5.5 *The lower bound estimator (40), associated with $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$ set in (39), can be computed as*

$$\varepsilon_{\perp}^{low} = \left(\sum_{i=1}^N a_{\phi_i}(\boldsymbol{\eta}_i^{low}, \boldsymbol{\eta}_i^{low}) \right)^{\frac{1}{2}}$$

where the functions $\boldsymbol{\eta}_i^{low}$ are the solutions of the local constrained minimization problems:
find $\boldsymbol{\eta}_i^{low} \in \overset{\circ}{\mathbf{W}}(\omega_i)$, with $b_{\phi_i}(\boldsymbol{\eta}_i^{low}, q) = \mathcal{R}_h^c(q\phi_i),\ \forall q \in Z(\omega_i)$, s.t.

$$a_{\phi_i}(\boldsymbol{\eta}_i^{low}, \boldsymbol{\eta}_i^{low}) \le a_{\phi_i}(\mathbf{v}_i, \mathbf{v}_i), \qquad \forall \mathbf{v}_i, \in \overset{\circ}{\mathbf{W}}(\omega_i),\ \ b_{\phi_i}(\mathbf{v}_i, q) = \mathcal{R}_h^c(q\phi_i)\ \ \forall q \in Z(\omega_i). \tag{41}$$

Proof: $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$ is a closed, convex, non empty subset of $\boldsymbol{\mathcal{M}}$ and $\tilde{a}(\cdot,\cdot)$ is a continuous, coercive and symmetric bilinear form. Thus, the minimization problem: *Find $\mathcal{H} = \{\boldsymbol{\eta}_i^{low}\}_{i=1}^N \in \boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$ such that*

$$\tilde{a}(\mathcal{H}, \mathcal{H}) \le \tilde{a}(\mathcal{O}, \mathcal{O}), \qquad \forall \mathcal{O} \in \boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$$

admits a unique solution and the estimator $\varepsilon_\perp^{low}$ can be computed as

$$\varepsilon_\perp^{low} = \sqrt{\tilde{a}(\mathcal{H}, \mathcal{H})} = \left(\sum_{i=1}^{N} a_{\phi_i}(\boldsymbol{\eta}_i^{low}, \boldsymbol{\eta}_i^{low}) \right)^{\frac{1}{2}}.$$

The previous minimization problem can be written in the equivalent form

$$\sum_{i=1}^{N} a_{\phi_i}(\boldsymbol{\eta}_i^{low}, \boldsymbol{\eta}_i^{low}) \leq \sum_{i=1}^{N} a_{\phi_i}(\mathbf{v}_i, \mathbf{v}_i), \qquad \forall \{\mathbf{v}_i\}_{i=1}^N \in \boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$$

Since the functions $\mathbf{v}_i$ are completely independent one to the other, each function $\boldsymbol{\eta}_i^{low}$, on each subdomain ω_i, is the solution of the local minimization problem stated in (41). □

Remark 5.5 *As in Remark (5.3), if the form b_{ϕ_i} satisfies an* inf-sup *condition, we can add the non homogeneous constraint explicitly by means of a Lagrange multiplier in (41) and write the first-order variation conditions. We obtain, in this case, the set of N local weighted Stokes problems:* find $\boldsymbol{\eta}_i^{low} \in \overset{\circ}{\mathbf{W}}(\omega_i)$ and $\xi_i^{low} \in Z(\omega_i)$ such that

$$\begin{cases} a_{\phi_i}(\boldsymbol{\eta}_i^{low}, \mathbf{v}) + b_{\phi_i}(\mathbf{v}, \xi_i^{low}) = 0, & \forall \mathbf{v} \in \overset{\circ}{\mathbf{W}}(\omega_i) \\ b_{\phi_i}(\boldsymbol{\eta}_i^{low}, q) = \mathcal{R}_h^c(q\phi_i) & \forall q \in Z(\omega_i) \end{cases} \tag{42}$$

Again, this will be the formulation adopted in Section 7 to define a computable error estimator.

Remark 5.6 *Also in this case, we could have taken as a broken space $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c} = \boldsymbol{\mathcal{M}}$. It is clear, indeed, that $\mathcal{I}(\mathbf{V}_{\mathcal{R}_h^c}) \subset \boldsymbol{\mathcal{M}}$. Yet, the solution to the minimization problem: Find $\mathcal{H} = \{\boldsymbol{\eta}_i^{low}\}_{i=1}^N \in \boldsymbol{\mathcal{M}}$ such that*

$$\tilde{a}(\mathcal{H}, \mathcal{H}) \leq \tilde{a}(\mathcal{O}, \mathcal{O}), \qquad \forall \mathcal{O} \in \boldsymbol{\mathcal{M}}$$

is the trivial solution $\boldsymbol{\eta}_i^{low} = \mathbf{0}$ for all $i = 1, \ldots, N$ and the corresponding error estimator would be $\varepsilon_\perp^{low} = 0$. Therefore, this choice is of no use in practice.

Remark 5.7 *The two local Stokes problems (37) and (42), defined on each patch of elements, are identical except for the right-hand side. From the numerical point of view this means that, whenever they are approximated in some discrete spaces, they could be solved simultaneously when using a direct solver.*

6 Patch-wise computed functions $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_\perp$

In the previous section we have defined the two estimators ε_0^{up} and $\varepsilon_\perp^{low}$, based on the broken spaces $\boldsymbol{\mathcal{M}}_0$ and $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}$, respectively. In this section, instead, we consider the other

two estimators ε_0^{low} and $\varepsilon_\perp^{up}$ that are based on the construction of two particular functions $\boldsymbol{\psi}_0 \in \mathbf{V}_{\mathrm{div}}$ and $\boldsymbol{\psi}_\perp \in \mathbf{V}_{\mathcal{R}_h^c}$, according to Proposition 4.1. The goal is to construct such functions by avoiding to solve a global problem. By similarity with the estimators proposed in the previous section, we present, here, a way to compute the functions $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_\perp$ that is based on the solution of local problems on patches of elements. Yet, this time, we will not make use of weighted bilinear forms.
We denote by $a_{\omega_i}(\cdot,\cdot)$ and $b_{\omega_i}(\cdot,\cdot)$ the restrictions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ to the subdomain ω_i (without any weight). Then, we introduce the two sets of local Stokes problems, defined on each patch $\omega_i,\ i = 1,\dots,N$:

find $\boldsymbol{\psi}_i^0 \in [H_0^1(\omega_i)]^d$ *and* $\zeta_i^0 \in L_0^2(\omega_i)$ *such that*

$$\begin{cases} a_{\omega_i}(\boldsymbol{\psi}_i^0,\mathbf{v}) + b_{\omega_i}(\mathbf{v},\zeta_i^0) = \mathcal{R}_h^m(\mathbf{v}) & \forall \mathbf{v} \in [H_0^1(\omega_i)]^d \\ b_{\omega_i}(\boldsymbol{\psi}_i^0, q) = 0 & \forall q \in L_0^2(\omega_i) \end{cases} \tag{43}$$

find $\boldsymbol{\psi}_i^\perp \in [H_0^1(\omega_i)]^d$ *and* $\zeta_i^\perp \in L_0^2(\omega_i)$ *such that*

$$\begin{cases} a_{\omega_i}(\boldsymbol{\psi}_i^\perp,\mathbf{v}) + b_{\omega_i}(\mathbf{v},\zeta_i^\perp) = 0 & \forall \mathbf{v} \in [H_0^1(\omega_i)]^d \\ b_{\omega_i}(\boldsymbol{\psi}_i^\perp, q) = \mathcal{R}_h^c(q\phi_i) & \forall q \in L_0^2(\omega_i) \end{cases} \tag{44}$$

Problems (43) and (44) are standard Stokes problems, set on each subdomain ω_i, with homogeneous Dirichlet boundary conditions and bounded functionals on the right hand side. Thus, they admit a unique solution.
We set, now

$$\boldsymbol{\psi}_0 = \sum_{i=1}^N \boldsymbol{\psi}_i^0 \qquad \text{and} \qquad \boldsymbol{\psi}_\perp = \sum_{i=1}^N \boldsymbol{\psi}_i^\perp \tag{45}$$

where $\boldsymbol{\psi}_i^0$ and $\boldsymbol{\psi}_i^\perp$ are extended by zero outside ω_i, and define the error estimators ε_0^{low} and $\varepsilon_\perp^{up}$ as

$$\varepsilon_0^{low} = \frac{\mathcal{R}_h^m(\boldsymbol{\psi}_0)}{\|\boldsymbol{\psi}_0\|_a} = \sum_{i=1}^N \frac{a_{\omega_i}(\boldsymbol{\psi}_i^0,\boldsymbol{\psi}_i^0)}{\|\boldsymbol{\psi}_0\|_a} \qquad \text{and} \qquad \varepsilon_\perp^{up} = \|\boldsymbol{\psi}_\perp\|_a. \tag{46}$$

The following result holds

Lemma 6.6 *The functions* $\boldsymbol{\psi}_0$ *and* $\boldsymbol{\psi}_\perp$*, defined in (45), satisfy*

$$\boldsymbol{\psi}_0 \in \mathbf{V}_{\mathrm{div}}, \qquad \textit{and} \qquad \boldsymbol{\psi}_\perp \in \mathbf{V}_{\mathcal{R}_h^c}$$

and the two estimators ε_0^{low} *and* $\varepsilon_\perp^{up}$ *satisfy*

$$\varepsilon_0^{low} \leq \|\mathbf{e}_0\|_a \qquad \textit{and} \qquad \varepsilon_\perp^{up} \geq \|\mathbf{e}_\perp\|_a$$

Proof: Both functions $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_\perp$ are sums of H_0^1 functions whose support have finite intersections, thus they belong to $\mathbf{V}$. Now, let us observe that the second equation in (43) and in (44) holds for all functions $q \in L^2(\omega_i)$ and not only for all $q \in L_0^2(\omega_i)$. Indeed, given a constant function $\bar{q}$ on ω_i, we have on one hand

$$b_{\omega_i}(\mathbf{v}_i, \bar{q}) = -\int_{\omega_i} \operatorname{div} \mathbf{v}_i \, \bar{q} = -\bar{q} \int_{\partial \omega_i} \mathbf{v}_i \cdot \mathbf{n} = 0, \qquad \text{for both } \mathbf{v}_i = \boldsymbol{\psi}_i^0 \text{ and } \mathbf{v}_i = \boldsymbol{\psi}_i^\perp$$

and on the other hand

$$\mathcal{R}_h^c(\bar{q}\phi_i) = \bar{q}\, \mathcal{R}_h^c(\phi_i) = 0$$

thanks to property (8). Then, for $\boldsymbol{\psi}_0$ we have:

$$b(\boldsymbol{\psi}_0, q) = \sum_{i=1}^N b(\boldsymbol{\psi}_i^0, q) = \sum_{i=1}^N b_{\omega_i}(\boldsymbol{\psi}_i^0, q_{|\omega_i}) = 0, \qquad \forall q \in L_0^2(\Omega),$$

the last equality holding since $q_{|\omega_i} \in L^2(\omega_i)$. On the other hand, for $\boldsymbol{\psi}_\perp$ we have

$$\begin{aligned} b(\boldsymbol{\psi}_\perp, q) &= \sum_{i=1}^N b(\boldsymbol{\psi}_i^\perp, q) = \sum_{i=1}^N b_{\omega_i}(\boldsymbol{\psi}_i^\perp, q_{|\omega_i}) \\ &= \sum_{i=1}^N \mathcal{R}_h^c(q_{|\omega_i}\phi_i) = \mathcal{R}_h^c(q \sum_{i=1}^N \phi_i) = \mathcal{R}_h^c(q) \qquad \forall q \in L_0^2(\Omega) \end{aligned}$$

and this achieves the proof of the first assertion in the Lemma. The second assertion comes immediately from Proposition 4.1. □

Remark 6.8 *The presence of the weight ϕ_i in the right-hand side of (44) is necessary to guarantee that the mass equation is satisfied for all $q \in L^2(\omega_i)$. Observe that, given a constant function $\bar{q}$ on ω_i, we always have $\mathcal{R}_h^c(\bar{q}\phi_i) = 0$, whereas, in general, the quantity $\mathcal{R}_h^c(\bar{q})$ does not vanish unless a discontinuous finite element space is used for the pressure field.*

Remark 6.9 *Whenever a discontinuous finite element space is used for the pressure field, the divergence constraint can be localized more easily element-wise without introducing a partition of unity. Indeed, in this case, on each element K of the mesh $\mathcal{T}_h$ we have*

$$\mathcal{R}_h^c(q) \equiv -\int_K \operatorname{div} \mathbf{u}_h \, q \, dK = 0, \qquad \forall q \textit{ constant on } K \textit{ and 0 elsewhere}.$$

and the local problems (43) and (44), set on each element K (instead of each patch ω_i) are well posed and lead to exact bounds for $\|\mathbf{e}_0\|_a$ and $\|\mathbf{e}_\perp\|_a$.

Remark 6.10 *What has been said in Remark 5.7 holds also in this case. The two local problems (43) and (44), set on each patch of elements, are identical except for the right-hand side. At the numerical level, they can be solved at the same time by using a direct solver.*

7 Computable a posteriori error estimators

The estimators introduced in the previous section involve the solution of infinite dimensional problems, although defined only locally on each patch of elements. Therefore, those estimators are not directly employable in applications. We can overcome this difficulty by approximating the local problems in some finite dimensional spaces. Yet, the choice of the approximation spaces is quite delicate since, on one hand, we need to guarantee that the local discrete problems thus obtained are well posed and, on the other hand, we would like to have a computable estimator that is still a good error estimator, i.e., it provides exact upper and lower bounds for the error up to higher order terms.
To derive a proper discretization of the local problems (37), (42), (43) and (44), we proceed as follows. We introduce two global enriched finite element spaces $\mathbf{V}_H \supset \mathbf{V}_h$ and $Q_H \supset Q_h$ and denote by $(\mathbf{u}_H, p_H)$ the finite element solution of the Stokes problem (2) in $(\mathbf{V}_H, Q_H)$:

$$\begin{cases} a(\mathbf{u}_H, \mathbf{v}) + b(\mathbf{v}, p_H) = (\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{V}_H, \\ b(\mathbf{u}_H, q) = 0 & \forall q \in Q_H. \end{cases} \tag{47}$$

We will call problem (47) the *fine* model and we expect it to provide a solution that is a much better approximation of the exact solution $(\mathbf{u}, p)$ than the *coarse* solution $(\mathbf{u}_h, p_h)$. (some authors [16, 15] call the fine problem *truth-mesh* discretization in opposition to the *working* approximation, corresponding to our coarse FE problem). Of course, the coarse FE solution $(\mathbf{u}_h, p_h)$ can be seen as an approximation of the fine solution $(\mathbf{u}_H, p_H)$ as well. If our goal were to solve only the fine model (47), instead of the true (infinite dimensional) Stokes problem (2), we could follow the general approach proposed in Section 4 to estimate the error $(\mathbf{e}^{Hh}, E^{Hh}) = (\mathbf{u}_H - \mathbf{u}_h, p_H - p_h)$. In particular, by splitting the errors $(\mathbf{e}^{Hh}, E^{Hh})$ in $\mathbf{e}^{Hh} = \mathbf{e}_0^{Hh} + \mathbf{e}_\perp^{Hh}$, and $E^{Hh} = E_0^{Hh} + E_\perp^{Hh}$ that satisfy, respectively, the problems

$$\begin{cases} a(\mathbf{e}_0^{Hh}, \mathbf{v}) + b(\mathbf{v}, E_0^{Hh}) = \mathcal{R}_h^m(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}_H \\ b(\mathbf{e}_0^{Hh}, q) = 0, & \forall q \in Q_H \end{cases}$$

$$\begin{cases} a(\mathbf{e}_\perp^{Hh}, \mathbf{v}) + b(\mathbf{v}, E_\perp^{Hh}) = 0, & \forall \mathbf{v} \in \mathbf{V}_H \\ b(\mathbf{e}_\perp^{Hh}, q) = \mathcal{R}_h^c(q), & \forall q \in Q_H, \end{cases}$$

we could derive estimators $\varepsilon_{0,H}^{up}$, $\varepsilon_{0,H}^{low}$, $\varepsilon_{\perp,H}^{up}$, $\varepsilon_{\perp,H}^{low}$, for the quantities $\|\mathbf{e}_0^{Hh}\|_a$ and $\|\mathbf{e}_\perp^{Hh}\|_a$. The characterization of the velocity error given in Lemmas 3.1 and 3.2, as well as the assertions in Propositions 4.1 and 4.2, are still valid, in this case, provided we replace the continuous spaces $\mathbf{V}_{\mathrm{div}}$ and $\mathbf{V}_{\mathcal{R}_h^c}$ with their discrete counterparts

$$\begin{aligned} \mathbf{V}_{\mathrm{div}}^H &= \{\mathbf{v} \in \mathbf{V}_H, \quad b(\mathbf{v}, q) = 0 \ \ \forall q \in Q_H\} \\ \mathbf{V}_{\mathcal{R}_h^c}^H &= \{\mathbf{v} \in \mathbf{V}_H, \quad b(\mathbf{v}, q) = \mathcal{R}_h^c(q) \ \ \forall q \in Q_H\}. \end{aligned}$$

It follows that those estimators will provide guaranteed upper and lower bounds for the quantities $\mathbf{e}_0^{Hh}$ and $\mathbf{e}_\perp^{Hh}$.

Observe that, now, the spaces $\mathbf{V}^H_{\rm div}$ and $\mathbf{V}^H_{\mathcal{R}^c_h}$ are finite dimensional. Thus, the estimators $\varepsilon^{up}_{0,H}$, $\varepsilon^{low}_{0,H}$, $\varepsilon^{up}_{\perp,H}$, $\varepsilon^{low}_{\perp,H}$ will be obtained by solving local problems in locally, yet finite dimensional, enriched spaces, and therefore they are *computable*.
Since the true error is $\mathbf{e} = \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{u}_H) + \mathbf{e}^{Hh}$, an estimator for the quantity $\|\mathbf{e}^{Hh}\|_a$ will also be an estimator for the true error $\|\mathbf{e}\|_a$ up to the term $\|\mathbf{u} - \mathbf{u}_H\|_a$, which, under reasonable assumptions on the regularity of the solution, is a higher order term with respect to $\|\mathbf{e}\|_a$.
The enriched spaces $(\mathbf{V}_H, Q_H)$ might be obtained by either refine the mesh or increasing the polynomial degree, or both. In the following of this section we detail the case where we only increase the polynomial degree. For any integer $p > 0$, we consider the enriched finite element spaces $\mathbf{V}_H = \mathbf{V}^{k+p}_h$ and $Q_H = Q^{m+p}_h$ obtained from the spaces $\mathbf{V}^k_h$ and Q^m_h by adding internal or edge *bubbles* up to the degree $k+p$ for the velocity field and $m+p$ for the pressure (here, p represents the "extra" degree that will be used in the solution of the local problems). We make the assumption that ($\mathbf{V}^{k+p}_h$,Q^{m+p}_h) satisfy the *inf-sup* condition (4) and the *local inf-sup* condition (6). Given any subset $\sigma \in \Omega$, we will, furthermore, indicate with $\mathbf{V}^{k+p}_h(\sigma) = \{\mathbf{v}_{|\sigma},\ \forall \mathbf{v} \in \mathbf{V}^{k+p}_h\}$ (similarly for $Q^{m+p}_h(\sigma)$).

7.1 Computable functions $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_\perp$

Keeping in mind the discrete framework we have just set, the functions $\boldsymbol{\psi}_0$ and $\boldsymbol{\psi}_\perp$ should belong, respectively, to the finite dimensional spaces $\mathbf{V}^H_{\rm div}$ and $\mathbf{V}^H_{\mathcal{R}^c_h}$.
To compute those functions, we introduce the local enriched finite dimensional spaces

$$\begin{aligned}
&\overset{\circ}{\mathbf{V}}{}^{k+p}_h(\omega_i) = \mathbf{V}^{k+p}_h(\omega_i) \cap [H^1_0(\omega_i)]^d, && \text{for each patch } \omega_i, \quad i = 1, \dots, N \\
&\overset{\circ}{Q}{}^{m+p}_h(\omega_i) = Q^{m+p}_h(\omega_i) \cap L^2_0(\omega_i), && \text{“}
\end{aligned}$$

and the local discrete problems:

find $\boldsymbol{\psi}^0_i \in \overset{\circ}{\mathbf{V}}{}^{k+p}_h(\omega_i)$ *and* $\zeta^0_i \in \overset{\circ}{Q}{}^{m+p}_h(\omega_i)$ *such that*

$$\begin{cases}
a_{\omega_i}(\boldsymbol{\psi}^0_i, \mathbf{v}) + b_{\omega_i}(\mathbf{v}, \zeta^0_i) = \mathcal{R}^m_h(\mathbf{v}) & \forall \mathbf{v} \in \overset{\circ}{\mathbf{V}}{}^{k+p}_h(\omega_i) \\
b_{\omega_i}(\boldsymbol{\psi}^0_i, q) = 0 & \forall q \in \overset{\circ}{Q}{}^{m+p}_h(\omega_i)
\end{cases} \tag{48}$$

find $\boldsymbol{\psi}^\perp_i \in \overset{\circ}{\mathbf{V}}{}^{k+p}_h(\omega_i)$ *and* $\zeta^\perp_i \in \overset{\circ}{Q}{}^{m+p}_h(\omega_i)$ *such that*

$$\begin{cases}
a_{\omega_i}(\boldsymbol{\psi}^\perp_i, \mathbf{v}) + b_{\omega_i}(\mathbf{v}, \zeta^\perp_i) = 0 & \forall \mathbf{v} \in \overset{\circ}{\mathbf{V}}{}^{k+p}_h(\omega_i) \\
b_{\omega_i}(\boldsymbol{\psi}^\perp_i, q) = \mathcal{R}^c_h(q\phi_i) & \forall q \in \overset{\circ}{Q}{}^{m+p}_h(\omega_i)
\end{cases} \tag{49}$$

We set, as in Section 6,

$$\boldsymbol{\psi}_{0,H} = \sum_{i=1}^N \boldsymbol{\psi}^0_i \qquad \text{and} \qquad \boldsymbol{\psi}_{\perp,H} = \sum_{i=1}^N \boldsymbol{\psi}^\perp_i \tag{50}$$

and define the computable error estimators $\varepsilon_{0,H}^{low}$ and $\varepsilon_{\perp,H}^{up}$ as

$$\varepsilon_{0,H}^{low} = \frac{\mathcal{R}_h^m(\boldsymbol{\psi}_{0,H})}{\|\boldsymbol{\psi}_{0,H}\|_a} = \sum_{i=1}^{N} \frac{a_{\omega_i}(\boldsymbol{\psi}_i^0, \boldsymbol{\psi}_i^0)}{\|\boldsymbol{\psi}_{0,H}\|_a} \qquad \text{and} \qquad \varepsilon_{\perp,H}^{up} = \|\boldsymbol{\psi}_{\perp,H}\|_a. \tag{51}$$

It is easy to show that the local problems (48) and (49) are well posed and that the computable functions $\boldsymbol{\psi}_{0,H}$ and $\boldsymbol{\psi}_{\perp,H}$ satisfy

$$\boldsymbol{\psi}_{0,H} \in \mathbf{V}_{\mathrm{div}}^H \qquad \text{and} \qquad \boldsymbol{\psi}_{\perp,H} \in \mathbf{V}_{\mathcal{R}_h^c}^H.$$

(The proof is identical to the one of Lemma 6.6). Thus, the following result holds

Lemma 7.7 *The computable estimators $\varepsilon_{0,H}^{low}$ and $\varepsilon_{\perp,H}^{up}$ satisfy*

$$\varepsilon_{0,H}^{low} \leq \|\mathbf{e}_0^{Hh}\|_a \qquad \textit{and} \qquad \varepsilon_{\perp,H}^{up} \geq \|\mathbf{e}_{\perp}^{Hh}\|_a$$

7.2 Discrete broken spaces $\boldsymbol{\mathcal{M}}^H$, $\boldsymbol{\mathcal{M}}_0^H$ and $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}^H$

Let us introduce the local weighted enriched spaces, on each patch $\omega_i,\ i = 1, \ldots, N$,

$$\mathring{\mathbf{W}}_h^{k+p}(\omega_i) = \mathbf{V}_h^{k+p}(\omega_i) \cap \mathring{\mathbf{W}}(\omega_i).$$

Then, the discrete broken space $\boldsymbol{\mathcal{M}}^H$ can be defined as

$$\boldsymbol{\mathcal{M}}^H = \prod_{i=1}^{N} \mathring{\mathbf{W}}_h^{k+p}(\omega_i)$$

Following the proof of Lemma 5.3, it can be shown that $\boldsymbol{\mathcal{M}}^H$ is a broken space for $\mathbf{V}_H$, i.e. the application $\mathcal{I}$ injects isometrically $\mathbf{V}^H$ into $\boldsymbol{\mathcal{M}}^H$.
To define the other two subspaces, we introduce the local enriched space for the pressure

$$Z_h^{m+p-1}(\omega_i) = Q_h^{m+p-1}(\omega_i) \cap Z(\omega_i).$$

The reason why we take polynomials of degree $m+p-1$, instead of the more natural choice $m+p$ will be clear later. Then, the two subspaces $\boldsymbol{\mathcal{M}}_0^H$ and $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}^H$ are defined as

$$\begin{aligned}
\boldsymbol{\mathcal{M}}_{0,S}^H &= \{\mathcal{O} = \{\mathbf{v}_i\}_{i=1}^N \in \boldsymbol{\mathcal{M}}^H,\ \ b_{\phi_i}(\mathbf{v}_i, q) = 0\ \ \forall q \in Z_h^{m+p-1}(\omega_i)\},\\
\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}^H &= \{\mathcal{O} = \{\mathbf{v}_i\}_{i=1}^N \in \boldsymbol{\mathcal{M}}^H,\ \ b_{\phi_i}(\mathbf{v}_i, q) = \mathcal{R}_h^c(q\phi_i)\ \ \forall q \in Z_h^{m+p-1}(\omega_i)\}.
\end{aligned}$$

The following result holds

Lemma 7.8 *Let $\mathcal{I}$ be the linear application defined in (30). We have:*

$$\mathcal{I}(\mathbf{V}_{\mathrm{div}}^H) \subset \boldsymbol{\mathcal{M}}_{0,S}^H, \qquad \textit{and} \qquad \mathcal{I}(\mathbf{V}_{\mathcal{R}_h^c}^H) \subset \boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}^H.$$

Proof: For any $q \in Q_h^{m+p-1}(\omega_i)$, the function $\tilde{q} = q\phi_i$, extended by zero outside ω_i, belongs to Q_h^{m+p}, since ϕ_i is a first order Lagrange basis function. Then, given any $\mathbf{v} \in \mathbf{V}_h^{k+p}$, we have

$$b_{\phi_i}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i, q) = b(\mathbf{v}, q\phi_i) = b(\mathbf{v}, \tilde{q}).$$

Thus, if $\mathbf{v} \in \mathbf{V}_{\mathrm{div}}^H$, we have $b_{\phi_i}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i, q) = b(\mathbf{v}, \tilde{q}) = 0$ whereas, if $\mathbf{v} \in \mathbf{V}_{\mathcal{R}_h^c}^H$, we have $b_{\phi_i}(\mathbf{v}_{|\omega_i} - \bar{\mathbf{v}}_i, q) = b(\mathbf{v}, \tilde{q}) = \mathcal{R}_h^c(q\phi_i)$ and these relations hold for all $q \in Q_h^{m+p-1}(\omega_i)$. □

Remark 7.11 *From the proof of Lemma 7.8 it should be clear that, if we consider the local enriched space $Z_h^{m+p}(\omega_i)$, instead of $Z_h^{m+p-1}(\omega_i)$ in the definition of either $\boldsymbol{\mathcal{M}}_{0,S}^H$ or $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}^H$, we will not have, in general the inclusions stated in Lemma 7.8, because of the presence of the weight ϕ_i in the bilinear form $b_{\phi_i}(\cdot,\cdot)$. On the other hand, with this choice of space, the local Stokes problems are even more constrained and we may expect to obtain better numerical results. We have tested numerically both cases and the results, presented in Section 8, confirm our expectation.*

Remark 7.12 *The presence of the weight ϕ_i in the form b_{ϕ_i} is necessary to localize the constraint, whenever a continuous finite element space for the pressure is considered (see also Remark 6.9). If, instead, a space of discontinuous pressures is employed, a localization of the continuity equation on each patch ω_i can be achieved by simply taking a pressure test function*

$$\tilde{q} \in Q_h^{m+p}, \qquad \tilde{q} = \begin{cases} q & \text{in } \omega_i \\ 0 & \text{in } \Omega \setminus \omega_i \end{cases}, \qquad \text{with } q \in Q_h^{m+p}(\omega_i).$$

Thus, in this case, we could define the broken spaces as

$$\boldsymbol{\mathcal{M}}_{0,S}^H = \{\mathcal{O} = \{\mathbf{v}_i\}_{i=1}^N, \ \ \mathbf{v}_i \in \mathring{\mathbf{W}}_h^{k+p}(\omega_i), \ \ b_{\omega_i}(\mathbf{v}_i, q) = 0 \ \ \forall q \in Q_h^{m+p}(\omega_i)\}$$
$$\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}^H = \{\mathcal{O} = \{\mathbf{v}_i\}_{i=1}^N, \ \ \mathbf{v}_i \in \mathring{\mathbf{W}}_h^{k+p}(\omega_i), \ \ b_{\omega_i}(\mathbf{v}_i, q) = \mathcal{R}_h^c(q) \ \ \forall q \in Q_h^{m+p}(\omega_i)\},$$

without the need to weight the local bilinear form $b(\cdot,\cdot)$. It is easy to show that, in this case, the result of Lemma 7.8 is still valid.

Following the same arguments as those presented in Sections 5.1 and 5.2, it can be shown that the estimators associated with the broken spaces $\boldsymbol{\mathcal{M}}_{0,S}^H$ and $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}^H$ can be computed by solving the local problems

find $\boldsymbol{\eta}_i^{up} \in \mathring{\mathbf{W}}_h^{k+p}(\omega_i)$ and $\xi_i^{up} \in Z_h^{m+p-1}(\omega_i)$ such that

$$\begin{cases} a_{\phi_i}(\boldsymbol{\eta}_i^{up}, \mathbf{v}) + b_{\phi_i}(\mathbf{v}, \xi_i^{up}) = \mathcal{R}_h^m(\mathbf{v}\phi_i), & \forall \mathbf{v} \in \mathring{\mathbf{W}}_h^{k+p}(\omega_i) \\ b_{\phi_i}(\boldsymbol{\eta}_i^{up}, q) = 0 & \forall q \in Z_h^{m+p-1}(\omega_i) \end{cases} \tag{52}$$

find $\boldsymbol{\eta}_i^{low} \in \overset{\circ}{\mathbf{W}}{}_h^{k+p}(\omega_i)$ *and* $\xi_i^{low} \in Z_h^{m+p-1}(\omega_i)$ *such that*

$$\begin{cases} a_{\phi_i}(\boldsymbol{\eta}_i^{low}, \mathbf{v}) + b_{\phi_i}(\mathbf{v}, \xi_i^{low}) = 0, & \forall \mathbf{v} \in \overset{\circ}{\mathbf{W}}{}_h^{k+p}(\omega_i) \\ b_{\phi_i}(\boldsymbol{\eta}_i^{low}, q) = \mathcal{R}_h^c(q\phi_i) & \forall q \in Z_h^{m+p-1}(\omega_i) \end{cases} \tag{53}$$

The computable error estimators $\varepsilon_{0,S,H}^{up}$ and $\varepsilon_{\perp,H}^{low}$ are then defined as

$$\varepsilon_{0,S,H}^{up} = \left(\sum_{i=1}^{N} a_{\phi_i}(\boldsymbol{\eta}_i^{up}, \boldsymbol{\eta}_i^{up}) \right)^{\frac{1}{2}} \qquad \text{and} \qquad \varepsilon_{\perp,H}^{low} = \left(\sum_{i=1}^{N} a_{\phi_i}(\boldsymbol{\eta}_i^{low}, \boldsymbol{\eta}_i^{low}) \right)^{\frac{1}{2}} \tag{54}$$

Lemma 7.9 *The local problems (52) and (53) admit a unique solution.*

Proof: As already observed, the form $a_{\phi_i}(\cdot,\cdot)$ is continuous and coercive in $\overset{\circ}{\mathbf{W}}(\omega_i)$ and the right-hand sides $\mathcal{R}_h^m(\mathbf{v}\phi_i)$ and $\mathcal{R}_h^m(q\phi_i)$ are continuous functionals on $\overset{\circ}{\mathbf{W}}(\omega_i)$ and $Z(\omega_i)$, respectively. The existence and uniqueness of the solution is then proved if the bilinear form $b_{\phi_i}(\cdot,\cdot)$ satisfies the discrete *inf-sup* condition

$$\forall q \in Q_h^{m+p-1}(\omega_i) \qquad \sup_{\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}_h^{k+p}(\omega_i)} \frac{b_{\phi_i}(\mathbf{v}, q)}{\|\mathbf{v}\|_{a,\phi_i}} \geq C\|q\|_{0,\phi_i}$$

Now, let us remark that for any $q \in Q_h^{m+p-1}(\omega_i)$, the function $\tilde{q} = q\phi_i$ belongs to $Q_h^{m+p}(\omega_i)$. Since, according to our assumptions, the spaces $\mathbf{V}_h^{k+p}$ and Q_h^{m+p} satisfy the local discrete inf-sup condition (6), we have that $\forall q \in Q_h^{m+p-1}(\omega_i), \quad \exists \tilde{\mathbf{v}} \in \mathbf{V}_h^{k+p}(\omega_i)$ such that

$$\left| \int_{\omega_i} \operatorname{div} \tilde{\mathbf{v}} \tilde{q} \, d\omega \right| \geq \beta_h^* \sqrt{\nu} \|\nabla \tilde{\mathbf{v}}\|_{L^2(\omega_i)} \|\tilde{q}\|_{L^2(\omega_i)}.$$

By setting $\mathbf{v} = \tilde{\mathbf{v}} - \mathbf{c}$, where the constant $\mathbf{c}$ is chosen in such a way that $\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}_h^{k+p}(\omega_i)$, we conclude that

$$\begin{aligned} &\forall q \in Q_h^{m+p-1}(\omega_i), \ \ \exists \mathbf{v} \in \overset{\circ}{\mathbf{W}}{}_h^{k+p}(\omega_i) \ \text{ such that} \\ &\qquad |b_{\phi_i}(\mathbf{v}, q)| = \left| \int_{\omega_i} \operatorname{div} \tilde{\mathbf{v}} \tilde{q} \, d\omega \right| \geq \beta_h^* \sqrt{\nu} \|\nabla \tilde{\mathbf{v}}\|_{L^2(\omega_i)} \|\tilde{q}\|_{L^2(\omega_i)} \geq C\|\mathbf{v}\|_{a,\phi_i} \|q\|_{0,\phi_i} \end{aligned}$$

where, in the last inequality we have exploited the fact that $\|\mathbf{v}\|_{a,\phi_i} \leq \sqrt{\nu}\|\nabla\tilde{\mathbf{v}}\|_{L^2(\omega_i)}$ and that in a finite-dimensional space all the norms are equivalent; thus $\|q\phi_i^{1/2}\|_{L^2(\omega_i)} \leq C_1\|q\phi_i\|_{L^2(\omega_i)}$ (observe that both the quantities $\|q\phi_i^{1/2}\|_{L^2(\omega_i)}$ and $\|q\phi_i\|_{L^2(\omega_i)}$ are norms for $q \in Q_h^{m+p-1}(\omega_i)$). This achieves the proof. □

Remark 7.13 *Following Remark 5.4, we can also take as $\boldsymbol{\mathcal{M}}_0^H$ the broken space $\boldsymbol{\mathcal{M}}_{0,P}^H = \boldsymbol{\mathcal{M}}^H$. This choice leads to the solution of local discrete Poisson problems in the space $\mathbf{V}_h^{k+p}(\omega_i)$. The two results presented here below are still valid also with this choice of broken space.*

We conclude this section with two results whose proof is an immediate consequence of the developments done so far:

Lemma 7.10 *The computable estimators $\varepsilon_{0,S,H}^{up}$ and $\varepsilon_{\perp,H}^{low}$ satisfy*

$$\varepsilon_{0,S,H}^{up} \geq \|\mathbf{e}_0^{Hh}\|_a \qquad \textit{and} \qquad \varepsilon_{\perp,H}^{low} \leq \|\mathbf{e}_\perp^{Hh}\|_a$$

Lemma 7.11 *We define the upper and lower estimators for the quantity $\|\mathbf{e}\|_a$ as*

$$\varepsilon_H^{up} = \sqrt{\left(\varepsilon_{0,S,H}^{up}\right)^2 + \left(\varepsilon_{\perp,H}^{up}\right)^2} \qquad \textit{and} \qquad \varepsilon_H^{low} = \sqrt{\left(\varepsilon_{0,H}^{low}\right)^2 + \left(\varepsilon_{\perp,H}^{low}\right)^2}.$$

Then we have

$$\varepsilon_H^{up} \geq \|\mathbf{e}^{Hh}\|_a, \qquad \varepsilon_H^{low} \leq \|\mathbf{e}^{Hh}\|_a \tag{55}$$

and

$$\varepsilon_H^{low} - \|\mathbf{u} - \mathbf{u}_H\|_a \leq \|\mathbf{e}\|_a \leq \varepsilon_H^{up} + \|\mathbf{u} - \mathbf{u}_H\|_a.$$

8 Numerical assessment

The first test case we consider is the classical example of the *driven cavity* for the Stokes regime. The domain Ω is the unit square and we consider homogeneous Dirichlet boundary conditions at the bottom, left and right boundaries and non-homogeneous conditions on the top side, namely

$$\mathbf{u} = (u_1, u_2), \qquad u_1 = 4x(1-x), \qquad u_2 = 0.$$

Figure 1 shows the magnitude and the streamlines of the velocity field on the left and the pressure field on the right.
The software we have utilized uses quadrilateral meshes and hp H^1-conformal finite elements whose degree can be chosen in the range $k = 2, \ldots, 8$ for the velocity field. The polynomial degree for the pressure field is then taken equal to $m = k-2$ for the interior bubbles and $m = k-1$ for the edge bubbles. Here we have always chosen $k = 2$, which corresponds to the classic Taylor-Hood finite elements $\mathbb{Q}^2/\mathbb{Q}^1$.
Concerning the error estimators, we have implemented the four estimators $\varepsilon_{0,S}^{up}$, $\varepsilon_\perp^{up}$, ε_0^{low}, $\varepsilon_\perp^{low}$ defined in (51) and (54) (we omit here and in the following the subscript H, to simplify the notation). We have also considered the estimator $\varepsilon_{0,P}^{up}$, associated with the choice $\boldsymbol{\mathcal{M}}_{0,P}^H$ for the broken space $\boldsymbol{\mathcal{M}}_0^H$ (see Remark 7.13), which relies on the solution of local Poisson problems, as well as the two estimators $\tilde{\varepsilon}_{0,S}^{up}$ and $\tilde{\varepsilon}_\perp^{low}$ obtained when we choose the local

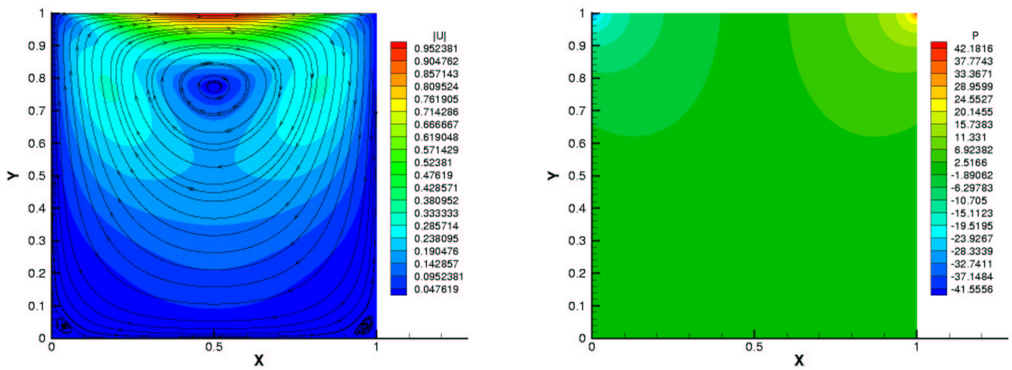

Figure 1: Driven cavity: magnitude and streamlines of the velocity field on the left; pressure field on the right.

enriched space $Z_h^{m+p}(\omega_i)$ instead of $Z_h^{m+p-1}(\omega_i)$ in the definition of the broken spaces $\boldsymbol{\mathcal{M}}_0^H$ and $\boldsymbol{\mathcal{M}}_{\mathcal{R}_h^c}^H$ (see Remark 7.11).

We have considered a sequence of uniform meshes of 2×2, 4×4, 8×8, 16×16 elements. We have computed the error estimators with either $p = 1$ or $p = 2$. We have also computed a very accurate approximation of the errors $\mathbf{e}_0$, $\mathbf{e}_\perp$ and E, as well as of the exact solution $(\mathbf{u}, p)$, by using finite elements of degree $k = 6$ for the velocity and $m = 4, 5$ for the pressure.

Table 1 shows the effectivity index for the estimators of the quantity $\|\mathbf{e}_0\|_a$ (i.e. the ratio $\varepsilon_0^{up,low}/\|\mathbf{e}_0\|_a$) in the two cases $p = 1$ and $p = 2$. We remind that the upper bound estimator is based on the solution of local weighted problems with natural (*Neumann-type*) boundary conditions, while the lower bound estimator is based on the solution of local problems with homogeneous *Dirichlet* boundary conditions. We observe that all the estimators give relatively good results even on the very coarse mesh of 2×2 elements. The estimator $\varepsilon_{0,P}^{up}$, based on the solution of local Poisson problems, gives an effectivity index bigger, although still acceptable, than the estimator $\varepsilon_{0,S}^{up}$, as it was expected theoretically. Moreover, for $p = 1$ the estimator ε_0^{low} does not provide for a lower bound of the error $\mathbf{e}_0$, although the effectivity index is very close to one. This is not in contrast with the result stated in Lemma 7.11.

Table 2 shows similar results for the estimators of the quantity $\|\mathbf{e}_\perp\|_a$. We recall that, in this case, the lower bound estimator is based on the solution of local Neumann-type problems, while the upper bound estimator relies on the solution of Dirichlet local problems. Again, we observe that the Dirichlet estimator does not provide an upper bound neither when $p = 1$, nor when $p = 2$. Yet, the effectivity index remains very close to 1. On the other hand, the Neumann estimator $\varepsilon_{\perp,S}^{low}$ for $p = 1$ is very poor, and improves significantly going from $p = 1$ to $p = 2$. In this case, the variant $\tilde{\varepsilon}_{\perp,S}^{low}$, which uses a richer local pressure space, gives much better results.

We consider, now, a posteriori error estimators for the energy norm of the whole velocity field, by combining estimators for $\|\mathbf{e}_0\|_a$ and for $\|\mathbf{e}_\perp\|_a$. Two of them have already been

$\mathbf{p=1}$		Upper (Neum.)			Lower (Dir.)
h	$\|\mathbf{e}_0\|_a/\|\mathbf{u}\|_a$	$\varepsilon^{up}_{0,S}$	$\tilde{\varepsilon}^{up}_{0,S}$	$\varepsilon^{up}_{0,P}$	ε^{low}_0
0.50000	0.2200	1.18761	1.09821	1.41413	1.01550
0.25000	0.1154	1.18760	1.09867	1.41844	1.00308
0.12500	0.0582	1.18888	1.10015	1.42049	1.00306
0.06250	0.0291	1.18897	1.10017	1.42074	1.00312

$\mathbf{p=2}$		Upper (Neum.)			Lower (Dir.)
h	$\|\mathbf{e}_0\|_a/\|\mathbf{u}\|_a$	$\varepsilon^{up}_{0,S}$	$\tilde{\varepsilon}^{up}_{0,S}$	$\varepsilon^{up}_{0,P}$	ε^{low}_0
0.50000	0.2200	1.14492	1.10644	1.42424	0.99009
0.25000	0.1154	1.14961	1.11535	1.42887	0.98764
0.12500	0.0582	1.15139	1.11719	1.43099	0.98815
0.06250	0.0291	1.15144	1.11723	1.43124	0.98828

Table 1: Driven cavity: effectivity index for the estimators of the quantity $\|\mathbf{e}_0\|_a$. On the top $p = 1$, on the bottom $p = 2$.

introduced in Lemma 7.11, namely

$$\varepsilon^{up} = \sqrt{\left(\varepsilon^{up}_{0,S}\right)^2 + \left(\varepsilon^{up}_{\perp}\right)^2} \qquad \text{and} \qquad \varepsilon^{low} = \sqrt{\left(\varepsilon^{low}_{0}\right)^2 + \left(\varepsilon^{low}_{\perp}\right)^2}$$

and for them the result stated in (55) holds. We can also introduce the two variants

$$\tilde{\varepsilon}^{up} = \sqrt{\left(\tilde{\varepsilon}^{up}_{0,S}\right)^2 + \left(\varepsilon^{up}_{\perp}\right)^2} \qquad \text{and} \qquad \tilde{\varepsilon}^{low} = \sqrt{\left(\varepsilon^{low}_{0}\right)^2 + \left(\tilde{\varepsilon}^{low}_{\perp}\right)^2}$$

as well as the two other estimators

$$\varepsilon^{neu} = \sqrt{\left(\tilde{\varepsilon}^{up}_{0,S}\right)^2 + \left(\tilde{\varepsilon}^{low}_{\perp}\right)^2} \qquad \text{and} \qquad \varepsilon^{dir} = \sqrt{\left(\varepsilon^{low}_{0}\right)^2 + \left(\varepsilon^{up}_{\perp}\right)^2}$$

These last two estimators are of some interest since they rely only on the solution of either Neumann or Dirichlet local problems. Thus, they are half less expensive than the previous ones (which, instead, rely on the solution of both Neumann and Dirichlet subproblems). Table 3 shows the effectivity indices for the six global estimators just introduced.

We can see that all the estimators provide very good results. In particular, the two “cheap” estimators ε^{neu} and ε^{dir}, although they do not provide upper or lower bounds on the error, yield effectivity indices very close to one.

We conclude these numerical results by introducing two estimators for the L^2-norm of the error on the pressure. Taking inspiration from inequality (15) and recalling Remark 5.4, we define the following estimator for the pressure

$$\mathcal{E}^{neu/dir} = \varepsilon^{up}_{0,P} + \varepsilon^{up}_{\perp}.$$

$\mathbf{p=1}$		Upper (Dir.)	Lower (Neum.)	
h	$\|\mathbf{e}_\perp\|_a/\|\mathbf{u}\|_a$	$\varepsilon_\perp^{up}$	$\varepsilon_\perp^{low}$	$\tilde{\varepsilon}_\perp^{low}$
0.50000	0.3370	0.96148	0.64537	0.93448
0.25000	0.1654	0.96478	0.63539	0.93543
0.12500	0.0826	0.96492	0.63400	0.93576
0.06250	0.0413	0.96493	0.63396	0.93576

$\mathbf{p=2}$		Upper (Dir.)	Lower (Neum.)	
h	$\|\mathbf{e}_\perp\|_a/\|\mathbf{u}\|_a$	$\varepsilon_\perp^{up}$	$\varepsilon_\perp^{low}$	$\tilde{\varepsilon}_\perp^{low}$
0.50000	0.3370	0.98869	0.87815	0.93252
0.25000	0.1654	0.98648	0.87760	0.93299
0.12500	0.0826	0.98658	0.87793	0.93332
0.06250	0.0413	0.98658	0.87793	0.93333

Table 2: Driven cavity: effectivity index for the estimators of the quantity $\|\mathbf{e}_\perp\|_a$. On the top $p = 1$, on the bottom $p = 2$.

From inequality (15) we have indeed

$$\|E\|_0 \leq \frac{1}{\beta}\left\{\|\mathcal{R}_h^m\|_{\mathbf{V}'} + \|\mathbf{e}_\perp\|_a\right\} \leq \frac{1}{\beta}\left(\varepsilon_{0,P}^{up} + \varepsilon_\perp^{up}\right).$$

Thus, $\mathcal{E}^{neu/dir}$ will be a reasonable estimator if the constant β, appearing in the inf-sup condition, is close to one. Similarly, we can define the estimator

$$\mathcal{E}^{neu/neu} = \varepsilon_{0,P}^{up} + \tilde{\varepsilon}_\perp^{low}.$$

This second estimator uses only Neumann local problems, although of different type: Poisson local problems to compute $\varepsilon_{0,P}^{up}$ and Stokes ones to compute $\tilde{\varepsilon}_\perp^{low}$.
Table 4 shows the effectivity indices of the estimators for the pressure introduced so far, for the two cases $p = 1$ and $p = 2$.

As a second example we propose the test case of the *backward facing step*. We have solved, in this case, the problem on the two meshes shown in Figure 2. The first one is a very coarse mesh (47 vertices) while the second one is finer (185 vertices excluding the hanging nodes) and has been refined around the reentrant corner to catch the singularity in the pressure that develops there. As in the previous example we have solved the problem using $\mathbb{Q}^2/\mathbb{Q}^1$ finite elements and computed an accurate solution using polynomials of degree six for the velocity. Tables 5, 6, 7, 8 show the effectivity indices of the different estimators previously introduced for the two meshes considered and the two cases $p = 1$ and $p = 2$.
The results are comparable with those obtained in the previous test case. As a general comment we point out that the estimator $\varepsilon_\perp^{low}$ has the poorest effectivity index. Actually, the quantity $\|\mathbf{e}_\perp\|_a$ seems to be the most critical to estimate. Moreover, we remark that the estimators for the pressure error are not reliable, at least for coarse meshes.

$\mathbf{p = 1}$

h	$\|\mathbf{e}\|_a/\|\mathbf{u}\|_a$	ε^{up}	$\tilde{\varepsilon}^{up}$	ε^{low}	$\tilde{\varepsilon}^{low}$	ε^{neu}	ε^{dir}
0.5000	0.4025	1.0343	1.0043	0.7748	0.9594	0.9863	0.9779
0.2500	0.2017	1.0430	1.0105	0.7752	0.9581	0.9918	0.9775
0.1250	0.1011	1.0446	1.0118	0.7762	0.9586	0.9934	0.9778
0.0625	0.0505	1.0446	1.0118	0.7762	0.9586	0.9933	0.9777

$\mathbf{p = 2}$

h	$\|\mathbf{e}\|_a/\|\mathbf{u}\|_a$	ε^{up}	$\tilde{\varepsilon}^{up}$	ε^{low}	$\tilde{\varepsilon}^{low}$	ε^{neu}	ε^{dir}
0.5000	0.4025	1.0379	1.0253	0.9131	0.9501	0.9877	0.9891
0.2500	0.2017	1.0427	1.0304	0.9151	0.9512	0.9963	0.9869
0.1250	0.1011	1.0442	1.0318	0.9160	0.9519	0.9981	0.9871
0.0625	0.0505	1.0441	1.0317	0.9160	0.9519	0.9981	0.9871

Table 3: Driven cavity: effectivity index for the estimators of the quantity $\|\mathbf{e}\|_a$. On the top $p = 1$, on the bottom $p = 2$.

		$\mathcal{E}^{neu,dir}$		$\mathcal{E}^{neu,neu}$	
h	$\|E\|_0/\|p\|_0$	$p = 1$	$p = 2$	$p = 1$	$p = 2$
0.5000	0.2628	1.5279	1.5553	1.5060	1.5098
0.2500	0.1215	1.6872	1.7122	1.6619	1.6660
0.1250	0.0593	1.7358	1.7614	1.7100	1.7144
0.0625	0.0296	1.7367	1.7624	1.7109	1.7153

Table 4: Driven cavity: effectivity index for the estimators of the L^2-norm of the error on the pressure

		Upper (Neum.)			Lower (Dir.)
	$\|\mathbf{e}_0\|_a/\|\mathbf{u}\|_a$	$\varepsilon^{up}_{0,S}$	$\tilde{\varepsilon}^{up}_{0,S}$	$\varepsilon^{up}_{0,P}$	ε^{low}_0
$\mathbf{p = 1}$					
MESH 1	0.0451	1.0704	1.0336	1.1379	0.8434
MESH 2	0.0098	1.1472	1.1038	1.2209	0.9203
$\mathbf{p = 2}$					
MESH 1	0.0451	1.1056	1.0836	1.1954	0.9108
MESH 2	0.0098	1.1407	1.1167	1.2521	0.9377

Table 5: Backward facing step: effectivity index for the estimators of the quantity $\|\mathbf{e}_0\|_a$.

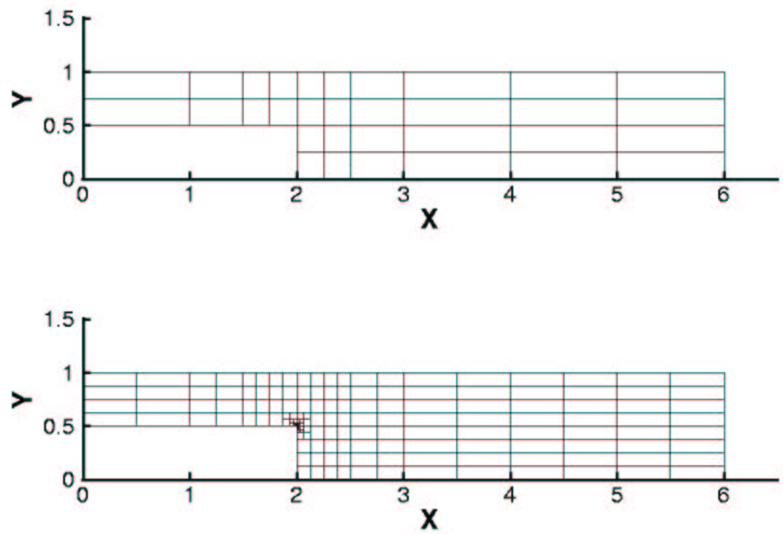

Figure 2: Backward facing step: coarse mesh on the top, refined mesh on the bottom.

		Upper (Dir.)	Lower (Neum.)	
	$\|\mathbf{e}_\perp\|_a/\|\mathbf{u}\|_a$	$\varepsilon_\perp^{up}$	$\varepsilon_\perp^{low}$	$\tilde{\varepsilon}_\perp^{low}$
		$\mathbf{p=1}$		
MESH 1	0.1006	0.8877	0.5578	0.7234
MESH 2	0.0169	0.9001	0.5608	0.7392
		$\mathbf{p=2}$		
MESH 1	0.1006	1.0052	0.7087	0.7686
MESH 2	0.0169	1.0185	0.7288	0.7919

Table 6: Backward facing step: effectivity index for the estimators of the quantity $\|\mathbf{e}_\perp\|_a$

	$\|\mathbf{e}\|_a/\|\mathbf{u}\|_a$	ε^{up}	$\tilde{\varepsilon}^{up}$	ε^{low}	$\tilde{\varepsilon}^{low}$	ε^{neu}	ε^{dir}
			$\mathbf{p=1}$				
MESH 1	0.1103	0.9208	0.9137	0.6149	0.7448	0.7839	0.8804
MESH 2	0.0195	0.9684	0.9556	0.6700	0.7889	0.8462	0.9052
			$\mathbf{p=2}$				
MESH 1	0.1103	1.0227	1.0187	0.7463	0.7941	0.8297	0.9900
MESH 2	0.0195	1.0507	1.0442	0.7868	0.8311	0.8852	0.9987

Table 7: Backward facing step: effectivity index for the estimators of the quantity $\|\mathbf{e}\|_a$.

		$\mathcal{E}^{neu,dir}$		$\mathcal{E}^{neu,neu}$	
	$\|E\|_0/\|p\|_0$	$p=1$	$p=2$	$p=1$	$p=2$
MESH 1	0.0100	0.4683	0.5163	0.4132	0.4370
MESH 2	0.0008	1.2007	1.3025	1.0807	1.1335

Table 8: Backward facing step: effectivity index for the estimators of the L^2-norm of the error on the pressure.

9 Conclusions and future work

In this work we have proposed a general approach to obtain upper and lower bounds on the error in the velocity field measured in the energy norm. We have also pointed out that, in general, estimates for the error in the pressure involve the unknown constant appearing in the *inf-sup* condition.

The a posteriori estimators analyzed in Sections 5 and 6 are based on the solution of local Stokes problems on patches of elements and are well suited for a finite element discretization involving continuous pressure spaces. In the case of a discontinuous pressure space, other options are available, as pointed out in several Remarks throughout the text, eventually leading to the solution of local problems on each element instead of patches of elements.

Some questions deserve further investigation. First, the extendibility of this technique to other problems like the Oseen or Navier-Stokes equations. The analysis carried out here relies on the symmetry of the bilinear form $a(\cdot,\cdot)$. Therefore, the extension to non symmetric problems is not straightforward. Yet, if for instance the unsteady Navier-Stokes equations are discretized with a time marching scheme that treats explicitly the convective non-linear term, at each time step we are faced with a symmetric Stokes-like problem and the proposed technique for error estimation could be applied in each time slab.

Another issue concerns the extension of this technique to the case where we wish to estimate the error in specific quantities of interest, and, in particular, quantities that might depend on the pressure field, on which we do not have a reliable estimator. This issue will be the subject of future work.

Finally, it would certainly be interesting to extend the present technique to other saddle point problems such as mixed formulations for elliptic or elasticity equations.

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