Infinite dimensional weak Dirichlet processes, stochastic PDEs and optimal control

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Abstract

The present paper continues the study of infinite dimensional calculus via regularization, started by C. Di Girolami and the second named author, introducing the notion of *weak Dirichlet process* in this context. Such a process $\mathbb{X}$, taking values in a Hilbert space H, is the sum of a local martingale and a suitable *orthogonal* process.

The new concept is shown to be useful in several contexts and directions. On one side, the mentioned decomposition appears to be a substitute of an Itô's type formula applied to $f(t, \mathbb{X}(t))$ where $f : [0, T] \times H \to \mathbb{R}$ is a $C^{0,1}$ function and, on the other side, the idea of weak Dirichlet process fits the widely used notion of *mild solution* for stochastic PDE. As a specific application, we provide a verification theorem for stochastic optimal control problems whose state equation is an infinite dimensional stochastic evolution equation.

KEY WORDS AND PHRASES: Covariation and Quadratic variation; Calculus via regularization; Infinite dimensional analysis; Tensor analysis; Dirichlet processes; Generalized Fukushima decomposition; Stochastic partial differential equations; Stochastic control theory.

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1 Introduction

Stochastic calculus constitutes one of the basic tools for stochastic optimal control theory; in particular the classical Itô formula allows to relate the solution of the problem in closed loop form with the (smooth enough) solutions of the related Hamilton-Jacobi-Bellman (HJB) equation via some suitable *verification theorems*.

For the finite dimensional systems the literature presents quite precise and general results, see e.g. [22, 56]. If the system is infinite dimensional, for instance if it is driven by a stochastic partial differential (SPDEs) or a stochastic delay differential equation, the situation is more complex especially when the value function of the problem is not regular enough.

This paper contributes to the subject providing an efficient (infinite dimensional) stochastic calculus which fits the structure of a mild solution of a SPDE and allows to prove a verification theorem for a class of stochastic optimal control problems with non-regular value function, refining previous results.

The contributions of the paper can be ascribed to the following three "labels": infinite dimensional stochastic calculus, SPDEs and dynamic programming. In the next subsection we describe the state of the art, while in the following we will concentrate on the new results.

State of the art

Stochastic calculus.

Stochastic calculus via regularization for real processes was initiated in [48] and [49]. It is an efficient calculus for non-semimartingales whose related literature is surveyed in [52].

Given a locally bounded real process Y and a continuous real process X, the forward integral of Y with respect to X and the covariation are defined as follows. Suppose that for every $t \in [0,T]$, the limit $I(t)$ [resp. $C(t)$] in probability exists:

$$\begin{aligned} I(t) : &= \lim_{\epsilon\to 0^+} \int_0^t X(r) \left(\frac{Y(r+\epsilon) - Y(r)}{\epsilon} \right) \mathrm{d}r \\ C(t) : &= \lim_{\epsilon\to 0^+} \int_0^t \frac{(X(r+\epsilon) - X(r))\,(Y(r+\epsilon) - Y(r))}{\epsilon} dr. \end{aligned} \tag{1}$$

If the random function I (resp. C) admits a continuous version, that will be denoted with $\int_0^\cdot Y d^- X$ (resp. $[X,Y]$). It is the *forward integral* of Y with respect to X (resp. the *covariation* of X and Y). If X is a real continuous semimartingale and Y is a càdlàg process which is progressively measurable (resp. a semimartingale), the integral $\int_0^\cdot Y d^- X$ (resp. the covariation $[X,Y]$) is the same as the classical Itô's integral (resp. covariation).

The definition of $[X, Y]$ given above is slightly more general (weak) than in [52]. There the authors supposed that the convergence in (1) holds in the ucp (uniformly convergence in probability) topology. In this work we use the weak definition for the real case, i.e. when both X and Y are real, and the strong definition, via ucp convergence, when either X or Y is not one-dimensional. When $X = Y$ the two definitions are equivalent taking into account Lemma 2.1 of [52].

Real processes X for which $[X, X]$ exists are called *finite quadratic variation processes*. A rich class of finite quadratic variation processes is provided by Dirichlet processes. Let $(\mathscr{F}_t, t \in [0, T])$ be a fixed filtration. A real process X is said to be *Dirichlet* (or *Föllmer-Dirichlet*) if it is the sum of a local martingale M and a *zero quadratic variation process* A, i.e. such that $[A, A] = 0$. Those processes were defined by H. Föllmer [23] using limits of discrete sums. A significant generalization, due to [21, 33], is the notion of *weak Dirichlet process*, extended to the case of jump processes in [8].

Definition 1.1. *A real process $X\colon [s, T] \times \Omega \to \mathbb{R}$ is called weak Dirichlet process if it can be written as*

$$X = M + A, \tag{2}$$

where

(i) M is an local martingale,

(ii) A is a process such that $[A, N] = 0$ for every continuous local martingale N and $A(0) = 0$.

Obviously a semimartingale is a weak Dirichlet process. In Remark 3.5 of [33], appears the following important statement.

Proposition 1.2. *The decomposition described in Definition 1.1 is unique.*

Elements of calculus via regularization were extended to Banach space valued processes in a series of papers, see e.g. [16, 15, 14, 17].

We start introducing two classical notions of stochastic calculus in Banach spaces, which appear in [42] and [19]: the scalar and tensor quadratic variations. We propose here a regularization approach, even though, originally appeared in a discretization framework. The two monographs above use the term *real* instead of *scalar*; we have decided to change it to avoid confusion with the quadratic variation of real processes.

Definition 1.3. *Consider a separable Banach spaces B. We say that a process $\mathbb{X}\colon [s, T] \times \Omega \to B$ admits a* scalar quadratic variation *if, for any $t \in [s, T]$, the limit, for $\epsilon \searrow 0$ of*

$$[\mathbb{X}, \mathbb{X}]^{\epsilon, \mathbb{R}}(t) := \int_s^t \frac{|\mathbb{X}(r + \epsilon) - \mathbb{X}(r)|_B^2}{\epsilon} dr$$

exists in probability and it admits a continuous version. The limit process is called scalar quadratic variation of $\mathbb{X}$ and it is denoted with $[\mathbb{X}, \mathbb{X}]^{\mathbb{R}}$.

Proposition 1.4. *Let B be a separable Banach space. A continuous B-valued process with bounded variation admits a zero scalar quadratic variation. In particular, a process $\mathbb{X}$ of the type $\mathbb{X}(t) = \int_s^t b(s)ds$, where b is an B-valued strong measurable process has a zero scalar quadratic variation.*

Proof. The proof is very similar to the one related to the case when $B = \mathbb{R}$, which was the object of Proposition 1 7) b), see [52]. □

Remark 1.5. *The definition above is equivalent to the one contained in [14]. In fact, previous convergence in probability implies the ucp convergence, since the ϵ-approximation processes are increasing and so Lemma 3.1 in [51] can be applied.*

From [14] we borrow the following definition.

Definition 1.6. *Consider two separable Banach spaces B_1 and B_2. Suppose that either B_1 or B_2 is different from $\mathbb{R}$. Let $\mathbb{X}\colon [s,T]\times\Omega \to B_1$ and $\mathbb{Y}\colon [s,T]\times\Omega \to B_2$ be two strongly measurable processes. We say that $(\mathbb{X},\mathbb{Y})$ admits a* tensor covariation *if the limit, for $\epsilon \searrow 0$ of the $B_1\hat{\otimes}_\pi B_2$-valued processes*

$$[\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon} := \int_s^{\cdot} \frac{(\mathbb{X}(r+\epsilon)-\mathbb{X}(r))\otimes(\mathbb{Y}(r+\epsilon)-\mathbb{Y}(r))}{\epsilon} dr$$

exists ucp. The limit process is called tensor covariation of $(\mathbb{X},\mathbb{Y})$ and is denoted with $[\mathbb{X},\mathbb{Y}]^{\otimes}$. The tensor covariation $[\mathbb{X},\mathbb{X}]^{\otimes}$ is called tensor quadratic variation of $\mathbb{X}$ and denoted with $[\mathbb{X}]^{\otimes}$.

The concepts of scalar and tensor quadratic variation are however too strong for the applications: indeed several interesting examples of Banach (or even Hilbert) valued processes have no tensor quadratic variation. A Banach space valued example is the $C([-\tau,0])$-valued process $\mathbb{X}$ defined as the frame (or window) of a standard Brownian motion W: $\mathbb{X}(t)(x) := W(t+x),\ x \in [-\tau,0]$. It is neither a semimartingale, nor a process with scalar quadratic variation process, see considerations after Remarks 1.9 and Proposition 4.5 of [14]. A second (more general) example among $C([-\tau,0])$-valued processes is given by the windows of a real Dirichlet (resp. weak Dirichlet) process, see the consideration below Definition 1.8 in [17]. A third example that constitutes a main motivation for the present paper is given by mild solutions of classical stochastic PDEs of evolution type: they have no scalar quadratic variation even if driven by a one-dimensional Brownian motion.

The idea of Di Girolami and Russo was to introduce a suitable space χ continuously embedded into the dual of the projective tensor space $B_1\hat{\otimes}_\pi B_2$, called *Chi-subspace.* χ is a characteristics of their notion of quadratic variation, recalled in Section 4.1 when B_i are Hilbert spaces. When χ is the full space $(B_1\hat{\otimes}_\pi B_2)^*$ the χ-quadratic variation is called global quadratic variation. Following the approach of Di Girolami and Russo, see for instance Definition 3.4 of [17], we make use of a the notion of χ-covariation $[\mathbb{X},\mathbb{Y}]_\chi$ when $\chi \subseteq (H_1\hat{\otimes}_\pi H_2)^*$

for two processes $\mathbb{X}$ and $\mathbb{Y}$ with values respectively in separable Hilbert spaces H_1 and H_2. That notion is recalled in Definition 4.4.

[15] introduces a (real valued) forward integral, denoted by $\int_0^t {}_{B^*}\langle \mathbb{Y}(s), d^-\mathbb{X}(s)\rangle_B$, in the case when the integrator $\mathbb{X}$ takes values in a Banach space B and the integrand $\mathbb{Y}$ is B^*-valued. This appears as a natural generalization of the first line of (1). That notion is generalized in Definition 3.1 for operator-valued integrands; in that case, this produces a Hilbert valued forward integral.

The Itô formula for processes $\mathbb{X}$ having a χ-quadratic variation is given in Theorem 5.2 of [14]. Let $F : [0,T] \times B \to \mathbb{R}$ or class $C^{1,2}$ such that $\partial^2_{xx} F \in C([0,T] \times B; \chi)$. Then, for every $t \in [0,T]$,

$$F(t, \mathbb{X}(t)) = F(0, \mathbb{X}(0)) + \int_0^t \partial_s F(s, \mathbb{X}(s)) ds + \int_0^t {}_{B^*}\langle \partial_x F(s, \mathbb{X}(s)), d^-\mathbb{X}(s)\rangle_B + \frac{1}{2}\int_0^t {}_{\chi}\langle \partial^2_{xx} F(s, \mathbb{X}(s)), d\widetilde{[\mathbb{X}, \mathbb{X}]}_s\rangle_{\chi^*} \ a.s. \quad (3)$$

We will introduce the notation $\widetilde{[\mathbb{X}, \mathbb{X}]}_s$ in Section 3.

Stochastic partial differential equations (SPDEs).

A stochastic PDE is generally a deterministic PDE perturbed by a Gaussian noise involving an infinite dimensional Wiener process $\mathbb{W}_Q$ as a multiplicative factor and/or an additive term. Many stochastic partial differential equations can be rewritten as evolution equations in Hilbert spaces. Consequently many related optimal control problems can be reformulated in the abstract infinite dimensional formulation. Among the equations, we remind heat and parabolic equations (even with boundary controls or noise), wave, reaction-diffusion, and (with different formalism) Burgers, Navier-Stokes and Duncan-Mortensen-Zakai equations, even though not all of them can be represented in the abstract framework we use in this work. We remark that functional stochastic differential equations, as delay or neutral differential equations, can be included in the same formalism. First elements and references can be found for example in Part III of the book [11]. The abstract formulation of the stochastic evolution equation introduced in Section 5 is characterized by an abstract generator of a C_0-semigroup A and Lipschitz coefficients b and σ. It appears as

$$\begin{cases} \mathrm{d}\mathbb{X}(t) = (A\mathbb{X}(t) + b(t, \mathbb{X}(t)))\, \mathrm{d}t + \sigma(t, \mathbb{X}(t))\, \mathrm{d}\mathbb{W}_Q(t) \\ \mathbb{X}(0) = x, \end{cases} \quad (4)$$

where $\mathbb{W}$ is a Q-Wiener process with respect to some covariance operator Q.

This describes in fact a significant range of systems modeling phenomena arising in very different fields (physics, economics, physiology, population growth and migration...).

There are several different possible ways to define the solution of an SPDE: strong solutions (see e.g. [10] Section 6.1), variational solutions (see e.g. [46]), martingale solutions (see [44])...

We make use of the notion of *mild solution* (see [10] Chapter 7 or [28] Chapter 3) where the solution of the SPDE (4) is defined (using, formally, a "variations of parameters" arguments) as the solution of the integral equation

$$\mathbb{X}(t) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} b(r, \mathbb{X}(r))\,\mathrm{d}r + \int_s^t e^{(t-r)A}\sigma(r, \mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r).$$

This concept is widely used in the literature.

Optimal control: dynamic programming and verification theorems.
As in the study of finite-dimensional stochastic (and non-stochastic) optimal control problem, the dynamic programming approach connects the study of the the minimization problem with the analysis of the related Hamilton-Jacobi-Bellman (HJB) equation: given a solution of the HJB and a certain number of hypotheses, the optimal control can be found in feedback form (i.e. as a function of the state) through a so called *verification theorem*.

When the state equation of the optimal control problem is an infinite dimensional stochastic evolution equation, the related HJB is of second order and infinite dimensional. The verification theorem depends on the way the HJB is settled; the simplest procedure consists in considering the *regular case* where the solution is assumed to have all the regularity needed to give meaning to all the terms appearing in the HJB. For defining weaker solutions (the HJB does not admit a regular solution in general) in the literature there are several possibilities, the various approaches being classified as follows.

Strong solutions: in this approach, first introduced in [3], the solution is defined as a proper limit of solutions of regularized problems. Verification results in this framework are given for example in [30] and [31], see [5, 6] for the reaction-diffusion case.

Viscosity solutions: in this case the solution is defined using test functions that locally "touch" the candidate solution. The viscosity solution approach was first adapted to the second order Hamilton Jacobi equation in Hilbert space in [37, 38, 39] and then, for the "unbounded" case (i.e. including the unbounded operator A appearing in (71)) in [55]. As far as we know, differently from the finite-dimensional case there are no verification theorems available for the infinite-dimensional case.

L^2_μ *approach*: it was introduced in [2, 29], see also [7, 1]. In this case the value function is found in the space $L^2_\mu(H)$, where μ is an invariant measure for an associated uncontrolled process. The paper [29] contains as well an excellent literature survey.

Backward approach: it can be applied when the mild solution of the HJB can be represented using the solution of a forward-backward system and allows to find an optimal control in feedback form. It was introduced in [47] and developed in [26, 27, 24], see [12, 25] for other particular cases.

The method we use in the present work to prove the verification theorem does not belong to any of the previous categories even if we use a strong solution approach to define the solution of the HJB. In the sequel of this introduction, we will be more precise.

The contributions of the work

The novelty of the present paper arises at the three levels mentioned above: stochastic calculus, SPDEs and stochastic optimal control. Indeed stochastic optimal control for infinite dimensional problems is a motivation to complete the theory of calculus via regularizations.

The stochastic calculus part starts (Sections 3) with a natural extension (Definition 3.1) of the notion of forward integral in Hilbert spaces introduced in [15] and with the proof of its equivalence with the classical notion of integral when we integrate a predictable process w.r.t. a Q-Wiener process (Theorem 3.4) and w.r.t. a general local martingale (Theorem 3.6).

In Section 4, we extend the notion of Dirichlet process to infinite dimension. Let $\mathbb{X}$ be an H-valued stochastic process. According to the literature, $\mathbb{X}$ can be naturally considered to be an *infinite dimensional Dirichlet process* if it is the sum of a local martingale and a zero energy process. A *zero energy process* (with some light sophistications) $\mathbb{X}$ is a process such that the expectation of the quantity in Definition 1.3 converges to zero when ε goes to zero. This happens for instance in [13], over though that decomposition also appears in [40] Chapter VI Theorem 2.5, for processes $\mathbb{X}$ associated with an infinite-dimensional Dirichlet form.

Extending Föllmer's notion of Dirichlet process to infinite dimension, a process $\mathbb{X}$ taking values in a Hilbert space H, could be called *Dirichlet* if it is the sum of a local martingale $\mathbb{M}$ plus a process $\mathbb{A}$ having a zero scalar quadratic variation. However that natural notion is not suitable for an efficient stochastic calculus for SPDEs.

Similarly to the notion of χ-finite quadratic variation process we introduce the notion of χ-Dirichlet process as the sum of a local martingale $\mathbb{M}$ and a process $\mathbb{A}$ having a zero χ-quadratic variation.

A completely new notion in the present paper is the one of Hilbert valued *ν-weak Dirichlet process* which is again related to a Chi-subspace ν of the dual projective tensor produce $H\hat{\otimes}_\pi H_1$ where H_1 is another Hilbert space, see Definition 4.23. It is of course an extension of the notion of real-valued weak Dirichlet process, see Definition 1.1. We illustrate that notion in the simple case when $H_1 = \mathbb{R}$, $\nu = \nu_0 \hat{\otimes}_\pi \mathbb{R} \equiv \nu_0$ and ν_0 is a Banach space continuously embedded in H^*: a process $\mathbb{X}$ is called ν-weak Dirichlet process if it is the sum of a local martingale $\mathbb{M}$ and a process $\mathbb{A}$ such that $[\mathbb{A}, N]_{\nu_0} = 0$ for every real local martingale N. This happens e.g. under the following assumptions:

(i) $\frac{1}{\epsilon}\int_0^T |\mathbb{A}(r+\epsilon) - \mathbb{A}(r)|_{\nu_0^*} |N(r+\epsilon) - N(r)| \, \mathrm{d}r$ is bounded in probability for all the small ϵ.

(ii) For all $h \in \nu_0$, $\lim_{\epsilon \to 0+} \frac{1}{\epsilon} \int_0^t {}_{\nu_0}\langle \mathbb{A}(r+\epsilon) - \mathbb{A}(r), h \rangle_{\nu_0^*} (N(r+\epsilon) - N(r))\, \mathrm{d}r = 0$, $\forall t \in [0,T]$.

At the level of pure stochastic calculus, the most important result, is Theorem 4.32. It generalizes to the Hilbert values framework, Proposition 3.10 of [33] which states that given $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ and X weak Dirichlet process with finite quadratic variation then $Y(t) = f(t, X(t))$ is a real weak Dirichlet process. This result is a Fukushima decomposition result in the spirit of Doob-Meyer. It can also be seen as a substitution-tool of Itô's formula if f is not smooth. Besides Theorem 4.32, an interesting general Itô's formula in the application to mild solutions of SPDEs is Theorem 5.4. The stochastic calculus theory developed in Sections 3 and 4, allows to prove that a mild solution of an SPDE of type (4) is a χ-Dirichlet process and a ν-weak-Dirichlet process; this is done in Corollary 5.3. Observe that many notions and results in this work may be written in a Banach framework. We decided however to keep the Hilbert formulation in order to simplify the readability.

As far as stochastic control is concerned, the main issue is the verification result stated in Theorem 6.10. As we said, the method we used does not belong to any of the described families even if we define the solution of the HJB in line with a strong solution approach. Since the solution $v : [0,T] \times H \to \mathbb{R}$ of the HJB equation is only of class $C^{0,1}$ (with derivative in $C(H, D(A^*))$), we cannot apply a Itô formula of class $C^{1,2}$. The substitute of such a formula is given in Theorem 6.7, which is based on the uniqueness character of the decomposition of the real weak Dirichlet process $v(t, \mathbb{X}(t))$, where $\mathbb{X}$ is a solution of the state equation $\mathbb{X}$. The fact that $v(t, \mathbb{X}(t))$ is weak Dirichlet follows by Theorem 4.32 because $\mathbb{X}$ is a ν-weak Dirichlet process for some suitable space ν. This is the first work that employs this method in infinite dimensions. A similar approach was used to deal with the finite dimensional case in [32] but of course in the infinite-dimensional case the situation is much more complicated since the state equation is not a semimartingale and so it indeed requires the introduction of the concept of ν-weak Dirichlet process.

For the reasons listed below, Theorem 6.10 is more general than the results obtained with the the classical strong solutions approach, see e.g. [30, 31], and, in a context slightly different than ours, [5, 6].

(1) The state equation is more general.

 (a) In equation (66) the coefficient σ depends on time and on the state while in classical strong solutions literature, it is constant and equal to identity.

 (b) In classical strong solutions contributions, the coefficient b appearing in equation (66) is of the particular form $b(t, \mathbb{X}, a) = b_1(\mathbb{X}) + a$ so it "separates" the control and the state parts.

(2) We only need the Hamiltonian to be well-defined and continuous without any particular differentiability, as in the classical strong solutions literature.

(3) We use a milder definition of solution than in [30, 31]; indeed we work with a bigger set of approximating functions: in particular (a), our domain $D(\mathscr{L}_0)$, does not require the functions and their derivatives to be be uniformly bounded; (b), the convergence of the derivatives $\partial_x v_n \to \partial_x v$ in (6.6) is not necessary and it is replaced by the weaker condition (78).

However, we have to pay a price: we assume that the gradient of the solution of the HJB $\partial_x v$, is continuous from H to $D(A^*)$, instead of simply continuous from H to H.

In comparison to the strong solutions approach, the L^2_μ method used in [29] allows weaker assumptions on the data and enlarges the range of possible applications. However, the authors still require σ to be the identity, the Hamiltonian to be Lipschitz and the coefficient b to be in a "separated" form as in (1)(b) above. In the case treated by [2], the terms containing the control in the state variable is more general but the author assumes that A and Q have the same eigenvectors. So, in both cases, the assumptions on the state equation are for several aspects more demanding than ours.

The backward approach used e.g. in [26, 27, 24] can treat degenerate cases in which the transition semigroup has no smoothing properties. Still, in the verification results proved in this context, the Hamiltonian has to be differentiable, the dependence on the control in the state equation is assumed to be linear and its coefficient needs to have a precise relation with $\sigma(t, \mathbb{X}(t))$. All those hypotheses are stronger than ours.

One important feature of Theorem 6.10 is that that we do not need to assume any hypothesis to ensure the integrability of the target. This is due to the fact that we apply the expectation operator only at the last moment.

The scheme of the work is the following. After some preliminaries in Section 2, in Section 3 we introduce the definition of forward integral in Hilbert spaces and we discuss the relation with the Da Prato-Zabczyk integral. Section 4, devoted to stochastic calculus, is the core of the paper: we introduce the concepts of χ-Dirichlet processes, ν-weak-Dirichlet processes and we study their general properties. In Section 5, the developed theory is applied to the case of mild solutions of SPDEs, while Section 6 contains the application to stochastic optimal control problems in Hilbert spaces.

2 Preliminaries and notations

Consider two separable Hilbert spaces, H and U. Denote $|\cdot|$ and $\langle \cdot, \cdot \rangle$ [resp. $|\cdot|_U$ and $\langle \cdot, \cdot \rangle_U$] the norm and the inner product on H [resp. U]. Denote by $\mathcal{L}(U, H)$ the set of all linear bounded operators from U to H. $\mathcal{L}(U, H)$, equipped with the operator norm $\|T\|_{\mathcal{L}(U,H)} := \sup_{u \in U} \frac{|Tu|_H}{|u|_U}$, is a Banach space. If $H = U$ we denote $\mathcal{L}(U, H)$ simply by $\mathcal{L}(U)$. Recall that, if U and H are both infinite dimensional, then $\mathcal{L}(U, H)$ is not a separable space.

Fix $T > 0$ and $s \in [0, T)$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and $\{\mathscr{F}_t^s\}_{t \geq s}$ a filtration on $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying the usual conditions. Each time we use expressions as "adapted", "predictable" etc... we will always mean "with respect to the filtration $\{\mathscr{F}_t^s\}_{t \geq s}$". Denote with $\mathscr{P}$ the predictable σ-field on $[s, T] \times \Omega$.

Given a subset $\tilde{\Omega} \in \mathscr{F}$ we denote with $I_{\tilde{\Omega}} \colon \Omega \to \{0, 1\}$ the characteristic function of the set $\tilde{\Omega}$, i.e. $I_{\tilde{\Omega}}(\omega) = 1$ if and only if $\omega \in \tilde{\Omega}$.

Notation 2.1. *The* blackboard bold *letters* $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{M}$... *are used for Hilbert-valued (or Banach-valued) processes, while notations* X *(or* Y, M...*) are reserved for real valued processes.*

The dependence of a process on the variable $\omega \in \Omega$ is emphasized only if needed by the context. When we say that a process is continuous (resp. left continuous, right continuous, càdlàg, càglàd ...) we mean that almost all paths are continuous (resp. left continuous, right continuous, càdlàg, càglàd...).

Let B be a Banach space. The Borel σ-field on B is denoted with $\mathscr{B}(B)$. By default we assume that all the processes $\mathbb{X} \colon [s, T] \times \Omega \to B$ are measurable functions with respect to the product σ-algebra $\mathscr{B}([s, T]) \times \mathscr{F}$ with values in $(B, \mathscr{B}(B))$. Let $\mathscr{G}$ be a sub-σ-field of $\mathscr{B}([s, T]) \otimes \mathscr{F}$. A process $\mathbb{X} : ([s, T] \times \Omega, \mathscr{G}) \to B$ is said to be strongly (Bochner) measurable if it is the limit of $\mathscr{G}$-measurable countable-valued functions. If B is separable, $\mathbb{X}$ measurable and càdlàg (or càglàd) then $\mathbb{X}$ is strongly measurable. If B is finite dimensional then any measurable process $\mathbb{X}$ is strongly measurable. We always suppose that all the processes $\mathbb{X}$ measurable on $(\Omega \times [s, T], \mathcal{P})$ we consider are strongly measurable. In that case we will also say that those processes are *predictable*.

Notation 2.2. *We always assume the following convention: when needed all the Banach space processes (or functions) indexed by* $[s, T]$ *are extended setting* $\mathbb{X}(t) = \mathbb{X}(s)$ *for* $t \leq s$ *and* $\mathbb{X}(t) = \mathbb{X}(T)$ *for* $t \geq T$.

As already emphasized in the introduction many of the concepts appearing in this paper could be generalized from the Hilbert space to the Banach space environment without major changes. In any case, to make the arguments more transparent, we distinguish between H and its dual H^* and use the following convention.

Notation 2.3. *If* H *is a Hilbert space with every element* $h \in H$ *we associate* $h^* \in H^*$ *through Riesz Theorem.*

2.1 Reasonable norms on tensor products

Consider two real Banach spaces B_1 and B_2. Denote, for $i = 1, 2$, with $| \cdot |_i$ the norm on B_i. $B_1 \otimes B_2$ stands for the *algebraic tensor product* i.e. the set of the elements of the form $\sum_{i=1}^{n} x_i \otimes y_i$ where x_i and y_i are respectively elements of B_1 and B_2. On $B_1 \otimes B_2$ we identify all the expressions we need in order to ensure that the product $\otimes \colon B_1 \times B_2 \to B_1 \otimes B_2$ is bilinear.

On $B_1 \otimes B_2$ we introduce the projective norm π defined, for all $u \in B_1 \otimes B_2$, as

$$\pi(u) := \inf \left\{ \sum_{i=1}^{n} |x_i|_1 |y_i|_2 \ : \ u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

The *projective tensor product of* B_1 *and* B_2, $B_1 \hat{\otimes}_\pi B_2$, is the Banach space obtained as completion of $B_1 \otimes B_2$ for the norm π, see [53] Section 2.1, or [15] for further details.

For $u \in B_1 \otimes B_2$ of the form $u = \sum_{i=1}^n x_i \otimes y_i$ we define

$$\varepsilon(u) := \sup \left\{ \left| \sum_{i=1}^{n} \Phi(x_i) \Psi(y_i) \right| \ : \ \Phi \in B_1^*,\ \Psi \in B_2^*,\ |\Phi|_{B_1^*} = |\Psi|_{B_2^*} = 1 \right\}$$

and denote with $B_1 \hat{\otimes}_\epsilon B_2$ the completion of $B_1 \otimes B_2$ for such a norm: it is the *injective tensor product of* B_1 *and* B_2. We remind that $\epsilon(u)$ does not depend on the representation of u and that, for any $u \in B_1 \otimes B_2$, $\varepsilon(u) \leq \pi(u)$.

A norm α on $B_1 \otimes B_2$ is said to be *reasonable* if for any $u \in B_1 \otimes B_2$,

$$\varepsilon(u) \leq \alpha(u) \leq \pi(u). \tag{5}$$

We denote with $B_1 \hat{\otimes}_\alpha B_2$ the completion of $B_1 \otimes B_2$ w.r.t. the norm α.

For any reasonable norm α on $B_1 \otimes B_2$, for any $x \in B_1$ and $y \in B_2$ one has $\alpha(x \otimes y) = |x|_1 |y|_2$. See [53] Chapter 6.1 for details.

Lemma 2.4. *Let* B_1 *and* B_2 *be two reflexive Banach spaces and* α *a reasonable norm on* $B_1 \otimes B_2$. *We denote* $B_1 \hat{\otimes}_\alpha B_2$ *by* B. *Choose* $a^* \in B_1^*$ *and* $b^* \in B_2^*$. *One can associate to* $a^* \otimes b^*$ *the elements* $i(a^* \otimes b^*)$ *of* B^* *acting as follows on a generic element* $u = \sum_{i=1}^n x_i \otimes y_i \in B_1 \otimes B_2$*:*

$$\langle i(a^* \otimes b^*), u \rangle = \sum_{i}^{n} \langle a^*, x_i \rangle \langle b^*, y_i \rangle .$$

Then $i(a^* \otimes b^*)$ *extends by continuity to the whole* B *and its norm in* B^* *equals* $|a^*|_{B_1^*} |b^*|_{B_2^*}$.

Proof. We first prove the $\leq$ inequality. It follows directly by the definition of injective tensor norm ε that

$$\langle i(a^* \otimes b^*), u \rangle \leq \varepsilon(u) |a^*|_{B_1^*} |b^*|_{B_2^*}.$$

By (5) ε is the less rough of the all reasonable tensorial norms so $\langle i(a^* \otimes b^*), u \rangle \leq \alpha(u) |a^*|_{B_1^*} |b^*|_{B_2^*}$ and the claim is proved.

Concerning the converse inequality, we have

$$|a^*|_{B_1^*} = \sup_{|\phi|_{B_1} = 1} {}_{B_1^*} \langle a^*, \phi \rangle_{B_1}$$

and similarly for b^*. So, chosen $\delta > 0$, there exist $\phi_1 \in B_1$ and $\phi_2 \in B_2$ with $|\phi_1|_{B_1} = |\phi_2|_{B_2} = 1$ and

$$|a^*|_{B_1^*} \leq \delta + {}_{B_1^*}\langle a^*, \phi_1\rangle_{B_1}, \qquad |b^*|_{B_2^*} \leq \delta + {}_{B_2^*}\langle b^*, \phi_2\rangle_{B_2}.$$

We set $u := \phi_1 \otimes \phi_2$. We obtain

$$\begin{aligned} |i(a^* \otimes b^*)|_{B^*} \geq \frac{{}_{B^*}\langle i(a^* \otimes b^*), u\rangle_B}{|u|_B} &= \frac{{}_{B^*}\langle i(a^* \otimes b^*), u\rangle_B}{|\phi_1|_{B_1}|\phi_2|_{B_2}} \\ &= {}_{B_1^*}\langle a^*, \phi_1\rangle_{B_1} \; {}_{B_2^*}\langle b^*, \phi_2\rangle_{B_2} \\ &\geq (|a^*|_{B_1^*} - \delta)(|b^*|_{B_2^*} - \delta). \end{aligned} \tag{6}$$

Since $\delta > 0$ is arbitrarily small we finally obtain

$$|i(a^* \otimes b^*)|_{B^*} \geq |a^*|_{B_1^*}|b^*|_{B_2^*}.$$

This gives the second inequality and concludes the proof. □

Notation 2.5. *When $B_1 = B_2 = B$ and $x \in B$ we denote with $x \otimes^2$ the element $x \otimes x \in B \otimes B$.*

The dual of the projective tensor product $B_1 \hat{\otimes}_\pi B_2$, denoted by $(B_1 \hat{\otimes}_\pi B_2)^*$, can be identified isomorphically with the linear space of bounded bilinear forms on $B_1 \times B_2$ denoted with $\mathcal{B}(B_1, B_2)$. If $u \in (B_1 \hat{\otimes}_\pi B_2)^*$ and ψ_u is the associated form in $\mathcal{B}(B_1, B_2)$, we have

$$|u|_{(B_1 \hat{\otimes}_\pi B_2)^*} = \sup_{|a|_1 \leq 1, |b|_1 \leq 1} |\psi_u(a, b)|.$$

See for this [53] Theorem 2.9 Section 2.2, page 22 and also the discussion after the proof of the theorem, page 23.

If $B_1 = H_1$ and $B_2 = H_2$ where H_1, H_2 are separable Hilbert spaces, we denote by $\mathcal{L}_2(H_1, H_2)$ (resp. $\mathcal{L}(H_1, H_2)$) the Hilbert space of the Hilbert-Schmidt (resp. the Banach space of linear continuous) operators, from H_1 to H_2. For more details about those operators, see [10] Appendix C.

Suppose now that $H = H_1 = H_2$; we denote with by $\mathcal{L}_1(H, H)$ the Banach space of nuclear operators, see [10] Appendix C. The product of a map $T \in \mathcal{L}(H, H)$ with a map $B \in \mathcal{L}_1(H, H)$ belongs to $\mathcal{L}_1(H, H)$, see Corollary C.2 Appendix C in [10]. If $T \in \mathcal{L}_1(H, H)$ we denote $\mathrm{Tr}T$ the Trace of T: more particularly, if (e_n) is an orthonormal basis of H we have $\mathrm{Tr}T = \sum_n \langle Te_n, e_n\rangle_H$ and that quantity does not depend on the choice of the orthonormal basis, see Proposition C.1 Appendix C in [10].
Every element $u \in H \hat{\otimes}_\pi H$ is isometrically associated with an element T_u in the space of nuclear operators $\mathcal{L}_1(H, H)$, defined, for u of the form $\sum_{i=1}^\infty a_n \otimes b_n$, as follows:

$$T_u(x) := \sum_{i=1}^\infty \langle x, a_n\rangle\, b_n,$$

see for instance [53] Corollary 4.8 Section 4.1 page 76.

Since T_u is nuclear, in particular (see Appendix C of [10]), there exist a sequence of real number λ_n and an orthonormal basis h_n of H such that T_u can be written as

$$T_u(x) = \sum_{n=1}^{+\infty} \lambda_n \langle h_n, x \rangle h_n \qquad \text{for all } x \in H; \tag{7}$$

moreover u can be written as

$$u = \sum_{i=1}^{+\infty} \lambda_n h_n \otimes h_n. \tag{8}$$

To each element ψ of $(H\hat{\otimes}_\pi H)^*$ we associate a bilinear continuous operator B_ψ and a linear continuous operator $L_\psi : H \to H$ (see [53] page 24, the discussion before Proposition 2.11 Section 2.2) such that

$$\langle L_\psi(x), y \rangle = B_\psi(x, y) = \psi(x \otimes y) \qquad \text{for all } x, y \in H. \tag{9}$$

Proposition 2.6. *Let $u \in H\hat{\otimes}_\pi H$ and $\psi \in (H\hat{\otimes}_\pi H)^*$ with associated maps $T_u \in \mathcal{L}(H, H), B_\psi \in \mathcal{L}_1(H, H)$. Then*

$${}_{(H\hat{\otimes}_\pi H)^*}\langle \psi, u \rangle_{H\hat{\otimes}_\pi H} = \mathrm{Tr}\left(T_u B_\psi\right).$$

Proof. The claim follows from what we have recalled above. Indeed, using (8) and (9) we have

$${}_{(H\hat{\otimes}_\pi H)^*}\langle \psi, u \rangle_{H\hat{\otimes}_\pi H} = \psi(u) = \psi\left(\sum_{i=1}^{+\infty} \lambda_n h_n \otimes h_n \right) = \sum_{n=1}^{+\infty} \langle L_\psi(\lambda_n h_n), h_n \rangle$$

and last expression is exactly $\mathrm{Tr}\left(T_u B_\psi\right)$ when we compute it using the basis h_n. □

3 Stochastic integrals

We adopt the notations introduced in Section 2.

Definition 3.1. *Let $\mathbb{X} \colon \Omega \times [s, T] \to \mathcal{L}(U, H)$ and $\mathbb{Y} \colon \Omega \times [s, T] \to U$ be two stochastic processes. Assume that $\mathbb{Y}$ is continuous and $\mathbb{X}$ is strongly measurable.*

If for almost every $t \in [s, T]$ the following limit exists in probability

$$\int_s^t \mathbb{X}(r)\, \mathrm{d}^-\mathbb{Y}(r) := \lim_{\epsilon \to 0^+} \int_s^t \mathbb{X}(r) \left(\frac{\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)}{\epsilon} \right) \mathrm{d}r$$

and the process $t \mapsto \int_s^t \mathbb{X}(r)\, \mathrm{d}^-\mathbb{Y}(r)$ admits a continuous version, we say that $\mathbb{X}$ is forward integrable with respect to $\mathbb{Y}$. That version of $\int_s^\cdot \mathbb{X}(r)\, \mathrm{d}^-\mathbb{Y}(r)$ is called forward integral of $\mathbb{X}$ with respect to $\mathbb{Y}$.

Remark 3.2. *The definition above is a natural generalization of that given in [15] Definition 3.4; there the forward integral is a real valued process.*

3.1 The case of Q-Wiener process

Consider a positive and self-adjoint operator $Q \in \mathcal{L}(U, U)$. Even if not necessary, we assume Q to be injective; this allows to avoid formal complications. However Theorem 3.4 below holds without this restriction.

Define $U_0 := Q^{1/2}(U)$: U_0 is a Hilbert space for the inner product $\langle x, y\rangle_{U_0} := \left\langle Q^{-1/2}x, Q^{-1/2}y\right\rangle_U$ and, clearly $Q^{1/2}\colon U \to U_0$ is an isometry, see e.g. [10] Section 4.3 for details. We remind that, given $A \in \mathcal{L}_2(U_0, H)$, we have $\|A\|^2_{\mathcal{L}_2(U_0,H)} = Tr\left(AQ^{1/2}(AQ^{1/2})^*\right)$.

Let $\mathbb{W}_Q = \{\mathbb{W}_Q(t) : s \le t \le T\}$ be an U-valued $\mathscr{F}^s_t$-Q-Wiener process with $\mathbb{W}_Q(s) = 0$, $\mathbb{P}$ a.s. The definition and properties of Q-Wiener processes are presented for instance in [28] Chapter 2.1.

If $\mathbb{Y}$ is predictable with some integrability properties, $\int_s^t \mathbb{Y}(r)\,\mathrm{d}\mathbb{W}(r)$ denotes the classical Itô-type integral with respect to $\mathbb{W}$, defined e.g. in [10].

In the sequel it is shown to be equal to the forward integral so that the forward integral happens to be an extension of the Itô integral. In the next subsection we introduce the Itô integral $\int_s^t \mathbb{Y}(r)\,\mathrm{d}\mathbb{M}(r)$ with respect to a local martingale $\mathbb{M}$. If $H = \mathbb{R}$, previous integral will also denoted $\int_s^t \langle \mathbb{Y}(r),\, \mathrm{d}\mathbb{M}(r)\rangle_U$.

Definition 3.3. *We say that a sequence of $\mathscr{F}^s_t$-stopping times $\tau_n\colon \Omega \to [0, +\infty]$ is* suitable *if, called*

$$\Omega_n := \{\omega \in \Omega \,:\, \tau_n(\omega) > T\},$$

we have

$$\Omega_n \subseteq \Omega_{n+1} \qquad a.s.\ for\ all\ n$$

and

$$\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega \qquad a.s.$$

In the sequel, we will use the terminology "stopping times" without mention to the underlying filtration $\mathscr{F}^s_t$.

Theorem 3.4. *Let $\mathbb{X}\colon [s, T] \times \Omega \to \mathcal{L}_2(U_0, H)$ be a predictable process satisfying*

$$\int_s^T \|\mathbb{X}(r)\|^2_{\mathcal{L}_2(U_0,H)}\,\mathrm{d}r < +\infty \qquad a.s. \tag{10}$$

Then the the forward integral

$$\int_s^{\cdot} \mathbb{X}(r)\,\mathrm{d}^-\mathbb{W}_Q(r).$$

exists and coincides with the classical Itô integral (defined for example in [10] Chapter 4)

$$\int_s^{\cdot} \mathbb{X}(r)\,\mathrm{d}\mathbb{W}_Q(r).$$

Proof. We fix $t \in [s, T]$. In the proof we follow the arguments related to the finite-dimensional case, see Theorem 2 of [52].

As a first step we consider $\mathbb{X}$ with

$$\mathbb{E}\left(\int_s^T \|\mathbb{X}(\eta)\|^2_{\mathcal{L}_2(U_0,H)}\,\mathrm{d}\eta\right) < +\infty. \tag{11}$$

This fact ensures that the hypotheses in the stochastic Fubini Theorem 4.18 of [10] are satisfied.

We have

$$\int_s^t \mathbb{X}(r)\frac{\mathbb{W}_Q(r+\epsilon) - \mathbb{W}_Q(r)}{\epsilon}\,\mathrm{d}r = \int_s^t \mathbb{X}(r)\frac{1}{\epsilon}\left(\int_r^{r+\epsilon} \mathrm{d}\mathbb{W}_Q(\theta)\right)\mathrm{d}r;$$

applying the stochastic Fubini Theorem, the expression above is equal to

$$\int_s^t \left(\frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} \mathbb{X}(r)\,\mathrm{d}r\right)\mathrm{d}\mathbb{W}_Q(\theta) + R_\epsilon(t)$$

where $R_\epsilon(t)$ is a boundary term that converges to 0 ucp, so that we can ignore it. We can apply now the maximal inequality stated in [54], Theorem 1: there exists a universal constant $C > 0$ such that, for every $f \in L^2([s,t];\mathbb{R})$,

$$\int_s^t \left(\sup_{\epsilon\in(0,1]}\left|\frac{1}{\epsilon}\int_{(r-\epsilon)}^{r} f(\xi)\,\mathrm{d}\xi\right|\right)^2 \mathrm{d}r \leq C\int_s^t f^2(r)\,\mathrm{d}r. \tag{12}$$

According to the vector valued version of the Lebesgue differentiation Theorem (see Theorem II.2.9 in [18]), the following quantity

$$\frac{1}{\epsilon}\int_{(r-\epsilon)}^{r} \mathbb{X}(\xi)\,\mathrm{d}\xi$$

converges $\mathrm{d}\mathbb{P}\otimes\mathrm{d}r$ a.e. to $\mathbb{X}(r)$. Consequently (12) and dominated convergence theorem imply

$$\mathbb{E}\int_s^t \left\|\left(\frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} \mathbb{X}(r)\,\mathrm{d}r\right) - \mathbb{X}(\theta)\right\|^2_{\mathcal{L}_2(U_0,H)} \mathrm{d}\theta \xrightarrow{\epsilon\to 0} 0.$$

Finally, the convergence

$$J_\epsilon := \int_s^t \left(\frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} \mathbb{X}(r)\,\mathrm{d}r\right)\mathrm{d}\mathbb{W}_Q(\theta) \xrightarrow[L^2(\Omega,H)]{\epsilon\to 0} J := \int_s^t \mathbb{X}(\theta)\,\mathrm{d}\mathbb{W}_Q(\theta), \tag{13}$$

justifies the claim.

If (11) is not satisfied we proceed by localization. Denote again with J_ϵ and J the processes defined in (13). Call τ_n the stopping times given by

$$\tau_n := \inf\left\{t \in [s,T] \;:\; \int_s^t \|\mathbb{X}(r)\|^2_{\mathcal{L}_2(U_0,H)}\,\mathrm{d}r \geq n\right\}$$

(and $+\infty$ if the set is void) and call Ω_n the sets

$$\Omega_n := \{\omega \in \Omega \ : \ \tau_n(\omega) > T\}.$$

It is easy to see that the stopping times τ_n are suitable in the sense of Definition 3.3.

For each fixed n, the process $I_{[s,\tau_n]}\mathbb{X}$ verifies (11) and from the first step

$$J_n^\epsilon := \int_s^t \left(\frac{1}{\epsilon} \int_{\theta-\epsilon}^{\theta} I_{[s,\tau_n]}(r)\mathbb{X}(r)\,\mathrm{d}r \right) \mathrm{d}\mathbb{W}(\theta) \xrightarrow[L^2(\Omega,H)]{\epsilon\to 0} J_n := \int_s^t I_{[s,\tau_n]}(\theta)\mathbb{X}(\theta)\,\mathrm{d}\mathbb{W}(\theta). \quad (14)$$

so

$$I_{\Omega_n} J_\epsilon = I_{\Omega_n} J_n^\epsilon \xrightarrow[L^2(\Omega,H)]{\epsilon\to 0} I_{\Omega_n} J_n = I_{\Omega_n} J.$$

Consequently, for all n, $I_{\Omega_n} J_\epsilon$ converges to $I_{\Omega_n} J$ in probability and finally J_ϵ converges to J in probability as well. This fact concludes the proof. □

3.2 The semimartingale case

Consider now the case when the integrator is a more general local martingale. Let H and U be two separable Hilbert spaces and we adopt the notations introduced in Section 2.

An U-valued process $\mathbb{M}\colon [s,T]\times\Omega \to U$ is called martingale if, for all $t \in [s,T]$, $\mathbb{M}$ is $\mathscr{F}_t^s$-adapted with $\mathbb{E}\left[|\mathbb{M}(t)|\right] < +\infty$ and $\mathbb{E}\left[\mathbb{M}(t_2)|\mathscr{F}_{t_1}^s\right] = \mathbb{M}(t_1)$ for all $s \le t_1 \le t_2 \le T$. The concept of (conditional) expectation for B-valued processes, for a separable Banach space B, are recalled for instance in [10] Section 1.3. All the considered martingales will be continuous.

We denote with $\mathcal{M}^2(s,T;H)$ the linear space of square integrable martingales equipped with the norm

$$|\mathbb{M}|_{\mathcal{M}^2(s,T;U)} := \left(\mathbb{E} \sup_{t\in[s,T]} |\mathbb{M}(t)|^2 \right)^{1/2}.$$

It is a Banach space as stated in [10]Proposition 3.9.

An U-valued process $\mathbb{M}\colon [s,T]\times\Omega \to U$ is called local martingale if there exists a non-decreasing sequence of stopping times $\tau_n\colon \Omega \to [s,T]\cup\{+\infty\}$ such that $\mathbb{M}(t\wedge\tau_n)$ for $t\in[s,T]$ is a martingale and $\mathbb{P}\left[\lim_{n\to\infty}\tau_n = +\infty\right] = 1$. All the considered local martingales are continuous.

Given a continuous local martingale $\mathbb{M}\colon [s,T]\times\Omega\to U$, the process $|\mathbb{M}|^2$ is a real local sub-martingale, see Theorem 2.11 in [35]. The increasing predictable process, vanishing at zero, appearing in the Doob-Meyer decomposition of $|\mathbb{M}|^2$ will be denoted by $[\mathbb{M}]^{\mathbb{R},cl}(t), t\in[s,T]$. It is of course uniquely determined and continuous.

We remind some properties of the Itô stochastic integral with respect to a local martingale $\mathbb{M}$. Call $\mathcal{I}_{\mathbb{M}}(s,T;H)$ the set of the processes $\mathbb{X}\colon [s,T]\times\Omega\to\mathcal{L}(U;H)$ that are strongly measurable from $([s,T]\times\Omega,\mathscr{P})$ to $\mathcal{L}(U;H)$ and such that

$$|\mathbb{X}|_{\mathcal{I}_{\mathbb{M}}(s,T;H)} := \left(\mathbb{E}\int_s^T \|\mathbb{X}(r)\|^2_{\mathcal{L}(U;H)}\,\mathrm{d}[\mathbb{M}]^{\mathbb{R},cl}(r)\right)^{1/2} < +\infty.$$

$\mathcal{I}_{\mathbb{M}}(s,T;H)$ endowed with the norm $|\cdot|_{\mathcal{I}_{\mathbb{M}}(s,T;H)}$ is a Banach space.

The linear map

$$\begin{cases} I\colon \mathcal{I}_{\mathbb{M}}(s,T;H)\to\mathcal{M}^2(s,T;H)\\ \mathbb{X}\mapsto\int_s^T\mathbb{X}(r)\,\mathrm{d}\mathbb{M}(r)\end{cases}$$

is a contraction, see e.g. [41] Section 20.4 (above Theorem 20.5). As illustrated in [35] Section 2.2 (above Theorem 2.14), the stochastic integral w.r.t. $\mathbb{M}$ extends to the integrands $\mathbb{X}$ which are strongly measurable from $([s,T]\times\Omega,\mathscr{P})$ to $\mathcal{L}(U;H)$ and such that

$$\int_s^T \|\mathbb{X}(r)\|^2_{\mathcal{L}(U;H)}\,\mathrm{d}[\mathbb{M}]^{\mathbb{R},cl}(r) < +\infty \qquad a.s. \tag{15}$$

We denote by $\mathcal{J}^2(s,T;U,H)$ such a family of integrands w.r.t. $\mathbb{M}$. Actually, the integral can be even defined for a wider class of integrands, see e.g. [42]. For instance, according to Section 4.7 of [10], let

$$\mathbb{M}_t = \int_s^t \mathbb{A}(r)\,\mathrm{d}\mathbb{W}_Q(r),\, t\in[s,T], \tag{16}$$

and $\mathbb{A}$ be an $\mathcal{L}(U,H)$-valued predictable process such that $\int_s^T \mathrm{Tr}[\mathbb{A}(r)Q^{1/2}(\mathbb{A}(r)Q^{1/2})^*]\,\mathrm{d}r<\infty$ a.s.

If $\mathbb{X}$ is an H-valued (or H^*-valued using Riesz identification) predictable process such that

$$\int_s^T \langle\mathbb{X}(r),\mathbb{A}(r)Q^{1/2}(\mathbb{A}Q^{1/2})^*\mathbb{X}(r)\rangle_H\,\mathrm{d}r<\infty,\ \text{a.s.}, \tag{17}$$

then, as argued in Section 4.7 of [10],

$$N(t) = \int_s^t \langle\mathbb{X}(r),\,\mathrm{d}\mathbb{M}(r)\rangle_H,\, t\in[s,T], \tag{18}$$

is well-defined and it equals $N(t)=\int_s^t\langle\mathbb{X}(r),\mathbb{A}(r)\,\mathrm{d}\mathbb{W}_Q(r)\rangle_H$ for $t\in[s,T]$.

The stochastic integral with respect to local martingales, fulfills some significant properties recalled in the following proposition.

Proposition 3.5. *Let $\mathbb{M}$ be a continuous $(\mathcal{F}_t^s)$-local martingale, $\mathbb{X}$ verifying (15). We set $\mathbb{N}(t)=\int_s^t\mathbb{X}(r)d\mathbb{M}(r)$.*

(i) $\mathbb{N}$ *is an* $(\mathcal{F}_t^s)$*-local martingale.*

(ii) Let $\mathbb{K}$ *be an* $(\mathcal{F}_t^s)$*- predictable process such that* $\mathbb{K}\mathbb{X}$ *fulfills* (15)*. Then the Itô-type stochastic integral* $\int_s^t \mathbb{K} d\mathbb{N}$ *for* $t \in [s,T]$ *is well-defined and it equals* $\int_s^t \mathbb{K}\mathbb{X} d\mathbb{M}$*.*

(iii) If $\mathbb{M}$ *is a* Q*-Wiener process* W_Q*, then, whenever* $\mathbb{X}$ *is such that*

$$\int_s^T Tr\left[\left(\mathbb{X}(s)Q^{1/2}\right)\left(\mathbb{X}(s)Q^{1/2}\right)^*\right] \mathrm{d}s < +\infty \text{ a.s.} \tag{19}$$

then $\mathbb{N}(t) = \int_s^t \mathbb{X}(s) d\mathbb{W}_Q(s)$ *is local martingale and*

$$[N]^{\mathbb{R},cl}(t) = \int_s^t \left(\mathbb{X}(s)Q^{1/2}\right)\left(\mathbb{X}(s)Q^{1/2}\right)^* \mathrm{d}s.$$

(iv) If in item (iii), the expectation of the quantity (19) *is finite, then* $\mathbb{N}(t) = \int_s^t \mathbb{X}(s) d\mathbb{W}_Q(s)$ *is square integrable continuous martingale.*

(v) If $\mathbb{M}$ *is defined as in* (16) *and* $\mathbb{X}$ *fulfills* (17)*, then* $\mathbb{M}$ *is a real local martingale. If moreover, the expectation of* (17) *is finite, then* N*, defined in (18), is a square integrable martingale.*

Proof. For (i) see [35] Theorem 2.14 page 14-15. For (ii) see [42], proof of Proposition 2.2 Section 2.4. (iii) and (iv) are contained in [10] Theorem 4.12 Section 4.4. (v) is a consequence of (iii) and (iv) and of the considerations before the statement of Proposition 3.5. □

Theorem 3.6. *Let us consider a continuous local martingale* $\mathbb{M}\colon [s,T] \times \Omega \to U$ *and a càglàd process predictable* $\mathcal{L}(U,H)$*-valued process. Then the forward integral*

$$\int_s^\cdot \mathbb{X}(r)\, \mathrm{d}^-\mathbb{M}(r).$$

defined in Definition 3.1 exists and coincides with the Itô integral

$$\int_s^\cdot \mathbb{X}(r)\, \mathrm{d}\mathbb{M}(r).$$

Remark 3.7. *Any càglàd adapted process is a.s. bounded and therefore it belongs to* $J^2(s,T;U,H)$*.*

Proof of Theorem 3.6. Replacing $\mathbb{X}$ with $(\mathbb{X} - \mathbb{X}(s))$ there is not restriction to generality to suppose that $\mathbb{X}(s) = 0$.

The proof follows partially the lines of Theorem 3.4. Similarly we first localize the problem using the stopping time

$$\tau_n := \inf\left\{t \in [s,T] \ : \ \|\mathbb{X}(t)\|^2_{\mathcal{L}(U,H)} + [\mathbb{M}]^{\mathbb{R},cl}(t) \geq n\right\}$$

(and $+\infty$ if the set is void); the localized process belongs to $\mathcal{I}_{\mathbb{M}}(s,T;H)$ and satisfies the hypotheses of the stochastic Fubini theorem in the form given in [36]. Since the integral is a contraction from $\mathcal{I}_{\mathbb{M}}(s,T;H)$ to $\mathcal{M}^2(s,T;H)$, it only remains to show that

$$\mathbb{E}\int_s^t \left\| \left(\frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} \mathbb{X}(r)\,\mathrm{d}r \right) - \mathbb{X}(\xi) \right\|^2_{\mathcal{L}(U,H)} \mathrm{d}[\mathbb{M}]^{\mathbb{R},cl}(\xi) \xrightarrow{\epsilon\to 0} 0 \tag{20}$$

when $\mathbb{X}$ belongs to $\mathcal{I}_{\mathbb{M}}(s,T;H)$. (20) holds, taking into account Lebesgue dominated theorem, because $\mathbb{X}$ is left continuous and both $\mathbb{X}$ and $[M]^{\mathbb{R},cl}$ are bounded. $\square$

An easier but still important statement concerns the integration with respect bounded variation processes.

Proposition 3.8. *Let us consider a continuous bounded variation process $\mathbb{V}\colon [s,T]\times\Omega \to U$ and let $\mathbb{X}$ be a càglàd measurable process $[s,T]\times\Omega \to \mathcal{L}(U,H)$. Then the forward integral*

$$\int_s^{\cdot} \mathbb{X}(r)\,\mathrm{d}^-\mathbb{V}(r).$$

defined in Definition 3.1 exists and coincides with the Lebesgue-Bochner integral

$$\int_s^{\cdot} \mathbb{X}(r)\,\mathrm{d}\mathbb{V}(r).$$

Proof. The proof is similar to the one of Theorem 3.6; one proceeds via Fubini theorem. $\square$

4 χ-quadratic variation and χ-Dirichlet processes

4.1 χ-quadratic variation processes

Denote with $\mathscr{C}([s,T])$ the space of the real continuous processes equipped with the ucp (uniform convergence in probability) topology.

Consider two real separable Hilbert spaces H_1 and H_2 equipped with norms $|\cdot|_1$ and $|\cdot|_2$. Following [15, 14] a **Chi-subspace** (of $(H_1\hat{\otimes}_\pi H_2)^*$) is defined as any Banach subspace $(\chi, |\cdot|_\chi)$ which is continuously embedded into $(H_1\hat{\otimes}_\pi H_2)^*$: in other words, there is some constant C such that

$$|\cdot|_{(H_1\hat{\otimes}_\pi H_2)^*} \le C|\cdot|_\chi.$$

Lemma 4.1. *Let us consider a Banach space ν_1 [resp. ν_2] continuously embedded in H_1^* [resp. H_2^*]. Define $\bar{\chi} := \nu_1\hat{\otimes}_\pi\nu_2$. Then $\bar{\chi}$ can be continuously embedded in $(H_1\hat{\otimes}_\pi H_2)^*$. In particular, after there exists a constant $C>0$ such that for all $u\in\bar{\chi}$,*

$$|u|_{(H_1\hat{\otimes}_\pi H_2)^*} \le C|u|_{\bar{\chi}}, \tag{21}$$

after having identified an element of χ with an element of $(H_1\hat{\otimes}_\pi H_2)^*$. In other words $\bar{\chi}$ is a Chi-subspace of $(H_1\hat{\otimes}_\pi H_2)^*$.

Remark 4.2. *In particular $H_1^* \hat{\otimes}_\pi H_2^*$ is a a Chi-subspace of $(H_1 \hat{\otimes}_\pi H_2)^*$*

Proof of Lemma 4.1. To simplify the notations assume the norm of the injections $\nu_1 \hookrightarrow H_1^*$ and $\nu_2 \hookrightarrow H_2^*$ to be less or equal than 1. We remind that $(H_1 \hat{\otimes}_\pi H_2)^*$ is isometrically identified with the Banach space of the bilinear bounded forms from $H_1 \times H_2$ to $\mathbb{R}$, denoted by $\mathcal{B}(H_1, H_2)$.

Consider first an element $u \in \bar{\chi}$ of the form $u = \sum_{i=1}^n a_i^* \otimes b_i^*$ for some $a_i^* \in \nu_1$ and $b_i^* \in \nu_2$. u can be identified with an element of $\mathcal{B}(H_1, H_2)$ acting as follows

$$u(\phi, \psi) := \sum_{i=1}^n \langle a_i^*, \phi \rangle \langle b_i^*, \psi \rangle .$$

We can choose $a_i^* \in \nu_1$ and $b_i^* \in \nu_2$ such that $u = \sum_{i=1}^n a_i^* \otimes b_i^*$ and

$$|u|_{\bar{\chi}} = \inf \left\{ \sum_{i=1}^n |x_i|_{\nu_1} |y_i|_{\nu_2} \ : \ u = \sum_{i=1}^n x_i \otimes y_i, \qquad x_i \in \nu_1, \ y_i \in \nu_2 \right\} > -\epsilon + \sum_{i=1}^n |a_i^*|_{\nu_1} |b_i^*|_{\nu_2}. \quad (22)$$

Using such an expression for u we have that

$$\begin{aligned} \|u\|_{\mathcal{B}(H_1,H_2)} &= \sup_{|\phi|_1, |\psi|_2 \le 1} \left| \sum_{i=1}^n \langle a_i^*, \phi \rangle \langle b_i^*, \psi \rangle \right| \\ &\le \sup_{|\phi|_1, |\psi|_2 \le 1} \sum_{i=1}^n |a_i|_1 |\phi|_1 |b_i|_2 |\psi|_2 \le \sum_{i=1}^n |a_i|_1 |b_i|_2 \\ &\le \sum_{i=1}^n |a_i^*|_{\nu_1} |b_i^*|_{\nu_2} \le \epsilon + |u|_{\bar{\chi}}. \quad (23) \end{aligned}$$

Since ϵ is arbitrary, we conclude that $\|u\|_{\mathcal{B}(H_1,H_2)} \le |u|_{\bar{\chi}}$.

Since this proves that the application that associates to $u \in \nu_1 \hat{\otimes}_\pi \nu_2$ its corresponding element in $\mathcal{B}(H_1, H_2)$, has norm 1 on the dense subset $\nu_1 \hat{\otimes}_\pi \nu_2$, then the claim is proved. □

Remark 4.3. *Even though the Chi-subspaces of tensor product type, described in Lemma 4.1 are natural, there are examples of Chi-subspace not of that form, see e.g. Section 2.6 in [15].*

Let χ be a generic Chi-subspace. We introduce the following definition.

Definition 4.4. *Given $\mathbb{X}$ [resp. $\mathbb{Y}$] a H_1-valued [resp. H_2-valued] process, we say that $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-covariation if the two following conditions are satisfied.*

H1 *For any sequence of positive real numbers $\epsilon_n \searrow 0$ there exists a subsequence ϵ_{n_k} such that*

$$\sup_k \int_s^T \frac{|(J(\mathbb{X}(r+\epsilon_{n_k}) - \mathbb{X}(r)) \otimes (\mathbb{Y}(r+\epsilon_{n_k}) - \mathbb{Y}(r)))|_{\chi^*}}{\epsilon_{n_k}} ds \; < \infty \; a.s. \tag{24}$$

H2 *If we denote by $[X,\mathbb{Y}]^{\epsilon}_{\chi}$ the application*

$$\begin{cases} [\mathbb{X},\mathbb{Y}]^{\epsilon}_{\chi} : \chi \longrightarrow \mathscr{C}([s,T]) \\ \phi \mapsto \int_s^{\cdot} {}_{\chi}\left\langle \phi, \frac{J((\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)))}{\epsilon} \right\rangle_{\chi^*} dr, \end{cases} \tag{25}$$

where $J : H_1\hat{\otimes}_{\pi}H_2 \longrightarrow (H_1\hat{\otimes}_{\pi}H_2)^{**}$ is the canonical injection between a space and its bidual, the following two properties hold.

(i) There exists an application, denoted by $[\mathbb{X},\mathbb{Y}]_{\chi}$, defined on χ with values in $\mathscr{C}([s,T])$, satisfying

$$[\mathbb{X},\mathbb{Y}]^{\epsilon}_{\chi}(\phi) \xrightarrow[\epsilon\longrightarrow 0_+]{ucp} [\mathbb{X},\mathbb{Y}]_{\chi}(\phi) \tag{26}$$

for every $\phi \in \chi \subset (H_1\hat{\otimes}_{\pi}H_2)^*$.

(ii) There exists a strongly measurable process $\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi} : \Omega \times [s,T] \longrightarrow \chi^*$, such that

- *for almost all $\omega \in \Omega$, $\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi}(\omega,\cdot)$ is a (càdlàg) bounded variation process,*
- *$\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi}(\cdot,t)(\phi) = [\mathbb{X},\mathbb{Y}]_{\chi}(\phi)(\cdot,t)$ a.s. for all $\phi \in \chi$, $t \in [s,T]$.*

Remark 4.5. *Since, $(H_1\hat{\otimes}_{\pi}H_2)^{**}$ is continuously embedded in χ^*, then $J(a\otimes b)$ can be considered as an element of χ^*. Therefore we have*

$$|J(a\otimes b)|_{\chi^*} = \sup_{\phi\in\chi, |\phi|_{\chi}\le 1} \langle J(a\otimes b), \phi\rangle = \sup_{\phi\in\chi, |\phi|_{\chi}\le 1} |\phi(a\otimes b)|.$$

We can apply this fact to the expression (24) *considering $a = \mathbb{X}(r+\epsilon_{n_k}) - \mathbb{X}(r)$ and $b = \mathbb{Y}(r+\epsilon_{n_k}) - \mathbb{Y}(r)$.*

Remark 4.6. *An easy consequence of Remark 3.10 and Lemma 3.18 in [14] is the following. We set*

$$A(\varepsilon) := \int_s^T \frac{|(J(\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)))|_{\chi^*}}{\epsilon} dr. \tag{27}$$

1. *If $\lim_{\epsilon\to 0} A(\epsilon)$ exists in probability then Condition* **H1** *of Definition 4.4 is verified.*

2. *If* $\lim_{\epsilon\to 0} A(\epsilon) = 0$ *in probability then* $\mathbb{X}$ *and* $\mathbb{Y}$ *admit a* χ*-covariation and* $\widetilde{[\mathbb{X},\mathbb{Y}]}$ *vanishes.*

If $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-covariation we call χ-covariation of $\mathbb{X}$ and $\mathbb{Y}$ the χ^*-valued process $\widetilde{[\mathbb{X},\mathbb{Y}]}_\chi$ defined for every $\omega \in \Omega$ and $t \in [s,T]$ by $\phi \mapsto \widetilde{[\mathbb{X},\mathbb{Y}]}_\chi(\omega,t)(\phi) = [\mathbb{X},\mathbb{Y}]_\chi(\phi)(\omega,t)$. By abuse of notation, $[\mathbb{X},\mathbb{Y}]_\chi$ will also be often called χ-covariation and it will be confused with $\widetilde{[\mathbb{X},\mathbb{Y}]}_\chi$.

Definition 4.7. *If* $\chi = (H_1\hat{\otimes}_\pi H_2)^*$ *the* χ*-covariation is called* global covariation. *In this case we omit the index* $(H_1\hat{\otimes}_\pi H_2)^*$ *using the notations* $[\mathbb{X},\mathbb{Y}]$ *and* $\widetilde{[\mathbb{X},\mathbb{Y}]}$.

Remark 4.8. *The notions of scalar and tensor covariation have been defined in Definitions 1.3 and 1.6.*

- *Suppose that* $\mathbb{X}$ *and* $\mathbb{Y}$ *admits a scalar quadratic variation and* $(\mathbb{X},\mathbb{Y})$ *has a tensor covariation, denoted with* $[\mathbb{X},\mathbb{Y}]^\otimes$. *Then* $\mathbb{X}$ *and* $\mathbb{Y}$ *admit a global covariation* $[\mathbb{X},\mathbb{Y}]$. *In particular, recalling that* $H_1\hat{\otimes}_\pi H_2$ *is embedded in* $(H_1\hat{\otimes}_\pi H_2)^{**}$, *we have* $\widetilde{[\mathbb{X},\mathbb{Y}]} = [\mathbb{X},\mathbb{Y}]^\otimes$. *The proof is a slight adaptation of the one of Proposition 3.14 in [14]. In particular condition* **H1** *holds using Cauchy-Schwarz inequality.*

- *If* $\mathbb{X}$ *admits a scalar zero quadratic variation then, by definition, the tensor covariation of* $(\mathbb{X},\mathbb{X})$ *also vanishes. Consequently, by item (i)* $\mathbb{X}$ *also admits a global quadratic variation.*

Remark 4.9. *The following properties hold.*

1. *If* $\mathbb{M}$ *is a continuous local martingale with values in* H *then* $\mathbb{M}$ *has a scalar quadratic variation, see Proposition 1.7 in [14].*

2. *If* $\mathbb{M}$ *is a continuous local martingale with values in* H *then* $\mathbb{M}$ *has a tensor quadratic variation. This fact is proved in Proposition 1.6 of [14]. Using similar arguments one can see that if* $\mathbb{M}_1$ *[resp.* $\mathbb{M}_2$*] is a continuous local martingale with values in* H_1 *[resp.* H_2*] then* $(\mathbb{M}_1,\mathbb{M}_2)$ *admits a tensor covariation, see [14].*

Remark 4.10. *If* $\mathbb{X}$ *and* $\mathbb{Y}$ *admit a global covariation then they admit a* χ*-covariation for any Chi-subspace* χ. *Moreover* $[\mathbb{X},\mathbb{Y}]_\chi(\phi) = [\mathbb{X},\mathbb{Y}](\phi)$ *for all* $\phi \in \chi$.

We say that a process $\mathbb{X}$ admits a χ-quadratic variation if $\mathbb{X}$ and $\mathbb{X}$ admit a χ-covariation. The process $\widetilde{[\mathbb{X},\mathbb{X}]}_\chi$, often denoted by $\widetilde{[\mathbb{X}]}_\chi$, is also called χ-quadratic variation of $\mathbb{X}$.

Remark 4.11. *In [14] the definition and the concepts that we have recalled are introduced when* H_1 *and* H_2 *are Banach spaces. For our purposes we can restrict ourself to separable Hilbert spaces.*

Remark 4.12. *For the global covariation case (i.e. for $\chi = (H_1 \hat{\otimes}_\pi H_2)^*$) the condition **H1** reduces to*

$$\sup_k \int_s^T \frac{1}{\epsilon_{n_k}} \left|(\mathbb{X}(r+\epsilon_{n_k}) - \mathbb{X}(r))\right|_1 \left|\mathbb{Y}(r+\epsilon_{n_k}) - \mathbb{Y}(r))\right|_2 ds \ < \infty \ a.s.$$

In fact the embedding of $(H_1 \hat{\otimes}_\pi H_2)$ in its bi-dual is isometric and, for $x \in H_1$ and $y \in H_2$, $|x \otimes y|_{(H_1 \hat{\otimes}_\pi H_2)} = |x|_1 |y|_2$.

The product of a real finite quadratic variation process and a zero real quadratic variation process is again a zero quadratic variation processes. Under some conditions this can be generalized to the infinite dimensional case.

Proposition 4.13. *Let $i = 1,2$ and ν_i be a continuous embedded Hilbert subspace of H_i^*. Let consider the Chi-subspace of the type $\chi_1 = \nu_1 \hat{\otimes}_\pi H_2^*$ and $\chi_2 = H_1^* \hat{\otimes}_\pi \nu_2$, $\hat{\chi}_i = \nu_i \hat{\otimes}_\pi \nu_i$, $i = 1,2$. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a process with values in H_1 (resp. H_2).*

1. *Suppose that $\mathbb{X}$ admits a $\hat{\chi}_1$-quadratic variation and $\mathbb{Y}$ a zero scalar quadratic variation. Then $[\mathbb{X}, \mathbb{Y}]_{\chi_1} = 0$.*

2. *Similarly suppose that $\mathbb{Y}$ admits a $\hat{\chi}_2$-quadratic variation and $\mathbb{X}$ a zero scalar quadratic variation. Then $[\mathbb{X}, \mathbb{Y}]_{\chi_2} = 0$.*

Proof. We remark that Lemma 4.1 imply that χ_i and $\hat{\chi}_i$, $i = 1,2$ are indeed Chi-subspaces. By item 2. of Remark 4.6, it is enough to show that $A(\varepsilon)$ defined in (27) converge to zero, with $\chi = \chi_i, i = 1,2$. By symmetrical reasons it is enough to show item 1.
We set $\chi = \chi_1$. Via Riesz, we identify H_1 as a subspace of ν_1^*. We remark that, by the usual identifications, we have

$$J(H_1 \hat{\otimes}_\pi H_2) \subset (H_1 \hat{\otimes}_\pi H_2)^{**} \subset \chi^*.$$

Let $a \in H_1 \subset \nu_1^*, b \in H_2$, so $a^* \in \nu_1$. By definition and by Lemma 2.4, we have

$$|J(a \otimes b)|_{\chi^*} = |i(a^* \otimes b^*)|_{\chi^*} = |a^*|_{\nu_1} |b|_{H_2}.$$

Consequently, using also Riesz identification, with $a = \mathbb{X}(r+\varepsilon) - \mathbb{X}(r)$ and

$b = \mathbb{Y}(r+\varepsilon) - \mathbb{Y}(r)$ for $r \in [s,T]$, we have

$$
\begin{aligned}
A(\varepsilon) &= \int_s^T \frac{|(J(\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)))|_{\chi^*}}{\epsilon} dr. \\
&= \int_s^T |\mathbb{X}(r+\varepsilon) - \mathbb{X}(r)|_{\nu_1^*} |\mathbb{Y}(r+\varepsilon) - \mathbb{Y}(r)|_{H_2} \frac{dr}{\varepsilon} \\
&\le \left(\int_s^T |\mathbb{X}(r+\varepsilon) - \mathbb{X}(r)|^2_{\nu_1^*} \frac{dr}{\varepsilon} \int_s^T |\mathbb{Y}(r+\varepsilon) - \mathbb{Y}(r)|^2_{H_2} \frac{dr}{\varepsilon} \right)^{1/2} \\
&= \left(\int_s^T \frac{|(J(\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes (\mathbb{X}(r+\epsilon) - \mathbb{X}(r)))|_{\hat{\chi}_1^*}}{\epsilon} dr \right)^{1/2} \\
&\quad \left(\int_s^T \frac{|\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r))|^2_{H_2}}{\epsilon} dr \right)^{1/2}.
\end{aligned}
$$

The fourth line is explained by the fact that, setting $a = \mathbb{X}(r+\epsilon) - \mathbb{X}(r)$ so $a \in \nu_1^*$, $J(a \otimes a)$ being assimilated to a bounded bilinear form on $\hat{\chi}_1$, we have

$$
\begin{aligned}
|J(a \otimes a)| &= \sup_{|\varphi|_{\nu_1} \le 1, |\psi|_{\nu_1}| \le 1} |{}_{\hat{\chi}_1^*} \langle J(a \otimes a), \varphi \otimes \psi \rangle_{\hat{\chi}_1}| \\
&= \sup_{|\varphi|_{\nu_1} \le 1} |{}_{\nu_1} \langle \varphi, a \rangle_{\nu_1^*}| \sup_{|\psi|_{\nu_1} \le 1} |{}_{\nu_1} \langle \psi, a \rangle_{\nu_1^*}| = |a|^2_{\nu_1^*}.
\end{aligned}
$$

The condition **H1** related to the $\hat{\chi}_1$-quadratic variation of $\mathbb{X}$ and the zero scalar quadratic variation of $\mathbb{Y}$, imply that previous expression converges to zero. □

When one of the processes is real the formalism of global covariation can be simplified as shown in the following proposition. Observe that, according to our conventions, $|\cdot|$ represents both the norm in H and the absolute value in $\mathbb{R}$.

Proposition 4.14. *Let be $\mathbb{X}\colon [s,T] \times \Omega \to H$ a Bochner integrable process and $Y\colon [s,T] \times \Omega \to \mathbb{R}$ a real valued process. Suppose the following.*

(a) For any ϵ, $\frac{1}{\epsilon}\int_s^T |\mathbb{X}(r+\epsilon) - \mathbb{X}(r)||Y(r+\epsilon) - Y(r)|\,\mathrm{d}r$ is bounded by a r.v. A_ϵ such that A_ϵ converges in probability when $\epsilon \to 0$.

(b) For every $h \in H$ the following limit

$$
C(t)(h) := \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_s^t \langle h, \mathbb{X}(r+\epsilon) - \mathbb{X}(r) \rangle (Y(r+\epsilon) - Y(r))\,\mathrm{d}r
$$

exists ucp and there exists a continuous process $\tilde{C}\colon [s,T] \times \Omega \to H$ s.t.

$$
\left\langle \tilde{C}(t,\omega), h \right\rangle = C(t)(h)(\omega) \qquad \textit{for } \mathbb{P}\textit{-a.s. } \omega \in \Omega,
$$

for all $t \in [s,T]$ and $h \in H$.

If we identify H with $(H\hat{\otimes}_\pi\mathbb{R})^*$, then $\mathbb{X}$ and Y admit a global covariation and $\tilde{C} = \widetilde{[\mathbb{X}, Y]}$.

Proof. Taking into account the identification of H with $(H\hat{\otimes}_\pi\mathbb{R})^*$ the result is a consequence of Corollary 3.26 of [14]. In particular condition **H1** follows from Remark 4.12.

□

4.2 Relations with tensor covariation and classical tensor covariation

The notions of tensor covariation recalled in Definition 1.6 concerns general processes. In the specific case when H_1 and H_2 are two separable Hilbert spaces and $\mathbb{M}\colon [s,T]\times\Omega \to H_1$, $\mathbb{N}\colon [s,T]\times\Omega \to H_2$ are two continuous local martingales, another (classical) notion of tensor covariation is defined, see for instance in Section 23.1 of [41]. This is denoted by $[\mathbb{M},\mathbb{N}]^{cl}$. Recall that the notion introduced in Definition 1.6 is denoted with $[\mathbb{M},\mathbb{N}]^{\otimes}$.

Remark 4.15. *We observe the following facts.*

(i) According to Chapter 22 and 23 in [41], given a H_1-values [resp. H_2-valued] continuous local martingale $\mathbb{M}$ [resp. $\mathbb{N}$], $[\mathbb{M},\mathbb{N}]^{cl}$ is an $(H_1\hat{\otimes}_\pi H_2)$-valued process. Recall that $(H_1\hat{\otimes}_\pi H_2) \subseteq (H_1\hat{\otimes}_\pi H_2)^{**}$.

(ii) Taking into account Lemma 2.4 we know that, given $h \in H_1$ and $k \in H_2$, $h^*\otimes k^*$ can be considered as an element of $(H_1\hat{\otimes}_\pi H_2)^*$. One has

$$[\mathbb{M},\mathbb{N}]^{cl}(t)(h^*\otimes k^*) = [\langle \mathbb{M}, h\rangle \, \langle \mathbb{N}, k\rangle]^{cl}(t), \tag{28}$$

where h^* [resp. k^*] is associated with h [resp. k] via Riesz theorem. This property characterizes $[\mathbb{M},\mathbb{N}]^{cl}$, see e.g. [10], Section 3.4 after Proposition 3.11.

(iii) If $H_2 = \mathbb{R}$ and $\mathbb{N} = N$ is a real continuous local martingale then, identifying $H_1\hat{\otimes}_\pi H_2$ with H_1, $[\mathbb{M},\mathbb{N}]^{cl}$ can be considered as a H_1-valued process. The characterization (28) can be translated into

$$[\mathbb{M},\mathbb{N}]^{cl}(t)(h^*) = [\langle \mathbb{M}, h\rangle, N]^{cl}(t). \tag{29}$$

By inspection, this allows us to see that the classical covariation between $\mathbb{M}$ and $\mathbb{N}$ can be expressed as

$$[\mathbb{M}, N]^{cl}(t) := \mathbb{M}(t)N(t) - \mathbb{M}(s)N(s) - \int_s^t N(r)\,\mathrm{d}\mathbb{M}(r) - \int_s^t \mathbb{M}(r)\,\mathrm{d}N(r). \tag{30}$$

Lemma 4.16. *Let H be a separable Hilbert space. Let $\mathbb{M}$ [resp. $\mathbb{N}$] be a continuous local martingale with values in H. Then $(\mathbb{M},\mathbb{N})$ admits a tensor covariation and*

$$[\mathbb{M},\mathbb{N}]^{\otimes} = [\mathbb{M},\mathbb{N}]^{cl}. \tag{31}$$

In particular $\mathbb{M}$ *and* $\mathbb{N}$ *admit a global covariation and*

$$\widetilde{[\mathbb{M},\mathbb{N}]} = [\mathbb{M},\mathbb{N}]^{cl}. \tag{32}$$

Proof. Thanks to Remark 4.9 $\mathbb{M}$ and $\mathbb{N}$ admit a scalar quadratic variation and $(\mathbb{M},\mathbb{N})$ a tensor covariation. By Remark 4.8 they admit a global covariation. It is enough to show that they are equal as elements of $(H_1\hat{\otimes}_\pi H_2)^{**}$ so one needs to prove that

$$[\mathbb{M},\mathbb{N}]^{\otimes}(\phi) = [\mathbb{M},\mathbb{N}]^{cl}(\phi) \tag{33}$$

for every $\phi \in (H_1\hat{\otimes}_\pi H_2)^*$.

Given $h \in H_1$ and $k \in H_2$, we consider (via Lemma 2.4) $h^* \otimes k^*$ as an element of $(H_1\hat{\otimes}_\pi H_2)^*$. According to Lemma 4.17 below, $H_1^*\hat{\otimes}_\pi H_2^*$ is dense in $(H_1\hat{\otimes}_\pi H_2)^*$ in the weak-* topology. Therefore, taking into account item (ii) of Remark 4.15 we only need to show that

$$[\mathbb{M},\mathbb{N}]^{\otimes}(h^* \otimes k^*) = [\langle \mathbb{M}, h\rangle\langle \mathbb{N}, k\rangle]^{cl}, \tag{34}$$

for every $h \in H_1, k \in H_2$. By the usual properties of Bochner integral the left hand side of (34) is the limit of

$$\begin{aligned}\frac{1}{\epsilon}\int_s^{\cdot} &(M(r+\epsilon)-M(r)) \otimes (N(r+\epsilon)-N(r))(h^*\otimes k^*)\,\mathrm{d}r \\ &= \frac{1}{\epsilon}\int_s^{\cdot} \langle (M(r+\epsilon)-M(r)), h\rangle\langle \otimes (N(r+\epsilon)-N(r)), k\rangle\,\mathrm{d}r. \end{aligned}\tag{35}$$

Since $\langle \mathbb{M}, h\rangle$ and $\langle \mathbb{N}, k\rangle$ are real local martingales, the covariation $[\langle \mathbb{M}, h\rangle\langle \mathbb{N}, k\rangle]$ exists and equals the classical covariation of local martingales because of Proposition 2.4(3) of [50]. □

Lemma 4.17. *Let* H_1, H_2 *be two separable Hilbert spaces. Then* $H_1^*\hat{\otimes}_\pi H_2^*$ *is dense in* $(H_1\hat{\otimes}_\pi H_2)^*$ *in the weak-* topology.*

Proof. Let (e_i) and (f_i) be respectively two orthonormal bases of H_1 and H_2. We denote with $\mathcal{D}$ the linear span of finite linear combinations of $e_i \otimes f_i$. Let $T \in (H_1\hat{\otimes}_\pi H_2)^*$, which is a linear continuous functional on $H_1\hat{\otimes}_\pi H_2$. Using the identification of $(H_1\hat{\otimes}_\pi H_2)^*$ with $\mathcal{B}(H_1, H_2)$, for each $n \in \mathbb{N}$, we define the bilinear form

$$T_n(a,b) := \sum_{i=1}^{i=n} \langle a, e_i\rangle_{H_1} \langle b, f_i\rangle_{H_2} T(e_i, f_i).$$

It defines an element of $H_1^*\hat{\otimes}_\pi H_2^* \subset (H_1\hat{\otimes}_\pi H_2)^*$. It remains to show that

$$_{(H_1\hat{\otimes}_\pi H_2)^*}\langle T_n, l\rangle_{H_1\hat{\otimes}_\pi H_2} \xrightarrow{n\to\infty} {}_{(H_1\hat{\otimes}_\pi H_2)^*}\langle T, l\rangle_{H_1\hat{\otimes}_\pi H_2}, \quad \text{for all } l \in H_1\hat{\otimes}_\pi H_2.$$

We show now the following.

(i) $T_n(a,b) \xrightarrow{n\to\infty} T(a,b)$ for all $a \in H_1$, $b \in H_2$.

(ii) For a fixed $l \in H_1\hat{\otimes}_\pi H_2$, the sequence $T_n(l)$ is bounded.

Let us prove first (i). Let $a \in H_1$ and $b \in H_2$. We write

$$T_n(a,b) = T\left(\sum_{i=1}^n \langle a, e_i\rangle_{H_1} e_i, \quad \sum_{i=1}^n \langle b, f_i\rangle_{H_2} f_i\right). \tag{36}$$

Since

$$\sum_{i=1}^n \langle a, e_i\rangle e_i \xrightarrow{n\to+\infty} a \quad \text{in } H_1$$

and

$$\sum_{i=1}^n \langle b, f_i\rangle f_i \xrightarrow{n\to+\infty} b \quad \text{in } H_2,$$

(i) follows, being T a bounded bilinear form.

Let us prove now (ii). Let $\epsilon > 0$ fixed and $l_0 \in \mathcal{D}$ such that $|l-l_0|_{H_1\hat{\otimes}_\pi H_2} \leq \epsilon$. Then

$$|T_n(l)| \leq |T_n(l-l_0)| + |T_n(l_0)| \leq |T_n|_{(H_1\hat{\otimes}_\pi H_2)^*}|l-l_0|_{H_1\hat{\otimes}_\pi H_2} + |T_n(l_0)|. \tag{37}$$

So (37) is bounded by

$$|T|_{(H_1\hat{\otimes}_\pi H_2)^*}\epsilon + \sup_n |T_n(l_0)|$$

recalling that the sequence $(T_n(l_0))$ is bounded, since it is convergent. Finally (ii) is also proved.

At this point (i) implies that

$${}_{(H_1\hat{\otimes}_\pi H_2)^*}\langle T_n, l\rangle_{H_1\hat{\otimes}_\pi H_2} \xrightarrow{n\to\infty} {}_{(H_1\hat{\otimes}_\pi H_2)^*}\langle T, l\rangle_{H_1\hat{\otimes}_\pi H_2}, \quad \text{for all } l \in \mathcal{D}.$$

Since $\mathcal{D}$ is dense in $H_1\hat{\otimes}_\pi H_2$, the conclusion follows by Banach-Steinhaus theorem, see Theorem 18, Chapter II in [20]. □

We recall the following fact that concerns the classical tensor covariation.

Lemma 4.18. *Let $\mathbb{W}_Q$ be a Q-Wiener process as in Subsection 3.1. Let $\Psi\colon ([s,T]\times\Omega, \mathscr{P}) \to \mathcal{L}_2(U_0, H)$ be a strongly measurable process satisfying condition (10). Consider the local martingale*

$$\mathbb{M}(t) := \int_s^t \Psi(r)\,\mathrm{d}\mathbb{W}_Q(r).$$

Then

$$[\mathbb{M},\mathbb{M}]^{cl}(t) = \int_s^t g(r)\,\mathrm{d}r.$$

where $g(r)$ is the element of $H\hat{\otimes}_\pi H$ associated with the nuclear operator $G_g(r) := \left(\Psi(r)Q^{1/2}\right)\left(\Psi(r)Q^{1/2}\right)^*$.

Proof. See [10] Section 4.7. □

Lemma 4.19. *Let* $\mathbb{M}\colon [s,T]\times\Omega \to H$ *be a continuous local martingale and* $\mathbb{Z}$ *a strongly measurable process from* $([s,T]\times\Omega,\mathscr{P})$ *to* H *and such that* $\int_s^T \|\mathbb{Z}(r)\|^2\,\mathrm{d}[\mathbb{M}]^{\mathbb{R},cl}(r) < +\infty$ *a.s. Of course* $\mathbb{Z}$ *can be Riesz-identified with an element of* $\mathcal{J}^2(s,T;H^*,\mathbb{R})$*. We define*

$$X(t) := \int_s^t \langle \mathbb{Z}(r),\, \mathrm{d}\mathbb{M}(r)\rangle\,. \tag{38}$$

Then, X *is a real continuous local martingale and for every continuous real local martingale* N*, the (classical, one-dimensional) covariation process* $[X,N]^{cl}$ *is given by*

$$[X,N]^{cl}(t) = \int_s^t \big\langle \mathbb{Z}(r),\, \mathrm{d}[\mathbb{M},N]^{cl}(r)\big\rangle\,; \tag{39}$$

in particular the integral in the right side is-well defined.

Proof. The fact that X is a local martingale is part of the result of Theorem 2.14 in [35]. For the other claim we can reduce, using a sequence of suitable stopping times as in the proof of Theorem 3.6, to the case in which $\mathbb{Z}$, $\mathbb{M}$ and N are square integrable martingales.Taking into account the characterization (29) and the discussion developed in [43], page 456, (39) follows. □

Proposition 4.20. *If* $\mathbb{M}\colon [s,T]\times\Omega\to H$ *and* $N\colon [s,T]\times\Omega\to\mathbb{R}$ *are continuous local martingales. Then* $\mathbb{M}$ *and* N *admit a global covariation and* $\widetilde{[\mathbb{M},N]} = [\mathbb{M},N]^{cl}$*.*

Proof. We have to check the conditions stated in Proposition 4.14 for $\tilde{C}$ equal to the right side of (30). Concerning (a), by Cauchy-Schwarz inequality we have

$$\frac{1}{\epsilon}\int_s^T |N(r+\epsilon)-N(r)||\mathbb{M}(r+\epsilon)-\mathbb{M}(r)|\,\mathrm{d}r \le [N,N]^{\epsilon,\mathbb{R}}[\mathbb{M},\mathbb{M}]^{\epsilon,\mathbb{R}}.$$

Since both N and $\mathbb{M}$ are local martingales they admit a scalar quadratic variation (as recalled in Remark 4.9), the result is established. Concerning (b), taking into account (29) we need to prove that for any $h\in H$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_s^{\cdot} (M^h(r+\epsilon)-M^h(r))(N(r+\epsilon)-N(r))\,\mathrm{d}r = [\langle M,h\rangle\,,N]^{cl} \tag{40}$$

ucp, where M^h is the real local martingale $\langle\mathbb{M},h\rangle$. (40) follows by Proposition 2.4(3) of [50]. □

4.3 χ-Dirichlet and ν-weak Dirichlet processes

We have now at our disposal all the elements we need to introduce the concept of χ-Dirichlet process and ν-weak Dirichlet process.

Definition 4.21. *Let $\chi \subseteq (H\hat{\otimes}_\pi H)^*$ be a Chi-subspace. A continuous H-valued process $\mathbb{X}\colon ([s,T] \times \Omega, \mathscr{P}) \to H$ is called* χ-Dirichlet process *if there exists a decomposition $\mathbb{X} = \mathbb{M} + \mathbb{A}$ where*

(i) $\mathbb{M}$ *is a continuous local martingale,*

(ii) $\mathbb{A}$ *is a continuous χ-zero quadratic variation process,*

(iii) $\mathbb{A}(0) = 0$.

Definition 4.22. *Let H and H_1 be two separable Hilbert spaces. Let $\nu \subseteq (H\hat{\otimes}_\pi H_1)^*$ be a Chi-subspace. A continuous adapted H-valued process $\mathbb{A}\colon [s,T] \times \Omega \to H$ is said to be $\mathscr{F}_t^s$-ν-*martingale-orthogonal *if*

$$[\mathbb{A}, \mathbb{N}]_\nu = 0$$

for any H_1-valued continuous local martingale $\mathbb{N}$.

As we have done for the expressions "stopping time", "adapted", "predictable"... since we always use the filtration $\mathscr{F}_t^s$, we simply write *ν-martingale-orthogonal* instead of $\mathscr{F}_t^s$-ν-martingale-orthogonal.

Definition 4.23. *Let H and H_1 be two separable Hilbert spaces. Let $\nu \subseteq (H\hat{\otimes}_\pi H_1)^*$ be a Chi-subspace. A continuous H-valued process $\mathbb{X}\colon [s,T]\times\Omega \to H$ is called* ν-weak-Dirichlet process *if it is adapted and there exists a decomposition $\mathbb{X} = \mathbb{M} + \mathbb{A}$ where*

(i) $\mathbb{M}$ *is a H-valued continuous local martingale,*

(ii) $\mathbb{A}$ *is an ν-martingale-orthogonal process,*

(iii) $\mathbb{A}(0) = 0$.

Remark 4.24. *The sum of two ν-martingale-orthogonal processes is again a ν-martingale-orthogonal process.*

Proposition 4.25.

1. *Any process admitting a zero scalar quadratic variation (for instance a bounded variation process) is a ν-martingale-orthogonal process.*
2. *Let $\mathbb{Q}$ be an equivalent probability to $\mathbb{P}$. Any ν-weak Dirichlet process under $\mathbb{P}$ is a ν-weak Dirichlet process under $\mathbb{Q}$.*

Proof. 1. follows from Proposition 4.13 1. setting $\nu_1 = H_1^*$. In fact, any local martingale has a global quadratic variation because of Remark 4.8 and Remark 4.9. So it has a $\nu_1\hat{\otimes}_\pi\nu_1$-quadratic variation by Remark 4.2 and Remark 4.10. Concerning 2., by Theorem in Section 30.3, page 208 of [41], a local martingale under $\mathbb{P}$ is a local martingale under $\mathbb{Q}$ plus a bounded variation process. The result follows by item 1. and Remark 4.24 since the ν-covariation remains unchanged under an equivalent probability measure. $\square$

We recall that the decomposition of a real weak Dirichlet process is unique, see Remark 3.5 of [33]. For the infinite dimensional case we now establish the uniqueness of the decomposition of a ν-weak-Dirichlet process in two cases: when $H_1 = H$ and when $H_1 = \mathbb{R}$.

Proposition 4.26. *Let $\nu \subseteq (H\hat{\otimes}_\pi H)^*$ be a Chi-subspace. Suppose that ν is dense in $(H\hat{\otimes}_\pi H)^*$. Then any decomposition of a ν-weak-Dirichlet process $\mathbb{X}$ is unique.*

Proof. Assume that $\mathbb{X} = \mathbb{M}^1 + \mathbb{A}^1 = \mathbb{M}^2 + \mathbb{A}^2$ are two decompositions where $\mathbb{M}^1$ and $\mathbb{M}^2$ are continuous local martingales and $\mathbb{A}^1, \mathbb{A}^2$ are ν-martingale-orthogonal processes. If we call $\mathbb{M} := \mathbb{M}^1 - \mathbb{M}^2$ and $\mathbb{A} := \mathbb{A}^1 - \mathbb{A}^2$ we have $0 = \mathbb{M} + \mathbb{A}$.

By Lemma 4.16, $\mathbb{M}$ has a global quadratic variation. In particular it also has a ν-quadratic variation and, thanks to the bilinearity of the ν-covariation,

$$0 = [\mathbb{M}, 0]_\nu = [\mathbb{M}, \mathbb{M} + \mathbb{A}]_\nu = [\mathbb{M}, \mathbb{M}]_\nu + [\mathbb{M}, \mathbb{A}]_\nu = [\mathbb{M}, \mathbb{M}]_\nu + 0 = [\mathbb{M}, \mathbb{M}]_\nu.$$

We prove now that $\mathbb{M}$ has also zero global quadratic variation. We denoted with $\mathcal{C}([s,T])$ the space of the real continuous processes defined on $[s,T]$. We introduce, for $\epsilon > 0$, the operators

$$\begin{cases} [\mathbb{M}, \mathbb{M}]^\epsilon \colon (H\hat{\otimes}_\pi H)^* \to \mathcal{C}([s,T]) \\ ([\mathbb{M}, \mathbb{M}]^\epsilon(\phi))(t) := \dfrac{1}{\epsilon} \displaystyle\int_s^t {}_{(H\hat{\otimes}_\pi H)}\big\langle (\mathbb{M}_{r+\epsilon} - \mathbb{M}_r)\otimes^2, \phi \big\rangle_{(H\hat{\otimes}_\pi H)^*} \, \mathrm{d}r. \end{cases} \tag{41}$$

Observe the following

(a) $[\mathbb{M}, \mathbb{M}]^\epsilon$ are linear and bounded operators.

(b) For $\phi \in (H\hat{\otimes}_\pi H)^*$ the limit $[\mathbb{M}, \mathbb{M}](\phi) := \lim_{\epsilon\to 0}[\mathbb{M}, \mathbb{M}]^\epsilon(\phi)$ exists.

(c) If $\phi \in \nu$ we have $[\mathbb{M}, \mathbb{M}](\phi) = 0$.

Thanks to (a) and (b) and Banach-Steinhaus theorem (see Theorem 17, Chapter II in [20]) we know that $[\mathbb{M}, \mathbb{M}]$ is linear and bounded. Thanks to (c) and the fact that the inclusion $\nu \subseteq (H\hat{\otimes}_\pi H)^*$ is dense it follows $[\mathbb{M}, \mathbb{M}] = 0$. By Lemma 4.16 $[\mathbb{M}, \mathbb{M}]$ coincides with the classical quadratic variation $[\mathbb{M}, \mathbb{M}]^{cl}$ and it is characterized by

$$0 = [\mathbb{M}, \mathbb{M}]^{cl}(h^*, k^*) = [\langle \mathbb{M}, h\rangle, \langle \mathbb{M}, k\rangle]^{cl}.$$

Since $\mathbb{M}(0) = 0$ and therefore $\langle \mathbb{M}, h\rangle(0) = 0$ it follows that $\langle \mathbb{M}, h\rangle \equiv 0$ for any $h \in H$. Finally $\mathbb{M} \equiv 0$, which concludes the proof. □

Proposition 4.27. *Let H be a separable Hilbert space. Let $\nu \subseteq H^* \equiv (H\hat{\otimes}_\pi \mathbb{R})^*$ be a Banach space with continuous and dense inclusion. Then any decomposition of a ν-weak-Dirichlet process $\mathbb{X}$ with values in H is unique.*

Remark 4.28. *Taking into account the identification of H^* with $(H\hat{\otimes}_\pi \mathbb{R})^*$ it is possible to consider ν as a dense subset of $(H\hat{\otimes}_\pi \mathbb{R})^*$ which is a Chi-subspace.*

Proof of Proposition 4.27. We denote again the H inner product by $\langle \cdot, \cdot \rangle$. We show that the unique decomposition of the 0 process is trivial. Assume that $0 = \mathbb{X} = \mathbb{M} + \mathbb{A}$. Since ν is dense in H^* it is possible to choose an orthonormal basis e_i^* in ν. We introduce $M^i := \langle \mathbb{M}, e_i \rangle$, they are continuous real local martingales and then, thanks to the properties of $\mathbb{A}$ we have

$$0 = [\mathbb{X}, M^i]_\nu = [\mathbb{M}, M^i]_\nu + [\mathbb{A}, M^i]_\nu = [\mathbb{M}, M^i]_\nu.$$

By Remark 4.10, Proposition 4.20 and Remark 4.15 (iii) we know that

$$\left\langle [\mathbb{M}, M^i]_\nu, e_i \right\rangle = \left\langle [\mathbb{M}, M^i]^{cl}, e_i \right\rangle = [M^i, M^i]^{cl};$$

so $M^i = 0$ for all i and then $\mathbb{M} = 0$. This concludes the proof. □

Proposition 4.29. *Let H and H_1 be two separable Hilbert spaces. Let $\chi = \chi_0 \hat{\otimes}_\pi \chi_0$ for some χ_0 Banach space continuously embedded in H^*. Define $\nu = \chi_0 \hat{\otimes}_\pi H_1^*$. Then an H-valued continuous zero χ-quadratic variation process $\mathbb{A}$ is a ν-martingale-orthogonal process.*

Proof. Taking into account Lemma 4.1, χ is a Chi-subspace of $(H \hat{\otimes}_\pi H)^*$ and ν is a Chi-subspace of $(H \hat{\otimes}_\pi H_1)^*$. Let $\mathbb{N}$ be a continuous local martingale with values in H_1. We need to show that $[\mathbb{A}, \mathbb{N}]_\nu = 0$. We consider the random maps $T^\epsilon \colon \nu \times \Omega \to \mathcal{C}([s,T])$ defined by

$$T^\epsilon(\phi) := [\mathbb{A}, \mathbb{N}]^\epsilon_\nu(\phi) = \frac{1}{\epsilon} \int_s^\cdot {}_{\nu^*}\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)) \otimes (\mathbb{N}(r+\epsilon) - \mathbb{N}(r)), \phi \rangle_\nu \, \mathrm{d}r,$$

for $\phi \in \nu$.

Step 1:

Suppose that $\phi = h^* \otimes k^*$ for $h^* \in \chi_0$ and $k \in H_1$. Then

$$\begin{aligned}
T^\epsilon(\phi)(t) &= \frac{1}{\epsilon} \int_s^t {}_{\chi_0^*}\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)), h^* \rangle_{\chi_0} \, \langle (\mathbb{N}(r+\epsilon) - \mathbb{N}(r)), k \rangle_{H_1} \, \mathrm{d}r \\
&\leq \left[\frac{1}{\epsilon} \int_s^t {}_{\chi_0^*}\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)), h^* \rangle^2_{\chi_0} \, \mathrm{d}r \int_s^t \langle (\mathbb{N}(r+\epsilon) - \mathbb{N}(r)), k \rangle^2_{H_1} \, \mathrm{d}r \right]^{1/2} \\
&= \left[\frac{1}{\epsilon} \int_s^t {}_{\chi^*}\left\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)) \otimes^2, h^* \otimes h^* \right\rangle_\chi \, \mathrm{d}r \right]^{\frac{1}{2}} \\
&\quad \times \left[\frac{1}{\epsilon} \int_s^t {}_{H_1 \hat{\otimes}_\pi H_1}\left\langle (\mathbb{N}(r+\epsilon) - \mathbb{N}(r)) \otimes^2, k^* \otimes k^* \right\rangle_{(H_1 \hat{\otimes}_\pi H_1)^*} \, \mathrm{d}r \right]^{\frac{1}{2}} \qquad (42)
\end{aligned}$$

that converges ucp to

$$([\mathbb{A}, \mathbb{A}](t)(h^* \otimes h^*)[\mathbb{N}, \mathbb{N}]_\chi(t)(k^* \otimes k^*))^{1/2} = 0.$$

Step 2:

We denote with $\mathcal{D}$ the linear combinations of elements of the form $h^* \otimes k^*$ for $h^* \in \chi_0$ and $k \in H_1$. We remark that $\mathcal{D}$ is dense in ν. From the convergence found in *Step 1*, it follows that, for every $\phi \in \mathcal{D}$, ucp we have

$$T^\epsilon(\phi) \xrightarrow{\epsilon \to 0} 0.$$

Step 3:

We consider a generic $\phi \in \nu$. By Lemma 2.4, for $t \in [s, T]$ it follows

$$\begin{aligned}
|T^\epsilon(\phi)(t)| &\leq |\phi|_\nu \int_s^t \frac{|(\mathbb{A}(r+\epsilon) - \mathbb{A}(r)) \otimes (\mathbb{N}(r+\epsilon) - \mathbb{N}(r))|_{\nu^*}}{\epsilon} \,\mathrm{d}r \\
&= |\phi|_\nu \frac{1}{\epsilon} \int_s^t |(\mathbb{N}(r+\epsilon) - \mathbb{N}(r))|_{H_1} |(\mathbb{A}(r+\epsilon) - \mathbb{A}(r))|_{\chi_0^*} \,\mathrm{d}r \\
&\leq |\phi|_\nu \left(\frac{1}{\epsilon} \int_s^t |(\mathbb{N}(r+\epsilon) - \mathbb{N}(r))|_{H_1}^2 \,\mathrm{d}r \,\frac{1}{\epsilon} \int_s^t |(\mathbb{A}(r+\epsilon) - \mathbb{A}(r))|_{\chi_0^*}^2 \,\mathrm{d}r \right)^{\frac{1}{2}} \\
&= |\phi|_\nu \left(\frac{1}{\epsilon} \int_s^t |(\mathbb{N}(r+\epsilon) - \mathbb{N}(r))|_{H_1}^2 \,\mathrm{d}r \right. \\
&\qquad \left. \times \frac{1}{\epsilon} \int_s^t \left|(\mathbb{A}(r+\epsilon) - \mathbb{A}(r))^{\otimes 2}\right|_{\chi^*} \,\mathrm{d}r \right)^{\frac{1}{2}}. \qquad (43)
\end{aligned}$$

To prove that $[\mathbb{A}, \mathbb{N}]_\nu = 0$ we check the corresponding conditions **H1** and **H2** of the Definition 4.4. By Lemma 4.16 we know that $\mathbb{N}$ admits a global quadratic variation i.e. a $(H_1 \otimes H_1)^*$-quadratic variation. By condition **H1** of the Definition 4.4 related to $(H_1 \otimes H_1)^*$-quadratic variation, for any sequence (ϵ_n) converging to zero, there is a subsequence (ϵ_{n_k}) such that the sequence $T^{\epsilon_{n_k}}(\phi)$ is bounded for any ϕ in the $\mathcal{C}[s, T]$ metric a.s. Moreover the P-null set does not depend on Φ. This in particular shows the condition **H1** of the ν-covariation. By Banach-Steinhaus for F-spaces (Theorem 17, Chapter II in [20]) it follows that $T^\epsilon(\phi) \xrightarrow{\epsilon \to 0} 0$ ucp for all $\phi \in \nu$ and so condition **H2** and the final result follows. ☐

Corollary 4.30. *Assume that the hypotheses of Proposition 4.29 are satisfied. If $\mathbb{X}$ is a χ-Dirichlet process then we have the following.*

(i) *$\mathbb{X}$ is a ν-weak-Dirichlet process.*

(ii) *$\mathbb{X}$ is a χ-weak Dirichlet process.*

(iii) *$\mathbb{X}$ is a χ-finite-quadratic-variation process.*

Proof. (i) follows by Proposition 4.29.
As far as (ii) is concerned, let $\mathbb{X} = \mathbb{M} + \mathbb{A}$ be a χ-Dirichlet process decomposition, where and $\mathbb{M}$ be a local martingale. Setting $H_1 = H$, then χ is included in ν, so Proposition 4.29 implies that $\mathbb{A}$ is a χ-orthogonal process and so (ii) follows. We prove now (iii). By Lemma 4.16 and Remark 4.10 $\mathbb{M}$ admits a χ-quadratic

variation. By the bilinearity of the χ-covariation, it is enough to show that $[\mathbb{M}, \mathbb{A}]_{\chi} = 0$. This follows from item (ii). $\square$

Proposition 4.31. *Let B_1 and B_2 two Banach spaces and χ a Chi-subspace of $(B_1 \hat{\otimes}_{\pi} B_2)^*$. Let $\mathbb{X}$ and $\mathbb{Y}$ be two stochastic processes with values respectively in B_1 and B_2 admitting a χ-covariation. Let G be a continuous measurable process $G : [s, T] \times \Omega \to \mathcal{K}$ where $\mathcal{K}$ is a closed separable subspace of χ. Then for every $t \in [s, T]$*

$$\int_s^t {}_{\chi}\langle G(\cdot, r), [\mathbb{X}, \mathbb{Y}]^{\epsilon}(\cdot, r)\rangle_{\chi^*} \, \mathrm{d}r \xrightarrow[\epsilon \longrightarrow 0]{} \int_s^t {}_{\chi}\langle G(\cdot, r), d\widetilde{[\mathbb{X}, \mathbb{Y}]}(\cdot, r)\rangle_{\chi^*} \tag{44}$$

in probability.

Proof. See [17] Proposition 3.7. $\square$

We state below the most important result related to the stochastic calculus part of the paper. It generalizes the finite dimensional result contained in [33] Theorem 4.14. The definition of real weak Dirichlet process is recalled in Definition 1.1.

Theorem 4.32. *Let ν_0 be a Banach subspace continuously embedded in H^*. Define $\nu := \nu_0 \hat{\otimes}_{\pi} \mathbb{R}$ and $\chi := \nu_0 \hat{\otimes}_{\pi} \nu_0$. Let $F : [s, T] \times H \to \mathbb{R}$ be a $C^{0,1}$-function with $(t, x) \mapsto \partial_x F(t, x)$ continuous from $[s, T] \times H$ to ν_0. Let $\mathbb{X}(t) = \mathbb{M}(t) + \mathbb{A}(t)$ for $t \in [s, T]$ be an ν-weak-Dirichlet process with finite χ-quadratic variation. Then $Y(t) := F(t, \mathbb{X}(t))$ is a (real) weak Dirichlet process with local martingale part*

$$R(t) = F(s, \mathbb{X}(s)) + \int_s^t \langle \partial_x F(r, \mathbb{X}(r)), \, \mathrm{d}\mathbb{M}(r)\rangle \, .$$

Remark 4.33. *Indeed the condition $\mathbb{X}$ having a χ-quadratic variation may be replaced with the weaker condition $\mathbb{X}$ and M having a $\nu_0 \hat{\otimes}_{\pi} \mathbb{R}$-covariation for any real local martingale M.*

Proof of Theorem 4.32. By definition $\mathbb{X}$ can be written as the sum of a continuous local martingale $\mathbb{M}$ and a ν-martingale-orthogonal process $\mathbb{A}$.

Let N be a real-valued local martingale. Taking into account Lemma 4.19 and that the covariation of two real local martingales defined in (1), coincide with the classical covariation, it is enough to prove that

$$[F(\cdot, \mathbb{X}(\cdot)), N](t) = \int_s^t \left\langle \partial_x F(r, \mathbb{X}(r)), \, \mathrm{d}[\mathbb{M}, N]^{cl}(r) \right\rangle, \qquad \text{for all } t \in [s, T].$$

Let $t \in [s, T]$. We evaluate the ϵ-approximation of the covariation, i.e.

$$\frac{1}{\epsilon} \int_s^t \left(F(r + \epsilon, \mathbb{X}(r + \epsilon)) - F(r, \mathbb{X}(r))\right) \left(N(r + \epsilon) - N(r)\right) \mathrm{d}r.$$

It equals

$$I_1(t, \epsilon) + I_2(t, \epsilon)$$

where

$$I_1(t,\epsilon) = \int_s^t \left(F(r+\epsilon,\mathbb{X}(r+\epsilon)) - F(r+\epsilon,\mathbb{X}(r))\right) \frac{(N(r+\epsilon)-N(r))}{\epsilon}\,\mathrm{d}r$$

and

$$I_2(t,\epsilon) = \int_s^t \left(F(r+\epsilon,\mathbb{X}(r)) - F(r,\mathbb{X}(r))\right) \frac{(N(r+\epsilon)-N(r))}{\epsilon}\,\mathrm{d}r.$$

We prove now that

$$I_1(t,\epsilon) \xrightarrow{\epsilon\to 0} \int_s^t \left\langle \partial_x F(r,\mathbb{X}(r)),\, \mathrm{d}[\mathbb{M},N]^{cl}(r)\right\rangle \tag{45}$$

in probability; in fact

$$I_1(t,\epsilon) = I_{11}(t,\epsilon) + I_{12}(t,\epsilon)$$

where

$$I_{11}(t,\epsilon) := \int_s^t \frac{1}{\epsilon}\left\langle \partial_x F(r,\mathbb{X}(r)), \mathbb{X}(r+\epsilon)-\mathbb{X}(r)\right\rangle (N(r+\epsilon)-N(r))\,\mathrm{d}r,$$

$$\begin{aligned} I_{12}(t,\epsilon) := \int_0^1 \int_s^t \frac{1}{\epsilon}\, &\langle \partial_x F(r+\epsilon, a\mathbb{X}(r)+(1-a)\mathbb{X}(r+\epsilon)) - \partial_x F(r,\mathbb{X}(r)), \\ &\mathbb{X}(r+\epsilon)-\mathbb{X}(r)\rangle\,(N(r+\epsilon)-N(r))\,\mathrm{d}r\,\mathrm{d}a. \end{aligned}$$

Now we apply Proposition 4.31 with $B_1 = H$, $B_2 = \mathbb{R}$, $\mathbb{X} = \mathbb{M}$, $\mathbb{Y} = N$, $\chi = \nu$ so that

$$I_{11}(t,\epsilon) \xrightarrow{\epsilon\to 0} \int_s^t {}_{\nu}\left\langle \partial_x F(r,\mathbb{X}(r)),\, \mathrm{d}\widetilde{[\mathbb{X},N]}(r)\right\rangle_{\nu^*}. \tag{46}$$

Recalling that $\mathbb{X} = \mathbb{M} + \mathbb{A}$, we remark that $[\mathbb{X},N]_\nu$ exists and the ν^*-valued process $\widetilde{[\mathbb{X},N]}_\nu$ equals

$$\widetilde{[\mathbb{M},N]}_\nu + \widetilde{[\mathbb{A},N]}_\nu = \widetilde{[\mathbb{M},N]}_\nu$$

since $\mathbb{A}$ is a ν-martingale orthogonal process. Taking into account the formalism of Proposition 4.14, Remark 4.10 and Proposition 4.20 if $\Phi \in H \equiv H^*$, we have

$$\begin{aligned} {}_{\nu}\left\langle \Phi, \widetilde{[\mathbb{M},N]}_\nu \right\rangle_{\nu^*} &= {}_{\nu_0}\left\langle \Phi, \widetilde{[\mathbb{M},N]}_\nu \right\rangle_{\nu_0^*} = {}_{H^*}\left\langle \Phi, \widetilde{[\mathbb{M},N]} \right\rangle_{H^{**}} \\ &= {}_{H^*}\left\langle \Phi, [\mathbb{M},N]^{cl} \right\rangle_{H}. \end{aligned}$$

Consequently, it is not difficult to show that the right-hand side of (46) gives

$$\int_s^t \left\langle \partial_x F(r,\mathbb{X}(r)),\, \mathrm{d}[\mathbb{M},N]^{cl}(r)\right\rangle.$$

For a fixed $\omega \in \Omega$ we consider the function $\partial_x F$ restricted to $[s,T] \times K$ where K is the (compact) subset of H obtained as convex hull of $\{a\mathbb{X}(r_1) + (1-a)\mathbb{X}(r_2) \,:\, r_1, r_2 \in [s,T]\}$. $\partial_x F$ restricted to $[s,T] \times K$ is uniformly continuous with values in ν_0. As a result, ω-a.s.

$$|I_{12}(t,\epsilon)| \le \int_s^T \delta\left(\partial_x F_{|[s,T]\times K}; \epsilon + \sup_{|r-t|\le\epsilon} |\mathbb{X}(r) - \mathbb{X}(t)|_{\nu_0^*}\right) \times |\mathbb{X}(r+\epsilon) - \mathbb{X}(r)|_{\nu_0^*} \frac{1}{\epsilon} |N(r+\epsilon) - N(r)| \,\mathrm{d}r \quad (47)$$

where, for a uniformly continuous function $g : [s,T] \times H \to \nu_0$, $\delta(g;\epsilon)$ is the modulus of continuity $\delta(g;\epsilon) := \sup_{|s-t|\le\epsilon} |g(s) - g(t)|_{\nu_0}$. In previous formula we have identified H with H^{**} so that $|x|_H \le |x|_{\nu_0^*}, \forall x \in H$. So (47) is lower than

$$\le \delta\left(\partial_x F_{|[s,T]\times K}; \epsilon + \sup_{|s-t|\le\epsilon} |\mathbb{X}(s) - \mathbb{X}(t)|_{\nu_0^*}\right) \times \left(\int_s^T \frac{1}{\epsilon}|N(r+\epsilon) - N(r)|^2 \,\mathrm{d}r \int_s^T \frac{1}{\epsilon}|(\mathbb{X}(r+\epsilon) - \mathbb{X}(r))|^2_{\nu_0^*} \,\mathrm{d}r\right)^{1/2}$$

$$= \delta\left(\partial_x F_{|[s,T]\times K}; \epsilon + \sup_{|s-t|\le\epsilon} |\mathbb{X}(s) - \mathbb{X}(t)|_{\nu_0^*}\right) \times \left(\int_s^T \frac{1}{\epsilon}|N(r+\epsilon) - N(r)|^2 \,\mathrm{d}r \int_s^T \frac{1}{\epsilon}|(\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes^2 |_{\chi^*} \,\mathrm{d}r\right)^{1/2}, \quad (48)$$

where we have used Lemma 2.4. This of course converges to zero since $\mathbb{X}$ [resp. N] is a χ-finite quadratic variation process [resp. a real finite quadratic variation process] and X is also continuous as a ν_0^*-valued process.

To conclude the proof of the proposition we only need to show that $I_2(t,\epsilon) \xrightarrow{\epsilon\to 0} 0$. This is relatively simple since

$$I_2(t,\epsilon) = \frac{1}{\epsilon}\int_s^t \Gamma(u,\epsilon) \,\mathrm{d}N(u) + R(t,\epsilon)$$

where $R(t,\epsilon)$ is a boundary term s.t. $R(t,\epsilon) \xrightarrow{\epsilon\to 0} 0$ ucp and

$$\Gamma(u,\epsilon) = \frac{1}{\epsilon}\int_{(u-\epsilon)_+}^u |F(r+\epsilon, \mathbb{X}(r)) - F(r, \mathbb{X}(r))| \,\mathrm{d}r.$$

Since

$$\int_s^T (\Gamma(u,\epsilon))^2 \,\mathrm{d}[N](u) \to 0$$

in probability, Problem 2.27, chapter 3 of [34] implies that $I_2(\cdot,\epsilon) \to 0$ ucp. The result finally follows. □

5 The case of stochastic PDEs

This section concerns applications of the stochastic calculus via regularization to mild solutions of stochastic partial differential equations (SPDEs).

Assume, as in Subsection 3.1 that H and U are real separable Hilbert spaces, $Q \in \mathcal{L}(U)$, $U_0 := Q^{1/2}(U)$. Assume that $\mathbb{W}_Q = \{\mathbb{W}_Q(t) : t \leq s \leq T\}$ is an U-valued $\mathscr{F}_s^t$-Q-Wiener process (with $\mathbb{W}_Q(s) = 0$, $\mathbb{P}$ a.s.) and denote with $\mathcal{L}_2(U_0, H)$ the Hilbert space of the Hilbert-Schmidt operators from U_0 to H. We adopt the conventions of the mentioned subsection.

We denote with $A\colon D(A) \subseteq H \to H$ the generator of the C_0-semigroup e^{tA} (for $t \geq 0$) on H. The reader may consult for instance [4] Part II, Chapter 1 for basic properties of C_0-semigroups. A^* denotes the adjoint of A, $D(A)$ and $D(A^*)$ are Banach (even Hilbert) spaces when endowed with the graph norm.

Denote with $C(H)$ the space of the continuous functions from H to $\mathbb{R}$, with $C^{1,2}([0,T] \times H)$ the set of all the $C^{1,2}$-Fresh differentiable functions from $[0,T] \times H$ to $\mathbb{R}$ and with $C(H; D(A^*))$ the set of all the continuous functions from H to $D(A^*)$, endowed with the graph norm.

Let b be a predictable process with values in H and σ be a predictable process with values in $\mathcal{L}_2(U_0, H)$ such that

$$\mathbb{P}\left[\int_0^T |b(t)| + \|\sigma(t)\|^2_{\mathcal{L}_2(U_0,H)} \, \mathrm{d}t < +\infty\right] = 1. \tag{49}$$

We introduce the process

$$\mathbb{X}(t) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} b(r) \, \mathrm{d}r + \int_s^t e^{(t-r)A} \sigma(r) \, \mathrm{d}\mathbb{W}_Q(r). \tag{50}$$

A mild solution to an SPDE of type (4) is a particular case as (50). We define

$$\mathbb{Y}(t) := \mathbb{X}(t) - \int_s^t b(r) \, \mathrm{d}r - \int_s^t \sigma(r) \, \mathrm{d}\mathbb{W}_Q(r) - x. \tag{51}$$

Lemma 5.1. *Let b be a predictable process with values in H and σ a predictable process with values in $\mathcal{L}_2(U_0, H)$ such that (49) is satisfied. Let $\mathbb{X}(t)$ be defined by (50) and $\mathbb{Y}$ defined by (51). If $z \in D(A^*)$ we have*

$$\langle \mathbb{Y}(t), z \rangle = \int_s^t \langle \mathbb{X}(r), A^* z \rangle \, \mathrm{d}r. \tag{52}$$

Proof. See [45] Theorem 12. □

We want now to prove that $\mathbb{Y}$ has zero-χ-quadratic variation for a suitable space χ. We will see that the space

$$\bar{\chi} := D(A^*) \hat{\otimes}_\pi D(A^*). \tag{53}$$

does the job. We set $\bar{\nu}_0 := D(A^*)$ which is clearly continuously embedded into H^*.

By Lemma 4.1, $\bar{\chi}$ is a Chi-subspace of $(H \hat{\otimes}_\pi H)^*$.

Proposition 5.2. *The process $\mathbb{Y}$ has zero $\bar{\chi}$-quadratic variation.*

Proof. Observe that, thank to Lemma 3.18 in [14] it will be enough to show that

$$I(\epsilon) := \frac{1}{\epsilon}\int_s^T |(\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)) \otimes^2 |_{\bar{\chi}^*} \, \mathrm{d}r \xrightarrow{\epsilon\to 0} 0, \qquad \text{in probability.}$$

In fact, identifying $\bar{\chi}^*$ with $B(\bar{\nu}_0, \bar{\nu}_0; \mathbb{R})$, we get

$$\begin{aligned} I(\epsilon) &= \frac{1}{\epsilon}\int_s^T \sup_{|\phi|_{\bar{\nu_0}}, |\psi|_{\bar{\nu_0}} \le 1} |\langle (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)), \phi\rangle \, \langle (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)), \psi\rangle| \, \mathrm{d}r \\ &\le \frac{1}{\epsilon}\int_s^T \sup_{|\phi|_{\bar{\nu_0}}, |\psi|_{\bar{\nu_0}} \le 1} \left\{ \left| \int_r^{r+\epsilon} \langle (\mathbb{X}(\xi), A^*\phi\rangle \, \mathrm{d}\xi \right| \left| \int_r^{r+\epsilon} \langle (\mathbb{X}(\xi), A^*\psi\rangle \, \mathrm{d}\xi \right| \right\} \mathrm{d}r \end{aligned} \tag{54}$$

where we have used Lemma 5.1. This is smaller than

$$\frac{1}{\epsilon}\int_s^T \left| \int_r^{r+\epsilon} |\mathbb{X}(\xi)| \, \mathrm{d}\xi \right|^2 \mathrm{d}r \le \epsilon \sup_{\xi\in[s,T]} |\mathbb{X}(\xi)|^2$$

which converges to zero almost surely. □

Corollary 5.3. *The process $\mathbb{X}$ is a $\bar{\chi}$-Dirichlet process. Moreover it is also a $\bar{\chi}$ finite quadratic variation process and a $\bar{\nu}_0 \hat{\otimes}_\pi \mathbb{R}$-weak-Dirichlet process.*

Proof. For $t \in [s,T]$, we have $\mathbb{X}(t) = \mathbb{M}(t) + \mathbb{A}(t)$, where

$$\begin{aligned} \mathbb{M}(t) &= x + \int_s^t \sigma(r) \, \mathrm{d}\mathbb{W}_Q(r) \\ \mathbb{A}(t) &= \mathbb{V}(t) + \mathbb{Y}(t) \\ \mathbb{V}(t) &= \int_s^t b(r) dr. \end{aligned}$$

$\mathbb{M}$ is a local martingale by Proposition 3.5 (i) and $\mathbb{V}$ is a bounded variation process. By Proposition 4.13 and Remark 4.10, we get

$$[\mathbb{V}, \mathbb{V}]_{\bar{\chi}} = [\mathbb{V}, \mathbb{Y}]_{\bar{\chi}} = [\mathbb{Y}, \mathbb{V}]_{\bar{\chi}} = 0.$$

By Proposition 5.2 and the bilinearity of the $\bar{\chi}$-covariation, it yields that $\mathbb{A}$ has a zero $\bar{\chi}$-quadratic variation and so $\mathbb{X}$ is a $\bar{\chi}$-Dirichlet process. The second part of the statement is a consequence of Corollary 4.30. □

The theorem below generalizes for some aspects the Itô formula of [14], i.e. their Theorem 5.2, to the case when the second derivatives do not necessarily belong to the Chi-subspace χ.

Theorem 5.4. *Let $\mathbb{X}$ be a $\bar{\chi}$-finite quadratic variation H-valued process. Let $F\colon [s,T]\times H \to \mathbb{R}$ of class $C^{1,2}$ such that $(t,x)\to \partial_x F(t,x)$ is continuous from $[s,T]\times H$ to $D(A^*)$. Suppose moreover the following.*

(i) There exists a (càdlàg) bounded variation process $C\colon [s,T]\times\Omega \to (H\hat{\otimes}_\pi H)$ such that, for all t in $[s,T]$ and $\phi\in\bar{\chi}$,

$$C(t,\cdot)(\phi) = [\mathbb{X},\mathbb{X}]_{\bar{\chi}}(\phi)(t,\cdot) \qquad a.s.$$

(ii) For every continuous function $\Gamma\colon [s,T]\times H\to D(A^*)$ the integral

$$\int_s^t \left\langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle \tag{55}$$

exists.

Then

$$\begin{aligned} F(t,\mathbb{X}(t)) &= F(s,\mathbb{X}(s)) + \int_s^t \left\langle \partial_r F(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle \\ &+ \frac{1}{2}\int_s^t {}_{(H\hat{\otimes}_\pi H)^*}\left\langle \partial^2_{xx} F(r,\mathbb{X}(r)),\, \mathrm{d}C(s)\right\rangle_{H\hat{\otimes}_\pi H} + \int_s^t \partial_r F(r,\mathbb{X}(r))\,\mathrm{d}r. \end{aligned} \tag{56}$$

Before the proof of the theorem we make some comments.

Remark 5.5. *A consequence of assumption (i) of Theorem 5.4 is the existence of a $\mathbb{P}$-null set O such that, for every $t\in[s,T]$, $\omega\notin O$,*

$$\widetilde{[\mathbb{X},\mathbb{X}]}_{\bar{\chi}}(\phi)(t,\omega) = C(t,\omega)(\phi)$$

for every $\phi\in D(A^*)\hat{\otimes}_\pi D(A^*)$. In other words the $\bar{\chi}$-quadratic variation of $\mathbb{X}$ coincides with C.

Remark 5.6. *The conditions (i) and (ii) of Theorem 5.4 are verified if for instance $\mathbb{X}=\mathbb{M}+\mathbb{V}+\mathbb{S}$, where $\mathbb{M}$ is a local martingale, $\mathbb{V}$ is an H-valued bounded variation process, and $\mathbb{S}$ is a process verifying*

$$\langle \mathbb{S}, h\rangle(t) = \int_s^t \langle \mathbb{Z}(r), A^*h\rangle\,\mathrm{d}r \qquad \text{for all } h\in D(A^*)$$

for some Bochner measurable process $\mathbb{Z}$ with $\int_s^T |\mathbb{Z}(r)|^2\,\mathrm{d}r<+\infty$ a.s.

Indeed, by Lemma 4.16, $\mathbb{M}$ admits a global quadratic quadratic variation which can be identified with $[\mathbb{M},\mathbb{M}]^{cl}$.
On the other hand $\mathbb{A}=\mathbb{V}+\mathbb{S}$ has a zero $\bar{\chi}$-quadratic variation, by Proposition 4.13 and the bilinearity character of the $\bar{\chi}$-covariation. $\mathbb{X}$ is therefore a $\bar{\chi}$-Dirichlet process. By Corollary 4.30 and again the bilinearity of the $\bar{\chi}$-covariation, we obtain that $\mathbb{X}$ has a finite $\bar{\chi}$-quadratic variation. Taking also into account Lemma 4.16 and Remark 4.10, we get $\widetilde{[\mathbb{X},\mathbb{X}]}_{\bar{\chi}}(\Phi)(\cdot) = \langle [\mathbb{M},\mathbb{M}]^{cl},\Phi\rangle$

if $\Phi \in \bar{\chi}$*. Consequently, we can set* $C = [\mathbb{M},\mathbb{M}]^{cl}$ *and condition (i) is verified.*
To prove (ii) consider a continuous function $\Gamma\colon [s,T] \times H \to D(A^*)$*. The integral of* $(\Gamma(r,\mathbb{X}(r)))$ *w.r.t. the semimartingale* $\mathbb{M}+\mathbb{V}$ *where* $\mathbb{M}(t) = x + \int_s^t \sigma(r)\,\mathrm{d}W_Q(r)$ *exists and equals the classical Itô integral, by Proposition 3.6 and Proposition 3.8. Therefore, we only have to prove that*

$$\int_s^t \left\langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{S}(r)\right\rangle, \qquad t \in [s,T]$$

exists. For every $t \in [s,T]$ *the* ϵ*-approximation of such an integral gives, up to a remainder boundary term* $C(\epsilon,t)$ *which converges ucp to zero,*

$$\begin{aligned}\frac{1}{\epsilon}\int_s^t &\langle \Gamma(r,\mathbb{X}(r)), \mathbb{S}(r+\epsilon) - \mathbb{S}(r)\rangle\,\mathrm{d}r \\ &= \frac{1}{\epsilon}\int_s^t \int_r^{r+\epsilon} \langle \mathbb{Z}(u), A^*\Gamma(r,\mathbb{X}(r))\rangle\,\mathrm{d}u\,\mathrm{d}r \\ = \frac{1}{\epsilon}\int_s^t \int_{u-\epsilon}^{u} &\langle \mathbb{Z}(u), A^*\Gamma(r,\mathbb{X}(r))\rangle\,\mathrm{d}r\,\mathrm{d}u \xrightarrow{\epsilon\to 0} \int_s^t \langle \mathbb{Z}(u), A^*\Gamma(u,\mathbb{X}(u))\rangle\,\mathrm{d}u\end{aligned} \tag{57}$$

in probability by classical Lebesgue integration theory. The right-hand side of (57) has obviously a continuous modification so (55) *exists by definition and condition (ii) is fulfilled.*
In particular we have proved that

$$\begin{aligned}\int_s^t \left\langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle &= \int_s^t \langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}\mathbb{M}(r)\rangle + \int_s^t \langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}\mathbb{V}(r)\rangle \\ &+ \int_s^t \langle \mathbb{Z}(u), A^*\Gamma(u,\mathbb{X}(u))\rangle\,\mathrm{d}u\end{aligned}$$

Proof of Theorem 5.4.
Step 1:
Let $\{e_i^*\}_{i\in\mathbb{N}}$ be an orthonormal basis of H^* made of elements of $D(A^*) \subseteq H^*$. For $N \geq 1$ we denote by $P_N\colon H \to H$ the orthogonal projection on the span of the vectors $\{e_1,..,e_N\}$. $P_\infty\colon H \to H$ will simply denote the identity.
We define $F_N\colon H \to \mathbb{R}$ as $F_N(x) := F(P_N(x))$. We have

$$\partial_x F_N(x) = P_N \partial_x F(P_N X(x))$$

and

$$\partial^2_{xx} F_N(x) = (P_N \otimes P_N)\partial^2_{xx} F(P_N X(x))$$

where the last expression has to be understood as

$$\begin{aligned}{}_{(H\hat{\otimes}_\pi H)^*}&\left\langle \partial^2_{xx} F_N(x), h_1 \otimes h_2\right\rangle_{(H\hat{\otimes}_\pi H)} \\ &= {}_{(H\hat{\otimes}_\pi H)^*}\left\langle \partial^2_{xx} F(P_N(x)), (P_N(h_1)) \otimes (P_N(h_2))\right\rangle_{(H\hat{\otimes}_\pi H)}\end{aligned} \tag{58}$$

for all $h_1, h_2 \in H$. $\partial^2_{xx} F_N(x)$ is an element of $(H\hat{\otimes}_\pi H)^*$ but it belongs to $(D(A^*)\hat{\otimes}_\pi D(A^*))$ as well; indeed it can be written as

$$\sum_{i,j=1}^{N} {}_{(H\hat{\otimes}_\pi H)^*}\left\langle \partial^2_{xx} F(P_N(x)), e_i \otimes e_j \right\rangle_{(H\hat{\otimes}_\pi H)} \left(e_i^* \otimes e_j^*\right)$$

and $e_i^* \otimes e_j^*$ are in fact elements of $(D(A^*)\hat{\otimes}_\pi D(A^*))$.

So one can apply the Itô formula proved in [14], Theorem 5.2, and with the help of Assumption (i), we find

$$F_N(t, \mathbb{X}(t)) = F_N(s, \mathbb{X}(s)) + \int_s^t \left\langle \partial_x F_N(r, \mathbb{X}(r)),\, \mathrm{d}^- \mathbb{X}(r) \right\rangle + \frac{1}{2} \int_s^t \left\langle \partial^2_{xx} F_N(r, \mathbb{X}(r)),\, \mathrm{d}C(s) \right\rangle + \int_s^t \partial_r F_N(r, \mathbb{X}(r))\, \mathrm{d}r. \quad (59)$$

Step 2:

We consider, for fixed $\epsilon > 0$, the map

$$\begin{cases} T_\epsilon \colon C([s,T] \times H; D(A^*)) \to L^0(\Omega) \\ T_\epsilon \colon G \mapsto \displaystyle\int_s^t \left\langle G(r, \mathbb{X}(r)), \frac{\mathbb{X}(r+\epsilon) - \mathbb{X}(r)}{\epsilon} \right\rangle \mathrm{d}r \end{cases}$$

where $C([s,T] \times H; D(A^*))$ is equipped with the uniform convergence on compact sets and $L^0(\Omega)$ (the set of the real random variables) with the convergence in probability. Assumption (ii) implies that $\lim_{\epsilon \to 0} T_\epsilon G$ exists for every G. By Banach-Steinhaus for F-spaces (see Theorem 17, Chapter II in [20]) it follows that the map

$$\begin{cases} C([s,T] \times H; D(A^*)) \to L^0(\Omega) \\ G \mapsto \displaystyle\int_s^t \left\langle G(r, \mathbb{X}(r)),\, \mathrm{d}^- \mathbb{X}(r) \right\rangle \end{cases}$$

is linear and continuous.

Step 3:

If $K \subseteq H$ is a compact set then the set

$$P(K) := \{P_N(y) \ : \ y \in K,\ N \in \mathbb{N} \cup +\infty\}$$

is compact as well. Indeed, consider $\{P_{N_l}(y_l)\}_{l \geq 1}$ be a sequence in $P(K)$. We look for a subsequence convergence to an element of $P(K)$.

Since K is compact we can assume, without restriction of generality, that y_l converges, for $l \to +\infty$, to some $y \in K$. If $\{N_l\}$ assumes only a finite number of values then (passing if necessary to a subsequence) $N_l \equiv \bar{N}$ for some $\bar{N} \in \mathbb{N} \cup +\infty$ and then $P_{N_l}(y_l) \xrightarrow{l \to +\infty} P_{\bar{N}}(y)$. Otherwise we can assume (passing if necessary to a subsequence) that $N_l \xrightarrow{l\to+\infty} +\infty$ and then it is not difficult to prove that $P_{N_l}(y_l) \xrightarrow{l\to+\infty} y$, which belongs to $P(K)$ since $y = P_\infty y$.

In particular, being $\partial_x F$ continuous,

$$\mathcal{D} := \{\partial_x F(P_N(x)) \ : \ x \in K, \ N \in \mathbb{N} \cup \{+\infty\}\}$$

is compact in $D(A^*)$. Since the sequence of maps $\{P_N\}$ is uniformly continuous it follows that

$$\sup_{x\in\mathcal{D}} |(P_N - I)(x)| \xrightarrow{N\to\infty} 0. \tag{60}$$

Step 4:

We show now that

$$\lim_{N\to\infty} \int_s^t \left\langle \partial_x F_N(r, \mathbb{X}(r)),\, \mathrm{d}^- \mathbb{X}(r) \right\rangle = \int_s^t \left\langle \partial_x F(r, \mathbb{X}(r)),\, \mathrm{d}^- \mathbb{X}(r) \right\rangle. \tag{61}$$

holds in probability for every $t \in [s, T]$.

Let K be a compact subset of H. In fact

$$\sup_{x\in K} |\partial_x F(P_N x) - \partial_x F(x)| \xrightarrow{N\to\infty} 0,$$

since $\partial_x F$ is continuous. On the other hand

$$\sup_{x\in K} |(P_N - I)(\partial_x F(P_N x))| \xrightarrow{N\to\infty} 0,$$

because of (60). Consequently

$$\partial_x F_N \to \partial_x F \tag{62}$$

uniformly on each compact, with values in H. This yields that ω-a.s.

$$\partial_x F_N(r, \mathbb{X}(r)) \to \partial_x F(r, \mathbb{X}(r))$$

uniformly on each compact. By step 2, then (61) follows.

Step 5:

Finally, we prove that

$$\lim_{N\to\infty} \frac{1}{2} \int_s^t \left\langle \partial^2_{xx} F_N(r, \mathbb{X}(r)),\, \mathrm{d}C(s) \right\rangle = \frac{1}{2} \int_s^t \left\langle \partial^2_{xx} F(r, \mathbb{X}(r)),\, \mathrm{d}C(s) \right\rangle. \tag{63}$$

For a fixed $\omega \in \Omega$ we define $K(\omega)$ the compact set as

$$K(\omega) := \{\mathbb{X}(t)(\omega) \ : \ t \in [s, T]\}.$$

We write

$$\left|\int_s^t \left\langle \partial_{xx}^2 F_N(r,\mathbb{X}(x)) - \partial_{xx}^2 F(r,\mathbb{X}(x)),\, \mathrm{d}C(r)\right\rangle\right|(\omega)$$
$$\leq \sup_{\substack{y\in K(w)\\ t\in[s,T]}} \left\|\partial_{xx}^2 F_N(t,y) - \partial_{xx}^2 F(t,y)\right\|_{(H\hat{\otimes}_\pi H)^*} \int_s^T \mathrm{d}|C(r)|(\omega). \quad (64)$$

Using arguments similar to those used in proving (62) one can see that

$$\partial_{xx}^2 F_N \xrightarrow{N\to\infty} \partial_{xx}^2 F$$

uniformly on each compact. Consequently

$$\sup_{r\in[s,T]} |(\partial_{xx}^2 F_N - \partial_{xx}^2 F)(r,\mathbb{X}(r))|_{(H\hat{\otimes}_\pi H)^*} \xrightarrow{N\to\infty} 0.$$

Since C has bounded variation, finally (63) holds.

Step 6:

Since F_N [resp. $\partial_r F_N$] converges uniformly on each compact to F [resp. $\partial_r F$], when $N\to\infty$, then

$$\int_s^t \partial_r F_N(r,\mathbb{X}(r))\,\mathrm{d}r \xrightarrow{N\to\infty} \int_s^t \partial_r F(r,\mathbb{X}(r))\,\mathrm{d}r.$$

Taking the limit when $N\to\infty$ in (59) finally provides (56). $\square$

Next result can be considered a Itô formula for *mild type processes*, essentially coming out from mild solutions of SPDEs. An interesting contribution in this direction, but in a different spirit appears in [9].

Corollary 5.7. *Assume that b is a predictable process with values in H and σ is a predictable process with values in $\mathcal{L}_2(U_0,H)$ satisfying (49). Define $\mathbb{X}$ as in (50). Let x be an element of H. Assume that $f\in C^{1,2}(H)$ with $\partial_x f\in C(H,D(A^*))$. Then*

$$f(t,\mathbb{X}(t)) = f(s,x) + \int_s^t \partial_s f(r,\mathbb{X}(r))\,\mathrm{d}r$$
$$+\int_s^t \langle A^*\partial_x f(r,\mathbb{X}(r)),\mathbb{X}(r)\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x f(r,\mathbb{X}(r)), b(r)\rangle\,\mathrm{d}r$$
$$+\frac{1}{2}\int_s^t Tr\left[\left(\sigma(r)Q^{1/2}\right)\left(\sigma(r)Q^{1/2}\right)^*\partial_{xx}^2 f(r,\mathbb{X}(r))\right]\,\mathrm{d}r$$
$$+\int_s^t \langle \partial_x f(r,\mathbb{X}(r)), \sigma(r)\,\mathrm{d}\mathbb{W}_Q(r)\rangle. \qquad \mathbb{P}-a.s., \quad (65)$$

Remark 5.8. *We remark that in* (65)*, the partial derivative* $\partial^2_{xx} f(r,x)$ *for any* $r \in [s,T]$ *and* $x \in H$ *stands in fact for its associated linear bounded operator in the sense of* (9)*. From now on we will make this natural identification.*

Proof. It is a consequence of Theorem 5.4 taking into account Remark 5.6: we have $\mathbb{M}(t) = x + \int_s^t \sigma(r)\, \mathrm{d}\mathbb{W}_Q(r), t \in [0,T]$, $\mathbb{V}(t) = \int_s^t b(r)dr$, $\mathbb{S} = \mathbb{Y}$ with $\mathbb{Z}(r) = \mathbb{X}(r)$. According to that Remark, in Theorem 5.4 we set $C = [\mathbb{M}, \mathbb{M}]^{cl}$. We also use the chain rule for Itô's integrals in Hilbert spaces, see the considerations before Proposition 3.5, together with Lemma 4.18. The fourth integral in the right-hand side of (65) appears from the second integral in (56) together with Proposition 2.6 and again Lemma 4.18. □

6 The optimal control problem

In this section we illustrate the utility of the tools of stochastic calculus via regularization in the study of optimal control problems driven by SPDEs. We will prove a decomposition result for the strong solutions of the Hamilton-Jacobi-Bellman equation related to the optimal control problem and we use that decomposition to derive a verification theorem.

U, U_0, $\mathbb{W}_Q = \{\mathbb{W}_Q(t) : t \leq s \leq T\}$, $\mathcal{L}_2(U_0, H)$ and $A \colon D(A) \subseteq H \to H$ were defined as in Section 5.

6.1 The setting of the problem

We consider a Polish space Λ and we formulate the following standard assumption that will ensure existence and uniqueness for the solution of the state equation.

Hypothesis 6.1. $b \colon [0,T] \times H \times \Lambda \to H$ *is a continuous function and satisfies, for some* $C > 0$*,*

$$|b(s,x,a) - b(s,y,a)| \leq C|x-y|,$$
$$|b(s,x,a)| \leq C(1+|x|)$$

for all $x, y \in H$*,* $s \in [0,T]$*,* $a \in \Lambda$*.* $\sigma \colon [0,T] \times H \to \mathcal{L}_2(U_0,H)$ *is continuous and, for some* $C > 0$*,*

$$\|\sigma(s,x) - \sigma(s,y)\|_{\mathcal{L}_2(U_0,H)} \leq C|x-y|,$$
$$\|\sigma(s,x)\|_{\mathcal{L}_2(U_0,H)} \leq C(1+|x|)$$

for all $x, y \in H$*,* $s \in [0,T]$*.*

Let us fix for the moment a predictable process $a = a(\cdot) : [s,T] \times \Omega \to \Lambda$, where the dot refers to the time variable. a will indicate in the sequel an admissible control in a sense to be specified.

We consider the state equation

$$\begin{cases} \mathrm{d}\mathbb{X}(t) = (A\mathbb{X}(t) + b(t, \mathbb{X}(t), a(t)))\, \mathrm{d}t + \sigma(t, \mathbb{X}(t))\, \mathrm{d}\mathbb{W}_Q(t) \\ \mathbb{X}(s) = x. \end{cases} \tag{66}$$

The solution of (66) is understood in the mild sense, so an H-valued adapted strongly measurable process $\mathbb{X}(\cdot)$ is a solution if

$$\mathbb{P}\left(\int_s^T |\mathbb{X}(r)| + |b(r,\mathbb{X}(r),a(r))| + \|\sigma(r,\mathbb{X}(r))\|^2_{\mathcal{L}_2(U_0,H)}\,\mathrm{d}r < +\infty\right) = 1$$

and

$$\mathbb{X}(t) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} b(r,\mathbb{X}(r),a(r))\,\mathrm{d}r + \int_s^t e^{(t-r)A}\sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r) \tag{67}$$

$\mathbb{P}$-a.s. for every $t \in [s,T]$,

Thanks to Theorem 3.3 of [28],given Hypothesis 6.1, there exists a unique solution $\mathbb{X}(\cdot;s,x,a(\cdot))$ of (66), which admits a continuous modification. So for us $\mathbb{X}$ can always be considered as a continuous process.

Setting $b := b(\cdot,\mathbb{X}(\cdot;s,x,a(\cdot)),a(\cdot))$, $\sigma := \sigma(\cdot,\mathbb{X}(\cdot;s,x,a(\cdot)))$ then $\mathbb{X}$ fulfills (49) and it is of type (50). The following corollary is just a particular case of Corollary 5.7, which is reformulated here for the reader convenience.

Corollary 6.2. *Assume that b and σ satisfy the Hypothesis 6.1. Let $a : [s,T] \times \Omega \to \Lambda$ be predictable and $x \in H$. Let $\mathbb{X}(\cdot)$ denote $\mathbb{X}(\cdot;s,x,a(\cdot))$. Assume that $f \in C^{1,2}(H)$ with $\partial_x f \in C(H, D(A^*))$.Then*

$$\begin{aligned} f(t,\mathbb{X}(t)) &= f(s,x) + \int_s^t \partial_r f(r,\mathbb{X}(r))\,\mathrm{d}r \\ &+ \int_s^t \langle A^*\partial_x f(r,\mathbb{X}(r)), \mathbb{X}(r)\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x f(r,\mathbb{X}(r)), b(r,\mathbb{X}(r),a(r))\rangle\,\mathrm{d}r \\ &+ \frac{1}{2}\int_s^t Tr\left[\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)^* \partial^2_{xx} f(r,\mathbb{X}(r))\right]\mathrm{d}r \\ &+ \int_s^t \langle \partial_x f(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)\rangle. \qquad \mathbb{P}-a.s. \end{aligned} \tag{68}$$

Let $l\colon [0,T]\times H \times \Lambda \to \mathbb{R}$ be a measurable function and $g\colon H \to \mathbb{R}$ a continuous function. l is called the running cost and g the terminal cost.
We introduce now the class $\mathcal{U}_s$ of admissible controls. It is constituted by $a : [s,T]\times\Omega \to \Lambda$ such that for ω a.s. $(r,\omega) \mapsto l(r,\mathbb{X}(r,s,x,a(\cdot)),a(r)) + g(\mathbb{X}(T,s,x,a(\cdot)))$ is $\mathrm{d}r \otimes \mathrm{d}\mathbb{P}$- is quasi-integrable. This means that, either its positive or negative part are integrable.
We want to determine a *minimum* over all $a(\cdot) \in \mathcal{U}_s$, of the cost functional

$$J(s,x;a(\cdot)) = \mathbb{E}\left[\int_s^T l(r,\mathbb{X}(r;s,x,a(\cdot)),a(r))\,\mathrm{d}r + g(\mathbb{X}(T;s,x,a(\cdot)))\right]. \tag{69}$$

The value function of this problem is defined as

$$V(s,x) = \inf_{a(\cdot)\in\mathcal{U}_s} J(s,x;a(\cdot)). \tag{70}$$

Definition 6.3. *If $a^*(\cdot) \in \mathcal{U}_s$ minimizes (69) among the controls in $\mathcal{U}_s$, i.e. if $J(s,x;a^*(\cdot)) = V(s,x)$, we say that the control $a^*(\cdot)$ is* optimal *at (s,x). In this case the pair $(a^*(\cdot), \mathbb{X}^*(\cdot))$, where $\mathbb{X}^*(\cdot) := \mathbb{X}(\cdot; s, x, a^*(\cdot))$, is called an* optimal couple *(or optimal pair) at (s,x).*

6.2 The HJB equation

The HJB equation associated to the minimization problem above is

$$\begin{cases} \partial_s v + \langle A^* \partial_x v, x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial^2_{xx} v\right] \\ \qquad\qquad + \inf_{a\in\Lambda} \left\{ \langle \partial_x v, b(s,x,a)\rangle + l(s,x,a) \right\} = 0, \\ v(T,x) = g(x). \end{cases} \tag{71}$$

In the above equation $\partial_x v$ [resp. $\partial^2_{xx} v$] is the [second] Fréchet derivatives of v w.r.t. the x variable; it is identified with elements of H [resp. with a symmetric bounded operator on H]. $\partial_s v$ is the derivative w.r.t. the time variable. For $(t,x,p,a) \in [0,T] \times H \times H \times \Lambda$, the term

$$F_{CV}(t,x,p,a) := \langle p, b(t,x,a)\rangle + l(t,x,a) \tag{72}$$

is called the *current value Hamiltonian* of the system and its infimum over $a \in \Lambda$

$$F(t,x,p) := \inf_{a\in\Lambda} \{\langle p, b(t,x,a)\rangle + l(t,x,a)\} \tag{73}$$

is called the *Hamiltonian.* Using this notation the HJB equation (71) can be rewritten as

$$\begin{cases} \partial_s v + \langle A^* \partial_x v, x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial^2_x v\right] + F(s,x,\partial_x v) = 0, \\ v(T,x) = g(x). \end{cases} \tag{74}$$

Hypothesis 6.4. *The value function is always finite and the Hamiltonian $F(t,x,p)$ is well-defined and finite for all $(t,x,p) \in [0,T] \times H \times H$. Moreover it is supposed to be continuous.*

We introduce the operator $\mathscr{L}_0$ on $C([0,T] \times H)$ defined as

$$\begin{cases} D(\mathscr{L}_0) := \left\{\varphi \in C^{1,2}([0,T] \times H) \ : \ \partial_x \varphi \in C([0,T] \times H; D(A^*))\right\} \\ \mathscr{L}_0(\varphi)(s,x) := \partial_s \varphi(s,x) \\ \qquad\qquad + \langle A^* \partial_x \varphi(s,x), x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial^2_{xx}\varphi(s,x)\right]. \end{cases} \tag{75}$$

The HJB equation (74) can be rewritten as

$$\begin{cases} \mathscr{L}_0(v)(s,x) + F(s,x,\partial_x v(s,x)) = 0 \\ v(T,x) = g(x). \end{cases}$$

6.3 Strict and strong solutions

For some $h \in C([0,T] \times H)$ and $g \in C(H)$ we consider the following Cauchy problem

$$\begin{cases} \mathscr{L}_0(v)(s,x) = h(s,x) \\ v(T,x) = g(x). \end{cases} \tag{76}$$

Definition 6.5. *We say that $v \in C([0,T] \times H)$ is a* strict solution *of (76) if $v \in D(\mathscr{L}_0)$ and (76) is satisfied.*

Definition 6.6. *Given $h \in C([0,T] \times H)$ and $g \in C(H)$ we say that $v \in C^{0,1}([0,T] \times H)$ with $\partial_x v \in C([0,T] \times H; D(A^*))$ is a* strong solution *of (76) if there exist three sequences: $\{v_n\} \subseteq D(\mathscr{L}_0)$, $\{h_n\} \subseteq C([0,T] \times H)$ and $\{g_n\} \subseteq C(H)$ fulfilling the following.*

(i) For any $n \in \mathbb{N}$, v_n is a strict solution of the problem

$$\begin{cases} \mathscr{L}_0(v_n)(s,x) = h_n(s,x) \\ v_n(T,x) = g_n(x). \end{cases} \tag{77}$$

(ii) The following convergences hold:

$$\begin{cases} v_n \to v & \text{in } C([0,T] \times H) \\ h_n \to h & \text{in } C([0,T] \times H) \\ g_n \to g & \text{in } C(H) \end{cases}$$

6.4 Decomposition for solutions of the HJB equation

Theorem 6.7. *Consider $h \in C([0,T] \times H)$ and $g \in C(H)$. Assume that Hypothesis 6.1 is satisfied. Suppose that $v \in C^{0,1}([0,T] \times H)$ with $\partial_x v \in C(H; D(A^*))$ is a strong solution of (76). Let $\mathbb{X}(\cdot) := \mathbb{X}(\cdot; t, x, a(\cdot))$ be the solution of (66) starting at time s at some $x \in H$ and driven by some control $a(\cdot) \in \mathcal{U}_s$. Assume that b is of the form*

$$b(t,x,a) = b_g(t,x,a) + b_i(t,x,a) \tag{78}$$

where b_g and b_i satisfy the following conditions.

(i) $\sigma(t, \mathbb{X}(t))^{-1} b_g(t, \mathbb{X}(t), a(t))$ is bounded (being $\sigma(t, \mathbb{X}(t))^{-1}$ the pseudo-inverse of σ);

(ii) b_i satisfies

$$\lim_{n\to\infty} \int_s^\cdot \langle \partial_x v_n(r, \mathbb{X}(r)) - \partial_x v(r, \mathbb{X}(r)), b_i(r, \mathbb{X}(r), a(r)) \rangle \, \mathrm{d}r = 0 \qquad \textit{ucp}. \tag{79}$$

Then

$$v(t,\mathbb{X}(t)) - v(s,\mathbb{X}(s)) = v(t,\mathbb{X}(t)) - v(s,x) = \int_s^t h(r,\mathbb{X}(r))\,\mathrm{d}r + \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), b(r,\mathbb{X}(r),a(r))\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)\rangle\,. \quad (80)$$

Example 6.8. *Hypothesis (i) and (ii) of Theorem 6.7 are satisfied if the approximating sequence v_n converges to v in a stronger way. For example if v is a strong solution of the HJB in the sense of Definition 6.6 and, moreover, $\partial_x v_n$ converges to $\partial_x v$ in $C([0,T]\times H)$, then the convergence in point (ii) can be easily checked. The convergence of the spatial partial derivative is the typical assumption required in the standard strong solutions literature.*

Example 6.9. *The assumptions of Theorem 6.7 are fulfilled if the following assumption is satisfied.*

$$\sigma(t,\mathbb{X}(t))^{-1}b(t,\mathbb{X}(t),a(t)) \text{ is bounded}$$

for all choice of admissible controls $a(\cdot)$. In this case we apply Theorem 6.7 with $b_i = 0$ and $b = b_g$.

Proof of Theorem 6.7. We denote by v_n the sequence of smooth solutions of the approximating problems prescribed by Definition 6.6, which converges to v. Thanks to Corollary 6.2, every v_n verifies,

$$v_n(t,\mathbb{X}(t)) = v_n(s,x) + \int_s^t \partial_r v_n(r,\mathbb{X}(r))\,\mathrm{d}r + \int_s^t \langle A^*\partial_x v_n(r,\mathbb{X}(r)),\mathbb{X}(r)\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b(r,\mathbb{X}(r),a(r))\rangle\,\mathrm{d}r + \frac{1}{2}\int_s^t \mathrm{Tr}\left[\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)^* \partial^2_{xx} v_n(r,\mathbb{X}(r))\right]\mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)\rangle\,. \qquad \mathbb{P}-a.s. \quad (81)$$

Using Girsanov's Theorem (see [10] Theorem 10.14) we can observe that

$$\beta_Q(t) := W_Q(t) + \int_s^t \sigma(r,\mathbb{X}(r))^{-1} b_g(r,\mathbb{X}(r),a(r))\,\mathrm{d}r$$

is a Q-Wiener process w.r.t. a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$. We can rewrite

(81) as

$$v_n(t,\mathbb{X}(t)) = v_n(s,x) + \int_s^t \partial_r v_n(r,\mathbb{X}(r))\,\mathrm{d}r$$
$$+ \int_s^t \langle A^*\partial_x v_n(r,\mathbb{X}(r)), \mathbb{X}(r)\rangle\ \mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle\ \mathrm{d}r$$
$$+ \frac{1}{2}\int_s^t \mathrm{Tr}\left[\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)^* \partial^2_{xx} v_n(r,\mathbb{X}(r))\right]\ \mathrm{d}r$$
$$+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\beta_Q(r)\rangle\,. \qquad \mathbb{P}-a.s. \tag{82}$$

Since v_n is a strict solution of (77), the expression above gives

$$v_n(t,\mathbb{X}(t)) = v_n(s,x) + \int_s^t h_n(r,\mathbb{X}(r))\,\mathrm{d}r$$
$$+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle\ \mathrm{d}r$$
$$+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\beta_Q(r)\rangle\,. \tag{83}$$

Since we wish to take the limit for $n \to \infty$, we define

$$M_n(t) := v_n(t,\mathbb{X}(t)) - v_n(s,x) - \int_s^t h_n(r,\mathbb{X}(r))\,\mathrm{d}r$$
$$- \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle\ \mathrm{d}r. \tag{84}$$

$\{M_n\}_{n\in\mathbb{N}}$ is a sequence of real $\mathbb{Q}$-local martingales converging ucp, thanks to the definition of strong solution and Hypothesis (79), to

$$M(t) := v(t,\mathbb{X}(t)) - v(s,x) - \int_s^t h(r,\mathbb{X}(r))\,\mathrm{d}r$$
$$- \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle\ \mathrm{d}r. \tag{85}$$

Since the space of real continuous local martingales equipped with the ucp topology is closed (see e.g. Proposition 4.4 of [33]) then M is a continuous $\mathbb{Q}$-local martingale.

We have now gathered all the ingredients to conclude the proof. As in Section 5, we set $\bar{\nu}_0 = D(A^*)$, $\nu = \bar{\nu}_0\hat{\otimes}_\pi\mathbb{R}$, $\bar{\chi} = \bar{\nu}_0\hat{\otimes}_\pi\bar{\nu}_0$.

Corollary 5.3 ensures that $\mathbb{X}(\cdot)$ is a ν-weak Dirichlet process with finite $\bar{\chi}$-quadratic variation with decomposition $\mathbb{M}+\mathbb{A}$ where $\mathbb{M}$ is the local martingale (with respect to $\mathbb{P}$) defined by $\mathbb{M}(t) = x + \int_s^t \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r)$ and $\mathbb{A}$ is a ν-martingale-orthogonal process. Now

$$\mathbb{X}(t) = \tilde{\mathbb{M}}(t) + \mathbb{V}(t) + \mathbb{A}(t), t \in [0, T],$$

where $\tilde{\mathbb{M}}(t) = x + \int_s^t \sigma(r, \mathbb{X}(r))\, \mathrm{d}\beta_Q(r)$ and $\mathbb{V}(t) = -\int_s^t b_g(r, \mathbb{X}(r), a(r)) dr$, $t \in [0, T]$ is a bounded variation process. So by Proposition 3.5 (i), $\tilde{\mathbb{M}}$ is a $\mathbb{Q}$-local martingale and by Proposition 4.25 1., $\mathbb{V}$ is a $\mathbb{Q} - \nu$-martingale orthogonal process. By Remark 4.24 $\mathbb{V} + \mathbb{A}$ is a $\mathbb{Q} - \nu$-martingale orthogonal process and $\mathbb{X}$ is a ν-weak Dirichlet process with local martingale part $\tilde{M}$, with respect to $\mathbb{Q}$. Still under $\mathbb{Q}$, Theorem 4.32 ensures that the process $v(\cdot, \mathbb{X}(\cdot))$ is a real weak Dirichlet process whose local martingale part being equal to

$$N(t) = \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\beta_Q(r) \rangle .$$

On the other hand, with respect to $\mathbb{Q}$, (85) implies that

$$v(t, \mathbb{X}(t)) = \left[v(s, x) + \int_s^t h(r, \mathbb{X}(r))\, \mathrm{d}r \right. \\ \left. + \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), b_i(r, \mathbb{X}(r), a(r)) \rangle\, \mathrm{d}r \right] + N(t). \quad (86)$$

is a decomposition of $v(\cdot, \mathbb{X}(\cdot))$ as $\mathbb{Q}$- semimartingale, which is also in particular, a $\mathbb{Q}$-weak Dirichlet process. By Proposition 1.2 such a decomposition is unique and so

$$M(t) = N(t) = \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\beta_Q(r) \rangle \\ = \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), b_g(r, \mathbb{X}(r), a(r))\, \mathrm{d}r \rangle \\ + \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle \quad (87)$$

This concludes the proof of Theorem 6.7. □

6.5 Verification Theorem

Theorem 6.10. *Assume that Hypotheses 6.1 and 6.4 are satisfied. Let* $v \in C^{0,1}([0, T] \times H)$ *with* $\partial_x v \in C(H; D(A^*))$ *be a strong solution of (71). Assume that for all initial data* $(s, x) \in [0, T] \times H$ *and every control* $a(\cdot) \in \mathcal{U}_s$ b *can be written as* $b(t, x, a) = b_g(t, x, a) + b_i(t, x, a)$ *with* b_i *and* b_g *satisfying hypotheses (i) and (ii) of Theorem 6.7. Let* v *such that* $\partial_x v$ *has most polynomial growth in the* x *variable. Then*

(i) $v \leq V$ *on* $[0, T] \times H$.

(ii) *Suppose that, for some $s \in [0, T)$, there exists a predictable process $a(\cdot) = a^*(\cdot) \in \mathcal{U}_s$ such that, denoting $\mathbb{X}(\cdot; s, x, a^*(\cdot))$ simply by $\mathbb{X}^*(\cdot)$, we have*

$$F\left(t, \mathbb{X}^*(t), \partial_x v\left(t, \mathbb{X}^*(t)\right)\right) = F_{CV}\left(t, \mathbb{X}^*(t), \partial_x v\left(t, \mathbb{X}^*(t)\right); a^*(t)\right), \quad (88)$$

$dt \otimes \mathrm{d}\mathbb{P}$ a.e. Then $a^*(\cdot)$ is optimal at (s, x); moreover $v(s, x) = V(s, x)$.

Proof. We choose a control $a(\cdot) \in \mathcal{U}_s$ and call $\mathbb{X}$ the related trajectory. Thanks to Theorem 6.7 we can write

$$\begin{aligned} g(\mathbb{X}(T)) = v(T, \mathbb{X}(T)) = v(s, x) - \int_s^T F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)))\, \mathrm{d}r \\ + \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), b(r, \mathbb{X}(r), a(r)) \rangle \, \mathrm{d}r \\ + \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle . \quad (89) \end{aligned}$$

Since both sides of (89) are a. s. finite, we can add $\int_s^T l(r, \mathbb{X}(r), a(r))\, \mathrm{d}r$ to them, obtaining

$$\begin{aligned} g(\mathbb{X}(T)) + \int_s^T l(r, \mathbb{X}(r), a(r))\, \mathrm{d}r = v(s, x) \\ + \int_s^T \left(-F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) + F_{CV}(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)))\right) \mathrm{d}r \\ + \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle . \quad (90) \end{aligned}$$

Observe now that, by definition of F and F_{CV} we know that

$$-F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) + F_{CV}(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)))$$

is always positive. So its expectation always exists even if it could be $+\infty$, but not $-\infty$ on an event of positive probability. This shows a posteriori that $\int_s^T l(r, \mathbb{X}(r), a(r))\, \mathrm{d}r$ cannot be $-\infty$ on a set of positive probability.
By [10] Theorem 7.4, all the momenta of $\sup_{r \in [s,T]} |\mathbb{X}(r)|$ are finite. On the other hand, σ is Lipschitz-continuous, $v(s, x)$ is deterministic and, since $\partial_x v$ has polynomial growth, then

$$\mathbb{E} \int_s^T \left\langle \partial_x v(r, \mathbb{X}(r)), \left(\sigma(r, \mathbb{X}(r)) Q^{1/2}\right) \left(\sigma(r, \mathbb{X}(r)) Q^{1/2}\right)^* \partial_x v(r, \mathbb{X}(r)) \right\rangle \mathrm{d}r.$$

is finite. Consequently, by Proposition 3.5 (v)

$$\int_s^{\cdot} \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle$$

is a true martingale vanishing at s. Consequently, its expectation is zero.
So the expectation of the right-hand side of (90) exists even if it could be $+\infty$;

consequently the same holds for the left-hand side.
By definition of J, we have

$$J(s,x,a(\cdot)) = \mathbb{E}\Big[g(\mathbb{X}(T)) + \int_s^T l(r,\mathbb{X}(r),a(r))\,\mathrm{d}r\Big] = v(s,x)$$
$$+\,\mathbb{E}\int_s^T \Big(-F(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r))) + F_{CV}(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r)),a(r))\Big)\,\mathrm{d}r. \tag{91}$$

So minimizing $J(s,x,a(\cdot))$ over $a(\cdot)$ is equivalent to minimize

$$\mathbb{E}\int_s^T \Big(-F(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r))) + F_{CV}(r,\mathbb{X}(r),\partial_x v(r,\mathbb{X}(r)),a(r))\Big)\,\mathrm{d}r. \tag{92}$$

As mentioned above, the integrand of such an expression is always nonnegative and then a lower bound for (92) is 0. If conditions of point (ii) are satisfied such a bound is attained by the control $a^*(\cdot)$, that in this way is proved to be optimal.

Concerning the proof of (i), since the integrand in (92) is nonnegative, (91) gives

$$J(s,x,a(\cdot)) \geq v(s,x).$$

Taking the inf over $a(\cdot)$ we get $V(s,x) \geq v(s,x)$, which concludes the proof. □

Remark 6.11. *1. The first part of the proof does not make use that a belongs to $\mathcal{U}_s$, but only that $r \mapsto l(r,\mathbb{X}(\cdot,s,x,a(\cdot)),a(\cdot))$ is a.s. strictly bigger then $-\infty$. Under that only assumption, $a(\cdot)$ is forced to be admissible, i.e. to belong to $\mathcal{U}_s$.*

2. Let v be a strong solution of HJB equation. Observe that the condition (88) can be rewritten as

$$a^*(t) \in \arg\min_{a\in\Lambda}\Big[F_{CV}\left(t,\mathbb{X}^*(t),\partial_x v\left(t,\mathbb{X}^*(t)\right);a\right)\Big].$$

Suppose that for any $(t,y) \in [0,T]\times H$, $\phi(t,y) = \arg\min_{a\in\Lambda}\big(F_{CV}\left(t,y,\partial_x v(t,y);a\right)\big)$ is measurable and single-valued. Suppose moreover that

$$\int_s^T l(r,\mathbb{X}^*(r),a^*(r))dr > -\infty \text{ a.s.} \tag{93}$$

Suppose that the equation

$$\begin{cases} \mathrm{d}\mathbb{X}(t) = (A\mathbb{X}(t) + b(t,\mathbb{X}(t),\phi(t,\mathbb{X}(t))\;\mathrm{d}t + \sigma(t,\mathbb{X}(t))\,\mathrm{d}\mathbb{W}_Q(t) \\ \mathbb{X}(s) = x. \end{cases} \tag{94}$$

admits a unique mild solution $\mathbb{X}^*$. Now (93) *and Remark 6.11 imply that $a(\cdot)^*$ is admissible. Then $\mathbb{X}^*$ is the optimal trajectory of the state variable* *and* $a^*(t) = \phi(t, \mathbb{X}^*(t)), t \in [0,T]$ *is the optimal control. The function* ϕ *is the* optimal feedback *of the system since it gives the optimal control as a function of the state.*

Remark 6.12. *Observe that, using exactly the same arguments we used in this section one could treat the (slightly) more general case in which* b *has the form:*

$$b(t,x,a) = b_0(t,x) + b_g(t,x,a) + b_i(t,x,a).$$

where b_g *and* b_i *satisfy condition of Theorem 6.7 and* $b_0 : [0,T] \times H \to H$ *is continuous. In this case the addendum* b_0 *can be included in the expression of* $\mathscr{L}_0$ *that becomes the following*

$$\begin{cases} D(\mathscr{L}_0^{b_0}) := \left\{\varphi \in C^{1,2}([0,T]\times H) \ : \ \partial_x \varphi \in C([0,T]\times H; D(A^*))\right\} \\ \mathscr{L}_0^{b_0}(\varphi) := \partial_s \varphi + \langle A^* \partial_x \varphi, x\rangle + \langle \partial_x \varphi, b_0(t,x)\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial^2_{xx}\varphi\right]. \end{cases} \tag{95}$$

Consequently in the definition of regular solution the operator $\mathscr{L}_0^{b_0}$ *appears instead* $\mathscr{L}_0$.

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