

GKW representation theorem and linear BSDEs under restricted information. An application to risk-minimization.

Claudia Ceci* Alessandra Cretarola† Francesco Russo‡

Abstract

In this paper we provide Galtchouk-Kunita-Watanabe representation results in the case where there are restrictions on the available information. This allows to prove existence and uniqueness for linear backward stochastic differential equations driven by a general càdlàg martingale under partial information. Furthermore, we discuss an application to risk-minimization where we extend the results of Föllmer and Sondermann (1986) to the partial information framework and we show how our result fits in the approach of Schweizer (1994).

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1 Introduction

This paper provides two main contributions. First, we prove Galtchouk-Kunita-Watanabe representation results in the case where there are restrictions on the available information and we show an application to risk-minimization. Second, as an important consequence, we prove existence and uniqueness for linear backward stochastic differential equations (in short BSDEs) driven by a general càdlàg martingale under partial information. For BSDEs driven by a general càdlàg martingale beyond the Brownian setting, there exist very few results in literature (see [5] and more recently [1] and [2], as far as we are aware). Here we study for the first time such a general case in the situation where there

*Dipartimento di Economia, Università degli Studi “G. D’Annunzio” di Chieti-Pescara, Viale Pindaro 42, I-65127 Pescara, Italy. E-mail: ceci@sci.unich.it.

†Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, via Vanvitelli, 1, I-06123 Perugia, Italy. E-mail: alessandra.cretarola@dmf.unipg.it.

‡Ecole Nationale Supérieure des Techniques Avancées, ENSTA-ParisTech Unité de Mathématiques appliquées, 32 Boulevard Victor, F-75739 Paris Cedex 15, France and INRIA Rocquencourt & Cermics Ecole des Ponts, Projet MathFi. E-mail: francesco.russo@ensta-paristech.fr.

are restrictions on the available information, that represents an interesting issue arising in many financial problems. Mathematically, this means to consider an additional filtration \mathbb{H} smaller than the full information flow \mathbb{F} . A typical example arises when $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}$ where $\tau \in (0, T)$ is a fixed delay and $(t - \tau)^+ := \max\{0, t - \tau\}$ and T denotes a time horizon.

We start our investigation by considering BSDEs of the form

$$Y_t = \xi - \int_t^T Z_s dM_s - (O_T - O_t), \quad 0 \leq t \leq T, \quad (1.1)$$

driven by a square-integrable (càdlàg) martingale $M = (M_t)_{0 \leq t \leq T}$, where $T > 0$ is a fixed time horizon, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ ¹ denotes the terminal condition and $O = (O_t)_{0 \leq t \leq T}$ is a square-integrable \mathbb{F} -martingale with $O_0 = 0$, satisfying a suitable orthogonality condition that we will make more precise in the next section.

We look for a solution (Y, Z) to equation (1.1) under partial information, where $Y = (Y_t)_{0 \leq t \leq T}$ is a càdlàg \mathbb{F} -adapted process and $Z = (Z_t)_{0 \leq t \leq T}$ is an \mathbb{H} -predictable process such that $\mathbb{E} \left[\int_0^T |Z_t|^2 d\langle M \rangle_t \right] < \infty$.

To this aim, we prove a Galtchouk-Kunita-Watanabe decomposition in the case where there are restrictions on the available information. More precisely, we obtain that every random variable $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ can be uniquely written as

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T, \quad \mathbb{P} - \text{a.s.}, \quad (1.2)$$

where $H^{\mathcal{H}} = (H_t^{\mathcal{H}})_{0 \leq t \leq T}$ is an \mathbb{H} -predictable process such that $\mathbb{E} \left[\int_0^T |H_t^{\mathcal{H}}|^2 d\langle M \rangle_t \right] < \infty$. To the authors' knowledge such a decomposition has not been proved yet in the existing literature. We will see that decomposition (1.2) allows to construct a solution to the BSDE (1.1) and ensures its uniqueness in this setting.

Moreover, we are able to provide an explicit characterization of the integrand process $H^{\mathcal{H}}$ given in decomposition (1.2) in terms of the one appearing in the classical Galtchouk-Kunita-Watanabe decomposition, by using \mathbb{H} -predictable (dual) projections.

Finally, we discuss a financial application. More precisely, we study the problem of hedging a contingent claim in the case where investors acting in the market have partial information. Since the market is incomplete we choose the risk-minimization approach, a quadratic hedging method which keeps the replicability constraint and relaxes the self financing condition, see [6] and [14] for further details. As in [6] and [13], we consider the case where the price process is a martingale under the real world probability measure. In [6], under the case of full information, the authors provide the risk-minimizing hedging strategy in terms of the classical Galtchouk-Kunita-Watanabe decomposition. Here, by using the Galtchouk-Kunita-Watanabe decomposition under partial information, we extend this result to the case where there are restrictions on the available information. Finally, thanks to the explicit representation of the integrand process $H^{\mathcal{H}}$ appearing in

¹The space $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ denotes the set of all \mathcal{F}_T -measurable real-valued random variables H such that $\mathbb{E} [|H|^2] = \int_{\Omega} |H|^2 d\mathbb{P} < \infty$.

decomposition (1.2), we find the same expression for the optimal strategy in terms of the Radon-Nikodym derivative of two \mathbb{H} -predictable dual projections, that is proved in [13]. The paper is organized as follows. In Section 2 we give the definition of solution to BSDEs under partial information. Section 3 is devoted to prove existence and uniqueness results for the solutions, which are obtained by applying the Galtchouk-Kunita-Watanabe decomposition adapted to the restricted information setting. The explicit representation of the integrand process $H^{\mathcal{H}}$ appearing in (1.2) can be found in Section 4. Finally, an application to risk-minimization is given in Section 5.

2 Setting

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$, where \mathcal{F}_t represents the full information at time t . We assume that $\mathcal{F}_T = \mathcal{F}$. Then we consider a subfiltration $\mathbb{H} := (\mathcal{H}_t)_{0 \leq t \leq T}$ of \mathbb{F} , i.e. $\mathcal{H}_t \subseteq \mathcal{F}_t$, for each $t \in [0, T]$, corresponding to the available information level. We remark that both filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity, see e.g. [12].

For simplicity we only consider the one-dimensional case. Extensions to several dimensions are straightforward and left to the reader. The data of the problem are:

- an \mathbb{R} -valued square-integrable (càdlàg) \mathbb{F} -martingale $M = (M_t)_{0 \leq t \leq T}$ with \mathbb{F} -predictable quadratic variation process denoted by $\langle M \rangle = (\langle M, M \rangle)_{0 \leq t \leq T}$;
- a terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Definition 2.1. *A solution of the BSDE*

$$Y_t = \xi - \int_t^T Z_s dM_s - (O_T - O_t), \quad 0 \leq t \leq T, \quad (2.1)$$

with data (ξ, \mathbb{H}) under partial information, where $O = (O_t)_{0 \leq t \leq T}$ is a square-integrable \mathbb{F} -martingale with $O_0 = 0$, satisfying the orthogonality condition

$$\mathbb{E} \left[O_T \int_0^T \varphi_t dM_t \right] = 0, \quad (2.2)$$

for all \mathbb{H} -predictable processes $\varphi = (\varphi_t)_{0 \leq t \leq T}$ such that $\mathbb{E} \left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$, is a couple (Y, Z) of processes with values in $\mathbb{R} \times \mathbb{R}$, satisfying (2.1) such that

- $Y = (Y_t)_{0 \leq t \leq T}$ is a càdlàg \mathbb{F} -adapted process;
- $Z = (Z_t)_{0 \leq t \leq T}$ is an \mathbb{H} -predictable process such that $\mathbb{E} \left[\int_0^T |Z_t|^2 d\langle M \rangle_t \right] < \infty$.

Remark 2.2. *The orthogonality condition given in (2.2) is weaker than the classical strong orthogonality condition, see e.g [11] or [12]. Indeed, set $N_t = \int_0^t \varphi_s dM_s$, for each $t \in [0, T]$, where φ is an \mathbb{H} -predictable process such that $\mathbb{E} \left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$. If*

$$\langle O, M \rangle_t = 0 \quad \mathbb{P} - \text{a.s.}, \quad \forall t \in [0, T],$$

then

$$\langle O, N \rangle_t = \int_0^t \varphi_s d\langle O, M \rangle_s = 0 \quad \mathbb{P} - \text{a.s.}, \quad \forall t \in [0, T].$$

Consequently, $O \cdot N$ is an \mathbb{F} -martingale null at zero, that implies

$$\mathbb{E}[O_t N_t] = 0, \quad \forall t \in [0, T],$$

and in particular condition (2.2).

Remark 2.3. Since for any \mathbb{H} -predictable process φ , the process $\mathbf{1}_{(0,t]}(s)\varphi_s$, with $t \leq T$, is \mathbb{H} -predictable, condition (2.2) implies that for every $t \in [0, T]$

$$\mathbb{E}\left[O_T \int_0^t \varphi_s dM_s\right] = 0,$$

and by conditioning with respect to \mathcal{F}_t (note that O is an \mathbb{F} -martingale), we have

$$\mathbb{E}\left[O_t \int_0^t \varphi_s dM_s\right] = \mathbb{E}\left[\int_0^t \varphi_s d\langle M, O \rangle_s\right] = 0 \quad \forall t \in [0, T].$$

From this last equality, we can argue that in the case of full information, i.e., $\mathcal{H}_t = \mathcal{F}_t$, for each $t \in [0, T]$, condition (2.2) is equivalent to the strong orthogonality condition between O and M (see e.g. Lemma 2 and Theorem 36, Chapter IV, page 180 of [12] for a rigorous proof).

3 Existence and uniqueness for linear BSDEs under partial information

Our aim is to investigate existence and uniqueness of a solution to the BSDE (2.1) with data (ξ, \mathbb{H}) driven by the general martingale M in the sense of Definition 2.1. This requires to prove a Galtchouk-Kunita-Watanabe representation result under restricted information.

We introduce the linear subspace $\mathcal{L}_T^{\mathcal{H}}$ of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ given by all random variables η of the form

$$\left\{ U_0 + \int_0^T \varphi_t dM_t \mid U_0 \in \mathcal{H}_0, \varphi \text{ is } \mathbb{H} - \text{predictable with } \mathbb{E}\left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t\right] < \infty \right\}. \quad (3.1)$$

Lemma 3.1. The set $\mathcal{L}_T^{\mathcal{H}}$ is a closed subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Proof. Let $U_0^n \in \mathcal{H}_0$ and $(\varphi^n)_{n \in \mathbb{N}}$, with $\varphi^n = (\varphi_t^n)_{0 \leq t \leq T}$, be a sequence of \mathbb{H} -predictable processes satisfying $\mathbb{E}\left[\int_0^T |\varphi_t^n|^2 d\langle M \rangle_t\right] < \infty$ such that the sequence

$$\eta^n = U_0^n + \int_0^T \varphi_t^n dM_t, \quad n \in \mathbb{N},$$

converges to some random variable $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, as n goes to infinity. By taking the conditional expectation with respect to \mathcal{H}_0 , we have

$$U_0^n = \mathbb{E} [\eta^n | \mathcal{H}_0] \longrightarrow \mathbb{E} [\eta | \mathcal{H}_0], \quad \text{as } n \rightarrow \infty.$$

We set $U_0 = \mathbb{E} [\eta | \mathcal{H}_0]$. Since $(\eta^n - U_0^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, it follows that

$$\mathbb{E} \left[\int_0^T (\varphi_t^n - \varphi_t^m)^2 d\langle M \rangle_t \right] \longrightarrow 0, \quad \text{as } n, m \rightarrow \infty.$$

Consequently, $(\varphi^n)_{n \in \mathbb{N}}$ converges in $L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})^2$ to some process $\varphi = (\varphi_t)_{0 \leq t \leq T} \in L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})$. Finally, since there is a subsequence converging $d\langle M \rangle \otimes d\mathbb{P}$ -a.e., the limit φ is necessarily an \mathbb{H} -predictable process. \square

Proposition 3.2. *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. There exists a unique decomposition of the form*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T, \quad \mathbb{P} - \text{a.s.}, \quad (3.2)$$

where $U_0 \in \mathcal{H}_0$, $H^{\mathcal{H}}$ is an \mathbb{H} -predictable process such that $\mathbb{E} \left[\int_0^T |H_t^{\mathcal{H}}|^2 d\langle M \rangle_t \right] < \infty$ and O is a square-integrable \mathbb{F} -martingale with $O_0 = 0$ such that $\mathbb{E} [O_T \cdot \eta] = 0$, for every $\eta \in \mathcal{L}_T^{\mathcal{H}}$. Moreover $U_0 = \mathbb{E} [\xi | \mathcal{H}_0]$ and $\mathbb{E} [O_T | \mathcal{H}_0] = 0$.

Proof. The existence and uniqueness property of decomposition (3.2) is clearly ensured by the orthogonal projection of the random variable ξ onto the space $\mathcal{L}_T^{\mathcal{H}}$, that is closed in virtue of Lemma 3.1. Since $(U_0 + \int_0^\cdot H_t^{\mathcal{H}} dM_t)$ is an \mathbb{F} -martingale, by taking the conditional expectation of ξ with respect to \mathcal{H}_0 in (3.2), we have

$$\begin{aligned} \mathbb{E} [\xi | \mathcal{H}_0] &= \mathbb{E} \left[U_0 + \int_0^T H_t^{\mathcal{H}} dM_t \middle| \mathcal{H}_0 \right] + \mathbb{E} [O_T | \mathcal{H}_0] \\ &= \mathbb{E} \left[\mathbb{E} \left[U_0 + \int_0^T H_t^{\mathcal{H}} dM_t \middle| \mathcal{F}_0 \right] \middle| \mathcal{H}_0 \right] + \mathbb{E} [\mathbb{E} [O_T | \mathcal{F}_0] | \mathcal{H}_0] \\ &= \mathbb{E} [U_0 | \mathcal{H}_0], \end{aligned}$$

where in the last equality we have used the fact that $\mathbb{E} [O_T | \mathcal{F}_0] = O_0 = 0$. Consequently $\mathbb{E} [O_T | \mathcal{H}_0] = 0$ and $U_0 = \mathbb{E} [U_0 | \mathcal{H}_0] = \mathbb{E} [\xi | \mathcal{H}_0]$. This concludes the proof. \square

Theorem 3.3. *Given data (ξ, \mathbb{H}) , there exists a unique couple (Y, Z) which solves the BSDE (2.1) according to Definition 2.1.*

²The space $L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})$ denotes the set of all \mathbb{F} -adapted processes $\varphi = (\varphi_t)_{0 \leq t \leq T}$ such that

$$\|\varphi\|_{L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})} := \left(\mathbb{E} \left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] \right)^{\frac{1}{2}} < \infty.$$

Proof. Existence. Let $\mathcal{L}_T^{\mathcal{H}}$ be the linear subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ introduced in (3.1). Given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, we know by Proposition 3.2 that there exists a unique decomposition of the form

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + A_T, \quad \mathbb{P} - \text{a.s.},$$

where in particular A is a square-integrable \mathbb{F} -martingale with $A_0 = 0$ orthogonal to all the elements in $\mathcal{L}_T^{\mathcal{H}}$. We use this result to construct a solution to the BSDE (2.1). We consider the orthogonal projection of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ onto this space:

$$P_{\mathcal{L}_T^{\mathcal{H}}}(\xi) := U_0 + \int_0^T H_t^{\mathcal{H}} dM_t.$$

The couple (U_0, H) , where $U_0 \in \mathcal{H}_0$ and $H^{\mathcal{H}}$ is an \mathbb{H} -predictable process in $L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})$, uniquely identifies the projection, that exists and it is well-defined since $\mathcal{L}_T^{\mathcal{H}}$ is closed. We set

$$A_T := \xi - P_{\mathcal{L}_T^{\mathcal{H}}}(\xi) \in (\mathcal{L}_T^{\mathcal{H}})^{\perp},$$

where $(\mathcal{L}_T^{\mathcal{H}})^{\perp}$ denotes the orthogonal subspace of $\mathcal{L}_T^{\mathcal{H}}$. Here A_T corresponds to the final value of a square-integrable \mathbb{F} -martingale A with zero initial value, that implies $\mathbb{E}[\xi - U_0 | \mathcal{F}_0] = 0$. Clearly, we have

$$\mathcal{L}_T^{\mathcal{H}} \oplus (\mathcal{L}_T^{\mathcal{H}})^{\perp} = L^2(\Omega, \mathcal{F}_T, \mathbb{P}).$$

Now we define the process Y as follows:

$$\begin{aligned} Y_t &:= \mathbb{E}[\xi | \mathcal{F}_t] \\ &= \mathbb{E}\left[U_0 + \int_0^T H_s^{\mathcal{H}} dM_s + A_T \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}[U_0 | \mathcal{F}_0] + \int_0^t H_s^{\mathcal{H}} dM_s + A_t \\ &= Y_0 + \int_0^t H_s^{\mathcal{H}} dM_s + A_t, \quad 0 \leq t \leq T, \end{aligned}$$

and we set $Z_t := H_t^{\mathcal{H}}$ and $O_t := A_t$, for every $t \in [0, T]$. Then we get

$$Y_t = \xi - \int_t^T Z_s dM_s - (O_T - O_t), \quad 0 \leq t \leq T.$$

Uniqueness. Let (Y, Z) , (Y', Z') be two solutions to the BSDE (2.1) under partial information associated to the terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. We set $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$. Then (\bar{Y}, \bar{Z}) satisfies the BSDE

$$\bar{Y}_t = - \int_t^T \bar{Z}_s dM_s - (\bar{O}_T - \bar{O}_t), \quad 0 \leq t \leq T, \quad (3.3)$$

with final condition $\bar{Y}_T = 0$. In addition, we have set $\bar{O} := O - O'$ in (3.3), where O and O' denote the square-integrable \mathbb{F} -martingales with $O_0 = O'_0 = 0$ satisfying the orthogonality condition

$$\mathbb{E} \left[O_T \int_0^T \varphi_t dM_t \right] = \mathbb{E} \left[O'_T \int_0^T \varphi_t dM_t \right] = 0,$$

for all \mathbb{H} -predictable processes φ such that $\mathbb{E} \left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$. Since (\bar{Y}, \bar{Z}) is a solution of (3.3), then

$$\bar{Y}_t = \bar{Y}_0 + \int_0^t \bar{Z}_s dM_s + \bar{O}_t, \quad 0 \leq t \leq T. \quad (3.4)$$

Since the process \bar{Y} is an \mathbb{F} -martingale such that $\bar{Y}_T = 0$, we have

$$\bar{Y}_t = \mathbb{E} [\bar{Y}_T | \mathcal{F}_t] = 0, \quad \text{for all } t \in [0, T].$$

Thus $Y_t = Y'_t$ \mathbb{P} -a.s., for every $t \in [0, T]$. Then we can rewrite (3.4) as follows

$$0 = \int_0^t \bar{Z}_s dM_s + \bar{O}_t, \quad 0 \leq t \leq T.$$

By computing the predictable covariation of $\int_0^t \bar{Z}_s dM_s + \bar{O}$ and \bar{O} and by taking the expectation of both sides in the equality, for each $t \in [0, T]$, we obtain

$$\begin{aligned} 0 &= \int_0^t \bar{Z}_s d\langle M, \bar{O} \rangle_s + \langle \bar{O} \rangle_t \\ &= \mathbb{E} \left[\int_0^t \bar{Z}_s d\langle M, \bar{O} \rangle_s \right] + \mathbb{E} [\langle \bar{O} \rangle_t]. \end{aligned}$$

Since \bar{Z} and \bar{O} are differences of solutions to the BSDE (2.1), then $\mathbb{E} \left[\int_0^t \bar{Z}_s d\langle M, \bar{O} \rangle_s \right] = 0$ for $t \in [0, T]$, and it follows

$$\mathbb{E} [\langle \bar{O} \rangle_t] = 0, \quad 0 \leq t \leq T. \quad (3.5)$$

By Theorem 4.2 of [11], since \bar{O} is a square-integrable \mathbb{F} -martingale null at zero, we have that $\bar{O}^2 - \langle \bar{O} \rangle$ is an \mathbb{F} -martingale null at zero. Then by (3.5)

$$\mathbb{E} [\bar{O}_t^2] = \mathbb{E} [\langle \bar{O} \rangle_t] = 0, \quad 0 \leq t \leq T,$$

that implies $\bar{O}_t^2 = 0$ \mathbb{P} -a.s. for every $t \in [0, T]$ and then $O_t = O'_t$ \mathbb{P} -a.s. for every $t \in [0, T]$. Now, let Y be the unique solution of (2.1) for a certain \mathbb{H} -predictable Z such that $\mathbb{E} \left[\int_0^T |Z_t|^2 d\langle M \rangle_t \right] < \infty$, i.e.

$$Y_t = Y_0 + \int_0^t Z_s dM_s + O_t, \quad 0 \leq t \leq T. \quad (3.6)$$

It only remains to prove that Z is unique. For $t = T$ equation (3.6) becomes

$$Y_T = \xi = Y_0 + \int_0^T Z_s dM_s + O_T.$$

By Proposition 3.2, $Z_t = H_t^{\mathcal{H}}$ \mathbb{P} -a.s., for each $t \in [0, T]$ and then Z is univocally determined. This concludes the proof. \square

4 Galtchouk-Kunita-Watanabe representation under partial information

We now wish to provide an explicit characterization of the integrand process $H^{\mathcal{H}}$ appearing in the representation (3.2) in terms of the one given in the classical Galtchouk-Kunita-Watanabe decomposition, by means in particular of the concept of \mathbb{H} -predictable dual projection.

Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. We consider the well-known Galtchouk-Kunita-Watanabe decomposition of ξ with respect to M :

$$\xi = \tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t + \tilde{O}_T, \quad \mathbb{P} - \text{a.s.}, \quad (4.1)$$

where $\tilde{U}_0 \in \mathcal{F}_0$, the integrand $H^{\mathcal{F}} = (H_t^{\mathcal{F}})_{0 \leq t \leq T}$ is an \mathbb{F} -predictable process such that $\mathbb{E} \left[\int_0^T |H_t^{\mathcal{F}}|^2 d\langle M \rangle_t \right] < \infty$ and $\tilde{O} = (\tilde{O}_t)_{0 \leq t \leq T}$ is a square-integrable \mathbb{F} -martingale with $\tilde{O}_0 = 0$ such that $\langle \tilde{O}, M \rangle_t = 0$, for every $t \in [0, T]$. Moreover, let us observe that $\tilde{U}_0 = \mathbb{E} [\xi | \mathcal{F}_0]$.

In the sequel we will denote by ${}^p X$ the \mathbb{H} -predictable projection of a (generic) integrable process $X = (X_t)_{0 \leq t \leq T}$, defined as the unique \mathbb{H} -predictable process such that

$$\mathbb{E} [X_\tau \mathbf{1}_{\{\tau < \infty\}} | \mathcal{H}_{\tau-}] = {}^p X_\tau \mathbf{1}_{\{\tau < \infty\}} \quad \mathbb{P} - \text{a.s.}$$

for every \mathbb{H} -predictable stopping time τ .

First we give a preliminary result under the additional assumption that the predictable quadratic variation $\langle M \rangle$ of the \mathbb{F} -martingale M is an \mathbb{H} -predictable process. In Theorem 4.7 we extend such result to the general case.

Proposition 4.1. *Let $(\tilde{U}_0, H^{\mathcal{F}}, \tilde{O}_T)$ be the triplet corresponding to decomposition (4.1) of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Suppose that the predictable quadratic variation $\langle M \rangle$ of the \mathbb{F} -martingale M is an \mathbb{H} -predictable process. Then*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T, \quad \mathbb{P} - \text{a.s.},$$

with

$$U_0 = \mathbb{E} \left[\tilde{U}_0 \middle| \mathcal{H}_0 \right], \quad (4.2)$$

$$H_t^{\mathcal{H}} = {}^p(H_t^{\mathcal{F}}), \quad 0 \leq t \leq T, \quad (4.3)$$

and O is a square-integrable \mathbb{F} -martingale with $O_0 = 0$ such that $\mathbb{E}[O_T \cdot \eta] = 0$, for every $\eta \in \mathcal{L}_T^{\mathcal{H}}$.

Proof. Let

$$\xi = \tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t + \tilde{O}_T, \quad \mathbb{P} - \text{a.s.}$$

be the classical Galtchouk-Kunita-Watanabe decomposition of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. By taking the expectation of ξ with respect to \mathcal{H}_0 , we have:

$$\mathbb{E}[\xi | \mathcal{H}_0] = \mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t \middle| \mathcal{H}_0\right] + \mathbb{E}[\tilde{O}_T | \mathcal{H}_0]. \quad (4.4)$$

Since $(\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t)$ is an \mathbb{F} -martingale, it follows:

$$\begin{aligned} \mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t \middle| \mathcal{H}_0\right] &= \mathbb{E}\left[\mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t \middle| \mathcal{F}_0\right] \middle| \mathcal{H}_0\right] \\ &= \mathbb{E}[\tilde{U}_0 | \mathcal{H}_0], \end{aligned}$$

so that we can rewrite (4.4) as follows:

$$\mathbb{E}[\xi | \mathcal{H}_0] = \mathbb{E}[\tilde{U}_0 | \mathcal{H}_0] + \mathbb{E}[\tilde{O}_T | \mathcal{H}_0].$$

Moreover, since \tilde{O} is an \mathbb{F} -martingale null at zero, we have

$$\mathbb{E}[\tilde{O}_T | \mathcal{H}_0] = \mathbb{E}\left[\mathbb{E}[\tilde{O}_T | \mathcal{F}_0] \middle| \mathcal{H}_0\right] = 0.$$

This implies equality (4.2). To prove equality (4.3), we need to calculate the orthogonal projection of ξ onto the space $\mathcal{L}_T^{\mathcal{H}}$, see (3.1). For the sake of brevity, we suppose that $\tilde{U}_0 = 0$. Thanks to Proposition 3.2, this means we need to check the following condition:

$$\mathbb{E}\left[\xi \int_0^T \varphi_t dM_t\right] = \mathbb{E}\left[\int_0^T p(H_t^{\mathcal{F}}) dM_t \int_0^T \varphi_t dM_t\right],$$

for every \mathbb{H} -predictable process φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t\right] < \infty$. Taking decomposition (4.1) into account, this corresponds to the following equality:

$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \varphi_t d\langle M \rangle_t\right] = \mathbb{E}\left[\int_0^T p(H_t^{\mathcal{F}}) \varphi_t d\langle M \rangle_t\right], \quad (4.5)$$

for every \mathbb{H} -predictable process φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t\right] < \infty$. If we write the process φ as follows

$$\varphi = \varphi^+ - \varphi^-,$$

where φ^+ and φ^- denote the positive and the negative part of φ respectively, and define the \mathbb{F} -martingales

$$R_t^+ = \int_0^t \sqrt{\varphi_s^+} dM_s, \quad R_t^- = \int_0^t \sqrt{\varphi_s^-} dM_s, \quad 0 \leq t \leq T,$$

equality (4.5) is equivalent to the following relationships:

$$\begin{aligned} \mathbb{E} \left[\int_0^T H_t^{\mathcal{F}} d\langle R^+ \rangle_t \right] &= \mathbb{E} \left[\int_0^T p(H_t^{\mathcal{F}}) d\langle R^+ \rangle_t \right] \\ \mathbb{E} \left[\int_0^T H_t^{\mathcal{F}} d\langle R^- \rangle_t \right] &= \mathbb{E} \left[\int_0^T p(H_t^{\mathcal{F}}) d\langle R^- \rangle_t \right]. \end{aligned}$$

Hence, we can reduce the problem by assuming directly $\varphi_t = 1$ in (4.5), for each $t \in [0, T]$. Then, it is enough to prove the equality

$$\mathbb{E} \left[\int_0^T H_t^{\mathcal{F}} d\langle M \rangle_t \right] = \mathbb{E} \left[\int_0^T p(H_t^{\mathcal{F}}) d\langle M \rangle_t \right]. \quad (4.6)$$

Since $\langle M \rangle$ is \mathbb{H} -predictable, Theorem VI.57 in [4] guarantees that equality (4.6) holds, once we have the positivity of the process $H^{\mathcal{F}}$. By writing

$$H^{\mathcal{F}} = (H^{\mathcal{F}})^+ - (H^{\mathcal{F}})^-,$$

and applying the above theorem to the positive and negative parts of $H^{\mathcal{F}}$, $(H^{\mathcal{F}})^+$ and $(H^{\mathcal{F}})^-$ respectively, and to the associated \mathbb{H} -predictable projections, we can get the result by setting

$$H^{\mathcal{H}} := p(H^{\mathcal{F}}) = p((H^{\mathcal{F}})^+) - p((H^{\mathcal{F}})^-).$$

□

Example 4.2. Let us consider the particular case where M is a square-integrable \mathbb{F} -martingale that is in addition a Lévy process, $\mathcal{F}_t = \mathcal{F}_t^M$ and $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}^M$, with $\tau \in (0, T)$ a fixed delay. We assume $\xi = h(M_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, for some measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$.

In this framework, by Lemma A.1 (see Appendix), we know that the integrand appearing in the Galtchouk-Kunita-Watanabe decomposition (4.1) can be written as

$$H_t^{\mathcal{F}} = F(t, M_{t-}), \quad t \in [0, T],$$

where the function F is such that the condition $\mathbb{E} \left[\int_0^T |F(t, M_{t-})|^2 d\langle M \rangle_t \right] < \infty$ is satisfied. Since in this case $\langle M \rangle$ is a deterministic process, we can apply Proposition 4.1 and get

$$H_t^{\mathcal{H}} = pF(t, M_{t-}) = \mathbb{E}[F(t, M_{t-}) | \mathcal{H}_{t-}], \quad t \in [0, T].$$

Then, it is easy to derive the following:

$$H_t^{\mathcal{H}} = \begin{cases} c(t, M_{(t-\tau)^-}) & \text{if } t > \tau \\ c(t, M_0) & \text{if } t \leq \tau, \end{cases}$$

where the function c is given by

$$c(t, y) = \int_{\mathbb{R}} F(t, y + z) d\rho_{t \wedge \tau}(z),$$

with ρ_t denoting the law of M_t , for every $t \in [0, T]$.

4.1 The \mathbb{H} -predictable dual projection

It is possible to extend the result of Proposition 4.1 by using the concept of \mathbb{H} -predictable dual projection. For reasons of clarity, we provide a self-contained discussion about this kind of projection in presence of more than one filtration. Let $G = (G_t)_{0 \leq t \leq T}$ be a càdlàg \mathbb{F} -adapted process of integrable variation, that is, $\mathbb{E}[\|G\|_T] < \infty$. Here the process $\|G\| = (\|G\|)_{0 \leq t \leq T}$ defined, for each $t \in [0, T]$, by

$$\|G\|_t(\omega) = \sup_{\Delta} \sum_{i=0}^{n(\Delta)-1} |G_{t_{i+1}}(\omega) - G_{t_i}(\omega)|,$$

where $\Delta = \{t_0 = 0 < t_1 < \dots < t_{n(\Delta)} = t\}$ is a partition of $[0, t]$, denotes the total variation of the function $t \rightsquigarrow G_t(\omega)$.

Proposition 4.3. *Let $G = (G_t)_{0 \leq t \leq T}$ be a càdlàg \mathbb{F} -adapted process of integrable variation. Then there exists a unique \mathbb{H} -predictable process $G^{\mathbb{H}} = (G_t^{\mathbb{H}})_{0 \leq t \leq T}$ of integrable variation, such that*

$$\mathbb{E} \left[\int_0^T \varphi_t dG_t^{\mathbb{H}} \right] = \mathbb{E} \left[\int_0^T \varphi_t dG_t \right],$$

for every \mathbb{H} -predictable (bounded) process φ . The process $G^{\mathbb{H}}$ is called the \mathbb{H} -predictable dual projection of G .

Proof. Without loss of generality, we can restrict our attention to the case where G is an increasing process and prove the statement on the generators φ of the form $\varphi_u = \mathbf{1}_{(s,t]}(u) \mathbf{1}_B$, with $B \in \mathcal{H}_s$ and $s, t \in [0, T]$ with $s < t$. Indeed, decomposing the process G as $G = G^+ - G^-$, where both the positive and negative parts of G are assumed to be increasing integrable processes, we can suppose G to be increasing such that

$$\mathbb{E}[G_T] = \mathbb{E}[\|G\|_T] < \infty.$$

If G is a càdlàg, increasing, integrable \mathbb{F} -adapted process, we will prove that there exists a unique increasing, integrable \mathbb{H} -predictable process $G^{\mathbb{H}}$ such that for every $s, t \in [0, T]$ with $s < t$ and $B \in \mathcal{H}_s$, the following relationship holds

$$\mathbb{E}[\mathbf{1}_B(G_t - G_s)] = \mathbb{E}[\mathbf{1}_B(G_t^{\mathbb{H}} - G_s^{\mathbb{H}})].$$

Let $\tilde{G} = (\tilde{G}_t)_{0 \leq t \leq T}$ be the \mathbb{H} -optional projection of G , such that for fixed times $t \in (0, T]$

$$\tilde{G}_t = \mathbb{E}[G_t | \mathcal{H}_t] \quad \mathbb{P} - \text{a.s.}$$

We observe that for every $s, t \in [0, T]$ with $s < t$ and $B \in \mathcal{H}_s$, we have

$$\mathbb{E} [\mathbf{1}_B(G_t - G_s)] = \mathbb{E} [\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)].$$

Indeed,

$$\begin{aligned} \mathbb{E} [\mathbf{1}_B(G_t - G_s)] &= \mathbb{E} [\mathbb{E} [\mathbf{1}_B(G_t - G_s) | \mathcal{H}_s]] = \mathbb{E} [\mathbf{1}_B (\mathbb{E} [G_t | \mathcal{H}_s] - \tilde{G}_s)] \\ &= \mathbb{E} [\mathbf{1}_B (\mathbb{E} [\tilde{G}_t | \mathcal{H}_s] - \tilde{G}_s)] = \mathbb{E} [\mathbb{E} [\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s) | \mathcal{H}_s]] \\ &= \mathbb{E} [\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)]. \end{aligned}$$

Furthermore, since G is increasing, then \tilde{G} is an \mathbb{H} -submartingale, that is

$$\mathbb{E} [\tilde{G}_t | \mathcal{H}_s] = \mathbb{E} [\mathbb{E} [G_t | \mathcal{H}_t] | \mathcal{H}_s] = \mathbb{E} [G_t | \mathcal{H}_s] \geq \mathbb{E} [G_s | \mathcal{H}_s] = \tilde{G}_s, \quad 0 \leq s \leq t \leq T.$$

Thanks to Doob-Meyer Theorem on decomposition of submartingales, see e.g. Theorem 3.15 of [11], there exists a unique increasing, integrable \mathbb{H} -predictable process $G^{\mathbb{H}}$ such that $\tilde{G} - G^{\mathbb{H}}$ is an \mathbb{H} -martingale, that is, for every $s, t \in [0, T]$ with $s < t$ and $B \in \mathcal{H}_s$, we have

$$\mathbb{E} [\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)] = \mathbb{E} [\mathbf{1}_B(G_t^{\mathbb{H}} - G_s^{\mathbb{H}})].$$

□

Remark 4.4. *If G is an \mathbb{H} -predictable process of integrable variation and X is an \mathbb{F} -adapted process satisfying $\mathbb{E} \left[\int_0^T X_t dG_t \right] < \infty$, then*

$$(X_t dG_t)^{\mathbb{H}} = {}^p X_t dG_t, \quad \mathbb{P} - \text{a.s.}, \text{ for every } t \in [0, T].$$

Indeed, by Theorem VI.57 in [4], for any \mathbb{H} -predictable (bounded) process φ we can prove that

$$\mathbb{E} \left[\int_0^T \varphi_t X_t dG_t \right] = \mathbb{E} \left[\int_0^T \varphi_t {}^p X_t dG_t \right].$$

4.2 Explicit representation results

We now can apply the results of Subsection 4.1 to extend Proposition 4.1. Let $\mathcal{P}^{\mathbb{H}}$ and \mathcal{P} be the \mathbb{H} -predictable and \mathbb{F} -predictable σ -field respectively. We consider the measures $\mu^{\mathbb{H}}$ (respectively μ) defined on $\mathcal{P}^{\mathbb{H}}$ (respectively \mathcal{P}) such that

$$\mu^{\mathbb{H}}((s, t] \times B) = \mathbb{E} [\mathbf{1}_B(A_t^{\mathbb{H}} - A_s^{\mathbb{H}})], \quad B \in \mathcal{H}_s, \quad s, t \in [0, T], \quad s < t, \quad (4.7)$$

where $A^{\mathbb{H}}$ is the \mathbb{H} -predictable dual projection of $A := (\int_0^t H_s^{\mathbb{F}} d\langle M \rangle_s)_{0 \leq t \leq T}$, that exists thanks to Theorem 4.3, and

$$\mu((u, v] \times F) = \mathbb{E} [\mathbf{1}_F(\langle M \rangle_v^{\mathbb{H}} - \langle M \rangle_u^{\mathbb{H}})], \quad F \in \mathcal{F}_u, \quad u, v \in [0, T], \quad u < v. \quad (4.8)$$

Here $H^{\mathbb{F}}$ is the integrand appearing in the Galtchouk-Kunita-Watanabe decomposition (4.1).

Lemma 4.5. *Let $\mu^{\mathcal{H}}$ and μ measures satisfying conditions (4.7) and (4.8) respectively. Then $\mu^{\mathcal{H}} \ll \mu$ on $\mathcal{P}^{\mathcal{H}}$, that is, $\mu^{\mathcal{H}}$ is absolutely continuous with respect to the restriction of μ on $\mathcal{P}^{\mathcal{H}}$.*

Proof. By using the definition of absolute continuity, we wish to show that if whenever $\mu(E) = 0$ for $E \in \mathcal{P}^{\mathcal{H}}$, then $\mu^{\mathcal{H}}(E) = 0$. Let $\psi = (\psi_t)_{0 \leq t \leq T}$ be a nonnegative \mathbb{H} -predictable process such that

$$\mathbb{E} \left[\int_0^T \psi_t d\langle M \rangle_t^{\mathbb{H}} \right] = 0.$$

Then

$$\mathbb{E} \left[\int_0^T \psi_t d\langle M \rangle_t \right] = 0,$$

that implies that $\psi = 0$ $d\langle M \rangle \otimes d\mathbb{P}$ a.e., since ψ is nonnegative. Finally

$$\mathbb{E} \left[\int_0^T \psi_t dA_t^{\mathbb{H}} \right] = \mathbb{E} \left[\int_0^T \psi_t dA_t \right] = \mathbb{E} \left[\int_0^T \psi_t H_t^{\mathcal{F}} d\langle M \rangle_t \right] = 0.$$

□

Since $\mu^{\mathcal{H}} \ll \mu$ on $\mathcal{P}^{\mathcal{H}}$, thanks to Lemma 4.5, by the Radon-Nikodym theorem there exists a $\mathcal{P}^{\mathcal{H}}$ -measurable function g on $[0, T] \times \Omega$ such that

$$\mu^{\mathcal{H}}(E) = \int_E g(t, \omega) d\mu(t, \omega), \quad \forall E \in \mathcal{P}^{\mathcal{H}}.$$

This allows to identify the process $H^{\mathcal{H}}$ as the Radon-Nikodym derivative:

$$H_t^{\mathcal{H}}(\omega) := \left. \frac{d\mu^{\mathcal{H}}(t, \omega)}{d\mu(t, \omega)} \right|_{\mathcal{P}^{\mathcal{H}}}, \quad (t, \omega) \in [0, T] \times \Omega. \quad (4.9)$$

Finally, we are ready to state the following theorem.

Theorem 4.6. *For any nonnegative \mathbb{F} -measurable process $H^{\mathcal{F}}$, the following equality holds*

$$\mathbb{E} \left[\int_0^T \varphi_t H_t^{\mathcal{F}} d\langle M \rangle_t \right] = \mathbb{E} \left[\int_0^T \varphi_t H_t^{\mathcal{H}} d\langle M \rangle_t \right], \quad (4.10)$$

for every \mathbb{H} -predictable process φ such that $\mathbb{E} \left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$. Here $H^{\mathcal{H}}$ is given by (4.9).

Proof. By relationship (4.9) and definition of the measure μ , see (4.8), we have for every $s, t \in [0, T]$ with $s < t$ and $B \in \mathcal{H}_s$

$$\mu^{\mathcal{H}}((s, t] \times B) = \int_s^t \int_B H_u^{\mathcal{H}}(\omega) d\mu(u, \omega) = \mathbb{E} \left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} d\langle M \rangle_u^{\mathbb{H}} \right] = \mathbb{E} \left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} d\langle M \rangle_u \right].$$

On the other hand, by (4.7)

$$\mu^{\mathcal{H}}((s, t] \times B) = \mathbb{E} \left[\mathbf{1}_B(A_t^{\mathbb{H}} - A_s^{\mathbb{H}}) \right] = \mathbb{E} \left[\mathbf{1}_B \int_s^t H_u^{\mathcal{F}} d\langle M \rangle_u \right].$$

If φ is of the form $\varphi_u = \mathbf{1}_{(s, t]}(u) \mathbf{1}_B$, with $B \in \mathcal{H}_s$ and $s, t \in [0, T]$ with $s < t$, then the statement is proved since relationship (4.10) is verified on the generators of $\mathcal{P}^{\mathcal{H}}$. \square

We now give the analogous of Proposition 4.1, without assuming that the process $\langle M \rangle$ is \mathbb{H} -predictable.

Theorem 4.7. *Let $(\tilde{U}_0, H^{\mathcal{F}}, \tilde{O}_T)$ be the triplet corresponding to decomposition (4.1) of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Then*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T, \quad \mathbb{P} - \text{a.s.}, \quad (4.11)$$

with

$$U_0 = \mathbb{E} \left[\tilde{U}_0 \middle| \mathcal{H}_0 \right],$$

$$H_t^{\mathcal{H}} = \left. \frac{d\mu^{\mathcal{H}}(t, \omega)}{d\mu(t, \omega)} \right|_{\mathcal{P}^{\mathcal{H}}}, \quad (t, \omega) \in [0, T] \times \Omega,$$

where $\mu^{\mathcal{H}}$ and μ are given in (4.7) and (4.8) respectively, and O is a square-integrable \mathbb{F} -martingale with $O_0 = 0$ such that $\mathbb{E}[O_T \cdot \eta] = 0$, for every $\eta \in \mathcal{L}_T^{\mathcal{H}}$.

Proof. We proceed as in the proof of Proposition 4.1 by observing that condition (4.10) plays the same role of condition (4.5). \square

In the next proposition we give a useful result which allows us to compute $H^{\mathcal{H}}$ as the Radon-Nikodym derivative of the \mathbb{H} -predictable dual projection $A^{\mathbb{H}}$ of the process $A = (\int_0^t H_s^{\mathcal{F}} d\langle M \rangle_s)_{0 \leq t \leq T}$ with respect to the \mathbb{H} -predictable dual projection $\langle M \rangle^{\mathbb{H}}$ of the \mathbb{F} -predictable quadratic variation $\langle M \rangle$.

Proposition 4.8. *The process $A^{\mathbb{H}} = (\int_0^t H_s^{\mathcal{F}} d\langle M \rangle_s)^{\mathbb{H}}$ is absolutely continuous with respect to $\langle M \rangle^{\mathbb{H}}$ and it is given by*

$$A_t^{\mathbb{H}} = \int_0^t H_s^{\mathcal{H}} d\langle M \rangle_s^{\mathbb{H}}, \quad 0 \leq t \leq T.$$

As a consequence

$$H_t^{\mathcal{H}} = \frac{dA_t^{\mathbb{H}}}{d\langle M \rangle_t^{\mathbb{H}}}, \quad 0 \leq t \leq T. \quad (4.12)$$

Proof. Set $\tilde{A}_t := \int_0^t H_s^{\mathcal{H}} d\langle M \rangle_s^{\mathbb{H}}$, for each $t \in [0, T]$. It is sufficient to prove that

$$\mathbb{E} \left[\int_0^T \varphi_u dA_u \right] = \mathbb{E} \left[\int_0^T \varphi_u d\tilde{A}_u \right]$$

for every \mathbb{H} -predictable (bounded) process φ . As before, we can consider φ of the form $\varphi_u = \mathbf{1}_{(s,t]}(u)\mathbf{1}_B$, with $B \in \mathcal{H}_s$ and $s < t \in [0, T]$.

Then by the definitions of the measure μ and $\mu^{\mathcal{H}}$, see (4.8) and (4.7), and recalling (4.9) we get

$$\begin{aligned} \mathbb{E} \left[\int_0^T \varphi_u d\tilde{A}_u \right] &= \mathbb{E} \left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} d\langle M \rangle_u^{\mathbb{H}} \right] = \int_s^t \int_B H_u^{\mathcal{H}}(\omega) d\mu(u, \omega) = \mu^{\mathcal{H}}((s, t] \times B) = \\ &= \mathbb{E} \left[\mathbf{1}_B (A_t^{\mathbb{H}} - A_s^{\mathbb{H}}) \right] = \mathbb{E} \left[\mathbf{1}_B \int_s^t H_u^{\mathcal{F}} d\langle M \rangle_u \right] = \mathbb{E} \left[\int_0^T \varphi_u dA_u \right] \end{aligned}$$

which concludes the proof. \square

Example 4.9. Suppose that the process $\langle M \rangle$ is of the form

$$\langle M \rangle_t = \int_0^t a_s dG_s, \quad 0 \leq t \leq T$$

for some \mathbb{F} -predictable process $a = (a_t)_{0 \leq t \leq T}$ and an increasing deterministic function G . Then by Remark 4.4

$$\langle M \rangle_t^{\mathbb{H}} = \int_0^t p a_s dG_s, \quad A_t^{\mathbb{H}} = \int_0^t p (H_s^{\mathcal{F}} a_s) dG_s, \quad 0 \leq t \leq T,$$

and as a consequence of Proposition 4.8 we get

$$H_t^{\mathcal{H}} = \frac{p(H_t^{\mathcal{F}} a_t)}{p a_t}, \quad 0 \leq t \leq T.$$

Remark 4.10. Let us observe that if the process $\langle M \rangle$ is \mathbb{H} -predictable, then again by Remark 4.4

$$\langle M \rangle_t^{\mathbb{H}} = \langle M \rangle_t, \quad A_t^{\mathbb{H}} = \int_0^t p (H_s^{\mathcal{F}}) d\langle M \rangle_s, \quad 0 \leq t \leq T,$$

and by applying Proposition 4.8 we obtain that

$$H_t^{\mathcal{H}} = p(H_t^{\mathcal{F}}), \quad 0 \leq t \leq T.$$

5 Risk-minimization under restricted information

In relation to the connection between risk-minimization under full and partial information respectively, we now show how our result obtained in Proposition 4.8 fits in the approach of [13] of risk-minimization under restricted information.

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a financial market with one riskless asset with (discounted) price 1 and one risky asset whose (discounted) price is given by a square-integrable (càdlàg) martingale $M = (M_t)_{0 \leq t \leq T}$ adapted to a (large) filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$.

We will study the problem of hedging a contingent claim, whose final payoff is given by

a random variable $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, in the case where investors acting in the market can access only to the information flow $\mathbb{H} := (\mathcal{H}_t)_{0 \leq t \leq T}$ with $\mathcal{H}_t \subseteq \mathcal{F}_t$, for each $t \in [0, T]$. We choose the risk-minimization approach to solve the above hedging problem. In the case of full information, in [6] the authors proved that there exists a unique \mathbb{F} -risk-minimizing hedging strategy $\phi^* = (\theta^*, \eta^*)$, where $\theta^* = (\theta_t^*)_{0 \leq t \leq T}$ is given by the integrand with respect to M in the classical Galtchouk-Kunita-Watanabe decomposition of ξ , i.e. $\theta^* = H^{\mathcal{F}}$ (see equation (4.1)).

In this section we extend this result to the case where there are restrictions on the available information, by using the Galtchouk-Kunita-Watanabe decomposition under partial information (see equation (3.2)). More precisely, we prove that the \mathbb{H} -risk-minimizing hedging strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ (see Definition 5.2 below) is such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$.

Risk-minimization under restricted information was studied in [13] by using a different approach. We obtain the same explicit representation given in Theorem 3.1 of [13] by applying Proposition 4.8. About risk-minimization under partial information for defaultable markets via nonlinear filtering, we refer to [8]. In particular, they consider the case where the contingent claim ξ is \mathcal{H}_T -measurable, in which we can solve the risk-minimization problem by using the classical Galtchouk-Kunita-Watanabe decomposition.

We now assume that the agent has at her/his disposal the information flow \mathbb{H} about trading in stocks while a complete information about trading in the riskless asset.

Definition 5.1. *An \mathbb{H} -strategy is a pair $\phi = (\theta, \eta)$ (θ_t is the number of shares of the risky asset to be held at time t , while η_t is the amount invested in the riskless asset at time t) where θ is \mathbb{H} -predictable and η is \mathbb{F} -adapted and such that*

$$\mathbb{E} \left[\int_0^T \theta_s^2 d\langle M \rangle_s \right] < \infty$$

and the value process $V(\phi) := \theta M + \eta$ satisfies

$$\mathbb{E} \left[\left(\sup_{t \in [0, T]} |V_t(\phi)| \right)^2 \right] < \infty.$$

For any \mathbb{H} -strategy ϕ , the associated cost process $C(\phi)$ is given by

$$C_t(\phi) := V_t(\phi) - \int_0^t \theta_r dM_r, \quad 0 \leq t \leq T.$$

Finally the \mathbb{H} -risk process of ϕ is defined by

$$R_t(\phi) := \mathbb{E} \left[(C_T(\phi) - C_t(\phi))^2 \mid \mathcal{H}_t \right], \quad 0 \leq t \leq T.$$

Definition 5.2. *An \mathbb{H} -strategy $\phi = (\theta, \eta)$ is called \mathbb{H} -risk-minimizing if $V_T(\phi) = \xi$ \mathbb{P} -a.s. and if for any \mathbb{H} -strategy ψ such that $V_T(\psi) = \xi$ \mathbb{P} -a.s., we have $R_t(\phi) \leq R_t(\psi)$ \mathbb{P} -a.s. for every $t \in [0, T]$.*

Remark 5.3. By Corollary 3.1 in [13] we have that if $\phi = (\theta, \eta)$ is an \mathbb{H} -risk-minimizing strategy then ϕ is mean-self-financing, i.e. the cost process $C(\phi)$ is an \mathbb{F} -martingale. Moreover, if $\phi = (\theta, \eta)$ is a mean-self-financing \mathbb{H} -strategy, then $V(\phi)$ is an \mathbb{F} -martingale, hence for a given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, we have that $V_t(\phi) = \mathbb{E}[\xi | \mathcal{F}_t]$, for every $t \in [0, T]$.

To prove the main result of this section, we need the following Lemma.

Lemma 5.4. Let $O = (O_t)_{0 \leq t \leq T}$ be a square-integrable \mathbb{F} -martingale with $O_0 = 0$, satisfying the orthogonality condition

$$\mathbb{E} \left[O_T \int_0^T \varphi_t dM_t \right] = 0$$

for all \mathbb{H} -predictable processes $\varphi = (\varphi_t)_{0 \leq t \leq T}$ such that $\mathbb{E} \left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$. Then for any $t \in [0, T]$

$$\mathbb{E} \left[(O_T - O_t) \int_t^T \varphi_s dM_s \middle| \mathcal{H}_t \right] = 0 \quad \mathbb{P} - \text{a.s.}$$

Proof. Since for any \mathbb{H} -predictable process φ

$$\mathbf{1}_{(t, T]}(s) \mathbf{1}_B \varphi_s, \quad B \in \mathcal{H}_t, \quad t \in [0, T],$$

is \mathbb{H} -predictable, we get

$$\mathbb{E} \left[O_T \mathbf{1}_B \int_t^T \varphi_s dM_s \right] = \mathbb{E} \left[\mathbf{1}_B \mathbb{E} \left[O_T \int_t^T \varphi_s dM_s \middle| \mathcal{H}_t \right] \right] = 0, \quad \forall B \in \mathcal{H}_t,$$

and then

$$\mathbb{E} \left[O_T \int_t^T \varphi_s dM_s \middle| \mathcal{H}_t \right] = 0 \quad \mathbb{P} - \text{a.s.}$$

Finally, let us observe that

$$\mathbb{E} \left[O_t \int_t^T \varphi_s dM_s \middle| \mathcal{H}_t \right] = \mathbb{E} \left[O_t \int_0^T \varphi_s dM_s \middle| \mathcal{H}_t \right] - \mathbb{E} \left[O_t \int_0^t \varphi_s dM_s \middle| \mathcal{H}_t \right] = 0$$

since

$$\mathbb{E} \left[O_t \int_0^t \varphi_s dM_s \middle| \mathcal{H}_t \right] = \mathbb{E} \left[\mathbb{E} \left[O_t \int_0^t \varphi_s dM_s \middle| \mathcal{F}_t \right] \middle| \mathcal{H}_t \right] = \mathbb{E} \left[O_t \int_0^t \varphi_s dM_s \middle| \mathcal{H}_t \right],$$

and this concludes the proof. \square

We are now in the position to provide an alternative proof to that given in [13], concerning the explicit representation for an \mathbb{H} -risk-minimizing strategy, by applying the Galtchouk-Kunita-Watanabe decomposition under partial information and the representation result given in Proposition 4.8.

Theorem 5.5. For every $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, there exists a unique \mathbb{H} -risk-minimizing strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$, where $H^{\mathcal{H}}$ is given by (4.12) and $\eta_t^{\mathcal{H}} = \mathbb{E}[\xi | \mathcal{F}_t] - \theta_t^{\mathcal{H}} M_t$, for every $t \in [0, T]$.

Proof. The proof is similar to that of Theorem 2.4 of [14] performed in the full information case. Let $\phi = (\theta, \eta)$ be a mean-self-financing \mathbb{H} -strategy such that $V_T(\phi) = \xi$ \mathbb{P} -a.s.. Hence, by computing the Galtchouk-Kunita-Watanabe decomposition under partial information, see (4.11), we have

$$\begin{aligned} C_T(\phi) - C_t(\phi) &= V_T(\phi) - V_t(\phi) - \int_t^T \theta_s dM_s = \xi - V_t(\phi) - \int_t^T \theta_s dM_s \\ &= U_0 + \int_0^T H_s^{\mathcal{H}} dM_s + O_T - \int_t^T \theta_s dM_s - V_t(\phi), \end{aligned}$$

where $H^{\mathcal{H}}$ is given by (4.12). Since $V_t(\phi) = E[\xi | \mathcal{F}_t]$, for every $t \in [0, T]$, see Remark 5.3, we get that

$$V_t(\phi) = U_0 + \int_0^t H_s^{\mathcal{H}} dM_s + O_t$$

and

$$C_T(\phi) - C_t(\phi) = \int_t^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s + O_T - O_t.$$

By similar computation we get that

$$C_T(\phi^{\mathcal{H}}) - C_t(\phi^{\mathcal{H}}) = O_T - O_t.$$

Finally

$$\begin{aligned} (C_T(\phi) - C_t(\phi))^2 &= \left(C_T(\phi^{\mathcal{H}}) - C_t(\phi^{\mathcal{H}})\right)^2 + \left(\int_t^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s\right)^2 \\ &\quad + 2(O_T - O_t) \int_t^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s \end{aligned}$$

and by Lemma 5.4 we obtain that

$$R_t(\phi) = R_t(\phi^{\mathcal{H}}) + \mathbb{E} \left[\left(\int_t^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s \right)^2 \middle| \mathcal{H}_t \right] \geq R_t(\phi^{\mathcal{H}}).$$

Hence $\phi^{\mathcal{H}}$ is \mathbb{H} -risk-minimizing. If some other ϕ is also \mathbb{H} -risk-minimizing then

$$\mathbb{E} \left[\left(\int_0^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s \right)^2 \middle| \mathcal{H}_0 \right] = \mathbb{E} \left[\int_0^T \{H_s^{\mathcal{H}} - \theta_s\}^2 d\langle M \rangle_s \middle| \mathcal{H}_0 \right] = 0,$$

which implies $H^{\mathcal{H}} = \theta$. Since $V_t(\phi) = V_t(\phi^{\mathcal{H}}) = \mathbb{E}[\xi | \mathcal{F}_t]$ for each $t \in [0, T]$, we also obtain $\phi = \phi^{\mathcal{H}}$. \square

In the rest of the section we investigate the case where there is a relationship between the information flow \mathbb{H} and the filtration generated by the stock price M , that we denote by \mathbb{F}^M . A possible choice is the assumption that investors acting in the market have access only to the information contained in past asset prices, that is $\mathbb{H} = \mathbb{F}^M$. Such a situation has been studied for instance in [7] and [3] for stock price dynamics with jumps. In the sequel we will assume $\mathbb{H} \subseteq \mathbb{F}^M$, which takes also into account, for instance, the case where the asset price is only observed at discrete times or with a fixed delay $\tau \in (0, T)$, i.e. $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}^M$, for every $t \in (0, T)$.

In a such particular case, when in addition $\xi \in L^2(\Omega, \mathcal{F}_T^M, \mathbb{P}; \mathbb{R})$, we can find an \mathbb{H} -risk minimizing strategy, $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$, where $\theta^{\mathcal{H}}$ is \mathbb{H} -predictable and $\eta^{\mathcal{H}}$ is \mathbb{F}^M -adapted, while in the general case $\eta^{\mathcal{H}}$ has been taken \mathbb{F} -adapted. This means that we study the situation where the agent has at her/his disposal the information flow $\mathbb{H} \subseteq \mathbb{F}^M$ about trading in stocks and the filtration \mathbb{F}^M about trading in the riskless asset, and when $\mathbb{H} = \mathbb{F}^M$ the same information flow.

More precisely, from now on we restrict ourself to consider \mathbb{H} -strategies $\phi = (\theta, \eta)$ as in Definition 5.1 where η is chosen \mathbb{F}^M -adapted.

Remark 5.6. *Let us observe that given an \mathbb{H} -strategy $\phi = (\theta, \eta)$, the associated value process $V(\phi) := \theta M + \eta$ turns out to be \mathbb{F}^M -adapted. By Corollary 3.1 in [13], we have that if $\phi = (\theta, \eta)$ is an \mathbb{H} -risk-minimizing strategy according to this new definition, then ϕ is mean-self-financing, i.e. the cost process $C(\phi)$ is an \mathbb{F}^M -martingale. Moreover, if $\phi = (\theta, \eta)$ is a mean-self-financing \mathbb{H} -strategy, then $V(\phi)$ is an \mathbb{F}^M -martingale, hence $V_t(\phi) = E[\xi | \mathcal{F}_t^M]$, for every $t \in [0, T]$.*

We are now ready to give the following result.

Theorem 5.7. *For every $\xi \in L^2(\Omega, \mathcal{F}_T^M, \mathbb{P}; \mathbb{R})$, there exists a unique \mathbb{H} -risk-minimizing strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$, where $H^{\mathcal{H}}$ is given by (4.12) and $\eta^{\mathcal{H}} = E[\xi | \mathcal{F}_t^M] - \theta_t^{\mathcal{H}} M_t$, for every $t \in [0, T]$.*

Proof. Since ξ is \mathcal{F}_T^M -measurable, by decomposition (4.11) we obtain that

$$\xi = \mathbb{E}[\xi | \mathcal{F}_T^M] = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + \mathbb{E}[O_T | \mathcal{F}_T^M],$$

where $H^{\mathcal{H}}$ is given by (4.12). Set $\hat{O}_t := \mathbb{E}[O_T | \mathcal{F}_t^M]$, for each $t \in [0, T]$. It is known that \hat{O} is an \mathbb{F}^M -martingale and

$$\mathbb{E} \left[\hat{O}_T \int_0^T H_t^{\mathcal{H}} dM_t \right] = \mathbb{E} \left[\mathbb{E} \left[\hat{O}_T | \mathcal{F}_T^M \right] \int_0^T H_t^{\mathcal{H}} dM_t \right] = \mathbb{E} \left[O_T \int_0^T H_t^{\mathcal{H}} dM_t \right] = 0.$$

Therefore we obtain the Galtchouk-Kunita-Watanabe decomposition of ξ under restricted information with respect to the filtration \mathbb{F}^M , that is

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + \hat{O}_T, \quad \mathbb{P} - \text{a.s.}$$

The rest of the proof follows from Theorem 5.5 by replacing the filtration \mathbb{F} by \mathbb{F}^M and the \mathbb{F} -martingale O by the \mathbb{F}^M -martingale \hat{O} . \square

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APPENDIX

A Technical results

Recall that $M = (M_t)_{0 \leq t \leq T}$ is a square-integrable \mathbb{F} -martingale and assume that $\mathbb{F} = \mathbb{F}^M := (\mathcal{F}_t^M)_{0 \leq t \leq T}$, i.e. the information flow \mathbb{F} coincides with the canonical filtration \mathbb{F}^M of M .

Lemma A.1. *Let M be a Lévy process and $\xi = h(M_T) \in L^2(\Omega, \mathcal{F}_T^M, \mathbb{P}; \mathbb{R})$ for some measurable function $h : \mathbb{R} \rightarrow \mathbb{R}$. Then, there exists a measurable function $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that*

$$\xi = \mathbb{E}[\xi] + \int_0^T F(s, M_{s-}) dM_s + \tilde{O}_T, \quad \mathbb{P} - a.s.,$$

where $\tilde{O} = (\tilde{O}_t)_{0 \leq t \leq T}$ is a square-integrable \mathbb{F}^M -martingale null at zero such that $\langle \tilde{O}, M \rangle_t = 0$, for every $t \in [0, T]$. Moreover, the following integrability condition is satisfied

$$\mathbb{E} \left[\int_0^T |F(s, M_{s-})|^2 d\langle M \rangle_s \right] < \infty.$$

Proof. If ξ is given as a Fourier transform of M_T , that is, the function h is of the form

$$h(x) = \int_{\mathbb{R}} e^{iax} d\nu(a), \quad \text{for all } x \in \mathbb{R}, \quad (\text{A.1})$$

where ν is a finite measure, the result is contained in Proposition 4.3 of [9], which was an adaptation of [10].

As a consequence, the thesis follows once we show the existence of a sequence $(h_n)_{n \in \mathbb{N}}$ of functions of the kind (A.1) such that

$$\mathbb{E} \left[|h_n(M_T) - h(M_T)|^2 \right] \xrightarrow[n \rightarrow \infty]{} 0. \quad (\text{A.2})$$

To see that, denoting by $F_n(t, M_{t-})$ the integrand in the Galtchouk-Kunita-Watanabe decomposition of $h_n(M_T)$, $n \in \mathbb{N}$, we can proceed as in the proof of Lemma 3.1 and we get that the sequence $(F_n(t, M_{t-}))_{n \in \mathbb{N}}$ converges in $L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})$ to the integrand $H^{\mathcal{F}}$ in the Galtchouk-Kunita-Watanabe decomposition of $h(M_T)$. Now, there is a subsequence converging $d\langle M \rangle \otimes d\mathbb{P}$ -a.e. to the \mathbb{F} -predictable process $(H_t^{\mathcal{F}})_{0 \leq t \leq T}$ and for almost all $t \in [0, T]$, $H_t^{\mathcal{F}}$ is $\sigma(M_{t-})$ -measurable. Finally this implies the existence of a measurable function $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $H_t^{\mathcal{F}} = F(t, M_{t-})$.

It remains to show the existence of a sequence $(h_n)_{n \in \mathbb{N}}$ of functions of type (A.1) verifying (A.2). If ρ_T is the law of M_T , (A.2), translates into

$$\int_{\mathbb{R}} (h_n(y) - h(y))^2 d\rho_T(y) \xrightarrow{n \rightarrow \infty} 0. \quad (\text{A.3})$$

Since ρ_T is a finite non-negative measure, it is well-known that the space of smooth functions with compact support is dense in $L^2(\rho_T)$. This implies that the Schwartz space $\mathcal{S}(\mathbb{R})$ of the fast decreasing functions is dense in $L^2(\rho_T)$. Let $(h_n)_{n \in \mathbb{N}}$ belong to $\mathcal{S}(\mathbb{R})$ such that (A.3), and consequently, (A.2) holds. Since the inverse Fourier transform \mathcal{F}^{-1} maps $\mathcal{S}(\mathbb{R})$ into itself, then we observe that h_n are of the type (A.1) with $\nu(da) = \mathcal{F}^{-1}h_n(a)da$. \square

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