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Definite hermitian forms and the cancellation of simple knots

By

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Schubert has shown that every classical knot $\Sigma^1 \subset S^3$ factorises uniquely into the connected sum of finitely many indecomposable knots (cf. [12]). In particular cancellation holds for these knots. For higher dimensional simple knots factorisation is not always unique (cf. [5] and [1]), but in many cases we still have cancellation (see [2], Proposition 6.6).

In this note we shall give counter examples to the cancellation of non-singular hermitian and skew-hermitian forms. In order to obtain these examples we shall show that the extension of the $\mathbb{Z}$-lattice Γ_{4n}, $n \neq 1$, to certain orders is indecomposable.

Using the classification of simple $(2q-1)$-knots $\Sigma^{2q-1} \subset S^{2q+1}$, $q \neq 1$, in terms of $(-1)^{q+1}$-hermitian (Blanchfield) forms, we shall then prove that cancellation does not hold for higher odd-dimensional knots.

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1. Definite hermitian forms. Let K be a number field with a Q-involution which we shall denote by an overbar. Assume that K is totally imaginary and that the fixed field F of the involution is totally real. Let A be an order of K, and let L be a torsion free A-module of finite rank. We shall say that a hermitian form $h\colon L \times L \to A$ is *definite* if h is anisotropic at every real embedding of F. Otherwise we shall say that h is *indefinite*.

The following is a result of Eichler (cf. [3]).

Lemma 1. *Every definite hermitian form decomposes uniquely as an orthogonal sum of indecomposable forms.*

Sketch of proof (see Kneser [8] and O'Meara [11], § 105). We shall say that $x \in L$ is irreducible if x cannot be written as a sum $x = y + z$, $y \neq 0$, $z \neq 0$ and $h(y, z) = 0$. Then every $x \in L$ can be expressed as a finite sum of irreducible elements. Indeed, if $x = y + z$ with $h(y, z) = 0$ then $h(x, x) = h(y, y) + h(z, z)$. As h is anisotropic at every real place, $h(y, y)$ and $h(z, z)$ have the same sign at each real

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embedding of F. Therefore

$$N_{K/\mathbb{Q}}(h(y,y)) < N_{K/\mathbb{Q}}(h(x,x)), \quad \text{and} \quad N_{K/\mathbb{Q}}(h(z,z)) < N_{K/\mathbb{Q}}(h(x,x)).$$

As $N_{K/\mathbb{Q}}(h(x,x))$ is a natural number, we see by induction that x can be written as a finite sum of irreducibles. We shall say that two irreducible elements x and x' are equivalent if there exists a finite chain of irreducible elements

$$x = x_0, x_1, \ldots, x_k = x',$$

such that $h(x_i, x_{i+1}) \neq 0$. Every equivalence class generates a sublattice of L, and L is the orthogonal sum of these lattices. It is easy to see (cf. [11], § 105), that every orthogonal splitting of L into indecomposables is a permutation of these sublattices.

Let L be a free $\mathbb{Z}$-module of finite rank, and let $b\colon L \times L \to \mathbb{Z}$ be a symmetric $\mathbb{Z}$-bilinear form. Let $\hat{L} = A \otimes_{\mathbb{Z}} L$, and let $h\colon \hat{L} \times \hat{L} \to A$ be the hermitian form which is defined by $h(\alpha x, \beta y) = \alpha \bar{\beta} b(x,y)$ for $\alpha, \beta \in A$ and $x, y \in L$. If b is definite then h is also definite.

We shall apply this construction to the $\mathbb{Z}$-bilinear form $b\colon L \times L \to \mathbb{Z}$ which corresponds to the lattice Γ_{4n} (cf. [10], chap. II, § 6, or [11], § 106 E).

Proposition. *The hermitian form $A\Gamma_{4n}$ is indecomposable if $n > 1$.*

The following lemma is well known.

Lemma 2. *Let $m = [F:\mathbb{Q}]$. If $a \in F$ is a totally positive algebraic integer, then $Tr_{F/\mathbb{Q}}(a) \geqq m$. Moreover, if $Tr_{F/\mathbb{Q}}(a) = m$ then $a = 1$.*

This follows immediately from the inequality between arithmetic and geometric means.

Proof of Proposition. Let $V = Ke_1 \oplus \cdots \oplus Ke_{4n}$ with the hermitian form $h(e_i, e_j) = \delta_{ij}$. Then $A\Gamma_{4n}$ is the lattice in V which is generated by $e_i + e_j$ and $\frac{1}{2}(e_1 + \cdots + e_{4n})$. We shall prove that if $x \in A\Gamma_{4n}$ such that $h(x,x) = 2$, then x is irreducible.

Indeed, assume that $x = y + z$ with $y \neq 0$, $z \neq 0$ and $h(y,z) = 0$. Therefore $h(x,x) = h(y,y) + h(z,z)$, so we have

$$2m = Tr_{F/\mathbb{Q}}(h(x,x)) = Tr_{F/\mathbb{Q}}(h(y,y)) + Tr_{F/\mathbb{Q}}(h(z,z)),$$

where $m = [F:\mathbb{Q}]$. But $h(y,y)$ and $h(z,z)$ are both totally positive. By Lemma 2 this implies that $Tr_{F/\mathbb{Q}}(h(y,y)) = Tr_{F/\mathbb{Q}}(h(z,z)) = m$ (in fact, $h(y,y) = h(z,z) = 1$). Now we shall show that if $y \in A\Gamma_{4n}$, then $Tr_{F/\mathbb{Q}}(h(y,y)) = m$ is impossible. Indeed, if $y = \sum_{i=1}^{4n} a_i e_i \in A\Gamma_{4n}$, then $a_i \in \frac{1}{2}A$, $a_i - a_j \in A$ for every $i, j = 1, \ldots, 4n$ and $\sum_{i=1}^{4n} a_i \in 2A$ (cf. [11], § 106 E). We have $h(y,y) = \sum_{i=1}^{4n} a_i \bar{a}_i$, so $m = \sum_{i=1}^{4n} Tr_{F/\mathbb{Q}}(a_i \bar{a}_i)$. Two cases are possible: either all of the a_i's are in A, or $a_i = \frac{1}{2} b_i$ with $b_i \in A$ and $b_i \neq 0$, $i = 1, \ldots, 4n$. If we are in the first case, then Lemma 2 implies that all the a_i's except one, say a_1, are zero. But then $a_1 \in 2A$, which contradicts

$Tr_{F/\mathbb{Q}}(a_1 \bar{a}_1) = m$. In the second case we have $m = \frac{1}{4} \sum_{i=1}^{4n} b_i \bar{b}_i \geqq n \cdot m$, using Lemma 2. But $n > 1$ so this is impossible. Let $x_i = e_i - e_{i+1}$ for $i = 1, \ldots, 4n-1$ and let $x_{4n} = e_{4n-1} + e_{4n}$. We have $h(x_i, x_i) = 2$, so $x_1, \ldots, x_{4n}$ are irreducible. But $h(x_i, x_{i+1}) \neq 0$, so the x_i's are all in the same indecomposable component of $A\Gamma_{4n}$ (see Lemma 1). But the x_i's are linearly independent, so this component must be $A\Gamma_{4n}$.

Remark 1. The proposition can be generalized as follows: If (L, b) is definite, indecomposable, then $(\hat{L}, h)$ is also indecomposable. If K is a quadratic field, then this has been proved by L. Gerstein (cf. [4], Corollary 1.4) and R. Smith (cf. [13], Theorem 2.2). In the general case the analogue of this statement for quadratic forms has been proved by Y. Kitaoka (cf. [7], Corollary of Theorem 4). It is possible to adapt Kitaoka's proof to hermitian forms, only obvious changes are necessary.

Remark 2. Assume that A is integrally closed and that there exists an $\alpha \in A$ such that $\bar{\alpha} + \alpha = 1$. Then two indefinite non-singular hermitian forms are isometric if and only if they have the same rank, signatures and isometric determinants (cf. [2], Definition 1.9 and Corollary 4.10).

By contrast, the above proposition shows that the number of isometry classes of definite hermitian forms of rank $4n$ and determinant $\langle 1 \rangle$ is at least $p(n)$, where $p(n)$ is the number of partitions of n into a sum of positive integers. (See Gerstein [4], Theorem 3.9 for related results.)

2. Counter-examples to the cancellation of simple $(2q-1)$-knots, $q > 1$. Let $\lambda \in \mathbb{Z}[x]$ be an irreducible polynomial such that $\lambda(x) = x^{\deg \lambda} \cdot \lambda(x^{-1})$ and $\lambda(0) = \lambda(1) = \lambda(-1) = 1$.

Set $A = \mathbb{Z}[x]/(\lambda)$, $K = \mathbb{Q}[x]/(\lambda) = \mathbb{Q}(\tau)$. Then K has a $\mathbb{Q}$-involution which sends τ to τ^{-1}.

Let M be a torsion free A-module of finite rank. By results of Kearton, Levine and Trotter, we have: Every non-singular $(-1)^{q+1}$-hermitian form $h\colon M \times M \to A$ can be realized as the Blanchfield form of a simple $(2q-1)$-knot $\Sigma^{2q-1} \subset S^{2q+1}$ if $q > 2$. Two simple $(2q-1)$-knots are isotopic if and only if the associated Blanchfield forms are isometric, for $q > 1$ (cf. [6], [9], [14]). Therefore it is enough to show that cancellation does not always hold for non-singular hermitian and skew-hermitian forms.

Let us choose λ such that K is totally imaginary and that the fixed field F of the involution is totally real. (For instance, $\lambda(x) = x^4 - x^2 + 1$, the cyclotomic polynomial corresponding to the 12th roots of unity.)

We have:

$$(*) \quad A\Gamma_8 \perp A\Gamma_8 \perp \langle -1 \rangle \cong A\Gamma_{16} \perp \langle -1 \rangle$$

(where $\perp$ denotes orthogonal sum, and $\langle -1 \rangle$ is the hermitian form $Ae \times Ae \to A$ such that $ee = -1$). Indeed, this isomorphism already holds over $\mathbb{Z}$ (cf. [10], Chap. II, Theorem (4.3)). On the other hand, $A\Gamma_8 \perp A\Gamma_8$ is not isometric to $A\Gamma_{16}$ because the latter is indecomposable (see Section 1).

This gives the desired counter-example for q odd, $q \neq 1$.

Let $u = \tau - \tau^{-1}$. Then u is a unit of A because

$$N_{K/\mathbb{Q}}(u) = N_{K/\mathbb{Q}}(\tau^{-1}) \cdot N_{K/\mathbb{Q}}(\tau - 1) N_{K/\mathbb{Q}}(\tau + 1) = \lambda(0) \cdot \lambda(1) \cdot \lambda(-1) = 1 .$$

We have $\bar{u} = -u$, so multiplying (*) by u we obtain a counter-example to cancellation of non-singular skew-hermitian forms, i.e. for the case q even, $q \neq 2$.

We need a special argument for 3-knots. Let $h: M \times M \to A$ be a non-singular skew-hermitian form. There exists a simple 3-knot $\Sigma^3 \subset S^5$ such that the Blanchfield form of Σ^3 is isometric to h if and only if the signature of the intersection form corresponding to h is divisible by 16 (cf. [9], [14]).

Let Γ be the orthogonal sum of 16 copies of Γ_8. We have

$$A\Gamma \perp \langle 1 \rangle \perp \langle -1 \rangle \cong A\Gamma_{128} \perp \langle 1 \rangle \perp \langle -1 \rangle .$$

As before, we multiply by u in order to obtain skew-hermitian forms.

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