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A stability analysis for the arbitrary Lagrangian Eulerian formulation with finite elements

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Abstract — In this paper we present some theoretical results on the Arbitrary Lagrangian Eulerian (ALE) formulation. This formulation may be used when dealing with moving domains and consists in recasting the governing differential equation and the related weak formulation in a frame of reference moving with the domain.

The ALE technique is first presented in the whole generality for conservative equations and a result on the regularity of the underlying mapping is proven. In a second part of the work, the stability property of two types of finite element ALE schemes for parabolic evolution problems are analyzed and its relation with the so-called Geometric Conservation Laws is addressed.

1. INTRODUCTION

When dealing with the numerical computation of fluid problems in a moving domain, a possible way to proceed consists in adapting the mesh in order to follow the boundary movement, thus keeping a boundary conforming grid. In order to effectively apply the technique one has to rewrite the equations in a moving frame of reference, leading to the so called *Arbitrary Lagrangian Eulerian* (ALE) formulation. An early presentation of this technique may be found on a work of J. Donea [6]. It is based on the definition of an appropriate mapping from a reference configuration to the current, moving domain.

The ALE formulation has been used extensively for fluid structure interaction problems particularly for compressible fluid dynamics problems and aero–elasticity, using mainly finite difference and finite volume schemes. In some of those works numerical instabilities and oscillations were noted. The main cause has been related to a misrepresentation of the convective fluxes due to an inaccurate calculation of geometrical quantities such as surface normals and volumes of the control cells used in the finite-volume computations. Indeed, when dealing with a moving domain it is possible to write the differential equations governing the "evolution" of such geometrical quantities during the domain movement. A failure of the numerical scheme in correctly representing such an evolution may cause a loss of the conservation properties with possible resulting instabilities. Possible ways to overcome the problem have been devised for finite volume schemes and they result in an appropriate evaluation of the geometric quantities to be used in the time advancing scheme. This led to the development of the Geometric Conservation Laws (GCL). Unfortunately, no clear-cut analysis is so far available and the real significance of those conditions in terms of scheme stability properties has not yet been established. In [8], M. Lesoinne and C. Farhat analyze a particular finite volume ALE formulation and reconduce the GCL to a minimal condition on the precision of the scheme time quadrature

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formula. Notably, the limit depends on the number of space dimensions. In the same work, a preliminary analysis on the form of the GCL conditions for finite element schemes is given. In a later work [7], H. Guillard and C. Farhat have proved that the GCL are sufficient to the guarantee that particular finite volume based schemes remain at least first order time accurate, independently of the domain movement.

Nevertheless, a thorough analysis on the implication and possible limits of the ALE formulation in the context of finite element methods is still missing and this work is an attempt in that direction. In Section 2 of the paper we will describe in a rather abstract form the ALE approach for a generic conservation law of the type

$$\frac{\partial U}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{F}(U) = f. \tag{1.1}$$

The ALE approach consists in recasting the equations for a moving domain. We will present two possible ALE weak formulations of (1.1). The former will lead to non-conservative approximation schemes while the latter will maintain the conservation properties also at discrete level. Always in this Section 2, we will investigate the smoothness condition that the ALE mapping should satisfy if we wish to maintain a H^1 spatial regularity which is the one enjoyed by piecewise continuous finite element spaces. The objective is to find a condition compatible with an approximation of the ALE mapping by means of finite elements base functions, which is indeed the subject tackled next in the paper.

In Section 3, we will consider a linear transport-diffusion model problem and we will analyze the stability properties of the finite element approximation of the two weak formulations presented in Section 2. In this framework, we will introduce the Geometric Conservation Laws (GCL) which have been originally ideated for finite-volume schemes as a sort of "patch test" to which the discrete scheme should obey. We will recognize their formulations in a finite element scheme and we will assess their relevance for stability and, in particular, for obtaining a stability result independent of the domain movement law. It is shown in this paper that, although in general the GCL are neither necessary nor sufficient to that scope, for certain schemes the fulfillment of the GCL indeed ensures a stability result independent of the domain velocity.

2. THE ARBITRARY LAGRANGIAN EULERIAN (ALE) FORMULATION

The Arbitrary Lagrangian Eulerian frame of reference is adopted when the domain is moving. It may be defined in a way very similar to that of the Lagrangian frame widely used in continuum mechanics. Let $\mathcal{A}_t$ be a family of mappings, which at each $t \in (t_0, T)$ associate a point $\boldsymbol{Y}$ of a *reference configuration* Ω_0, taken to be equal to the domain configuration at time $t = t_0$, to a point $\boldsymbol{x}$ on the current domain configuration Ω_t. That is, for each $t \in (t_0, T)$,

$$\mathcal{A}_t : \Omega_0 \subset \mathbb{R}^d \to \Omega_t \subset \mathbb{R}^d, \qquad \boldsymbol{x}(\boldsymbol{Y}, t) = \mathcal{A}_t(\boldsymbol{Y}).$$

We will assume $\mathcal{A}_t$ to be an homeomorphism, that is $\mathcal{A}_t \in C^0(\overline{\Omega}_0)$ is invertible with continuous inverse $\mathcal{A}_t^{-1} \in C^0(\overline{\Omega}_t)$. Furthermore, we assume that the application

$$t \to \boldsymbol{x}(\boldsymbol{Y}, t), \quad \boldsymbol{Y} \in \Omega_0$$

is differentiable almost everywhere in $[t_0, T)$. In the following, we will denote by I the interval (t_0, T).

We will call $\boldsymbol{Y} \in \Omega_0$ the *ALE coordinate* while $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{Y}, t)$ will be indicated as the spatial (or Eulerian) coordinate.

In the following we will often have the necessity of switching between the different frame of reference. In order to avoid an excessive number of symbols we adopt the following convention. We will use the shorthand notation $\Omega_t \times I$ to indicate the set

$$\{(\boldsymbol{x}, t) \mid \quad \boldsymbol{x} \in \Omega_t, \, t \in I\}.$$

Let then $f: \Omega_t \times I \to \mathbb{R}$ be a function defined on the Eulerian frame and $\hat{f} := f \circ \mathcal{A}_t$ the corresponding function on the ALE frame, defined as

$$\hat{f}: \Omega_0 \times I \to \mathbb{R}, \quad \hat{f}(\boldsymbol{Y}, t) = f(\mathcal{A}_t(\boldsymbol{Y}), t).$$

We will indicate with the symbol $\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{Y}}$ the time derivative on the ALE frame, written in the spatial coordinate. It is defined as

$$\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{Y}} : \Omega_t \times I \to \mathbb{R}, \qquad \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{Y}} (\boldsymbol{x}, t) = \frac{\partial \hat{f}}{\partial t}(\boldsymbol{Y}, t), \quad \boldsymbol{Y} = \mathcal{A}_t^{-1}(\boldsymbol{x}). \tag{2.1}$$

For analogy, we will indicate by $\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}}$ the partial time derivative in the spatial frame.

We then define the domain velocity $\boldsymbol{w}$ as

$$\boldsymbol{w}(\boldsymbol{x}, t) = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{Y}}. \tag{2.2}$$

2.1. Derivation of ALE formulation for first order time evolution problems in conservative form

Let us consider a first order time evolution equation for a function

$$u: \Omega_t \times I \to \mathbb{R}$$

written as

$$\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}} + \mathcal{L}(u) = 0 \tag{2.3}$$

with appropriate initial and boundary conditions.

Here, $\mathcal{L}$ indicates a differential operator (linear or non linear) in the space variable $\boldsymbol{x}$. In order to find the equivalent equation for $u \circ \mathcal{A}_t$, a standard application of the chain rule to the time derivative gives

$$\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{Y}} = \left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}} + \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{Y}} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} u = \left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{w} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} u. \tag{2.4}$$

The symbol $\boldsymbol{\nabla}_{\boldsymbol{x}}$ is here used to indicate the gradient with respect to the $\boldsymbol{x}$ variable, while $\boldsymbol{\nabla}_{\boldsymbol{Y}}$ will be used when the gradient is taken with respect to the reference domain. The substitution of the previous result in equation (2.3) provides the following expression

$$\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{Y}} + \mathcal{L}(u) - \boldsymbol{w} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} u = 0 \tag{2.5}$$

which is the ALE counterpart of (2.3). It may be noted that the main difference with the original formulation is the appearance of a convective-type term due to the domain movement.

Often, PDE's governing continuum mechanics problems are written in conservative form, which reflects the fact that they express indeed conservation properties. Since the context in which the ALE technique is used is normally that of conservation laws, we will in the following always refer to equation written in conservative form. Nevertheless, large part of the results illustrated in this work may be readily extended to the more general case. The conservation equation for a quantity u is written as

$$\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{F} = f \tag{2.6}$$

where $\boldsymbol{F}$ indicates the flux vector which is generally a function of u and of its first and second space derivatives, while f is a possible *source term*. The application of relation (2.4) gives

$$\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{Y}} + \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot(\boldsymbol{F}) - \boldsymbol{w}\boldsymbol{\nabla}u = f. \tag{2.7}$$

Expression (2.7) represents one of the possible forms in which a conservation law may be cast in the ALE frame. Another possible ALE formulation may be directly derived from the integral formulation of the conservation equation. We indicate the Jacobian matrix of the ALE mapping as

$$\boldsymbol{J}_{\mathcal{A}_t} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{Y}}$$

and its determinant,

$$J_{\mathcal{A}_t} = \det(\boldsymbol{J}_{\mathcal{A}_t}).$$

We now make use in the following derivation of the *Euler expansion formula* [1], which relates the time evolution of $J_{\mathcal{A}_t}$ to the divergence of the domain velocity field, according to the following differential equation

$$\left.\frac{\partial J_{\mathcal{A}_t}}{\partial t}\right|_{\boldsymbol{Y}} = J_{\mathcal{A}_t}\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{w} \tag{2.8}$$

which is valid for $\boldsymbol{Y} \in \Omega_0$, $\boldsymbol{x} = \mathcal{A}_t(\boldsymbol{Y})$ and $t \in I$. Supplemented by the initial condition $J_{\mathcal{A}_t} = 1$ for $t = t_0$, expression (2.8) may be also interpreted as an evolution law for the Jacobian determinant, once the mesh velocity field is known. This interpretation is not the usual one, since expression (2.8) is normally regarded as an identity satisfied at each time during the domain evolution process. Yet, considering (2.8) as an evolution law may shed some light on a possible interpretation of the Geometric Conservation Laws, as it will be discussed later on. The derivation of expression (2.8), yet relative to a full Lagrangian frame, may be found in [1]. It may be readily extended to the ALE frame.

We wish now to find an expression for a term in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} u \,\mathrm{d}\Omega \tag{2.9}$$

where V_t is an arbitrary subdomain $V_t \subset \Omega_t$. We will indicate with V_0 the subset of Ω_0 such that $V_t = \mathcal{A}_t(V_0)$. We have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} u \,\mathrm{d}\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} u J_{\mathcal{A}_t} \,\mathrm{d}\Omega = \int_{V_0} \left.\frac{\partial(u J_{\mathcal{A}_t})}{\partial t}\right|_{\boldsymbol{Y}} \mathrm{d}\Omega \tag{2.10}$$

then, using expression (2.8), we finally obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} u\,\mathrm{d}\Omega = \int_{V_t}\left[\left.\frac{\partial u}{\partial t}\right|_{Y} + u\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\right]\mathrm{d}\Omega \tag{2.11}$$

which is a generalization of the well known *Reynolds transport formula* [1].

Consequently, the conservation equation (2.6) in integral form and ALE frame is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_t} u\,\mathrm{d}\Omega + \int_{V_t}\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{F}-\boldsymbol{w}u)\,\mathrm{d}\Omega = \int_{V_t} f\,\mathrm{d}\Omega. \tag{2.12}$$

In the previous relation all integrals have been expressed in the current frame of reference. However, one may choose to write the equation with respect to the reference domain. This would lead to the following expression:

$$\int_{V_0}\left\{\left.\frac{\partial(J_{A_t}u)}{\partial t}\right|_{Y} + J_{A_t}\left[\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{F}-\boldsymbol{w}u) - f\right]\right\}\mathrm{d}\Omega = 0 \tag{2.13}$$

and, due to the arbitrariness of V_0, we may write the following differential equation,

$$\left.\frac{\partial(J_{A_t}u)}{\partial t}\right|_{Y} + J_{A_t}\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{F}-\boldsymbol{w}u) = J_{A_t}f \tag{2.14}$$

which is another form in which the ALE equations are often found (see, for instance, [12])†. Relation (2.14) could have been derived directly from (2.7) by employing the Euler expansion formula (2.8). Indeed, it may be said that expression (2.14) is equivalent to (2.7) if the relation (2.8) is true.

This somehow lengthy presentation had the main objective to derive some of the different formulations in which the ALE equations are presented in the literature. Another approach may be followed by using a weak formulation as a starting point, which is the basis for the derivation of ALE finite elements.

2.2. Weak formulation in the ALE frame

The flux $\boldsymbol{F}(u)$ may be often decomposed into two parts

$$\boldsymbol{F}(u) = \boldsymbol{F}_e(u) + \boldsymbol{F}_v(u) \tag{2.15}$$

where $\boldsymbol{F}_e$ does not contain any derivative of u, while $\boldsymbol{F}_v(u)$ contains first order spatial derivatives of the unknown. A typical case is the Navier-Stokes equations which govern fluid dynamics, where $\boldsymbol{F}_e$ contains the convective terms, while $\boldsymbol{F}_v(u)$ represents the viscous fluxes.

A weak formulation of (2.6) may be formally obtained as

$$\int_{\Omega_t}\psi\left(\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{F}_e(u)\right)\mathrm{d}\Omega - \int_{\Omega_t}\boldsymbol{F}_v(u)\cdot\nabla_{\boldsymbol{x}}\psi\,\mathrm{d}\Omega = \int_{\Omega_t}\psi f\mathrm{d}\Omega \quad \forall\,\psi\in\mathcal{W}(\Omega_t) \tag{2.16}$$

where u is sought in an appropriate functional space and $\mathcal{W}(\Omega_t)$ is the space of test functions defined on Ω_t, with the required regularity at each time t. Relation (2.16) is formally the weak

†Relation (2.14) may be written completely on the reference domain transforming the divergence term by exploiting the Piola-Kirchoff theorem [5]. We omit the derivation here; it may be found in [12].

formulation for a moving boundary problem. Yet, in that form is impractical, since it contains a time derivative in the Eulerian frame, while it will be natural to work with variables that follow the domain evolution. The test functions moreover, cannot be taken constant with time since they should vanish on the part of the moving boundary where essential boundary conditions are applied. It is then natural to recast them in the moving frame of reference as well.

To that purpose, we consider a space of admissible test functions $\mathcal{Y}(\Omega_0)$, defined on the *reference* domain, and made of functions $\hat{\psi}: \Omega_0 \to \mathbb{R}$ that are smooth enough. The ALE mapping then identifies a corresponding set $\mathcal{X}(\Omega_t)$ of weighting functions on the "current configuration", defined as follows:

$$\mathcal{X}(\Omega_t) = \{\psi: \Omega_t \times I \to \mathbb{R}, \quad \psi = \hat{\psi} \circ \mathcal{A}_t^{-1}, \ \hat{\psi} \in \mathcal{Y}(\Omega_0)\}. \tag{2.17}$$

Clearly, we must have that, at each time t, $\mathcal{X}(\Omega_t) \subset \mathcal{W}(\Omega_t)$ in order that the function space is admissible. This condition will impose constraints on the regularity of the mapping, as it will be analyzed in a later section for the particular case of $\mathcal{W}(\Omega_t) \equiv H^1(\Omega_t)$.

In the following we will illustrate two possible ways of building a weak formulation in the ALE frame.

2.2.1. A non-conservative formulation

If we transform the Eulerian time derivative of relation (2.16) into its ALE counterpart the following weak formulation may be written

$$\int_{\Omega_t} \psi \left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{Y}} \mathrm{d}\Omega + \int_{\Omega_t} \psi\,(\nabla_{\boldsymbol{x}}\cdot \boldsymbol{F}_e - \boldsymbol{w}\nabla_{\boldsymbol{x}} u)\,\mathrm{d}\Omega - \int_{\Omega_t} \boldsymbol{F}_v(u)\cdot\nabla_{\boldsymbol{x}}\psi\,\mathrm{d}\Omega = \int_{\Omega_t} \psi f\,\mathrm{d}\Omega \quad \forall\, \psi \in \mathcal{X}(\Omega_t). \tag{2.18}$$

This equation stems directly from the weak formulation of the original problem, recast in the ALE frame.

2.2.2. A conservative ALE formulation

Another ALE weak formulation may be obtained bearing in mind that functions in $\mathcal{Y}(\Omega_0)$ do not depend on time. An immediate consequence is that the following important relation may be written:

$$0 = \left.\frac{\partial \psi}{\partial t}\right|_{\boldsymbol{Y}} = \left.\frac{\partial \psi}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{w}\cdot\nabla_{\boldsymbol{x}}\psi \quad \forall\, \psi \in \mathcal{X}(\Omega_t) \tag{2.19}$$

and then, for any time-differentiable function $g = g(\boldsymbol{x}, t)$, we have that

$$\left.\frac{\partial(\psi g)}{\partial t}\right|_{\boldsymbol{Y}} = \psi \left.\frac{\partial g}{\partial t}\right|_{\boldsymbol{Y}} \quad \forall\, \psi \in \mathcal{X}(\Omega_t). \tag{2.20}$$

By recalling expression (2.11), one may then write the following useful formulae, valid for any $\psi, \chi \in \mathcal{X}(\Omega_t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \psi\,\mathrm{d}\Omega = \int_{\Omega_t} \psi\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\,\mathrm{d}\Omega \tag{2.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} \psi u\,\mathrm{d}\Omega = \int_{\Omega_t} \psi\left(\left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{Y}} + u\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\right)\mathrm{d}\Omega \tag{2.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\psi\chi\,\mathrm{d}\Omega = \int_{\Omega_t}\psi\chi\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\,\mathrm{d}\Omega. \tag{2.23}$$

The alternative ALE weak formulation may then be obtained following two routes. The first starts from (2.16), taking $\psi \in \mathcal{X}(\Omega_t)$, expanding the time derivative using (2.4) and finally exploiting relation (2.22). The result is the following expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\psi u\,\mathrm{d}\Omega + \int_{\Omega_t}\psi(\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{F}_e(u) - \boldsymbol{w}u))\,\mathrm{d}\Omega - \int_{\Omega_t}\boldsymbol{F}_v(u)\cdot\nabla_{\boldsymbol{x}}\psi\,\mathrm{d}\Omega = \int_{\Omega_t}\psi f\,\mathrm{d}\Omega \;\;\forall\,\psi \in \mathcal{X}(\Omega_t). \tag{2.24}$$

The second route starts from the differential expression (2.14) which in weak form reads

$$\int_{\Omega_0}\hat{\psi}\left(\left.\frac{\partial(J_{\mathcal{A}_t}u)}{\partial t}\right|_Y\right)\mathrm{d}\Omega + \int_{\Omega_0}\hat{\psi}J_{\mathcal{A}_t}\left(\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{F}_e(u) - \boldsymbol{w}u)\right)\mathrm{d}\Omega - \int_{\Omega_0}\boldsymbol{F}_v(u)\cdot\nabla_{\boldsymbol{x}}\hat{\psi}\,\mathrm{d}\Omega$$
$$= \int_{\Omega_0}\hat{\psi}f\,\mathrm{d}\Omega \quad \forall\,\psi \in \mathcal{Y}(\Omega_0). \tag{2.25}$$

By exploiting the fact that $\left.\frac{\partial\psi}{\partial t}\right|_Y = 0$ the time derivative may be moved out of the integral sign and the integrals may be transformed on the current domain configuration, leading again to (2.24). In this formulation the transient term is expressed as a total time derivative while the ALE convection term appears in the form of the divergence of the product of the mesh velocity field and the solution u. The time derivative term accounts for both effects due to the variation of the solution u and the change in the grid nodes position. For conservation equations, it has the advantage that the ALE term is itself in "conservation form", therefore the modification of an existing "fixed-grid" code is (at least apparently) straightforward, as it is just required to change the definition of the fluxes. In addition, the formulation is "conservative", in the sense that, taking any $V \subset \Omega_t$ with Lipschitz continuous boundary, should $\psi|_V = \text{const}$ be admissible[‡], we derive from (2.24), taking $s = 0$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V u\,\mathrm{d}\Omega + \int_{\partial V}\boldsymbol{F}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma - \int_{\partial V}u\boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma = 0 \tag{2.26}$$

which indeed expresses the fact that, in absence of source terms, the variation of u over V is due only to contribution coming from the boundary of V. It can be noted that also the contribution of the ALE term to the conservation reduces to a boundary term, which is indeed related to the additional "flux" of u through the boundary as a consequence of its movement[§].

Formulations (2.18) and (2.24) are equivalent at the continuous level, but they lead to different discrete systems. In particular, the conservation property just mentioned, may not be satisfied by the discrete systems associated to (2.18).

2.3. Considerations on the regularity of the ALE mapping

We may note that the fulfillment of the appropriate regularity condition for the functions in $\mathcal{X}(\Omega_t)$ may impose a certain level of regularity to the ALE mapping. Since we are mainly

[‡]This is always the case if $\overline{V} \subset \Omega_t$.

[§]We may anticipate that for the Navier-Stokes equation the "ALE" fluxes will exactly balance the convective fluxes on the part of the boundary which moves at the same velocity as the fluid, reflecting the fact that there is no mass exchange through that portion of boundary.

interested in fluid flow and elasticity problems, we would deal with functions $u(\boldsymbol{x},t)$ so that

$$u(\cdot,t)\colon I \to V(\Omega_t) \subset H^1(\Omega_t).$$

Therefore, in the following we will investigate the regularity required on the mapping in order that if $\mathcal{Y}(\Omega_0) \subset H^1(\Omega_0)$ then, at each time t during the domain motion the 'transformed function space space' $\mathcal{X}(\Omega_t)$ remains a subspace of $H^1(\Omega_t)$.

2.3.1. Some additional nomenclature

In the following we will make use of standard function spaces. We will indicate by $L^p(\Omega)$, with $1 \le p \le \infty$, the set of measurable functions v defined on $\Omega \subset \mathbb{R}^d$ and such that

$$\int_\Omega |v(\boldsymbol{x})|^p \,\mathrm{d}\Omega < \infty. \tag{2.27}$$

The set $L^p(\Omega)$ forms a Banach space when equipped with the norm

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(\boldsymbol{x})|^p \,\mathrm{d}\Omega\right)^{\frac{1}{p}}. \tag{2.28}$$

In the case of $L^\infty(\Omega)$ the norm is defined as

$$\|v\|_{L^\infty(\Omega)} = \inf\{M| \quad |v(\boldsymbol{x})| < M \quad \text{a.e. in } \Omega\}. \tag{2.29}$$

With the Sobolev space $W^{k,p}(\Omega)$, with $k > 0$ integer and $1 \le p \le \infty$, we indicate the class of functions

$$W^{k,p}(\Omega) = \{v \in L^p(\Omega)| \quad D^\alpha v \in L^p(\Omega), \quad |\boldsymbol{\alpha}| \le k\} \tag{2.30}$$

being $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$, with $\alpha_i > 0$ integer, and $|\boldsymbol{\alpha}| = \alpha_1 + \cdot + \alpha_d$. D^α indicates the distributional partial derivative

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d}. \tag{2.31}$$

With $H^p(\Omega)$ it is indicated the Hilbert space $W^{p,2}(\Omega)$. Finally, $H^1_0(\Omega)$ indicates the set formed by functions of $H^1(\Omega)$ with zero trace on $\partial\Omega$.

For vector functions, we will use the bold symbols. For example $\boldsymbol{L}^p(\Omega)$ indicates the space of vector functions $\boldsymbol{v}$ such that each component v_i satisfies $v_i \in L^p(\Omega)$. The associated norm is then defined by

$$\|\boldsymbol{v}\|^2_{\boldsymbol{L}^p(\Omega)} = \sum \|v_i\|^2_{L^p(\Omega)}. \tag{2.32}$$

When dealing with space-time functions

$$v(t,\boldsymbol{x}) \quad \text{with} \quad (t,\boldsymbol{x}) \in I \times \Omega$$

we will make use of the spaces

$$L^2(I; H^p(\Omega)) = \{v\colon I \to H^p(\Omega)| \quad v \text{ measurable}, \int_I \|v(t)\|^2_{H^p(\Omega)}\,\mathrm{d}t < \infty\} \tag{2.33}$$

and

$$H^1(I; H^p(\Omega)) = \left\{v \in L^2(I; H^p(\Omega))| \quad \frac{\partial v}{\partial t} \in L^2(I; H^p(\Omega))\right\}. \tag{2.34}$$

The dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$, is formed by all continuous linear operators on functions belonging to $H_0^1(\Omega)$. We will indicate the duality paring between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ by

$$\langle f, v\rangle, \qquad f \in H^{-1}(\Omega), \quad v \in H_0^1(\Omega). \tag{2.35}$$

The space $H^{-1}(\Omega)$ is equipped with the norm

$$\|f\|_{H^{-1}(\Omega)} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{H^1(\Omega)} \neq 0}} \frac{\langle f, v\rangle}{\|v\|_{H^1(\Omega)}}. \tag{2.36}$$

All these definitions can naturally be extended to a moving domain Ω_t. For example,

$$L^2(I; H^p(\Omega_t)) = \{v: I \to H^p(\Omega_t) | \quad v \text{ measurable}, \int_I \|v(t)\|^2_{H^p(\Omega_t)}\, dt < \infty\}. \tag{2.37}$$

2.3.2. Mapping regularity condition

The following problem will be addressed.

Problem 2.1. Find sufficient conditions for the ALE mapping $\mathcal{A}_t$ so that if $\hat{v} \in H^1(\Omega_0)$ then $v = \hat{v} \circ \mathcal{A}_t^{-1} \in H^1(\Omega_t)$ and vice-versa.

Classical results [2] indicate that a sufficient condition is that $\mathcal{A}_t$ be a C^1-diffeomorphism, which implies that $\forall\, t \in I$

$$\mathcal{A}_t \in \boldsymbol{C}^1(\overline{\Omega}_0), \qquad \mathcal{A}_t^{-1} \in \boldsymbol{C}^1(\overline{\Omega}_t) \tag{2.38}$$

and, moreover,

$$\boldsymbol{J}_{\mathcal{A}_t} \in \boldsymbol{L}^\infty(\Omega_0), \qquad \boldsymbol{J}_{\mathcal{A}_t^{-1}} \in \boldsymbol{L}^\infty(\Omega_t). \tag{2.39}$$

Unfortunately, this requirement is too restrictive for our purposes. In fact, we would like to express the ALE mapping by means of finite element shape functions, which are required to be in $H^1(\Omega)$, but not necessarily in $C^1(\overline{\Omega})$. The reason is that in practical applications, we will reconstruct the ALE mapping from the boundary movement and, as we shall see later on, we will use for this purpose a finite element space discretisation. We need then to slightly relax the requirements stated above. This is possible by imposing some (quite reasonable) constraints on Ω_t.

We can then state the following Proposition.

Proposition 2.1. *Let Ω_0 be a bounded domain with Lipschitz continuous boundary and $\mathcal{A}_t$ be invertible in $\overline{\Omega}_0$ and satisfying the following conditions: for each $t \in I$*

- *$\Omega_t = \mathcal{A}_t(\Omega_0)$ is bounded and Lipschitz continuous*[¶],
- *we have that*

$$\mathcal{A}_t \in \boldsymbol{W}^{1,\infty}(\Omega_0), \qquad \mathcal{A}_t^{-1} \in \boldsymbol{W}^{1,\infty}(\Omega_t). \tag{2.40}$$

Then, $v \in H^1(\Omega_t)$ if and only if $\hat{v} = v \circ \mathcal{A}_t \in H^1(\Omega_0)$. Moreover, $\|v\|_{H^1(\Omega_t)}$ is equivalent to $\|\hat{v}\|_{H^1(\Omega_0)}$.

Proof. Under the assumption (2.40), Sobolev embedding theorem assures that

$$\mathcal{A}_t \in \boldsymbol{C}^0(\overline{\Omega}_0), \qquad \mathcal{A}_t^{-1} \in \boldsymbol{C}^0(\overline{\Omega}_t) \tag{2.41}$$

[¶] If $\mathcal{A}_t$ is a C^1-diffeomorphism this requirement would be automatically satisfied, see, for example, [5].

and, as consequence of the hypotheses, the Jacobian

$$\boldsymbol{J}_{\mathcal{A}_t}\colon \Omega_0 \to \mathbb{R}^{d\times d}$$

and

$$\boldsymbol{J}_{\mathcal{A}_t^{-1}}\colon \Omega_t \to \mathbb{R}^{d\times d}$$

are in $\boldsymbol{L}^\infty(\Omega_0)$ and $\boldsymbol{L}^\infty(\Omega_t)$, respectively, and their determinants, $J_{\mathcal{A}_t}$ and $J_{\mathcal{A}_t^{-1}}$, being made of products of L^∞ functions defined on a bounded domain, are themselves in $L^\infty(\Omega_0)$ and $L^\infty(\Omega_t)$, respectively. Moreover, we have

$$\|\boldsymbol{J}_{\mathcal{A}_t} \circ \mathcal{A}_t^{-1}\|_{L^\infty(\Omega_t)} = \|\boldsymbol{J}_{\mathcal{A}_t}\|_{L^\infty(\Omega_0)} \tag{2.42}$$

$$\|\boldsymbol{J}_{\mathcal{A}_t^{-1}} \circ \mathcal{A}_t\|_{L^\infty(\Omega_0)} = \|\boldsymbol{J}_{\mathcal{A}_t^{-1}}\|_{L^\infty(\Omega_t)}. \tag{2.43}$$

Finally, because of the invertibility of $\mathcal{A}_t$, there is no loss of generality in assuming that there are two positive constants c_1, c_2 such that

$$J_{\mathcal{A}_t}(\boldsymbol{Y}) \geq c_1 \qquad \forall\, \boldsymbol{Y} \in \Omega_0,\ \forall\, t \in I$$

and

$$J_{\mathcal{A}_t^{-1}}(\boldsymbol{x}) \geq c_2 \qquad \forall\, \boldsymbol{x} \in \Omega_t,\ \forall\, t \in I.$$

We then have

$$\|\hat{v}\|^2_{L_2(\Omega_0)} = \int\limits_{\Omega_0} \hat{v}^2\, \mathrm{d}\Omega = \int\limits_{\Omega_t} J_{\mathcal{A}_t^{-1}} v^2\, \mathrm{d}\Omega \leq \|J_{\mathcal{A}_t^{-1}}\|_{L_\infty(\Omega_t)} \|v\|^2_{L_2(\Omega_t)} \tag{2.44}$$

$$\|v\|^2_{L_2(\Omega_t)} = \int\limits_{\Omega_t} v^2\, \mathrm{d}\Omega = \int\limits_{\Omega_0} J_{\mathcal{A}_t} \hat{v}^2\, \mathrm{d}\Omega \leq \|J_{\mathcal{A}_t}\|_{L_\infty(\Omega_0)} \|\hat{v}\|^2_{L_2(\Omega_0)}. \tag{2.45}$$

In addition,

$$|\hat{v}|^2_{H^1(\Omega_0)} = \int\limits_{\Omega_0} \sum_{k=1}^{d} \left(\frac{\partial \hat{v}}{\partial Y_k}\right)^2 \mathrm{d}\Omega = \int\limits_{\Omega_t} J_{\mathcal{A}_t^{-1}} \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \frac{\partial v}{\partial x_j} \frac{\partial x_j}{\partial Y_k}\right)^2 \mathrm{d}\Omega. \tag{2.46}$$

We have

$$\begin{aligned} \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \frac{\partial v}{\partial x_j} \frac{\partial x_j}{\partial Y_k}\right)^2 &\leq \sum_{k=1}^{d} \left[\sum_{j=1}^{d} \left(\frac{\partial v}{\partial x_j}\right)^2\right] \left[\sum_{j=1}^{d} \left(\frac{\partial x_j}{\partial Y_k}\right)^2\right] \\ &\leq d \sum_{k=1}^{d} \max_j \left(\frac{\partial x_j}{\partial Y_k}\right)^2 \sum_{j=1}^{d} \left(\frac{\partial v}{\partial x_j}\right)^2 \leq d^2 \left(\max_{jk} \left|\frac{\partial x_j}{\partial Y_k}\right|\right)^2 \sum_{j=1}^{d} \left(\frac{\partial v}{\partial x_j}\right)^2. \end{aligned} \tag{2.47}$$

By using the previous relation into expression (2.46), we obtain

$$|\hat{v}|^2_{H^1(\Omega_0)} \leq d^2 \|J_{\mathcal{A}_t^{-1}}\|_{L_\infty(\Omega_t)} \|\boldsymbol{J}_{\mathcal{A}_t}\|^2_{L_\infty(\Omega_0)} |v|^2_{H^1(\Omega_t)}. \tag{2.48}$$

Analogously, by exchanging the role of v and $\hat{v}$ we have

$$|v|^2_{H^1(\Omega_t)} \leq d^2 \|J_{\mathcal{A}_t}\|_{L_\infty(\Omega_0)} \|\boldsymbol{J}_{\mathcal{A}_t^{-1}}\|^2_{L_\infty(\Omega_t)} |\hat{v}|^2_{H^1(\Omega_0)} \tag{2.49}$$

by which the theorem is proved. □

As for the time regularity of the mapping, we will assume that the function $\boldsymbol{x}(\boldsymbol{Y}, t)$ satisfies

$$\boldsymbol{x} \in H^1\left(I; \boldsymbol{W}^{1,\infty}(\Omega_0)\right). \tag{2.50}$$

Proposition 2.2. *Under the assumption* (2.50), *we have that if* $\hat{v} \in H^1(I, H^1(\Omega_0))$ *then,* $v = \hat{v} \circ \mathcal{A}_t^{-1} \in H^1(I, H^1(\Omega_t))$ *and*

$$\left.\frac{\partial v}{\partial t}\right|_{\boldsymbol{Y}} \in L^2(I, H^1(\Omega_t)). \tag{2.51}$$

Proof. Indeed, with the help of Proposition 2.1 we have established an isomorphism between $H^1(\Omega_0)$ and $H^1(\Omega_t)$. Being $H^1(\Omega_0)$ a separable Hilbert space, we may express any $\hat{v} \in H^1(I; H^1(\Omega_0))$ as

$$\hat{v}(\boldsymbol{Y}, t) = \sum_{i=1}^{\infty} \hat{v}_i(t) \Psi_i(\boldsymbol{Y}) \tag{2.52}$$

where $\{\Psi_i(\boldsymbol{Y})\}$ is an orthonormal basis of $H^1(\Omega_0)$ and $\hat{v}_i(t) = (v, \Psi_i)_{H^1(\Omega_0)}$ is the corresponding Fourier coefficient. We have indicated by $(\cdot, \cdot)_{H^1(\Omega_0)}$ the scalar product in $H^1(\Omega_0)$. Clearly, $\hat{v}_i(t) \in H^1(I)$. Then, we have

$$\left.\frac{\partial v}{\partial t}\right|_{\boldsymbol{Y}} \circ \mathcal{A}_t = \frac{\partial \hat{v}}{\partial t} = \sum_{i=1}^{\infty} \frac{\mathrm{d}\hat{v}_i}{\mathrm{d}t} \Psi_i. \tag{2.53}$$

Therefore,

$$\left.\frac{\partial v}{\partial t}\right|_{\boldsymbol{Y}} \circ \mathcal{A}_t \in L^2(I; H^1(\Omega_0)). \tag{2.54}$$

Finally, we note that the set $\{\Phi_i | \quad \Phi_i = \Psi_i \circ \mathcal{A}_t^{-1}\}$ forms a complete basis (not necessarily orthogonal) of $H^1(\Omega_t)$, thanks to the equivalence of norms in $H^1(\Omega_t)$ and $H^1(\Omega_0)$ proved in Proposition 2.1. Then we have

$$v = \hat{v} \circ \mathcal{A}_t^{-1} = \sum_{i=1}^{\infty} \hat{v}_i \Psi_i \circ \mathcal{A}_t^{-1} = \sum_{i=1}^{\infty} \hat{v}_i \Phi_i \tag{2.55}$$

thus $v \in H^1(I, H^1(\Omega_t))$. Furthermore,

$$\left.\frac{\partial v}{\partial t}\right|_{\boldsymbol{Y}} = \frac{\partial \hat{v}}{\partial t} \circ \mathcal{A}_t^{-1} = \sum_{i=1}^{\infty} \frac{\mathrm{d}\hat{v}_i}{\mathrm{d}t} \Psi_i \circ \mathcal{A}_t^{-1} = \sum_{i=1}^{\infty} \frac{\mathrm{d}\hat{v}_i}{\mathrm{d}t} \Phi_i. \tag{2.56}$$

Then, $\left.\frac{\partial v}{\partial t}\right|_{\boldsymbol{Y}} \in L^2(I; H^1(\Omega_t))$. □

2.4. A practical construction of the ALE mapping $\mathcal{A}_t$

In practice we are normally faced with the following problem.

Problem 2.2. Given the time evolution of the domain boundary

$$g: \partial\Omega_0 \times I \to \partial\Omega_t$$

find an ALE mapping $\mathcal{A}_t$ such that, at each time $t \in I$,

$$\mathcal{A}_t(\boldsymbol{Y}) = \boldsymbol{g}(\boldsymbol{Y}, t), \qquad \boldsymbol{Y} \in \partial\Omega_0.$$

Several techniques have been proposed in the literature. For instance, one may construct the domain motion by considering the domain as an "elastic" or viscoelastic solid and solve the stated problem by resorting to the equations of elastodynamics. This approach is used, for example in [9]. Yet, one may look to simplified models. Here, we present two possibilities without the pretension of being exhaustive.

2.4.1. Solving a parabolic system

It consists in finding a solution to the following problem.

Problem 2.3. Given the initial configuration Ω_0 and the law of evolution of the domain boundary $\boldsymbol{g}$ find

$$\boldsymbol{x}: \Omega_0 \times I \to \Omega_t$$

such that

$$\begin{cases} \dfrac{\partial \boldsymbol{x}}{\partial t} - \nabla_{\boldsymbol{Y}}\cdot(\kappa \nabla_{\boldsymbol{Y}} \boldsymbol{x}) = \boldsymbol{0}, & \boldsymbol{Y} \in \Omega_0,\ t \in I \\ \boldsymbol{x}(\boldsymbol{Y}, 0) = \boldsymbol{Y}, & \boldsymbol{Y} \in \Omega_0 \\ \boldsymbol{x}(\boldsymbol{Y}, t) = \boldsymbol{g}(\boldsymbol{Y}, t), & \boldsymbol{Y} \in \partial\Omega_0,\ t \in I. \end{cases} \tag{2.57}$$

Here κ is a positive constant (which may be taken equal to 1).

Remark 2.1. We may note that a more complex expression for κ, by letting it be a tensor function with coefficients depending on the numerical solution of the problem at hand, may allow to implement a mesh adaption scheme based on node movement, at very little computational price, since the domain movement has to be computed anyway.

2.4.2. Harmonic extension

Very often we need to know the ALE mapping only at discrete time levels, where the approximate solution of the problem at hand is sought. The data of the problem are the reference (initial) configuration and the new position of the boundary, which could be described as a function $\boldsymbol{h} : \partial\Omega_0 \to \partial\Omega_T$, being Ω_T the configuration at the given time T. In this case a simple alternative to the technique just presented consists in making an harmonic extension of $\boldsymbol{h}$ onto the whole Ω_0, in order to obtain the ALE mapping at time T. That is, one solves the following problem. following problem.

Problem 2.4. Given Ω_0 and $\boldsymbol{h}$, find $\boldsymbol{x} : \Omega_0 \to \Omega_T$ such that

$$\begin{cases} \nabla_{\boldsymbol{Y}}\cdot(\kappa \nabla_{\boldsymbol{Y}} \boldsymbol{x}) = \boldsymbol{0}, & \boldsymbol{Y} \in \Omega_0 \\ \boldsymbol{x}(\boldsymbol{Y}) = \boldsymbol{h}(\boldsymbol{Y}), & \boldsymbol{Y} \in \partial\Omega_0. \end{cases} \tag{2.58}$$

Again, if κ is a function of the numerical solution it may be used to drive an adaption type procedure.

2.5. Finite element discretisation of the ALE mapping

Our final objective will be the numerical solution of problems (2.18) or (2.24) by a finite element method. The choice of finite element space for the main variable u will undoubtedly affect the type of discretisation to be used for the ALE mapping. In particular, the discrete ALE mapping should be such that the domain triangulation maintains during its movement its suitability with respect to the chosen finite element space. For instance, if we use linear finite elements we need to ensure that the images of the mesh during the domain movement maintain straight edges.

There are thus two inter-related issues which must be faced.

1. Finding the appropriate discrete formulation for (2.18) and (2.24).

2. Finding a suitable finite element discretisation, $\mathcal{A}_{h,t}$, for the ALE mapping and, consequently, an algorithm for solving expression (2.57) or (2.58).

We will first briefly recall some basics of the finite element method which are necessary for our discussion. The domain Ω_0 is discretised by partitioning it into a finite number of (possibly curved) polyhedra called finite elements. The set of finite elements is called mesh and it is indicated by $\mathcal{T}_{0,h}$. The discretised domain $\Omega_{h,0}$, formed by the union of all mesh elements, may differ from Ω_0 because of the approximation of the boundary geometry. Yet, since this fact is not particularly relevant for our discussion, in the following we will assume $\Omega_{h,0} = \Omega_0$. We consider Lagrangian finite elements and the general case of a finite element function space $\mathcal{F}_{n,k}$ of degree n and parametric mapping degree k, defined as follows:

$$\mathcal{F}_{n,k}(\mathcal{T}_{0,h}) = \{\widehat{\psi}_h: \Omega_0 \to \mathbb{R} \mid \quad \widehat{\psi}_h \in C^0(\overline{\Omega}_0), \ \widehat{\psi}_h\big|_{K_0} \circ \mathcal{M}_k^{K_0} \in P_n(K_R) \quad \forall\, K_0 \in \mathcal{T}_{0,h}\} \tag{2.59}$$

where, $\widehat{\psi}_h\big|_{K_0}$ indicates the restriction of $\widehat{\psi}_h$ to the finite element K_0, $P_n(K_R)$ is the space of polynomials of degree n defined on a reference element K_R and $\mathcal{M}_k^{K_0} \in P_k(K_R)$ is a homeomorphic mapping from K_R to K_0. In general $k \leq n$ and in particular it is either equal to 1 (affine mapping) or n (isoparametric mapping). Since we wish to consider the general case, we will indicate with the term "vertices" the finite element nodes which are used for the parametric mapping. $\mathcal{M}_k^{K_0}$ is defined as follows:

$$\mathcal{M}_k^{K_0} : K_R \to K_0, \quad \boldsymbol{Y}(\boldsymbol{\eta}) = \mathcal{M}_k^{K_0}(\boldsymbol{\eta}) = \sum_{i \in \mathcal{N}^K} \boldsymbol{Y}_i \tilde{\phi}_i(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in K_R, \quad \tilde{\phi}_i \in P_k(K_R). \tag{2.60}$$

Here, $\tilde{\phi}_i$ is the base function associated to i-th vertex of the reference element, while $\boldsymbol{Y}_i$ is the coordinate of the correspondent vertex in $\mathcal{T}_{0,h}$. The sum extends over all vertices of K_R, here indicated by $\mathcal{N}^K$. It can be shown that $\mathcal{F}_{n,k}(\mathcal{T}_{0,h}) \in H^1(\Omega_0)$, and, in particular, $\mathcal{F}_{n,k}(\mathcal{T}_{0,h}) \in W^{1,\infty}(\Omega_0)$. In case of an affine mapping the finite element space $\mathcal{F}_{n,1}$ reduces to the more familiar expression:

$$\mathcal{F}_{n,1}(\mathcal{T}_{0,h}) = \{\widehat{\psi}_h: \Omega_0 \to \mathbb{R} \mid \quad \widehat{\psi}_h \in C^0(\overline{\Omega}_0), \quad \widehat{\psi}_h\big|_{K_0} \in P_n(K_0) \quad \forall\, K_0 \in \mathcal{T}_{0,h}\}. \tag{2.61}$$

Should we wish to utilize a finite element discretisation for our problem, the space of test functions $\mathcal{Y}(\Omega_0)$ will be approximated by $\mathcal{X}_h(\Omega_0) = \mathcal{F}_{n,k}(\mathcal{T}_{0,h})$; we remark that this implies, in particular, that $\forall\, K_0 \in \mathcal{T}_{0,h}, \ \ \mathcal{M}_k^{K_0} \in P_k(K_R)$. In Subsection 2.2 we have advocated that the proper ALE extension of the discrete test function space to a moving domain would be

$$\mathcal{X}_h(\Omega_t) = \{\psi_h: \Omega_t \times I \to \mathbb{R}, \quad \psi_h \circ \mathcal{A}_{h,t} = \widehat{\psi}_h, \ \widehat{\psi}_h \in \mathcal{X}_h(\Omega_0)\} \tag{2.62}$$

where now we use the discrete ALE mapping. Therefore, in order to be consistent with the chosen finite element discretisation, we should require that at any t $\mathcal{X}_h(\Omega_t) = \mathcal{F}_{n,k}(\mathcal{T}_{t,h})$ where $\mathcal{T}_{t,h}$ is the image under the ALE mapping of $\mathcal{T}_{0,h}$. If $\boldsymbol{x}_i(t)$ denotes the position of the i-th vertex at time t, we may formally define $\mathcal{M}_k^{K_t}$ on each element K_t of $\mathcal{T}_{t,h}$ as follows:

$$\mathcal{M}_k^{K_t}: K_R \to K_t, \quad \boldsymbol{x}(\boldsymbol{\eta}) = \mathcal{M}_k^{K_t}(\boldsymbol{\eta}) = \sum_{i \in \mathcal{N}^K} \boldsymbol{x}_i(t) \tilde{\phi}_i(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in K_R, \quad \tilde{\phi}_i \in P_k(K_R) \tag{2.63}$$

while $\mathcal{F}_{n,k}(\mathcal{T}_{t,h})$ is

$$\mathcal{F}_{n,k}(\mathcal{T}_{t,h}) = \{\psi_h: \Omega_t \to \mathbb{R} \mid \quad \psi_h \in C^0(\Omega_t), \ \psi_h|_{K_t} \circ \mathcal{M}_k^{K_t} \in P_n(K_R) \quad \forall\, K_t \in \mathcal{T}_{t,h}\}. \tag{2.64}$$

Proposition 2.3. *If, at any $t \in I$ the discrete* ALE *mapping satisfies*

$$\mathcal{A}_{h,t}|_{K_0} \circ \mathcal{M}_k^{K_0} = \mathcal{M}_k^{K_t} \quad \forall\, K_0 \in \mathcal{T}_{0,h}, \quad K_t = \mathcal{A}_t(K_0) \tag{2.65}$$

and, in particular, $\mathcal{A}_{h,t}|_{K_0} = \mathcal{M}_k^{K_t} \circ (\mathcal{M}_k^{K_0})^{-1}$ *and at time* $t = t_0$

$$\mathcal{X}_h(\Omega_0) = \mathcal{F}_{n,k}(\mathcal{T}_{0,h}) \tag{2.66}$$

then we have that $\mathcal{X}_h(\Omega_t)$*, as defined in* (2.62) *satisfies, at each* $t \in I$*,*

$$\mathcal{X}_h(\Omega_t) = \mathcal{F}_{n,k}(\mathcal{T}_{t,h})\,. \tag{2.67}$$

Proof. We just note that the under the given hypothesis

$$\mathcal{X}_h(\Omega_0) = \{\widehat{\psi}_h: \Omega_0 \to \mathbb{R} \mid \quad \widehat{\psi}_h \in C^0(\Omega_0),\ \widehat{\psi}_h\Big|_{K_0} \circ \mathcal{M}_k^{K_0} \in P_n(K_R) \quad \forall\, K_0 \in \mathcal{T}_{0,h}\}. \tag{2.68}$$

Then, by recalling the definition of $\mathcal{X}_h(\Omega_t)$ given in (2.62), we have that a if a function ψ_h satisfies $\psi_h \in \mathcal{X}_h(\Omega_t)$ then

$$(\psi_h \circ \mathcal{A}_{h,t})|_{K_0} \circ \mathcal{M}_k^{K_0} \in P_n(K_R).$$

By recalling the definition of $\mathcal{F}_{n,k}(\mathcal{T}_{t,h})$, given in (2.64), and by exploiting the continuity of the discrete ALE mapping, we finally obtain that

$$\psi_h \in \mathcal{F}_{n,k}(\mathcal{T}_{t,h})\,.$$

□

As a consequence of condition (2.65) it is easily verified that the appropriate finite element function space for the discrete ALE mapping is the isoparametric space $\mathcal{F}_{k,k}(\mathcal{T}_{0,h})$, since, by definition, if $\varphi \in \mathcal{F}_{k,k}(\mathcal{T}_{0,h})$ then $\varphi|_{K_0} \circ \mathcal{M}_k^{K_0} \in P_k(K_R)$, and condition (2.65) indeed implies that

$$\mathcal{A}_{h,t}|_{K_0} \circ \mathcal{M}_k^{K_0} \in P_k(K_R).$$

If we indicate by

$$\{\psi_i, \quad i \in \mathcal{N}^{\mathcal{T}}\}$$

the set of nodal basis function of $\mathcal{F}_{k,k}$, the discrete ALE mapping would then provide the following discretisation for the function $\boldsymbol{x}$ in (2.57) or (2.58)

$$\boldsymbol{x}_h(\boldsymbol{Y},t) = \mathcal{A}_{h,t}(\boldsymbol{Y}) = \sum_{i \in \mathcal{N}^{\mathcal{T}}} \boldsymbol{x}_i(t)\psi_i(\boldsymbol{Y}). \tag{2.69}$$

We can then proceed to the solution of (2.57) or (2.58) by standard finite element procedures.

Remark 2.2. We may note that the construction of the discrete ALE mapping depends on the degree k of the parametric mapping chosen for the finite element space where we wish to solve our problem, not on the degree n of the finite element representation chosen for our principal unknown u. For instance, if we decide to use quadratic element with an affine parametric mapping, the discrete ALE transformation should be build by using isoparametric linear elements.

Remark 2.3. By construction, the discrete ALE mapping satisfies the required regularity assumptions, and the mesh velocity field on Ω_0, can be expressed as

$$\hat{\boldsymbol{w}} = \sum_{i \in \mathcal{N}^{\mathcal{T}}} \phi_i \boldsymbol{w}_i \quad \text{in } \widehat{\Omega}_0 \times I. \tag{2.70}$$

A further discussion on the finite element spaces in the ALE frame will be carried out in the next section, with respect to a specific model problem.

3. A LINEAR ADVECTION DIFFUSION PROBLEM AND THE GEOMETRIC CONSERVATION LAWS

To analyze the properties of the discrete schemes resulting from the ALE formulation, we consider in this Section a model problem consisting of a linear advection diffusion equation of the type

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{\beta} u) - \mu\,\Delta u = f \qquad & \text{for } \boldsymbol{x} \in \Omega_t,\ t \in I \\ u = u_0 \qquad & \text{for } \boldsymbol{x} \in \Omega_0,\ t = t_0 \\ u = u_D \qquad & \text{for } \boldsymbol{x} \in \partial\Omega_t,\ t \in I \end{aligned} \tag{3.1}$$

where $\boldsymbol{\beta}$ is a convection velocity, which is assumed to satisfy $\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\beta} = 0$, μ a constant diffusivity, Δ indicates the Laplacian operator and u_D is an assigned boundary condition of Dirichlet type (in the case $\mu = 0$, the boundary condition should be applied only on the inflow part of the boundary). Here, we may note that equation (3.1) is a special case of (2.6), with

$$\boldsymbol{F}_e(u) = \boldsymbol{\beta} u, \quad \boldsymbol{F}_v(u) = -\mu\nabla_{\boldsymbol{x}} u.$$

We will first investigate the stability analysis of the continuous problem. We will then consider the discrete scheme and derive some necessary conditions by which the numerical approximation satisfies the so called "Geometric Conservation Laws". To do so, we will choose the type of finite elements to be employed for the spatial discretisation and the mesh movement law, and we will focus on the time discretisation aspects.

3.1. Stability analysis of the differential equation in ALE frame

We first verify that the differential equation written in the ALE frame maintains stability properties similar to those of an advection-diffusion problem on a fixed domain. We consider, in the following, homogeneous Dirichlet boundary conditions, and, for the sake of completeness, we will consider a general convective field $\boldsymbol{\beta}$, satisfying $\|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\beta}\|_{L_\infty(\Omega_t)} < \infty$. We write the differential equation on the ALE frame in the form

$$\begin{aligned} \left.\frac{\partial u}{\partial t}\right|_{\boldsymbol{Y}} + \nabla_{\boldsymbol{x}}\cdot\left[(\boldsymbol{\beta} - \boldsymbol{w})u\right] - \mu\,\Delta u + u\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w} = f \qquad & \text{for } \boldsymbol{x} \in \Omega_t,\ t \in I \\ u = u_0 \qquad & \text{for } \boldsymbol{x} \in \Omega_0,\ t = t_0 \\ u = 0 \qquad & \text{for } \boldsymbol{x} \in \partial\Omega_t,\ t \in I. \end{aligned} \tag{3.2}$$

We multiply the equation by u and integrate over Ω_t. Using the Reynolds transport formula, we have

$$\int_{\Omega_t} \left.\frac{\partial\,(u^2)}{\partial t}\right|_{\boldsymbol{Y}} \mathrm{d}\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L_2(\Omega_t)} - \int_{\Omega_t} u^2\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\,\mathrm{d}\Omega. \tag{3.3}$$

From (3.2) it may be derived that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L_2(\Omega_t)} + \mu\,\|\nabla_{\boldsymbol{x}} u\|^2_{L_2(\Omega_t)} + \int_{\Omega_t}\nabla_{\boldsymbol{x}}\cdot\left[(\boldsymbol{\beta} - \boldsymbol{w})u\right]u\,\mathrm{d}\Omega + \frac{1}{2}\int_{\Omega_t} u^2\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\,\mathrm{d}\Omega = \int_{\Omega_t} f u\,\mathrm{d}\Omega. \tag{3.4}$$

Thanks to the homogeneous boundary conditions, we have

$$\int_{\Omega_t}\nabla_{\boldsymbol{x}}\cdot\left[(\boldsymbol{\beta} - \boldsymbol{w})u\right]u\,\mathrm{d}\Omega = \frac{1}{2}\left[\int_{\Omega_t} u^2\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\beta}\,\mathrm{d}\Omega - \int_{\Omega_t} u^2\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\,\mathrm{d}\Omega\right] \tag{3.5}$$

and therefore relation (3.4) becomes:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{L_2(\Omega_t)} + \mu \|\nabla_{\boldsymbol{x}} u\|^2_{L_2(\Omega_t)} = \int\limits_{\Omega_t} f u \,\mathrm{d}\Omega - \frac{1}{2}\int\limits_{\Omega_t} u^2 \nabla_{\boldsymbol{x}}\cdot\boldsymbol{\beta}\,\mathrm{d}\Omega. \tag{3.6}$$

We may observe that all the terms containing a dependence on the grid velocity have been canceled out. Integrating in time between t_0 and t, we have:

$$\|u(t)\|^2_{L_2(\Omega_t)} + 2\mu \int\limits_{t_0}^{t} \|\nabla_{\boldsymbol{x}} u\|^2_{L_2(\Omega_s)}\,\mathrm{d}s = \|u(t_0)\|^2_{L_2(\Omega_{t_0})} + 2\int\limits_{t_0}^{t}\mathrm{d}s \int\limits_{\Omega_s} f u\,\mathrm{d}\Omega - \int\limits_{t_0}^{t}\mathrm{d}s\int\limits_{\Omega_s} u^2\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\beta}\,\mathrm{d}\Omega. \tag{3.7}$$

By standard arguments [13], we deduce that

$$\begin{aligned}\|u(t)\|^2_{L_2(\Omega_t)} + \mu \int\limits_{t_0}^{t} \|\nabla_{\boldsymbol{x}} u\|^2_{L_2(\Omega_s)}\,\mathrm{d}s \le \|u(t_0)\|^2_{L_2(\Omega_{t_0})} &+ \frac{1+C_\Omega}{\mu}\int\limits_{t_0}^{t} \|f\|^2_{H^{-1}(\Omega_s)}\,\mathrm{d}s \\ &+ \|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\beta}\|_{L_\infty(\Omega_{t_0})}\int\limits_{t_0}^{t}\|u\|^2_{L_2(\Omega_s)}\,\mathrm{d}s\end{aligned} \tag{3.8}$$

where C_Ω is the Poincaré constant.

We have assumed, for sake of simplicity, that $\boldsymbol{\beta}$ is constant in time, and we have exploited the equivalence between $\|\cdot\|_{L_\infty(\Omega_t)}$ and $\|\cdot\|_{L_\infty(\Omega_{t_0})}$ for all $t \in I$. Using the Gronwall lemma [13], we finally obtain

$$\|u(t)\|^2_{L_2(\Omega_t)} + \mu \int\limits_{t_0}^{t} \|\nabla_{\boldsymbol{x}} u\|^2_{L_2(\Omega_s)}\,\mathrm{d}s \le K \exp \gamma t \tag{3.9}$$

where

$$K = \|u(t_0)\|^2_{L_2(\Omega_{t_0})} + \frac{1+C_\Omega}{\mu}\int\limits_{t_0}^{t}\|f\|^2_{H^{-1}(\Omega_s)}\,\mathrm{d}s \tag{3.10}$$

and

$$\gamma = \|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\beta}\|_{L_\infty(\Omega_{t_0})}. \tag{3.11}$$

If $\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\beta} = 0$ then the stability expression simplifies furtherly. At continuous level, then, the stability properties of the problem are not affected by the domain velocity field. Clearly, we may expect that this will not be anymore true for the discrete problem.

3.2. Finite element spaces in the ALE frame

We will use Lagrangian type finite elements and a discrete ALE mapping, as described in Subection 2.5, and a Galerkin formulation. Our discrete function space on Ω_0, $\mathcal{X}_h(\Omega_0)$, will then coincide with $\mathcal{F}_{n,k}(\mathcal{T}_{0,h})$, defined in (2.59). The corresponding finite element space on Ω_t, $\mathcal{X}_h(\Omega_t)$, will be formed by functions of $\mathcal{F}_{n,k}(\mathcal{T}_{t,h})$.

We will now indicate with $\mathcal{N}$ the set of nodes of the finite element mesh, and with $\mathcal{N}_{\text{int}} \subset \mathcal{N}$ the set of internal nodes. We also introduce the set of finite element nodal basis functions

$$\{\psi_i, \quad \psi_i \in \mathcal{X}_h(\Omega_t), \quad i \in \mathcal{N}\}$$

which forms a basis of $\mathcal{X}_h(\Omega_t)$. Here, ψ_i is the finite element function associated with node i. With $\mathcal{X}_{0,h}(\Omega_t)$ we indicate the discrete function space $\mathcal{X}_h(\Omega_t) \cap H_0^1(\Omega_t)$. The set

$$\{\psi_i, \quad i \in \mathcal{N}_{\text{int}}\}$$

forms a basis of $\mathcal{X}_{0,h}(\Omega_t)$.

The numerical solution u_h will then be sought in the space $\mathcal{X}_h(\Omega_t)$. In particular, we have that u_h will be expressed as linear combination of nodal finite element basis functions,

$$u_h(\boldsymbol{x}, t) = \sum_{i \in \mathcal{N}} \psi_i(\boldsymbol{x}, t) u_i(t) \tag{3.12}$$

with time dependent coefficients $u_i(t)$. The set $\mathcal{X}_{0,h}(\Omega_t)$ will be used as test function space.

For the sake of simplicity, we will drop the subscript h when it is clear from the context that we are considering the discrete solution. This applies in particular to $\mathcal{A}_{h,t}$, which in the following will be indicated by simply $\mathcal{A}_t$. We wish to remind that, because of the ALE mapping, functions in $\mathcal{X}_h(\Omega_t)$ depend both on $\boldsymbol{x}$ and t, even if the functions in $\mathcal{X}_h(\Omega_0)$ do not depend on time. Thanks to relation (2.19) we may then write,

$$\left.\frac{\partial u_h}{\partial t}\right|_{Y} (\boldsymbol{x}, t) = \sum_{i \in \mathcal{N}} \psi_i(\boldsymbol{x}, t) \frac{\mathrm{d}u_i}{\mathrm{d}t}(t). \tag{3.13}$$

3.3. Some considerations on the significance of the Geometric Conservation Laws

Geometric Conservation Laws have been originally investigated in the context of finite difference and finite volume schemes for fluid dynamic problems. It stems from the basic idea that the solution should be minimally affected by the domain movement law. Indeed, at the continuous level, the ALE formulation is formally equivalent to the original problem; yet this is not generally true when the fully discrete system is considered. It has been proposed that some 'simple' solution of the differential problem should be also solutions of the discrete system. In particular, the attention has been concentrated on the capability of the discrete system of representing a constant solution, which is clearly a solution of the differential equation (in the absence of the source term and with the appropriate boundary and initial conditions). Following this approach we can state that *a numerical scheme satisfy the Geometric Conservation Laws if it is able to reproduce a constant solution*. It is therefore, similar to the "patch test" often used by finite element practitioners. As we will see, the GCL constraint gives a relation which involves only mesh geometrical quantities and the domain velocity field. The significance of this condition is still not clear. Recent results are available for special type of finite-volume schemes in [7] where the GCL have been linked to convergence properties of the proposed scheme.

In the following analysis, the GCL for a finite element scheme will be investigated in more detail and linked to the stability properties of some time evolution schemes.

3.4. Discrete system employing conservative formulation (2.24)

The finite element semi-discrete approximation of (2.24) then reads as follows:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} \psi_h u_h \,\mathrm{d}\Omega + \mu \int_{\Omega_t} \nabla_{\boldsymbol{x}} \psi_h \nabla_{\boldsymbol{x}} u_h \,\mathrm{d}\Omega + \int_{\Omega_t} \psi_h \nabla_{\boldsymbol{x}} \cdot [(\boldsymbol{\beta} - \boldsymbol{w}) u_h] \,\mathrm{d}\Omega \\ &\qquad = \int_{\Omega_t} f \psi_h \,\mathrm{d}\Omega \quad \forall\, \psi_h \in \mathcal{X}_{0,h}(\Omega_t), \quad t \in I \end{aligned} \tag{3.14}$$

with

$$u_h = u_D \qquad \text{for } \boldsymbol{x} \in \partial\Omega_t,\ \ t \in I$$
$$u_h = u_0 \qquad \text{for } \boldsymbol{x} \in \Omega_0,\ \ t = t_0.$$

Equation (3.14) may be equivalently written in algebraic form as follows:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{M}(t)\,U\right) + \left(\boldsymbol{H}(t) - \boldsymbol{A}(t,\boldsymbol{w})\right)U = \boldsymbol{F} \\ &\qquad\qquad\qquad\qquad u_i = u_D, \quad i \in \mathcal{N} \setminus \mathcal{N}_{\text{int}} \end{aligned} \tag{3.15}$$

where $U = \{u_i\}_{i\in\mathcal{N}}$ is the vector of the nodal values of the discrete solution:

$$\boldsymbol{M}(t) = \left\{ \int_{\Omega_t} \psi_i \psi_j \,\mathrm{d}\Omega \right\}_{i,j\in\mathcal{N}_{\text{int}}}$$

is the mass matrix, while $\boldsymbol{H}$ and $\boldsymbol{A}$ are defined as

$$\boldsymbol{H}(t) = \left\{ \int_{\Omega_t} \psi_i \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{\beta}\psi_j)\,\mathrm{d}\Omega + \mu \int_{\Omega_t} \nabla_{\boldsymbol{x}}\psi_i \cdot \nabla_{\boldsymbol{x}}\psi_j \,\mathrm{d}\Omega \right\}_{i,j\in\mathcal{N}_{\text{int}}}$$

and

$$\boldsymbol{A}(t,\boldsymbol{w}) = \left\{ \int_{\Omega_t} \psi_i \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{w}\psi_j)\,\mathrm{d}\Omega \right\}_{i,j\in\mathcal{N}_{\text{int}}}.$$

In this context the space integrals may be substituted by numerical quadrature. *In the following, we will always assume that the numerical quadrature rules employed are able to integrate exactly the terms involved.*

3.4.1. Stability analysis of the semi-discrete conservative scheme

For the stability analysis of the semi-discrete scheme we cannot take, as it is instead done in standard calculations for fixed domains, $\psi_h = u_h$ since the two functions have, in general, a different time evolution. Anyway, we can express the solution u_h as a linear combination of the test functions with time dependent coefficients, as indicated in (3.12). We then take $\psi_h = \psi_i$ and we multiply the equation for $u_i(t)$ obtaining:

$$u_i(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} u_h\psi_i\,\mathrm{d}\Omega + \int_{\Omega_t} \nabla_{\boldsymbol{x}}\cdot\left[(\boldsymbol{\beta}-\boldsymbol{w})u_h\right]u_i(t)\psi_i + \mu\int_{\Omega_t} \nabla_{\boldsymbol{x}}u_h \nabla_{\boldsymbol{x}}(u_i(t)\psi_i)\,\mathrm{d}\Omega = \int_{\Omega_t} f u_i(t)\psi_i\,\mathrm{d}\Omega. \tag{3.16}$$

The first term can be rewritten as

$$\begin{aligned} u_i(t)\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} u_h\psi_i\,\mathrm{d}\Omega &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} u_h u_i(t)\psi_i\,\mathrm{d}\Omega - \int_{\Omega_t} u_h\psi_i\frac{\mathrm{d}u_i(t)}{\mathrm{d}t}\,\mathrm{d}\Omega \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} u_h u_i(t)\psi_i\,\mathrm{d}\Omega - \int_{\Omega_t} u_h \left.\frac{\partial\psi_i u_i(t)}{\partial t}\right|_Y \mathrm{d}\Omega. \end{aligned} \tag{3.17}$$

Summing over i all the equations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2_{L_2(\Omega_t)} - \int_{\Omega_t} u_h \left.\frac{\partial u_h}{\partial t}\right|_Y \mathrm{d}\Omega + \int_{\Omega_t} \nabla_{\boldsymbol{x}}\cdot\left[(\boldsymbol{\beta}-\boldsymbol{w})u_h\right]u_h + \mu\,\|\nabla_{\boldsymbol{x}}u_h\|^2_{L_2(\Omega_t)} = \int_{\Omega_t} f u_h\,\mathrm{d}\Omega. \tag{3.18}$$

The term

$$\int_{\Omega_t} u_h \left.\frac{\partial u_h}{\partial t}\right|_Y \mathrm{d}\Omega = \frac{1}{2}\int_{\Omega_t} \left.\frac{\partial (u_h^2)}{\partial t}\right|_Y \mathrm{d}\Omega$$

can be manipulated as in (3.3), leading to equation (3.4) written for the semi-discrete solution u_h of the problem. We can then proceed as in Subsection 3.1, obtaining a stability inequality that is not affected by the domain velocity field.

3.4.2. The discrete scheme

System (3.15) is a system of ODE's which has to be integrated in time. In the following we will consider the following time integration schemes:

$$\boldsymbol{M}_{t_{n+1}} U_{n+1} - \boldsymbol{M}_{t_n} U_n + \mathcal{INT}_1{}_{t_n}^{t_{n+1}}[\boldsymbol{H}\,U] - \mathcal{INT}_2{}_{t_n}^{t_{n+1}}[\boldsymbol{A}\,U] = \boldsymbol{F} \qquad (3.19)$$

where $\mathcal{INT}_1$ and $\mathcal{INT}_2$ represent two quadrature formulae used to integrate numerically in time the terms $\boldsymbol{H}\,U$ and $\boldsymbol{A}\,U$:

$$\mathcal{INT}_1{}_{t_n}^{t_{n+1}}[\boldsymbol{H}\,U] \approx \int_{t_n}^{t_{n+1}} \boldsymbol{H}\,U\,\mathrm{d}t$$

$$\mathcal{INT}_2{}_{t_n}^{t_{n+1}}[\boldsymbol{A}\,U] \approx \int_{t_n}^{t_{n+1}} \boldsymbol{A}\,U\,\mathrm{d}t.$$

In particular, the unknown U in the these two terms may be taken equal to U^{n+1} (implicit scheme) or U^n (explicit scheme).

3.4.3. The Geometric Conservation Laws

In case of $\boldsymbol{F} = 0$, if we substitute a constant function into the discrete system (3.19), it is verified that a constant field is a solution of the numerical scheme if, for each time step interval $(t_n, t_{n+1}) \subset I$, the following relations hold

$$\sum_{j\in\mathcal{N}} \int_{\Omega_t} \psi_i \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{\beta}\psi_j)\,\mathrm{d}\Omega\,\mathrm{d}t = 0 \quad \forall\, i \in \mathcal{N}_{\text{int}} \qquad (3.20)$$

$$\sum_{j\in\mathcal{N}} \int_{\Omega_t} \nabla_{\boldsymbol{x}}\psi_i \cdot \nabla_{\boldsymbol{x}}\psi_j\,\mathrm{d}\Omega\,\mathrm{d}t = 0 \quad \forall\, i \in \mathcal{N}_{\text{int}} \qquad (3.21)$$

and

$$\int_{\Omega_{t_{n+1}}} \psi_i(\boldsymbol{x}, t_{n+1})\,\mathrm{d}\Omega - \int_{\Omega_{t_n}} \psi_i(\boldsymbol{x}, t_n)\,\mathrm{d}\Omega = \mathcal{INT}_2{}_{t_n}^{t_{n+1}}\left[\int_{\Omega_t} \psi_i \nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\,\mathrm{d}\Omega\right] \quad \forall\, i \in \mathcal{N}_{\text{int}}. \qquad (3.22)$$

Relation (3.20) and (3.21) are satisfied by the finite element shape functions, indeed they satisfy the following expressions:

$$\sum_{i\in\mathcal{N}} \psi_i(\boldsymbol{x}) = 1 \quad \forall\, \boldsymbol{x} \in \Omega_t \qquad (3.23)$$

and consequently, being $\nabla_x \psi_i \in L^2(\Omega_t)$,

$$\sum_{i\in\mathcal{N}} \nabla_x \psi_i = 0 \quad \text{in } \Omega_t. \tag{3.24}$$

Remark 3.1. In a finite volume context, a relation equivalent to (3.20) would furnish a condition for the computation of the normal of the moving control volumes. That condition is often called *surface conservation law* [3]. The reader may refer to [10] for a discussion about the relationship between Lagrangian finite element and finite volume schemes for conservation laws.

Condition (3.22) expresses the *Geometric Conservation Laws* written for the FEM schemes which employs formulation (3.19). It states that at each time step the identity

$$\int_{\Omega_{t_{n+1}}} \psi_h \, d\Omega - \int_{\Omega_{t_n}} \psi_h \, d\Omega = \mathcal{INT}_2{}_{t_n}^{t_{n+1}} \left[\int_{\Omega_t} \psi_h \nabla_x \cdot \boldsymbol{w} \, d\Omega \right] \quad \forall\, \psi_h \in \mathcal{X}_{0,h}(\Omega_t) \tag{3.25}$$

must hold *for the time integration scheme used in the right hand side.*

Remark 3.2. Relation (3.25) may also be interpreted as the finite element discretisation of the weak form associated to relation (2.8). Indeed, we have already observed that relation (2.8) must be identically satisfied in order that the differential equation (2.14) (of which relation (2.24) is the weak form) is equivalent to the original differential problem (2.7). The GCL enforce such condition at the discrete level.

Remark 3.3. In a finite element framework, it may be useful to consider another identity, namely relation (2.23). Requiring its fulfillment at discrete level would lead to the following relation:

$$\int_{\Omega_{t_{n+1}}} \psi_i \psi_j \, d\Omega - \int_{\Omega_{t_n}} \psi_i \psi_j \, d\Omega = \mathcal{INT}_2{}_{t_n}^{t_{n+1}} \left[\int_{\Omega_t} \psi_i \psi_j \nabla_x \cdot \boldsymbol{w} \, d\Omega \right] \quad \forall\, i, j \in \mathcal{N}_{\text{int}}. \tag{3.26}$$

This relation may be considered as another form of the GCL for scheme (3.19), suited for a finite element approximation.

Proposition 3.1. *A sufficient condition for the satisfaction of* (3.25) *and* (3.26) *is to use a time integration scheme for the* ALE *term of degree* $d \cdot s - 1$, *where* d *is the number of space dimensions and* s *is the degree of the polynomial used to represent the time evolution of the nodal displacement within each time step.*

Proof. In order to find out the degree of exactness of the time integration scheme necessary for the fulfillment of the GCL, we consider the time interval $[t_n, t_{n+1}]$ and we take Ω_{t_n} as the reference configuration. In the following, we indicate with $\mathcal{A}_{t_n,t_{n+1}}$ the ALE mapping between the two time levels, that is

$$\mathcal{A}_{t_n,t_{n+1}} = \mathcal{A}_{t_{n+1}} \circ \mathcal{A}_{t_n}^{-1}. \tag{3.27}$$

The following identity holds for all $\psi_h \in \mathcal{X}_{0,h}(\Omega_t)$:

$$\int_{\Omega_{t_{n+1}}} \psi_h \nabla_x \cdot \boldsymbol{w} \, d\Omega = \int_{\Omega_{t_n}} \widehat{\psi}_h(\boldsymbol{Y}) (\boldsymbol{J}_{\text{cof}} \nabla_{\boldsymbol{Y}}) \cdot \hat{\boldsymbol{w}} \, d\Omega \tag{3.28}$$

where $\widehat{\psi}_h(\boldsymbol{Y}) = \psi_h \circ \mathcal{A}_{t_n,t_{n+1}}$ and $\hat{\boldsymbol{w}} = \boldsymbol{w} \circ \mathcal{A}_{t_n,t_{n+1}}$, while $\boldsymbol{J}_{\text{cof}}$ isthe co-factor matrix of the

ALE mapping Jacobian $J_{A_{t_n,t_{n+1}}}$. If the domain displacement law is taken to be a piecewise polynomial in time, then J_{cof} would be a polynomial in time in $[t_n, t_{n+1}]$, whose degree will depend also on the number of space dimensions. Expression (3.28) allows to determine the degree of the polynomial which has to be numerically integrated in time in the right hand side of relation (3.25). Assuming that the space integral is computed exactly, the satisfaction of the GCL conditions imposes some restrictions on the time integration rule employed for the ALE convection term.

Function $\hat{\psi}$ is constant in time. If the nodal displacement is represented on each time step by a polynomial in time of degree s then $\boldsymbol{w}$ is a polynomial of degree $s-1$, and J_{cof} of degree $(d-1)s$, being d the number of space dimensions. Consequently, a sufficient condition for the fulfillment of the GCL is obtained by employing a time advancing scheme of exactness (at least) $d \cdot s - 1$. Similar arguments apply for formulation (3.26). □

For instance, if we assume a linear time variation for $\boldsymbol{x}(\boldsymbol{Y}, t)$ on each time step ($s = 1$) the GCL are satisfied in $2D$ if we use the mid-point rule, which exactly integrates a linear function. These results are in agreement with what has been found by Leisoinne and Farhat in [8], in the context of finite volume formulation.

Remark 3.4. It may be noted that the condition on the time advancing scheme just found for the satisfaction of the Geometry Conservation Laws involves only the numerical discretisation of the ALE convective term, without any direct involvement of u. Indeed, only terms related to the mesh movement are present. Consequently, as suggested in [8], we may in principle use a separate (possibly less accurate) time-integration rule for the other terms. For example, for a two dimensional problem and piecewise linear time evolution of mesh displacement, we may adopt an explicit treatment for the term $\boldsymbol{H}U$, using, for instance, a first-order forward Euler scheme, while adopting a mid-point rule just for the time integral of the ALE term.

Remark 3.5. Since there is no one-point integration scheme of exactness 2, in three dimensional problems the mesh quantities should be evaluated at (at least) two intermediate points. For example, we may use a 2 points Gauss quadrature formula for the ALE convective term.

Remark 3.6. For the case of linear finite elements, B. Nkonga and H. Guillard [11] have exploited the equivalence between Galerkin finite element discretisation and finite volumes on the dual grid in order to integrate exactly the ALE convection term, thus assuring the satisfaction of the GCL for their three dimensional numerical scheme.

Remark 3.7. The GCL formula for a finite volume scheme is readily inferred by noting that if we take $\psi = 1$ on a patch of elements $\Omega_i(t)$ expression (3.25) becomes

$$\int_{\Omega_i(t_{n+1})} \mathrm{d}\Omega - \int_{\Omega_i(t_n)} \mathrm{d}\Omega = \mathcal{INT}_2{}_{t_n}^{t_{n+1}} \left[\int_{\Omega_i(t)} \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{w} \,\mathrm{d}\Omega \right] = \mathcal{INT}_2{}_{t_n}^{t_{n+1}} \left[\oint_{\partial\Omega_i(t)} \boldsymbol{w} \cdot \boldsymbol{n} \,\mathrm{d}\Gamma \right] \quad (3.29)$$

which is the form which has been proposed in [4, 8].

3.4.4. A stability result for the implicit Euler method applied to the conservative scheme

Let us consider the following time discretisation of the semi-discrete problem (2.24) for a two dimensional problem. We assume a piecewise constant in time mesh velocity field and we adopt a mid-point time integration rule, thus satisfying the GCL. Yet, we will adopt an implicit

Euler time discretisation for u_h. We obtain the following expression

$$\int_{\Omega_{t^{n+1}}} {u_h}^{n+1}\psi_h \,\mathrm{d}\Omega - \int_{\Omega_{t^n}} {u_h}^{n}\psi_h \,\mathrm{d}\Omega + \Delta t \int_{\Omega_{t^{n+1/2}}} \mu \nabla_{\boldsymbol{x}} {u_h}^{n+1} \nabla_{\boldsymbol{x}} \psi_h \,\mathrm{d}\Omega$$
$$+ \Delta t \int_{\Omega_{t^{n+1/2}}} \psi_h \nabla_{\boldsymbol{x}} \cdot [(\boldsymbol{\beta} - \boldsymbol{w}) {u_h}^{n+1}] \,\mathrm{d}\Omega = \Delta t \int_{\Omega_{t^{n+1/2}}} f^{n+1/2} \psi_h \,\mathrm{d}\Omega \quad \forall \psi_h \in \mathcal{X}_{0,h}(\Omega_t) \tag{3.30}$$

with

$$\begin{aligned} {u_h}^i &= 0 \qquad \text{on } \partial\Omega_t, \; i = 1, 2, \ldots \\ {u_h}^0 &= u_0 \qquad \text{in } \Omega_0. \end{aligned}$$

Taking $\psi_h = {u_h}^{n+1}$ it can be shown, integrating by parts the convective terms as in (3.5), that

$$\|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\, \mu \|\nabla_{\boldsymbol{x}} {u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1/2}})} - \frac{1}{2}\Delta t \int_{\Omega_{t^{n+1/2}}} |{u_h}^{n+1}|^2 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{w} \,\mathrm{d}\Omega$$
$$\leq \int_{\Omega_{t^n}} {u_h}^n {u_h}^{n+1} \,\mathrm{d}\Omega + \Delta t \int_{\Omega_{t^{n+1/2}}} f^{n+1/2} {u_h}^{n+1} \,\mathrm{d}\Omega \leq \frac{1}{2}\|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^n})} \tag{3.31}$$
$$+ \frac{1}{2}\|{u_h}^n\|^2_{L_2(\Omega_{t^n})} + \Delta t \frac{\mu}{2} \|\nabla_{\boldsymbol{x}} {u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1/2}})} + \Delta t \frac{1 + C_\Omega}{2\mu} \|f^{n+1/2}\|^2_{H^{-1}(\Omega_{t^{n+1/2}})}.$$

Since this scheme satisfies the GCL, the following equality holds (we exploit equation (3.26), where we set $\psi_i = \psi_j = {u_h}^{n+1}$):

$$\begin{aligned} \|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} - \|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^n})} &= \int_{t^n}^{t^{n+1}} \int_{\Omega_t} |{u_h}^{n+1}|^2 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{w} \,\mathrm{d}\Omega \,\mathrm{d}t \\ &= \Delta t \int_{\Omega_{t^{n+1/2}}} |{u_h}^{n+1}|^2 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{w} \,\mathrm{d}\Omega. \end{aligned} \tag{3.32}$$

Substituting this equation in the previous inequality, we obtain the stability result:

$$\begin{aligned} &\|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\, \mu \|\nabla_{\boldsymbol{x}} {u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1/2}})} \\ &\leq \|{u_h}^n\|^2_{L_2(\Omega_{t^n})} + \Delta t \frac{1 + C_\Omega}{\mu} \|f^{n+1/2}\|^2_{H^{-1}(\Omega_{t^{n+1/2}})}. \end{aligned} \tag{3.33}$$

Finally, summing over all the time steps:

$$\begin{aligned} &\|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\, \mu \sum_{i=0}^{n} \|\nabla_{\boldsymbol{x}} {u_h}^{i+1}\|^2_{L_2(\Omega_{t^{i+1/2}})} \\ &\leq \|{u_h}^0\|^2_{L_2(\Omega_{t^0})} + \Delta t \frac{1 + C_\Omega}{\mu} \sum_{i=0}^{n} \|f^{i+1/2}\|^2_{H^{-1}(\Omega_{t^{i+1/2}})}. \end{aligned} \tag{3.34}$$

In this case, one may observe that the stability result does not depend on the domain velocity field and that this property has been obtained thanks to the fulfillment of the GCL condition written in the form (3.26).

3.5. Discrete system employing formulation (2.18)

The semi-discrete counterpart of (2.18), based on the pure Galerkin finite element method, reads as follows:

$$\int_{\Omega_t} \left.\frac{\partial u_h}{\partial t}\right|_Y \psi_h \,\mathrm{d}\Omega + \int_{\Omega_t} \psi_h \,(\boldsymbol{\beta} - \boldsymbol{w})\, \boldsymbol{\nabla}_{\boldsymbol{x}} u_h \,\mathrm{d}\Omega + \mu \int_{\Omega_t} \boldsymbol{\nabla}_{\boldsymbol{x}} u_h \boldsymbol{\nabla}_{\boldsymbol{x}} \psi_h \,\mathrm{d}\Omega = \int_{\Omega_t} f \psi_h \,\mathrm{d}\Omega \quad \forall\, \psi_h \in \mathcal{X}_{0,h}(\Omega_t) \tag{3.35}$$

with

$$u_h = u_D \qquad \text{for } \boldsymbol{x} \in \partial\Omega_t,\; t \in I$$
$$u_h = u_0 \qquad \text{for } \boldsymbol{x} \in \Omega_0,\; t = t_0$$

where we have written $\boldsymbol{\beta}\boldsymbol{\nabla}_{\boldsymbol{x}} u_h$ instead of $\boldsymbol{\nabla}_{\boldsymbol{x}}\cdot(\boldsymbol{\beta} u_h)$ thanks to the hypothesis of incompressibility of the convective field $\boldsymbol{\beta}$. System (3.35) may be equivalently written in algebraic form as we have done for the conservative scheme:

$$\boldsymbol{M}(t)\frac{\mathrm{d}U}{\mathrm{d}t} + \boldsymbol{H}(t)U - \boldsymbol{B}(t, \boldsymbol{w})U = \boldsymbol{F} \tag{3.36}$$

and

$$u_i(t_0) = u(\boldsymbol{x}_i(t_0), t_0) \qquad i \in \mathcal{N} \tag{3.37}$$

$$u_i(t) = u_D(\boldsymbol{x}_i, t) \qquad i \in \mathcal{N} \setminus \mathcal{N}_{\text{int}}. \tag{3.38}$$

Here,

$$\boldsymbol{B}(t, \boldsymbol{w}) = \left\{ \int_{\Omega_t} \psi_i (\boldsymbol{w} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}) \psi_j \,\mathrm{d}\Omega \right\}_{i,j \in \mathcal{N}_{\text{int}}} \tag{3.39}$$

If the space integrals are computed exactly, it is immediately verified that a constant function will satisfy (3.36) (in the absence of forcing terms) independently of the numerical time integration formula adopted. Indeed,

$$\sum_{j \in \mathcal{N}} \int_{\Omega_t} \psi_i (\boldsymbol{w} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}) \psi_j \,\mathrm{d}\Omega = 0 \quad \forall\, i \in \mathcal{N}_{\text{int}} \tag{3.40}$$

because of relation (3.24). Therefore, this scheme automatically satisfies the GCL since it is able to represent a constant solution. Unfortunately, while the discrete system (3.14), maintains the conservation property of the original problem this is not immediately true for relation (3.35).

3.5.1. Stability analysis of the semi-discrete scheme

As for the conservative scheme (3.14) analyzed in the previous section, we cannot take $\psi_h = u_h$. Since each term in (3.35) is linear in ψ_h, if we take $\psi_h = \psi_i$, we multiply each term for $u_i(t)$ and sum over the index i, we get:

$$\int_{\Omega_t} \left.\frac{\partial u_h}{\partial t}\right|_Y u_h \,\mathrm{d}\Omega + \int_{\Omega_t} (\boldsymbol{\beta} - \boldsymbol{w})\, \boldsymbol{\nabla}_{\boldsymbol{x}} u_h u_h \,\mathrm{d}\Omega + \mu \int_{\Omega_t} |\boldsymbol{\nabla}_{\boldsymbol{x}} u_h|^2 \,\mathrm{d}\Omega = \int_{\Omega_t} f u_h \,\mathrm{d}\Omega. \tag{3.41}$$

The first term can be transformed exploiting (3.3) and obtaining:

$$\int_{\Omega_t} \left.\frac{\partial u_h}{\partial t}\right|_Y u_h \,\mathrm{d}\Omega = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h\|^2_{L_2(\Omega_t)} - \frac{1}{2}\int_{\Omega_t} u_h^2 \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\boldsymbol{w} \,\mathrm{d}\Omega \tag{3.42}$$

while the second one becomes:

$$\begin{aligned}\int_{\Omega_t} (\boldsymbol{\beta}-\boldsymbol{w})\,\nabla_{\boldsymbol{x}} u_h u_h \,\mathrm{d}\Omega &= \frac{1}{2}\int_{\Omega_t} (\boldsymbol{\beta}-\boldsymbol{w})\,\nabla_{\boldsymbol{x}} |u_h|^2 \,\mathrm{d}\Omega \\ &= -\frac{1}{2}\int_{\Omega_t} \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{\beta}-\boldsymbol{w})|u_h|^2\,\mathrm{d}\Omega = \frac{1}{2}\int_{\Omega_t}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|u_h|^2\,\mathrm{d}\Omega.\end{aligned} \tag{3.43}$$

Combining these two results into equation (3.41), we obtain exactly equation (3.6) written for the solution u_h of the semi-discrete problem at hand. Also in this case, we can proceed as in Subsection 3.1, obtaining stability inequality without any intervention of the domain velocity field.

3.5.2. Stability result for the implicit Euler method

Let us consider the implicit Euler discretisation of problem (3.35):

$$\begin{aligned}\int_{\Omega^{t^{n+1}}} {u_h}^{n+1}\psi_h\,\mathrm{d}\Omega - \int_{\Omega^{t^{n+1}}} {u_h}^{n}\psi_h\,\mathrm{d}\Omega + \Delta t \int_{\Omega^{t^{n+1}}} \left[\psi_h(\boldsymbol{\beta}-\boldsymbol{w})\nabla_{\boldsymbol{x}}{u_h}^{n+1} + \mu\nabla_{\boldsymbol{x}}{u_h}^{n+1}\nabla_{\boldsymbol{x}}\psi_h\right]\mathrm{d}\Omega \\ = \Delta t \int_{\Omega^{t^{n+1}}} f^{n+1}\psi_h\,\mathrm{d}\Omega \quad \forall\,\psi_h \in \mathcal{X}_{0,h}(\Omega_t)\end{aligned} \tag{3.44}$$

with

$$\begin{aligned} u_h &= 0 && \text{on } \partial\Omega_t,\ i=1,2,\ldots \\ {u_h}^0 &= u_0 && \text{in } \Omega_0. \end{aligned}$$

Again, we take $\psi_h = {u_h}^{n+1}$; by exploiting equation (3.43) for the treatment of the convective terms, we obtain

$$\begin{aligned} &\|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\mu\|\nabla_{\boldsymbol{x}}{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} \\ &\quad\le -\frac{1}{2}\Delta t\int_{\Omega^{t^{n+1}}}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|{u_h}^{n+1}|^2\,\mathrm{d}\Omega + \Delta t\int_{\Omega^{t^{n+1}}} f^{n+1}{u_h}^{n+1}\,\mathrm{d}\Omega + \int_{\Omega^{t^{n+1}}}{u_h}^n{u_h}^{n+1}\,\mathrm{d}\Omega \\ &\quad\le -\frac{1}{2}\Delta t\int_{\Omega^{t^{n+1}}}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|{u_h}^{n+1}|^2\,\mathrm{d}\Omega + \Delta t\frac{1+C_\Omega}{2\mu}\|f^{n+1}\|^2_{H^{-1}(\Omega_{t^{n+1}})} \\ &\qquad + \Delta t\frac{\mu}{2}\|\nabla_{\boldsymbol{x}}{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \frac{1}{2}\|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \frac{1}{2}\|{u_h}^{n}\|^2_{L_2(\Omega_{t^{n+1}})} \end{aligned} \tag{3.45}$$

where the last term is evaluated on the configuration at time t^{n+1}; such term can be modified as in (3.32):

$$\|{u_h}^n\|^2_{L_2(\Omega_{t^{n+1}})} = \|{u_h}^n\|^2_{L_2(\Omega_{t^n})} + \int_{t^n}^{t^{n+1}}\int_{\Omega_t}|{u_h}^n|^2\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\,\mathrm{d}\Omega\,\mathrm{d}t. \tag{3.46}$$

Consequently we have

$$\begin{aligned} &\|{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\,\mu\,\|\nabla_{\boldsymbol{x}}{u_h}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} \\ &\quad\le \left(\int_{t^n}^{t^{n+1}}\int_{\Omega_t}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|{u_h}^n|^2\,\mathrm{d}\Omega\,\mathrm{d}t - \Delta t\int_{\Omega^{t^{n+1}}}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|{u_h}^{n+1}|^2\,\mathrm{d}\Omega\right) \\ &\qquad + \|{u_h}^n\|^2_{L_2(\Omega_{t^n})} + \Delta t\,\frac{1+C_\Omega}{\mu}\,\|f^{n+1}\|^2_{H^{-1}(\Omega_{t^{n+1}})}. \end{aligned} \tag{3.47}$$

In this case we cannot obtain a stability result independent on $\boldsymbol{w}$, because of the presence of the term

$$\left(\int_{t^n}^{t^{n+1}}\int_{\Omega_t} \nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|u_h{}^n|^2 \,\mathrm{d}\Omega\,\mathrm{d}t - \Delta t \int_{\Omega_{t^{n+1}}} \nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|u_h{}^{n+1}|^2 \,\mathrm{d}\Omega\right). \tag{3.48}$$

Remark 3.8. We may note that even if the scheme satisfies condition (3.26), i.e. in the 2D case we compute the second integral in the mid-configuration and consequently we can write:

$$\Delta t \int_{\Omega_{t^{n+1/2}}} \nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|u_h{}^{n+1}|^2 \,\mathrm{d}\Omega = \int_{t^n}^{t^{n+1}}\int_{\Omega_t} \nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}|u_h{}^{n+1}|^2 \,\mathrm{d}\Omega\,\mathrm{d}t \tag{3.49}$$

the term (3.48) is equal to

$$\int_{t^n}^{t^{n+1}}\int_{\Omega_t} \nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}(|u_h{}^n|^2 - |u_h{}^{n+1}|^2) \,\mathrm{d}\Omega\,\mathrm{d}t \tag{3.50}$$

which is, in general, different than zero.

We can finally obtain a stability inequality, which depends on $\boldsymbol{w}$, from (3.47):

$$\begin{aligned}
&\|u_h{}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\,\mu\,\|\nabla_{\boldsymbol{x}} u_h{}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})}\\
&\quad\le \Delta t\|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}(t^{n+1})\|_{L_\infty(\Omega_{t^{n+1}})}\|u_h{}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\,\frac{1+C_\Omega}{\mu}\,\|f^{n+1}\|^2_{H^{-1}(\Omega_{t^{n+1}})}\\
&\qquad + \left(1 + \Delta t \sup_{t\in(t^n,t^{n+1})} \|J_{\mathcal{A}_{t_n,t_{n+1}}}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\|_{L_\infty(\Omega_t)}\right)\|u_h{}^n\|^2_{L_2(\Omega_{t^n})}.
\end{aligned} \tag{3.51}$$

Using the notations

$$\begin{aligned}
\gamma_1^i &= \|\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}(t^i)\|_{L_\infty(\Omega_{t^i})}\\
\gamma_2^i &= \sup_{t\in(t^i,t^{i+1})} \|J_{\mathcal{A}_{t_i,t_{i+1}}}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}\|_{L_\infty(\Omega_t)}
\end{aligned}$$

we can rewrite the previous inequality as

$$\begin{aligned}
&\|u_h{}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\,\mu\,\|\nabla_{\boldsymbol{x}} u_h{}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})}\\
&\quad\le \Delta t\,\gamma_1^{n+1}\|u_h{}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + (1+\Delta t\,\gamma_2^n)\|u_h{}^n\|^2_{L_2(\Omega_{t^n})} + \Delta t\,\frac{1+C_\Omega}{\mu}\,\|f^{n+1}\|^2_{H^{-1}(\Omega_{t^{n+1}})}.
\end{aligned} \tag{3.52}$$

Summing over the index n we obtain:

$$\begin{aligned}
&\|u_h{}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\,\mu\sum_{i=1}^{n+1}\|\nabla_{\boldsymbol{x}} u_h{}^i\|^2_{L_2(\Omega_{t^i})}\\
&\quad\le \Delta t\,\gamma_1^{n+1}\|u_h{}^{n+1}\|^2_{L_2(\Omega_{t^{n+1}})} + \Delta t\sum_{i=1}^{n}(\gamma_1^i+\gamma_2^i)\|u_h{}^i\|^2_{L_2(\Omega_{t^i})} + (1+\Delta t\gamma_2^0)\|u_h{}^0\|^2_{L_2(\Omega_{t^0})}\\
&\qquad + \Delta t\,\frac{1+C_\Omega}{\mu}\sum_{i=1}^{n+1}\|f^i\|^2_{H^{-1}(\Omega_{t^i})}\\
&\quad\le \Delta t\sum_{i=1}^{n+1}(\gamma_1^i+\gamma_2^i)\|u_h{}^i\|^2_{L_2(\Omega_{t^i})} + (1+\Delta t\,\gamma_2^0)\|u_h{}^0\|^2_{L_2(\Omega_{t^0})} + \Delta t\,\frac{1+C_\Omega}{\mu}\sum_{i=1}^{n+1}\|f^i\|^2_{H^{-1}(\Omega_{t^i})}.
\end{aligned} \tag{3.53}$$

Let us recall the following discrete Gronwall lemma (for the proof see for instance [13]).

Lemma 3.1. *Given $\delta, g_0, a_n, b_n, c_n, \gamma_n$ sequences of non negative numbers for $n \geq 0$, if the following inequality holds:*

$$a_n + \delta \sum_{j=0}^{n} b_j \leq \delta \sum_{j=0}^{n} \gamma_j a_j + \delta \sum_{j=0}^{n} c_j + g_0$$

then, for all $n \geq 0$

$$a_n + \delta \sum_{j=0}^{n} b_j \leq \exp\left(\delta \sum_{j=0}^{n} \sigma_j \gamma_j\right) \left[\delta \sum_{j=0}^{n} c_j + g_0\right]$$

where $\sigma_j = \frac{1}{1-\gamma_j \delta}$ and $\gamma_j \delta < 1$ for all j.

Then we conclude that

$$\begin{aligned}
&\|{u_h}^{n+1}\|^2_{L_2(\Omega_{t_{n+1}})} + \Delta t\, \mu \sum_{i=1}^{n+1} \|\nabla_{\boldsymbol{x}} {u_h}^i\|^2_{L_2(\Omega_{t^i})} \\
&\leq \left((1+\Delta t\, \gamma_2^0)\|{u_h}^0\|^2_{L_2(\Omega_{t^0})} + \Delta t \frac{1+C_\Omega}{\mu} \sum_{i=1}^{n+1} \|f^i\|^2_{H^{-1}(\Omega_{t^i})}\right) \\
&\quad \times \exp\left\{\Delta t \sum_{i=1}^{n+1} \frac{\gamma_1^i + \gamma_2^i}{1 - \Delta t(\gamma_1^i + \gamma_2^i)}\right\}
\end{aligned} \tag{3.54}$$

provided that:

$$\begin{aligned}
\Delta t &< \frac{1}{\gamma_1^i + \gamma_2^i} \\
&= \left(\|\nabla_{\boldsymbol{x}} \cdot \boldsymbol{w}(t^i)\|_{L_\infty(\Omega_{t^i})} + \sup_{t \in (t^i, t^{i+1})} \|J_{\mathcal{A}_{t_i,t_{i+1}}} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{w}\|_{L_\infty(\Omega_t)}\right)^{-1} \quad \forall\, i = 1, \ldots, n+1.
\end{aligned}$$

According to this stability analysis, we conclude that the scheme is only conditionally stable even if it is based on an implicit Euler method. Moreover, the maximum allowable time step will depend on the speed at which the domain is deforming.

4. CONCLUSIONS

We have presented some results for Arbitrary Lagrangian Eulerian finite element formulations. We have developed a general set-up for the finite element discretisation together with investigating the required regularity for the mapping. Some analysis on a linear convection diffusion equation has been carried out. We have assessed that the so-called geometric conservation laws, yet far for furnishing a general stability result, do provide a sufficient condition for stability independently of the domain movement law for a specific approximation scheme. It is believed that this result may be extended to other classes of schemes as well, such as those based on the conservative weak form and backward difference time integration formulae.

In a follow-up report, we plan to extend the analysis to the incompressible Navier-Stokes equations in the context of fractional step projection schemes, where the problem is furtherly complicated by the equation splitting and the possible presence of different approximation spaces.

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