A CODIMENSION TWO STABLE MANIFOLD OF NEAR SOLITON EQUIVARIANT WAVE MAPS

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ABSTRACT. We consider finite energy equivariant solutions for the wave map problem from $\mathbb{R}^{2+1}$ to $\mathbb{S}^2$ which are close to the soliton family. We prove asymptotic orbital stability for a codimension two class of initial data which is small with respect to a stronger topology than the energy.

1. INTRODUCTION

We consider Wave Maps $U : \mathbb{R}^{2+1} \to \mathbb{S}^2$ which are equivariant with co-rotation index 1. In particular, they satisfy $U(t, \omega x) = \omega U(t, x)$ for $\omega \in SO(2, \mathbb{R})$, where the latter group acts in standard fashion on $\mathbb{R}^2$, and the action on $\mathbb{S}^2$ is induced from that on $\mathbb{R}^2$ via stereographic projection. Wave maps are characterized by being critical with respect to the functional

$$U \to \int_{\mathbb{R}^{2+1}} \langle \partial_\alpha U, \partial^\alpha U \rangle \, d\sigma, \qquad \alpha = 0, 1, 2$$

with Einstein's summation convention being in force, $\partial^\alpha = m^{\alpha\beta} \partial_\beta$, $m_{\alpha\beta} = (m^{\alpha\beta})^{-1}$ the Minkowski metric on $\mathbb{R}^{2+1}$, and $d\sigma$ the associated volume element. Also, $\langle \cdot, \cdot \rangle$ refers to the standard inner product on $\mathbb{R}^3$ if we use ambient coordinates to describe u, $\partial_\alpha u$ etc. Recall that the energy is preserved:

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \langle DU(\cdot, t), DU(\cdot, t) \rangle \, dx = \text{const}$$

The problem at hand is *energy critical*, meaning that the conserved energy is invariant under the natural re-scaling $U \to U(\lambda t, \lambda x)$.

We focus on a particular subset of equivariant maps characterized by the additional property that $U(t, r, \theta) = (u(t, r), \theta)$ in spherical coordinates, where, on the right-hand side, u stand for the longitudinal angle and θ stands for the latitudinal angle, while, on the left-hand side, r, θ are the polar coordinates on $\mathbb{R}^2$. Now $u(t, r)$, a scalar function, satisfies the equation

$$(1.1) \qquad -u_{tt} + u_{rr} + \frac{u_r}{r} = \frac{\sin(2u)}{2r^2}$$

Then the energy has the form

$$(1.2) \qquad \mathcal{E}(u) = \pi \int_{\mathbb{R}^2} (|u_t|^2 + |u_r|^2 + \frac{\sin^2(u)}{r^2}) r \, dr$$

We shall be interested in co-rotational maps that are topologically non-trivial, namely with

$$u(t, 0) = 0, \qquad u(t, \infty) = \pi.$$

A natural space adapted to the elliptic part of this energy is $\dot{H}_e^1$

$$\|f\|_{\dot{H}_e^1}^2 = \|\partial_r f\|_{L^2}^2 + \|\frac{f}{r}\|_{L^2}^2$$

This is the equivariant translation of the usual two dimensional space $\dot{H}^1$. The size of the elliptic part of the energy of u in (1.2) and its $\dot{H}^1_e$ norm are comparable provided that u is small pointwise. This is not true directly for u but it true after we subtract from u the "nearby" soliton which we describe below.

The solitons for (1.1) have the form

$$Q_\lambda(r) = Q(\lambda r), \qquad Q(r) = 2 \arctan r, \qquad \lambda \in \mathbb{R}_+ = (0, \infty)$$

and are global minimizers of the energy $\mathcal{E}$ within their homotopy class, $\mathcal{E}(Q_\lambda) = 4\pi$.

We consider solutions u which are close to the soliton in the sense that

$$(1.3) \qquad \mathcal{E}(u) - \mathcal{E}(Q) \ll 1$$

As it turns out, such solutions must stay close to the soliton family $\{Q_\lambda\}$ due to the bound

$$(1.4) \qquad \inf_\lambda \|(u(t) - Q_\lambda)\|^2_{\dot{H}^1_e} + \|u_t(t)\|^2_{L^2} \sim \mathcal{E}(u) - \mathcal{E}(Q)$$

Indeed, this follows for example from [3]. Thus at any given t one can choose some $\lambda(t)$ so that

$$(1.5) \qquad \|(u(t) - Q_\lambda)\|^2_{\dot{H}^1_e} + \|u_t(t)\|^2_{L^2} \sim \mathcal{E}(u) - \mathcal{E}(Q)$$

Such a parameter λ is uniquely determined up to an error of size $O((\mathcal{E}(u) - \mathcal{E}(Q))^{\frac{1}{2}})$. One can for instance choose λ to be the minimizer in (1.4) though there are no obvious benefits to be derived from that. Another equivalent choice is more direct, namely by the relation

$$(1.6) \qquad u(t, \lambda^{-1}(t)) = \frac{\pi}{2}$$

and this still satisfies (1.5), see for instance [1]. Since this problem is locally well-posed in the energy space, scaling considerations show that (for well chosen $\lambda(t)$) we have

$$(1.7) \qquad \left| \frac{d}{dt} \lambda(t) \right| \lesssim \lambda^{-2}$$

so at least locally λ stays bounded. Then the main question to ask is as follows:

Open Problem. *What is the behavior of the function $\lambda(t)$ for equivariant maps satisfying* (1.3) *?*

We can distinguish several interesting plausible scenarios:

- *Type 1:* $\lambda(t) \to \infty$ as $t \to t_0$ (finite time blow-up). By (1.7) this can only happen at rates $\lambda(t) \gtrsim |t - t_0|^{-1}$. The above extreme corresponds to self-similar concentration; this can be thought also as a consequence of the finite speed of propagation. In effect, by the important work [10], it is known that such blow up can only occur with speed strictly faster than self-similar:
$$\lambda(t)|t - t_0| \to \infty$$
- *Type 2:* $\lambda(t) \to \infty$ as $t \to \infty$ (infinite time focusing).
- *Type 3:* $\lambda(t) \to 0$ as $t \to \infty$ (infinite time relaxation). By (1.7) this can only happen at rates $\lambda(t) \gtrsim t^{-1}$, which corresponds to self-similar relaxation.
- *Type 4:* $\lambda(t)$ stays in a compact set globally in time. Then we have a global solution, and possibly a resolution into a soliton plus a dispersive part.

Blow-up solutions of Type 1 were constructed not long ago in two quite different papers [6] and [8], and the result of the latter paper was significantly strengthened and generalized in [9]. The behavior of $\lambda(t)$ in [6] as $t \to 0$ is given by

$$\lambda(t) = t^{-1-\nu}, \qquad \nu \geq 1$$

(here the restriction $\nu \geq 1$ seems technical, should really be $\nu > 0$) while that in [9] is

$$\lambda(t) \sim t^{-1} e^{c\sqrt{\log t}}$$

The latter solutions were also proved to be stable with respect to class of small smooth perturbations. It is not implausible that the set of all blow-up solutions is open in a suitable topology, although numerical evidence in [2] appears to suggest the existence of a co-dimension one manifold of data leading to an unstable blow up, which separates scattering solutions from a stable regime of finite time blow up solutions.

Up to this point we are not aware of any examples of solutions of type 2, 3 and also of type 4 other than the Q_λ's in the wave maps context, although recent work [4] revealed unusual solutions of this type in the context of the Landau-Lifshitz equation. Earlier work [7] showed the existence of type 4 solutions for the critical focusing nonlinear wave equation on $\mathbb{R}^{3+1}$.

Understanding the general picture for data in the energy space seems out of reach for now. However, there is a simpler question one may ask, namely what happens for data which is close to a soliton in a stronger topology, which includes both extra regularity and extra decay at infinity. Neither the results of [6] nor the ones in [9] apply in this context. A good starting point for this investigation is the following

Conjecture. *There exists a codimension one set of (small) data leading to Type 4 solutions, which separates Type 1 and Type 3 solutions.*

One should take this only as a rough guide; some fine adjustments may be needed. Our main result is to construct a large class of Type 4 solutions:

Theorem 1.1. *There exists a codimension two set of Type 4 equivariant wave maps satisfying* (1.3).

For a more precise formulation of the theorem we refer the reader to Section 2.1. Compared with the conjecture above, one can see that we are one dimension short. At this point it is not clear if this is a technical issue, or something new happens. A plausible scenario might be that the missing dimension may include Type 2 solutions, as well as slowly relaxing Type 4 solutions.

One should also compare this result with the related problem for Schrödinger maps. Although the solitons are the same and the operator H arising below in the linearization is also the same for Schrödinger maps, in [1] it is shown that the solitons are stable with respect to small localized perturbations. One way to explain this is that the linear growth in the resonant direction occuring in the H-wave equation has a stronger destabilizing effect than the corresponding lack of decay in the H-Schrödinger equation.

1.1. **Notations.** Here we introduce a few notation which will be used throughout the paper. We slightly modify the use of $\langle \cdot \rangle$ in the following sense

$$\langle x \rangle = \sqrt{4 + x^2}, \qquad x \in \mathbb{R}$$

For a real number a we define $a^+ = \max\{0, a\}$ and $a^- = \min\{0, a\}$.

We will use a dyadic partition of $\mathbb{R}_+$ into sets $\{A_m\}_{m \in \mathbb{Z}}$ given by

$$A_m = \{2^{m-1} < r < 2^{m+1}\}.$$

For given $M > 0$, we use smooth localization functions $\chi_{\lesssim M}, \chi_{\gtrsim M}$ forming a partition of unity for $\mathbb{R}_+$ and such that

$$|(r\partial_r)^\alpha \chi_{\lesssim M}| + |(r\partial_r)^\alpha \chi_{\gtrsim M}| \lesssim_\alpha 1$$

2. The gauge derivative and linearizations

The linearized equation (1.1) around the soliton Q has the form

$$(2.1) \qquad -v_{tt} - Hv = 0, \qquad H = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\cos(2Q)}{r^2}$$

The elliptic operator H admits the factorization

$$(2.2) \qquad H = L^*L, \qquad L = h_1\partial_r h_1^{-1} = \partial_r + \frac{h_3}{r}, \qquad L^* = -h_1^{-1}\partial_r h_1 - \frac{1}{r} = -\partial_r + \frac{h_3 - 1}{r}.$$

where[1] $h_1 = \sin Q = \frac{2r}{1+r^2}, h_3 = -\cos Q = \frac{r^2-1}{r^2+1}$. H is nonnegative and has a zero resonance

$$\phi_0 = h_1 = \frac{2r}{1+r^2}$$

This resonance is the reason why (2.1) does not have good dispersive estimates. Since ϕ_0 fails to be an eigenvalue, we cannot project it away as it is usually done in standard modulation theory. This suggests that working with the variable u and its equation (1.1) runs into problems due to the lack of good linear estimates needed to treat the nonlinearity. Therefore, instead of working with the solution u we introduce a new variable

$$(2.3) \qquad w = \partial_r u - \frac{1}{r}\sin u$$

which has the nice property that

$$w = 0 \Longleftrightarrow u = Q_\lambda$$

for some $\lambda \in \mathbb{R}_+$. Indeed, by rearranging (1.2) and using $u(0) = 0, u(\infty) = \pi$, we obtain

$$\mathcal{E}(u) = \pi \int_0^\infty (|u_t|^2 + |w|^2)r\,dr + \pi \int_0^\infty 2\sin u \cdot \partial_r u\,dr = \pi \int_0^\infty (|u_t|^2 + |w|^2)r\,dr + 4\pi$$

from which the above observation follows. This type of change of variables originates at least with the work [5]. If $\lambda(t)$ is chosen such that (1.5) holds, then using (1.3), a direct computation shows that

$$(2.4) \qquad \|u - Q_\lambda\|_{\dot{H}_e^1} \approx \|w\|_{L^2}^2.$$

Then a direct computation shows that w solves

$$(2.5) \qquad w_{tt} - \Delta w + \frac{2(1 + \cos u)}{r^2}w = \frac{1}{r}\sin u(u_t^2 - w^2)$$

The function u appears in this equation, but it can be recovered from w by solving the ode (2.3) with Q-like "data" at $r = \infty$.

We remark that the linearized form of (2.3) near Q is

$$(2.6) \qquad z = (\partial_r - \frac{1}{r}\cos Q)v = Lv$$

where L was introduced above in (2.2).

On the other hand the linearized equation for w near Q has the form

$$(2.7) \qquad z_{tt} - \Delta z + \frac{2(1 + \cos Q)}{r^2}z = 0$$

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[1] throughout this paper we use $\sin Q, \cos Q$ instead of h_1, h_3; however the reader may need this correspondence in order to relate to the work [1].

This wave equation is governed by the operator

$$\tilde{H} = -\Delta + \frac{2(1 + \cos Q)}{r^2} = -\Delta + \frac{4}{r^2(1 + r^2)} = LL^*$$

This operator is better behaved compared to H, in particular its zero mode ψ_0 grows logarithmically at infinity.

The plan is to treat the equation (2.5) in a perturbative manner for the most part. To fix things, we will rewrite it in the form

$$(2.8) \qquad (\partial_t^2 + \tilde{H})w = \frac{2(\cos Q - \cos u)}{r^2}w + \frac{1}{r}\sin u(u_t^2 - w^2) := N(w, u)$$

and work with this from here on. The equation (2.8) for w is preferable due to the nice dispersive properties of its linear part. However, as u occurs in the w equation, one has to also keep track of it through the elliptic equation (2.3).

In order to study this equation we need to understand better the structure of its linear part, and, in particular, the spectral theory for the operator $\tilde{H}$. This is the subject of section 3.

2.1. **Setup of the problem.** The starting point is to consider $\bar{w}$ to be an exact real solution to the linear homogeneous equation

$$(2.9) \qquad (\partial_t^2 + \tilde{H})\bar{w} = 0, \qquad w(0) = w_0, \qquad w_t(0) = w_1$$

where w_0 and w_1 are real Schwartz functions which are assumed to satisfy the nonresonance conditions

$$(2.10) \qquad \langle w_0, \psi_0 \rangle = 0, \qquad \langle w_1, \psi_0 \rangle = 0$$

We denote by $\bar{u}$ the corresponding map, see (2.3) (this will be made precise in Proposition 5.2), obtained by solving the ode

$$(2.11) \qquad \partial_r \bar{u} - \frac{1}{r}\sin \bar{u} = \bar{w}, \qquad \bar{u} \sim Q \text{ as } r \to \infty$$

Now we seek a solution to the nonlinear equation u and its associated gauge derivative w close to $\bar{u}, \bar{w}$ respectively,

$$(2.12) \qquad u = \bar{u} + \varepsilon, \qquad w = \bar{w} + \gamma$$

so that u and w match $\bar{u}$ and $\bar{w}$ asymptotically as $t \to \infty$.

By a slight abuse of notation we use $\|\cdot\|_S$ to denote a norm obtained by adding sufficiently many seminorms of the Schwartz space S. We also use $\lesssim_S$ for inequalities where the implicit constant depends on $\|(w_0, w_1)\|_S$. Modulo defining the X and LX norms, we are now in a position to restate our main result in a more detailed fashion.

Theorem 2.1. *Let w_0, w_1 be Schwartz functions satisfy the nonresonance conditions* (2.10). *Let $\bar{u}$ and $\bar{w}$ be defined as above. Then there exists $T \lesssim_S 1$ and a unique wave map u in $[T, \infty)$ so that u and w match $\bar{u}$ and $\bar{w}$ as $t \to \infty$ in the following asymptotic fashion for $t \in [T, \infty)$:*

$$(2.13) \qquad \|\gamma(t)\|_{LX} \lesssim_S t^{-\frac{3}{2}}, \qquad \|\partial_t \gamma(t)\|_{LX} \lesssim_S t^{-\frac{5}{2}}, \qquad \|\gamma(t)\|_{\dot{H}^1} \lesssim_S t^{-\frac{5}{2}}$$

respectively

$$(2.14) \qquad \|\varepsilon(t)\|_X \lesssim_S t^{-\frac{3}{2}}, \qquad \|\partial_t \varepsilon(t)\|_{LX} \lesssim_S t^{-\frac{5}{2}}$$

Furthermore, the map u and its corresponding gauge derivative w have a Lipschitz dependence on (w_0, w_1) with respect to the above norms.

One would expect the above result to be in terms of L^2 and $\dot{H}^1_e$ spaces. However these spaces are very disconnected from the spectral structure of H and $\tilde{H}$, particularly at low frequencies, and this makes them unsuitable. The spaces $X \subset \dot{H}^1_e$ and $LX \subset L^2$ have been introduced in [1] to address exactly this issue: they are low frequency corrections of $\dot{H}^1_e$, respectively L^2. Their exact definition is provided in the next section.

In view of the equation (2.8), the function γ solves

$$(2.15) \qquad\qquad (\partial_t^2 + \tilde{H})\gamma = N(\bar{w} + \gamma, \bar{u} + \varepsilon)$$

with zero Cauchy data at infinity. By (2.3), (2.12) and (2.11), the function ε is determined from the equation

$$(2.16) \qquad\qquad \gamma = \partial_r \varepsilon - \frac{\sin(\varepsilon + \bar{u}) - \sin \bar{u}}{r}$$

We proceed as follows. In the next section we recall from [1] the spectral theory for H (which in fact originates in [6]) and $\tilde{H}$ and the definitions and some properties of the spaces X and LX. Then, in Section 4 we provide linear estimates for the linear (inhomogenous) wave equation corresponding to (2.9). In Section 5 we analyze the first approximations $\bar{w}$ and $\bar{u}$ using (2.11). Then, in Section 6, we continue with the study of the relation between ε and γ based on the the equation (2.16). All the analysis carried in Sections 4-6 is done in the context of X and LX spaces. In the end, in Section 7 we study the solvability of equation (2.15) using perturbative methods in LX based spaces.

3. THE MODIFIED FOURIER TRANSFORM

In this section we recall the spectral theory associated with the operators $H, \tilde{H}$. The spectral theory for H was developed in [6] and the one for $\tilde{H}$ was derived from the one for H in [1]. In this paper, we follow closely the exposition in [1].

3.1. **Generalized eigenfunctions.** We consider H acting as an unbounded selfadjoint operator in $L^2(rdr)$. Then H is nonnegative, and its spectrum $[0, \infty)$ is absolutely continuous. H has a zero resonance, namely $\phi_0 = h_1$,

$$Hh_1 = 0.$$

For each $\xi > 0$ one can choose a normalized generalized eigenfunction ϕ_ξ,

$$H\phi_\xi = \xi^2 \phi_\xi.$$

These are unique up to a ξ dependent multiplicative factor, which is chosen as described below.

To these one associates a generalized Fourier transform $\mathcal{F}_H$ defined by

$$\mathcal{F}_H f(\xi) = \int_0^\infty \phi_\xi(r) f(r) r dr$$

where the integral above is considered in the singular sense. This is an L^2 isometry, and we have the inversion formula

$$f(r) = \int_0^\infty \phi_\xi(r) \mathcal{F}_H f(\xi) d\xi$$

The functions ϕ_ξ are smooth with respect to both r and ξ. To describe them one considers two distinct regions, $r\xi \lesssim 1$ and $r\xi \gtrsim 1$.

In the first region $r\xi \lesssim 1$ the functions ϕ_ξ admit a power series expansion of the form

$$(3.1) \qquad\qquad \phi_\xi(r) = q(\xi)\left(\phi_0 + \frac{1}{r}\sum_{j=1}^\infty (r\xi)^{2j}\phi_j(r^2)\right), \qquad r\xi \lesssim 1$$

where $\phi_0 = h_1$ and the functions ϕ_j are analytic and satisfy

$$(3.2) \qquad |(r\partial_r)^\alpha \phi_j| \lesssim_\alpha \frac{C^j}{(j-1)!} \log(1+r)$$

This bound is not spelled out in [6], but it follows directly from the integral recurrence formula for f_j's (page 578 in the paper). The smooth positive weight q satisfies

$$(3.3) \qquad q(\xi) \approx \begin{cases} \dfrac{1}{\xi^{\frac{1}{2}}|\log \xi|}, & \xi \ll 1 \\[2ex] \xi^{\frac{3}{2}}, & \xi \gg 1 \end{cases} \qquad , \qquad |(\xi\partial_\xi)^\alpha q| \lesssim_\alpha q$$

Defining the weight

$$(3.4) \qquad m_k^1(r) = \begin{cases} \min\{1, r2^k \dfrac{\log(1+r^2)}{\langle k \rangle}\}, & k < 0 \\[2ex] \min\{1, r^3 2^{3k}\}, & k \geq 0 \end{cases}$$

it follows that the nonresonant part of ϕ_ξ satisfies

$$(3.5) \qquad |(\xi\partial_\xi)^\alpha (r\partial_r)^\beta (\phi_\xi(r) - q(\xi)\phi_0(r))| \lesssim_{\alpha\beta} 2^{\frac{k}{2}} m_k^1(r), \qquad \xi \approx 2^k, \ r\xi \lesssim 1$$

In the other region $r\xi \gtrsim 1$ we begin with the functions

$$(3.6) \qquad \phi_\xi^+(r) = r^{-\frac{1}{2}} e^{ir\xi} \sigma(r\xi, r), \qquad r\xi \gtrsim 1$$

solving

$$H\phi_\xi^+ = \xi^2 \phi_\xi^+$$

where for σ we have the following asymptotic expansion

$$\sigma(q, r) \approx \sum_{j=0}^\infty q^{-j} \phi_j^+(r), \qquad \phi_0^+ = 1, \qquad \phi_1^+ = \frac{3i}{8} + O(\frac{1}{1+r^2})$$

with

$$\sup_{r>0} |(r\partial_r)^k \phi_j^+| < \infty$$

in the following sense

$$\sup_{r>0} |(r\partial r)^\alpha (q\partial_q)^\beta [\sigma(q, r) - \sum_{j=0}^{j_0} q^{-j} \phi_j^+(r)]| \leq c_{\alpha,\beta,j_0} q^{-j_0-1}$$

Then we have the representation

$$(3.7) \qquad \phi_\xi(r) = a(\xi)\phi_\xi^+(r) + \overline{a(\xi)\phi_\xi^+(r)}$$

where the complex valued function a satisfies

$$(3.8) \qquad |a(\xi)| = \sqrt{\frac{2}{\pi}}, \qquad |(\xi\partial_\xi)^\alpha a(\xi)| \lesssim_\alpha 1$$

The spectral theory for $\tilde{H}$ is derived from the spectral theory for H due to the conjugate representations

$$H = L^*L, \qquad \tilde{H} = LL^*$$

This allows us to define generalized eigenfunctions ψ_ξ for $\tilde{H}$ using the generalized eigenfunctions ϕ_ξ for H,

$$(3.9) \qquad\qquad \psi_\xi = \xi^{-1} L\phi_\xi, \qquad L^*\psi_\xi = \xi\phi_\xi$$

It is easy to see that ψ_ξ are real, smooth, vanish at $r = 0$ and solve

$$\tilde{H}\psi_\xi = \xi^2\psi_\xi$$

With respect to this frame we can define the generalized Fourier transform adapted to $\tilde{H}$ by

$$\mathcal{F}_{\tilde{H}}f(\xi) = \int_0^\infty \psi_\xi(r)f(r)rdr$$

where the integral above is considered in the singular sense. This is an L^2 isometry, and we have the inversion formula

$$(3.10) \qquad\qquad f(r) = \int_0^\infty \psi_\xi(r)\mathcal{F}_{\tilde{H}}f(\xi)d\xi$$

To see this we compute, for a Schwartz function f:

$$\mathcal{F}_{\tilde{H}}Lf(\xi) = \int_0^\infty \psi_\xi(r)Lf(r)rdr = \int_0^\infty L^*\psi_\xi(r)f(r)rdr$$
$$= \int_0^\infty \xi\phi_\xi(r)f(r)rdr = \xi\mathcal{F}_H f(\xi)$$

Hence

$$\|\mathcal{F}_{\tilde{H}}Lf\|_{L^2}^2 = \|\xi\mathcal{F}_H f(\xi)\|_{L^2}^2 = \langle Hf, f\rangle_{L^2(rdr)} = \|Lf\|_{L^2}^2$$

which suffices since Lf spans a dense subset of L^2.

The representation of ψ_ξ in the two regions $r\xi \lesssim 1$ and $r\xi \gtrsim 1$ is obtained from the similar representation of ϕ_ξ. In the first region $r\xi \lesssim 1$ the functions ψ_ξ admit a power series expansion of the form

$$(3.11) \qquad\qquad \psi_\xi = \xi q(\xi)\left(\psi_0(r) + \sum_{j\geq 1}(r\xi)^{2j}\psi_j(r^2)\right)$$

where

$$\psi_j(r) = (h_3 + 1 + 2j)\phi_{j+1}(r) + r\partial_r\phi_{j+1}(r)$$

From (3.2), it follows that

$$|(r\partial_r)^\alpha\psi_j| \lesssim_\alpha \frac{C^j}{(j-1)!}\log\left(1 + r^2\right)$$

In addition, ψ_0 solves $L^*\psi_0 = \phi_0$ therefore a direct computation shows that

$$\psi_0 = \frac{1}{2}\left(\frac{(1+r^2)\log(1+r^2)}{r^2} - 1\right)$$

In particular, defining the weights

$$(3.12) \qquad\qquad m_k(r) = \begin{cases} \min\{1, \dfrac{\log\left(1 + r^2\right)}{\langle k\rangle}\}, & k < 0 \\[3mm] \min\{1, r^2 2^{2k}\}, & k \geq 0 \end{cases}$$

we have the pointwise bound for ψ_ξ

$$(3.13) \qquad\qquad |(r\partial_r)^\alpha(\xi\partial_\xi)^\beta\psi_\xi(r)| \lesssim_{\alpha\beta} 2^{\frac{k}{2}}m_k(r), \qquad \xi \approx 2^k, \ r\xi \lesssim 1$$

On the other hand in the regime $r\xi \gtrsim 1$ we define

$$\psi^+ = \xi^{-1} L \phi^+$$

and we obtain the representation

(3.14)
$$\psi_\xi(r) = a(\xi)\psi_\xi^+(r) + \overline{a(\xi)\psi_\xi^+(r)}$$

For ψ^+ we obtain the expression

(3.15)
$$\psi_\xi^+(r) = r^{-\frac{1}{2}} e^{ir\xi} \tilde{\sigma}(r\xi, r), \qquad r\xi \gtrsim 1$$

where $\tilde{\sigma}$ has the form

$$\tilde{\sigma}(q,r) = i\sigma(q,r) - \frac{1}{2}q^{-1}\sigma(q,r) + \frac{\partial}{\partial q}\sigma(q,r) + \xi^{-1}L\sigma(q,r)$$

therefore it has exactly the same properties as σ. In particular, for fixed ξ, we obtain that

(3.16)
$$\tilde{\sigma}(r\xi, r) = i - \frac{7}{8}r^{-1}\xi^{-1} + O(r^{-2})$$

We conclude our description of the generalized eigenfunctions and of the associated Fourier transforms with a bound on the $\tilde{H}$ Fourier transforms of Schwartz functions.

Lemma 3.1. *If f is a Schwartz function satisfying $\langle f, \psi_0 \rangle = 0$ then*

(3.17)
$$|(\xi\partial_\xi)^\alpha \mathcal{F}_{\tilde{H}} f(\xi)| \lesssim_{\alpha,N} \begin{cases} \dfrac{\xi^{\frac{5}{2}}}{\langle \log \xi \rangle}, & \xi \lesssim 1 \\ \langle \xi \rangle^{-N}, & \xi \gtrsim 1 \end{cases}$$

Proof. We start from the definition of modified Fourier transform and use that $\langle f, \psi_0 \rangle = 0$

$$|\mathcal{F}_{\tilde{H}} f(\xi)| \lesssim \left(\left| \int_0^{\xi^{-1}} \psi_\xi(r) f(r) r \, dr \right| + \left| \int_{\xi^{-1}}^\infty \psi_\xi(r) f(r) r \, dr \right| \right)$$

$$\lesssim \xi q(\xi) \left(\int_{\xi^{-1}}^\infty |\psi_0(r) f(r)| r \, dr + \int_0^{\xi^{-1}} \sum_{j \geq 1} (r\xi)^{2j} \psi_j(r^2) f(r) r \, dr \right) + \int_{\xi^{-1}}^\infty |f(r)| r^{\frac{1}{2}} \, dr$$

$$\lesssim \xi^3 q(\xi)$$

A similar argument takes care of the case $\alpha > 0$.

$\square$

3.2. The spaces X and LX.

The operator L maps $\dot{H}_e^1$ into L^2. Conversely one would like that, given some $f \in L^2$, we could solve $Lu = f$ and we obtain a solution u which is in $\dot{H}_e^1$ and satisfies

$$\|u\|_{\dot{H}_e^1} \lesssim \|f\|_{L^2}$$

However, this is not the case. The first observation is that the solution is only unique modulo a multiple of the resonance ϕ_0. Moreover the inequality above is not expected to be true, even assuming that somehow we choose the "best" u from all candidates.

The spaces X and LX are in part introduced in order to remedy both the ambiguity in the inversion of L and the failing inequality.

Definition 3.2. *a) The space X is defined as the completion of the subspace of $L^2(rdr)$ for which the following norm is finite*

$$\|u\|_X = \left(\sum_{k \geq 0} 2^{2k} \|P_k^H u\|_{L^2}^2 \right)^{\frac{1}{2}} + \sum_{k < 0} \frac{1}{|k|} \|P_k^H u\|_{L^2}$$

where P_k^H is the Littlewood-Paley operator localizing at frequency $\xi \approx 2^k$ in the H calculus.

b) LX is the space of functions of the form $f = Lu$ with $u \in X$, with norm $\|f\|_{LX} = \|u\|_X$. Expressed in the $\tilde{H}$ calculus, the LX norm is written as

$$\|f\|_{LX} = \left(\sum_{k \geq 0} \|P_k^{\tilde{H}} f\|_{L^2}^2 \right)^{\frac{1}{2}} + \sum_{k < 0} \frac{2^{-k}}{|k|} \|P_k^{\tilde{H}} f\|_{L^2}$$

In this article we work with equivariant wave maps u for which $\|u - Q\|_X \ll 1$. This corresponds to functions w which satisfy $\|w\|_{LX} \ll 1$. The simplest properties of the space X are summarized as follows, see Proposition 4.2 in [1]:

Proposition 3.3. *The following embeddings hold for the space X:*

(3.18) $$H_e^1 \subset X \subset \dot{H}_e^1$$

In addition for f in X we have the following bounds:

(3.19) $$\|\langle r \rangle^{\frac{1}{2}} f\|_{L^\infty} \lesssim \|f\|_X$$

(3.20) $$\left\| \frac{f}{\log(1+r)} \right\|_{L^2} \lesssim \|f\|_X$$

(3.21) $$\left\| \langle r \rangle^{\frac{1}{2}} f \right\|_{L^4} \lesssim \|f\|_X$$

Now we turn our attention to the space LX. From [1], Lemma 4.4 and Proposition 4.5, we have

Lemma 3.4. *If $f \in L^2$ is localized at $\tilde{H}$- frequency 2^k then*

(3.22) $$|f(r)| \lesssim 2^k m_k(r)(1 + 2^k r)^{-\frac{1}{2}} \|f\|_{L^2}$$

Proposition 3.5. *The following embeddings hold for LX:*

(3.23) $$L^1 \cap L^2 \subset LX \subset L^2$$

4. LINEAR ESTIMATES FOR THE $\tilde{H}$ WAVE EQUATION

In this section we prove estimates for the linear equation

(4.1) $$(\partial_t^2 + \tilde{H})\psi = f$$

with zero Cauchy data at infinity. The solution is given by $\psi = Kf$, where

$$Kf(r,t) = -\mathcal{F}_{\tilde{H}}^{-1} \int_t^\infty \frac{\sin(t-s)\xi}{\xi} \mathcal{F}_{\tilde{H}} f(\xi,s) ds$$

We also need its time derivative, which is given by

$$\partial_t Kf = -\mathcal{F}_{\tilde{H}}^{-1} \int_t^\infty \cos(t-s)\xi \cdot \mathcal{F}_{\tilde{H}} f(\xi,s) ds$$

Finally we need the following formula, which follows from (3.9)

$$L^* Kf = -\mathcal{F}_H^{-1} \int_t^\infty \sin(t-s)\xi \cdot \mathcal{F}_{\tilde{H}} f(\xi,s) ds$$

The following result is a modification of the standard energy estimate for the wave equation:

Lemma 4.1. *Assume that $f(s) \in LX$. Then for every $\alpha > 0$, the solution of* (4.1) *with zero data at ∞ satisfies*

$$(4.2) \qquad t^\alpha \|\psi(t)\|_{LX} + t^{\alpha+1}(\|\partial_t \psi(t)\|_{LX} + \|\psi(t)\|_{\dot{H}^1_e}) \lesssim \sup_s s^{\alpha+2} \|f(s)\|_{LX}$$

Proof. The solution of (4.1) with zero data at ∞ is given by $\psi = Kf$. The estimate for the first term follows from the bound $|\frac{\sin(t-s)\xi}{\xi}| \lesssim |t-s|$ and the representation of the spaces LX on the Fourier side. The estimate for the second term is similar.

The argument for the third term is more involved. We denote

$$\mathcal{F}_{\tilde{H}} g(t, \xi) = - \int_t^\infty \sin((t-s)\xi) \mathcal{F}_{\tilde{H}} f(\xi, s) ds$$

Then

$$\xi \mathcal{F}_{\tilde{H}} \psi(t, \xi) = \mathcal{F}_{\tilde{H}} g(t, \xi)$$

We estimate as above

$$\|g(t)\|_{LX} \lesssim \int_t^\infty \|f(s)\|_{LX} ds \lesssim t^{-\alpha-1} \sup_s s^{\alpha+2} \|f(s)\|_{LX}$$

Hence it suffices to show that for ψ and g related as above we have

$$(4.3) \qquad \|\psi\|_{\dot{H}^1_e} \lesssim \|g\|_{LX}$$

Here the time variable plays no role and is discarded. Recalling the form of L^* from (2.2), namely $L^* = -\partial_r + \frac{h_3 - 1}{r}$, it follows that

$$\|\psi\|_{\dot{H}^1_e} \lesssim \|L^* \psi\|_{L^2} + \|\frac{\psi}{r}\|_{L^2}$$

For the first term we use Plancherel to write

$$\|L^* \psi(t)\|_{L^2}^2 = \langle \psi(t), \tilde{H} \psi(t) \rangle = \|\xi \mathcal{F}_{\tilde{H}} \psi(\xi)\|_{L^2}^2 = \|g\|_{L^2} \lesssim \|g\|_{LX}^2$$

For the second term the L^2 bound for g no longer suffices, and we need to use the LX norm of g. We consider a Littlewood-Paley decomposition for both ψ and g, and denote their dyadic pieces by ψ_k, respectively g_k. Then

$$\|\psi_k\|_{L^2} \approx 2^{-k} \|g_k\|_{L^2}$$

By using (3.13), (3.14) and the Cauchy-Schwartz inequality we obtain pointwise bounds for ψ_k, namely

$$|\psi_k| \lesssim \frac{m_k(r)}{\langle 2^k r \rangle^{\frac{1}{2}}} 2^k \|\psi_k\|_{L^2} \lesssim \frac{m_k(r)}{\langle 2^k r \rangle^{\frac{1}{2}}} \|g_k\|_{L^2}$$

with m_k as in (3.12). For $k \geq 0$ the contributions are almost orthogonal and we obtain

$$\|\frac{\psi_{\geq 0}}{r}\|_{L^2} \lesssim \|g_{\geq 0}\|_{L^2}$$

However, if $k < 0$ then the weaker logarithmic decay for small r no longer suffices for such an argument. Instead by direct computation we obtain a weaker bound,

$$\|\frac{\psi_k}{r}\|_{L^2} \lesssim |k|^{\frac{1}{2}} \|g_k\|_{L^2} \lesssim |k|^{\frac{3}{2}} 2^k \|g\|_{LX}$$

Then the k summation is easily accomplished.

$\square$

5. ANALYSIS OF THE FIRST APPROXIMATIONS $\bar{w}$ AND $\bar{u}$

5.1. **Pointwise bounds for $\bar{w}$.** We denote $f_0 = \mathcal{F}_{\tilde{H}} w_0$ and $f_1 = \mathcal{F}_{\tilde{H}} w_1$. Then for $\bar{w}$ we have the representation

$$\bar{w}(t,r) = \int_0^\infty \psi_\xi(r)(f_0(\xi)\cos(t\xi) + \frac{1}{\xi}f_1(\xi)\sin(t\xi))d\xi$$

Since w_0, w_1 are Schwartz functions satisfying (2.10), from (3.17) we obtain

$$(5.1) \qquad |(\xi\partial_\xi)^\alpha f_0(\xi)| + |(\xi\partial_\xi)^\alpha f_1(\xi)| \lesssim_{\alpha,N} \|(w_0,w_1)\|_S \begin{cases} \frac{\xi^{\frac{5}{2}}}{\langle\log\xi\rangle}, & \xi \lesssim 1 \\ \langle\xi\rangle^{-N}, & \xi \gtrsim 1 \end{cases}$$

Here by a slight abuse of notation we use $\|.\|_S$ to denote a finite collection of the S seminorms. This will allow us to obtain pointwise bounds for $\bar{w}$:

Lemma 5.1. *If w_0, w_1 are Schwartz functions satisfying the moment conditions* (2.10) *then $\bar{w}$ satisfies*

$$(5.2) \qquad |\bar{w}(r,t)| \lesssim \frac{\log(1+r^2)}{\log\langle r+t\rangle} \frac{1}{\langle t+r\rangle^{\frac{1}{2}}\langle t-r\rangle^{\frac{5}{2}}\log\langle r-t\rangle}\|(w_0,w_1)\|_S$$

Proof. We fix k and consider

$$\bar{w}_k(t,r) = \int_0^\infty \psi_\xi(r)(f_0(\xi)\cos(t\xi) + \frac{1}{\xi}f_1(\xi)\sin(t\xi))\chi_k(\xi)d\xi$$

For $\psi_\xi(r)$ we use the representation (3.11) in the region $\{r\xi \lesssim 1\}$, respectively (3.14) in the region $\{r\xi \gtrsim 1\}$. Then via a standard stationary phase argument we obtain

$$|w_k(r,t)| \lesssim_N \frac{2^{\frac{k}{2}}\langle 2^k r\rangle^{-\frac{1}{2}} m_k(r)}{\langle 2^k|r-t|\rangle^N\langle k^-\rangle} 2^{\frac{5k}{2}} 2^{-Nk^+}.$$

The desired estimate (5.2) follows by summing these bounds with respect to k. $\qquad \square$

5.2. **Bounds for $\bar{u}$, $\bar{u}_t$.** Next we consider $\bar{u}$, which is recovered from $\bar{w}$ via (2.11). This equation contains a nonlinear part coming from the sine function. Consequently, we split $\bar{u}$ into a linear and a nonlinear part:

$$\bar{u} = Q + \bar{u}^l + \bar{u}^{nl}$$

where $\bar{u}^l$ solves the linear part of (2.11)

$$L\bar{u}^l = \bar{w}$$

and $\bar{u}^{nl}$ solves

$$(5.3) \qquad L\bar{u}^{nl} = N(\bar{u}^l, \bar{u}^{nl})$$

where

$$N(u,v) = \frac{1}{r}\Big[\sin Q \cdot (\cos(u+v) - 1) + \cos Q \cdot (\sin(u+v) - (u+v))\Big]$$

Both of the above ode's are taken with zero Cauchy data at infinity or, equivalently, can be interpreted via the diffeomorphism $L : X \to LX$. The linear part $\bar{u}^l$ is recovered from the explicit formula

$$\bar{u}^l := L^{-1}\bar{w} = \int_0^\infty \xi^{-1}\phi_\xi(r)(f_0(\xi)\cos(t\xi) + \frac{1}{\xi}f_1(\xi)\sin(t\xi))d\xi$$

and will be split into a resonant and a nonresonant part $\bar{u}^l = \bar{u}^{l,r} + \bar{u}^{l,nr}$.

For the nonlinear part we use an iterative argument based on the fact that there is enough decay on the right-hand side so that we can recover it via

$$(5.4) \qquad \bar{u}^{nl} = h_1(r) \int_r^\infty \frac{N(\bar{u}^l, \bar{u}^{nl})}{h_1(s)} ds$$

At this stage we also want to keep track of the differences of solutions. For this we denote by $\delta w_0, \delta w_1, \delta \bar{w}, \delta \bar{u}$ the corresponding differences.

Proposition 5.2. *a) Assume that w_0, w_1 are small Schwartz functions satisfying (2.10). Then*

$$(5.5) \qquad \bar{u}^l = \bar{u}^{l,r} + \bar{u}^{l,nr},$$

where $\bar{u}^{l,r}$ and $\bar{u}^{l,nr}$ satisfy the following bounds

$$(5.6) \qquad |\bar{u}^{l,r}| + r|\partial_r \bar{u}^{l,r}| + \langle r+t \rangle |\partial_t \bar{u}^{l,r}| \lesssim \frac{h_1(r)}{\langle t+r \rangle \log^2 \langle t+r \rangle} \|(w_0, w_1)\|_S,$$

$$|\bar{u}^{l,nr}| + \frac{r \langle r-t \rangle}{\langle t+r \rangle} |\partial_r \bar{u}^{l,nr}| + \langle r-t \rangle |\partial_t \bar{u}^{l,nr}| \lesssim \frac{r}{r + \langle t \rangle} \frac{1}{\langle t+r \rangle^{\frac{1}{2}} \langle t-r \rangle^{\frac{3}{2}} \log \langle t-r \rangle} \|(w_0, w_1)\|_S.$$

In addition,

$$(5.7) \qquad |(\partial_r + \partial_t) \bar{u}^l + \frac{1}{2r} \bar{u}^l| \lesssim \frac{1}{t^{\frac{5}{2}} \langle r-t \rangle^{\frac{1}{2}} \log \langle t-r \rangle} \|(w_0, w_1)\|_S, \qquad r \sim t$$

b) For $t \gtrsim_S 1$ the nonlinear part $\bar{u}^{nl}$ satisfies the bounds

$$(5.8) \qquad |\bar{u}^{nl}(r,t)| \lesssim_S h_1(r) t^{-1.5} \|(w_0, w_1)\|_S,$$

$$|\partial_t \bar{u}^{nl} + \frac{1}{6} h_1 (\bar{u}^l)^3| \lesssim_S h_1(r) t^{-2} \|(w_0, w_1)\|_S$$

c) The above estimates hold true for $\delta \bar{u}^{nl}$ and $\delta \partial_t \bar{u}_l$,

$$(5.9) \qquad |\delta \bar{u}^{nl}(r,t)| \lesssim_S h_1(r) t^{-1.5} \|(\delta w_0, \delta w_1)\|_S,$$

$$|\partial_t \delta \bar{u}^{nl} + \frac{1}{6} h_1 \delta(\bar{u}^l)^3| \lesssim_S h_1(r) t^{-2} \|(\delta w_0, \delta w_1)\|_S$$

Remark 5.3. By finite speed of propagation arguments it is not difficult to show that $\bar{u}^l$ decays rapidly outside the cone. However, for our purposes the decay established in the above proposition suffices.

Remark 5.4. The bound (5.7) shows that a double cancellation occurs on the light cone, as opposed to the expected single cancellation. This is a consequence of the exact decay properties at infinity for the potential in $\tilde{H}$.

Remark 5.5. The second estimate in part (b) is the outcome of a more subtle nonlinear cancellation, rather then a brute force computation.

Proof. a) We first split $\bar{u}^l$ into two parts,

$$\bar{u}^l(r,t) = \sum_k \bar{u}_k^l(r,t) = \sum_{2^k \lesssim r^{-1}} \bar{u}_k^l(r,t) + \sum_{2^k \gtrsim r^{-1}} \bar{u}_k^l(r,t) := \bar{u}_{low}^l(r,t) + \bar{u}_{hi}^l(r,t)$$

where

$$\bar{u}_k^l = \int \xi^{-1} \phi_\xi(r) \chi_k(\xi) \left[\cos(t\xi) \cdot \widehat{f_0}(\xi) + \frac{\sin(t\xi)}{\xi} \widehat{f_1}(\xi) \right] d\xi$$

Further, using the power series (3.1), we can write

$$\bar{u}^l_k = \int \xi^{-2} q(\xi) \sin(t\xi)(\phi_0(r) + \frac{1}{r}\sum_{j\geq 1}(r\xi)^{2j}\phi_j(r^2))\widehat{f_1}(\xi)\chi_k(\xi)d\xi, \qquad 2^k r \lesssim 1$$

which leads to a corresponding decomposition

$$\bar{u}^l_{low} = \bar{u}^{l,0}_{low} + \sum_{j\geq 1}\bar{u}^{l,j}_{low}$$

Then we set

(5.10) $$\bar{u}^{l,r} = \bar{u}^{l,0}_{low}, \qquad \bar{u}^{l,nr} = \bar{u}^l_{hi} + \sum_{j\geq 1}\bar{u}^{l,j}_{low}$$

and proceed to estimate all of the above components of $\bar{u}^l$.

The terms in $\bar{u}^l_{hi}$ are estimated by stationary phase using (5.1) and the ϕ_ξ representation in (3.7). This yields

(5.11) $$|\bar{u}^l_k| \lesssim \frac{r^{-\frac{1}{2}}2^{\frac{3k}{2}}}{\langle 2^k|r-t|\rangle^N \langle k^-\rangle}2^{-Nk^+}, \qquad 2^k r \gtrsim 1$$

which, after summation with respect to k gives the bound

$$|\bar{u}^l_{hi}| \lesssim \sum_{2^k \gtrsim r^{-1}} |\bar{u}^l_k(r,t)| \lesssim \left(\frac{r}{\langle r+t\rangle}\right)^N \frac{1}{\langle r+t\rangle^{\frac{1}{2}}\langle r-t\rangle^{\frac{3}{2}}\log\langle r-t\rangle}.$$

The bounds for the time derivative are obtained from the explicit formula

$$\partial_t \bar{u}^l = \int_0^\infty \phi_\xi(r)(-f_0(\xi)\sin(t\xi) + \frac{1}{\xi}f_1(\xi)\cos(t\xi))d\xi$$

which shows that we produce an extra 2^k factor in (5.11). Similarly, an r derivative applied to ϕ_ξ yields an additional 2^k factor in the asymptotic expansion. Thus we obtain

(5.12) $$|\partial_t \bar{u}^l_k| + |\partial_r \bar{u}^l_k| \lesssim \frac{r^{-\frac{1}{2}}2^{\frac{5k}{2}}}{\langle 2^k|r-t|\rangle^N \langle k^-\rangle}2^{-Nk^+}, \qquad 2^k r \gtrsim 1$$

which leads to

$$|\partial_t \bar{u}^l_{hi}| + |\partial_r \bar{u}^l_{hi}| \lesssim \left(\frac{r}{\langle r+t\rangle}\right)^N \frac{1}{\langle r+t\rangle^{\frac{1}{2}}\langle r-t\rangle^{\frac{5}{2}}\log\langle r-t\rangle}.$$

We now consider the terms in $\bar{u}^{l,j}_{low}$. The main contribution comes from f_1, so we take $f_0 = 0$ for convenience. For $j = 0$ we have

$$\bar{u}^{l,0}_{low} = \phi_0(r)\sum_k \chi_{\lesssim 2^{-k}}(r)\int \xi^{-2}q(\xi)\sin(t\xi)\widehat{f_1}(\xi)\chi_k(\xi)d\xi := \phi_0(r)\sum_k \chi_{\lesssim 2^{-k}}(r)g^0_k(t) := \phi_0(r)g^0(r,t)$$

Using stationary phase and the properties of q we have

$$|g^0_k(t)| + 2^{-k}|\partial_t g^0_k(t)| \lesssim \frac{2^k}{\langle k^-\rangle^2 \langle 2^k t\rangle^N}2^{-Nk^+}$$

By summing with respect to k we obtain

(5.13) $$|g^0(r,t)| + \langle t+r\rangle(|\partial_r g^0(r,t)| + |\partial_t g^0(r,t)|) \lesssim \frac{1}{\langle t+r\rangle \log^2\langle t+r\rangle}.$$

which yields the $\bar{u}^{l,r}$ bound in (5.6).

For $j \geq 1$ we have

$$u_{low}^{l,j} = \sum_k \chi_{\{r \lesssim 2^{-k}\}} \frac{1}{r} \int \xi^{-2} q(\xi) \sin(t\xi) \sum_{j \geq 1} (r\xi)^{2j} \phi_j(r^2) \widehat{f_1}(\xi) \chi_k(\xi) d\xi$$

$$:= r^{2j-1} \phi_j(r^2) \sum_k \chi_{\lesssim 2^{-k}}(r) g_k^j(t) := r^{2j-1} \phi_j(r^2) g^j(r,t)$$

By stationary phase and the properties of q and $\widehat{f_1}$ we have

$$|g_k^j(r,t)| + 2^{-k}(|\partial_t g_k^j(r,t)| + |\partial_r g_k^j(r,t)|) \lesssim \frac{2^{(2j+1)k}}{\langle k^- \rangle^2 \langle 2^k t \rangle^N} 2^{-Nk+}$$

Summing up over k we obtain

(5.14)
$$|g^j(r,t)| + \langle t+r \rangle(|\partial_r g^j(r,t)| + |\partial_t g^j(r,t)|) \lesssim \frac{1}{\langle t+r \rangle^{2j+1} \log^2 \langle t+r \rangle}$$

Hence, using the bound (3.2) for ϕ_j we obtain a bound for $\bar{u}_{low}^{l,j}$, namely

(5.15)
$$|\bar{u}_{low}^{l,j}(r,t)| + |r\partial_r \bar{u}_{low}^{l,j}(r,t)| + \langle t+r \rangle|\partial_t \bar{u}_{low}^{l,j}(r,t)| \lesssim \frac{C^j}{j!} \frac{r^{2j-1} \log(1+r^2)}{\langle t+r \rangle^{2j+1} \log^2 \langle t+r \rangle}$$

Thus these contributions satisfy the bounds required of $\bar{u}^{l,nr}$.

We now turn our attention to the estimate (5.7), which applies in the region where $r \approx t$. By (5.6) (for $\bar{u}^l$) and by (5.15), the contributions of the term $\bar{u}_{low}^l$ are all below the required threshold, so it remains to consider $\bar{u}_{hi}^l$. We have

$$\bar{u}_{hi}^l(r,t) = \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi) \xi^{-1} \phi_\xi(r)(f_0(\xi) \cos(t\xi) + \frac{1}{\xi} f_1(\xi) \sin(t\xi)) d\xi$$

For ϕ_ξ we use the representation (3.7) with ϕ_ξ^+ as in (3.6),

$$\phi_\xi = r^{-\frac{1}{2}}(a(\xi)\sigma(r\xi,r)e^{ir\xi} + \bar{a}(\xi)\bar{\sigma}(r\xi,r)e^{-ir\xi}), \qquad r\xi \gtrsim 1$$

We notice that the operator $\partial_r + \partial_t$ kills the resonant factors $e^{\pm i(r-t)\xi}$ factors. Precisely, we have

$$(\partial_r + \partial_t + \frac{1}{2r})\phi_\xi(r) \sin(t\xi) = 2r^{-\frac{1}{2}} \operatorname{Re}\left(e^{i\xi(r+t)} \xi a(\xi)\sigma(r\xi,r)\right) - r^{-\frac{1}{2}} \operatorname{Re}\left(e^{ir\xi} a(\xi)\partial_r\sigma(r\xi,r)\right) \sin(t\xi)$$

and a similar computation where $\sin(t\xi)$ is replaced by $\cos(t\xi)$. This leads to

$$(\partial_r + \partial_t + \frac{1}{2r})\bar{u}_{hi}^l = \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi) r^{-\frac{1}{2}} \operatorname{Re}\left(2i\xi e^{i(r+t)\xi} a(\xi)\sigma(r\xi,r) + e^{ir\xi} a(\xi)\partial_r\sigma(r\xi,r) \cos(t\xi)\right) \frac{f_0(\xi)}{\xi} d\xi$$

$$+ \int_0^\infty \chi_{\gtrsim r^{-1}}(\xi) r^{-\frac{1}{2}} \operatorname{Re}\left(2i\xi e^{i(r+t)\xi} a(\xi)\sigma(r\xi,r) + e^{ir\xi} a(\xi)\partial_r\sigma(r\xi,r) \sin(t\xi)\right) \frac{f_1(\xi)}{\xi^2} d\xi$$

The two integrals above are treated as before, using stationary phase. The first term in each of the last integrals has a nonresonant phase, therefore each integrating by parts gains a factor of $(\xi t)^{-1}$. Thus, taking (5.1) into account, their contributions can be estimated by

$$\int_0^\infty \chi_{\gtrsim t^{-1}}(\xi) t^{-\frac{1}{2}} \xi(t\xi)^{-N} \frac{\xi^{\frac{5}{2}}}{\xi^2 \log \xi} d\xi \approx \frac{1}{t^3 \log t}$$

The second term contains the expression $\partial_r\sigma(r\xi,r)$ which (see the description of σ in Section 3) brings an additional factor of $r^{-1}(r\xi)^{-1} \approx t^{-2}\xi^{-1}$. The contribution of the part with phase $e^{i\xi(r+t)}$

is better than above, while the contribution of the part with phase $e^{i\xi(r-t)}$ is of the form

$$\int_0^\infty \chi_{\gtrsim t^{-1}}(\xi) a(\xi) t^{-\frac{1}{2}} t^{-1}(t\xi)^{-1} e^{i\xi(t-r)} \frac{\xi^{\frac{5}{2}}}{\xi^2 \log \xi} d\xi \approx \frac{1}{t^{\frac{5}{2}} \langle t-r \rangle^{\frac{1}{2}} \log\langle t-r \rangle}$$

as desired.

b) We find u^{nl} from the equation (5.4) using a fixed point argument in the Banach space Z^{nl} with norm

$$\|f\|_{Z^{nl}} = \|h_1^{-1} t^{1.5} f\|_{L^\infty}$$

Denoting by Z^l the Banach space of functions of the form $\bar{u}^{l,r} + \bar{u}^{l,nr}$ with norm as in (5.5)-(5.6), we will show that the map

$$T : (u, v) \to L^{-1} N(u, v) = h_1(r) \int_r^\infty \frac{N(u, v)}{h_1(s)} ds$$

is locally Lipschitz from $Z^l \times Z^{nl}$ into Z^{nl}, with a Lipschitz constant which can be made small if either both arguments are small or v is small and the time t is large enough, depending on the size of u. This would imply the existence and uniqueness of $\bar{u}^{nl}$, as well as its Lipschitz dependence on $\bar{u}^l$ and implicitly on (w_0, w_1). Recall that

$$N(u, v) = \frac{1}{r} \Big[\sin Q \cdot (\cos(u + v) - 1) + \cos Q \cdot (\sin(u + v) - (u + v)) \Big]$$

Then

$$|N(u, v)| \lesssim \frac{1}{r^2 + 1}(|u|^2 + |v|^2) + \frac{1}{r}(|u|^3 + |v|^3)$$

$$|\nabla N(u, v)| \lesssim \frac{1}{r^2 + 1}(|u| + |v|) + \frac{1}{r}(|u|^2 + |v|^2)$$

Hence it remains to show that

$$\int_0^\infty \frac{1}{r}(|u|^2 + |v|^2) + \frac{r^2 + 1}{r^2}(|u|^3 + |v|^3) dr \lesssim t^{-1.5}(\|u\|_{Z^l}^2 + \|v\|_{Z^{nl}}^2 + \|u\|_{Z^l}^3 + \|v\|_{Z^{nl}}^3)$$

For u we have two components u^r and u^{nr}, therefore we need to consider the following six integrals:

$$\int_0^\infty \frac{1}{r}|u^r|^2 dr \lesssim \int_0^\infty \frac{1}{r} \frac{h_1^2(r)}{(t \log^2 t)^2} dr \cdot \|u\|_{Z^l}^2 \approx \frac{1}{t^2 \log^4 t} \|u\|_{Z^l}^2$$

$$\int_0^\infty \frac{1}{r}|u^{nr}|^2 dr \lesssim \int_0^\infty \frac{1}{r} \frac{r^2}{(t+r)^2 t \langle t-r \rangle^3 \log^2 \langle t-r \rangle} dr \cdot \|u\|_{Z^l}^2 \approx \frac{1}{t^2} \|u\|_{Z^l}^2$$

$$\int_0^\infty \frac{1}{r}|v|^2 dr \lesssim \int_0^\infty \frac{1}{r} h_1^2(r) t^{-3} dr \cdot \|v\|_{Z^{nl}}^2 \approx \frac{1}{t^3} \|v\|_{Z^{nl}}^2$$

$$\int_0^\infty \frac{r^2 + 1}{r^2}|u^r|^3 dr \lesssim \int_0^\infty \frac{r^2 + 1}{r^2} \frac{h_1^3(r)}{(t \log^2 t)^3} dr \cdot \|u\|_{Z^l}^3 \approx \frac{1}{t^3 \log^6 t} \|u\|_{Z^l}^3$$

$$\int_0^\infty \frac{r^2 + 1}{r^2}|u^{nr}|^3 dr \lesssim \int_0^\infty \frac{r^2 + 1}{r^2} \frac{r^3}{(t+r)^3 t^{\frac{3}{2}} \langle t-r \rangle^{\frac{9}{2}} \log^3 \langle t-r \rangle} dr \cdot \|u\|_{Z^l}^3 \approx \frac{1}{t^{1.5}} \|u\|_{Z^l}^3$$

$$\int_0^\infty \frac{r^2 + 1}{r^2}|v|^3 dr \lesssim \int_0^\infty \frac{r^2 + 1}{r^2} h_1^3(r) t^{-4.5} dr \cdot \|v\|_{Z^{nl}}^3 \approx \frac{1}{t^{4.5}} \|v\|_{Z^{nl}}^3$$

We remark that the worst decay $t^{-1.5}$ comes from the fifth integral above; all other terms are better.

The argument for $\partial_t \bar{u}^{nl}$ is more involved. Differentiating the equation (5.3) we obtain

$$L(\partial_t \bar{u}^{nl} + \frac{h_1}{6}(\bar{u}^l)^3) = N_u(\bar{u}^l, \bar{u}^{nl})\partial_t \bar{u}^l + N_v(\bar{u}^l, \bar{u}^{nl})\partial_t \bar{u}^{nl} + \frac{h_1}{6}\partial_r(\bar{u}^l)^3$$

(5.16)
$$= N_v(\bar{u}^l, \bar{u}^{nl})(\partial_t \bar{u}^{nl} + \frac{h_1}{6}(\bar{u}^l)^3) + [N_u(\bar{u}^l, \bar{u}^{nl}) - \frac{h_1}{2}(\bar{u}^l)^2]\partial_t \bar{u}^l$$

$$- \frac{1}{6}N_v(\bar{u}^l, \bar{u}^{nl})h_1(\bar{u}^l)^3 + \frac{h_1}{6}(\partial_t + \partial_r)(\bar{u}^l)^3$$

The approach is similar to what we have done before. We adjust the base space to

$$\|f\|_{\tilde{Z}^{nl}} = \|h_1^{-1}t^2 f\|_{L^\infty}$$

and continue with the same steps. By the previous computation the first term on the right is perturbative. The main cancellation occurs in the second term, where the $(\bar{u}^l)^2$ term disappears. Precisely, we have

$$N_u(u, v) - \frac{1}{2}h_1 u^2 = \frac{2}{1 + r^2}\sin(u + v) - \frac{1 - r^2}{r(1 + r^2)}(1 - \cos(u + v)) - \frac{r}{1 + r^2}u^2$$

therefore

$$|N_u(u, v) - \frac{1}{2}h_1 u^2| \lesssim \frac{1}{1 + r^2}(|u| + |v|) + \frac{1}{r}(|u|^3 + |u||v| + |v|^2) + \frac{1}{r(1 + r^2)}|u|^2$$

For $\partial_t \bar{u}^l$ we use the same bounds as for $\bar{u}^l$. Then, compared with the previous computation, we need to reestimate the terms involving $|u|^3$, $|u||v|$ and $|u|^2$. The resonant part of u yields better bounds, so we only estimate terms involving u^{nr}:

$$\int_0^\infty \frac{r^2 + 1}{r^2}|u^{nr}|^4 dr \lesssim \|u\|_{Z^l}^4 \cdot \int_0^\infty \frac{r^2 + 1}{r^2}\frac{r^4}{(t + r)^4 t^2 \langle t - r\rangle^6 \log^4 \langle t - r\rangle} dr \approx \frac{1}{t^2}\|u\|_{Z^l}^4$$

$$\int_0^\infty \frac{r^2 + 1}{r^2}|u^{nr}|^2|v|\, dr \lesssim \|u\|_{Z^l}^2\|v\|_{Z^{nl}} \cdot \int_0^\infty \frac{r^2 + 1}{r^2}\frac{r^2}{(t + r)^2 t^{2.5}\langle t - r\rangle^3 \log^2\langle t - r\rangle} dr \approx \frac{1}{t^{2.5}}\|u\|_{Z^l}^2\|v\|_{Z^{nl}}$$

$$\int_0^\infty \frac{1}{r^2}|u^{nr}|^3 dr \lesssim \|u\|_{Z^l}^3 \cdot \int_0^\infty \frac{1}{r^2}\frac{r^3}{(t + r)^3 t^{1.5}\langle t - r\rangle^{4.5}\log^3\langle t - r\rangle}dr \approx \frac{1}{t^{3.5}}\|u\|_{Z^l}^4$$

The third term on the right in (5.16) is better behaved than the second. Finally, for the last term in (5.16) we invoke (5.7) so that we use the same bounds for $(\partial_t + \partial_r)(\bar{u}^l)$ as for $r^{-1}\bar{u}^l$. Then the integral to estimate is

$$\int_0^\infty \frac{1}{r}|u|^3 dr \lesssim \frac{1}{t^{2.5}}\|u\|_{Z^l}^3$$

c) In the case of $\bar{u}^l$ this part follows from the linearity. In the case of $\bar{u}^{nl}$ the Lipschitz dependence on $\bar{u}^l$ has already been discussed above. An additional argument is required for $\delta\partial_t \bar{u}^{nl}$. However, nothing new happens there, and the details are left for the reader.

$\square$

6. THE TRANSITION BETWEEN γ AND ε

In this section study the transition from γ to ε, which were both introduced in (2.12). This transition is described by (2.16), which we recall for convenience

$$\gamma = \partial_r\varepsilon - \frac{\sin(\varepsilon + \bar{u}) - \sin\bar{u}}{r}$$

The main result of this section is the following

Proposition 6.1. *a) Assume that $\gamma \in LX$ is small and $\bar{u}, \bar{w}$ are as in Proposition 5.2. Then for t large enough there exists a unique solution $\varepsilon \in X$ of (2.16) which satisfies*

$$(6.1) \qquad \qquad \|\varepsilon\|_X \lesssim_S \|\gamma\|_{LX}$$

Furthermore, ε has a Lipschitz dependence on both γ and on the linear data (w_0, w_1) for $\bar{w}$,

$$(6.2) \qquad \|\delta\varepsilon\|_X \lesssim_S \|\delta\gamma\|_{LX} + \frac{1}{t\log^2 t}\|(\delta w_0, \delta w_1)\|_S \|\gamma\|_{LX},$$

Also, if γ is a function of t then

$$(6.3) \qquad \|\partial_t \varepsilon\|_X \lesssim_S \|\partial_t \gamma\|_{LX} + \frac{1}{t\log^2 t}\|\gamma\|_{LX}$$

b) Assume in addition that $\gamma \in L^\infty$. Then

$$(6.4) \qquad |\varepsilon(r)| \lesssim_S r\log r \|\gamma\|_{LX \cap L^\infty}, \qquad r \ll 1$$

Proof. a) The equation (2.16) is rewritten as

$$(6.5) \qquad L\varepsilon = \gamma + \frac{\sin(\varepsilon + \bar{u}) - \sin\bar{u} - \cos Q \cdot \varepsilon}{r} := \gamma + F(\varepsilon, \bar{u} - Q)$$

Hence in order to prove both (6.1) and (6.2) it suffices to show that at fixed large enough time the map F is Lipschitz

$$F : X \times (Z^l + Z^{nl}) \to LX$$

with a small Lipschitz constant in the second variable. For the X norm we use the embeddings (3.18)-(3.21). For the LX norm we use (3.23), which shows that is enough to estimate $F(\bar{u}, \varepsilon)$ in $L^1 \cap L^2$. We expand F as follows:

$$F(\beta, v) = \frac{\sin(\beta + Q + v) - \sin(Q + v) - \cos Q \cdot \beta}{r}$$

$$= \frac{(\cos(Q + v) - \cos Q) \cdot \beta}{r} - \frac{\sin(Q + v) \cdot \beta^2}{2r} + \frac{O(\beta^3)}{r}$$

$$= \frac{\sin Q \cdot v\beta}{r} + \frac{\sin Q \cdot \beta^2}{2r} + \frac{O(v^2\beta)}{r} + \frac{O(\beta^3)}{r}$$

Hence

$$(6.6) \qquad |F(\beta, v)| \lesssim \frac{|v||\beta|}{1 + r^2} + \frac{|\beta|^2}{1 + r^2} + \frac{|\beta|^3}{r} + \frac{|v|^2|\beta|}{r}$$

By using (3.20), (3.18) and (5.6), we bound this first in L^2

$$\|F(\beta, v)\|_{L^2} \lesssim \|\frac{\beta}{\log(1 + r)}\|_{L^2} \left(\|\beta\|_{L^\infty} + \|\beta\|_{L^\infty}^2 + \|\frac{v}{1 + r}\|_{L^\infty} + \|\frac{v^2\log(1 + r)}{r}\|_{L^\infty} \right)$$

$$\lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X \left(\frac{1}{t\log^2 t}\|v\|_{Z^l + Z^{nl}} + \frac{\log t}{t^2}\|v\|_{Z^l + Z^{nl}}^2 \right)$$

and then in L^1,

$$\|F(\beta, v)\|_{L^1} \lesssim \|\frac{\beta}{\log(1 + r)}\|_{L^2}^2 (1 + \|\beta\|_{L^\infty})$$

$$+ \|\frac{\beta}{\log(1 + r)}\|_{L^2} \left(\|\frac{v\log(1 + r)}{1 + r^2}\|_{L^2} + \|\frac{v^2\log(1 + r)}{r}\|_{L^2} \right)$$

$$\lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X \left(\frac{1}{t\log^2 t}\|v\|_{Z^l + Z^{nl}} + \frac{\log t}{t^{\frac{3}{2}}}\|v\|_{Z^l + Z^{nl}}^2 \right)$$

Hence we obtain

$$\|F(\beta,v)\|_{LX} \lesssim \|\beta\|_X^2 + \|\beta\|_X^3 + \|\beta\|_X \big(\frac{1}{t\log^2 t}\|v\|_{Z^l+Z^{nl}} + \frac{\log t}{t^{\frac{3}{2}}}\|v\|_{Z^l+Z^{nl}}^2\big)$$

A similar analysis yields

$$\|\beta_1 F_\beta(\beta,v)\|_{LX} \lesssim \|\beta_1\|_X\big(\|\beta\|_X + \|\beta\|_X^2 + \frac{1}{t\log^2 t}\|v\|_{Z^l+Z^{nl}} + \frac{\log t}{t^{\frac{3}{2}}}\|v\|_{Z^l+Z^{nl}}^2\big)$$

respectively

$$\|v_1 F_v(\beta,v)\|_{LX} \lesssim \|v_1\|_{Z^l+Z^{nl}} \|\beta\|_X \big(\frac{1}{t\log^2 t} + \frac{\log t}{t^{\frac{3}{2}}}\|v\|_{Z^l+Z^{nl}}\big)$$

By the contraction principle this proves both (6.1) and (6.2). The time decaying factors guarantee that for any size of $\bar{u} - Q$ the problem can be solved for large enough time.

To prove (6.3) we differentiate with respect to t in (6.5),

$$L\partial_t\varepsilon = \partial_t\gamma + F_\varepsilon(\varepsilon,\bar{u})\partial_t\varepsilon + F_{\bar{u}}(\varepsilon,\bar{u})\partial_t\bar{u}$$

Since $\partial_t\bar{u}$ satisfies the same pointwise bounds as $\bar{u}$, the last two estimates above show that the contraction principle still applies.

b) Due to the embedding $X \subset \dot{H}_e^1 \subset L^\infty$ we already have a small uniform bound for ε. We solve the ode (6.5) in $[0,1]$ with Cauchy data at $r = 1$. Making the bootstrap assumption

(6.7)
$$|\varepsilon| \leq Mr|\log(r/2)|$$

we rewrite the equation (6.5) in the form

$$|L\varepsilon - \gamma| \leq M^3 r^2 |\log^3(r/2)| + C, \qquad C \approx_S \|\varepsilon\|_{L^\infty}$$

Then solving the linear L evolution we have

$$|\varepsilon| \lesssim r(|\gamma(1)| + M^3) + Cr|\log(r/2)| \lesssim_S M^3 r + r|\log(r/2)|\|\varepsilon\|_{L^\infty}$$

If $\|\varepsilon\|_{L^\infty}$ is sufficiently small then we can choose M small enough so that the above bound is stronger than our bootstrap assumption (6.7). The proof of (6.4) is concluded.

$\square$

7. THE NONLINEARITY IN THE γ EQUATION

Our main goal is to solve the equation (2.15) for γ with zero Cauchy data at $t = \infty$. For the linear $\tilde{H}$ wave equation we use the LX bounds in Lemma 4.1. The auxiliary function ε is uniquely determined by γ via Proposition 6.1. In this section we estimate the nonlinear contribution in (2.15), namely

$$N(w,u) = \frac{2(\cos Q - \cos u)}{r^2}w + \frac{1}{r}\sin u(u_t^2 - w^2)$$

In light of the decompositions $u = \bar{u}+\varepsilon, w = \bar{w}+\gamma$, this nonlinearity has three types of contributions,

$$N(w,u) = N(\bar{w},\bar{u}) + N^l(\bar{w},\bar{u},\gamma,\varepsilon) + N^n(\bar{w},\bar{u},\gamma,\varepsilon)$$

The first one, $N(\bar{w},\bar{u})$, should be seen as an inhomogeneous term. The reason we need to consider this separately is that $\bar{u},\bar{w}$ have a different behavior compared to ε,γ as t goes to infinity. The term N^l contains the linear contributions in ε,γ in the difference $N(w,u) - N(\bar{w},\bar{u})$,

$$N^l = \frac{2(\cos Q - \cos\bar{u})}{r^2}\gamma + \frac{2\sin\bar{u}\cdot\varepsilon}{r^2}\bar{w} + \frac{\sin\bar{u}(2\bar{u}_t\varepsilon_t - 2\bar{w}\gamma) + \cos\bar{u}\cdot\varepsilon(\bar{u}_t^2 - \bar{w}^2)}{r}$$

The remaining term N^l contains the genuinely nonlinear contributions in ε, γ in the difference $N(w, u) - N(\bar{w}, \bar{u})$,

$$N^n = \frac{2(\cos\bar{u} - \cos u - \sin\bar{u}\cdot\varepsilon)}{r^2}\bar{w} + \frac{2(\cos\bar{u} - \cos(\bar{u}+\varepsilon))}{r^2}\gamma + \frac{\sin\bar{u}(\varepsilon_t^2 - \gamma^2)}{r}$$
$$+ \frac{(\sin u - \sin\bar{u})(2\bar{u}_t\varepsilon_t - 2\bar{w}\gamma + \bar{u}_t^2 - \bar{w}^2)}{r} + \frac{(\sin u - \sin\bar{u} - \cos\bar{u}\cdot\varepsilon)(\bar{u}_t^2 - \bar{w}^2)}{r}$$

Our main result is the following

Proposition 7.1. *a) If $\bar{u}, \bar{w}$ are as in Proposition 5.2 then for $t \gtrsim_S 1$ we have*

$$(7.1) \qquad t^{1.5}\|KN(\bar{u}, \bar{w})\|_{LX} + t^{2.5}(\|\partial_t KN(\bar{u}, \bar{w})\|_{LX} + \|KN(\bar{u}, \bar{w})\|_{\dot{H}_e^1}) \lesssim_S 1$$

with Lipschitz dependence on the initial data (w_0, w_1) for $\bar{w}$.
  b) If we assume the following

$$\sup_t t^{1.5}(\|\gamma(t)\|_{LX} + \|\varepsilon(t)\|_X) + \sup_t t^{2.5}(\|\partial_t\gamma(t)\|_{LX} + \|\gamma\|_{\dot{H}_e^1} + \|\partial_t\varepsilon(t)\|_X) \lesssim M$$

then

$$(7.2) \qquad t^{1.5}\|KN^l\|_{LX} + t^{2.5}(\|\partial_t KN^l\|_{LX} + \|KN^l\|_{\dot{H}_e^1}) \lesssim_S M(\log t)^{-2}$$

$$(7.3) \qquad t^{1.5}\|KN^n\|_{LX} + t^{2.5}(\|\partial_t KN^n\|_{LX} + \|KN^n\|_{\dot{H}_e^1}) \lesssim_S (M^2 + M^3)t^{-.5}\log t$$

In addition, the maps $(\varepsilon, \gamma) \to (KN^l, KN^n)$ satisfy similar Lipschitz bounds.

When combined with Proposition 6.1 this result allows us to treat the problem in γ, ε perturbatively. The additional gains in t decay in (7.2) and (7.3) allows us to consider large Schwartz perturbations of the soliton. We note that in the case of KN^l we gain only logarithms. This implies that for large Schwartz data (w_0, w_1) in the linear equation our solutions are only defined for $t > T$ with T exponentially large.

7.1. **The term $N(\bar{w}, \bar{u})$.** Our goal here is to prove the estimate

$$(7.4) \qquad \|N(\bar{w}, \bar{u})\|_{LX} \lesssim_S t^{-3.5}$$

Then the bound (7.1) is a consequence of Lemma 4.1. To prove this we split

$$N(\bar{w}, \bar{u}) = \chi_{r \ll t}N(\bar{w}, \bar{u}) + \chi_{r \gg t}N(\bar{w}, \bar{u}) + \chi_{r \approx t}N(\bar{w}, \bar{u}) = N_1 + N_2 + N_3$$

For the first two terms it suffices to use a direct estimate

$$|N(\bar{w}, \bar{u})| \lesssim \frac{\sin Q}{r^2}|\bar{u} - Q||\bar{w}| + \frac{1}{r^2}|\bar{u} - Q|^2|\bar{w}| + \frac{1}{r}(\sin Q + |\bar{u}|)(|\bar{u}_t|^2 + |\bar{w}|^2)$$

Using the bounds (5.6) and (5.8) for $\bar{u} - Q$, as well as the bound (5.2) for $\bar{w}$, this gives

$$|N_1(\bar{w}, \bar{u})| \lesssim_S \chi_{r \ll t}\frac{1}{\langle r\rangle^4 t^4}$$

where the leading contribution comes from $u^{l,r}$. This implies that

$$\|N_1\|_{L^1 \cap L^2} \lesssim_S t^{-4}$$

which suffices for (7.4) in view of the embedding (3.23). Similarly

$$|N_2| \lesssim_S \chi_{r \gg t}\frac{1}{\langle r\rangle^8}$$

which also gives

$$\|N_2\|_{L^1 \cap L^2} \lesssim_S t^{-4}$$

However, a similar direct computation for N_3 only gives

$$|N_3(\bar{w}, \bar{u})| \lesssim_S \chi_{r \sim t} \frac{1}{t^{2.5} \langle t - r \rangle^{5.5}}$$

which fails by two units,

$$\|N_3\|_{L^1 \cap L^2} \lesssim_S t^{-1.5}$$

Hence in order to conclude the proof of (7.4) we need to better exploit the structure of N and capture a double cancellation on the null cone. In the computations below (through the end of the subsection) we work in the regime $r \approx t$. We expand $N(\bar{w}, \bar{u})$ as

$$N(\bar{w}, \bar{u}) = 2\frac{\sin Q}{r^2}(\bar{u} - Q)\bar{w} + \frac{\cos Q}{r^2}(\bar{u} - Q)^2\bar{w} + \frac{\sin Q}{r}(\bar{u}_t^2 - \bar{w}^2) + \frac{\cos Q}{r}(\bar{u}_t^2 - \bar{w}^2)(\bar{u} - Q)$$

$$+ \frac{\sin Q}{r^2}wO((\bar{u} - Q)^3) + \frac{\cos Q}{r^2}(wO((\bar{u} - Q)^4)$$

$$+ \frac{\sin Q}{r}(\bar{u}_t^2 - \bar{w}^2)O((\bar{u} - Q)^2) + \frac{\cos Q}{r}(\bar{u}_t^2 - \bar{w}^2)O((\bar{u} - Q)^3)$$

The terms on the second line are already acceptable, i.e. can be estimated by $t^{-4.5}\langle t - r \rangle^{-3.5}$. For further progress we observe that by (5.8) we have

$$\bar{u}^{nl} = O_S(\langle t \rangle)^{-2.5}, \qquad \partial_t \bar{u}^{nl} = O_S(t^{-2.5}\langle t - r \rangle^{-0.5})$$

and that by (5.7) we can write

(7.5) $$\partial_t \bar{u} + \bar{w} = \partial_t \bar{u}^{nl} + \partial_t \bar{u}^l + \partial_r \bar{u}^l + \frac{\cos Q}{r}\bar{u}^l = O_S(t^{-1.5}\langle t - r \rangle^{-1.5})$$

The first relation above allows us to dispense with $\bar{u}^{nl}$ everywhere and replace $\bar{u} - Q$ by $\bar{u}^l$, and the second allows us to estimate the third line in $N(\bar{w}, \bar{u})$. We are left with

$$N(\bar{w}, \bar{u}) = 2\frac{\sin Q}{r^2}\bar{u}^l\bar{w} + \frac{\cos Q}{r^2}(\bar{u}^l)^2\bar{w} + \frac{\sin Q}{r}((\bar{u}_t^l)^2 - \bar{w}^2) + \frac{\cos Q}{r}((\bar{u}_t^l)^2 - \bar{w}^2)\bar{u}^l + O_S(t^{-4.5}\langle t - r \rangle^{-3.5})$$

To advance further we substitute $\bar{w} = \partial_r \bar{u}^l - \frac{\cos Q}{r}\bar{u}^l$ everywhere. The $\frac{\cos Q}{r}\bar{u}^l$ is acceptable in the first two terms of N, i.e. it gives contributions of $O_S(t^{-4.5}\langle t - r \rangle^{-3.5})$, and we discard it. For the last two terms we use the better approximation from (5.7)

$$\bar{u}_t^l = -\partial_r \bar{u}^l - \frac{1}{2r}\bar{u}^l + O(t^{-2.5}\langle t - r \rangle^{-0.5})$$

Then we can write

$$(\bar{u}_t^l)^2 - \bar{w}^2 = (\partial_r \bar{u}^l + \frac{1}{2r}\bar{u}^l)^2 - (\partial_r \bar{u}^l - \frac{\cos Q}{r}\bar{u}^l)^2 + O_S(t^{-3}\langle t - r \rangle^{-3})$$

$$= -\frac{1}{r}\bar{u}^l\partial_r \bar{u}^l + O_S(t^{-3}\langle t - r \rangle^{-3})$$

It is also harmless to replace $\sin Q$ by r^{-1} and $\cos Q$ by -1 everywhere. Returning to N we obtain

$$N(\bar{w}, \bar{u}) = \frac{2}{r^3}\bar{u}^l\partial_r \bar{u}^l - \frac{1}{r^2}(\bar{u}^l)^2\partial_r \bar{u}^l - \frac{1}{r^3}u^l\partial_r u^l + \frac{1}{r^2}(\bar{u}^l)^2\partial_r \bar{u}^l + O_S(\langle t \rangle^{-4.5}\langle t - r \rangle^{3.5})$$

$$= \frac{1}{2r^3}\partial_r(\bar{u}^l)^2 + O_S(t^{-4.5}\langle t - r \rangle^{-3.5})$$

in the region $r \approx t$, which we rewrite as

$$N_3 = Lg + \chi_{r \approx t}O_S(t^{-4.5}\langle t - r \rangle^{-3.5}), \qquad g = \chi_{r \approx t}\frac{1}{2r^3}(\bar{u}^l)^2$$

The last term can be directly estimated in $L^1 \cap L^2$. For the leading term Lg we estimate g in H_e^1 and use the embedding (3.18). We have

$$|g| \lesssim_S \frac{1}{t^4 \langle t-r \rangle^3}, \qquad |\partial_r g| \lesssim_S \frac{1}{t^4 \langle t-r \rangle^4}$$

therefore

$$\|g\|_{H_e^1} \lesssim_S \frac{1}{t^{3.5}}$$

This concludes the proof of (7.4).

7.2. **The bound for N^l.** Our goal here is to establish the bound

(7.6)
$$\|N^l\|_{LX} \lesssim_S \frac{1}{t^{3.5} \log^2 t} M$$

We recall that

$$N^l = \frac{2(\cos Q - \cos \bar{u})}{r^2} \gamma + \frac{2 \sin \bar{u} \cdot \varepsilon}{r^2} \bar{w} + \frac{\sin \bar{u}(2\bar{u}_t \varepsilon_t - 2\bar{w}\gamma) + \cos \bar{u} \cdot \varepsilon(\bar{u}_t^2 - \bar{w}^2)}{r}$$

The pointwise estimate

$$\left| \frac{2(\cos Q - \cos \bar{u})}{r} \right| \lesssim \frac{1}{r^2+1} |\bar{u} - Q| + \frac{1}{r} |\bar{u} - Q|^2$$

combined with the pointwise bounds for $\bar{u}$ from (5.6) leads to

$$\left\| \frac{2(\cos Q - \cos \bar{u})}{r} \right\|_{L^\infty \cap L^2} \lesssim_S \frac{1}{t \log^2 t}$$

with the worst contribution arising from the resonant part of $\bar{u}$. From (3.23) it follows that

$$\left\| \frac{2(\cos Q - \cos \bar{u})}{r^2} \gamma \right\|_{LX} \lesssim \left\| \frac{2(\cos Q - \cos \bar{u})}{r} \right\|_{L^\infty \cap L^2} \cdot \left\| \frac{\gamma}{r} \right\|_{L^2} \lesssim_S \frac{1}{t^{3.5} \log^2 t} M.$$

Next, from (5.6) and (5.2), it follows that

$$\left\| \frac{\bar{u} \cdot \bar{w}}{r^2} \log(2+r) \right\|_{L^\infty \cap L^2} \lesssim_S \frac{\log t}{t^{2.5}}$$

which combined with (recall (3.20))

$$\left\| \frac{\varepsilon}{\log(2+r)} \right\|_{L^2} \lesssim \|\varepsilon\|_X \lesssim t^{-1.5} M$$

gives

$$\left\| \frac{2 \sin \bar{u} \cdot \varepsilon}{r^2} \bar{w} \right\|_{LX} \lesssim_S \frac{\log t}{t^4} M$$

Using (5.6) we obtain

$$\left\| \frac{\bar{u} \bar{u}_t}{r} \log(2+r) \right\|_{L^\infty \cap L^2} \lesssim_S \frac{\log t}{t^{1.5}}$$

therefore by invoking (3.23) and (3.20), it follows that

$$\left\| \frac{\sin(\bar{u}) \cdot \bar{u}_t \varepsilon_t}{r} \right\|_{LX} \lesssim \left\| \frac{\bar{u} \bar{u}_t}{r} \log(2+r) \right\|_{L^\infty \cap L^2} \left\| \frac{\varepsilon_t}{\log(2+r)} \right\|_{L^2} \lesssim_S \frac{\log t}{t^4} M$$

The following term requires some extra work. Using (5.6) and (5.2), we note that away from the cone we have $|\sin(\bar{u})| \lesssim \sin Q$ and continue with

$$\left\| \chi_{r \not\approx t} \frac{\bar{w} \sin \bar{u}}{r} \right\|_{L^1 \cap L^2} \lesssim_S t^{-2}$$

followed by

$$\|\chi_{r\not\approx t}\frac{\sin(\bar{u})\cdot\bar{w}\gamma}{r}\|_{LX} \lesssim \|\chi_{r\not\approx t}\frac{\bar{u}\bar{w}}{r}\|_{L^1\cap L^2}\|\gamma\|_{L^\infty} \lesssim_S t^{-4.5}M$$

Near the cone we write

$$\chi_{r\approx t}\frac{\bar{w}\sin\bar{u}}{r} = \chi_{r\approx t}\left(\frac{2\bar{w}}{1+r^2} - \frac{\bar{w}(\bar{u}-Q)}{r}\cos Q + \frac{\bar{w}O((\bar{u}-Q)^2)}{1+r^2} + \frac{\bar{w}O((\bar{u}-Q)^3)}{r}\right))$$

$$= \chi_{r\approx t}\frac{\bar{w}(\bar{u}-Q)}{r} + O_S(t^{-2.5}\langle t-r\rangle^{-2.5})$$

$$= L(\chi_{r\approx t}r^{-1}(\bar{u}^l)^2) + O_S(t^{-2.5}\langle t-r\rangle^{-2.5})$$

The second term is estimated as above in $L^1\cap L^2$ and yields a contribution of $t^{-4}M$ to the $\|N^l\|_{LX}$ bound. For the first term we write its contribution to N^l in the form

$$L(\chi_{r\approx t}r^{-1}(\bar{u}^l)^2)\gamma = L(\chi_{r\approx t}r^{-1}(\bar{u}^l)^2\gamma) + \chi_{r\approx t}r^{-1}(\bar{u}^l)^2\partial_r\gamma$$

Then, using (3.18) for the first term and (3.23) for the second term, we have

$$\|L(\chi_{r\approx t}r^{-1}(\bar{u}^l)^2)\gamma\|_{LX} \lesssim \|\chi_{r\approx t}r^{-1}(\bar{u}^l)^2\gamma\|_{H_e^1} + \|\chi_{r\approx t}r^{-1}(\bar{u}^l)^2\partial_r\gamma\|_{L^1\cap L^2}$$

$$\lesssim \|\chi_{r\approx t}r^{-1}(\bar{u}^l)^2\|_{H_e^1}\|\gamma\|_{\dot{H}_e^1} + \|\chi_{r\approx t}r^{-1}(\bar{u}^l)^2\|_{L^2\cap L^\infty}\|\partial_r\gamma\|_{L^2}$$

$$\lesssim_S t^{-1.5}\|\gamma\|_{\dot{H}_e^1} \lesssim_S t^{-4}M$$

It remains to bound the last term in N^l. For this we take advantage of the first order cancellation on the cone in the expression $\bar{u}_t - \bar{w}$, see (7.5), which combined with (5.6) and (5.2), gives

$$\|\frac{\cos\bar{u}(\bar{u}_t^2 - \bar{w}^2)\log(2+r)}{r}\|_{L^2\cap L^\infty} \lesssim_S \frac{\log t}{t^{2.5}}.$$

This leads to

$$\|\varepsilon\frac{\cos\bar{u}(\bar{u}_t^2 - \bar{w}^2)}{r}\|_{L^1\cap L^2} \lesssim_S \frac{\log t}{t^{2.5}}\|\frac{\varepsilon}{\log(2+r)}\|_{L^2} \lesssim_S \frac{\log t}{t^{2.5}}\|\varepsilon\|_X \lesssim_S \frac{\log t}{t^4}M$$

This concludes the proof of the N^l bound (7.6).

7.3. **The bound for N^n.** Our goal here will be to prove the bound

(7.7) $$\|N^n\|_{LX} \lesssim_S \frac{\log t}{t^4}(M^2 + M^3)$$

We recall the expression of N^n:

$$N^n = \frac{2(\cos\bar{u} - \cos u - \sin\bar{u}\cdot\varepsilon)}{r^2}\bar{w} + \frac{2(\cos\bar{u} - \cos(\bar{u}+\varepsilon))}{r^2}\gamma + \frac{\sin u(\varepsilon_t^2 - \gamma^2)}{r}$$

$$+ \frac{(\sin u - \sin\bar{u})(2\bar{u}_t\varepsilon_t - 2\bar{w}\gamma)}{r} + \frac{(\sin u - \sin\bar{u} - \cos\bar{u}\cdot\varepsilon)(\bar{u}_t^2 - \bar{w}^2)}{r}$$

We successively consider the terms on the right. For the first one we start with

$$|\frac{2(\cos\bar{u} - \cos u - \sin\bar{u}\cdot\varepsilon)}{r^2}\bar{w}| \lesssim \frac{\varepsilon^2|\bar{w}|}{r^2}.$$

Then, using (5.2) and (3.20), we obtain

$$\|\frac{\varepsilon^2\bar{w}}{r^2}\|_{L^1\cap L^2} \lesssim \|\frac{\varepsilon}{\log(2+r)}\|_{L^\infty\cap L^2}\|\frac{\varepsilon}{\log(2+r)}\|_{L^2}\|\frac{\bar{w}}{r^2}\log^2(2+r)\log\|_{L^\infty} \lesssim_S \frac{\log^2 t}{t^{5.5}}M^2$$

The second term in N^n is estimated by

$$|\frac{\cos\bar{u}-\cos(\bar{u}+\varepsilon)}{r^2}\gamma| \lesssim \frac{|\sin\bar{u}\cdot\varepsilon\gamma|}{r^2} + \frac{|\varepsilon^2\gamma|}{r^2} \lesssim \frac{|\varepsilon\gamma|}{r\langle r\rangle^2} + \frac{|(\bar{u}-Q)\varepsilon\gamma|}{r^2} + \frac{|\varepsilon^2\gamma|}{r^2}.$$

The first two terms can be estimated in $L^1\cap L^2$ as before,

$$\|\frac{\varepsilon\gamma}{r\langle r\rangle^2}\|_{L^1\cap L^2} \lesssim \|\frac{\gamma}{r}\|_{L^2}\|\frac{\varepsilon}{\langle r\rangle^2}\|_{L^\infty\cap L^2} \lesssim_S t^{-4}M^2$$

$$\|\frac{(\bar{u}-Q)\varepsilon\gamma}{r^2}\|_{L^1\cap L^2} \lesssim \|\frac{\gamma}{r}\|_{L^2}\|\frac{\bar{u}-Q}{r}\|_{L^2\cap L^\infty}\|\varepsilon\|_{L^\infty} \lesssim_S t^{-5}M^2$$

For the last term we first get the L^1 bound

$$\|\frac{\varepsilon^2\gamma}{r^2}\|_{L^1} \lesssim \|\varepsilon\|_{L^\infty}\|\frac{\varepsilon}{r}\|_{L^2}\|\frac{\gamma}{r}\|_{L^2} \lesssim \frac{1}{t^{5.5}}M^3$$

However, getting the L^2 bound is more delicate:

$$\|\frac{\varepsilon^2\gamma}{r^2}\|_{L^2} \lesssim \|\frac{\varepsilon}{\sqrt{r}}\|_{L^\infty}^2\|\frac{\gamma}{r}\|_{L^2} \lesssim \frac{1}{t^{5.5}}M^3$$

where the pointwise bound for $\frac{\varepsilon}{\sqrt{r}}$ near $r=0$ comes from (6.4).

The third term in N is estimated by using (5.6)

$$|\frac{\sin u(\varepsilon_t^2-\gamma^2)}{r}| \lesssim \frac{|\varepsilon_t|^2}{1+r} + \frac{|\gamma^2|}{1+r}$$

We successively consider all terms:

$$\|\frac{|\varepsilon_t|^2}{1+r}\|_{L^1\cap L^2} \lesssim \|\frac{\varepsilon_t}{\log(2+r)}\|_{L^2\cap L^\infty}\|\frac{\varepsilon_t}{\log(2+r)}\|_{L^2} \lesssim \frac{1}{t^5}M^2$$

$$\|\frac{|\gamma|^2}{1+r}\|_{L^1\cap L^2} \lesssim \|\gamma\|_{L^2\cap L^\infty}\|\frac{\gamma}{r}\|_{L^2} \lesssim \frac{1}{t^4}M^2$$

Next we estimate the fourth term in N^n,

$$|\frac{(\sin u-\sin\bar{u})(2\bar{u}_t\varepsilon_t-2\bar{w}\gamma)}{r}| \lesssim \frac{|\varepsilon|(|\bar{u}_t\varepsilon_t|+|\bar{w}\gamma|)}{r}$$

On behalf of (5.2), (5.6) and (3.20), we have

$$\|\frac{\varepsilon\bar{u}_t\varepsilon_t}{r}\|_{L^1\cap L^2} \lesssim \|\varepsilon_t\|_{L^\infty}\|\frac{\varepsilon}{\log(2+r)}\|_{L^2}\|\frac{\bar{u}_t}{r}\log(2+r)\|_{L^\infty\cap L^2} \lesssim_S \frac{\log t}{t^4}M^2$$

$$\|\frac{\varepsilon\bar{w}\gamma}{r}\|_{L^1\cap L^2} \lesssim \|\varepsilon\|_{L^\infty}\|\bar{w}\|_{L^2\cap L^\infty}\|\frac{\gamma}{r}\|_{L^2} \lesssim_S t^{-4}M^2$$

Finally we consider the last term in N^n,

$$|\frac{(\sin u-\sin\bar{u}-\cos\bar{u}\cdot\varepsilon)(\bar{u}_t^2-\bar{w}^2)}{r}| \lesssim \frac{\varepsilon^2(\bar{u}_t^2+\bar{w}^2)}{r}$$

which, by using (5.2), (5.6) and (3.20), we further bound as follows

$$\|\frac{\varepsilon^2(\bar{u}_t^2+\bar{w}^2)}{r}\|_{L^1\cap L^2} \lesssim \|\frac{\varepsilon}{\log(2+r)}\|_{L^2}\|\frac{\varepsilon}{\log(2+r)}\|_{L^2\cap L^\infty}\|\frac{\bar{u}_t^2+\bar{w}^2}{r}\log^2(2+r)\|_{L^\infty}$$

$$\lesssim_S \frac{\log^2 t}{t^5}M^2$$

7.4. **Conclusion.** The proof of Proposition 7.1 is a direct consequence of all the estimates in the previous subsections. Indeed, the result in part a) follows from (7.4) and (4.2). The results in part b) follow from (7.6), (7.7) and (4.2).

We are also ready to prove our main result.

Proof of Theorem 2.1. Based on the results in Proposition 7.1, one can iterate the equation (2.15) in the following space

$$\|\gamma\|_Y = t^{\frac{3}{2}}\|\gamma(t)\|_{LX} + t^{\frac{5}{2}}\|\partial_t\gamma(t)\|_{LX} + t^{\frac{5}{2}}\|\gamma(t)\|_{\dot{H}^1}$$

The size of ε is controlled by using the results of Proposition 6.1. $\square$

REFERENCES

[1] I. Bejenaru, D. Tataru, *Near soliton evolution for equivariant Schrödinger Maps in two spatial dimensions*, preprint (2010)

[2] Bizon, P., Tabor, Z., *Formation of singularities for equivariant $2 + 1$-dimensional wave maps into the 2-sphere*, Nonlinearity 14(2001), no. 5, 1041-1053

[3] Cote, R., *Instability of non-constant harmonic maps for the $2 + 1$-dimensional equivariant wave map system*, preprint

[4] Gustafson, S., Nakanishi, K., Tsai, T.-P., *Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schroedinger maps on R^2*, preprint (2009)

[5] *Asymptotic stability of harmonic maps under the Schrdinger flow*, Duke Math. J. 145 (2008), no. 3, 537583.

[6] J. Krieger, D. Tataru, W. Schlag, *Renormalization and blow up for charge one equivariant critical wave maps.* Invent. Math. 171 (2008), no. 3, 543–615.

[7] Krieger, J., Schlag, W., *On the focusing critical semi-linear wave equation*, American Journal Of Mathematics, 129, 843-913 (2007)

[8] Rodnianski, I., Sterbenz, J., *On the formation of singularities in the critical $O(3)$ σ-model*,Ann. of Math. (2) 172 (2010), no. 1, 187 - 242.

[9] Raphael, P., Rodnianski, I., *Stable blow up dynamics for the critical co-rotational Wave Maps and equivariant Yang-Mills problems*, preprint (2009)

[10] Struwe, M., *Equivariant wave maps in two space dimensions*, Comm. Pure Appl. Math. 56(2003), no. 7, 815-823.

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