

Universal Sets for Straight-Line Embeddings of Bicolored Graphs^{*†}

Josef Cibulka¹, Jan Kynčl², Viola Mészáros^{2,3},
Rudolf Stolař¹ and Pavel Valtr²

¹ Department of Applied Mathematics,
Charles University, Faculty of Mathematics and Physics,
Malostranské nám. 25, 118 00 Prague, Czech Republic;
cibulka@kam.mff.cuni.cz, ruda@kam.mff.cuni.cz

² Department of Applied Mathematics and Institute for Theoretical Computer Science,
Charles University, Faculty of Mathematics and Physics,
Malostranské nám. 25, 118 00 Prague, Czech Republic;
kync1@kam.mff.cuni.cz

³ Bolyai Institute, University of Szeged,
Aradi vértanúk tere 1, 6720 Szeged, Hungary;
viola@math.u-szeged.hu

Abstract

A set S of n points is *2-color universal* for a graph G on n vertices if for every proper 2-coloring of G and for every 2-coloring of S with

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the same sizes of color classes as G has, G is straight-line embeddable on S .

We show that the so-called double chain is 2-color universal for paths if each of the two chains contains at least one fifth of all the points, but not if one of the chains is more than approximately 28 times longer than the other.

A 2-coloring of G is *equitable* if the sizes of the color classes differ by at most 1. A bipartite graph is *equitable* if it admits an equitable proper coloring. We study the case when S is the double-chain with chain sizes differing by at most 1 and G is an equitable bipartite graph. We prove that this S is not 2-color universal if G is not a forest of caterpillars and that it is 2-color universal for equitable caterpillars with at most one half non-leaf vertices. We also show that if this S is equitably 2-colored, then equitably properly 2-colored forests of stars can be embedded on it.

1 Introduction

1.1 Previous Results

It is frequently asked in geometric graph theory whether a given graph G can be drawn without edge crossings on a given planar point set S under some additional constraints on the drawing. In this paper, we always assume $|V(G)| = |S|$.

One possibility is to prescribe a fixed position for each vertex of G . If the edges are allowed to be arbitrary curves, then we can obtain a planar drawing of an arbitrary graph G by moving vertices from an arbitrary planar drawing of G to the given points. Pach and Wenger showed [17] that every planar G with prescribed vertex positions can be drawn so that each edge is a piecewise-linear curve with $O(n)$ bends and that this bound is tight even if G is a path.

In another setting, we are only given the graph G and the set of points S , but we are allowed to choose the mapping between $V(G)$ and S . Kaufmann and Wiese [15] showed that two bends per edge are then always enough and for some graphs necessary. An *outerplanar graph* is a graph admitting a planar drawing where one face contains all vertices. By a result of Gritzmann et al. [11], outerplanar graphs are exactly the graphs with a straight-line planar drawing on an arbitrary set of points in general position.

In this paper we are dealing with a combination of these two versions. The vertices and points are colored with two colors and each vertex has to be placed on a point of the same color. An obvious necessary condition to find such a drawing is that each color class has the same number of vertices as points. We then say that the 2-coloring of S is *compatible* with the 2-coloring of $V(G)$.

A *caterpillar* is a tree in which the non-leaf vertices induce a path. The coloring of $V(G)$ is *proper* if it doesn't create any monochromatic edge. It is known that drawing some bicolored planar graphs on some bicolored point sets requires at least $\Omega(n)$ bends per edge [9], but caterpillars can be drawn with two bends per edge [9] and one bend per edge is enough for paths [10] and properly colored caterpillars [9].

We restrict our attention to the proper 2-colorings of a bipartite graph G . Then the question of embeddability of G on a bicolored point set is very similar to finding a non-crossing copy of G in a complete geometric graph from which we removed edges of the two complete subgraphs on points of the two color classes. The only difference is that in the latter case, we can swap colors on some connected components of G . A related question was posed by Micha Perles on DIMACS Workshop on Geometric Graph Theory in 2002. He asked what is the maximum number $h(n)$ such that if we remove arbitrary $h(n)$ edges from a complete geometric graph on an arbitrary set of n points in general position, we can still find a non-crossing Hamiltonian path. Černý et al. [6] showed that $h(n) = \Omega(\sqrt{n})$ and also that it is safe to remove the edges of an arbitrary complete graph on $\Omega(\sqrt{n})$ vertices and that this bound is asymptotically optimal. Aichholzer et al. [3] summarize history and results of this type also for graphs different from the path.

It is thus impossible to find a non-crossing Hamiltonian alternating (that is, properly colored) path (*NHAP* for short) on some bicolored point sets. Kaneko et al. [14] proved that the smallest such point set has 16 points if we allow only even number n of points and 13 for arbitrary n .

Several sufficient conditions are known under which a bicolored point set admits an NHAP. An NHAP exists whenever the two color classes are separable by a line [1] or if one of them is composed of the points of the convex hull of S [1].

The result on sets with color classes separable by a line readily implies that any 2-colored set S with each color class of size $n/2$ admits a non-crossing alternating path (NAP) on at least $n/2$ points of S . It is an open problem if this lower bound can be improved to $n/2 + f(n)$, where $f(n)$ is unbounded

(see also Chapter 9.7 of the book [5]). On the other hand, there are such 2-colored sets admitting no NAP of length more than $\approx 2n/3$ [2, 16]. This upper bound is proved for certain colorings of points in convex position. The above general lower bound $n/2$ can be slightly improved for sets in convex position [16, 12], the best bound is currently $n/2 + \Omega(\sqrt{n})$ by Hajnal and Mészáros [12].

The main result of this paper is that some point sets contain an NHAP for any equitable 2-coloring of their points. We call such point sets *2-color universal* for a path.

See the survey of Kaneko and Kano [13] for more results on embedding graphs on bicolored point sets. One of the results mentioned in the survey is the possibility to embed some graphs with a fixed 2-coloring on an arbitrary compatibly 2-colored point set. Let G be a forest of stars with centers colored black and leaves white and let S be a 2-colored point set. If we map the centers of stars arbitrarily and then we map the leaves so that the sum of lengths of edges is minimized, then no two edges cross.

Previously, a different notion of universality was considered in the context of embedding colored graphs on colored point sets. A k -colored set S of n points is *universal* for a class \mathcal{G} of graphs if every (not necessarily proper) coloring of vertices of $G \in \mathcal{G}$ on n vertices admits an embedding on S . Brandes et al. [4] find, for example, universal k -colored point sets for the class of caterpillars for every $k \leq 3$.

1.2 Our Results

A *convex* or a *concave chain* is a finite set of points in the plane lying on the graph of a strictly convex or a strictly concave function, respectively. A *double-chain* (C_1, C_2) consists of a convex chain C_1 and a concave chain C_2 such that each point of C_2 lies strictly below every line determined by C_1 and similarly, each point of C_1 lies strictly above every line determined by C_2 (see Fig. 1). Double-chains were first considered in [8].

The size of a chain C_i is the number $|C_i|$ of its points. Note that we allow different sizes of the chains C_1 and C_2 . If the sizes $|C_1|$, $|C_2|$ of the chains differ by at most 1, we say that the double-chain is *balanced*.

We consider only 2-colorings and we use *black* and *white* as the colors. A 2-coloring of a set S of n points in the plane is *compatible* with a 2-coloring of a graph G on n vertices if the number of black points of S is the same as the number of black vertices in G . This implies the equality of numbers of

white points and white vertices as well.

A graph G with 2-colored vertices is *embeddable* on a 2-colored point set S if the vertices of G can be mapped to the points of S so that the colors match and no two edges cross if they are drawn as straight-line segments.

A set S of points is *2-color universal* for a bipartite graph G if for every proper 2-coloring of G and for every compatible 2-coloring of S , G is embeddable on S .

If the properly colored path on n vertices, P_n , can be embedded on a 2-colored set S of n points, then we say that S has an NHAP (non-crossing Hamiltonian alternating path).

A 2-coloring of a set S of n points is *equitable* if it is compatible with some proper 2-coloring of P_n , that is, if the sizes of the two color classes differ by at most 1.

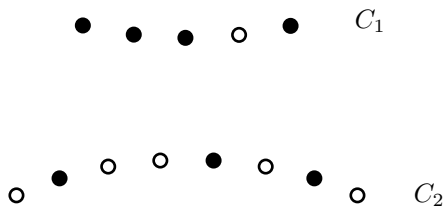


Figure 1: an equitable 2-coloring of a double-chain (C_1, C_2)

Section 2 contains the proof of the following theorem.

Theorem 1.1. *Let (C_1, C_2) be a double-chain satisfying $|C_i| \geq 1/5(|C_1| + |C_2|)$ for $i = 1, 2$. Then (C_1, C_2) is 2-color universal for P_n , where $n = |C_1| + |C_2|$. Moreover, an NHAP on an equitably colored (C_1, C_2) can be found in linear time.*

Note that this doesn't always hold if we don't require the coloring of the path to be proper. For example, if we color first two vertices of P_4 black and the other two white, then it cannot be embedded on the double-chain with both chains of size 2 if the two black points are the top-left one and the bottom-right one.

In Section 3, we show that double-chains with highly unbalanced sizes of chains do not admit an NHAP for some equitable 2-colorings.

Theorem 1.2. *Let (C_1, C_2) be a double-chain, and let C_1 be periodic with the following period of length 16: 2 black, 4 white, 6 black and 4 white points. If $|C_1| \geq 28(|C_2| + 1)$, then (C_1, C_2) has no NHAP.*

An *equitable coloring* of a graph is a coloring where the sizes of any two color classes differ by at most 1. A bipartite graph is *equitable* if it admits a proper equitable 2-coloring.

Section 4 mainly studies other graphs for which the balanced double-chain is 2-color universal.

Theorem 1.3. *The balanced double-chain is 2-color universal for equitable caterpillars with at least as many leaves as non-leaf vertices.*

If a forest of stars is 2-colored equitably and properly, then it can be embedded on every compatibly 2-colored balanced double-chain.

If the balanced double chain is 2-color universal for an equitable bipartite graph G , then G is a forest of caterpillars.

We also present examples of equitable bipartite planar graphs for which no set of points is 2-color universal.

2 Proof of Theorem 1.1

The main idea of our proof is to cover the chains C_i by a special type of pairwise non-crossing paths, so called hedgehogs, and then to connect these hedgehogs into an NHAP by adding some edges between C_1 and C_2 .

2.1 Notation Used in the Proof

For $i = 1, 2$, let b_i be the number of black points of C_i and let $w_i := |C_i| - b_i$ denote the number of white points of C_i .

Since the coloring is equitable, we may assume that $b_1 \geq w_1$ and $w_2 \geq b_2$. Then black is *the major color of C_1* and *the minor color of C_2* , and white is *the major color of C_2* and *the minor color of C_1* . Points in the major color, i.e., black points on C_1 and white points on C_2 , are called *major points*. Points in the minor color are called *minor points*.

Points on each C_i are linearly ordered according to the x -coordinate. An *interval* of C_i is a sequence of consecutive points of C_i . An *inner point* of an interval I is any point of I which is neither the leftmost nor the rightmost point of I .

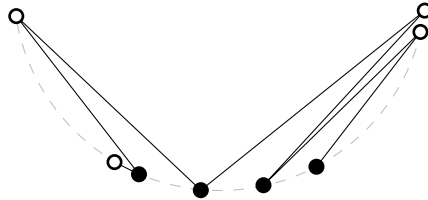


Figure 2: a hedgehog in C_1

A *body* D is a non-empty interval of a chain C_i ($i = 1, 2$) such that all inner points of D are major. If the leftmost point of D is minor, then we call it a *head* of D . Otherwise D has no head. If the rightmost point of D is minor, then we call it a *tail* of D . Otherwise D has no tail. If a body consists of just one minor point, this point is both the head and the tail.

Bodies are of the following four types. A *00-body* is a body with no head and no tail. A *11-body* is a body with both head and tail. The bodies of remaining two types have exactly one endpoint major and the other one minor. We will call the body a *10-body* or a *01-body* if the minor endpoint is a head or a tail, respectively.

Let D be a body on C_i . A *hedgehog* (built on the body $D \subseteq C_i$) is a non-crossing alternating path H with vertices in C_i satisfying the following three conditions: (1) H contains all points of D , (2) H contains no major points outside of D , (3) the endpoints of H are the first and the last point of D . A hedgehog built on an $\alpha\beta$ -body is an $\alpha\beta$ -*hedgehog* ($\alpha, \beta = 0, 1$). If a hedgehog H is built on a body D , then D is *the body of* H and the points of H that do not lie in D are *spines*. Note that each spine is a minor point. All possible types of hedgehogs can be seen on Fig. 3 (for better lucidity, we will draw hedgehogs with bodies on a horizontal line and spines indicated only by a “peak” from now on).

On each C_i , maximal intervals containing only major points are called *runs*. Clearly, runs form a partition of major points. For $i = 1, 2$, let r_i denote the number of runs in C_i .

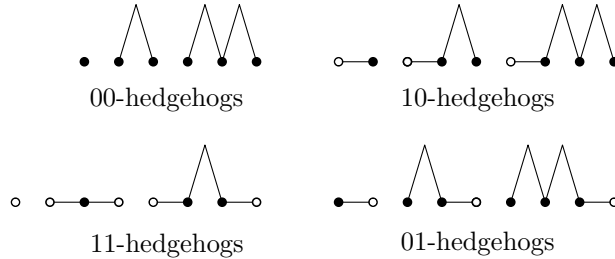


Figure 3: types of hedgehogs (sketch)

2.2 Proof in the Even Case

Throughout this subsection, (C_1, C_2) denotes a double-chain with $|C_1| + |C_2|$ even. Since the coloring is equitable, we have $b_1 + b_2 = w_1 + w_2$. Set

$$\Delta := b_1 - w_1 = w_2 - b_2.$$

First we give a lemma characterizing collections of bodies on a chain C_i that are bodies of some pairwise non-crossing hedgehogs covering the whole chain C_i .

Lemma 2.1. *Let $i \in \{1, 2\}$. Let all major points of C_i be covered by a set \mathcal{D} of pairwise disjoint bodies. Then the bodies of \mathcal{D} are the bodies of some pairwise non-crossing hedgehogs covering the whole C_i if and only if $\Delta = d_{00} - d_{11}$, where $d_{\alpha\alpha}$ is the number of $\alpha\alpha$ -bodies in \mathcal{D} .*

Proof. An $\alpha\beta$ -hedgehog containing t major points contains $(t - 1) + \alpha + \beta$ minor points. It follows that the equality $\Delta = d_{00} - d_{11}$ is necessary for the existence of a covering of C_i by disjoint hedgehogs built on the bodies of \mathcal{D} .

Suppose now that $\Delta = d_{00} - d_{11}$. Let F be the set of minor points on C_i that lie in no body of \mathcal{D} , and let M be the set of the mid-points of straight-line segments connecting pairs of consecutive major points lying in the same body. It is easily checked that $|F| = |M|$. Clearly $F \cup M$ is a convex or a concave chain. Now it is easy to prove that there is a non-crossing perfect matching formed by $|F| = |M|$ straight-line segments between F and M (for the proof, take any segment connecting a point of F with a neighboring point of M , remove the two points, and continue by induction); see Fig. 4.

If $f \in F$ is connected to a point $m \in M$ in the matching, then f will be a spine with edges going from it to those two major points that determined m .

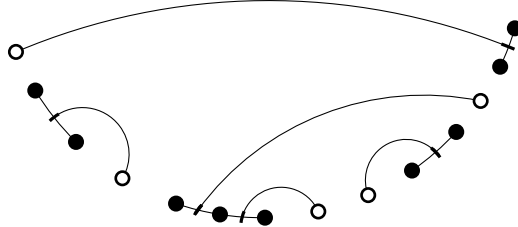


Figure 4: a non-crossing matching of minor points and midpoints (in C_1)

Obviously, these spines and edges define non-crossing hedgehogs with bodies in \mathcal{D} and with all the required properties. ■

The following three lemmas and their proofs show how to construct an NHAP in some special cases.

Lemma 2.2. *If $\Delta \geq \max\{r_1, r_2\}$ then (C_1, C_2) has an NHAP.*

Proof. Let $i \in \{1, 2\}$. Since $r_i \leq \Delta \leq \max(b_i, w_i)$, the runs in C_i may be partitioned into Δ 00-bodies. By Lemma 2.1, these 00-bodies may be extended to pairwise non-crossing hedgehogs covering C_i . This gives us 2Δ hedgehogs on the double-chain. They may be connected into an NHAP by $2\Delta - 1$ edges between the chains in the way shown in Fig. 5. ■

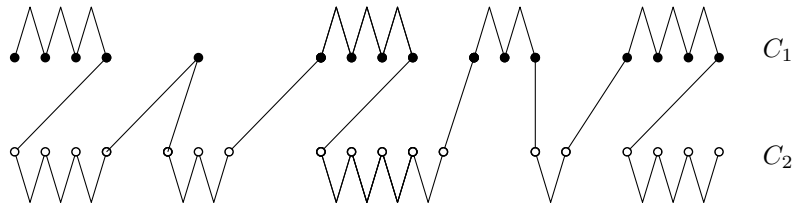


Figure 5: 00-hedgehogs connected to an NHAP

Lemma 2.3. *If $r_1 = r_2$ then (C_1, C_2) has an NHAP.*

Proof. Set $r := r_1 = r_2$. If $r \leq \Delta$ then we may apply Lemma 2.2. Thus, let $r > \Delta$.

Suppose first that $\Delta \geq 1$. We cover each run on each C_i by a single body whose type is as follows. On C_1 we take Δ 00-bodies followed by $(r - \Delta)$ 10-bodies. On C_2 we take (from left to right) $(\Delta - 1)$ 00-bodies, $(r - \Delta)$ 01-bodies, and one 00-body. By Lemma 2.1, the r bodies on each C_i can be extended to hedgehogs covering C_i . Altogether we obtain $2r$ hedgehogs. They can be connected to an NHAP by $2r - 1$ edges between C_1 and C_2 (see Fig. 6).

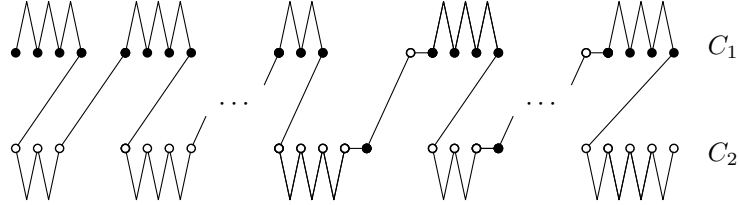


Figure 6: an NHAP in the case $r_1 = r_2 > \Delta \geq 1$

Suppose now that $\Delta = 0$. We add one auxiliary major point on each C_i as follows. On C_1 , the auxiliary point extends the leftmost run on the left. On C_2 , the auxiliary point extends the rightmost run on the right. This does not change the number of runs and increases Δ to 1. Thus, we may proceed as above. The NHAP obtained has the two auxiliary points on its ends. We may remove the auxiliary points from the path, obtaining an NHAP for (C_1, C_2) . ■

A *singleton* $s \in C_i$ is an inner point of C_i ($i = 1, 2$) such that its two neighbors on C_i are colored differently from s .

Lemma 2.4. *Suppose that C_1 has no singletons and C_2 can be covered by $r_1 - 1$ pairwise disjoint hedgehogs. Then (C_1, C_2) has an NHAP.*

Proof. For simplicity of notation, set $r := r_1$. We denote the $r - 1$ hedgehogs on C_2 by P_1, P_2, \dots, P_{r-1} in the left-to-right order in which the bodies of these hedgehogs appear on C_2 . For technical reasons, we enlarge the leftmost run of C_1 from the left by an auxiliary major point σ .

Our goal is to find r hedgehogs H_1, H_2, \dots, H_r on $C_1 \cup \{\sigma\}$ such that they may be connected with the hedgehogs P_1, P_2, \dots, P_{r-1} into an NHAP. For each $j = 1, \dots, r$, the body of the hedgehog H_j will be denoted by D_j .

For each $j = 1, \dots, r$, D_j covers the j -th run of $C_1 \cup \{\sigma\}$ (in the left-to-right order). We now finish the definition of the bodies D_j by specifying for each D_j if it has a head and/or a tail. The body D_1 is without head. For $j > 1$, D_j has a head if and only if P_{j-1} has a tail. The last body D_r is without tail and $D_j, j < r$, has a tail if and only if P_j has a head.

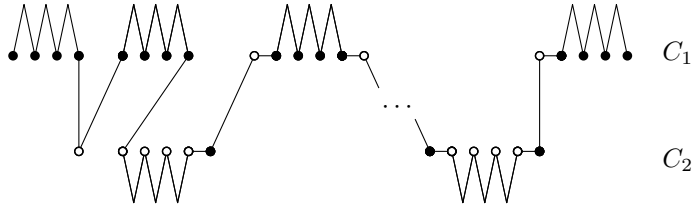


Figure 7: an NHAP in the case of no singletons on C_1

It follows from Lemma 2.1 that we may add or remove some minor points on $C_1 \cup \{\sigma\}$ so that D_1, \dots, D_r can then be extended to pairwise non-crossing hedgehogs H_1, \dots, H_r covering the “new” C_1 . More precisely, there is a double-chain (C'_1, C_2) such that D_1, \dots, D_r can be extended to pairwise non-crossing hedgehogs H_1, \dots, H_r covering C'_1 , where either $C'_1 = C_1 \cup \{\sigma\}$ or C'_1 is obtained from $C_1 \cup \{\sigma\}$ by adding some minor (white) points on the left of $C_1 \cup \{\sigma\}$ (say) or C'_1 is obtained from $C_1 \cup \{\sigma\}$ by removal of some minor (white) points lying in none of the bodies D_1, \dots, D_r . Then the concatenation $H_1 P_1 H_2 P_2 \dots H_{r-1} P_{r-1} H_r$ shown in Fig. 7 gives an NHAP on (C'_1, C_2) . This NHAP starts with the point σ . Removal of σ from it gives an NHAP P for the double-chain $(C'_1 \setminus \{\sigma\}, C_2)$. The endpoints of P have different colors. Thus, P covers the same number of black and white points. Black points on P are the $(|C_1| + |C_2|)/2$ black points of (C_1, C_2) . Thus, P covers exactly $|C_1| + |C_2|$ points. It follows that $|C'_1 \setminus \{\sigma\}| = |C_1|$ and thus $C'_1 \setminus \{\sigma\} = C_1$. The path P is an NHAP on the double-chain (C_1, C_2) . ■

The following lemma will be used to find a covering needed in Lemma 2.4.

Lemma 2.5. *Suppose that $|C_i| \geq k$, $r_i \leq k$ and $\Delta \leq k$ for some $i \in \{1, 2\}$ and for some integer k . Then C_i can be covered by k pairwise disjoint hedgehogs.*

Proof. The idea of the proof is to start with the set \mathcal{D} of $|C_i|$ bodies, each of them being a single point, and then gradually decrease the number of bodies

in \mathcal{D} by joining some of the bodies together. We see that $\Delta = d_{00} - d_{11}$, where $d_{\alpha\alpha}$ is the number of $\alpha\alpha$ -bodies in \mathcal{D} . If we join two neighboring 00-bodies to one 00-body and withdraw a single-point 11-body from \mathcal{D} (to let the minor point become a spine) at the same time, the difference between the number of 00-bodies and the number of 11-bodies remains the same and $|\mathcal{D}|$ decreases by two. We can reduce $|\mathcal{D}|$ by one while preserving the difference $d_{00} - d_{11}$ by joining a 00-body with a neighboring single-point 11-body into a 01- or a 10-body. Similarly we can join a 01- or a 10-body with a neighboring (from the proper side) single-point 11-body into a new 11-body to decrease $|\mathcal{D}|$ by one as well. When we are joining two 00-bodies, we choose the single-point 11-body to remove in such a way to keep as many single-point 11-bodies adjacent to 00-bodies as possible. This guarantees that we can use up to r_i of them for heads and tails.

We start with joining neighboring 00-bodies and we do this as long as $|\mathcal{D}| > k + 1$ and $d_{00} > r_i$. Note that by the assumption $\Delta \leq k$, we will have enough single-point 11-bodies to do that. When we end, one of the following conditions holds: $|\mathcal{D}| = k$, $|\mathcal{D}| = k + 1$ or $d_{00} = r_i$. In the first case we are done. If $|\mathcal{D}| = k + 1$, we just add one head or one tail (we can do this since $d_{00} + d_{11} = |\mathcal{D}| = k + 1 \geq d_{00} - d_{11} + 1$, which implies $d_{11} > 0$). If $d_{00} = r_i$, then each run is covered by just one 00-body. We need to add $|\mathcal{D}| - k$ heads and tails. We have enough single-point 11-bodies to do that since $d_{11} = |\mathcal{D}| - d_{00} = |\mathcal{D}| - r_i \geq |\mathcal{D}| - k$. On the other hand, $r_i - d_{11} = \Delta \geq 0$, so the number of heads and tails needed is at most r_i . Therefore, all the single-point 11-bodies are adjacent to 00-bodies and we can use them to form heads and tails.

In all cases we get a set \mathcal{D} of k bodies. Now we can apply Lemma 2.1 to obtain k pairwise disjoint hedgehogs covering C_i . \blacksquare

By a *contraction* we mean removing a singleton with both its neighbors and putting a point of the color of its neighbors in its place instead. It is easy to verify that if there is an NHAP in the new double-chain obtained by this contraction, it can be expanded to an NHAP in the original double-chain.

Now we can prove our main theorem in the even case.

Proof of Theorem 1.1 (even case) Without loss of generality we may assume that $r_1 \geq r_2$. In the case $r_1 = r_2$, we get an NHAP by Lemma 2.3. In the case $\Delta \geq r_1$, we get an NHAP by Lemma 2.2. Therefore, the only case left is $r_1 > r_2$, $r_1 > \Delta$.

If there is a singleton on C_1 , we make a contraction of it. By this we decrease r_1 by one and both r_2 and Δ remain unchanged. If now $r_1 = r_2$ or $r_1 = \Delta$, we again get an NHAP, otherwise we keep making contractions until one of the previous cases appears or there are no more singletons to contract.

If there is no more singleton to contract on C_1 and still $r_1 > r_2$ and $r_1 > \Delta$, we try to cover C_2 by $r_1 - 1$ pairwise disjoint paths. Before the contractions, $|C_2| \geq |C_1|/4$ did hold and by the contractions we could just decrease $|C_1|$, therefore it still holds.

All the maximal intervals on the chain C_1 (with possible exception of the first and the last one) have now length at least two, which implies that $r_1 \leq |C_1|/4 + 1$. Hence $|C_2| \geq |C_1|/4 \geq r_1 - 1$, so we can create $r_1 - 1$ pairwise disjoint hedgehogs covering C_2 using Lemma 2.5. Then we apply Lemma 2.4 and expand the NHAP obtained by Lemma 2.4 to an NHAP on the original double-chain.

There is a straightforward linear-time algorithm for finding an NHAP on (C_1, C_2) based on the above proof. ■

2.3 Proof in the Odd Case

In this subsection we prove Theorem 1.1 for the case when $|C_1| + |C_2|$ is odd. We set $\Delta = w_2 - b_2$ and proceed similarly as in the even case. On several places in the proof we will add one auxiliary point ω to get the even case (its color will be chosen to equalize the numbers of black and white points). We will be able to apply one of the Lemmas 2.2–2.4 to obtain an NHAP. The point ω will be at some end of the NHAP and by removing ω we obtain an NHAP for (C_1, C_2) .

Without loss of generality we may assume that $r_1 \geq r_2$. In the case $r_1 = r_2$, we add an auxiliary major point ω , which is placed either as the left neighbor of the leftmost major point on C_1 or as the right neighbor of the rightmost major point on C_2 . Then we get an NHAP by Lemma 2.3 and the removal of ω gives us an NHAP for (C_1, C_2) .

In the case $\Delta \geq r_1$, we add an auxiliary point ω to the same place and we get an NHAP by Lemma 2.2. Again, the removal of ω gives us an NHAP for (C_1, C_2) .

Now, the only case left is $r_1 > r_2$, $r_1 > \Delta$. If there are any singletons on C_1 , we make the contractions exactly the same way as in the proof of the even case. If Lemma 2.2 or 2.3 needs to be applied, we again add an auxiliary point ω and proceed as above.

If there is no more singleton to contract on C_1 and still $r_1 > r_2$ and $r_1 > \Delta$, we have $|C_2| \geq |C_1|/4 \geq r_1 - 1$ as in the proof of the even case and we can use Lemma 2.5 to get $r_1 - 1$ pairwise disjoint hedgehogs covering C_2 . Now we need to consider two cases: (1) If $b_1 + b_2 > w_1 + w_2$, then we find an NHAP for (C_1, C_2) in the same way as in the proof of Lemma 2.4, except we do not add the auxiliary point σ . (2) If $b_1 + b_2 < w_1 + w_2$, we add an auxiliary point ω as the right neighbor of the rightmost major point on C_1 . The number r_1 didn't change so Lemma 2.4 gives us an NHAP. Again, the removal of ω gives us an NHAP for (C_1, C_2) .

There is a straightforward linear-time algorithm for finding an NHAP on (C_1, C_2) based on the above proof. \blacksquare

3 Unbalanced Double-Chains with no NHAP

In this section we prove Theorem 1.2. Let (C_1, C_2) be a double-chain whose points are colored by an equitable 2-coloring, and let C_1 be periodic with the following period: 2 black, 4 white, 6 black and 4 white points. Let $|C_1| \geq 28(|C_2| + 1)$. We want to show that (C_1, C_2) has no NHAP.

Suppose on the contrary that (C_1, C_2) has an NHAP. Let P_1, P_2, \dots, P_t denote the maximal subpaths of the NHAP containing only points of C_1 . Since between every two consecutive paths P_i, P_j in the NHAP there is at least one point of C_2 , we have $t \leq |C_2| + 1$. In the following we think of C_1 as of a cyclic sequence of points on the circle. Note that we get more intervals in this way. Theorem 1.2 now directly follows from the following theorem.

Theorem 3.1. *Let C_1 be a set of points on a circle periodically 2-colored with the following period of length 16: 2 black, 4 white, 6 black and 4 white points. Suppose that all points of C_1 are covered by a set of t non-crossing alternating and pairwise disjoint paths P_1, P_2, \dots, P_t . Then $t > |C_1|/28$.*

Proof. Each maximal interval spanned by a path P_i on the circle is called a *base*. Let $b(P_i)$ denote the number of bases of P_i . A path with one base only is called a *leaf*. We consider the following special types of edges in the paths. *Long edges* connect points that belong to different bases. *Short edges* connect consecutive points on C_1 . Note that short edges cannot be adjacent to each other. A maximal subpath of a path P_i spanning two subintervals of two different bases and consisting of long edges only is called a *zig-zag*. A path is *separated* if all of its edges can be crossed by a line. Note that each

zig-zag is a separated path. A maximal separated subpath of P_i that contains an endpoint of P_i and spans one interval only is a *rainbow*. We find all the zig-zags and rainbows in each P_i , $i = 1, 2, \dots, t$. Note that two zig-zags, or a zig-zag and a rainbow, are either disjoint or share an endpoint. A *branch* is a maximal subpath of P_i that spans two intervals and is induced by a union of zig-zags.

For each path P_i that is not a leaf construct the following graph G_i . The vertices of G_i are the bases of P_i . We add an edge between two vertices for each branch that connects the corresponding bases. If G_i has a cycle (including the case of a “2-cycle”), then one of the corresponding branches consists of a single edge that lies on the convex hull of P_i . We delete such an edge from P_i and don’t call it a branch anymore. By deleting a corresponding edge from each cycle of G_i we obtain a graph G'_i , which is a spanning tree of G_i . The *branch graph* G' is a union of all graphs G'_i .

Let \mathcal{L} denote the set of leaves and \mathcal{B} the set of branches. Let $\mathcal{P} = \{P_1, P_2, \dots, P_t\}$.

Observation 3.2. *The branch graph G' is a forest with components G'_i . Therefore,*

$$|\mathcal{B}| = \sum_{i, P_i \notin \mathcal{L}} (b(P_i) - 1).$$

■

The branches and rainbows in P_i do not necessarily cover all the points of P_i . Each point that is not covered is adjacent to a deleted long edge and to a short edge that connects this point to a branch or a rainbow. It follows that between two consecutive branches (and between a rainbow and the nearest branch) there are at most two uncovered points, that are endpoints of a common deleted edge. By an easy case analysis it can be shown that this upper bound can be achieved only if one of the nearest branches consists of a single zig-zag.

In the rest of the paper, a *run* will be a maximal monochromatic interval of any color. In the following we will count the runs that are spanned by the paths P_i . The *weight* of a path P , $w(P)$, is the number of runs spanned by P . If P spans a whole run, it adds one unit to $w(P)$. If P partially spans a run, it adds half a unit to $w(P)$.

Observation 3.3. *The weight of a zig-zag or a rainbow is at most 1.5. A branch consists of at most two zig-zags, hence it weights at most three units.* ■

Lemma 3.4. *A path P_i that is not a leaf weights at most $3.5k + 3.5$ units where k is the number of branches in P_i .*

Proof. According to the above discussion, for each pair of uncovered points that are adjacent on P_i we can join one of them to the adjacent branch consisting of a single zig-zag. To each such branch we join at most two uncovered points, hence its weight increases by at most one unit to at most 2.5 units. The number of the remaining uncovered points is at most $k + 1$. Therefore, $w(P_i) \leq 3k + 3 + 0.5 \cdot (k + 1) = 3.5k + 3.5$. ■

Lemma 3.5. *A leaf weights at most 3.5 units.*

Proof. Let L be a leaf spanning at least two points. Consider the interval spanned by L . Cut this interval out of C_1 and glue its endpoints together to form a circle. Take a line l that crosses the first and the last edge of L . Note that the line l doesn't separate any of the runs. Exactly one of the arcs determined by l contains the gluing point γ .

Each of the ending edges of L belongs to a rainbow, all of whose edges cross l . It follows that if L has only one rainbow, then this rainbow covers the whole leaf L and $w(L) \leq 1.5$. Otherwise L has exactly two rainbows, R_1 and R_2 . We show that R_1 and R_2 cover all edges of L that cross the line l . Suppose there is an edge s in L that crosses l and does not belong to any of the rainbows R_1, R_2 . Then one of these rainbows, say R_1 , is separated from γ by s . Then the edge of L that is the second nearest to R_1 also has the same property as the edge s . This would imply that R_1 spans two whole runs, a contradiction. It follows that all the edges of L that are not covered by the rainbows are consecutive and connect adjacent points on the circle. There are at most three such edges; at most one connecting the points adjacent to γ , the rest of them being short on C_1 . But this upper bound of three cannot be achieved since it would force both rainbows to span two whole runs. Therefore, there are at most two edges and hence at most one point in L uncovered by the rainbows. The lemma follows. ■

Lemma 3.6. $|\mathcal{L}| \geq \sum_{i, P_i \notin \mathcal{L}} (b(P_i) - 2) + 2$.

Proof. The number of runs in C_1 is at least 4. By Lemma 3.5, if all the paths P_i are leaves, then at least 2 of them are needed to cover C_1 and the lemma follows.

If not all the paths are leaves, we order the paths so that all the leaves come at the end of the ordering. The path P_1 spans $b(P_1)$ bases. Shrink these bases to points. These points divide the circle into $b(P_1)$ arcs each of which contains at least one leaf. If P_2 is not a leaf then continue. The path P_2 spans $b(P_2)$ intervals on one of the previous arcs. Shrink them to points. These points divide the arc into $b(P_2) + 1$ subarcs. At least $b(P_2) - 1$ of them contain leaves. This increased the number of leaves by at least $b(P_2) - 2$. The case of $P_i, i > 2$, is similar to P_2 . The lemma follows by induction. ■

Corollary 3.7. $|\mathcal{B}| \leq |\mathcal{P}| - 2$.

Proof. Combining Lemma 3.6 and Observation 3.2 we get the following:

$$|\mathcal{B}| = \sum_{i, P_i \notin \mathcal{L}} (b(P_i) - 1) = \sum_{i, P_i \notin \mathcal{L}} (b(P_i) - 2) + |\mathcal{P}| - |\mathcal{L}| + 2 - 2 \leq |\mathcal{P}| - 2.$$

■

Now we are in position to finish the proof of Theorem 3.1. If the whole C_1 is covered by the paths P_i , then $\sum_{i=1}^t w(P_i) \geq |C_1|/4$. Therefore,

$$|C_1| \leq 4 \cdot (3.5|\mathcal{B}| + 3.5(|\mathcal{P}| - |\mathcal{L}|) + 3.5|\mathcal{L}|) < 4 \cdot 7|\mathcal{P}| = 28|\mathcal{P}|.$$

■

4 Embedding equitable bipartite graphs

4.1 Embedding on balanced double-chains

We already know that the balanced double-chain is 2-color universal for the path P_n . In this subsection, we further study the class of graphs for which the balanced double-chain is 2-color universal. The three lemmas of this subsection prove the three claims of Theorem 1.3.

Lemma 4.1. *If the balanced double-chain is 2-color universal for an equitable bipartite graph G then G is a forest of caterpillars.*

Proof. Let $K_{1,3}^+$ be the 3-star with subdivided edges (see Fig. 8). A connected graph is a caterpillar if and only if it contains no cycle and no $K_{1,3}^+$ as a subgraph.

We will color all points of one chain white and points of the other chain black so that the resulting coloring is compatible with the 2-coloring of G . We assume for contradiction that G can be embedded on it and that it contains either a cycle or $K_{1,3}^+$.

As the double-chain has monochromatic chains, all the edges connect the two chains. Because the embedding has no edge crossings, we can consider the leftmost edge of the cycle and let u and v be its endvertices. Then the two edges of the cycle incident to exactly one of u and v cross.

We now assume that $K_{1,3}^+$ can be embedded on the double-chain and let the color of its root vertex be white. Let ω_1 be the white point where the root of $K_{1,3}^+$ is mapped and let $\beta_1, \beta_2, \beta_3$ be (from left to right) the three black points where the middle vertices are mapped. Then β_2 is connected by an edge to some white leaf vertex of $K_{1,3}^+$, but this edge is crossed either by the segment $\omega_1\beta_1$ or by $\omega_1\beta_3$. See Fig. 8. ■

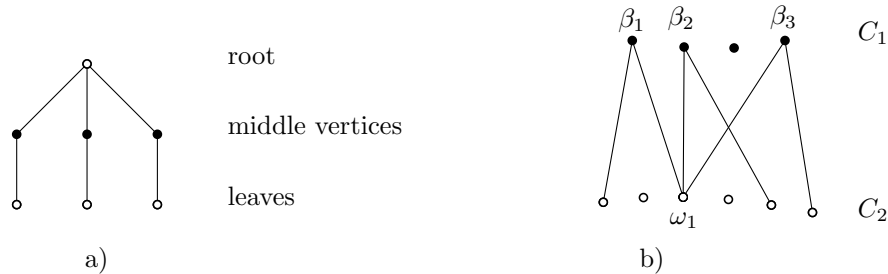


Figure 8: a) $K_{1,3}^+$ and b) impossibility of its embedding on the double chain with monochromatic chains

The *central path* in a caterpillar is the set of non-leaf vertices.

Lemma 4.2. *If an equitable bipartite graph G on n vertices is a caterpillar with at most $\lfloor n/2 \rfloor$ vertices on the central path, then the balanced double-chain is 2-color universal for G .*

Proof. Let b_i be the number of black points on the chain C_i and w_i the number of white points on C_i . Since the coloring is equitable, we may assume that $b_1 \geq w_1$ and $w_2 \geq b_2$. Then black is the *major color* of C_1 and the *minor*

color of C_2 , and white is *the major color* of C_2 and *the minor color* of C_1 . Points in the major color are called *major points*. Points in the minor color are called *minor points*.

Observe that the number of minor points on each chain is at most the number of leaves of G of that color. Let G' be the graph with b_1 black and w_1 white vertices obtained from G by removing some b_2 black and w_1 white leaves.

In the first phase, we embed G' on the set of major points of the two chains. We take the vertices of the central path of G' starting from one of its ends. A vertex v of the central path is placed on the leftmost unused major point on the chain where the color of v is major. The leaves of v in G' are then successively placed on the leftmost unused major points of the other chain.

In the second phase, we map all the leaves removed in the first phase on minor points. In the same greedy way as in the proof of Lemma 2.1, we keep connecting the closest pair of an unused white point of C_1 and a black point of the central path that still misses at least one leaf. The same is done on C_2 .

This guarantees that no crossing appears and that every vertex is mapped to some point. See Fig. 9. ■

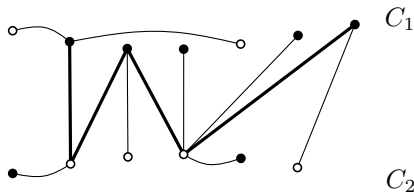


Figure 9: embedding a caterpillar on a balanced double-chain; the bold edges form the central path

Lemma 4.3. *If a forest of stars G is 2-colored equitably and properly, then G can be embedded on every compatibly 2-colored balanced double-chain.*

Proof. We take some fixed proper equitable 2-coloring of G .

We show that by adding edges to G , we are able to create a properly 2-colored caterpillar on the set of all vertices of G and with at most $\lfloor n/2 \rfloor$ non-leaf vertices. By Lemma 4.2 this caterpillar can be embedded on every compatibly 2-colored balanced double-chain and thus G can be embedded.

The cases when $n \leq 3$ and when G has no edges are trivial.

For every $i \geq 3$, let k_i (h_i) be the number of stars on i vertices and with black (white) central vertex. In case of 2-vertex components of G , we cannot distinguish the central vertex and we let n_2 be their number. We also let n_1 be the total number of 1-vertex components of G as it is not necessary for the proof to count black and white ones separately.

We assume without loss of generality, that at least half of the stars on at least three vertices have black central vertex. We start connecting the central vertices of stars on at least three vertices into an alternating path starting with a black vertex. At some point we run out of stars with white central vertex. We then use the stars on two vertices as stars with white central vertex. If we run out of stars with black central vertex, we use every second star on two vertices as a star with black central vertex. Otherwise we run out of stars on two vertices. Then we start connecting each of the remaining stars on at least three vertices by an edge between one of its white leaves and the last black vertex on the path.

The resulting graph is composed of a connected graph T and all 1-vertex components of G . The graph T is a properly colored caterpillar and the created path is its central path P . Since G has some edges, P is not empty.

If P contains only one vertex v , we pick one of its leaves, u , and connect every 1-vertex component of G either to u or to v , depending on its color. The central path then has 2 vertices, which is at most $\lfloor n/2 \rfloor$.

If P has at least two vertices, we connect every 1-vertex component of G to a vertex of the other color on the central path.

See Fig. 10.

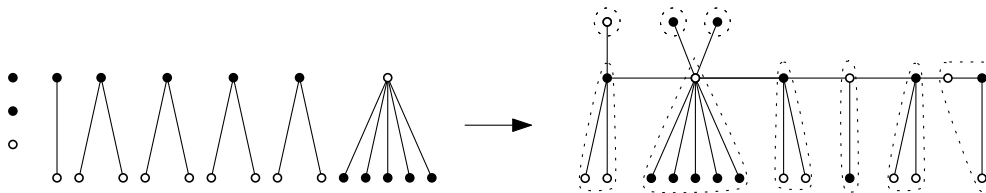


Figure 10: connecting stars to form a caterpillar

It remains to show that the central path is not too long. The total number of vertices is

$$n = n_1 + 2n_2 + \sum_{i=3}^n i(k_i + h_i). \quad (1)$$

If $\sum_{i=3}^n k_i \leq n_2 + \sum_{i=3}^n h_i$, then every vertex of the central path has at least one leaf and thus the caterpillar has at most $\lfloor n/2 \rfloor$ non-leaf vertices.

Otherwise, the central path starts and ends with a black vertex and the black vertices of the central path are exactly the black centers of stars on at least three vertices. The central path thus has $2 \sum_{i=3}^n k_i - 1$ vertices.

Because the 2-coloring of G is equitable, the number of black vertices of G is at least $\lfloor n/2 \rfloor$ and thus

$$\sum_{i=3}^n k_i + \sum_{i=3}^n (i-1)h_i + n_2 + n_1 \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

At most $\lfloor n/2 \rfloor + 1$ vertices of G are white, which leads to

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 \geq \sum_{i=3}^n h_i + \sum_{i=3}^n (i-1)k_i + n_2.$$

Putting the two inequalities together gives us

$$\sum_{i=3}^n (i-2)h_i + n_1 \geq \sum_{i=3}^n (i-2)k_i - 1. \quad (2)$$

The number of vertices of the central path is at most $\lfloor n/2 \rfloor$, because

$$\begin{aligned} 2 \left(2 \sum_{i=3}^n k_i - 1 \right) &\leq \sum_{i=3}^n 2(i-1)k_i - 2 \\ &< \sum_{i=3}^n ik_i + \sum_{i=3}^n (i-2)k_i - 1 \\ &\leq \sum_{i=3}^n ik_i + \sum_{i=3}^n (i-2)h_i + n_1 \quad \text{by Eq. 2} \\ &\leq \sum_{i=3}^n i(k_i + h_i) + n_1 \\ &\leq n \quad \text{by Eq. 1.} \end{aligned}$$

■

4.2 Open problems

It seems plausible that the balanced double-chain is 2-color universal for all equitable forests of caterpillars.

Conjecture. *The balanced double-chain is 2-color universal for an equitable bipartite graph G if and only if G is a forest of caterpillars.*

Some graphs for which the balanced double-chain is not 2-color universal have a different 2-color universal set. For example, the balanced double-chain is not 2-color universal for $K_{1,3}^+$ by Lemma 4.1, but it is easy to verify that the double-chain with one chain composed of only one vertex is.

There even exist graphs with a 2-color universal set of points, but no double chain is 2-color universal for them. Consider the properly colored $K_{1,4}^+$ with black central vertex. It is not embeddable on double-chains colored as in Fig. 11. But a modification of the double chain in Fig. 12 is 2-color universal for $K_{1,4}^+$.

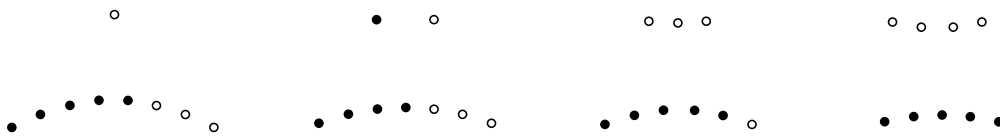


Figure 11: colorings of double-chains not admitting $K_{1,4}^+$

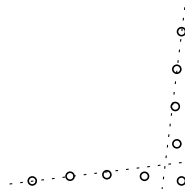


Figure 12: a 2-color universal point set for $K_{1,4}^+$

Some equitable bipartite planar graphs have no 2-color universal set of points.

Claim 4.4. *If G is a bipartite planar quadrangulation on at least five vertices, then G has no 2-color universal set of points.*

Proof. Because the bipartite graph G has no 3-cycle, each of its faces has at least four vertices. Then, by Euler's formula, every planar drawing of G is a quadrangulation.

Take a set S of points and let $H(S)$ be the set of points of S on its convex hull. In a straight-line planar drawing of a graph on a set S of points, the points of $H(S)$ lie all on the outer face of the drawing. Thus G can only be drawn on S if $3 \leq |H(S)| \leq 4$. In a proper coloring of G on at least five vertices, one color class contains at least three vertices. If we color three points of $H(S)$ by this color and the rest arbitrarily, G cannot be embedded, because no face in a drawing of G can contain three vertices of one color. ■

The results of this paper solve only a few particular cases of the following general question.

Question. *Which planar bipartite graphs have a 2-color universal set of points?*

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