ON THE RAMSEY MULTIPLICITY OF THE ODD CYCLES

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Abstract. The Ramsey multiplicity $R(G)$ of a graph G is the minimum number of monochromatic copies of G in any two-colouring of the edges of $K_{r(G)}$, where $r(G)$ denotes the Ramsey number of G. Here we prove that odd cycles have super-exponentially large Ramsey multiplicity: If C_n is an odd cycle of length n, then $\log R(C_n) = \Theta(n \log n)$.

1. Introduction

Let G denote a simple graph without isolated vertices. The Ramsey number $r(G)$ of G is the smallest positive integer r with the property that any two-colouring of the edges of the complete graph K_r of order r induces a monochromatic copy of G. The Ramsey multiplicity $R(G)$ of G is the minimum number of monochromatic copies of G in any two-colouring of the edges of $K_{r(G)}$. The concept was introduced by Harary and Prins in [6], who made the conjecture that $R(G)$ is usually large and attains the smallest possible value 1 if and only if G is a star on m vertices, where $m = 2$ or $m > 1$ is an odd integer. It is already quite difficult

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to determine the Ramsey multiplicity of the complete graph on 4 vertices, the exact value $R(K_4) = 9$ was obtained in [9] aided by computer search.

More generally, let $R(G, n)$ denote the minimum number of monochromatic copies of G in any two-coloring of K_n. For a survey on early results we refer to [2]. In case when G is a cycle of length k, Sidorenko [13] proved, using functional inequalities, the asymptotic result

$$R(G, n) = \frac{1}{k}\left(\frac{n}{2}\right)^k \left(1 + O\left(\frac{1}{n}\right)\right).$$

See [3, 5, 7] for more recent developments.

To obtain good lower estimates on $R(G) = R(G, r(G))$ is apparently more difficult. For any integer $n \geq 3$ let C_n denote the cycle of length n. The Ramsey multiplicity of the triangle and the quadrangle is $R(C_3) = R(C_4) = 2$. For odd values of n, an exponential lower bound for $R(C_n)$ was proved by Rosta and Surányi [12]. An unpublished superexponential bound is due to Rosta [11]. The aim of this paper is to improve upon these results.

Theorem 1. *For odd integers $n > 3$, the Ramsey multiplicity of C_n satisfies*

$$e^{(\frac{1}{24} - o(1))n \log n} < R(C_n) < e^{n \log n}.$$

With a more sophisticated argument based on a similar basic idea we obtain the following stronger result.

Theorem 2. *For odd integers $n > 3$, the Ramsey multiplicity of C_n satisfies*

$$e^{(\frac{1}{20} - o(1))n \log n} < R(C_n).$$

We present both proofs, firstly because the proof of Theorem 1 helps the complicated proof of Theorem 2 to become more transparent, and secondly because we feel that along the first proof strategy, a strong general version of Lemma 10 may probably lead to further improvements.

2. The Structure of the Proofs

Let $n > 3$ be an odd integer; then $r(C_n) = 2n - 1$, see [1, 4, 8, 10]. The upper bound is trivial: Consider the disjoint union of a red clique C of order n and another red clique D of order $n - 1$, and colour every edge connecting a vertex in

C to a vertex in D blue. In the two-coloured complete graph on $2n-1$ vertices thus obtained, every blue cycle has an even length. On the other hand, every red cycle of length n is contained in C. Since C_n has $2n$ automorphisms, the graph has

$$\frac{n!}{2n} < e^{n \log n}$$

monochromatic cycles of length n.

For the lower bound, let G be a complete graph on a set V of $2n-1$ vertices whose edge set is partitioned into a set $\mathcal{R}$ of red edges and a set $\mathcal{B}$ of blue edges. One of the colour classes, say $\mathcal{R}$ contains a cycle of length n. So let $C = \{v_1, v_2, \dots, v_n\}$, where $v_1 v_2 \dots v_n v_1$ is a red cycle, and let $D = V \setminus C$. If D contains a large red clique, then it is easy to construct many monochromatic cycles of length n.

Lemma 3. *Suppose that D contains a red clique K of order larger than $n/2$. Then either $\mathcal{R}$ or $\mathcal{B}$ contains at least*

$$\left(\left\lfloor \frac{n}{4} \right\rfloor - 2\right)! > e^{(\frac{1}{4} - o(1)) n \log n}$$

cycles of length n.

For $1 \le i \le m = \lfloor n/6 \rfloor$, consider

$$V_i = \{v_{6i-5}, v_{6i-4}, v_{6i-3}, v_{6i-2}, v_{6i-1}, v_{6i}\}.$$

Denote by R_i the set of vertices $v \in D$ for which $vw \in \mathcal{R}$ for at least 3 different vertices $w \in V_i$. Similarly, let B_i denote the set of vertices $v \in D$ for which $vw \in \mathcal{B}$ for at least 4 different vertices $w \in V_i$. This way we obtain a partition of D for every possible value of i, thus

$$|R_i| + |B_i| = |D| = n - 1.$$

Denote by r the number of i's for which $|R_i| \ge \frac{n}{4} - 1$, and by b the number of i's for which $|B_i| \ge \frac{3n}{4}$. Clearly $r + b = m$. If r is sufficiently large, then it is possible to construct a lot of red cycles of length n:

Lemma 4. *If $r \ge \lfloor n/24 \rfloor - 2$, then $\mathcal{R}$ contains at least*

$$\left(\left\lfloor \frac{n}{24} \right\rfloor - 5\right)! > e^{(\frac{1}{24} - o(1)) n \log n}$$

different cycles of length n.

On the other hand, if neither of the previous two lemmas can be applied, then it is possible to construct many blue cycles of length n:

Lemma 5. *Suppose that D does not contain a red clique of order larger than $n/2$. If $b \geq \lfloor n/8 \rfloor + 2$, then $\mathcal{B}$ contains at least*

$$\left(\left\lfloor \frac{n}{8} \right\rfloor + 1\right)! > e^{(\frac{1}{8} - o(1)) n \log n}$$

different cycles of length n.

This completes the proof of Theorem 1. To prove Theorem 2 we once again start with Lemma 3, but instead of the 6-element V_i's we have to introduce several families of pairwise disjoint 3-element subsets of C. For $1 \leq i \leq m = \lfloor n/6 \rfloor$ and $0 \leq \sigma \leq 5$, consider (indices meant modulo n)

$$V_{i,\sigma} = \{v_{6i-5+\sigma}, v_{6i-4+\sigma}, v_{6i-3+\sigma}, v_{6i-2+\sigma}, v_{6i-1+\sigma}, v_{6i+\sigma}\}.$$

Partition each $V_{i,\sigma}$ in two different ways into 3-element subsets as follows. Let $V_{i,\sigma} = U_{i,\sigma,1} \cup U_{i,\sigma,2} = W_{i,\sigma,1} \cup W_{i,\sigma,2}$, where

$$U_{i,\sigma,1} = \{v_{6i-5+\sigma}, v_{6i-4+\sigma}, v_{6i-3+\sigma}\}, \quad U_{i,\sigma,2} = \{v_{6i-2+\sigma}, v_{6i-1+\sigma}, v_{6i+\sigma}\},$$

$$W_{i,\sigma,1} = \{v_{6i-5+\sigma}, v_{6i-3+\sigma}, v_{6i-1+\sigma}\}, \quad W_{i,\sigma,2} = \{v_{6i-4+\sigma}, v_{6i-2+\sigma}, v_{6i+\sigma}\}.$$

Finally define the families

$$\mathcal{V}_\sigma = \{V_{i,\sigma} \mid 1 \leq i \leq m\}, \quad \mathcal{U}_\sigma = \{U_{i,\sigma,\tau} \mid 1 \leq i \leq m,\ 1 \leq \tau \leq 2\},$$

$$\mathcal{W}_\sigma = \{W_{i,\sigma,\tau} \mid 1 \leq i \leq m,\ 1 \leq \tau \leq 2\}.$$

For each $X \in (\mathcal{U}_1 \cup \mathcal{W}_1) \cup \ldots \cup (\mathcal{U}_6 \cup \mathcal{W}_6)$, denote by R_X the set of vertices $v \in D$ for which $vw \in \mathcal{R}$ for at least 2 different vertices $w \in X$. Similarly, let B_X denote the set of vertices $v \in D$ for which $vw \in \mathcal{B}$ for at least 2 different vertices $w \in X$. Denote by $r(\mathcal{U}_\sigma)$ the number of such sets $U \in \mathcal{U}_\sigma$ for which $|R_U| \geq \frac{n}{4} - 1$, and by $b(\mathcal{U}_\sigma)$ the number of such sets $U \in \mathcal{U}_\sigma$ for which $|B_U| \geq \frac{3n}{4}$. Define also $r(\mathcal{W}_\sigma)$ and $b(\mathcal{W}_\sigma)$ in a similar way. Clearly $r(\mathcal{U}_\sigma) + b(\mathcal{U}_\sigma) = r(\mathcal{W}_\sigma) + b(\mathcal{W}_\sigma) = 2m$ holds for every $0 \leq \sigma \leq 5$.

Now the proof of Theorem 2 can be completed juxtaposing the following analogues of Lemmas 5 and 4, respectively.

Lemma 6. *Suppose that D does not contain a red clique of order larger than $n/2$. If $b(\mathcal{U}_\sigma) \geq \lfloor n/4 \rfloor + 1$ or $b(\mathcal{W}_\sigma) \geq \lfloor n/4 \rfloor + 1$ holds for some $\sigma \in \{0, \ldots, 5\}$, then $\mathcal{B}$ contains at least*

$$\left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right)! > e^{(\frac{1}{4} - o(1)) n \log n}$$

different cycles of length n.

Lemma 7. *If $r(\mathcal{U}_\sigma) \geq \lfloor n/12 \rfloor - 1$ and $r(\mathcal{W}_\sigma) \geq \lfloor n/12 \rfloor - 1$ holds for every $\sigma \in \{0, \dots, 5\}$, then $\mathcal{R}$ contains at least*

$$\left(\left\lfloor \frac{n}{20} \right\rfloor - 10 \right)! > e^{(\frac{1}{20} - o(1)) n \log n}$$

different cycles of length n.

We organize the content of this paper as follows. In Section 3 we prove Lemma 3. For the construction of the blue cycles we need a simple observation on the existence of certain alternating paths in bipartite graphs that we present in Section 4. This is applied to the constructions of the blue cycles in Section 5; here we first prove the simpler Lemma 6 and then indicate what modifications are necessary to obtain Lemma 5. Section 6 contains the constructions of the red cycles; we start with the relatively simple proof of Lemma 4 and then prove Lemma 7. We conclude the paper with a remark concerning even cycles. Finally we must mention that some of our constructions and estimates do not work for very small values of n. Since our results are asymptotic in nature, we did not make the effort to elaborate on the precise bounds and include them in the statements. Accordingly, throughout the whole paper we tacitly assume that n is large enough, say $n > 1000$.

3. The Simple Cases

To prove Lemma 3, we consider $\frac{n+3}{2}$ consecutive vertices along C in every possible way. That is, we define

$$J_i = \{v_i, v_{i+1}, \dots, v_{i+\frac{n+1}{2}}\}$$

for every $1 \leq i \leq n$. Once again, when necessary, indices are understood modulo n. We distinguish three different cases.

Case 1. For every $1 \leq i \leq n$ there exist two vertex disjoint red edges connecting J_i to K. Let $v_{j(i)}s_{j(i)}$ and $v_{k(i)}s_{k(i)}$ denote two such edges such that $\ell(i) = k(i) - j(i)$ is as large as possible; here the indices are not taken modulo n. Choose and fix an index i for which $\ell = \ell_i$ is maximum, then $\frac{n+3}{4} \leq \ell \leq \frac{n+1}{2}$. Every path of length $\ell - 2$ which connects $s_{j(i)}$ to $s_{k(i)}$ in K can be completed along the path $s_{k(i)}v_{k(i)}v_{k(i)+1} \dots v_{j(i)-1}v_{j(i)}s_{j(i)}$ to a red cycle of length n. Since $\ell \leq |D|$, the number of such paths is

$$(|D| - 2)(|D| - 3)(|D| - \ell + 1) \geq (\ell - 3)! \geq \left(\left\lceil \frac{n}{4} \right\rceil - 3 \right)!.$$

This way we obtain the desired number of different red cycles of length n.

Case 2. There is an index $1 \le i \le n$ such that all red edges connecting J_i to K are incident to the same vertex v and the two-coloured complete graph $G[J_i]$ induced by G on J_i contains two disjoint blue edges; note that one of them necessarily avoids the vertex v. Assume that $v \in J_i$. All edges that connect $J_i \setminus \{v\}$ to K are blue, thus we obtain a complete bipartite graph H isomorphic to $K_{\frac{n+1}{2},\frac{n+1}{2}}$. Moreover, there is a blue edge xy that connects two vertices in $J_i \setminus \{v\}$. Any path of length $n-1$ that connects x to y in H can be completed along the edge xy to a blue cycle of length n. This way we obtain $\left(\frac{n-3}{2}\right)!\left(\frac{n+1}{2}\right)!$ different blue cycles of length n. In the case when $v \in K$, a similar argument shows that there exist at least $\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!$ different blue cycles of length n.

Case 3. There is an index $1 \le i \le n$ such that $G[J_i]$ does not contain two disjoint blue edges. Then either all blue edges in $G[J_i]$ are incident to the same vertex v, or they form a triangle xyz. In the first case denote by v^- and v^+ the neighbours of v along C. One of them may not belong to J_i; in that special case we only consider the other one. Omit the blue edges and merge the vertices v^-, v, v^+ to the one vertex v to obtain a red complete graph on the vertex set $J_i \setminus \{v^-, v^+\}$. In this graph, consider any Hamiltonian path $v_i \dots svt \dots v_{i+\frac{n+1}{2}}$. Replacing the subpath svt by sv^-vv^+t, it can be completed along C to a red cycle of length n. This way we obtain at least $\left(\frac{n+3}{2}-4\right)!$ different red cycles of length n. Now suppose that the blue edges form a triangle xyz. Let $x', y' \in J_i$ be neighbours of x resp. y along C such that the five vertices x, y, z, x', y' are all different, and neither x', nor y' coincide with v_i or $v_{i+\frac{n+1}{2}}$. Omit the blue edges and merge these five vertices to the one vertex z to obtain a red complete graph on the vertex set $J_i \setminus \{x, y, x', y'\}$. It is clear how to complete a Hamiltonian path in this graph that connects v_i to $v_{i+\frac{n+1}{2}}$, with the help of the red path $xx'yy'z$, to a red cycle of length n in order to obtain at least $\left(\frac{n+3}{2}-6\right)!$ different red cycles of length n. Note that it may happen that one of the vertices v_i, $v_{i+\frac{n+1}{2}}$, or even both, have been merged to v; in such cases similar arguments work whose details we leave to the reader.

4. Alternating Paths is Bipartite Graphs

For a triple of positive integers (x, y, z), denote by $\mathcal{G}(x, y, z)$ the family of all bipartite graphs in which the two parts X and Y have cardinalities x and y, respectively, and in which every vertex belonging to X has degree at least z. $\mathcal{G}(x,y,z)$ is not empty if and only if $z \leq y$. Let us denote by $\Phi(x,y,z)$ the following statement: *If a graph $G = G(X,Y;E)$ belongs to $\mathcal{G}(x,y,z)$, then for every vertex $v \in X$ of G there exists a path of length $2(x-1)$ starting at v*; such an alternating path visits every vertex of X and ends in X. The statement $\Phi(x,y,z)$ cannot be valid if either $x = 1$, or $y < x - 1$ or $y \geq 2z$. On the other hand, it follows from the pigeonhole principle that $\Phi(2,y,z)$ is valid for every $1 \leq z \leq y \leq 2z - 1$.

Observation 8. *$\Phi(x,y,z)$ implies $\Phi(x+1,y+1,z+1)$.*

Proof. Consider a bipartite graph $G = G(X,Y;E) \in \mathcal{G}(x+1,y+1,z+1)$. Let v denote any vertex in X. If $y < 2z-1$, then $y+1 < 2(z+1)-1$. The pigeonhole principle implies the existence of a vertex $w \in X$ such that v and w have a common neighbour u in Y. Let G' denote the bipartite graph induced by G on the vertex set $X \cup Y \setminus \{u,v\}$, then $G' \in \mathcal{G}(x,y,z)$. If $\Phi(x,y,z)$ is true, then G' contains a part of length $2(x-1)$ starting at w, which the path vuw extends to a path in G starting at v and of length $2x$. $\square$

In order to prove Lemma 6 we need the following.

Observation 9. *$\Phi(3,3,2)$ is true.*

Proof. Consider a bipartite graph $G = G(X,Y;E) \in \mathcal{G}(3,3,2)$. Let $X = \{v,s,t\}$. If the vertices in X do not have a common neighbour, then G is a cycle of length 6, thus G contains a path of length 4 starting at v. Assume that u is a common neighbour of v,s,t. G contains an edge $vw \neq vu$. If either s or t is incident to w, then by symmetry we may assume $sw \in E$. In this case $vwsut$ is a path of length 4 starting at v. Finally, if $p \notin \{u,w\}$ is a common neighbour of s and t, then G contains the path $vuspt$, which will do . $\square$

A somewhat more complicated case analysis reveals that the statements $\Phi(3,5,3)$ and $\Phi(4,5,3)$ are also true. Putting all this together we arrive at the following observation that we need for the proof of Lemma 5.

Lemma 10. *$\Phi(x,6,4)$ is true for every $2 \leq x \leq 5$.*

5. Finding Many Blue Cycles

To prove Lemma 6, let $\mathcal{X} \in \{\mathcal{U}_\sigma, \mathcal{W}_\sigma\}$ be a family of disjoint 3-element subsets of C satisfying $b(\mathcal{X}) \geq s = \lfloor n/4 \rfloor + 1$. Choose and fix s different 3-element sets $X \in \mathcal{X}$ satisfying $|B_X| \geq \frac{3n}{4}$, and consider any permutation $X_1, X_2, \ldots, X_s$ of them. For any such permutation we are going to construct a different blue cycle of length n.

For simplicity, let us assume first that $n = 4(s-1)+3$. Since B_{X_1} and B_{X_s} are both subsets of D,

$$|B_{X_1} \cap B_{X_s}| \geq |B_{X_1}| + |B_{X_s}| - |D| \geq \frac{n+1}{2}.$$

D does not contain a red clique of size $\frac{n+1}{2}$, therefore there is a blue edge yy_0 such that $y, y_0 \in B_{X_1} \cap B_{X_s}$. We start to build the blue cycle at y_0; it will be completed in the end with the edge yy_0. Put $X_0 = X_s$. Assume that $i \leq s-2$ and that we have already constructed a sequence $Y_0 = \{y, y_0\}, Y_1, \ldots, Y_i$ of pairwise disjoint subsets of D and a blue path $P_i = y_0 a_1 b_1 c_1 y_1 \ldots y_{i-1} a_i b_i c_i y_i$ of length $4i$ with the following properties:

(*i*) $y_j \in Y_j$ for $0 \leq j \leq i$;
(*ii*) $a_j, c_j \in X_j$, $b_j \in Y_j$ for $1 \leq j \leq i$;
(*iii*) $|Y_j| = 2$ for $1 \leq j \leq i$;
(*iv*) $Y_j \subseteq B_{X_j} \cap B_{X_{j+1}}$ for $0 \leq j \leq i$.

This is certainly possible for $i = 0$. The construction can be extended form i to $i+1$ as follows. We have $|B_{X_i} \cap B_{X_{i+1}}| \geq \frac{n+1}{2}$, hence

$$|(B_{X_i} \cap B_{X_{i+1}}) \setminus \bigcup_{j=0}^{i-1} F_j| \geq \frac{n+1}{2} - 2i \geq \frac{n+1}{2} - 2\left(\left\lfloor \frac{n}{4} - 1 \right\rfloor\right) \geq 2.$$

Therefore there exists a set $Y_{i+1} \subset D$ disjoint from $Y_0, Y_1, \ldots, Y_i$, which satisfies (*iii*) and (*iv*) with $j = i+1$. Put $X = Y_{i+1} \cup \{y_i\}$ and $Y = X_{i+1}$, then $|X| = |Y| = 3$. Since $X \subseteq B_Y$, every vertex of X is connected to Y with at least two blue edges. According to Observation 9, there is a blue path $y_i a_{i+1} b_{i+1} c_{i+1} y_{i+1}$ such that $a_{i+1}, c_{i+1} \in Y$, $b_{i+1}, y_{i+1} \in X$. This path extends P_i to a blue path P_{i+1} of length $4(i+1)$ so that (*i*) and (*ii*) also hold with $j = i+1$.

By induction we can construct a sequence $Y_0, Y_1, \ldots, Y_{s-1}$ and a blue path P_{s-1} of length $4(s-1) = n-3$ such that conditions (*i*)–(*iv*) hold with $i = s-1$. Note that y, y_{s-1} are in B_{X_s}, so they have a common neighbour y_s in B_{X_s}. This means that the blue path P_{s-1} can be completed to a blue cycle of length n

along the blue path $y_{s-1}y_syy_0$. Moreover, for any blue cycle obtained by this construction one can easily identify the vertices y_0 and y, and thus reconstruct the whole permutation $X_1, \dots, X_s$. This way we obtain

$$s! = \left(\left\lfloor \frac{n}{4} + 1 \right\rfloor\right)!$$

different blue cycles of length n.

In the case when $n = 4(s-1)+1$, we apply the above construction with a minor modification. Namely, in the last step we find a set Y_{s-1} of cardinality 1 instead of 2, and instead of Observation 9 we use the fact that $\Phi(2,3,2)$ is true to extend P_{s-2} to a blue path $P_{s-1} = P_{s-2}a_{s-1}y_{s-1}$ of length $4(s-1)-2$ with $a_{s-1} \in X_{s-1}$, $y_{s-1} \in Y_{s-1}$. This completes the proof of Lemma 6.

The proof of Lemma 5 is quite similar. Here we work with a family $\mathcal{V} = \{V_1, \dots, V_m\}$ of 6-element subsets of C which satisfy $b(\mathcal{V}) \geq \lfloor n/8 \rfloor + 2$. We write $n = 8(s-1) + 3 - 2q$ where $q \in \{0,1,2,3\}$ and $s = \lfloor n/8 \rfloor + 1$ or $s = \lfloor n/8 \rfloor + 2$. Since

$$\frac{n+1}{2} - 4(s-2) \geq 4 - q,$$

we can construct $Y_0, Y_1, \dots, Y_{s-1}$ with $|Y_i| = 4$ for $1 \leq i \leq s-2$ and $|Y_{s-1}| = 4-q$, so that with the help of Lemma 10 a blue path P_{s-1} of length $8(s-1)-2q$ can be built, which along the blue path $y_{s-1}y_syy_0$ completes to a blue cycle of length n. The details can be left to the reader.

6. Finding Many Red Cycles

Proof of Lemma 4. Without any loss of generality we may suppose that $|R_i| \geq \frac{n}{4} - 1$ holds for $1 \leq i \leq s = \lfloor n/24 \rfloor - 2$. For such an i we call a vertex $u \in R_i$ of *Type A* with respect to V_i if there exist $v_j, v_{j+2} \in V_i$ such that $uv_j, uv_{j+2} \in \mathcal{R}$. The vertex $u \in R_i$ is of *Type B* with respect to V_i if it is not of Type A and there exist $v_j, v_{j+3} \in V_i$ such that $uv_j, uv_{j+3} \in \mathcal{R}$. Finally it is of *Type C*, if it is neither of Type A, nor of Type B, thus there exist $v_j, v_{j+4} \in V_i$ such that $uv_j, uv_{j+4} \in \mathcal{R}$. Note that for Type B and Type C vertices u there also exist $v_l, v_{l+1} \in V_i$ such that $uv_l, uv_{l+1} \in \mathcal{R}$.

We say that V_i is of Type X if the majority of the vertices in R_i are of Type X; in case of ties we may decide either way. Thus, if V_i is of Type X, then there are at least $\lfloor n/12 \rfloor$ vertices in R_i which are of Type X with respect to V_i. Let us denote by n_X the number of indices $i \in \{1, 2, \dots, s\}$ for which V_i is of Type X.

Put $a = n_A$, $b = \lfloor n_B/2 \rfloor$ and $c = \lfloor n_C/3 \rfloor$, then $s - 3 \le a + 2b + 3c \le s$. Once again without any loss of generality we may suppose that $V_1, \ldots, V_a$ are of Type A, $V_{a+1}, \ldots, V_{a+2b}$ are of Type B and $V_{a+2b+1}, \ldots, V_{a+2b+3c}$ are of Type C.

Now we are going to construct

$$N = \left\lfloor \frac{n}{12} \right\rfloor \left(\left\lfloor \frac{n}{12} \right\rfloor - 1 \right) \ldots \left(\left\lfloor \frac{n}{12} \right\rfloor - a - 2b - 3c + 1 \right) \ge (s-3)!$$

different red cycles of length n as follows. There are at least N different ways to select pairwise disjoint vertices $u_1, u_2, \ldots, u_{a+2b+3c} \in D$ so that $u_i \in R_i$ is of Type A (with respect to V_i) for $1 \le i \le a$, is of Type B for $a + 1 \le i \le a + 2b$, and is of Type C for $a + 2b + 1 \le i \le a + 2b + 3c$. For each such selection it will be enough to construct a red cycle of length n which contains the vertices $u_1, u_2, \ldots, u_{a+2b+3c}$ and all whose other vertices belong to C. We do it according to the following rules.

We start with the red cycle C. V_i is of Type A for $1 \le i \le a$. For such an index i, take a red path $P_i = v_j u_i v_{j+2}$ such that $v_j, v_{j+2} \in V_i$, and replace the arc $v_j v_{j+1} v_{j+2}$ of C by P_i. Such a replacement has no effect on the length of the cycle. Next, for $a + 1 \le i \le a + b$, take a red path $P_i = v_j u_i v_{j+3}$ such that $v_j, v_{j+3} \in V_i$, and replace the arc $v_j v_{j+1} v_{j+2} v_{j+3}$ of C by P_i. The length of the cycle is then shortened by b. This effect can be compensated by taking, for every $a + b + 1 \le i \le a + 2b$, a red path $P_i = v_j u_i v_{j+1}$ such that $v_j, v_{j+1} \in V_i$, and replacing the edge $v_j v_{j+1}$ of C by P_i. Finally, we do the same for $a+2b+1 \le i \le a+2b+2c$, thus making the cycle $2c$ longer, and compensate this effect by taking, for every $a + 2b + 2c + 1 \le i \le a + 2b + 3c$, a red path $P_i = v_j u_i v_{j+4}$ such that $v_j, v_{j+4} \in V_i$, and replacing the arc $v_j v_{j+1} v_{j+2} v_{j+3} v_{j+4}$ of C by P_i.

It is clear that the (at least) N red cycles obtained this way are all different. This completes the proof of Lemma 4.

The proof of Lemma 7 is based on a similar basic idea. Put $N = \left\lfloor \frac{n}{20} \right\rfloor - 10$. For a path $P_i = v_{j_i} v_{j_i+1} \ldots v_{j_i+\ell_i}$ denote by $U(P_i)$ the set of vertices $u \in D$ for which $uv_{j_i}, uv_{j_i+\ell_i} \in \mathcal{R}$. If we find $N' \ge N$ pairwise edge-disjoint paths $P_1, \ldots, P_{N'}$ of total length

$$\ell_1 + \ell_2 + \ldots + \ell_{N'} = 2N'$$

such that $|U(P_i)| \ge N'$ holds for every $i \in \{1, 2, \ldots, N'\}$, then for any selection of pairwise disjoint vertices $u_1 \in U(P_1), \ldots, u_{N'} \in U(P_{N'})$ we can construct a red cycle of length n replacing in the cycle C the path P_i by the path $v_{j_i} u_i v_{j_i+\ell_i}$

for every $i \in \{1, 2, \ldots, N'\}$, thus obtaining at least $N'!$ different red cycles of length n. The crucial part of the proof will be the construction of the suitable paths $P_1, \ldots, P_{N'}$.

Proof of Lemma 7. A set $Q = \{v_j, v_{j+\ell}\}$ is called an *ℓ-replacer* if there exist at least $\lfloor n/12 \rfloor$ different vertices $u \in D$ such that $uv_j, uv_{j+\ell} \in \mathcal{R}$. The *support* of such a replacer is the path $p(Q) = v_j v_{j+1} \ldots v_{j+\ell}$. Two replacers are *compatible* if their supports do not share a common edge. It will be enough to find 2-replacers $Q_1, \ldots, Q_x$, 4-replacers $Q_{x+1}, \ldots, Q_{x+y}$ and 1-replacers $Q_{x+y+1}, \ldots, Q_{x+3y}$ such that they are pairwise compatible and $N \le x + 3y \le \lfloor n/12 \rfloor$.

First we consider 2-replacers. If a pair of 2-replacers is not compatible, we call them *interlacing*. A set $V_{i,\sigma}$ is called compatible, if it contains two compatible 2-replacers. Thus, any $V_{i,\sigma}$ which contains at least three different 2-replacers is compatible. If $V_{i,\sigma}$ contains only two 2-replacers and they are not compatible, we say that $V_{i,\sigma}$ is interlacing. Denote by M the maximum number of pairwise compatible 2-replacers. Clearly there cannot be more than $2M - 1$ pairs of interlacing 2-replacers, but we need something stronger.

Lemma 11. *There exists a $\sigma \in \{0, \ldots, 5\}$ such that*

$$|\{i \mid 1 \le i \le m,\ V_{i,\sigma} \textit{ is interlacing}\}| \le \frac{M}{2}.$$

Proof. Consider a maximal chain of 2-replacers

$$\{v_j, v_{j+2}\}, \{v_{j+1}, v_{j+3}\}, \ldots, \{v_{j+k-1}, v_{j+k+1}\};$$

the length of such a chain is k, and it contains $k-1$ interlacing pairs. If $k \ge 3$, then there are (at most) two interlacing sets $V_{i,\sigma}$, namely

$$\{v_{j-2}, \ldots, v_{j+3}\} \quad \text{and} \quad \{v_{j+k-2}, \ldots, v_{j+k+3}\},$$

which contain one (and thus only one) interlacing pair out of these. If $k = 2$, then the number of such interlacing sets is (at most) three.

Now partition the 2-replacers into maximal chains. Let α denote the number of such chains, and let $k_1, \ldots, k_\alpha$ denote their lengths. Then the number I of interlacing sets $V_{i,\sigma}$ satisfies

$$I \le \frac{3k_1}{2} + \frac{3k_2}{2} + \ldots + \frac{3k_\alpha}{2}.$$

On the other hand, the maximum number of pairwise compatible 2-replacers is

$$M = \left\lceil \frac{k_1}{2} \right\rceil + \left\lceil \frac{k_2}{2} \right\rceil + \ldots + \left\lceil \frac{k_\alpha}{2} \right\rceil.$$

It follows that $I \leq 3M$. The statement follows noting that the interlacing sets are distributed among the 6 families $\mathcal{V}_\sigma$. $\square$

If there are N pairwise compatible 2-replacers, then the conclusion of Lemma 7 is true. Accordingly, for the rest of the proof we may assume that $M < N$. In view of Lemma 11, there is a $\sigma \in \{0, \ldots, 5\}$ such that

$$|\{i \mid 1 \leq i \leq m,\ V_{i,\sigma} \textit{ is interlacing}\}| \leq \frac{N-1}{2}.$$

Without any loss of generality we may suppose that $\sigma = 0$. Thus, $V_{i,\sigma} = V_i$. It will be convenient to write $\mathcal{U} = \mathcal{U}_\sigma$, $\mathcal{V} = \mathcal{V}_\sigma$ and $\mathcal{W} = \mathcal{W}_\sigma$. Denote by $\mathcal{A}$ and $\mathcal{B}$ the family of compatible resp. interlacing V_i's, and put $a = |\mathcal{A}|$, $b = |\mathcal{B}|$. If $2a + 2b \geq \lfloor n/12 \rfloor - 1$, then we find

$$2a + b = 2(a+b) - b \geq \left\lfloor \frac{n}{12} \right\rfloor - 1 - \frac{N-1}{2} > N$$

pairwise compatible 2-replacers, and we are done. Thus, we may assume that

$$2a + 2b < \left\lfloor \frac{n}{12} \right\rfloor - 1 \leq \min\{r(\mathcal{U}), r(\mathcal{W})\}.$$

Let $\mathcal{V}_4$ denote the family of such sets $V \in \mathcal{V} \setminus (\mathcal{A} \cup \mathcal{B})$ which contain a 4-replacer, and denote by $\mathcal{V}_2$ the family of such sets $V \in \mathcal{V} \setminus (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$ which contain a (unique) 2-replacer. Note that if $|R_W| \geq \frac{n}{4} - 1$ holds for some $W \in \mathcal{W}$, then W necessarily contains either a 4-replacer, or a 2-replacer. Since $r(\mathcal{W}) \geq \lfloor n/12 \rfloor - 1$, this way we find at least $\lfloor n/12 \rfloor - 1$ different 2- and 4-replacers such that each of them is contained in some V_i, and each V_i contains at most two of them. It follows that there exist nonnegative integers c, d and families $\mathcal{C} \subseteq \mathcal{V}_4$, $\mathcal{D} \subseteq \mathcal{V}_2$ such that $|\mathcal{C}| = c$, $|\mathcal{D}| = d$ and

$$\left\lfloor \frac{n}{12} \right\rfloor - 2 \leq 2a + 2b + 2c + d \leq \left\lfloor \frac{n}{12} \right\rfloor - 1.$$

Now we turn our attention to the family $\mathcal{U}$. If $|R_U| \geq \frac{n}{4} - 1$ holds for some $U \in \mathcal{U}$, then U necessarily contains either a 1-replacer, or a 2-replacer. Since $r(\mathcal{W}) \geq \lfloor n/12 \rfloor - 1$, there exists a family $\mathcal{Q}$ of $\lfloor n/12 \rfloor - 1 - 2a - 2b \geq 2c + d$ pairwise compatible 1- and 2-replacers such that each of them is contained in some element of $\mathcal{U}$, but none of them is contained in an element of $\mathcal{A} \cup \mathcal{B}$. Note that no element of $\mathcal{C}$ can contain two different 2-replacers from $\mathcal{Q}$. Accordingly, we can partition $\mathcal{C}$ as $\mathcal{C}_{20} \cup \mathcal{C}_{11} \cup \mathcal{C}_{10} \cup \mathcal{C}_{01} \cup \mathcal{C}_{00}$, where $\mathcal{C}_{\mu\nu}$ denotes the family of sets $V \in \mathcal{C}$ which contain exactly μ 1-replacers and ν 2-replacers from $\mathcal{Q}$. Put $c_{\mu\nu} = |\mathcal{C}_{\mu\nu}|$, then $c = c_{20} + c_{11} + c_{10} + c_{01} + c_{00}$. Denote by $\mathcal{Q}^*$ the set of those elements of $\mathcal{Q}$ which are not contained in any element of $\mathcal{C}$. Thus,

$$|\mathcal{Q}^*| = |\mathcal{Q}| - 2c_{20} - 2c_{11} - c_{10} - c_{01} \geq c_{10} + c_{01} + 2c_{00} + d.$$

Claim 12. *There exists two disjoint families $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of replacers with the following properties:*

(i) *every element of $\mathcal{Q}_1$ is a 1- or 2-replacer;*
(ii) *every element of $\mathcal{Q}_2$ is a 2-replacer;*
(iii) *the elements of $\mathcal{Q}_1 \cup \mathcal{Q}_2$ are pairwise compatible and none of them is contained in an element of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$;*
(iv) $|\mathcal{Q}_1| = c_{10} + c_{01} + 2c_{00}$*;*
(v) $|\mathcal{Q}_2| = d$.

Proof. Each element of $\mathcal{D}$ contains a unique 2-replacer. These are pairwise compatible and none of them is contained in an element of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. They form a family $\mathcal{Q}_2$ which comply with (ii) and (v). The family $\mathcal{Q}_1$ we obtain from $\mathcal{Q}^*$ after omitting elements which interfere with $\mathcal{Q}_2$; then (i) and (iii) will be also guaranteed. First, omit form $\mathcal{Q}^*$ all 2-replacers which are already contained in $\mathcal{Q}_2$. Note that these elements of Q_2 are compatible with all elements in $\mathcal{Q}^*$. Because each element of $\mathcal{D}$ contains a unique 2-replacer, all other elements of $\mathcal{Q}_2$ are compatible with the 2-replacers in $\mathcal{Q}^*$. Such elements of $\mathcal{Q}_2$ may be incompatible with 1-replacers in $\mathcal{Q}^*$, but each of them is incompatible with at most one. After omitting those incompatible 1-replacers from $\mathcal{Q}^*$ as well, the remaining family will have at least $c_{10}+c_{01}+2c_{00}$ elements, for we have omitted at most d elements of $\mathcal{Q}^*$. Therefore it contains a subfamily $\mathcal{Q}_1$, which together with $\mathcal{Q}_2$ satisfies all the requirements. $\square$

Now we are in the position to construct the replacers Q_i which comply with the requirements set up at the beginning of the proof. First, choose two compatible 2-replacers from each element of $\mathcal{A}$ and one 2-replacer from each element of $\mathcal{B}$. Next, select $\lfloor c_{20}/2 \rfloor$ elements of $\mathcal{C}_{20}$ and choose a 4-replacer from each of them. Compensate this by choosing two 1-replacers from each of $\lfloor c_{20}/2 \rfloor$ remaining elements of $\mathcal{C}_{20}$. Similarly, select $\lfloor c_{11}/3 \rfloor$ elements of $\mathcal{C}_{11}$ and choose a 4-replacer from each of them. This we compensate by choosing one 1-replacer and one 2-replacer from each of $2\lfloor c_{11}/3 \rfloor$ remaining elements of $\mathcal{C}_{11}$.

Turning to the sophisticated part, denote by α the largest integer $\omega \le \lfloor c_{10}/2 \rfloor$ such that there exist ω different 1-replacers in $\mathcal{Q}_1$. Choose a 4-replacer from each of α different elements of $\mathcal{C}_{10}$ and compensate by choosing a 1-replacer from α other elements of $\mathcal{C}_{10}$ plus choosing α 1-replacers from $\mathcal{Q}_1$. In case $\alpha < \lfloor c_{10}/2 \rfloor$ put $\beta = \lfloor (c_{10} - 2\alpha)/3 \rfloor$, choose a 4-replacer from each of β still unused elements of $\mathcal{C}_{10}$ and a 1-replacer from each of 2β other elements of $\mathcal{C}_{10}$. In addition, choose 3β 2-replacers from $\mathcal{Q}_1$; it is possible because $|\mathcal{Q}_1| \ge c_{10} \ge \alpha + 3\beta$.

Next, denote by γ the largest integer $\omega \le c_{01}$ such that there still exist ω different 2-replacers in $\mathcal{Q}_1$. Choose a 2-replacer from each of γ different elements of $\mathcal{C}_{01}$ and in addition choose γ 2-replacers from $\mathcal{Q}_1$. In case $\gamma < c_{01}$ put $\delta = \lfloor (c_{10} - \gamma)/2 \rfloor$, choose a 4-replacer from each of δ still unused elements of $\mathcal{C}_{10}$ and a 2-replacer from each of δ other elements of $\mathcal{C}_{10}$, then compensate by choosing 2δ 1-replacers from $\mathcal{Q}_1$. It is again possible, for $|\mathcal{Q}_1| \ge c_{10} + c_{01} \ge \alpha + 3\beta + \gamma + 2\delta$.

To handle $\mathcal{C}_{00}$, denote by ε the largest integer $\omega \le c_{00}$ such that there still exist 2ω different 1-replacers in $\mathcal{Q}_1$. Choose a 4-replacer from each of ε elements of $\mathcal{C}_{00}$ and compensate by choosing 2ε 1-replacers from $\mathcal{Q}_1$. Since $|\mathcal{Q}_1| = c_{10} + c_{01} + 2c_{00}$, there are still at least $2(c_{00} - \varepsilon)$ remaining elements from $\mathcal{Q}_1$, of which at most one can be a 1-replacer. It follows that we can still add to the list $2c_{00} - 2\varepsilon - 1$ different 2-replacers from $\mathcal{Q}_1$. Finally, complete the list of 2-replacers with the d elements of $\mathcal{Q}_2$.

It is clear that the replacers selected by the above procedure are pairwise compatible. The number of selected 2-replacers is

$$x = 2a + b + 2\left\lfloor \frac{c_{11}}{3} \right\rfloor + 3\beta + 2\gamma + \delta + (2c_{00} - 2\varepsilon - 1) + d.$$

The number of selected 4-replacers is

$$y = \left\lfloor \frac{c_{20}}{2} \right\rfloor + \left\lfloor \frac{c_{11}}{3} \right\rfloor + \alpha + \beta + \delta + \varepsilon,$$

whereas the number of 1-replacers we selected is $2y$. Doing the arithmetic we obtain that the number of selected replacers is

$$x + 3y = 2a + b + 3\left\lfloor \frac{c_{20}}{2} \right\rfloor + 5\left\lfloor \frac{c_{11}}{3} \right\rfloor + 3\alpha + 6\beta + 2\gamma + 4\delta + 2c_{00} + \varepsilon - 1 + d.$$

In view of the inequalities $2\alpha + 3\beta \ge c_{10} - 2$ and $\gamma + 2\delta \ge c_{01} - 1$ we get

$$\begin{aligned} x+3y &\geq 2a+b+\frac{3}{2}c+d-10 \\ &\geq \frac{3}{4}(2a+2b+2c+d)-\frac{b}{2}-10 \\ &\geq \frac{3}{4}\left(\left\lfloor\frac{n}{12}\right\rfloor-2\right)-\frac{N-1}{4}-10 \\ &\geq N. \end{aligned}$$

Should the value of $x+3y$ exceed $\lfloor n/12 \rfloor$, we can reduce the value by omitting some 2-replacers and, if necessary, some additional triples, each consisting of a 4-replacer and two 1-replacers, to achieve $N \leq x+3y \leq \lfloor n/12 \rfloor$. This completes the proof of Lemma 7.

7. A Concluding Remark Concerning Even Cycles

Finding a good lower bound on the Ramsey multiplicity of the even cycles is apparently more difficult, because the Ramsey number is much smaller: $r(C_n) = 3n/2-1$ for n even, see [4, 8, 10]. With the technique presented here nevertheless it is possible to prove for example the following.

Theorem 13. *For even integers $n > 2$,*

$$R(C_n, 2n-1) > e^{(\frac{1}{20}-o(1))n \log n}.$$

The proof is almost literally the same as that of Theorem 2. The only difference is the following. The counterpart of Lemma 6 is valid without the assumption that D does not contain a red clique of order larger than $n/2$. This is because in this case we do not need the blue edge yy_0 for the construction of the blue cycles, it is enough to guarantee that y_0 is a common vertex in B_{X_1} and B_{X_s}, and to complete the blue cycle with the path $y_{s-1}y_sy_0$. Therefore we do not even have to discuss simple cases as in Section 3 separately; there is no need for a counterpart of Lemma 3 is the even case.

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