ON THE SPECTRAL ESTIMATES FOR THE SCHRÖDINGER-LIKE OPERATORS: THE CASE OF SMALL LOCAL DIMENSION

G. ROZENBLUM AND MICHAEL SOLOMYAK

Abstract. The behavior of the discrete spectrum of the Schrödinger operator $-\Delta - V$, in quite a general setting, up to a large extent is determined by the behavior of the corresponding heat kernel $P(t; x, y)$ as $t \to 0$ and $t \to \infty$. If this behavior is powerlike, i.e.,

$$\|P(t;\cdot,\cdot)\|_{L^\infty} = O(t^{-\delta}),\ t \to 0; \qquad \|P(t;\cdot,\cdot)\|_{L^\infty} = O(t^{-D}),\ t \to \infty,$$

then it is natural to call the exponents δ, D "*the local dimension*" and "*the dimension at infinity*" respectively. The character of spectral estimates depends on the relations between these dimensions. In the paper we analyze the case where $\delta < D$ that was insufficiently studied before. Our applications concern the combinatorial and the metric graphs.

1. Introduction

One of the most influential papers by M.Sh.Birman has been [1] (1961). Since then, the approach developed there is, under the name 'The Birman-Schwinger Principle', the source of inspiration and one of the main tools in the study of the spectral distribution for Schrödinger-like operators.

In the paper [11] this tool was applied to obtaining eigenvalue estimates for such operators in a very abstract setting, and it turned out that these estimates depend essentially on two numerical characteristics of the operator, δ and D, that can be called the local dimension and the dimension at infinity. In the case of the standard Schrödinger operator on $\mathbb{R}^d$, these characteristics coincide with the dimension; generally they differ.

In the survey paper [12] the case $\delta \neq D$ had been discussed, with the main attention given to the effects appearing when $\delta > D$. The simplest example of the situation where $\delta < D$ is given by the lattice $\mathbb{Z}^d$ (i.e., by the discrete Schrödinger operator); here $D = d$ and $\delta = 0$. For this case some peculiarities in the spectral distribution were discovered in [13]. In particular, it turned out that large coupling

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constant eigenvalue estimates, that are order sharp in $\mathbb{R}^d$, are not sharp in $\mathbb{Z}^d$ any more. Deeper deliberations on the effects found in [13] lead the authors to understanding that these peculiarities are common to all situations when $\delta < D$. In the present paper we consider the cases of rather general combinatorial and quantum (metric) graphs, where $\delta = 0$ or $\delta = 1$; we restrict ourselves to the situation where $D > 2$. We find a range of spectral estimates for Schrödinger-like operators on such graphs. For quantum graphs, in particular, it turns out that such estimates are determined by the corresponding estimates on the associated combinatorial graph, which looks rather unexpected since the quantum graph contains much more 'flesh'. We find also criteria for the Birman-Schwinder operator to belong to different Schatten, or "weak" Schatten classes, and conditions for the validity of the Weyl type eigenvalue asymptotics.

2. General setting

Let (X, σ) be a measure space with sigma-finite measure. We denote $L^q(X) = L^q(X, \sigma)$ and $\|\cdot\|_q = \|\cdot\|_{L^q(X)}$. Often we drop the symbol X in these notations. Let $\mathbf{A}$ be a non-negative self-adjoint operator in $L^2(X)$ and $\mathbf{P}(t) = \exp(-\mathbf{A}t)$ be the corresponding semigroup. We suppose that for any $t > 0$ the operator $\mathbf{P}(t)$ is positivity preserving and is bounded as acting from L^1 to L^∞. It is well known that under these assumptions $\mathbf{P}(t)$ can be represented as an integral operator whose kernel $P(t; x, y)$ (heat kernel) is well-defined as a function in $L^\infty(\mathbb{R}_+ \times X \times X)$. We denote

$$M_{\mathbf{A}}(t) = \|P(t; \cdot, \cdot)\|_{L^\infty(X \times X)}. \tag{2.1}$$

The above described class of operators $\mathbf{A}$ (we use the notation $\mathcal{P}$ for it) includes the Laplacian, both in its continuous and discrete versions, and also many other important examples; see e.g. [11, 8]. For simplicity, we use below the term "Laplacian" for any operator $\mathbf{A} \in \mathcal{P}$. We denote by $\mathbf{a}[u]$ the quadratic form of $\mathbf{A}$, and by $H_{\mathbf{a}}$ the domain of $\mathbf{a}$, completed with respect to the "$\mathbf{a}$-norm" $\|u\|_{\mathbf{a}} = \sqrt{\mathbf{a}[u]}$.

Let $V \geq 0$ be a measurable function on X. Under some additional assumptions the operator $\mathbf{A} - V$ can be well-defined (via its quadratic form $\mathbf{a}[u] - \int_X V|u|^2 d\sigma$) and is a self-adjoint operator, whose negative spectrum is finite, that is, it consists of a finite number of eigenvalues, each of finite multiplicity. Below $N_-(\mathbf{A} - V)$ stands for the total multiplicity of the negative spectrum. It is proved in [11] that the number $N_-(\mathbf{A} - V)$ can be conveniently estimated in terms of the function $M_{\mathbf{A}}(t)$. This estimate is an abstract version of Lieb's approach to the proof of the Rozenblum – Lieb – Cwikel (RLC) estimate.

The function $M_{\mathbf{A}}(t)$ is non-increasing. Its main characteristics are the behavior as $t \to 0$ and as $t \to \infty$. In particular, suppose that there are two non-negative exponents δ, D, such that

$$M_{\mathbf{A}}(t) = O(t^{-\delta}),\ t \to 0; \qquad M_{\mathbf{A}}(t) = O(t^{-D/2}),\ t \to \infty. \tag{2.2}$$

The estimates for $N_-(\mathbf{A} - V)$ depend on which of these exponents is greater than the other. In particular, the following version of RLC estimate is valid (see, e.g., [12], Remark 1 in section 3.2). We write it down for the operator with a large parameter $\alpha > 0$ (the coupling constant) incorporated.

Theorem 2.1. *Suppose that $D \geq \delta$, $D > 2$. Then for any $V \in L^{D/2}(X)$ and any $\alpha > 0$ the operator $\mathbf{A} - \alpha V$ is well-defined, its negative spectrum is finite, and the following estimate is satisfied:*

$$N_-(\mathbf{A} - \alpha V) \leq C\alpha^{D/2} \int_X V^{D/2} d\sigma, \qquad C = C(X, D). \tag{2.3}$$

It is convenient to formulate the estimates of this type in terms of the corresponding Birman – Schwinger operator $\mathbf{B}_V$. Let us recall (see, e.g., [1, 4] that $\mathbf{B}_V$ is the operator in the space $H_{\mathbf{a}}$, generated by the quadratic form

$$\mathbf{b}_V[u] = \int_X V|u|^2 d\sigma. \tag{2.4}$$

This operator is well-defined, since due to the assumption $D > 2$ the space $H_{\mathbf{a}}$ can be realized as a space of functions on X. The Rayleigh quotient for $\mathbf{B}_V$ is

$$\mathbf{b}_V[u]/\mathbf{a}[u], \qquad u \in H_{\mathbf{a}};$$

its eigenvalue counting function is denoted by $n(s, \mathbf{B}_V)$.

The next result is an equivalent reformulation of Theorem 2.1.

Theorem 2.2. *Under the assumptions of Theorem 2.1 we have $\mathbf{B}_V \in \Sigma_{D/2}$, and*

$$\|\mathbf{B}_V\|_{\Sigma_{D/2}} \leq C\|V\|_{D/2}. \tag{2.5}$$

We recall that Σ_p stands for the class of all compact operators with the powerlike behavior of the s-numbers, $s_n(T) = O(n^{-1/p})$, see [3], §11.6. The similar classes with o in place of O are denoted by $\Sigma_p^{(0)}$. We use below also the standard Neumann – Schatten classes $\mathfrak{S}_p, 0 < p \leq \infty$.

Suppose now that $\delta < D$. Then (2.2) implies $M_{\mathbf{A}}(t) = O(t^{-q}),\ t \to 0, \infty$, with any $q \in (\delta/2, D/2)$. Respectively, an estimate similar to

(2.5) but with any such exponent q instead of $D/2$ (and with a constant depending on q) is also valid:

$$\|\mathbf{B}_V\|_{\Sigma_q} \le C\|V\|_q, \qquad \max(\delta/2, 1) < q \le D/2. \tag{2.6}$$

In the case of the Euclidean Laplacian on $\mathbb{R}^d$, $d \ge 3$ (here $D = \delta = d$) the estimate (2.5) is known to be sharp, in the sense that for $V \not\equiv 0$ the operator $\mathbf{B}_V$ cannot belong to any class, narrower than $\Sigma_{d/2}$. It was shown in [13] that for the lattice Laplacian the situation is different: $V \in \ell^{d/2}(\mathbb{Z}^d)$, $d \ge 3$, yields $\mathbf{B}_V \in \Sigma^{(0)}_{d/2}$ (or, in other terms, $N_-(\mathbf{A} - \alpha V) = o(\alpha^{d/2})$). Our next result shows that the similar fact takes place in the general case, provided that $D > \delta$.

Theorem 2.3. *Suppose that in the assumptions of Theorem 2.1 we have $D > \delta$. Then $\mathbf{B}_V \in \Sigma^{(0)}_{D/2}$, or, equivalently,*

$$N_-(\mathbf{A} - \alpha V) = o(\alpha^{D/2}), \qquad \forall V \in L^{D/2}.$$

Proof. Fix a number $q \in (\max(\delta/2, 1), D/2)$. Functions $V \in L^{D/2} \cap L^{\delta/2}$ belong to L^q and form a dense subset in $L^{D/2}$. For such functions V the result of Theorem 2.2 applies with $2q$ instead of D, and gives $B_V \in \Sigma_q \subset \Sigma^{(0)}_{D/2}$. By continuity (see [3], Theorem 11.6.7), this inclusion carries over to any $V \in L^{D/2}$. $\square$

The following result shows that the estimate (2.6) with any $q < D/2$ is not sharp, in the sense that the class of admissible potentials can be considerably widened, and the operators $\mathbf{B}_V$ for $V \in L^q$ actually belong to much a better class than Σ_q. Below L^q_w stands for the weak L^q-space.

Theorem 2.4. *Suppose that in the assumptions of Theorem 2.1 we have $D > \delta$. Then for any $q \in (\max(\delta/2, 1), D/2)$*

$$V \in L^q_w \Longrightarrow \mathbf{B}_V \in \Sigma_q; \qquad V \in L^q \Longrightarrow \mathbf{B}_V \in \mathfrak{S}_q,$$

with the corresponding inequalities for the norms. In particular,

$$N_-(\mathbf{A} - \alpha V) \le C\alpha^q \|V\|^q_{L^q_w} \tag{2.7}$$

and $N_-(\mathbf{A} - \alpha V) = o(\alpha^q)$ for $V \in L^q$.

The result follows from (2.6) by the real interpolation method.

3. Combinatorial graphs

In the rest of the paper we consider the problems of the above type, concerning graphs. In this section combinatorial graphs (notation G), and in the next section metric graphs (notation Γ) are discussed. In both cases we assume that the graph is connected, has an infinite number of vertices, and has no loops, dead ends, or multiple edges. We denote the set of edges by $\mathcal{V}$ and the set of vertices by $\mathcal{E}$. The notation $v \sim v'$ means that the vertices v, v' are connected by an edge that we sometimes denote by (v, v'). We suppose that the degrees of the vertices are uniformly bounded,:

$$deg(v) = \#\{v' \in \mathcal{V}, v' \sim v\} \leq \mathbf{d}. \tag{3.1}$$

With each edge $e \in \mathcal{E}$ we associate a positive weight $g(e)$. We need such "weighted graphs" when dealing with the metric graphs in the next sections.

On the set $\mathcal{V}$ we consider the counting measure: $\sigma(v) = 1$ for any $v \in \mathcal{V}$. The space $\ell^2(\mathcal{V}) = L^2(\mathcal{V}, \sigma)$ is our basic Hilbert space in this section. The quadratic form

$$Q_G[f] = \sum_{e \in \mathcal{E}; e=(v,v')} g(e)|f(v) - f(v')|^2, \tag{3.2}$$

with the natural form domain, defines in $\ell^2(\mathcal{V})$ a nonnegative self-adjoint operator, $\mathbf{A} = \mathbf{A}_G = -\Delta_G$. Due to the inclusions $\ell^1(\mathcal{V}) \subset \ell^2(\mathcal{V}) \subset \ell^\infty(\mathcal{V})$, the operators $\exp(-\mathbf{A}t)$ are bounded as acting from ℓ^1 to ℓ^∞, so that

$$M_{\mathbf{A}}(t) \leq C. \tag{3.3}$$

This means that $\delta = 0$. Our main assumption is that

$$M_{\mathbf{A}}(t) = O(t^{-D/2}),\ t \to \infty, \qquad \text{with some } D > 2. \tag{3.4}$$

By [15], Theorems II.2.7, II.3.1, (3.4) is equivalent to the Sobolev inequality

$$\|f\|^2_{\ell^r} \leq CQ_G[f], \qquad r = r(D) = 2D(D-2)^{-1} \tag{3.5}$$

that holds for all f with finite support. Hence, the closure of the set of all such functions in the metric $Q_G[f]$ is a space of functions, and we denote it by $\mathcal{H}(G)$. In notation of section 2, it plays the role of the space $H_{\mathbf{a}}$. Later on, $\mathcal{H}_{\text{fin}}(G)$ stands for the set of finitely supported functions in $\mathcal{H}(G)$.

There are many geometrical and analytical criteria for the inequality $M(t) = O(t^{-D/2})$ to hold. Not going into details, we refer to [5] ,[7] [14] where such criteria are presented. An example of such graph is the integer lattice $\mathbb{Z}^d$; for each edge e we take $g(e) = 1$. Here $D = d$, which can be checked by direct computation of the heat kernel. This case was the object of our study in the paper [13]. Here we extend some of its results to general graphs.

The following result, in its main part, is just a special case of Theorems 1.2 – 1.4. The only novelty is that the condition $q > 2$ is no more necessary. Note that now $\mathbf{b}_V$ takes the form

$$\mathbf{b}_V[f] = \sum_{v\in\mathcal{V}} V(v)|f(v)|^2. \tag{3.6}$$

Theorem 3.1. *Let $D > 2$.*

(1) *Suppose $V \in \ell^{D/2}(\mathcal{V})$. Then*

$$N_-(\mathbf{A} - \alpha V) \le C\alpha^{D/2} \sum_{v\in\mathcal{V}} V^{D/2}(v), \tag{3.7}$$

or, equivalently, $\|\mathbf{B}_V\|_{\Sigma_{D/2}} \le C\|V\|_{\ell^{D/2}}$, and besides, $\mathbf{B}_V \in \Sigma^{(0)}_{D/2}$.

(2) *Suppose $V \in \ell^q_w(G)$ for some $q \in (0, D/2)$. Then $\mathbf{B}_V \in \Sigma_q$. If $V \in \ell^q(G)$, then $\mathbf{B}_V \in \mathfrak{S}_q$. In both cases, the corresponding estimates for the (quasi-)norm of $\mathbf{B}_V$ are satisfied.*

The proof (we need it only for $q \le 1$) is the same as for the case $G = \mathbb{Z}^d$, see [13], and we skip it.

In contrast to the general situation of section 2, for graphs it is also possible to obtain a lower estimate of $n_+(s, \mathbf{B}_V)$ in terms of the distribution function for V, i.e.,

$$\nu(\tau, V) = \#E(\tau, V);\ E(\tau, V) = \{v \in \mathcal{V} : V(v) > \tau\}, \qquad \tau > 0.$$

This estimate does not require any preliminary assumptions about V. We need, however, an additional assumption about the graph G: the weights g_e should be uniformly bounded:

$$|g_e| \le g_0. \tag{3.8}$$

Theorem 3.2. *Let for a graph G the condition* (3.8) *be satisfied. Then for any $V \ge 0$ the inequality holds:*

$$n(s, \mathbf{B}_V) \ge (\mathbf{d} + 1)^{-1}\nu(g_0(\mathbf{d} + 1)s, V), \tag{3.9}$$

and therefore, for any $q > 0$,

$$\|\mathbf{B}_V\|_{\Sigma_q} \ge c(q)\|V\|_{\ell^{D/2}_w}, \qquad \|\mathbf{B}_V\|_{\mathfrak{S}_q} \ge cq\|V\|_{\ell^{D/2}}, \qquad c(q) > 0. \tag{3.10}$$

Proof. It is well known that the set $\mathcal{V}$ can be split into the union of no more than $\mathbf{d} + 1$ disjoint subsets $\mathcal{V}_j$, so that no vertices belonging to the same subset are connected by an edge. Respectively, for a fixed $\tau > 0$, the set $E(\tau, V) = \{v \in \mathcal{V} : V(v) > \tau\}$ splits into the union of no more than $\mathbf{d} + 1$ disjoint subsets $\Omega_j = E(\tau, V) \cap \mathcal{V}_j$. For at least one of them, say Ω_1, we have $\#\Omega_1 \ge (\mathbf{d} + 1)^{-1}\nu(\tau, V)$. Now, consider the subspace $\mathcal{L} \subset \mathcal{H}(G)$ generated by the functions $f(v) = \delta^{v'}_v,\ v' \in \Omega_1$.

These functions are mutually orthogonal both in the metric (3.2) and with respect to the quadratic form $\mathbf{b}_V$ in (3.6). So, for any $f(v) = \sum_{v'\in\Omega_1} c_{v'}\delta_v^{v'} \in \mathcal{L}$ we have

$$Q_G[f] = \sum_{v\in\Omega_1} |c_v|^2 \sum_{e\ni v} g_e \le g_0(\mathbf{d}+1) \sum_{v\in\Omega_1} |c_v|^2, \tag{3.11}$$

while $\mathbf{b}_V[f] \ge \sum_{v\in\Omega_1} |c_v|^2 V(v)$. So we have constructed the subspace $\mathcal{L}$ of dimension not less than $(\mathbf{d}+1)^{-1}\nu(\tau, V)$, on which

$$\mathbf{b}_V[f] \ge \tau g_0^{-1}(\mathbf{d}+1)^{-1} Q_G[f].$$

This implies (3.9) by the variational principle. The estimates (3.10) follow in a standard way. □

4. Metric graphs: upper estimates

Each edge e of a metric graph Γ is considered as a line segment of the length $l_e > 0$. With Γ we associate the combinatorial graph $G = G(\Gamma)$, with the same set of vertices $\mathcal{V}$, the same set of edges $\mathcal{E}$, and the same connection relations. To any edge e of Γ_G we assign the weight $g_e = l_e^{-1}$. If $v \in \mathcal{V}$, then $\mathcal{S}(v)$ stands for its *star*: $\mathcal{S}(v) = \cup_{e\ni v} e$. If $e = (v, v')$, then $\mathcal{S}(e) = \mathcal{S}(v) \cup \mathcal{S}(v')$.

The Lebesgue measure on the edges induces a measure on Γ, and our basic Hilbert space is $L^2(\Gamma)$ with respect to this measure.

On the space $H^1(\Gamma)$ of continuous functions φ on Γ, such that $\varphi \in H^1(e)$ on each edge, and $\int_\Gamma (|\varphi'|^2 + \varphi|^2)dy < \infty$, we consider the quadratic form

$$Q_\Gamma[\varphi] := \int_\Gamma |\varphi'(y)|^2 dy. \tag{4.1}$$

The Laplacian $\mathbf{A}_\Gamma$ in $L^2(\Gamma)$ is determined by this quadratic form. Our first concern is about the relations between the exponents δ, D for the semigroups generated by the operators $\mathbf{A}_\Gamma$ in $L^2(\Gamma)$ and $\mathbf{A}_{G(\Gamma)}$ in $\ell^2(G(\Gamma))$.

To this end, let us consider two pre-Hilbert spaces that are linear subspaces in the space $H^1_{\mathrm{comp}}(\Gamma)$ of all compactly supported functions from $H^1(\Gamma)$. One of them is $H^1_{\mathrm{comp},pl}(\Gamma)$, formed by functions linear on each edge; the subscript *pl* stands for 'piecewise-linear'. Any function $\varphi \in H^1_{\mathrm{comp},pl}(\Gamma)$ is determined by its values $\varphi(v)$ at the vertices. Given a finite sequence $f = \{f(v)\}$, $v \in \mathcal{V}$, we denote by Jf the (unique) function in $H^1_{\mathrm{comp},pl}(\Gamma)$, such that $(Jf)(v) = f(v)$, $\forall v \in \mathcal{V}$. The mapping J defines an isometry between the pre-Hilbert spaces $H^1_{\mathrm{comp},pl}(\Gamma)$ equipped by the metric Q_Γ, and $\mathcal{H}_{\mathrm{fin}}(G)$, equipped by the metric Q_G

defined in (3.2). By means of this isometry we identify canonically these pre-Hilbert spaces.

Another subspace is $H^1_{\mathrm{comp},\mathcal{D}}$ that consists of all functions $\varphi \in H^1_{\mathrm{comp}}(\Gamma)$, such that $\varphi(v) = 0$ for all $v \in \mathcal{V}$. It is clear that

$$H^1_{\mathrm{comp}} = H^1_{\mathrm{comp},pl} \oplus H^1_{\mathrm{comp},\mathcal{D}} \tag{4.2}$$

(the orthogonal decomposition in the metric Q_Γ). We will denote by φ_{pl} and $\varphi_{\mathcal{D}}$ the components of a given element φ with respect to this decomposition.

Proposition 4.1. *Suppose that the degrees of the vertices of* Γ *and the lengths of the edges are uniformly bounded. Let* $G = G(\Gamma)$. *Then, if* $M_{\mathbf{A}_G}(t) = O(t^{-D/2})$, $D > 2$, *as* $t \to \infty$, *then also* $M_{\mathbf{A}_\Gamma}(t) = O(t^{-D/2})$ *as* $t \to \infty$, *and* $M_{\mathbf{A}_\Gamma}(t) = O(t^{-\delta/2})$ *as* $t \to 0$, *with any* $\delta > 2$.

Proof. By Theorem II.3.1 in [15], it is sufficient to prove that the Sobolev inequality

$$\|\varphi\|^2_{L^r} \le C Q_\Gamma[\varphi], \qquad r = 2D(D-2)^{-1}, \tag{4.3}$$

holds for any $\varphi \in H^1_{\mathrm{comp}}(\Gamma)$. Since the decomposition (4.2) is orthogonal, it is sufficient to establish (4.3) separately for the components φ_{pl} and $\varphi_{\mathcal{D}}$. As for the term $\varphi_{pl} = Jf$, its norm in $L^r(\Gamma)$ is majorized by the norm of f in $\ell^r(G)$, so the Sobolev inequality for φ_{pl} follows from the similar inequality for f.

For the term $\varphi_{\mathcal{D}}$, under our condition of the boundedness of the lengths of the edges, the required Sobolev inequality holds on each edge, with a common constant, and the summation gives (4.3). □

Remark 4.2. A somewhat more complicated reasoning proves that $\delta(\Gamma) = 1$, however we do not need this fact in the present paper.

We always suppose that $D > 2$. In this case, by the Sobolev inequality the Hilbert function space $\mathcal{H}^1 = \mathcal{H}^1(\Gamma)$ is well defined as the closure of $H^1_{\mathrm{comp}}(\Gamma)$ in the metric Q_Γ. By closing the terms in the decomposition (4.2) we obtain the Hilbert spaces $\mathcal{H}^1_{pl}$ and $\mathcal{H}^1_{\mathcal{D}}$, so that

$$\mathcal{H}^1_{\mathcal{D}}(\Gamma) = \sum_{e \in \mathcal{E}}{}^{\oplus} H^{1,0}(e)$$

and

$$\mathcal{H}^1(\Gamma) = \mathcal{H}^1_{pl} \oplus \mathcal{H}^1_{\mathcal{D}}. \tag{4.4}$$

The isometry J extends to the isometry of $\mathcal{H}(G)$ onto $\mathcal{H}^1_{pl}$.

The quadratic form (2.4) in our case is

$$\mathbf{b}_V[\varphi] = \int_\Gamma V(y)|\varphi(y)|^2 dy. \tag{4.5}$$

In general, the decomposition (4.4) does not reduce the corresponding operator $\mathbf{B}_V$. Still, we introduce the operators $\mathbf{B}_{V,pl}$ and $\mathbf{B}_{V,\mathcal{D}}$, acting in $\mathcal{H}^1_{pl}$ and $\mathcal{H}^1_{\mathcal{D}}$ respectively and generated by the quadratic form (4.5) restricted to the corresponding subspace. The spectral estimates for $\mathbf{B}_V$ easily reduce to the ones for these two operators. Indeed, it is clear that $\mathbf{B}_V$ is bounded (compact) if and only if these two operators meet this property. Moreover, due to the inequality

$$\mathbf{b}_V[\varphi] \le 2(\mathbf{b}_V[\varphi_{pl}] + \mathbf{b}_V[\varphi_{\mathcal{D}}]),$$

we have (in the case of compactness)

$$n_+(s, \mathbf{B}_{V,pl}),\ n_+(s, \mathbf{B}_{V,\mathcal{D}}) \le n_+(s, \mathbf{B}_V) \le n_+(s/2, \mathbf{B}_{V,pl}) + n_+(s/2, \mathbf{B}_{V,\mathcal{D}}). \tag{4.6}$$

The structure of the operator $\mathbf{B}_{V,\mathcal{D}}$ is simple:

$$\mathbf{B}_{V,\mathcal{D}} = \sum_{e\in\mathcal{E}}{}^{\oplus} \mathbf{B}_{V,e,\mathcal{D}} \tag{4.7}$$

where $\mathbf{B}_{V,e,\mathcal{D}}$ stands for the operator in $H^{1,0}(e)$, generated by the quadratic form similar to (4.5), with the integration over the edge e.

Consider now the quadratic form (4.5) for $\varphi \in \mathcal{H}^1_{pl}(\Gamma)$. Let $f = \{f(v)\}$ be the restriction of φ onto $\mathcal{V}$, i.e., $f(v) = \varphi(v)$, $\forall v \in \mathcal{V}$. Then

$$\mathbf{b}_V[\varphi_{pl}] = \mathbf{b}_V[Jf] = \sum_{e\in\mathcal{E}} \int_e V(y)|(Jf)(y)|^2 dy. \tag{4.8}$$

The corresponding operator on $\mathcal{H}^1_{pl}$ is $\mathbf{B}_{V,pl}$. Consider also the operator $\widehat{\mathbf{B}}_{V,pl}$ in $\mathcal{H}^1_{pl}(\Gamma)$, generated by the quadratic form $\mathbf{b}_V[Jf]$. It is clear that the operators $\mathbf{B}_{V,pl}$ and $\widehat{\mathbf{B}}_{V,pl}$ are unitary equivalent.

Now we discuss in turn the estimates for $\mathbf{B}_{V,\mathcal{D}}$ and for $\mathbf{B}_{V,pl}$.

4.1. **Operator $\mathbf{B}_{V,\mathcal{D}}$.** The orthogonal decomposition (4.7) reduces the study of the spectrum of $\mathbf{B}_{V,e,\mathcal{D}}$ to the similar problem for a family of finite intervals, and thus makes the task elementary.

We associate with V the number sequence

$$\boldsymbol{\eta}_V = \{\eta_V(e)\}, \qquad \eta_V(e) = l_e \int_e V dy,\ e \in \mathcal{E}. \tag{4.9}$$

It is well known (see, e.g., [2], §4.8), that

$$n(\lambda, \mathbf{B}_{V,e,\mathcal{D}}) \le C\lambda^{-1/2}\sqrt{\eta_V(e)}, \qquad \forall\lambda > 0. \tag{4.10}$$

and

$$\lambda^{1/2} n(\lambda, \mathbf{B}_{V,e,\mathfrak{D}}) \to \frac{1}{\pi} \int_e \sqrt{V} dx, \qquad \lambda \to 0.$$

Let $\nu(s, \boldsymbol{\eta}_V) = \#\{e : \eta_V(e) > s\},\ s > 0$, be the distribution function for the sequence (4.9). We say that $\eta_V(e) \to 0$ if $\nu(s, \boldsymbol{\eta}_V) < \infty$ for any $s > 0$.

The next statement follows from (4.7), due to the above results for a single interval.

Lemma 4.3. *1° If $\boldsymbol{\eta}_V \in \ell^\infty$, the operator $\mathbf{B}_{V,\mathfrak{D}}$ is bounded and*

$$\|\mathbf{B}_{V,\mathfrak{D}}\| \le C \|\boldsymbol{\eta}_V\|_{\ell^\infty}, \qquad c > 0. \tag{4.11}$$

If $\boldsymbol{\eta}_V \to 0$, the operator $\mathbf{B}_{V,\mathfrak{D}}$ is compact.

2° Let $\widetilde{\ell}$ stand for any space of number sequences ℓ^q or ℓ^q_w, and let $\widetilde{\mathfrak{S}}$ stand for the corresponding space of operators $\mathfrak{S}_q$ or Σ_q. Then for any $1/2 < q < \infty$

$$\|\mathbf{B}_{V,\mathfrak{D}}\|_{\widetilde{\mathfrak{S}}} \le C(q) \|\boldsymbol{\eta}_V\|_{\widetilde{\ell}}, \qquad c(q) > 0. \tag{4.12}$$

3° Let $\boldsymbol{\eta}_V \in \ell^{1/2}$. Then

$$\|\mathbf{B}_{V,\mathfrak{D}}\|_{\Sigma_{1/2}} \le C \sum_{e \in \mathcal{E}} \sqrt{\eta_V(e)} \tag{4.13}$$

and

$$\lambda^{1/2} n(\lambda, \mathbf{B}_{V,\mathfrak{D}}) \to \frac{1}{\pi} \int_\Gamma \sqrt{V} dy, \qquad \lambda \to 0.$$

Proof. The reasoning is rather standard, see, e.g., [10], and we prove only the statement 2° for the classes Σ_q. If $\boldsymbol{\eta}_V \in \ell^q_w$, then, after an appropriate numeration of edges, e_j, we have :

$$\eta_V(e_j) \le M j^{-1/q}.$$

Hence, by (4.10),

$$n(\lambda, B_{V,e_j,\mathfrak{D}}) \le C M^{1/2} \lambda^{-1/2} j^{-1/2q}.$$

In particular, $n(\lambda, \mathbf{B}_{V,e_j,\mathfrak{D}}) = 0$ if $j > C^{2q} M^q \lambda^{-q}$. Therefore,

$$n(\lambda, \mathbf{B}_{V,\mathfrak{D}}) = \sum_e n(\lambda, \mathbf{B}_{V,e,\mathfrak{D}}) \le C M^{1/2} \lambda^{-1/2} \sum_{j \le C^{2q} M^q \lambda^{-q}} j^{-1/2q}$$

and, since $2q > 1$,

$$n(\lambda, \mathbf{B}_{V,\mathfrak{D}}) \le C' M^q \lambda^{-q},$$

whence the result. □

4.2. **Operator $\mathbf{B}_{V,pl}$.** We compare our operator $\mathbf{B}_{V,pl}$ (or, equivalently, $\widehat{\mathbf{B}}_{V,pl}$) with the operator $\mathbf{B}_{\boldsymbol{\varkappa}_V}$, where the discrete potential $\boldsymbol{\varkappa}_V = \{\varkappa_V(v)\}$ is chosen in a special way:

$$\varkappa_V(v) = \int_{\mathcal{S}(v)} V dy = \sum_{e \ni v} \eta_V(e), \qquad \forall v \in \mathcal{V}. \tag{4.14}$$

It is clear that the (quasi-)norms of both sequences $\boldsymbol{\varkappa}_V$, $\boldsymbol{\eta}_V$ in any space ℓ^q, ℓ^q_w are equivalent to each other.

Let us return to the quadratic form in (4.8). Choose an edge $e = (v, v')$. Identifying e with the interval $(0, l_e)$, we have, for $f \in \mathcal{H}(G)$,

$$\int_e V(y)|(Jf)(y)|^2 dy = l_e^{-1} \int_0^{l_e} V(y)|f(v)(l_e - y) + f(v')y|^2 dy.$$

Summing up the integrals, we see that in the expression for $\mathbf{b}_{\boldsymbol{\varkappa}_V}[Jf]$ the term v_n appears in the combination

$$|f(v)|^2 \sum_{e \in \mathcal{S}(v)} l_e^{-1} \int_e V(y)(l_e - y)^2 dy + 2\,\mathrm{Re} \sum_{v' : e = (v,v')} f(v)\overline{f(v')} l_e^{-1} \int_e V(y) y (l_e - y) dy.$$

We conclude that

$$\Big|\widehat{\mathbf{b}}_V[f] - \sum_{v \in \mathcal{V}} |f(v)|^2 \sum_{e \in \mathcal{S}(v)} l_e^{-1} \int_e V(y)(l_e - y)^2 dy\Big| \le \mathbf{b}_{\boldsymbol{\varkappa}_V}[f], \tag{4.15}$$

and hence (since the integrals on the left do not exceed $\varkappa_V(v)$),

$$\widehat{\mathbf{b}}_V[f] \le 2\mathbf{b}_{\boldsymbol{\varkappa}_V}[f]. \tag{4.16}$$

This leads us to the following result.

Lemma 4.4. *Suppose the operator $\mathbf{B}_{\boldsymbol{\varkappa}_V}$ on the combinatorial graph G is bounded, compact, or lies in one of the classes $\mathfrak{S}_q$, Σ_q, $q > 0$. Then the same is true for the operator $\mathbf{B}_{V,pl}$, and the estimate*

$$\|\mathbf{B}_{V,pl}\| \le 2\|\mathbf{B}_{\boldsymbol{\varkappa}_V}\| \tag{4.17}$$

holds in the (*quasi*)*-norm of the corresponding class.*

5. Metric graphs: lower estimates and the final results

5.1. **Lower estimates.** We start by a simple graph-theoretical lemma.

Lemma 5.1. *Let for a graph G the condition* (3.1) *be satisfied. Then the set $\mathcal{E}$ can be split into no more than $2\mathbf{d}^2 + 1$ subsets $\mathcal{E}_j$, $1 \le j \le n_0(\mathbf{d}) \le 2\mathbf{d}^2 + 1$, so that $\mathcal{S}(e) \cap \mathcal{S}(e') = \varnothing$ for any $e \ne e'$ belonging to the same $\mathcal{E}_j$.*

Proof. Let us order the edges in $\mathcal{E}$ in an arbitrary way. We must color the edges so that the stars of the edges of the same color are disjoint. Suppose that we have already colored all edges $e_k, k < n$. The star of the edge e_n can have common edges with no more than $2\mathbf{d}^2$ stars of the previously colored edges. So we can apply the unused color to e_n $\square$

Now we are in a position to give some lower estimates for our original operator $\mathbf{B}_V$. Similarly to Theorem 3.2, they require an additional condition, namely, that the edge lengths l_e are bounded and separated from zero:

$$0 < l_0 \le l_e \le l_1, \qquad \forall e \in \mathcal{E}. \tag{5.1}$$

Note that the left inequality in (5.1) implies the condition (3.8) for the combinatorial graph $G(\Gamma)$.

Lemma 5.2. *Suppose the conditions* (3.1) *and* (5.1) *are satisfied. Then*
1° *If the operator* $\mathbf{B}_V$ *is bounded, then* $\boldsymbol{\eta}_V \in \ell_\infty$ *and*

$$(4d-2)\|\mathbf{B}_V\| \ge \|\boldsymbol{\eta}_V\|_\infty.$$

2° *If* $\mathbf{B}_V$ *is compact, then the sequence* $\boldsymbol{\eta}_V$ *tends to zero. Moreover, there exist constants* c', c'' *depending only on* $\mathbf{d}$*, such that*

$$n(s, \mathbf{B}_V) \ge c'\nu(c''s, \boldsymbol{\eta}_V), \qquad \forall s > 0. \tag{5.2}$$

Proof. It follows the scheme repeatedly used in the literature, see, e.g., [4, 10, 13]. Let $e = (v, v')$ be an edge. Take a function $\varphi_e \in \mathcal{H}^1_{pl}(\Gamma)$, such that $\varphi_e(y) = 1$ for $y \in e$ and $\varphi_e(y) = 0$ outside the set $\mathcal{S}(e) = \mathcal{S}(v) \cup \mathcal{S}(v')$. Such function is unique, and $\int_\Gamma |\varphi_e'|^2 dy \le 2(\mathbf{d}-1)l_0^{-1}$. Moreover, $\int_\Gamma V|\varphi_e|^2 dy \ge l_1 \eta_V(e)$, and hence,

$$\frac{\int_\Gamma V|\varphi_e|^2 dy}{\int_\Gamma |\varphi_e'|^2 dx} \ge 2(\mathbf{d}-1)^{-1} l_0 l_1^{-1} \eta_V(e). \tag{5.3}$$

The statement 1° immediately follows.

Further, if for two edges e_1, e_2 the sets $\mathcal{S}(e_1), \mathcal{S}(e_2)$ do not intersect, then the corresponding functions f_{e_1}, f_{e_2} are orthogonal both in $\mathcal{H}^1(\Gamma)$ and in the space L_2 with the weight V. Let us say that a subset $F \subset \mathcal{E}$ is nice, if the sets $\mathcal{S}(e)$, $e \in F$, are mutually disjoint. By restricting the quadratic form $\mathbf{b}_V$ onto the linear hull of the functions $u_e, e \in F$, we obtain an operator whose eigenvalues, up to the ordering, are exactly the numbers in the right-hand side of (5.3). Therefore, these numbers $\eta_V(e)$, after re-ordering by decrease, do not exceed the eigenvalues $\lambda_n(\mathbf{B}_V)$ (or even, $\lambda_n(\mathbf{B}_{V,pl})$), multiplied by $2(\mathbf{d}-1)^{-1} l_0 l_1^{-1}$. By Lemma 5.1 the set $\mathcal{E}$ can be split into no more than $2\mathbf{d}^2 + 1$ nice subsets, and this leads to the statement 2°. $\square$

Note that the similar lower estimate for the operator $\mathbf{B}_{V,D}$ (in place of $\mathbf{B}_V$) does not hold.

5.2. **The operator $\mathbf{B}_V$.** Now we easily obtain the following result for our original operator $\mathbf{B}_V$ in the space $\mathcal{H}^1(\Gamma)$, generated by the quadratic form (4.5). We compare it with the "discrete" operator $\mathbf{B}_{\boldsymbol{\varkappa}_V}$ in the space $\mathcal{H}(G)$, $G = G(\Gamma)$, where the discrete potential $\boldsymbol{\varkappa}_V$ is defined by (4.14).

Theorem 5.3. *Let Γ be a metric graph, such that the conditions* (3.1), *and* (5.1) *are fulfilled. Suppose also that for the combinatorial graph $G(\Gamma)$ we have $D > 2$. Then the operator $\mathbf{B}_V$ is bounded, compact, or lies in one of the classes $\mathfrak{S}_q$, $\mathfrak{S}_{q,w}$ with $q > 1/2$, if and only if the following two conditions hold.*

1° The operator $\mathbf{B}_{\boldsymbol{\varkappa}_V}$ belongs to the corresponding class;

2° $\boldsymbol{\varkappa}_V \in \widetilde{\ell}$ where $\widetilde{\ell}$ stands for ℓ_∞, its subspace of sequences tending to zero, ℓ_q, or $\ell_{q,w}$.

If $q < D/2$, the condition 2° follows from 1° and thus, can be removed.

Remark 5.4. In particular, $\mathbf{B}_V \in \mathfrak{S}_1$ if and only if $\int_{\Gamma^d} V dy < \infty$.

Proof. Part "if" follows from Lemmas 4.3 and 4.4, due to the right inequality in (4.6).

Part "only if": if $\mathbf{B}_V$ possesses one of the properties mentioned in the assumption, then the same is true for the operator $\mathbf{B}_{\boldsymbol{\varkappa}_V}$ due to the left inequality in (4.6). The condition 2° is fulfilled by Lemma 5.2. □

This result shows that the spectral properties of the operator $\mathbf{B}_V$ are basically determined by the ones for its discrete analogue. It is worth noting that the result for $q > D/2$ should be considered as "conditional": indeed, our results for general combinatorial graphs (Theorem 3.1) concern only the case $q \le D/2$. For more advanced results, one needs an additional information about the structure of G. For the important case $G = \mathbb{Z}^d$, $d \ge 3$, such results were obtained in [13].

In particular, a construction of "sparse potentials" was suggested there, that allows one to construct a discrete potential producing an operator $\mathbf{B}_V$ with an arbitrary prescribed asymptotic behavior of the spectrum. In this connection, we note that this construction extends to arbitrary combinatorial graphs with $D > 2$. This material will be presented elsewhere.

Theorem 5.3 does not include the borderline case $q = 1/2$. For this case, a simple sufficient condition for $\mathbf{B}_V \in \mathfrak{S}_{1/2,w}$ can be given.

Theorem 5.5. *Let $\eta(V) \in \ell^{1/2}$. Then*

$$\|\mathbf{B}_V\|_{\Sigma_{1/2}} \le C\|\boldsymbol{\eta}_V\|_{\ell^{1/2}}, \tag{5.4}$$

or, equivalently,

$$N_-(\mathbf{A} - \alpha V) \le C\alpha^{1/2} \sum_{e \in \mathcal{E}} \boldsymbol{\eta}_V(e)^{1/2}.$$

The Weyl-type asymptotic formula

$$N_-(\mathbf{A} - \alpha V) \sim \frac{1}{\pi} \int_\Gamma \sqrt{V} dy \tag{5.5}$$

is satisfied.

Proof. By Lemma 4.3, the estimate (5.4) and the asymptotics (5.5) hold for the operator $\mathbf{B}_{V,\mathfrak{D}}$. Since $\boldsymbol{\varkappa}_V$ lies in $\ell^{1/2}$ together with $\boldsymbol{\eta}_V$, the operator $\mathbf{B}_{V,pl}$ belongs to $\mathfrak{S}_{1/2}$, and thus to $\Sigma_{1/2}$, with the corresponding estimate for its quasi-norm. Hence, it it does not contribute to the asymptotics of order 1/2. Both statements of Theorem 5.4 follow from here. □

A necessary and sufficient condition for the validity of (5.4) and (5.5) can be obtained by analogy with [9]. We do not present it here.

References

[1] Birman, M.Sh.: *On the spectrum of singular boundary-value problems,* (Russian) Mat. Sb. (N.S.) **55(97)**:2, (1961), 125–174.
English translation in: Eleven Papers on Analysis, AMS Transl. **53**, 23–80, AMS, Providence, R.I., 1966.

[2] Birman, M.Sh., Solomyak, M.: *Quantitative analysis in Sobolev imbedding theorems and applications to spectral theory*, Tenth Mathem. School, Izd. Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev, 5–189 (1974) (Russian); transl. in Amer. Math. Soc. Transl. (2) **114** (1980).

[3] Birman, M.Sh., Solomyak, M.: Spectral theory of selfadjoint operators in Hilbert space, D. Reidel Publishing Co., Dordrecht, 1987.

[4] Birman, M.Sh., Solomyak, M.: *Estimates for the number of negative eigenvalues of the Schrödinger operator and its generalizations*, Adv. Soviet Math., **7**, 1–55, Amer. Math. Soc., Providence, RI (1991).

[5] Coulhon, T. Grigoryan, A.:*Random walks on graphs with regular volume growth.* Geom. Funct. Anal. **8** , no. 4, 656–701 (1998).

[6] Gohberg, I.C., Krein, M.G.: Introduction to the theory of linear non-selfadjoint operators in Hilbert space. Izdat. "Nauka", Moscow 1965 (Russian). English translation: Amer. Math. Soc., Providence (1969).

[7] Grigoryan, A., Hu, J.:*: Off-diagonal upper estimates for the heat kernel of the Dirichlet forms on metric spaces.* Invent. Math. **174** , no. 1, 81–126 (2008).

[8] Molchanov S.; Vainberg, B.: On general Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities. Around the Research of Vladimir Maz'ya III, International Mathematical Series, Vol. 13, Springer, 2010, 201-246

[9] Naimark, K.; Solomyak, M.: *Regular and pathological eigenvalue behaviour for the equation $-\lambda u'' = Vu$ on the semiaxis*, J. of Functional Analysis 151(2) (1997), 504-530.

[10] Naimark, K.; Solomyak, M.: *Eigenvalue estimates for the weighted Laplacian on metric trees.* Proc. London Math. Soc. (3) 80 (2000), no. 3, 690–724.

[11] Rozenblum, G., Solomyak, M.: *CLR-estimate for the generators of positivity preserving and positively dominated semigroups (Russian)*, Algebra i Analis v.9, No. 6 (1997), 214-236. Translation in St. Petersburg Math. J. **9** (1998), no. 6, 1195–1211.

[12] Rozenblum, G., Solomyak, M.: *Counting Schrödinger boundstates: semiclassics and beyond*, In: Sobolev Spaces in Mathematics. II. Applications in Analysis and Partial Differential Equations, International Mathematical Series, **8**, Springer, 2009, 329-354.

[13] Rozenblum, G., Solomyak, M.: *On the spectral estimates for the Schrödinger operator on $\mathbb{Z}^d$, $d \geq 3$*, Problems in mathematical analysis. No. 41. J. Math. Sci. (N. Y.) 159 (2009), no. 2, 241–263.

[14] Saloff-Coste, L.: *Sobolev inequalities in familiar and unfamiliar settings.* Sobolev spaces in mathematics. I, 299–343, Int. Math. Ser. (N. Y.), 8, Springer, New York, 2009

[15] Varopoulos, N.Th, Saloff-Coste, L., Coulhon, T.: *Analysis and Geometry on Groups*, Cambridge University Press, 1992.

Department of Mathematics, Chalmers University of Technology and The University of Gothenburg, S-412 96, Gothenburg, Sweden
E-mail address: `grigori@chalmers.se`

Department of Mathematics, Weizmann Institute, Rehovot, Israel
E-mail address: `solom@wisdom.weizmann.ac.il`