Generalized covariation for Banach space valued processes, Itô formula and applications

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Abstract

This paper concerns the notion of quadratic variation and covariation for Banach space valued processes (not necessarily semimartingales) and related Itô formula. If $\mathbb{X}$ and $\mathbb{Y}$ take respectively values in Banach spaces B_1 and B_2 and χ is a suitable subspace of the dual of the projective tensor product of B_1 and B_2 (denoted by $(B_1\hat{\otimes}_\pi B_2)^*$), we define the so-called χ-covariation of $\mathbb{X}$ and $\mathbb{Y}$. If $\mathbb{X}=\mathbb{Y}$ the χ-covariation is called χ-quadratic variation. The notion of χ-quadratic variation is a natural generalization of the one introduced by Métivier-Pellaumail and Dinculeanu which is too restrictive for many applications. In particular, if χ is the whole space $(B_1\hat{\otimes}_\pi B_1)^*$ then the χ-quadratic variation coincides with the quadratic variation of a B_1-valued semimartingale. We evaluate the χ-covariation of various processes for several examples of χ with a particular attention to the case $B_1=B_2=C([-\tau,0])$ for some $\tau>0$ and $\mathbb{X}$ and $\mathbb{Y}$ being *window processes*. If X is a real valued process, we call window process associated with X the $C([-\tau,0])$-valued process $\mathbb{X}:=X(\cdot)$ defined by $X_t(y)=X_{t+y}$, where $y\in[-\tau,0]$. The Itô formula introduced here happens to be an important instrument to establish a representation result of Clark-Ocone type for a class of path dependent random variables of type $h=H(X_T(\cdot))$, $H:C([-T,0])\longrightarrow\mathbb{R}$ for not-necessarily semimartingales X with finite quadratic variation. This representation will be linked to a function $u:[0,T]\times C([-T,0])\longrightarrow\mathbb{R}$ solving an infinite dimensional partial differential equation.

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1 Introduction

The present paper settles the basis for the calculus via regularization for processes with values in an infinite dimensional separable Banach space B. The extension of Itô stochastic integration theory for Hilbert valued processes dates only from the eighties, the results of which can be found in the monographs [18, 19, 5] and [31] with different techniques. However the discussion of this last approach is not the aim of this paper. Extension to nuclear valued spaces is simpler and was done in [15, 30]. One of the most

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natural but difficult situations arises when the processes are Banach space valued.
As for the real case, a possible tool of infinite dimensional stochastic calculus is the concept of quadratic variation, or more generally of covariation. The notion of covariation is historically defined for two real valued $(\mathcal{F}_t)$-semimartingales X and Y and it is denoted by $[X, Y]$. This notion was extended to the case of general processes by means of discretization techniques, for instance by [12], or via regularization, in [26, 28]. In this paper we will follow the language of regularization; for simplicity we suppose that either X or Y is continuous. In the whole paper T will be a fixed positive number. Every process will be indexed by $[0, T]$, but can be extended to the real line for convenience by setting $X_t = X_0$ if $t < 0$ and $X_t = X_T$ for $t \geq T$.

Definition 1.1. Let X and Y be two real processes such that X is continuous and Y has almost surely locally integrable paths. For $\epsilon > 0$, we denote

$$\begin{aligned} [X,Y]^\epsilon_t &= \int_0^t \frac{(X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s)}{\epsilon} ds \;, \quad t \in [0,T] \;, \\ I^-(\epsilon, Y, dX)_t &= \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds \;, \quad t \in [0,T] \;. \end{aligned}$$

1. We say that X and Y admit a **covariation** if $\lim_{\epsilon \to 0} [X, Y]^\epsilon_t$ exists in probability for every $t \in [0, T]$ and the limiting process admits a continuous version that will be denoted by $[X, Y]$. If $[X, X]$ exists, we say that X has a **quadratic variation** and it will also be denoted by $[X]$. If $[X] = 0$ we say that X is a **zero quadratic variation process**.

2. The **forward integral** $\int_0^t Y_s d^- X_s$ is a continuous process Z, such that whenever it exists, $\lim_{\epsilon \to 0} I^-(\epsilon, Y, dX)_t = Z_t$ in probability for every $t \in [0, T]$.

3. If $\int_0^t Y_s d^- X_s$ exists for any $0 \leq t < T$; $\int_0^T Y_s d^- X_s$ will symbolize the **improper forward integral** defined by $\lim_{t \to T} \int_0^t Y_s d^- X_s$, whenever it exists in probability.

Remark 1.2.

1. Lemma 3.1 in [27] allows to show that, whenever $[X, X]$ exists, then $[X, X]^\varepsilon$ also converges in the **uniform convergence in probability (ucp)** sense, see [26, 28]. The basic results established there are still valid here, see the following items.

2. If X and Y are $(\mathcal{F}_t)$-local semimartingales, then $[X, Y]$ coincides with the classical covariation, see Corollaries 2 and 3 in [28].

3. If X (resp. A) is a finite (resp. zero) quadratic variation process, then $[A, X] = 0$.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a fixed probability space, equipped with a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ fulfilling the usual conditions. If X is an $(\mathcal{F}_t)$-semimartingale and Y is $(\mathcal{F}_t)$-progressively measurable and càdlàg (resp. an $(\mathcal{F}_t)$-semimartingale) $\int_0^\cdot Y_s d^- X_s$ (resp. $[X, Y]$) coincides with the classical Itô integral $\int_0^\cdot Y dX$ (resp. the classical covariation). The class of real finite quadratic variation processes is much richer than the one of semimartingales. Typical examples of such processes are $(\mathcal{F}_t)$-Dirichlet processes. D is called $(\mathcal{F}_t)$-**Dirichlet process** if it admits a decomposition $D = M + A$ where M is an $(\mathcal{F}_t)$-local martingale and A is an $(\mathcal{F}_t)$-adapted zero quadratic variation process. A slight generalization of that notion is the one of weak Dirichlet process, which was introduced in [10]. Another interesting example is the bifractional Brownian motion $B^{H,K}$ with parameters $H \in]0, 1[$ and $K \in]0, 1]$ which has finite quadratic variation if and only if $HK \geq 1/2$, see [24]. Notice that if $K = 1$, then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in]0, 1[$. If $HK = 1/2$ it holds $[B^{H,K}]_t = 2^{1-K} t$; if $K \neq 1$ this process is not even Dirichlet with respect to its own filtration. One object of this paper consists in investigating a possible useful generalization of the notions of covariation and quadratic variation for Banach space valued processes. Particular emphasis will be devoted to *window processes* with values in the Banach space of real continuous functions defined on $[-\tau, 0]$. Given $0 < \tau \le T$ and a real continuous process $X = (X_t)_{t\in[0,T]}$ one can link to it a natural infinite dimensional valued process defined as follows.

Definition 1.3. We call **window process** associated with X, denoted by $X(\cdot)$, the $C([-\tau, 0])$-valued process

$$X(\cdot) = \left(X_t(\cdot)\right)_{t\in[0,T]} = \{X_t(u) := X_{t+u}; u \in [-\tau, 0], t \in [0, T]\} \ .$$

In the present paper, W will always denote a real standard Brownian motion. The window process $W(\cdot)$ associated with W will be called **window Brownian motion**.

Those processes naturally appear in functional dependent stochastic differential equations as delay equations. We emphasize that $C([-\tau, 0])$ is typical a non-reflexive Banach space. So we will introduce a notion of covariation for processes with values in general Banach spaces but which will be performing also for window processes. This paper settles the theoretical basis for the stochastic calculus part related to the first part of [7] and which partially appears in [6]. Let B_1, B_2 be two general Banach spaces. In this paper $\mathbb{X}$ (resp. $\mathbb{Y}$) will be a B_1 (resp. B_2) valued stochastic process. It is not obvious to define an exploitable notion of covariation (resp. quadratic variation) of $\mathbb{X}$ and $\mathbb{Y}$ (resp. of $\mathbb{X}$). When $\mathbb{X}$ is an H-valued martingale and $B_1 = B_2 = H$ is a separable Hilbert space, [5], Chapter 3 introduces an operational notion of quadratic variation for martingales with values in H. [8] introduces in Definitions A.1 in Chapter 2.15 and B.9 in Chapter 6.23 the notions of *semilocally summable* and *locally summable processes with respect to a given bilinear mapping on* $B \times B$; see also Definition C.8 in Chapter 2.9 for the definition of *summable* process. Similar notions appears in [20]. Those processes are very close to Banach space valued semimartingales. If B is a Hilbert space, a semimartingale is semilocally summable when the bilinear form is the inner product. For previous processes [8] defines two natural notions of quadratic variation: the real quadratic variation and the tensor quadratic variation. For avoiding confusion with the quadratic variation of real processes, we will use the terminology *scalar* instead of *real*. Even though [20, 8] make use of discretizations, we define here, for commodity, two very similar objects but in our regularization language, see Definition 1.4. Moreover, the notion below extends to the covariation of two processes $\mathbb{X}$ and $\mathbb{Y}$ for which we remove the assumption of *semilocally summable* or *locally summable*. Before that, we remind some properties related to tensor products of two Banach spaces E and F, see [29] for details. If E and F are Banach spaces, $E\hat{\otimes}_\pi F$ (resp. $E\hat{\otimes}_h F$) is a Banach space which denotes the **projective** (resp. **Hilbert**) **tensor product of** E **and** F. We recall that $E\hat{\otimes}_\pi F$ (resp. $E\hat{\otimes}_h F$) is obtained by a completion of the algebraic tensor product $E \otimes F$ equipped with the projective norm π (resp. Hilbert norm h). For a general element $u = \sum_{i=1}^n e_i \otimes f_i$ in $E \otimes F$, $e_i \in E$ and $f_i \in F$, it holds $\pi(u) = \inf \{\sum_{i=1}^n \|e_i\|_E \|f_i\|_F : u = \sum_{i=1}^n e_i \otimes f_i, e_i \in E\, , f_i \in F\}$. For the definition of the Hilbert tensor norm h the reader may refer [29], Chapter 7.4. We remind that if E and F are Hilbert spaces the Hilbert tensor product $E\hat{\otimes}_h F$ is also Hilbert and its inner product between $e_1 \otimes f_1$ and $e_2 \otimes f_2$ equals $\langle e_1, e_2\rangle_E \cdot \langle f_1, f_2\rangle_F$. Let $e \in E$ and $f \in F$, the symbol $e \otimes f$ (resp. $e\otimes^2$) will denote an elementary element of the algebraic tensor product $E \otimes F$ (resp. $E \otimes E$). The Banach space $(E\hat{\otimes}_\pi F)^*$ denotes the topological dual of the projective tensor product equipped with the operator norm. As announced we give now the two definitions of scalar and tensor covariation and quadratic variation.

Definition 1.4. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1 (resp. B_2) valued stochastic process.

1. $\mathbb{X}$ and $\mathbb{Y}$ are said to admit a **scalar covariation** if the limit for $\epsilon \downarrow 0$ of the sequence

$$[\mathbb{X}, \mathbb{Y}]^{\mathbb{R},\epsilon}_{\cdot} = \int_0^{\cdot} \frac{\|\mathbb{X}_{s+\epsilon} - \mathbb{X}_s\|_{B_1} \|\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s\|_{B_2}}{\epsilon} ds$$

exists ucp. That limit will be indeed called scalar covariation of $\mathbb{X}$ and $\mathbb{Y}$ and it will be simply denoted by $[\mathbb{X}, \mathbb{Y}]^{\mathbb{R}}$. The scalar covariation $[\mathbb{X}, \mathbb{X}]^{\mathbb{R}}$ will be called **scalar quadratic variation** of $\mathbb{X}$ and simply denoted by $[\mathbb{X}]^{\mathbb{R}}$.

2. $\mathbb{X}$ and $\mathbb{Y}$ admit a **tensor covariation** if there exists a $(B_1 \hat{\otimes}_\pi B_2)$-valued process denoted by $[\mathbb{X}, \mathbb{Y}]^{\otimes}$ such that the sequence of Bochner $(B_1 \hat{\otimes}_\pi B_2)$-valued integrals

$$[\mathbb{X}, \mathbb{Y}]^{\otimes,\epsilon}_{\cdot} = \int_0^{\cdot} \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)}{\epsilon} ds \tag{1.3}$$

converges ucp for $\epsilon \downarrow 0$ (according to the strong topology) to a $(B_1 \hat{\otimes}_\pi B_2)$-valued process $[\mathbb{X}, \mathbb{Y}]^{\otimes}$. $[\mathbb{X}, \mathbb{Y}]^{\otimes}$ will indeed be called tensor covariation of $\mathbb{X}$ and $\mathbb{Y}$. The tensor covariation $[\mathbb{X}, \mathbb{X}]^{\otimes}$ will be called **tensor quadratic variation** and simply denoted by $[\mathbb{X}]^{\otimes}$.

Remark 1.5. 1. According to Lemma 3.1 in [27], if $[\mathbb{X}, \mathbb{Y}]^{\mathbb{R},\epsilon}_{\cdot}$ converges for any $t \geq 0$ to Z_t, where Z is a continuous process, then the scalar covariation exists and $[\mathbb{X}, \mathbb{Y}]^{\mathbb{R}} = Z$.

2. If $\mathbb{X}$ and $\mathbb{Y}$ admit both scalar and tensor covariation, the tensor covariation process has bounded variation and its total variation is bounded by the scalar covariation which is clearly an increasing process.

3. If $\mathbb{X}$ and $\mathbb{Y}$ admit a tensor covariation we have in particular

$$\frac{1}{\epsilon} \int_0^{\cdot} \langle \phi, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle \, ds \xrightarrow[\epsilon \longrightarrow 0]{ucp} \langle \phi, [\mathbb{X}, \mathbb{Y}]^{\otimes} \rangle$$

for every $\phi \in (B_1 \hat{\otimes}_\pi B_2)^*$, $\langle \cdot, \cdot \rangle$ denoting the duality between $B_1 \hat{\otimes}_\pi B_2$ and its dual.

4. If $\mathbb{X}$ and $\mathbb{Y}$ are such that $[\mathbb{X}, \mathbb{Y}]^{\mathbb{R}} = 0$, then $\mathbb{X}$ and $\mathbb{Y}$ admits a tensor covariation which also vanishes.

A sketch of the proof of the two propositions below are given in the Appendix.

Proposition 1.6. Let $\mathbb{X}$ be an $(\mathcal{F}_t)$-adapted semilocally summable process with respect to the bilinear forms $B \times B \longrightarrow B \hat{\otimes}_\pi B$, $(a, b) \mapsto a \otimes b$ and $(a, b) \mapsto b \otimes a$. Then $\mathbb{X}$ admits a tensor quadratic variation.

Proposition 1.7. Let $\mathbb{X}$ be a Hilbert space valued continuous $(\mathcal{F}_t)$-semimartingale in the sense of [20], section 10.8. Then $\mathbb{X}$ admits a scalar quadratic variation.

Corollary 1.8. Let $\mathbb{X}$ be a Banach space valued process which is semilocally summable with respect to the tensor products. If $\mathbb{X}$ has a scalar quadratic variation, it admits a tensor quadratic variation process which has bounded variation.

Remark 1.9. The tensor quadratic variation can be linked to the one of [5]; see Chapter 6 in [6] for details. Let H be a separable Hilbert space. If $\mathbb{V}$ is an H-valued Q-Brownian motion with $Tr(Q) < +\infty$ (see [5] section 4), then $\mathbb{V}$ admits a scalar quadratic variation $[\mathbb{V}]^{\mathbb{R}}_t = t\,Tr(Q)$ and a tensor quadratic variation $[\mathbb{V}]^{\otimes}_t = tq$ where q is the tensor associated to the nuclear operator tQ.

Unfortunately, even the window process $W(\cdot)$ associated with a real Brownian motion W, does not admit a scalar quadratic variation. In fact the limit of

$$\int_0^t \frac{\|W_{s+\epsilon}(\cdot) - W_s(\cdot)\|^2_{C([-\tau,0])}}{\epsilon} \, ds \ , \quad t \in [0,T] \ , \tag{1.4}$$

for ϵ going to zero does not converge, as we will see in Proposition 4.7. This suggests that when $\mathbb{X}$ is a window process, the tensor quadratic variation is not the suitable object in order to perform stochastic calculus. On the other hand in Proposition 4.5, we remark that $W(\cdot)$ is not a $C([-\tau,0])$-valued semimartingale.
Let $\mathbb{X}$ (resp. $\mathbb{Y}$) a B_1 (resp. B_2)-valued process. In Definition 3.7 we will introduce a notion of covariation of $\mathbb{X}$ and $\mathbb{Y}$ (resp. quadratic variation when $\mathbb{X} = \mathbb{Y}$) which generalizes the tensor covariation (resp. tensor quadratic variation). This will be called χ-covariation (resp. χ-quadratic variation) in reference to a topological subspace χ of the dual of $B_1 \hat{\otimes}_\pi B_2$ (resp. when $B_1 = B_2$). According to our strategy, we will suppose that

$$\frac{1}{\epsilon} \int_0^t \langle \phi, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle \, ds \tag{1.5}$$

converges for every $\phi \in \chi$. If Ω were a singleton (the processes being deterministic) and χ would coincides with the whole space $(B_1 \hat{\otimes}_\pi B_2)^*$ then previous convergence is the one related to the weak star topology in $(B_1 \hat{\otimes}_\pi B_2)^{**}$.
Our χ-covariation generalizes the concept of tensor covariation at two levels.

• First we replace the (strong) convergence of (1.3) with a weak star type topology convergence of (1.5).

• Secondly the choice of a suitable subspace χ of $(B_1 \hat{\otimes}_\pi B_2)^*$ gives a degree of freedom. For instance, compatibly with (1.4), a window Brownian motion $\mathbb{X} = W(\cdot)$ admits a χ- quadratic variation only for strict subspaces χ.

When χ equals the whole space $(B_1 \hat{\otimes}_\pi B_2)^*$ (resp. $(B_1 \hat{\otimes}_\pi B_1)^*$) this will be called **global covariation** (resp. **global quadratic variation**). This situation corresponds for us to the *elementary situation*.
Let $B_1 = B_2$ be the finite dimensional space $\mathbb{R}^n$ and $\mathbb{X} = (X^1, \ldots, X^n)$ and $\mathbb{Y} = (Y^1, \ldots, Y^n)$ with values in $\mathbb{R}^n$, Corollary 3.27 says that $(\mathbb{X}, \mathbb{Y})$ admits all its mutual brackets (i.e. $[X^i, Y^j]$ exists for all $1 \leq i, j \leq n$) if and only if $\mathbb{X}$ and $\mathbb{Y}$ have a global covariation. It is well-known that, in that case, $(B_1 \hat{\otimes}_\pi B_2)^*$ can be identified with the space of matrix $\mathbb{M}_{n \times n}(\mathbb{R})$. If χ is finite dimensional, then Proposition 3.26 gives a simple characterization for $\mathbb{X}$ to have a χ-quadratic variation.
Propositions 1.6, 1.7, 3.14 and Remark 1.9 will imply that whenever $\mathbb{X}$ admits one of the classical quadratic variations (in the sense of [5, 20, 8]), it admits a global quadratic variation and they are essentially equal.
In this paper we calculate the χ-covariation of Banach space valued processes in various situations with a particular attention for window processes associated to real finite quadratic variation processes (for instance semimartingales, Dirichlet processes, bifractional Brownian motion).

The notion of covariation intervenes in Banach space valued stochastic calculus for semimartingales, especially via Itô type formula, see for [8] and [20]. An important result of this paper is an Itô formula for Banach space valued processes admitting a χ-quadratic variation, see Theorem 5.2. This generalizes the corresponding formula for real valued processes which is stated below, see [26]. Let X be a real finite quadratic variation process and $f \in C^{1,2}([0,T] \times \mathbb{R})$. Then the forward integral $\int_0^t \partial_x f(s, X_s) d^- X_s$ exists and

$$f(t, X_t) = f(0, X_0) + \int_0^t \partial_t f(s, X_s) ds + \int_0^t \partial_x f(s, X_s) d^- X_s + \frac{1}{2} \int_0^t \partial^2_{xx} f(s, X_x) d[X]_s \quad t \in [0,T].$$

[12] gives a similar formula in the discretization approach to pathwise stochastic integration.
For that purpose, let $\mathbb{Y}$ (resp. $\mathbb{X}$) be a B^*-valued strongly measurable with locally bounded paths (resp. B-valued continuous) process, B denoting a separable Banach space, we define a real valued forward-type integral $\int_0^t {}_{B^*}\langle Y, d^- X\rangle_B$, see Definition 5.1. We emphasize that Theorem 5.2 constitutes a generalization of the Itô formula in [20], section 3.7, (see also [8]) for two reasons. First, taking $\chi = (B\hat{\otimes}_\pi B)^*$, i.e. the full space, the integrator processes $\mathbb{X}$ that we consider is more general than those in the class considered in [20] or [8]. The second more important reason is the use of space χ which gives a supplementary degree of freedom.
In the last section 6, we introduce two applications of our infinite dimensional stochastic calculus. That section concentrates on window processes, which first motivated our general construction. In Section 6.1 we discuss an application of the Itô formula to anticipating calculus in a framework for which Malliavin calculus cannot be used necessarily. In Section 6.2, we discuss the application to a representation result of *type Clark-Ocone* for not necessarily semimartingales with finite quadratic variation, including zero quadratic variation. Let X be a continuous stochastic process with finite quadratic variation $[X]_t = \sigma^2 t, \sigma \geq 0$. Our Itô formula is one basic ingredient to prove a Clark-Ocone result for path dependent real random variables of the type $h := H(X_T(\cdot))$ with $H : C([-T, 0]) \longrightarrow \mathbb{R}$. We are interested in natural sufficient conditions to decompose h into the sum of a real number H_0 and a forward integral $\int_0^T \xi_t d^- X_t$. Suppose that $u \in C^{1,2}\left([0, T] \times C([-T, 0])\right)$ is a solution of an infinite dimensional partial differential equation (PDE) of the type

$$\begin{cases} \partial_t u(t,\eta) + \text{``}\displaystyle\int_{]-t,0]} D^\perp u\,(t,\eta)\; d\eta\text{''} + \dfrac{\sigma^2}{2}\langle D^2 u\,(t,\eta)\;,\; \mathbb{1}_{D_t}\rangle = 0 \\ u(T,\eta) = H(\eta) \end{cases} \tag{1.6}$$

where $\mathbb{1}_{D_t}(x,y) := \begin{cases} 1 & \text{if } x = y,\ x, y \in [-t, 0] \\ 0 & \text{otherwise} \end{cases}$ and $D^\perp u\,(t,\eta) := Du\,(t,\eta) - Du(\{0\})\delta_0\,(t,\eta)$.
$Du(t,\eta)$ (resp. $D^2u(t,\eta)$) denotes the first (resp. second) order Fréchet derivatives of u. A proper notion of solution for (1.6) will be given in Definition 6.8. The integral "$\int_{]-t,0]} D^\perp u\,(t,\eta)\; d\eta$" has to be suitably defined. At this stage we only say that supposing $D^\perp u\,(t,\eta)$ absolutely continuous with respect to Lebesgue measure and that its Radon–Nikodym derivative has bounded variation, then $\int_{]-t,0]} D^\perp u\,(t,\eta)\; d\eta$ is well-defined by an integration by parts, see Notation 6.1. The term $\langle D^2 u\,(t,\eta)\;,\; \mathbb{1}_{D_t}\rangle$ indicates the evaluation of the second order derivative on the increasing diagonal of the square $[-t, 0]^2$, provided that $D^2u(t,\eta)$ is a measure on $[-T, 0]^2$. Our Itô formula, i.e. Theorem 5.2, allows in fact to get the mentioned representation above with $H_0 = u(0, X_0(\cdot))$, $\xi_t = D^{\delta_0}u(t, X_t(\cdot)) := Du(t, X_t(\cdot))(\{0\})$. In Chapter 9 of [6] we construct explicitly solutions of the infinite dimensional PDE (1.6) when H has some smooth regularity in $L^2([-\tau, 0])$ or when it depends (even non smoothly) on a finite number of Wiener integrals.
A third application of Theorem 5.2 appears in [11]. In particular, those two authors calculate and they use the χ-quadratic variation of a mild solution of a stochastic PDE which generally is not a finite quadratic variation process in the sense of [5].

The paper is organized as follows. After this introduction, Section 2 contains general notations and some preliminary results. Section 3 will be devoted to the definition of χ-covariation and χ-quadratic variation and some related results. Section 5 is devoted to the definition of a forward integral for Banach space valued processes and related Itô formula. The final section is devoted to applications of our Itô formula to the case of window processes.

2 Preliminaries

Throughout this paper we will denote by $(\Omega, \mathbb{F}, \mathbb{P})$ a fixed probability space, equipped with a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ fulfilling the usual conditions. Let K be a compact space; $C(K)$ denotes the linear space of real continuous functions defined on K, equipped with the uniform norm denoted by $\|\cdot\|_\infty$. $\mathcal{M}(K)$ will denote the dual space $C(K)^*$, i.e. the set of finite signed Borel measures on K. In particular, if $a < b$ are two real numbers, $C([a,b])$ will denote the Banach linear space of real continuous functions. If E is a topological space, $\mathcal{B}or(E)$ will denote its Borel σ-algebra. The topological dual (resp. bidual) space of B will be denoted by B^* (resp. B^{**}). If ϕ is a linear continuous functional on B, we shall denote the value of ϕ of an element $b \in B$ either by $\phi(b)$ or $\langle \phi, b\rangle$ or even ${}_{B^*}\langle \phi, b\rangle_B$. Throughout the paper the symbols $\langle \cdot, \cdot\rangle$ will always denote some type of duality that will change depending on the context. Let E, F, G be Banach spaces. $L(E;F)$ stands for the Banach space of linear bounded maps from E to F. We shall denote the space of $\mathbb{R}$-valued bounded bilinear forms on the product $E \times F$ by $\mathcal{B}(E,F)$ with the norm given by $\|\phi\|_{\mathcal{B}} = \sup\{|\phi(e,f)| : \|e\|_E \leq 1; \|f\|_F \leq 1\}$. Our principal references about functional analysis and about Banach spaces topologies are [9, 1].
T will always be a positive fixed real number. The capital letters $\mathbb{X}, \mathbb{Y}, \mathbb{Z}$ (resp. X, Y, Z) will generally denote Banach space (resp. real) valued processes indexed by the time variable $t \in [0,T]$. A stochastic process $\mathbb{X}$ will also be denoted by $(\mathbb{X}_t)_{t\in[0,T]}$. A B-valued (resp. $\mathbb{R}$-valued) stochastic process $\mathbb{X} : \Omega \times [0,T] \to B$ (resp. $\mathbb{X} : \Omega \times [0,T] \to \mathbb{R}$) is said to be measurable if $\mathbb{X} : \Omega \times [0,T] \longrightarrow B$ (resp. $\mathbb{X} : \Omega \times [0,T] \to \mathbb{R}$) is measurable with respect to the σ-algebras $\mathcal{F} \otimes \mathcal{B}or([0,T])$ and $\mathcal{B}or(B)$ (resp. $\mathcal{B}or(\mathbb{R})$). We recall that $\mathbb{X} : \Omega \times [0,T] \longrightarrow B$ (resp. $\mathbb{R}$) is said to be **strongly measurable** (or **measurable in the Bochner sense**) if it is the limit of measurable countable valued functions. If $\mathbb{X}$ is measurable, càdlàg and B is separable then $\mathbb{X}$ is strongly measurable. If B is finite dimensional then a measurable process $\mathbb{X}$ is also strongly measurable. All the processes indexed by $[0,T]$ will be naturally prolonged by continuity setting $\mathbb{X}_t = \mathbb{X}_0$ for $t \leq 0$ and $\mathbb{X}_t = \mathbb{X}_T$ for $t \geq T$. A sequence $(\mathbb{X}^n)_{n\in\mathbb{N}}$ of continuous B-valued processes indexed by $[0,T]$, will be said to converge **ucp** (**uniformly convergence in probability**) to a process $\mathbb{X}$ if $\sup_{0\leq t\leq T} \|\mathbb{X}^n - \mathbb{X}\|_B$ converges to zero in probability when $n \to \infty$. The space $\mathscr{C}([0,T])$ will denote the linear space of continuous real processes; it is a Fréchet space (or F-space shortly) if equipped with the metric $d(X,Y) = \mathbb{E}\left[\sup_{t\in[0,T]} |X_t - Y_t| \wedge 1\right]$ which governs the ucp topology, see Definition II.1.10 in [9]. For more details about F-spaces and their properties see section II.1 in [9].
A fundamental property of the tensor product of Banach spaces which will be used in the whole paper is the following. If $\tilde{T} : E \times F \to \mathbb{R}$ is a continuous bilinear form, there exists a unique bounded linear operator $T : E\hat{\otimes}F \to \mathbb{R}$ satisfying ${}_{(E\hat{\otimes}_\pi F)^*}\langle T, e\otimes f\rangle_{E\hat{\otimes}_\pi F} = T(e\otimes f) = \tilde{T}(e,f)$ for every $e \in E$, $f \in F$. We observe moreover that there exists a canonical identification between $\mathcal{B}(E,F)$ and $L(E;F^*)$ which identifies $\tilde{T}$ with $\bar{T} : E \to F^*$ by $\tilde{T}(e,f) = \bar{T}(e)(f)$. Summarizing, there is an isometric isomorphism between the dual space of the projective tensor product and the space of bounded bilinear forms equipped with the usual norm, i.e.

$$(E\hat{\otimes}_\pi F)^* \cong \mathcal{B}(E,F) \cong L(E;F^*) \ . \tag{2.1}$$

With this identification, the action of a bounded bilinear form T as a bounded linear functional on $E\hat{\otimes}_\pi F$ is given by

$${}_{(E\hat{\otimes}_\pi F)^*}\langle T, \sum_{i=1}^n x_i \otimes y_i\rangle_{E\hat{\otimes}_\pi F} = T\left(\sum_{i=1}^n x_i\otimes y_i\right) = \sum_{i=1}^n \tilde{T}(x_i,y_i) = \sum_{i=1}^n \bar{T}(x_i)(y_i).$$

In the sequel that identification will often be used without explicit mention.
The importance of tensor product spaces and their duals is justified first of all by identification (2.1):

indeed the second order Fréchet derivative of a real function defined on a Banach space E belongs to $\mathcal{B}(E,E)$. We state a useful result involving Hilbert tensor products and Hilbert direct sums.

Proposition 2.1. Let X and Y_1, Y_2 be Hilbert spaces.We consider $Y = Y_1 \oplus Y_2$ equipped with the Hilbert direct norm. Then $X\hat{\otimes}_h Y = (X\hat{\otimes}_h Y_1) \oplus (X\hat{\otimes}_h Y_2)$.

Proof. Since $X \otimes Y_i \subset X \otimes Y$, $i = 1, 2$ we can write $X \otimes_h Y_i \subset X \otimes_h Y$ and so

$$(X\hat{\otimes}_h Y_1) \oplus (X\hat{\otimes}_h Y_2) \subset X\hat{\otimes}_h Y \tag{2.2}$$

Since we handle with Hilbert norms, it is easy to show that the norm topology of $X\hat{\otimes}_h Y_1$ and $X\hat{\otimes}_h Y_2$ is the same as the one induced by $X\hat{\otimes}_h Y$.
It remains to show the converse inclusion of (2.2). This follows because $X \otimes Y \subset X\hat{\otimes}_h Y_1 \oplus X\hat{\otimes}_h Y_2$. □

We recall another important property.

$$\mathcal{M}([-\tau,0]^2) = \big(C([-\tau,0]^2)\big)^* \subset \big(C([-\tau,0])\hat{\otimes}_\pi C([-\tau,0])\big)^* \cong \mathcal{B}(C([-\tau,0]),C([-\tau,0]))\,. \tag{2.3}$$

With every $\mu \in \mathcal{M}([-\tau,0]^2)$ we can associate a unique operator $T^\mu \in \mathcal{B}(C([-\tau,0]),C([-\tau,0]))$ defined by $T^\mu(f,g) = \int_{[-\tau,0]^2} f(x)g(y)\mu(dx,dy)$.
Let η_1, η_2 be two elements in $C([-\tau,0])$. The element $\eta_1 \otimes \eta_2$ in the algebraic tensor product $C([-\tau,0])\otimes^2$ will be identified with the element η in $C([-\tau,0]^2)$ defined by $\eta(x,y) = \eta_1(x)\eta_2(y)$ for all x, y in $[-\tau,0]$. So if μ is a measure on $\mathcal{M}([-\tau,0]^2)$, the pairing duality ${}_{\mathcal{M}([-\tau,0]^2)}\langle \mu, \eta_1 \otimes \eta_2\rangle_{C([-\tau,0]^2)}$ has to be understood as the following pairing duality:

$${}_{\mathcal{M}([-\tau,0]^2)}\langle \mu, \eta\rangle_{C([-\tau,0]^2)} = \int_{[-\tau,0]^2} \eta(x,y)\mu(dx,dy) = \int_{[-\tau,0]^2} \eta_1(x)\eta_2(y)\mu(dx,dy)\,. \tag{2.4}$$

In the Itô formula for B valued processes at Section 5, naturally appear the first and second order Fréchet derivatives of some functionals defined on a general Banach space B. When $B = C([-\tau,0])$, the first derivative belongs to $\mathcal{M}([-\tau,0])$ and second derivative mostly belongs to $\mathcal{M}([-\tau,0]^2)$. In particular in Sections 4 and 6 those spaces and their subsets appear in relation with window processes. We introduce a notation which has been already used in the Introduction.

Notation 2.2. Let μ be a measure on $\mathcal{M}([-\tau,0])$, $\tau > 0$. μ^{δ_0} will denote the scalar defined by $\mu(\{0\})$ and $\mu^\perp$ will denote the measure defined by $\mu - \mu^{\delta_0}\delta_0$. If $\mu^\perp$ is absolutely continuous with respect to Lebesgue measure, its density will be denoted with the same letter $\mu^\perp$.

Let B be a Banach space. A function $F : [0,T] \times B \longrightarrow \mathbb{R}$, is said to be Fréchet of class $C^{1,2}([0,T] \times B)$, or $C^{1,2}$ for short, if the following properties are fulfilled.

- F is once Fréchet continuously differentiable; the partial derivative with respect to t will be denoted by $\partial_t F : [0,T] \times B \longrightarrow \mathbb{R}$;
- for any $t \in [0,T]$, $x \mapsto DF(t,x)$ is of class C^1 where $DF : [0,T] \times B \longrightarrow B^*$ denotes the Fréchet derivative with respect to the second argument;
- the second order Fréchet derivative with respect to the second argument $D^2F : [0,T] \times B \to (B\hat{\otimes}_\pi B)^*$ is continuous.

3 Chi-covariation and Chi-quadratic variation

3.1 Notion and examples of Chi-subspaces

Definition 3.1. Let E be a Banach space. A Banach space χ included in E will be said a **continuously embedded Banach subspace** of E if the inclusion of χ into E is continuous.
If $E = (B_1 \hat{\otimes}_\pi B_2)^*$ then χ will be said **Chi-subspace** (of E).

Remark 3.2.
1. Let χ be a linear subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$ with Banach structure. χ is a Chi-subspace if and only if $\| \cdot \|_{(B_1 \hat{\otimes}_\pi B_2)^*} \leq \| \cdot \|_\chi$, where $\| \cdot \|_\chi$ is a norm related to the topology of χ.
2. Any continuously embedded Banach subspace of a Chi-subspace is a Chi-subspace.
3. Let $\chi_1, \cdots, \chi_n$ be Chi-subspaces such that, for any $1 \leq i \neq j \leq n$, $\chi_i \bigcap \chi_j = \{0\}$ where 0 is the zero of $(B_1 \hat{\otimes}_\pi B_2)^*$. Then the normed space $\chi = \chi_1 \oplus \cdots \chi_n$ is a Chi-subspace.

The last item allows to express a Chi-subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$ as direct sum of Chi-subspaces (of $(B_1 \hat{\otimes}_\pi B_2)^*$). This, together with Proposition 3.16, helps to evaluate the χ-covariations and the χ-quadratic variations of different processes.

Before providing the definition of the so-called χ-covariation between a B_1-valued and a B_2-valued stochastic processes, we will give some examples of Chi-subspaces that we will use in the paper.

Example 3.3. Let B_1, B_2 be two Banach spaces.

- $\chi = (B_1 \hat{\otimes}_\pi B_2)^*$. This appears in our elementary situation anticipated in the Introduction, see also Proposition 3.14.

Example 3.4. Let $B_1 = B_2 = C([-\tau, 0])$.
This is the natural value space for all the window of continuous processes. We list some examples of Chi-subspaces χ for which some window processes have a χ-covariation or a χ-quadratic variation. Moreover those χ-covariation and χ-quadratic variation will intervene in applications in Section 6. Our basic reference Chi-subspace of $(C([-\tau, 0] \hat{\otimes}_\pi C([-\tau, 0]))^*$ will be the Banach space $\mathcal{M}([-\tau, 0]^2)$ equipped with the usual total variation norm, denoted by $\| \cdot \|_{Var}$. The relation in item 1 of Remark 3.2 is verified since $\|T^\mu\|_{(B \hat{\otimes}_\pi B)^*} = \sup_{\|f\| \leq 1, \|g\| \leq 1} |T^\mu(f, g)| \leq \|\mu\|_{Var}$ for every $\mu \in \mathcal{M}([-\tau, 0]^2)$. All the other spaces considered in the sequel of the present example will be shown to be continuously embedded Banach subspaces of $\mathcal{M}([-\tau, 0]^2)$; by item 2 of Remark 3.2 they are Chi-subspaces. Here is a list. Let a, b two fixed given points in $[-\tau, 0]$.

- $L^2([-\tau, 0]^2) \cong L^2([-\tau, 0]) \hat{\otimes}_h^2$ is a Hilbert subspace of $\mathcal{M}([-\tau, 0]^2)$, equipped with the norm derived from the usual scalar product. The Hilbert tensor product $L^2([-\tau, 0]) \hat{\otimes}_h^2$ will be always identified with $L^2([-\tau, 0]^2)$, conformally to a quite canonical procedure, see [21], chapter 6.
- $\mathcal{D}_{a,b}([-\tau, 0]^2)$ (shortly $\mathcal{D}_{a,b}$) which denotes the one dimensional Hilbert space of the multiples of the Dirac measure concentrated at $(a, b) \in [-\tau, 0]^2$, i.e.
$$\mathcal{D}_{a,b}([-\tau, 0]^2) := \{\mu \in \mathcal{M}([-\tau, 0]^2);\ s.t. \mu(dx, dy) = \lambda\, \delta_a(dx) \delta_b(dy) \text{ with } \lambda \in \mathbb{R}\} \cong \mathcal{D}_a \hat{\otimes}_h \mathcal{D}_b\ . \quad (3.1)$$
If $\mu = \lambda\, \delta_a(dx) \delta_b(dy)$, $\|\mu\|_{Var} = |\lambda| = \|\mu\|_{\mathcal{D}_{a,b}}$.
- $\mathcal{D}_a([-\tau, 0]) \hat{\otimes}_h L^2([-\tau, 0])$ and $L^2([-\tau, 0]) \hat{\otimes}_h \mathcal{D}_a([-\tau, 0])$ where $\mathcal{D}_a([-\tau, 0])$ (shortly $\mathcal{D}_a$) denotes the one-dimensional space of multiples of the Dirac measure concentrated at $a \in [-\tau, 0]$, i.e.
$$\mathcal{D}_a([-\tau, 0]) := \{\mu \in \mathcal{M}([-\tau, 0]);\ s.t. \mu(dx) = \lambda\, \delta_a(dx) \text{ with } \lambda \in \mathbb{R}\}\ . \quad (3.2)$$

$\mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ (resp. $L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_a([-\tau,0])$) is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$ and for a general element in this space $\mu=\lambda\delta_a(dx)\phi(y)dy$ (resp. $\mu=\lambda\phi(x)dx\delta_a(dy)$), $\phi\in L^2([-\tau,0])$, we have $\|\mu\|_{Var}\le\|\mu\|_{\mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])}$(resp. $\|\mu\|_{L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_a([-\tau,0])}) = |\lambda|\cdot\|\phi\|_{L^2}$.

- $\chi^0([-\tau,0]^2)$, χ^0 shortly, which denotes the subspace of measures defined as $\chi^0([-\tau,0]^2):=(\mathcal{D}_0([-\tau,0])\oplus L^2([-\tau,0]))\hat{\otimes}_h^2$.

  Remark 3.5. An element μ in $\chi^0([-\tau,0]^2)$ can be uniquely decomposed as $\mu=\phi_1+\phi_2\otimes\delta_0+\delta_0\otimes\phi_3+\lambda\delta_0\otimes\delta_0$, where $\phi_1\in L^2([-\tau,0]^2)$, ϕ_2, ϕ_3 are functions in $L^2([-\tau,0])$ and λ is a real number. We have $\mu(\{0,0\})=\lambda$.

- $Diag([-\tau,0]^2)$ (shortly $Diag$), will denote the subset of $\mathcal{M}([-\tau,0]^2)$ defined as follows:

$$Diag([-\tau,0]^2):=\left\{\mu^g\in\mathcal{M}([-\tau,0]^2)\; s.t.\; \mu^g(dx,dy)=g(x)\delta_y(dx)dy;\; g\in L^\infty([-\tau,0])\right\}\;. \tag{3.3}$$

  $Diag([-\tau,0]^2)$, equipped with the norm $\|\mu^g\|_{Diag([-\tau,0]^2)}=\|g\|_\infty$, is a Banach space. Let f be a function in $C([-\tau,0]^2)$; the pairing duality (2.4) between f and $\mu(dx,dy)=g(x)\delta_y(dx)dy\in Diag$ gives

$$_{C([-\tau,0]^2)}\langle f,\mu\rangle_{Diag([-\tau,0]^2)}=\int_{[-\tau,0]^2} f(x,y)g(x)\delta_y(dx)dy=\int_{-\tau}^0 f(x,x)g(x)dx\;.$$

  The following are two closed subspaces of $Diag([-\tau,0]^2)$. We denote by $Diag_c([-\tau,0]^2)$ (resp. $Diag_d([-\tau,0]^2)$) the subset constituted by measures $\mu^g\in Diag([-\tau,0]^2)$ for which g belongs to $C([-\tau,0])$ (resp. in $D([-\tau,0])$). We remind that $D([-\tau,0])$ is the set of the (classes of) bounded functions $g:[-\tau,0]\longrightarrow\mathbb{R}$ admitting a càdlàg version.

3.2 Definition of χ-covariation and some related results

Let B_1, B_2 and B be three Banach spaces. In this subsection, we introduce the definition of χ-covariation between a B_1-valued stochastic process $\mathbb{X}$ and a B_2-valued stochastic process $\mathbb{Y}$. We remind that $\mathscr{C}([0,T])$ denotes the space of continuous processes equipped with the ucp topology.
Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be B_1 (resp. B_2) valued stochastic process. Let χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$ and $\epsilon>0$. We denote by $[\mathbb{X},\mathbb{Y}]^\epsilon$, the following application

$$[\mathbb{X},\mathbb{Y}]^\epsilon:\chi\longrightarrow\mathscr{C}([0,T])\quad\text{defined by}\quad \phi\mapsto\left(\int_0^t {}_\chi\langle\phi,\frac{J\left((\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\right)}{\epsilon}\rangle_{\chi^*}\,ds\right)_{t\in[0,T]} \tag{3.4}$$

where $J:B_1\hat{\otimes}_\pi B_2\longrightarrow(B_1\hat{\otimes}_\pi B_2)^{**}$ is the canonical injection between a space and its bidual. With application $[\mathbb{X},\mathbb{Y}]^\epsilon$ it is possible to associate another one, denoted by $\widetilde{[\mathbb{X},\mathbb{Y}]}^\epsilon$, defined by

$$\widetilde{[\mathbb{X},\mathbb{Y}]}^\epsilon(\omega,\cdot):[0,T]\longrightarrow\chi^*\quad\text{such that}\quad t\mapsto\left(\phi\mapsto\int_0^t {}_\chi\langle\phi,\frac{J\left((\mathbb{X}_{s+\epsilon}(\omega)-\mathbb{X}_s(\omega))\otimes(\mathbb{Y}_{s+\epsilon}(\omega)-\mathbb{Y}_s(\omega))\right)}{\epsilon}\rangle_{\chi^*}\,ds\right).$$

Remark 3.6. 1. We recall that $\chi\subset(B_1\hat{\otimes}_\pi B_2)^*$ implies $(B_1\hat{\otimes}_\pi B_2)^{**}\subset\chi^*$.

2. As indicated, ${}_\chi\langle\cdot,\cdot\rangle_{\chi^*}$ denotes the duality between the space χ and its dual χ^*. In fact by assumption, ϕ is an element of χ and element $J\left((\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\right)$ naturally belongs to $(B_1\hat{\otimes}_\pi B_2)^{**}\subset\chi^*$.

3. With a slight abuse of notation, in the sequel the injection J from $B_1\hat{\otimes}_\pi B_2$ to its bidual will be omitted. The tensor product $(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)$ has to be considered as the element $J\left((\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\right)$ which belongs to χ^*.

4. Suppose $B_1=B_2=B=C([-\tau,0])$ and let χ be a Chi-subspace.
An element of the type $\eta=\eta_1\otimes\eta_2$, η_1, $\eta_2\in B$, can be either considered as an element of the type $B\hat{\otimes}_\pi B\subset(B\hat{\otimes}_\pi B)^{**}\subset\chi^*$ or as an element of $C([-\tau,0]^2)$ defined by $\eta(x,y)=\eta_1(x)\eta_2(y)$. When χ is indeed a closed subspace of $\mathcal{M}([\tau,0]^2)$, then the pairing between χ and χ^* will be compatible with the pairing duality between $\mathcal{M}([\tau,0]^2)$ and $C([-\tau,0]^2)$ given by (2.4).

Definition 3.7. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1 (resp. B_2) valued stochastic process. We say that $\mathbb{X}$ **and** $\mathbb{Y}$ **admit a χ-covariation** if the following assumptions hold.

H1 For all sequence (ϵ_n) it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k\int_0^T\sup_{\|\phi\|_\chi\leq 1}\left|\langle\phi,\frac{(\mathbb{X}_{s+\epsilon_{n_k}}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon_{n_k}}-\mathbb{Y}_s)}{\epsilon_{n_k}}\rangle\right|ds=\sup_k\int_0^T\frac{\left\|(\mathbb{X}_{s+\epsilon_{n_k}}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon_{n_k}}-\mathbb{Y}_s)\right\|_{\chi^*}}{\epsilon_{n_k}}ds\ <\infty\ a.s.$$

H2 (i) There exists an application $\chi\longrightarrow\mathscr{C}([0,T])$, denoted by $[\mathbb{X},\mathbb{Y}]$, such that

$$[\mathbb{X},\mathbb{Y}]^\epsilon(\phi)\xrightarrow[\epsilon\longrightarrow 0_+]{ucp}[\mathbb{X},\mathbb{Y}](\phi)\tag{3.5}$$

for every $\phi\in\chi\subset(B_1\hat{\otimes}_\pi B_2)^*$.

(ii) There is a measurable process $\widetilde{[\mathbb{X},\mathbb{Y}]}:\Omega\times[0,T]\longrightarrow\chi^*$, such that

- for almost all $\omega\in\Omega$, $\widetilde{[\mathbb{X},\mathbb{Y}]}(\omega,\cdot)$ is a (càdlàg) bounded variation function,
- $\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,t)(\phi)=[\mathbb{X},\mathbb{Y}](\phi)(\cdot,t)$ a.s. for all $\phi\in\chi$, $t\in[0,T]$.

If $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-covariation we will call **χ-covariation** of $\mathbb{X}$ and $\mathbb{Y}$ the χ^*-valued process $(\widetilde{[\mathbb{X},\mathbb{Y}]})_{0\leq t\leq T}$. By abuse of notation, $[\mathbb{X},\mathbb{Y}]$ will also be called χ-covariation and it will be sometimes confused with $\widetilde{[\mathbb{X},\mathbb{Y}]}$.

Definition 3.8. Let $\mathbb{X}=\mathbb{Y}$ be a B-valued stochastic process and χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. The χ-covariation $[\mathbb{X},\mathbb{X}]$ (or $\widetilde{[\mathbb{X},\mathbb{X}]}$) will also be denoted by $[\mathbb{X}]$ and $\widetilde{[\mathbb{X}]}$; it will be called **χ-quadratic variation of** $\mathbb{X}$ and we will say that $\mathbb{X}$ has a χ-quadratic variation.

Remark 3.9. 1. For every fixed $\phi\in\chi$, the processes $\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,t)(\phi)$ and $[\mathbb{X},\mathbb{Y}](\phi)(\cdot,t)$ are indistinguishable. In particular the χ^*-valued process $\widetilde{[\mathbb{X},\mathbb{Y}]}$ is weakly star continuous, i.e. $\widetilde{[\mathbb{X},\mathbb{Y}]}(\phi)$ is continuous for every fixed ϕ.

2. The existence of $\widetilde{[\mathbb{X},\mathbb{Y}]}$ guarantees that $[\mathbb{X},\mathbb{Y}]$ admits a bounded variation version which allows to consider it as pathwise integrator.

3. The quadratic variation $\widetilde{[\mathbb{X}]}$ will be the object intervening in the second order term of the Itô formula expanding $F(\mathbb{X})$ for some C^2-Fréchet function F, see Theorem 5.2.

4. In Corollaries 3.24 and 3.25 we will show that, whenever χ is separable (most of the cases) the Condition **H2** can be relaxed in a significant way. In fact the Condition **H2**(i) reduces to the convergence in probability of (3.5) on a dense subspace and **H2**(ii) will be automatically granted.

Remark 3.10. 1. A practical criterion to verify Condition **H1** is

$$\frac{1}{\epsilon}\int_0^T \|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\|_{\chi^*}\,ds \le B(\epsilon)$$

where $B(\epsilon)$ converges in probability when ϵ goes to zero. In fact the convergence in probability implies the a.s. convergence of a subsequence.

2. A consequence of Condition **H1** is that for all $(\epsilon_n)\downarrow 0$ there exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \|\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon_{n_k}}\|_{Var([0,T])} < \infty \quad a.s.$$

In fact $\|\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}\|_{Var([0,T])} \le \frac{1}{\epsilon}\int_0^T \|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)\|_{\chi^*}ds$, which implies that $\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}$ is a χ^*-valued process with bounded variation on $[0,T]$. As a consequence, for a χ-valued continuous stochastic process $\mathbb{Z}$, $t\in[0,T]$, the integral $\int_0^t {}_{\chi}\langle \mathbb{Z}_s, d\widetilde{[\mathbb{X},\mathbb{Y}]}_s^{\epsilon_{n_k}}\rangle_{\chi^*}$ is a well-defined Lebesgue-Stieltjes type integral for almost all $\omega\in\Omega$.

Remark 3.11. 1. To a Borel function $G:\chi\longrightarrow C([0,T])$ we can associate $\tilde{G}:[0,T]\longrightarrow\chi^*$ setting $\tilde{G}(t)(\phi)=G(\phi)(t)$. By definition $\tilde{G}:[0,T]\longrightarrow\chi^*$ has bounded variation if $\|\tilde{G}\|_{Var([0,T])} := \sup_{\sigma\in\Sigma_{[0,T]}}\sum_{i|(t_i)_i=\sigma}\left\|\tilde{G}(t_{i+1})-\tilde{G}(t_i)\right\|_{\chi^*} = \sup_{\sigma\in\Sigma_{[0,T]}}\sum_{i|(t_i)_i=\sigma}\sup_{\|\phi\|_\chi\le 1}|G(\phi)(t_{i+1})-G(\phi)(t_i)|$ is finite, where $\Sigma_{[0,T]}$ is the set of all possible partitions $\sigma=(t_i)_i$ of the interval $[0,T]$. This quantity is the **total variation** of $\tilde{G}$. For example if $G(\phi)=\int_0^t \dot{G}_s(\phi)\,ds$ with $\dot{G}:\chi\to C([0,T])$ Bochner integrable, then $\|G\|_{Var[0,T]}\le\int_0^T\sup_{\|\phi\|_\chi\le 1}|\dot{G}_s(\phi)|\,ds$.

2. If $G(\phi)$, $\phi\in\chi$ is a family of stochastic processes, it is not obvious to find a good version $\tilde{G}:[0,T]\longrightarrow\chi^*$ of G. This will be the object of Theorem 3.22.

Definition 3.12. If the χ-covariation exists with $\chi=(B_1\hat{\otimes}_\pi B_2)^*$, we say that $\mathbb{X}$ and $\mathbb{Y}$ admit a **global covariation**. Analogously if $\mathbb{X}$ is B-valued and the χ-quadratic variation exists with $\chi=(B\hat{\otimes}_\pi B)^*$, we say that $\mathbb{X}$ admits a **global quadratic variation**.

Remark 3.13. 1. $\widetilde{[\mathbb{X},\mathbb{Y}]}$ takes values "a priori" in $(B_1\hat{\otimes}_\pi B_2)^{**}$.

2. If $[\mathbb{X},\mathbb{Y}]^{\mathbb{R}}$ exists then Condition **H1** follows by Remark 3.10.1.

Proposition 3.14. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) process such that $\mathbb{X}$ and $\mathbb{Y}$ admit scalar and tensor covariation. Then $\mathbb{X}$ and $\mathbb{Y}$ admit a global covariation. In particular the global covariation takes values in $B_1\hat{\otimes}_\pi B_2$ and $\widetilde{[\mathbb{X},\mathbb{Y}]}=[\mathbb{X},\mathbb{Y}]^{\otimes}$ a.s.

Proof. We set $\chi=(B_1\hat{\otimes}_\pi B_2)^*$. Taking into account Remark 3.13.2, it will be enough to verify Condition **H2**. Recalling the definition of $[\mathbb{X},\mathbb{Y}]^\epsilon$ at (3.4) and the definition of injection J we observe that

$$[\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(\cdot,t)=\int_0^t {}_{(B_1\hat{\otimes}_\pi B_2)^*}\langle\phi,\frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)}{\epsilon}\rangle_{B_1\hat{\otimes}_\pi B_2}\,ds\,. \tag{3.6}$$

Since Bochner integrability implies Pettis integrability, for every $\phi \in (B_1\hat{\otimes}_\pi B_2)^*$, we also have

$$ {}_{(B_1\hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_t \rangle_{B_1\hat{\otimes}_\pi B_2} = \int_0^t {}_{(B_1\hat{\otimes}_\pi B_2)^*}\langle \phi, \frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes(\mathbb{Y}_{s+\epsilon}-\mathbb{Y}_s)}{\epsilon} \rangle_{B_1\hat{\otimes}_\pi B_2} \, ds \, . \tag{3.7} $$

(3.6) and (3.7) imply that

$$ [\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(\cdot,t) = {}_{(B_1\hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_t \rangle_{B_1\hat{\otimes}_\pi B_2} \qquad \text{a.s.} \tag{3.8} $$

Concerning the validity of Condition **H2** we will show that

$$ \sup_{t\le T} \left| [\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(\cdot,t) - {}_{(B_1\hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes}_t \rangle_{B_1\hat{\otimes}_\pi B_2} \right| \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} 0 \, . \tag{3.9} $$

By (3.8) the left-hand side of (3.9) gives

$$ \sup_{t\le T} \left| {}_{(B_1\hat{\otimes}_\pi B_2)^*}\langle \phi, [\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_t - [\mathbb{X},\mathbb{Y}]^{\otimes}_t \rangle_{B_1\hat{\otimes}_\pi B_2} \right| \le \|\phi\|_{(B_1\hat{\otimes}_\pi B_2)^*} \sup_{t\le T} \left\| [\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon}_t - [\mathbb{X},\mathbb{Y}]^{\otimes}_t \right\|_{B_1\hat{\otimes}_\pi B_2} , $$

where the last quantity converges to zero in probability by Definition 1.4 item 2 of the tensor quadratic variation; this implies (3.9). The tensor quadratic variation has always bounded variation because of item 2 of Remark 1.5. In conclusion **H2**(ii) is also verified. □

Remark 3.15. We observe some interesting features related to the global covariation, i.e. the χ-covariation when $\chi = (B_1\hat{\otimes}_\pi B_2)^*$.

1. When χ is separable, for any $t \in [0,T]$, there exists a null subset N of Ω and a sequence (ϵ_n) such that $\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon_n}(\omega,t) \xrightarrow[\epsilon\longrightarrow 0]{} \widetilde{[\mathbb{X},\mathbb{Y}]}(\omega,t)$ weak star for $\omega \notin N$, see Lemma A.1. This confirms the relation between the global covariation and the weak star convergence in the space $(B_1\hat{\otimes}_\pi B_2)^{**}$ as anticipated in the Introduction.

2. We recall that $J(B_1\hat{\otimes}_\pi B_2)$ is isometrically embedded (and weak star dense) in $(B_1\hat{\otimes}_\pi B_2)^{**}$. In particular it is the case if B_1 or B_2 has infinite dimension. If the Banach space $B_1\hat{\otimes}_\pi B_2$ is not reflexive, then $(B_1\hat{\otimes}_\pi B_2)^{**}$ strictly contains $B_1\hat{\otimes}_\pi B_2$. The weak star convergence is weaker then the strong convergence in $J(B_1\hat{\otimes}_\pi B_2)$, required in the definition of the tensor quadratic variation, see Definition 1.4 item 2. The global covariation is therefore truly more general than the tensor covariation.

3. In general $B_1\hat{\otimes}_\pi B_2$ is not reflexive even if B_1 and B_2 are Hilbert spaces, see for instance [29] at Section 4.2.

We go on with some related results about the χ-covariation and the χ-quadratic variation.

Proposition 3.16. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) process and χ_1, χ_2 be two Chi-subspaces of $(B_1\hat{\otimes}_\pi B_2)^*$ with $\chi_1 \cap \chi_2 = \{0\}$. Let $\chi = \chi_1 \oplus \chi_2$. If $\mathbb{X}$ and $\mathbb{Y}$ admit a χ_i-covariation $[\mathbb{X},\mathbb{Y}]_i$ for $i = 1, 2$ then they admit a χ-covariation $[\mathbb{X},\mathbb{Y}]$ and it holds $[\mathbb{X},\mathbb{Y}](\phi) = [\mathbb{X},\mathbb{Y}]_1(\phi_1) + [\mathbb{X},\mathbb{Y}]_2(\phi_2)$ for all $\phi \in \chi$ with unique decomposition $\phi = \phi_1 + \phi_2$, $\phi_1 \in \chi_1$ and $\phi_2 \in \chi_2$.

Proof. χ is a Chi-subspace because of item 3 of Remark 3.2. It will be enough to show the result for a fixed norm in the space χ. We set $\|\phi\|_\chi = \|\phi_1\|_{\chi_1} + \|\phi_2\|_{\chi_2}$ and we remark that $\|\phi\|_\chi \geq \|\phi_i\|_{\chi_i}$, $i = 1, 2$. Condition **H1** follows immediately by inequality

$$\int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| {}_\chi\langle \phi, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle_{\chi^*} \right| ds \leq \int_0^T \sup_{\|\phi_1\|_{\chi_1} \leq 1} \left| {}_{\chi_1}\langle \phi_1, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle_{\chi_1^*} \right| ds +$$
$$+ \int_0^T \sup_{\|\phi_2\|_{\chi_2} \leq 1} \left| {}_{\chi_2}\langle \phi_2, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle_{\chi_2^*} \right| ds \,.$$

Condition **H2**(i) follows by linearity; in fact

$$[\mathbb{X}, \mathbb{Y}]^\epsilon(\phi) = \int_0^t {}_\chi\langle \phi_1 + \phi_2, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle_{\chi^*} ds =$$
$$= \int_0^t {}_{\chi_1}\langle \phi_1, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle_{\chi_1^*} ds + \int_0^t {}_{\chi_2}\langle \phi_2, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle_{\chi_2^*} ds$$
$$\xrightarrow[\epsilon \to 0]{ucp} [\mathbb{X}, \mathbb{Y}]_1(\phi_1) + [\mathbb{X}, \mathbb{Y}]_2(\phi_2) \,.$$

Concerning Condition **H2**(ii), for $\omega \in \Omega$, $t \in [0, T]$ we can obviously set $\widetilde{[\mathbb{X}, \mathbb{Y}]}(\omega, t)(\phi) = \widetilde{[\mathbb{X}, \mathbb{Y}]}_1(\omega, t)(\phi_1) + \widetilde{[\mathbb{X}, \mathbb{Y}]}_2(\omega, t)(\phi_2)$. □

Proposition 3.17. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) stochastic process.

1. Let χ_1 and χ_2 be two subspaces $\chi_1 \subset \chi_2 \subset (B_1 \hat{\otimes}_\pi B_2)^*$, χ_1 being a Banach subspace continuously embedded into χ_2 and χ_2 a Chi-subspace. If $\mathbb{X}$ and $\mathbb{Y}$ admit a χ_2-covariation $[\mathbb{X}, \mathbb{Y}]_2$, then they also admit a χ_1-covariation $[\mathbb{X}, \mathbb{Y}]_1$ and it holds $[\mathbb{X}, \mathbb{Y}]_1(\phi) = [\mathbb{X}, \mathbb{Y}]_2(\phi)$ for all $\phi \in \chi_1$.

2. In particular if $\mathbb{X}$ and $\mathbb{Y}$ admit a tensor quadratic variation, then $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-quadratic variation for any Chi-subspace χ.

Proof. 1. If Condition **H1** is valid for χ_2 then it is also verified for χ_1. In fact we remark that $(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)$ is an element in $(B_1 \hat{\otimes}_\pi B_2) \subset (B_1 \hat{\otimes}_\pi B_2)^{**} \subset \chi_2^* \subset \chi_1^*$. If $A := \{\phi \in \chi_1 \,; \|\phi\|_{\chi_1 \leq 1}\}$ and $B := \{\phi \in \chi_2 \,; \|\phi\|_{\chi_2 \leq 1}\}$, then $A \subset B$ and clearly $\int_0^t \sup_{\phi \in A} |\langle \phi, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle| ds \leq \int_0^t \sup_{\phi \in B} |\langle \phi, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s) \rangle| ds$. This implies the inequality $\|(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\|_{\chi_1^*} \leq \|(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\|_{\chi_2^*}$ and Assumption **H1** follows immediately. Assumption **H2**(i) is trivially verified because, by restriction, we have $[\mathbb{X}, \mathbb{Y}]^\epsilon(\phi) \xrightarrow[\epsilon \to 0]{ucp} [\mathbb{X}, \mathbb{Y}]_2(\phi)$ for all $\phi \in \chi_1$. We define $[\mathbb{X}, \mathbb{Y}]_1(\phi) = [\mathbb{X}, \mathbb{Y}]_2(\phi)$, $\forall\ \phi \in \chi_1$ and $\widetilde{[\mathbb{X}, \mathbb{Y}]}_1(\omega, t)(\phi) = \widetilde{[\mathbb{X}, \mathbb{Y}]}_2(\omega, t)(\phi)$, for all $\omega \in \Omega$, $t \in [0, T]$, $\phi \in \chi_1$. Condition **H2**(ii) follows because given $G : [0, T] \longrightarrow \chi_1$ we have $\|G(t) - G(s)\|_{\chi_1^*} \leq \|G(t) - G(s)\|_{\chi_2^*}$, $\forall\, 0 \leq s \leq t \leq T$.

2. It follows from 1. and Proposition 3.14.

□

We continue with some general properties of the χ-covariation.

Lemma 3.18. Let $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) stochastic process and χ be a Chi-subspace. Suppose that $\frac{1}{\epsilon} \int_0^T \|(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\|_{\chi^*} ds$ converges to 0 in probability when ϵ goes to zero.

1. Then $\mathbb{X}$ and $\mathbb{Y}$ admit a zero χ-covariation.
2. If $\chi = (B_1 \hat{\otimes}_\pi B_2)^*$, then $\mathbb{X}$ and $\mathbb{Y}$ admit a zero scalar and tensor covariation.

Proof. Concerning item 1 Condition **H1** is verified because of Remark 3.10 item 1. We verify **H2**(i) directly. For every fixed $\phi \in \chi$ we have

$$|[\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(t)| = \left| \int_0^t {}_\chi\langle \phi, \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)}{\epsilon} \rangle_{\chi^*} \, ds \right| \leq \int_0^T \left| {}_\chi\langle \phi, \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)}{\epsilon} \rangle_{\chi^*} \right| ds.$$

So we obtain

$$\sup_{t\in[0,T]} |[\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(t)| \leq \|\phi\|_\chi \frac{1}{\epsilon} \int_0^T \|(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon} - \mathbb{Y}_s)\|_{\chi^*} \, ds \xrightarrow[\epsilon\longrightarrow 0]{} 0$$

in probability by the hypothesis. Since condition **H2**(ii) holds trivially, we can conclude for the first result. Concerning item 2 the scalar covariation vanishes by hypothesis, which also forces the tensor covariation to be zero, see Remark 1.5, item 4. □

3.3 Technical issues

3.3.1 Convergence of infinite dimensional Stieltjes integrals

We state now an important technical result which will be used in the proof of the Itô formula appearing in Theorem 5.2.

Proposition 3.19. Let χ be a separable Banach space, a sequence $F^n : \chi \longrightarrow \mathscr{C}([0,T])$ of linear continuous maps and measurable random fields $\widetilde{F}^n : \Omega \times [0,T] \longrightarrow \chi^*$ such that $\widetilde{F}^n(\cdot,t)(\phi) = F^n(\phi)(\cdot,t)$ a.s. $\forall\, t \in [0,T]$, $\phi \in \chi$. We suppose the following.

i) For every n, $t \mapsto \widetilde{F}^n(\cdot,t)$ is a. s. of bounded variation and for all (n_k) there is a subsequence (n_{k_j}) such that $\sup_j \|\widetilde{F}^{n_{k_j}}\|_{Var([0,T])} < \infty$ a.s.

ii) There is a linear continuous map $F : \chi \longrightarrow \mathscr{C}([0,T])$ such that for all $t \in [0,T]$ and for every $\phi \in \chi$ $F^n(\phi)(\cdot,t) \longrightarrow F(\phi)(\cdot,t)$ in probability.

iii) There is measurable random field $\widetilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ of such that for ω a.s. $\widetilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ has bounded variation and $\widetilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t)$a.s. $\forall\, t \in [0,T]$ and $\phi \in \chi$.

iv) $F^n(\phi)(0) = 0$ for every $\phi \in \chi$.

Then for every $t \in [0,T]$ and every continuous process $H : \Omega \times [0,T] \longrightarrow \chi$

$$\int_0^t {}_\chi\langle H(\cdot,s), d\widetilde{F}^n(\cdot,s)\rangle_{\chi^*} \longrightarrow \int_0^t {}_\chi\langle H(\cdot,s), d\widetilde{F}(\cdot,s)\rangle_{\chi^*} \qquad \text{in probability.}$$

Proof. See Appendix A. □

Corollary 3.20. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$. Let $\mathbb{X}$ and $\mathbb{Y}$ be two stochastic processes with values in B_1 and B_2 admitting a χ-covariation and $\mathbb{H}$ be a continuous measurable process $\mathbb{H} : \Omega \times [0,T] \longrightarrow \mathcal{V}$ where $\mathcal{V}$ is a closed separable subspace of χ. Then, for every $t \in [0,T]$,

$$\int_0^t {}_\chi\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}^\epsilon(\cdot,s)\rangle_{\chi^*} \xrightarrow[\epsilon\longrightarrow 0]{} \int_0^t {}_\chi\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,s)\rangle_{\chi^*} \quad \text{in probability.} \tag{3.10}$$

Proof. By item 2 in Remark 3.2, $\mathcal{V}$ is a Chi-subspace. By Proposition 3.17, $\mathbb{X}$ and $\mathbb{Y}$ admit a $\mathcal{V}$-covariation $[\mathbb{X},\mathbb{Y}]_{\mathcal{V}}$ and $[\mathbb{X},\mathbb{Y}]_{\mathcal{V}}(\phi) = [\mathbb{X},\mathbb{Y}](\phi)$ for all $\phi \in \mathcal{V}$; in the sequel of the proof, $[\mathbb{X},\mathbb{Y}]_{\mathcal{V}}$ will be still denoted by $[\mathbb{X},\mathbb{Y}]$. Since the ucp convergence implies the convergence in probability for every $t \in [0,T]$, by Proposition 3.19 and definition of $\mathcal{V}$-covariation, it follows

$$\int_0^t {}_{\mathcal{V}}\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon}(\cdot,s)\rangle_{\mathcal{V}^*} \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} \int_0^t {}_{\mathcal{V}}\langle \mathbb{H}(\cdot,s), d\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,s)\rangle_{\mathcal{V}^*} \ .$$

Since the pairing duality between χ and χ^* is compatible with the one between $\mathcal{V}$ and $\mathcal{V}^*$, the result (3.10) is now established. □

3.3.2 Weaker conditions for the existence of the χ-covariation

An important and useful theorem which helps to find sufficient conditions for the existence of the χ-quadratic variation of a Banach space valued process is given below. It will be a consequence of a Banach-Steinhaus type result for Fréchet spaces, see Theorem II.1.18, pag. 55 in [9]. We start with a remark.

Remark 3.21.

1. Let $(\mathbb{Y}^n)$ be a sequence of random elements with values in a Banach space $(B, \|\cdot\|_B)$ such that $\sup_n \|\mathbb{Y}^n\|_B \leq Z$ a.s. for some real positive random variable Z. Then $(\mathbb{Y}^n)$ is **bounded**[1] in the F-space of random elements equipped with the convergence in probability which is governed by the metric $d(\mathbb{X},\mathbb{Y}) = \mathbb{E}\left[\|\mathbb{X}-\mathbb{Y}\|_B \wedge 1\right]$. In fact by Lebesgue dominated convergence theorem it follows $\lim_{\gamma\to 0} \mathbb{E}[\gamma Z \wedge 1] = 0$.

2. In particular taking $B = C([0,T])$ a sequence of continuous processes $(\mathbb{Y}^n)$ such that $\sup_n \|\mathbb{Y}^n\|_\infty \leq Z$ a.s. is bounded for the usual metric in $\mathscr{C}([0,T])$ equipped with the topology related to the ucp convergence.

Theorem 3.22. Let $F^n : \chi \longrightarrow \mathscr{C}([0,T])$ be a sequence of linear continuous maps such that $F^n(\phi)(0) = 0$ a.s. and there is $\tilde{F}^n : \Omega \times [0,T] \longrightarrow \chi^*$ having a.s. bounded variation. We formulate the following assumptions.

i) $F^n(\phi)(\cdot,t) = \tilde{F}^n(\cdot,t)(\phi)$ a.s. $\forall\, t \in [0,T]$, $\phi \in \chi$.

ii) $\forall\, \phi \in \chi$, $t \mapsto \tilde{F}^n(\cdot,t)(\phi)$ is càdlàg.

iii) $\sup_n \|\tilde{F}^n\|_{Var([0,T])} < \infty$ a.s.

iv) There is a subset $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$ and a linear application $F : \mathcal{S} \longrightarrow \mathscr{C}([0,T])$ such that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \mathcal{S}$.

1) Suppose that χ is separable.
Then there is a linear and continuous extension $F : \chi \longrightarrow \mathscr{C}([0,T])$ and there is a measurable random field $\tilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ such that $\tilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t)$ a.s. for every $t \in [0,T]$. Moreover the following properties hold.

[1] This notion plays a role in Banach-Steinhaus theorem in [9]. Let E be a Fréchet spaces, F-space shortly. A subset C of E is called **bounded** if for all $\epsilon > 0$ it exists δ_ϵ such that for all $0 < \alpha \leq \delta_\epsilon$, αC is included in the open ball $\mathcal{B}(0,\epsilon) := \{e \in E; d(0,e) < \epsilon\}$

a) For every $\phi \in \chi$, $F^n(\phi) \xrightarrow{ucp} F(\phi)$.

In particular for every $t \in [0,T]$, $\phi \in \chi$, $F^n(\phi)(\cdot,t) \xrightarrow{\mathbb{P}} F(\phi)(\omega,t)$.

b) $\tilde{F}$ has bounded variation and $t \mapsto \tilde{F}(\cdot,t)$ is weakly star continuous a.s.

2) Suppose the existence of a measurable $\tilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ such that a.s. $t \mapsto \tilde{F}(\cdot,t)$ has bounded variation and is weakly star càdlàg such that

$$\tilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t) \qquad \text{a.s.} \qquad \forall\, t \in [0,T],\ \forall\, \phi \in \mathcal{S}\ .$$

Then point **a)** still follows.

Remark 3.23. In point **2)** we do not necessarily suppose χ to be separable.

Proof. See Appendix A. □

Important implications of Theorem 3.22 are Corollaries 3.24 and 3.25, which give us easier conditions for the existence of the χ-covariation as anticipated in Remark 3.9 item 4.

Corollary 3.24. Let B_1 and B_2 be Banach spaces, $\mathbb{X}$ (resp. $\mathbb{Y}$) be a B_1-valued (resp. B_2-valued) stochastic process and χ be a separable Chi-subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$. We suppose the following.

H0' There is $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \le 1} \left| {}_\chi\langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes (\mathbb{Y}_{s+\epsilon_{n_k}} - \mathbb{Y}_s)}{\epsilon_{n_k}} \rangle_{\chi^*} \right| ds \quad < +\infty\ .$$

H2' There is $\mathcal{T} : \chi \longrightarrow \mathscr{C}([0,T])$ such that $[\mathbb{X},\mathbb{Y}]^\epsilon(\phi)(t) \to \mathcal{T}(\phi)(t)$ ucp for all $\phi \in \mathcal{S}$.

Then $\mathbb{X}$ and $\mathbb{Y}$ admit a χ-covariation and application $[\mathbb{X},\mathbb{Y}]$ is equal to $\mathcal{T}$.

Proof. Condition **H1** is verified by assumption. Conditions **H2**(i) and (ii) follow by Theorem 3.22 setting $F^n(\phi)(\cdot,t) = [\mathbb{X},\mathbb{Y}]^{\epsilon_n}(\phi)(t)$ and $\tilde{F}^n = \widetilde{[\mathbb{X},\mathbb{Y}]}^{\epsilon_n}$ for a suitable sequence (ϵ_n). □

In the case $\mathbb{X} = \mathbb{Y}$ and $B = B_1 = B_2$ we can further relax the hypotheses.

Corollary 3.25. Let B be a Banach space, $\mathbb{X}$ a be B-valued stochastic processes and χ be a separable Chi-subspace. We suppose the following.

H0" There are subsets $\mathcal{S}, \mathcal{S}^p$ of χ such that $\overline{Span(\mathcal{S})} = \chi$, $Span(\mathcal{S}) = Span(\mathcal{S}^p)$ and $\mathcal{S}^p$ is constituted by **positive definite** elements ϕ in the sense that $\langle \phi, b \otimes b\rangle \ge 0$ for all $b \in B$.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \le 1} \left| {}_\chi\langle \phi, \frac{(\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s)^{\otimes 2}}{\epsilon_{n_k}} \rangle_{\chi^*} \right| ds \quad < +\infty\ .$$

H2" There is $\mathcal{T} : \chi \longrightarrow \mathscr{C}([0,T])$ such that $[\mathbb{X}]^\epsilon(\phi)(t) \to \mathcal{T}(\phi)(t)$ in probability for every $\phi \in \mathcal{S}$ and for every $t \in [0,T]$.

Then $\mathbb{X}$ admits a χ-quadratic variation and application $[\mathbb{X}]$ is equal to $\mathcal{T}$.

Proof. We verify the conditions of Corollary 3.24. Conditions **H0'** and **H1** are verified by assumption. We observe that, for every $\phi \in \mathcal{S}^p$, $[\mathbb{X}]^\epsilon(\phi)$ is an increasing process. By linearity, it follows that for any $\phi \in \mathcal{S}^p$, $[\mathbb{X}]^\epsilon(\phi)(t)$ converges in probability to $\mathcal{T}(\phi)(t)$ for any $t \in [0, T]$. Lemma 3.1 in [27] implies that $[\mathbb{X}]^\epsilon(\phi)$ converges ucp for every $\phi \in \mathcal{S}^p$ and therefore in $\mathcal{S}$. Conditions **H2'** of Corollary 3.24 is now verified. □

When χ has finite dimension the notion of χ-quadratic variation becomes very natural.

Proposition 3.26. Let $\chi = Span\{\phi_1, \ldots, \phi_n\}$, $\phi_1, \ldots, \phi_n \in (B\hat{\otimes}_\pi B)^*$ of positive type and linearly independent. $\mathbb{X}$ has a χ-quadratic variation if and only if there are continuous processes Z^i such that $[\mathbb{X}]^\epsilon_t(\phi_i)$ converges in probability to Z^i_t for ϵ going to zero for all $t \in [0, T]$ and $i = 1, \ldots, n$.

Proof. We only need to show that the condition is sufficient, the converse implication resulting immediately. We verify the hypotheses of Corollary 3.25 taking $\mathcal{S} = \{\phi_1, \ldots, \phi_n\}$. Without restriction to generality we can suppose $\|\phi_i\|_{(B\hat{\otimes}_\pi B)^*} = 1$, for $1 \le i \le n$. Conditions **H0"** and **H2"** are straightforward. It remains to verify **H1**. Since χ is finite dimensional it can be equipped with the norm $\|\phi\|_\chi = \sum_{i=1}^n |a_i|$ if $\phi = \sum_{i=1}^n a_i \, \phi_i$ with $a_i \in \mathbb{R}$. For ϕ such that $\|\phi\|_\chi = \sum_{i=1}^n |a_i| \le 1$ we have

$$\frac{1}{\epsilon}\int_0^T \left|\langle \phi \, , \, \mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2\rangle\right| ds \le \sum_{i=1}^n \frac{1}{\epsilon}\int_0^T \left|\langle a_i \, \phi_i \, , \, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2\rangle\right| ds = \sum_{i=1}^n \frac{|a_i|}{\epsilon}\int_0^T \langle \phi_i \, , \, (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)\otimes^2\rangle ds$$

because ϕ_i are of positive type. Previous expression is smaller or equal than

$$\sum_{i=1}^n \frac{1}{\epsilon}\int_0^T \langle \phi_i \, , \, (X_{s+\epsilon} - X_s)\otimes^2\rangle = \sum_{i=1}^n [\mathbb{X}]^\epsilon_T(\phi_i)$$

because $|a_i| \le 1$ for $1 \le i \le n$. Taking the supremum over $\|\phi\|_\chi \le 1$ and using the hypothesis of convergence in probability of the quantity $[\mathbb{X}]^\epsilon_T(\phi_i)$ for $1 \le i \le n$, the result follows. □

Corollary 3.27. Let $B_1 = B_2 = \mathbb{R}^n$. $\mathbb{X}$ admits all its mutual brackets if and only if $\mathbb{X}$ admits a global quadratic variation.

4 Evaluations of χ-covariations for window processes

In this section we consider X and Y as real continuous processes as usual prolonged by continuity and $X(\cdot)$ and $Y(\cdot)$ their associated window processes. We set $B = C([-\tau, 0])$. We will proceed to the evaluation of some χ-covariations (resp. χ-quadratic variations) for window processes $X(\cdot)$ and $Y(\cdot)$ (resp. for process $X(\cdot)$) with values in $B = C([-\tau, 0])$. We start with some examples of χ-covariation calculated directly through the definition.

Proposition 4.1. Let X and Y be two real valued processes with Hölder continuous paths of parameters γ and δ such that $\gamma + \delta > 1$. Then $X(\cdot)$ and $Y(\cdot)$ admit a zero scalar and tensor covariation. In particular $X(\cdot)$ and $Y(\cdot)$ admit a zero global covariation.

Proof. By Remark 1.5 item 4 and Proposition 3.14 we only need to show that $X(\cdot)$ and $Y(\cdot)$ admit a zero scalar covariation, i.e. the convergence to zero in probability of following quantity.

$$\frac{1}{\epsilon}\int_0^T \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|_B \|Y_{s+\epsilon}(\cdot) - Y_s(\cdot)\|_B \, ds = \frac{1}{\epsilon}\int_0^T \sup_{u \in [-\tau, 0]} |X_{s+u+\epsilon} - X_{s+u}| \sup_{v \in [-\tau, 0]} |Y_{s+v+\epsilon} - Y_{s+v}| \, ds \, . \quad (4.1)$$

Since X (resp. Y) is a.s. γ-Hölder continuous (resp. δ-Hölder continuous), there is a non-negative finite random variable Z such that the right-hand side of (4.1) is bounded by a sequence of random variables $Z(\epsilon)$ defined by $Z(\epsilon) := \epsilon^{\gamma+\delta-1} Z T$. This implies that (4.1) converges to zero a.s. for $\gamma + \delta > 1$. □

Remark 4.2. As a consequence of previous proposition every window process $X(\cdot)$ associated with a continuous process with Hölder continuous paths of parameter $\gamma > 1/2$ admits zero real, tensor and global quadratic variation.

Remark 4.3. Let B^H (resp. $B^{H,K}$) be a real fractional Brownian motion with parameters $H \in]0,1[$ (resp. real bifractional Brownian motion with parameters $H \in]0,1[$, $K \in]0,1]$). See [24] and [14] for elementary facts about the bifractional Brownian motion. As immediate consequences of Proposition 4.1 we obtain the following results. 1) The fractional window Brownian motion $B^H(\cdot)$ with $H > 1/2$ admits a zero scalar, tensor and global quadratic variation. 2) The bifractional window Brownian motion $B^{H,K}(\cdot)$ with $KH > 1/2$ admits a zero scalar, tensor and global quadratic variation. 3) We recall that the paths of a Brownian motion W are a priori only a.s. Hölder continuous of parameter $\gamma < 1/2$ so that we can not use Proposition 4.1.

Propositions 4.5 and 4.7 show that the stochastic calculus developed by [5], [8] and [20] cannot be applied for $\mathbb{X}$ being a window Brownian motion $W(\cdot)$.

Definition 4.4. Let B be a Banach space and $\mathbb{X}$ be a B-valued stochastic process. We say that $\mathbb{X}$ is a **Pettis semimartingale** if, for every $\phi \in B^*$, $\langle \phi, \mathbb{X}_t \rangle$ is a real semimartingale.

We remark that if $\mathbb{X}$ is a B-valued semimartingale in the sense of Section 1.17, [20], then it is also a Pettis semimartingale.

Proposition 4.5. The $C([-\tau, 0])$-valued window Brownian $W(\cdot)$ motion is not a Pettis semimartingale.

Proof. It is enough to show that the existence of an element μ in $B^* = \mathcal{M}([-\tau, 0])$ such that $\langle \mu, W_t(\cdot) \rangle = \int_{[-\tau,0]} W_t(x)\mu(dx)$ is not a semimartingale with respect to any filtration. We will proceed by contradiction: we suppose that $W(\cdot)$ is a Pettis semimartingale, so that in particular if we take $\mu = \delta_0 + \delta_{-\tau}$, the process $\langle \delta_0 + \delta_{-\tau}, W_t(\cdot) \rangle = W_t + W_{t-\tau} := X_t$ is a semimartingale with respect to some filtration $(\mathcal{G}_t)$. Let $(\mathcal{F}_t)$ be the natural filtration generated by the real Brownian motion W. Now $W_t + W_{t-\tau}$ is $(\mathcal{F}_t)$-adapted, so by Stricker's theorem (see Theorem 4, pag. 53 in [23]), X is a semimartingale with respect to filtration $(\mathcal{F}_t)$. We recall that a $(\mathcal{F}_t)$-**weak Dirichlet** is the sum of a local martingale M and a process A which is adapted and $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$-local martingale N; A is called the $(\mathcal{F}_t)$-martingale orthogonal process. On the other hand $(W_{t-\tau})_{t\geq\tau}$ is a strongly predictable process with respect to $(\mathcal{F}_t)$, see Definition 3.5 in [4]. By Proposition 4.11 in [3], it follows that $(W_{t-\tau})_{t\geq\tau}$ is an $(\mathcal{F}_t)$-martingale orthogonal process. Since W is an $(\mathcal{F}_t)$-martingale, the process $X_t = W_t + W_{t-\tau}$ is an $(\mathcal{F}_t)$-weak Dirichlet process. By uniqueness of the decomposition for $(\mathcal{F}_t)$-weak Dirichlet processes, $(W_{t-\tau})_{t\geq\tau}$ has to be a bounded variation process. This generates a contradiction because $(W_{t-\tau})_{t\geq\tau}$ is not a zero quadratic variation process. In conclusion $\langle \mu, W_t(\cdot) \rangle$ is not a semimartingale. □

Remark 4.6. Process X defined by $X_t = W_t + W_{t-\tau}$ is an example of $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation which is not an $(\mathcal{F}_t)$-Dirichlet process.

Proposition 4.7. If W is a classical Brownian motion, then $W(\cdot)$ does not admit a scalar quadratic variation. In particular $W(\cdot)$ does not admit a global quadratic variation.

Proof. We can prove that

$$\int_0^T \frac{1}{\epsilon}\,\|W_{u+\epsilon}(\cdot)-W_u(\cdot)\|_B^2\,du \geq T\,A^2(\tilde\epsilon)\ln(1/\tilde\epsilon) \quad \text{where} \quad \tilde\epsilon = \frac{2\epsilon}{T} \tag{4.2}$$

and $(A(\epsilon))$ is a family of non negative r.v. such that $\lim_{\epsilon\to 0} A(\epsilon) = 1$ a.s. In fact the left-hand side of (4.2) gives

$$\int_0^T \frac{1}{\epsilon}\sup_{x\in[0,u]}|W_{x+\epsilon}-W_x|^2 du \geq \int_{T/2}^T \frac{1}{\epsilon}\sup_{x\in[0,u]}|W_{x+\epsilon}-W_x|^2 du \geq \int_{T/2}^T \frac{1}{\epsilon}\sup_{x\in[0,T/2-\epsilon]}|W_{x+\epsilon}-W_x|^2 du$$
$$= \frac{T}{2\epsilon}\sup_{x\in[0,T/2-\epsilon]}|W_{x+\epsilon}-W_x|^2 \,.$$

Clearly we have $W_t = \sqrt{\frac{T}{2}}B_{\frac{2t}{T}}$ where B is another standard Brownian motion. Previous expression gives

$$\frac{T^2}{4\epsilon}\sup_{x\in[0,T/2-\epsilon]}|B_{(x+\epsilon)\frac{2}{T}} - B_{\frac{2x}{T}}|^2 = \frac{T^2}{4\epsilon}\sup_{y\in[0,1-\frac{2\epsilon}{T}]}|B_{y+\frac{2\epsilon}{T}} - B_y|^2$$

We choose $\tilde\epsilon = \frac{2\epsilon}{T}$. Previous expression gives $T\ln(1/\tilde\epsilon)A^2(\tilde\epsilon)$ where

$$A(\epsilon) = \left(\frac{\sup_{x\in[0,1-\epsilon]}|B_{x+\epsilon}-B_x|}{\sqrt{2\,\epsilon\,\ln(1/\epsilon)}}\right) .$$

According to Theorem 1.1 in [2], $\lim_{\epsilon\to 0} A(\epsilon) = 1$ a.s. and the result is established. □

Below we will see that $W(\cdot)$, even if it does not admit a global quadratic variation, it admits a χ-quadratic variation for several Chi-subspaces χ. More generally we can state a significant existence result of χ-covariation for finite quadratic variation processes with the help of Corollaries 3.24 and 3.25. We remind that $\mathcal{D}_a([-\tau,0])$ and $\mathcal{D}_{a,b}([-\tau,0]^2)$ were defined at (3.2) and (3.1).

Proposition 4.8. Let X and Y be two real continuous processes with finite quadratic variation and $0<\tau\leq T$. Let a,b two given points in $[-\tau,0]$. The following properties hold true.

1. $X(\cdot)$ and $Y(\cdot)$ admit a zero χ-covariation, where $\chi = L^2([-\tau,0]^2)$.
2. $X(\cdot)$ and $Y(\cdot)$ admit zero χ-covariation where $\chi = L^2([-\tau,0])\hat\otimes_h\mathcal{D}_a([-\tau,0])$.

If moreover the covariation $[X_{\cdot+a},Y_{\cdot+b}]$ exists, the following statement is valid.

3. $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation, where $\chi = \mathcal{D}_{a,b}([-\tau,0]^2)$, and it equals

$$[X(\cdot),Y(\cdot)](\mu) = \mu(\{a,b\})[X_{\cdot+a},Y_{\cdot+b}], \quad \forall\mu\in\chi.$$

Proof. The proof will be similar in all the three cases. Example 3.4 says that the three involved sets χ are separable Chi-subspaces.
Let $\{e_j\}_{j\in\mathbb{N}}$ be a topological basis for $L^2([-\tau,0])$; $\{\delta_a\}$ is clearly a basis for $\mathcal{D}_a([-\tau,0])$. Then $\{e_i\otimes e_j\}_{i,j\in\mathbb{N}}$ is a basis of $L^2([-\tau,0]^2)$, $\{e_j\otimes\delta_a\}_{j\in\mathbb{N}}$ is a basis of $L^2([-\tau,0])\hat\otimes_h\mathcal{D}_a([-\tau,0])$ and $\{\delta_a\otimes\delta_b\}$ is a basis of $\mathcal{D}_{a,b}([-\tau,0]^2)$. The results will follow using Corollary 3.25. To verify Condition **H1** we consider

$$A(\epsilon) := \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_\chi\leq 1}\left|{}_\chi\langle\phi,(X_{s+\epsilon}(\cdot)-X_s(\cdot))\otimes(Y_{s+\epsilon}(\cdot)-Y_s(\cdot))\rangle_{\chi^*}\right| ds$$

for the three Chi-subspaces mentioned above. In all the three situations we will show the existence of a family of random variables $\{B(\epsilon)\}$ converging in probability to some random variable B, such that $A(\epsilon) \leq B(\epsilon)$ a.s. By Remark 3.10.1 this will imply Assumption **H1**.

1. Suppose $\chi = L^2([-\tau, 0]^2)$. By Cauchy-Schwarz inequality we have

$$
\begin{aligned}
A(\epsilon) &\leq \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_{L^2([-\tau,0]^2)}\leq 1} \|\phi\|^2_{L^2([-\tau,0]^2)} \cdot \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|_{L^2([-\tau,0])} \cdot \|Y_{s+\epsilon}(\cdot) - Y_s(\cdot)\|_{L^2([-\tau,0])}\, ds \\
&\leq \frac{1}{\epsilon}\int_0^T \sqrt{\int_0^s (X_{u+\epsilon} - X_u)^2 \, du}\sqrt{\int_0^s (Y_{v+\epsilon} - Y_v)^2 \, dv}\, ds \leq T\, B(\epsilon) \quad \text{where}
\end{aligned}
$$

$$
B(\epsilon) = \sqrt{\int_0^T \frac{(X_{u+\epsilon} - X_u)^2}{\epsilon}\, du \int_0^T \frac{(Y_{v+\epsilon} - Y_v)^2}{\epsilon}\, dv} \tag{4.3}
$$

which converges in probability to $\sqrt{[X]_T[Y]_T}$.

2. We proceed similarly for $\chi = L^2([-\tau, 0])\hat{\otimes}_h\mathcal{D}_a([-\tau, 0])$.

We consider ϕ of the form $\phi = \tilde{\phi} \otimes \delta_a$, where $\tilde{\phi}$ is an element of $L^2([-\tau, 0])$. We first observe

$$
\|\phi\|_{L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_a} = \left\|\tilde{\phi}\right\|_{L^2([-\tau,0])} \cdot \|\delta_a\|_{\mathcal{D}_a} = \sqrt{\int_{[-\tau,0]} \tilde{\phi}(s)^2\, ds}\,.
$$

Then

$$
\begin{aligned}
A(\epsilon) &= \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_{L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_a}\leq 1} \left| (X_{s+\epsilon}(a) - X_s(a)) \int_{[-\tau,0]} (Y_{s+\epsilon}(x) - Y_s(x))\, \tilde{\phi}(x)\, dx \right| ds \leq \\
&\leq \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|\leq 1} \left\{ \left(\sqrt{(X_{s+\epsilon}(a) - X_s(a))^2} \right) \cdot \right. \\
&\qquad \left. \cdot \left(\left\|\tilde{\phi}\right\|_{L^2([-\tau,0])} \sqrt{\int_{[-\tau,0]} (Y_{s+\epsilon}(x) - Y_s(x))^2 \, dx} \right) \right\} ds \leq \\
&\leq \int_0^T \sqrt{\frac{(X_{s+\epsilon}(a) - X_s(a))^2}{\epsilon}} \sqrt{\int_{[-T,0]} \frac{(Y_{s+\epsilon}(x) - Y_s(x))^2}{\epsilon}\, dx}\, ds \leq \sqrt{T} B(\epsilon)
\end{aligned}
$$

where $B(\epsilon)$ is the same family of r.v. defined in (4.3).

3. The last case is $\chi = \mathcal{D}_{a,b}([-\tau, 0]^2)$. A general element ϕ which belongs to χ admits a representation $\phi = \lambda\, \delta_{(a,b)}$, with norm equals to $\|\phi\|_{\mathcal{D}_{a,b}} = |\lambda|$. We have

$$
\begin{aligned}
A(\epsilon) &= \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_{\mathcal{D}_{a,b}}\leq 1} |\lambda\, (X_{s+a+\epsilon} - X_{s+a})\, (Y_{s+b+\epsilon} - Y_{s+b})| \, ds \\
&\leq \frac{1}{\epsilon}\int_0^T |(X_{s+a+\epsilon} - X_{s+a})\, (Y_{s+b+\epsilon} - Y_{s+b})| \, ds\,;
\end{aligned} \tag{4.4}
$$

using again Cauchy-Schwarz inequality, previous quantity is bounded by

$$\sqrt{\int_0^T \frac{(X_{s+a+\epsilon}-X_{s+a})^2}{\epsilon}\,ds}\sqrt{\int_0^T \frac{(Y_{v+b+\epsilon}-Y_{v+b})^2}{\epsilon}\,dv} \le B(\epsilon).$$

We verify now the Conditions **H0"** and **H2"**.

1. A general element in $\{e_i \otimes e_j\}_{i,j\in\mathbb{N}}$ is difference of two positive definite elements in the set $\mathcal{S}^p = \{e_i \otimes^2, (e_i+e_j)\otimes^2\}_{i,j\in\mathbb{N}}$. We also define $\mathcal{S} = \{e_i \otimes e_j\}_{i,j\in\mathbb{N}}$. The fact that $Span(\mathcal{S}) = Span(\mathcal{S}^p)$ implies **H0"**. To conclude we need to show the validity of Condition **H2"**. For this we have to verify

$$[X(\cdot), Y(\cdot)]^\epsilon (e_i \otimes e_j)(t) \xrightarrow[\epsilon\longrightarrow 0]{} 0 \tag{4.5}$$

in probability for any $i,j\in\mathbb{N}$. Clearly we can suppose $\{e_i\}_{i\in\mathbb{N}} \in C^1([-\tau,0])$. We fix $\omega\in\Omega$, outside some null set, fixed but omitted. We have

$$[X(\cdot), Y(\cdot)]^\epsilon (e_i \otimes e_j)(t) = \int_0^t \frac{\gamma_j(s,\epsilon)\ \gamma_i(s,\epsilon)}{\epsilon} ds \quad \text{where}$$

$$\gamma_j(s,\epsilon) = \int_{(-\tau)\vee(-s)}^0 e_j(y)\,(X_{s+y+\epsilon}-X_{s+y})\,dy \quad \text{and} \quad \gamma_i(s,\epsilon) = \int_{(-\tau)\vee(-s)}^0 e_i(x)\,(Y_{s+x+\epsilon}-Y_{s+x})\,dx.$$

Without restriction of generality, in the purpose not to overcharge notations, we can suppose from now on that $\tau = T$.
For every $s\in[0,T]$, we have

$$\begin{aligned}|\gamma_j(s,\epsilon)| &= \left|\int_{-s}^0 (e_j(y-\epsilon)-e_j(y))\,X_{s+y}dy + \int_0^\epsilon e_j(y-\epsilon)X_{s+y}dy - \int_{-s}^{-s+\epsilon} e_j(y-\epsilon)X_{s+y}dy\right| \\ &\le \epsilon\left(\int_{-T}^0 |\dot{e_j}(y)|dy + 2\|e_j\|_\infty\right)\sup_{s\in[0,T]}|X_s|\ . \end{aligned}\tag{4.6}$$

For $t\in[0,T]$, this implies that

$$\begin{aligned}\int_0^t \left|\frac{\gamma_j(s,\epsilon)\ \gamma_i(s,\epsilon)}{\epsilon}\right| ds &\le \int_0^T \left|\frac{\gamma_j(s,\epsilon)\ \gamma_i(s,\epsilon)}{\epsilon}\right| ds \\ &\le T\ \epsilon\ \left(\int_{-T}^0 |\dot{e_j}(y)|dy + 2\|e_j\|_\infty\right)\left(\int_{-T}^0 |\dot{e_i}(y)|dy + 2\|e_i\|_\infty\right)\left(\sup_{s\in[0,T]}|X_s|\right)\left(\sup_{u\in[0,T]}|Y_u|\right)\end{aligned}$$

which trivially converges a.s. to zero when ϵ goes to zero which yields (4.5).

2. A general element in $\{e_j\otimes\delta_a\}_{j\in\mathbb{N}}$ is difference of two positive definite elements of type $\{e_j\otimes^2, \delta_a\otimes^2, (e_j+\delta_a)\otimes^2\}_{j\in\mathbb{N}}$. This shows **H0"**. It remains to show that

$$[X(\cdot), Y(\cdot)]^\epsilon\,(e_j\otimes\delta_a)\,(t) \longrightarrow 0$$

in probability for every $j\in\mathbb{N}$. In fact the left-hand side equals

$$\int_0^t \frac{\gamma_j(s,\epsilon)}{\epsilon}\,(X_{s+a+\epsilon}-X_{s+a})\ ds\ .$$

Using estimate (4.6), we obtain

$$\int_0^t \left| \frac{\gamma_j(s,\epsilon)}{\epsilon} (Y_{s+a+\epsilon} - Y_{s+a}) \right| ds \le T \left(\int_{-T}^0 |\dot{e}_j(y)| dy + 2\|e_j\|_\infty \right) \left(\sup_{s\in[0,T]} |X_s| \right) \varpi_Y(\epsilon) \xrightarrow[\epsilon \longrightarrow 0]{a.s.} 0$$

where $\varpi_Y(\epsilon)$ is the usual (random in this case) continuity modulus, so the result follows.

3. A general element $\delta_a \otimes \delta_b$ is difference of two positive definite elements $(\delta_a + \delta_b)\otimes^2$ and $\delta_a \otimes^2 + \delta_b \otimes^2$. So that Condition **H0"** is fulfilled. Concerning Condition **H2"** we have

$$[X(\cdot), Y(\cdot)]^\epsilon (\delta_a \otimes \delta_b)(t) = \frac{1}{\epsilon} \int_0^t (X_{s+a+\epsilon} - X_{s+a}) (Y_{s+b+\epsilon} - Y_{s+b}) ds .$$

This converges to $[X_{\cdot+a}, Y_{\cdot+b}]$ which exists by hypothesis.

This finally concludes the proof of Proposition 4.8. □

Corollary 4.9. Let X and Y be two real continuous processes such that $[X]$, $[Y]$ and $[X,Y]$ exist and a is a given point in $[-\tau, 0]$.

1. $X(\cdot)$ and $Y(\cdot)$ admit a zero χ-covariation, where $\chi = \mathcal{D}_a([-\tau,0]) \hat{\otimes}_h L^2([-\tau,0])$.

2. $X(\cdot)$ and $Y(\cdot)$ admit a $\chi^0([-\tau,0]^2)$-covariation which equals $[X(\cdot),Y(\cdot)](\mu) = \mu(\{0,0\})[X,Y]$, $\forall \mu \in \chi^0$.

Proof. Using Proposition 2.1, it follows that $\chi^0([-\tau,0]^2)$ can be decomposed into the finite direct sum decomposition $L^2([-\tau,0]^2) \oplus L^2([-\tau,0]) \hat{\otimes}_h \mathcal{D}_0([-\tau,0]) \oplus \mathcal{D}_0([-\tau,0]) \hat{\otimes}_h L^2([-\tau,0]) \oplus \mathcal{D}_{0,0}([-\tau,0]^2)$. The results follow immediately applying Propositions 3.16 and 4.8. □

When $\chi = \mathcal{D}_{0,0}([-\tau,0]^2)$ the existence of a χ-covariation between X and Y holds even under more relaxed hypotheses.

Proposition 4.10. Let X, Y be continuous processes such that $[X,Y]$ exists and for every sequence $(\epsilon_n) \downarrow 0$, it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \frac{1}{\epsilon_{n_k}} \int_0^T \left| X_{s+\epsilon_{n_k}} - X_s \right| \cdot \left| Y_{s+\epsilon_{n_k}} - Y_s \right| ds \quad < +\infty . \tag{4.7}$$

Then 1) the scalar covariation process $[X,Y]$ has bounded variation and 2) $X(\cdot)$ and $Y(\cdot)$ admit a $\mathcal{D}_{0,0}([-\tau,0]^2)$-covariation and $[X(\cdot),Y(\cdot)]_t(\mu) = \mu(\{0,0\})[X,Y]_t$.

Proof. 1) The processes X and Y take values in $B = \mathbb{R}$ and the (separable) space $\chi = (B \hat{\otimes}_\pi B)^*$ coincides with $\mathbb{R}$. Taking into account Corollary 3.24, the processes X and Y admit therefore a global covariation which coincides with the classical covariation $[X,Y]$ defined in Definition 1.1 and in particular $[X,Y]$ has bounded variation. 2) The proof is again very similar to the one of Proposition 4.8. The only relevant difference consists in checking the validity of condition **H1**. This will be verified identically until (4.4); the next step will follow by (4.7). □

Before mentioning some examples, we give some information about the covariation structure of bifractional Brownian motion.

Proposition 4.11. Let $B^{H,K}$ be a bifractional Brownian motion with $HK = 1/2$. Then $[B^{H,K}]_t = 2^{1-K}t$ and $[B^{H,K}_{\cdot+a}, B^{H,K}_{\cdot+b}] = 0$ for $a \neq b \in [-\tau, 0]$.

Remark 4.12.

- If $K = 1$, then $H = 1/2$ and $B^{H,K}$ is a Brownian motion.
- In the case $K \neq 1$ we recall that the bifractional Brownian motion $B^{H,K}$ is not a semimartingale, see Proposition 6 from [24].

Proof of Proposition 4.11. Proposition 1 in [24] says that $B^{H,K}$ has finite quadratic variation which is equal to $[B^{H,K}]_t = 2^{1-K}t$. By Proposition 1 and Theorem 2 in [17] there are two constants α and β depending on K, a centered Gaussian process $X^{H,K}$ with absolutely continuous trajectories on $[0, +\infty[$ and a standard Brownian motion W such that $\alpha X^{H,K} + B^{H,K} = \beta W$. Then

$$[\alpha X^{H,K}_{\cdot+a} + B^{H,K}_{\cdot+a}, \alpha X^{H,K}_{\cdot+b} + B^{H,K}_{\cdot+b}] = \beta^2 [W_{\cdot+a}, W_{\cdot+b}]. \tag{4.8}$$

Using the bilinearity of the covariation, we expand the left-hand side in (4.8) into the sum of four terms

$$\alpha^2 [X^{H,K}_{\cdot+a}, X^{H,K}_{\cdot+b}] + \alpha [B^{H,K}_{\cdot+a}, X^{H,K}_{\cdot+b}] + \alpha [X^{H,K}_{\cdot+a}, B^{H,K}_{\cdot+b}] + [B^{H,K}_{\cdot+a}, B^{H,K}_{\cdot+b}] \tag{4.9}$$

Since $X^{H,K}$ has bounded variation then the first three terms of (4.9) vanish because of point 6) of Proposition 1 in [28]. On the other hand the right-hand side of (4.8) is equal to zero for $a \neq b$ since W is a semimartingale, see Example 4.13, item 1. We conclude that $[B^{H,K}_{\cdot+a}, B^{H,K}_{\cdot+b}] = 0$ if $a \neq b$. □

Example 4.13. We list some examples of processes X for which $X(\cdot)$ admits a χ-quadratic variation through Proposition 4.8 and Corollary 4.9 and it is explicitly given by the quadratic variation structure $[X]$ of the real process X.

1. All continuous real semimartingales S (for instance Brownian motion). In fact S is a finite quadratic variation process; moreover $[S_{\cdot+a}, S_{\cdot+b}] = 0$ for $a \neq b$, as it easily follows by Corollary 3.11 in [4].
2. Let $B^{H,K}$ be a bifractional Brownian motion with parameters H and K and such that $HK = 1/2$. As shown in Proposition 4.11, $B^{H,K}$ satisfies the hypotheses of the Corollary 4.9.
3. Let D be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $D = M + A$, M local martingale and A zero quadratic variation process. Then D satisfies the hypotheses of the Corollary 4.9. In fact $[D] = [M]$ and $[D_{\cdot+a}, D_{\cdot+b}] = 0$ for $a \neq b$.
4. Similar examples can be produced considering the window of a weak Dirichlet process with finite quadratic variation.

We go on evaluating other χ-covariations.

Proposition 4.14. Let V and Z be two real absolutely continuous processes such that $V', Z' \in L^2([0, T])$ ω-a.s. Then the associated window processes $V(\cdot)$ and $Z(\cdot)$ have zero scalar and tensor covariation. In particular they admit a zero global covariation.

Proof. Similarly to the proof of Proposition 4.1, by Remark 1.5 item 4. and Proposition 3.14 we only need to show that $V(\cdot)$ and $Z(\cdot)$ admit a zero scalar covariation, i.e. the convergence to zero in probability of the quantity

$$\int_0^T \frac{1}{\epsilon} \|V_{s+\epsilon}(\cdot) - V_s(\cdot)\|_B \, \|Z_{s+\epsilon}(\cdot) - Z_s(\cdot)\|_B \, ds. \tag{4.10}$$

By Cauchy-Schwarz, (4.10) is bounded by

$$\sqrt{\int_0^T \frac{1}{\epsilon} \sup_{x\in[-\tau,0]} |V_{s+\epsilon}(x) - V_s(x)|^2 \, ds} \cdot \sqrt{\int_0^T \frac{1}{\epsilon} \sup_{x\in[-\tau,0]} |Z_{u+\epsilon}(x) - Z_u(x)|^2 \, du} \, , \tag{4.11}$$

which will be shown to converge even a.s. to zero. The square of the first square root in (4.11) equals

$$\int_0^T \frac{1}{\epsilon} \sup_{x\in[-\tau,0]} \left| \int_{s+x}^{s+x+\epsilon} V'(y)dy \right|^2 ds \le \int_0^T \frac{1}{\epsilon} \max_{x\in[-\tau,0]} \int_{s+x}^{s+x+\epsilon} V'(y)^2 dyds \le T \, \varpi_{\int_0^{\cdot}(V'^2)(y)dy}(\epsilon) \xrightarrow[\epsilon\longrightarrow 0]{a.s.} 0 \, ,$$

since $\varpi_{\int_0^{\cdot}(V'^2)(y)dy}(\epsilon)$ denotes the modulus of continuity of the a.s. continuous function $t \mapsto \int_0^t (V'^2)(y)dy$. The square of the second square root in (4.11) can be treated analogously and the result is finally established. □

If X is a finite quadratic variation processes then $\mathbb{X} = X(\cdot)$ admits a $Diag([-\tau, 0]^2)$-quadratic variation, where $Diag([-\tau, 0]^2)$ was defined in (3.3). This is the object of Proposition 4.15.

Proposition 4.15. Let $0 < \tau \le T$. Let X and Y be two real continuous processes such that $[X, Y]$ exists and (4.7) is verified. Then $X(\cdot)$ and $Y(\cdot)$ admit a $Diag([-\tau, 0]^2)$-covariation. Moreover we have

$$\widetilde{[X(\cdot), Y(\cdot)]}_t(\mu) = \int_0^{t\wedge\tau} g(-x)[X, Y]_{t-x} dx \, , \qquad t \in [0, T] \, ,$$

where μ is a generic element in $Diag([-\tau, 0]^2)$ of the type $\mu(dx, dy) = g(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau, 0])$.

Remark 4.16. Taking into account the usual convention $[X, Y]_t = 0$ for $t < 0$, the process $\left(\int_0^{t\wedge\tau} g(-x)[X, Y]_{t-x}dx\right)_{t\ge 0}$ can also be written as $\left(\int_0^{\tau} g(-x)[X, Y]_{t-x}dx\right)_{t\ge 0}$.

Proof. We recall that, for a generic element μ, we have $\|\mu\|_{Diag} = \|g\|_\infty$.
First we verify Condition **H1**. We can write

$$\begin{aligned}
&\frac{1}{\epsilon} \int_0^T \sup_{\|\mu\|_{Diag}\le 1} |\langle \mu, (X_{s+\epsilon}(\cdot) - X_s(\cdot)) \otimes (Y_{s+\epsilon}(\cdot) - Y_s(\cdot)) \rangle| \, ds \\
&\le \frac{1}{\epsilon} \int_0^T \sup_{\|g\|_\infty\le 1} \left| \int_{-T}^0 g(x) \left(X_{s+\epsilon}(x) - X_s(x)\right) \left(Y_{s+\epsilon}(x) - Y_s(x)\right) dx \right| ds \\
&= \int_0^T \sup_{\|g\|_\infty\le 1} \left| \int_0^s \frac{\left(X_{x+\epsilon} - X_x\right)\left(Y_{x+\epsilon} - Y_x\right)}{\epsilon} g(x - s) \, dx \right| ds \, .
\end{aligned}$$

Condition **H1** is verified because of Hypothesis (4.7).

It remains to prove Condition **H2**. Using Fubini theorem, we write

$$
\begin{aligned}
[X(\cdot),Y(\cdot)]^{\epsilon}_t(\mu) &= \frac{1}{\epsilon}\int_0^t \langle \mu(dx,dy), (X_{s+\epsilon}(\cdot)-X_s(\cdot))\otimes(Y_{s+\epsilon}(\cdot)-Y_s(\cdot))\rangle\, ds \\
&= \frac{1}{\epsilon}\int_0^t \int_{[-\tau,0]} (X_{s+\epsilon}(x)-X_s(x))\,(Y_{s+\epsilon}(x)-Y_s(x))\, g(x)dx\; ds \\
&= \int_{(-t)\vee(-\tau)}^0 g(x) \int_{-x}^t \frac{(X_{s+x+\epsilon}-X_{s+x})\,(Y_{s+x+\epsilon}-Y_{s+x})}{\epsilon}\, ds\, dx \\
&= \int_{(-t)\vee(-\tau)}^0 g(x) \int_0^{t+x} \frac{(X_{s+\epsilon}-X_s)\,(Y_{s+\epsilon}-Y_s)}{\epsilon}\, ds\ dx \\
&= \int_0^{t\wedge\tau} g(-x) \int_0^{t-x} \frac{(X_{s+\epsilon}-X_s)\,(Y_{s+\epsilon}-Y_s)}{\epsilon}\, ds\ dx\ .
\end{aligned}
$$

To conclude the proof of **H2**(i) it remains to show that

$$
\left(\int_0^{t\wedge\tau} g(-x) \int_0^{t-x} \frac{(X_{s+\epsilon}-X_s)\,(Y_{s+\epsilon}-Y_s)}{\epsilon}\, ds\ dx\right)_{t\in[0,T]} \xrightarrow[\epsilon\longrightarrow 0]{ucp} \left(\int_0^{t\wedge\tau} g(-x)[X,Y]_{t-x}\, dx\right)_{t\in[0,T]}
$$

$$
\text{i.e.}\quad \sup_{t\leq T}\left|\int_0^{t\wedge\tau}\left(g(-x)\int_0^{t-x} \frac{(X_{s+\epsilon}-X_s)\,(Y_{s+\epsilon}-Y_s)}{\epsilon}\, ds - [X,Y]_{t-x}\right)\ dx\right| \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} 0\,. \qquad (4.12)
$$

The left-hand side of (4.12) is bounded by

$$
\begin{aligned}
&\int_0^T |g(-x)| \sup_{t\in[0,T]}\left|\int_0^{t-x} \frac{(X_{s+\epsilon}-X_s)\,(Y_{s+\epsilon}-Y_s)}{\epsilon}\, ds - [X,Y]_{t-x}\right| dx \\
&\leq T\,\|g\|_\infty \sup_{t\in[0,T]}\left|\int_0^{t} \frac{(X_{s+\epsilon}-X_s)\,(Y_{s+\epsilon}-Y_s)}{\epsilon}\, ds - [X,Y]_{t}\right|\ .
\end{aligned}
$$

Since X and Y admit a covariation, previous expression converges to zero. This shows Condition **H2**(i). Concerning Condition **H2**(ii), we have

$$
[X(\cdot),Y(\cdot)]_t(\mu) = \int_0^{t\wedge\tau} g(-x)[X,Y]_{t-x}\, dx = \begin{cases} \int_0^t g(-x)[X,Y]_{t-x}dx & 0\leq t\leq\tau \\ \int_0^\tau g(-x)[X,Y]_{t-x}dx & \tau< t\leq T\ . \end{cases}
$$

Previous expression has an obvious modification $\widetilde{[X(\cdot),Y(\cdot)]}$ which has finite variation with values in χ^*. The total variation is in fact easily dominated by $\int_0^T |[X,Y]_x|dx$. □

A useful proposition related to Proposition 4.15 is the following. We recall that $D([-\tau,0])$ denotes the space of càdlàg functions equipped with the uniform norm. $Diag_d([-\tau,0]^2)$ was introduced in (3.3).

Proposition 4.17. Let X be a finite quadratic variation process. Let $G:[0,T]\longrightarrow \chi := Diag_d([-\tau,0]^2)$, càdlàg. We have

$$
\int_0^T {}_{\chi}\langle G(s)\,,\, d\widetilde{[X(\cdot)]}_s\rangle_{\chi^*} = \int_0^\tau \left(\int_x^T g(s,-x)[X]_{ds-x}\right) dx = \int_0^\tau \left(\int_0^{T-x} g(s+x,-x)d[X]_s\right) dx, \quad (4.13)
$$

where $G(s) = g(s,x)\delta_y(dx)dy$ for some bounded Borel function $g : [0,T] \times [-\tau, 0] \longrightarrow \mathbb{R}$ and $[X]_{ds-x}$ represents the Radon-Nikodym derivative of the increasing function $s \mapsto [X]_{s+x}$.

Proof. We remark that $t \mapsto g(t,\cdot)$ is left continuous from $[0,T]$ to $D([-\tau,0])$ equipped with the $\|\cdot\|_\infty$ norm. By item 2 in Remark 3.2, Proposition 3.17 item 2 and Proposition 4.15, $X(\cdot)$ admits a χ-quadratic variation. The proof will be established fixing $\omega \in \Omega$. We first suppose that

$$G(s) = \sum_{i=0}^{N-1} A_i \mathbb{1}_{]t_i,t_{i+1}]}(s) + A_0 \mathbb{1}_{\{0\}}(s), \tag{4.14}$$

where, for some positive integer $N \in \mathbb{N}$, $0 = t_0 < \ldots < t_N = T$ is an element of subdivisions of $[0,T]$; $A_0, \ldots, A_N \in \chi$; in particular there are $a_0, \ldots, a_N \in D_d([-\tau, 0])$ with

$$A_i(dx, dy) = a_i(x)\delta_y(dx)dy \quad \text{for all } i \in \{0, \ldots, N\} . \tag{4.15}$$

Then (4.13) holds by use of Proposition 4.15.
To treat the general case we approach a general G by a sequence (G^n) of type (4.14), i.e.

$$G^n(s) = \sum_{i=0}^{N-1} A_i^n \mathbb{1}_{]t_i,t_{i+1}]}(s) + A_0^n \mathbb{1}_{\{0\}}(s)$$

where $A_i^n = G(t_i)$, $0 \le i \le (N-1)$, $0 = t_0 < \ldots < t_N = T$ is a an element of subdivisions of $[0,T]$ indexed by n whose mesh goes to zero when n diverges to infinity. Let $a_0^n, \ldots, a_N^n \in D([-\tau,0])$ related to $A_0^n, \ldots, A_N^n$ through relation (4.15). Consequently we have

$$\int_0^T {}_{\chi}\langle G^n(s)\, , \, d\widetilde{[X(\cdot)]}_s \rangle_{\chi^*} = \int_0^\tau \left(\int_x^T g^n(s,-x)[X]_{ds-x} \right) dx \tag{4.16}$$

with $g^n(s,x) = \sum_{i=0}^{N-1} a_i^n(x)\mathbb{1}_{]t_i,t_{i+1}]}(s) + a_0^n$. In particular $a_i^n = g(t_i, \cdot)$.
By assumption, for every $s \in [0,T]$ we have

$$\lim_{n\to+\infty} \sup_{x\in[-\tau,0]} |g^n(s,x) - g(s,x)| = 0 .$$

Consequently, for every $x \in [0,\tau]$, by Lebesgue dominated convergence theorem,

$$\lim_{n\to+\infty} \int_x^T \left(g^n(s,-x) - g(s,-x)\right)[X]_{ds-x} = 0 .$$

Moreover

$$\left| \int_x^T \left(g^n(s,-x) - g(s,-x)\right)[X]_{ds-x} \right| \le \left(\sup_n \|g^n\|_\infty + \|g\|_\infty \right) [X]_T.$$

Again by Lebesgue dominated convergence theorem, the right-hand side of (4.16) converges to the right-hand side of (4.13) and the result follows. □

Remark 4.18. If $[X]$ is absolutely continuous with respect to Lebesgue, (4.13) in the statement would be valid with $\chi = Diag([-\tau,0]^2)$.

5 Itô formula

We need now to formulate the definition of the forward type integral for B-valued integrator and B^*-valued integrand, where B is a separable Banach space.

Definition 5.1. Let $(\mathbb{X}_t)_{t\in[0,T]}$ (respectively $(\mathbb{Y}_t)_{t\in[0,T]}$) be a B-valued (respectively a B^*-valued) stochastic process. We suppose $\mathbb{X}$ to be continuous and $\mathbb{Y}$ to be strongly measurable such that $\int_0^T \|\mathbb{Y}_s\|_{B^*} ds < +\infty$ a.s. For every fixed $t \in [0,T]$ we define the **definite forward integral of $\mathbb{Y}$ with respect to $\mathbb{X}$** denoted by $\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s\rangle_B$ as the following limit in probability:

$$\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s\rangle_B := \lim_{\epsilon\to 0}\int_0^t {}_{B^*}\langle \mathbb{Y}_s, \frac{\mathbb{X}_{s+\epsilon}-\mathbb{X}_s}{\epsilon}\rangle_B ds \ .$$

We say that the **forward stochastic integral of $\mathbb{Y}$ with respect to $\mathbb{X}$** exists if the process

$$\left(\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s\rangle_B\right)_{t\in[0,T]}$$

admits a continuous version. In the sequel indices B and B^* will often be omitted.

We are now able to state an Itô formula for stochastic processes with values in a general separable Banach space.

Theorem 5.2. Let χ be a Chi-subspace and $\mathbb{X}$ a B-valued continuous process admitting a χ-quadratic variation. Let $F : [0,T]\times B \longrightarrow \mathbb{R}$ Fréchet of class $C^{1,2}$ such that $D^2F(t,\eta) \in \chi$ for all $t \in [0,T]$ and $\eta \in C([-T,0])$ and $D^2F : [0,T]\times B \longrightarrow \chi$ is continuous.
Then for every $t \in [0,T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B$$

exists and the following formula holds.

$$F(t,\mathbb{X}_t) = F(0,\mathbb{X}_0) + \int_0^t \partial_t F(s,\mathbb{X}_s)ds + \int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B + \frac{1}{2}\int_0^t {}_{\chi}\langle D^2F(s,\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s\rangle_{\chi^*}. \quad (5.1)$$

Proof. We fix $t \in [0,T]$ and we observe that the quantity

$$I_0(\epsilon,t) = \int_0^t \frac{F(s+\epsilon,\mathbb{X}_{s+\epsilon}) - F(s,\mathbb{X}_s)}{\epsilon} ds \quad (5.2)$$

converges ucp for $\epsilon \to 0$ to $F(t,\mathbb{X}_t) - F(0,\mathbb{X}_0)$ since $\big(F(s,\mathbb{X}_s)\big)_{s\geq 0}$ is continuous. At the same time, using Taylor's expansion, (5.2) can be written as the sum of the two terms:

$$I_1(\epsilon,t) = \int_0^t \frac{F(s+\epsilon,\mathbb{X}_{s+\epsilon}) - F(s,\mathbb{X}_{s+\epsilon})}{\epsilon} ds$$

and

$$I_2(\epsilon,t) = \int_0^t \frac{F(s,\mathbb{X}_{s+\epsilon}) - F(s,\mathbb{X}_s)}{\epsilon} ds \ , \quad \epsilon > 0, \quad t \in [0,T] \ . \quad (5.3)$$

We fix $t \in [0,T]$ and we prove that

$$I_1(\epsilon,t) \longrightarrow \int_0^t \partial_t F(s,\mathbb{X}_s)ds \tag{5.4}$$

in probability. In fact

$$I_1(\epsilon,t) = \int_0^t \partial_t F(s,\mathbb{X}_{s+\epsilon})ds + R_1(\epsilon,t) \tag{5.5}$$

where

$$R_1(\epsilon,t) = \int_0^t \int_0^1 \left(\partial_t F\big(s+\alpha\epsilon,\mathbb{X}_{s+\epsilon}\big) - \partial_t F(s,\mathbb{X}_{s+\epsilon})\right) d\alpha ds \ .$$

For fixed $\omega \in \Omega$ we denote by $\mathcal{V}(\omega) := \{\mathbb{X}_t(\omega);\, t \in [0,T]\}$ and

$$\mathcal{U} = \mathcal{U}(\omega) = \overline{conv(\mathcal{V}(\omega))}, \tag{5.6}$$

i.e. the set $\mathcal{U}$ is the closed convex hull of the compact subset $\mathcal{V}(\omega)$ of B. For $x \in \Omega$, we have

$$\sup_{t\in[0,T]} |R_1(\epsilon,t)| \le T\, \varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}(\epsilon)$$

where $\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}(\epsilon)$ is the continuity modulus in ϵ of the application $\partial_t F : [0,T]\times B \longrightarrow \mathbb{R}$ restricted to $[0,T]\times\mathcal{U}$. From the continuity of the $\partial_t F$ as function from $[0,T]\times B$ to $\mathbb{R}$, it follows that the restriction on $[0,T]\times\mathcal{U}$ is uniformly continuous and $\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero. Therefore we have proved that $R_1(\epsilon,\cdot) \to 0$ ucp as $\epsilon \to 0$.
On the other hand the first term in (5.5) can be rewritten as

$$\int_0^t \partial_t F(s,\mathbb{X}_s)ds + R_2(\epsilon,t)$$

where $R_2(\epsilon,t) \to 0$ ucp arguing similarly as for $R_1(\epsilon,t)$ and so the convergence (5.4) is established.
The second addend $I_2(\epsilon,t)$ in (5.3), can be approximated by Taylor's expansion and it can be written as the sum of the following three terms:

$$\begin{aligned}
I_{21}(\epsilon,t) &= \int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), \frac{\mathbb{X}_{s+\epsilon}-\mathbb{X}_s}{\epsilon}\rangle_B ds \ ,\\
I_{22}(\epsilon,t) &= \frac{1}{2}\int_0^t {}_{\chi}\langle D^2F(s,\mathbb{X}_s), \frac{(X_{s+\epsilon}-\mathbb{X}_s)\otimes^2}{\epsilon}\rangle_{\chi^*} ds \ ,\\
I_{23}(\epsilon,t) &= \int_0^t \left[\int_0^1 \alpha\, {}_{\chi}\langle D^2F\left(s,(1-\alpha)\mathbb{X}_{s+\epsilon}+\alpha\mathbb{X}_s\right) - D^2F(s,\mathbb{X}_s), \frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes^2}{\epsilon}\rangle_{\chi^*} d\alpha\right] ds \ .
\end{aligned}$$

Since $D^2F : [0,T]\times B \longrightarrow \chi$ is continuous and B separable, we observe that the process H defined by $H_s = D^2F(s,X_s)$ takes values in a separable closed subspace $\mathcal{V}$ of χ. Applying Corollary 3.20, it yields

$$I_{22}(\epsilon,t) \xrightarrow[\epsilon\to 0]{\mathbb{P}} \frac{1}{2}\int_0^t {}_{\chi}\langle D^2F(s,\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s\rangle_{\chi^*} \quad \text{for every } t\in[0,T].$$

We analyze now $I_{23}(\epsilon,t)$ and we show that $I_{23}(\epsilon,t) \xrightarrow[\epsilon\longrightarrow 0]{\mathbb{P}} 0$. In fact we have

$$\begin{aligned}
|I_{23}(\epsilon,t)| &\leq \frac{1}{\epsilon}\int_0^t\int_0^1 \alpha \left| {}_{\chi}\langle D^2F\left(s,(1-\alpha)\mathbb{X}_{s+\epsilon}+\alpha\mathbb{X}_s\right) - D^2F(s,\mathbb{X}_s), (\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes^2\rangle_{\chi^*}\right| d\alpha\, ds \\
&\leq \frac{1}{\epsilon}\int_0^t\int_0^1 \alpha \left\| D^2F\left(s,(1-\alpha)\mathbb{X}_{s+\epsilon}+\alpha\mathbb{X}_s\right) - D^2F(s,\mathbb{X}_s)\right\|_{\chi} \left\|(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes^2\right\|_{\chi^*} d\alpha\, ds \\
&\leq \varpi_{D^2F}^{[0,T]\times\mathcal{U}}(\epsilon)\int_0^t \sup_{\|\phi\|_{\chi}\leq 1}\left|\langle \phi, \frac{(\mathbb{X}_{s+\epsilon}-\mathbb{X}_s)\otimes^2}{\epsilon}\rangle\right| ds\ ,
\end{aligned}$$

where $\varpi_{D^2F}^{[0,T]\times\mathcal{U}}(\epsilon)$ is the continuity modulus of the application $D^2F : [0,T]\times B \longrightarrow \chi$ restricted to $[0,T]\times\mathcal{U}$ where $\mathcal{U}$ is the same random compact set introduced in (5.6). Again D^2F on $[0,T]\times\mathcal{U}$ is uniformly continuous and $\varpi_{D^2F}^{[0,T]\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero. Taking into account condition **H1** in the definition of χ-quadratic variation, $I_{23}(\epsilon,t)\to 0$ in probability when ϵ goes to zero.
Since $I_0(\epsilon,t)$, $I_1(\epsilon,t)$, $I_{22}(\epsilon,t)$ and $I_{23}(\epsilon,t)$ converge in probability for every fixed $t\in[0,T]$, it follows that $I_{21}(\epsilon,t)$ converges in probability when $\epsilon\to 0$. Therefore the forward integral

$$\int_0^t {}_{B^*}\langle DF(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B$$

exists by definition. This in particular implies the Itô formula (5.1). □

We make now some operational comments. The Chi-subspace χ of $(B\hat{\otimes}_\pi B)^*$ constitutes a degree of freedom in the statement of Itô formula. In order to find the suitable expansion for $F(t,\mathbb{X}_t)$ we may proceed as follows.

- Let $F : [0,T]\times B \longrightarrow \mathbb{R}$ of class $C^{1,1}([0,T]\times B)$ we compute the second order derivative D^2F if it exists.
- We look for the existence of a Chi-subspace χ for which the range of $D^2F : [0,T]\times B \longrightarrow (B\hat{\otimes}_\pi B)^*$ is included in χ and it is continuous with respect to the topology of χ.
- We verify that $\mathbb{X}$ admits a χ-quadratic variation.

We observe that whenever $\mathbb{X}$ admits a global quadratic variation, i.e. $\chi = (B\hat{\otimes}_\pi B)^*$, previous points reduce to check that $F\in C^{1,2}([0,T]\times B)$. When $\mathbb{X}$ is a semimartingale (or more generally a semilocally summable B-valued process with respect to the tensor product) then it admits a tensor quadratic variation and in particular previous result generalizes the classical Itô formula in [20], Section 3.7.

6 Applications of Itô formula for window processes

The scope of this section is to illustrate some applications of our Banach space valued Itô formula to window processes. In this section D^m denotes the classical Malliavin gradient and $\mathbb{D}^{1,2}\left(L^2([0,T])\right)$ (shortly $\mathbb{D}^{1,2}$) denotes the classical Malliavin-Sobolev space, related to the case when X is a classical Brownian motion. For more information the reader may consult for instance [22].
We go on fixing some notations. Let $0<\tau\leq T$, we set $B=C([-\tau,0])$. In this section we will deal with Fréchet derivatives of functionals on B. Let $F : [0,T]\times B\longrightarrow\mathbb{R}$ Fréchet of class $C^{1,2}([0,T]\times B)$. We

remind that the first order Fréchet derivative DF defined on $[0,T]\times B$ takes values in $B^*\cong\mathcal{M}([-\tau,0])$. For all $(t,\eta)\in[0,T]\times B$, we will denote by $D_{dx}F(t,\eta)$ the measure defined by

$$_{\mathcal{M}([-\tau,0])}\langle DF(t,\eta),h\rangle_{C([-\tau,0])}=DF(t,\eta)(h)=\int_{[-\tau,0]}h(x)D_{dx}F(t,\eta)\quad\text{for every } h\in C([-\tau,0]).$$

We remark that the second order Fréchet derivative D^2F defined on $[0,T]\times B$ takes values in $L\left(B;B^*\right)\cong\mathcal{B}(B,B)\cong\left(B\hat{\otimes}_\pi B\right)^*$. Recalling (2.3), if $D^2F\left(t,\eta\right)\in\mathcal{M}([-\tau,0]^2)$ for all $(t,\eta)\in[0,T]\times B$ (which will happen in most of the treated cases), we will denote with $D^2_{dx\,dy}F(t,\eta)$ the measure on $[-\tau,0]^2$ such that following duality holds for all $g\in C([-\tau,0]^2)$

$$_{\mathcal{M}([-\tau,0]^2)}\langle D^2F(t,\eta),g\rangle_{C([-\tau,0]^2)}=D^2F(t,\eta)(g)=\int_{[-\tau,0]^2}g(x,y)\,D^2_{dx\,dy}F(t,\eta)\ .$$

We recall also that $D^{\delta_0}F\left(t,\eta\right):=DF\left(t,\eta\right)(\{0\})$ and $D^{\perp}F(t,\eta)=DF\left(t,\eta\right)-D^{\delta_0}F\left(t,\eta\right)\delta_0$.

Notation 6.1. If $g,\ell:[a,b]\to\mathbb{R}$ are càdlàg and g has bounded variation we denote

$$\int_{]a,b]}g\,\ell\eta=g(b)\ell(b)-g(a)\ell(a)-\int_{]a,b]}\ell\,dg\quad\text{ and }\quad\int_{[a,b]}g\,d\ell=g(b)\,\ell(b)-\int_{[a,b]}\ell\,dg\ .$$

6.1 About anticipative integration with respect to finite quadratic variation process

This section aims at giving one application of infinite dimensional calculus to anticipative calculus in a situation in which Malliavin-Skorohod calculus could not be applied. On the other hand, as side-effect, our methods produce some identities involving path-dependent Itô or Skorohod integrals with forward integrals. Let X be a real finite quadratic variation process such that $X_0=0$ a.s. and prolonged as usual by continuity to the real line. One motivation is to express, for $\tau\in[0,T]$,

$$\int_0^{T-\tau}\left(\int_{-\tau}^0 g\left(X_{y+\tau+x},X_y\right)dx\right)d^-X_y=\int_0^{T-\tau}\left(\int_y^{y+\tau}g(X_x,X_y)dx\right)d^-X_y$$

for some smooth enough $g:\mathbb{R}^2\longrightarrow\mathbb{R}$. We remark that, even when X is a semimartingale, previous forward integral is not an Itô integral since the integrand is anticipating (not adapted). In this perspective we consider $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ of class $C^2(\mathbb{R}^2)$ such that $f(x,y)=\int_0^y g(x,z)dz$. In particular $g=\partial_2 f$. For this purpose, we start expanding

$$\int_{-\tau}^0 f\left(X_{x+t},X_{t-\tau}\right)dx$$

through our Banach space B-valued Itô formula.

Proposition 6.2. Let $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ be a function of class C^2. We have

$$\begin{aligned}\int_{-\tau}^0 f\left(X_{x+t},X_{t-\tau}\right)dx&=\tau\,f(0,0)+\int_0^T\left(\int_y^{(y+\tau)\wedge T}\partial_1 f\left(X_y,X_{t-\tau}\right)dt\right)d^-X_y\\&+\int_0^{T-\tau}\left(\int_{-\tau}^0\partial_2 f\left(X_{y+x+\tau},X_y\right)dx\right)d^-X_y+\frac{1}{2}\int_0^{T-\tau}\left(\int_{-\tau}^0\partial^2_{2\,2}f\left(X_{y+z+\tau},X_y\right)dz\right)d[X]_y\\&+\frac{1}{2}\int_{-\tau}^0\left(\int_{-x}^T\partial^2_{1\,1}f\left(X_{t+x},X_{t-\tau}\right)\,[X]_{dt+x}\right)dx\end{aligned}\qquad(6.1)$$

provided that at least one of the two forward integrals above exists.

Remark 6.3. If X is an $(\mathcal{F}_t)$-semimartingale the forward integral

$$\int_0^T \left(\int_y^{(y+\tau)\wedge T} \partial_1 f\left(X_y, X_{t-\tau}\right) dt \right) d^- X_y \tag{6.2}$$

coincides with the Itô integral

$$\int_0^T \left(\int_y^{(y+\tau)\wedge T} \partial_1 f\left(X_y, X_{t-\tau}\right) dt \right) dX_y.$$

Proof. We will apply Theorem 5.2 to $F\left(X_t(\cdot)\right)$ where $F : C([-\tau, 0]) \longrightarrow \mathbb{R}$ is the functional defined by $F(\eta) = \int_{-\tau}^0 f\left(\eta(x), \eta(-\tau)\right) dx$ which is of class $C^2(B)$. Below we express the first derivative

$$D_{dx} F\left(\eta\right) = \partial_1 f\left(\eta(x), \eta(-\tau)\right) \mathbb{1}_{[-\tau,0]}(x) dx + \int_{-\tau}^0 \partial_2 f\left(\eta(z), \eta(-\tau)\right) dz\, \delta_{-\tau}(dx)$$

and the second derivative

$$\begin{aligned} D^2_{dx,\,dy} F(\eta) &= \partial^2_{1\,1} f\left(\eta(x), \eta(-\tau)\right) \mathbb{1}_{[-\tau,0]}(x) \delta_y(dx)\, dy + \partial^2_{2\,1} f\left(\eta(x), \eta(-\tau)\right) \delta_{-\tau}(dx)\, \mathbb{1}_{[-\tau,0]}(y) dy \\ &\quad + \partial^2_{1\,2} f\left(\eta(x), \eta(-\tau)\right) \mathbb{1}_{[-\tau,0]}(x) dx\, \delta_{-\tau}(dy) + \int_{-\tau}^0 \partial^2_{2\,2} f\left(\eta(z), \eta(-\tau)\right) dz\, \delta_{-\tau}(dx)\, \delta_{-\tau}(dy)\,. \end{aligned}$$

The second order Fréchet derivative $D^2 F(\eta)$ belongs to χ with $\chi := Diag \oplus \mathcal{D}_{-\tau} \otimes_h L^2 \oplus L^2 \otimes_h \mathcal{D}_{-\tau} \oplus \mathcal{D}_{-\tau,-\tau}$. Since X is a finite quadratic variation process, Propositions 4.8, 4.15 and 3.16 imply that $X(\cdot)$ admits a χ-quadratic variation. We apply now Theorem 5.2 to $F\left(X_T(\cdot)\right)$. The forward integral appearing in the Itô formula

$$I_1 := \int_0^T \langle DF(X_t(\cdot))\,,\, d^- X_t(\cdot) \rangle$$

exists and it is given by $I_{11} + I_{12}$ where

$$\begin{aligned} I_{11} &= \lim_{\epsilon \to 0} \int_0^T \int_{-\tau}^0 \partial_1 f\left(X_{t+x}, X_{t-\tau}\right) \frac{X_{t+x+\epsilon} - X_{t+x}}{\epsilon} dx\, dt \qquad \text{and} \\ I_{12} &= \lim_{\epsilon \to 0} \int_0^T \left(\int_{-\tau}^0 \partial_2 f\left(X_{t+x}, X_{t-\tau}\right) dx \right) \frac{X_{t-\tau+\epsilon} - X_{t-\tau}}{\epsilon} dt, \end{aligned}$$

provided that previous limits in probability exist. We have

$$\begin{aligned} I_{11} &= \lim_{\epsilon \to 0} \int_0^T \int_{(-\tau)\vee(-t)}^0 \partial_1 f\left(X_{t+x}, X_{t-\tau}\right) \frac{X_{t+x+\epsilon} - X_{t+x}}{\epsilon} dx\, dt \\ &= \lim_{\epsilon \to 0} \int_0^T \int_{(t-\tau)\vee(0)}^t \partial_1 f\left(X_y, X_{t-\tau}\right) \frac{X_{y+\epsilon} - X_y}{\epsilon} dy\, dt\,. \end{aligned}$$

By Fubini theorem, previous limit equals (6.2), provided that previous forward limit exists.

We go on specifying I_{12}.

$$
\begin{aligned}
I_{12} &= \lim_{\epsilon\to 0}\int_\tau^T \left(\int_{-\tau}^0 \partial_2 f\,(X_{t+x}, X_{t-\tau})\,dx\right)\frac{X_{t-\tau+\epsilon}-X_{t-\tau}}{\epsilon}dt\\
&= \lim_{\epsilon\to 0}\int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\,(X_{y+x+\tau}, X_{y})\,dx\right)\frac{X_{y+\epsilon}-X_{y}}{\epsilon}dy\\
&= \int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\,(X_{y+x+\tau}, X_{y})\,dx\right)d^-X_y
\end{aligned}
$$

provided that previous forward integral exists.
We evaluate now the integrals involving the second order derivative of F, i.e.

$$
\frac{1}{2}\int_0^T {}_{\chi}\langle D^2F(X_t(\cdot))\,,\,d\widetilde{[X(\cdot)]}_t\rangle_{\chi^*}\ . \tag{6.3}
$$

We remind that $D^2F(\eta)$ takes values in $\chi := Diag \oplus \mathcal{D}_{-\tau}\otimes_h L^2 \oplus L^2\otimes_h \mathcal{D}_{-\tau}\oplus \mathcal{D}_{-\tau,-\tau}$. The term (6.3) splits into a sum of four terms. Since by Proposition 4.8 item 2, $X(\cdot)$ has zero $\mathcal{D}_{-\tau}\otimes_h L^2$ and $L^2\otimes_h \mathcal{D}_{-\tau}$-quadratic variation, the only non vanishing integrals are the two terms I_{21} and I_{22} given respectively by the $\mathcal{D}_{-\tau,-\tau}$ and the $Diag$-quadratic variation. Again by Proposition 4.8 item 3, expression (6.3) becomes $I_{21}+I_{22}$ where

$$
I_{21} = \frac{1}{2}\int_0^{T-\tau}\int_{-\tau}^0 \partial^2_{2\,2} f\,(X_{y+z+\tau}, X_y)\,dz\,d[X]_y\ , \qquad I_{22} = \frac{1}{2}\int_0^T {}_{Diag}\langle G(t)\,,\,d\widetilde{[X(\cdot)]}_t\rangle_{Diag^*}
$$

and $G(t) = g(t,x)\delta_y(dx)dy$, with $g(t,x) = \partial^2_{1\,1} f\,(X_{t+x}, X_{t-\tau})$. Since $\partial^2_{1\,1}f$ is a continuous function, Proposition 4.17 can be applied and we get

$$
I_{22} = \frac{1}{2}\int_{-\tau}^0 \left(\int_{-x}^T \partial^2_{1\,1} f\,(X_{t+x}, X_{t-\tau})\ [X]_{dt+x}\right)dx\ .
$$

In conclusion we obtain (6.1). □

Corollary 6.4. Let X be an $(\mathcal{F}_t)$-semimartingale and $g:\mathbb{R}^2\longrightarrow\mathbb{R}$ of class $C^{2,1}(\mathbb{R}\times\mathbb{R})$. Then the forward integral $\int_0^{T-\tau}\left(\int_{-\tau}^0 g\,(X_{y+\tau+x}, X_y)\,dx\right)d^-X_y$ exists and it can be explicitly given.

Proof. We set $f(x,y) = \int_0^y g(x,z)dz$. The first forward integral in Proposition 6.2 exists and it is an Itô integral. We apply finally Proposition 6.2. □

Corollary 6.5. Let $X = W$ be a classical Wiener process, $f\in C^2(\mathbb{R}^2)$. We have the following identity.

$$
\begin{aligned}
\int_{-\tau}^0 f\,(W_{x+t}, W_{t-\tau})\,dx &= \tau\, f(0,0) + \int_0^T \left(\int_y^{(y+\tau)\wedge T} \partial_1 f\,(W_y, W_{t-\tau})\,dt\right)dW_y\\
&+ \int_0^{T-\tau}\left(\int_{-\tau}^0 \partial_2 f\,(W_{y+x+\tau}, W_y)\,dx\right)\delta W_y + \int_0^{T-\tau}\left(\int_{-\tau}^0 \partial^2_{2\,1} f\,(W_{t+\tau+z}, W_t)\,dz\right)dt\\
&+ \frac{1}{2}\int_0^{T-\tau}\left(\int_{-\tau}^0 \partial^2_{2\,2} f\,(W_{y+z+\tau}, W_y)\,dz\right)dy + \frac{1}{2}\int_{-\tau}^0\left(\int_{-x}^T \partial^2_{1\,1} f\,(W_{t+x}, W_{t-\tau})\ dt\right)dx.
\end{aligned}
$$

Remark 6.6. If $Y \in \mathbb{D}^{1,2}\left(L^2([0,T])\right)$, $D^m Y$ represents the Malliavin derivative and $\int_0^t Y_s \delta W_s$, $t \in [0,T]$, is the Skorohod integral. We recall that, by [25] and [28]

$$\int_0^t Y_s d^- W_s = \int_0^t Y_s \delta W_s + \left(Tr^- D^m Y\right)(t) \quad \text{where} \tag{6.4}$$

$$\left(Tr^- D^m Y\right)(t) = \lim_{\epsilon \to 0} \int_0^t \left(\int_s^{s+\epsilon} \frac{D_r^m Y_s}{\epsilon} dr \right) ds \quad \text{in } L^2(\Omega).$$

Proof of Corollary 6.5. It follows from Proposition 6.2 provided we prove that

$$\int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\left(W_{y+x+\tau}, W_y\right) dx \right) d^- W_y \quad \text{equals}$$

$$\int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_2 f\left(W_{y+x+\tau}, W_y\right) dx \right) \delta W_y + \int_0^{T-\tau} \left(\int_{-\tau}^0 \partial_{2\,1}^2 f\left(W_{t+\tau+z}, W_t\right) dz \right) dt\ .$$

This follows by Remark 6.6 with

$$Y_s = \int_{-\tau}^0 \partial_2 f\left(W_{s+\tau+z}, W_s\right) dz\ .$$

In fact, for $r > s$, $D_r Y_s = \int_{r-s-\tau}^0 \partial_{2\,1}^2 f\left(W_{s+\tau+z}, W_s\right) dz$ and so

$$\left(Tr^- D^m Y\right)(t) = \lim_{r \downarrow s} \int_0^t D_r^m Y_s\, ds = \int_0^t \left(\int_{-\tau}^0 \partial_{2\,1}^2 f\left(W_{s+\tau+z}, W_s\right) dz \right) ds. \tag{6.5}$$

Combining (6.5) with (6.4) for $t = T - \tau$ the result is now established. □

Remark 6.7. Another example of exploitation of Proposition 6.2 arises when X is a Gaussian centered process with covariance $R(s,t) = \mathbb{E}\left[X_s X_t\right]$ such that $\frac{\partial^2 R}{\partial s \partial t}$ is a signed finite measure μ. We say in this case that the covariance of X has a measure structure, see [16]. We remind that in this case X is a finite quadratic variation process and $[X]_t = \mu(\{(s,s) | s \in [0,t]\})$. With some slight technical assumptions, the following relation holds:

$$\int_0^t Y_s d^- X_s = \int_0^t Y_s \delta X_s + \int_{[0,t]^2} D_{r^+} Y_s d\mu(r,s)\ . \tag{6.6}$$

This allows to show the existence of both the forward integrals in the statement of Proposition 6.2 using (6.6).

6.2 Infinite dimensional partial differential equation and Clark-Ocone type results

One natural application consists in obtaining a *Clark-Ocone type formula* for real finite quadratic variation processes. Let X be such a process and we assume again $X_0 = 0$ for simplicity. Given a real path dependent random variable h, the idea consists in finding $H_0 \in \mathbb{R}$ and an adapted (with respect to the natural filtration of X) process ξ such that $h = H_0 + \int_0^T \xi_s d^- X_s$. If X is a classical Brownian motion, previous forward integral is a Itô integral and the result holds by the martingale representation theorem.

If h belongs to $\mathbb{D}^{1,2}$, then $H_0 = \mathbb{E}[h]$ and $\xi_t = \mathbb{E}\left[D_t^m h|\mathcal{F}_t\right]$. This statement is the classical **Clark-Ocone formula**.
The proof of our Clark-Ocone type formula consists in two steps. 1) Given a solution of an infinite dimensional PDE of the type (1.6), an important ingredient of the proof is our infinite dimensional Itô formula, more precisely Theorem 5.2. This allows, almost immediately, to obtain an explicit representation of H_0 and ξ. This is the object of Theorem 6.9. 2) The second step consists in constructing indeed a solution of such PDE. For a large class of random variables h, Chapter 9 of [6] provides a solution of the PDE at least when $[X]_t = t$. The present section is devoted to step 1) which, among others, generalizes Theorem 7.1 of [7] and it expands its proof to the case when $[X]_t = \sigma^2 t$, $\sigma \geq 0$. In this subsection we set $\tau = T$ and therefore $B = C([-T,0])$.

Definition 6.8. Let $H : C([-T,0]) \longrightarrow \mathbb{R}$ be a Borel functional. $u : [0,T] \times B \longrightarrow \mathbb{R}$ of class $C^{1,2}\left([0,T[\times B\right) \cap C^0\left([0,T] \times B\right)$ is said to be a solution of (the infinite dimensional PDE)

$$\begin{cases} \partial_t u(t,\eta) + \int_{]-t,0]} D^{\perp} u\,(t,\eta)\; d\eta + \frac{\sigma^2}{2} \langle D^2 u\,(t,\eta)\;,\; \mathbb{1}_{D_t}\rangle = 0 \quad \text{for } t \in [0,T[\\ u(T,\eta) = H(\eta) \end{cases} \tag{6.7}$$

if the following conditions hold.
i) $D^{\perp}u(t,\eta)$ is absolutely continuous with respect to Lebesgue measure and its Radon-Nikodym derivative $x \mapsto D^{\perp}_x u\,(t,\eta)$ has bounded variation for any $t \in [0,T]$, $\eta \in B$; **ii)** $D^2 u(t,\eta)$ is a Borel finite measure on $[-T,0]^2$ for all $t \in [0,T]$ and $\eta \in B$; **iii)** u solves (6.7) where $\int_{]-t,0]} D^{\perp} u\,(t,\eta)\; d\eta$ in the sense of Notation 6.1 and $\langle D^2 u\,(t,\eta)\;,\; \mathbb{1}_{D_t}\rangle$ indicates the evaluation of the second order derivative on the diagonal $D_t = \{(s,s)|s \in [-t,0]\}$.

Theorem 6.9. Let $H : B \longrightarrow \mathbb{R}$ be a Borel functional and $u : [0,T] \times B \longrightarrow \mathbb{R}$ be a solution to (6.7). We set $\chi := \chi^0([-T,0]^2) \oplus Diag([-T,0]^2)$, (shortly $\chi^0 \oplus Diag$). We suppose the following.
i) $(t,\eta) \mapsto \|D^{\perp}u\;(t,\eta)\|_{BV} := |D^{\perp}_0 u\;(t,\eta)| + \int_{-T}^{0} |D^{\perp}_x u\;(t,\eta)| dx = |D^{\perp}_0 u\;(t,\eta)| + \|D^{\perp}u\;(t,\eta)\|_{Var}$ is bounded on $[0,T] \times K$ for each compact K of B.
ii) $D^2 u\,(t,\eta) \in \chi$ for every $t \in [0,T]$, $\eta \in B$ and that map $(t,\eta) \mapsto D^2 u\,(t,\eta)$ is continuous from $[0,T] \times B$ to χ.
Let X be a real continuous finite quadratic variation process with $[X]_t = \sigma^2 t$, $\sigma \geq 0$, and $X_0 = 0$.
Then the random variable $h := H(X_T(\cdot))$ admits the following representation

$$h = u(T, X_T(\cdot)) = H_0 + \int_0^T \xi_t d^- X_t \tag{6.8}$$

with $H_0 = u(0, X_0(\cdot))$, $\xi_t = D^{\delta_0} u\,(s, X_s(\cdot))$ and $\int_0^T \xi_t d^- X_t$ is an improper forward integral.

Proof. Since $u \in C^0\left([0,T] \times B\right)$, $H = u(T,\cdot)$ is automatically continuous. By Propositions 4.9 item 2, 4.15 and 3.16 $X(\cdot)$ admits a χ-quadratic variation which is the sum of the χ^0-quadratic variation and the $Diag$-quadratic variation. Applying Theorem 5.2 to $u\,(t, X_t(\cdot))$ for $t < T$ we obtain

$$\begin{aligned} u(t, X_t(\cdot)) = u(0, X_0(\cdot)) + \int_0^t \partial_t u(s, X_s(\cdot)) ds + \int_0^t {}_{\mathcal{M}([-T,0])}\langle Du(s, X_s(\cdot)), d^- X_s(\cdot)\rangle_{C([-T,0])} \\ + \frac{1}{2} \int_0^t {}_{\chi}\langle D^2 u(s, X_s(\cdot)), d\widetilde{[X(\cdot)]}_s\rangle_{\chi^*}. \end{aligned} \tag{6.9}$$

By Assumption **i)** it is possible to show that $\int_0^t {}_{\mathcal{M}([-T,0])}\langle D^\perp u(s, X_s(\cdot)), d^- X_s(\cdot)\rangle_{C([-T,0])}$ exists and equals $\int_0^t \left(\int_{]-s,0]} D^\perp u(s,\eta)d\eta\right)|_{\eta=X_s(\cdot)}ds$. We omit the technicalities. Consequently, by subtraction, $\int_0^t D^{\delta_0}u(s, X_s(\cdot))d^- X_s$ exists for $t \in [0, T[$. The Itô expansion (6.9) gives

$$u(t, X_t(\cdot)) = u(0, X_0(\cdot)) + \int_0^t D^{\delta_0} u\,(s, X_s(\cdot))d^- X_s + \int_0^t \mathcal{L}u\,(s, X_s(\cdot))ds \tag{6.10}$$

where

$$\mathcal{L}u\,(t, \eta) = \partial_t u(t, \eta) + \int_{]-t,0]} D^\perp u(t, \eta)\, d\eta + \frac{\sigma^2}{2}\langle D^2 u\,(t, \eta)\ ,\ \mathbb{1}_{D_t}\rangle$$

for $t \in [0, T[$, $\eta \in B$. By hypothesis $\mathcal{L}u\,(t, \eta) = 0$, so (6.10) gives

$$u(t, X_t(\cdot)) = u(0, X_0(\cdot)) + \int_0^t D^{\delta_0} u\,(s, X_s(\cdot))d^- X_s. \tag{6.11}$$

Now for every fixed ω, since $u \in C^0([0, T] \times B)$ and X is continuous, we have $\lim_{t\to T} u(t, X_t(\cdot)) = u(T, X_T(\cdot))$, which equals $H(X_T(\cdot))$ by (6.7). This forces the right-hand side of (6.11) to converge, so that the result follows. □

Remark 6.10. Previous theorem also applies in the case $\sigma = 0$, i.e. $[X] = 0$. To this purpose we observe the following.

1. Let
$$h = f\left(\int_0^T \varphi_1(s)d^- X_s, \ldots, \int_0^T \varphi_n(s)d^- X_s\right) \tag{6.12}$$
with $\varphi_i \in C^2([0, T])$ and $f \in C^2(\mathbb{R}^n)$. In that case PDE in (6.7) simplifies into $\partial_t u + \int_{]-t,0]} D^\perp u\,(t, \eta)\, d\eta = 0$ and it is easy to provide a solution u in the sense of Definition 6.8. That $u : [0, T]\times C([-T, 0]) \longrightarrow \mathbb{R}$ is given by
$$u(t, \eta) = f\left(\int_{[-t,0]} \varphi_1(s + t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s + t)d\eta(s)\right)\ .$$

2. Since $D^{\delta_0}u(t, \eta) = \sum_{i=1}^n \partial_i f\left(\int_{[-t,0]} \varphi_1(s + t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s + t)d\eta(s)\right)\varphi_i(t)$, by Theorem 6.9, we obtain representation (6.8) with $H_0 = f(0, \ldots, 0)$ and $\xi_t = D^{\delta_0}u(t, X_t(\cdot))$ The assumptions of Theorem 6.9 can be easily checked, but we omit the details. We remind only that $X(\cdot)$ admits χ^0-quadratic variation.

3. In that case $\sigma = 0$, representation (6.8) can be also established via an application of the finite dimensional Itô formula for finite quadratic variation processes, see Proposition 2.4 in [13].

4. The general case $\sigma \neq 0$ with the same r.v. h given by (6.12) was treated in Section 9.9 of [6].

Remark 6.11. The assumption $[X]_t = \sigma^2 t$ is not crucial. With some more work it is possible to obtain similar representations even if $[X]_t = \int_0^t a^2(s, X_s)ds$ for a large class of $a : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$.

A Appendix: Proofs of some technical results

Sketch of the proof of the Proposition 1.6. Let $\mathbb{V}$ (resp. $\mathbb{Y}$) be an H-valued bounded variation (resp. continuous) process. Proceeding as for real valued processes, see for instance [28], Proposition 1.7)b), one can show that $\mathbb{V}$ and $\mathbb{Y}$ has a zero scalar covariation. A semilocally summable process is the sum of a locally summable process and a bounded variation process. Therefore, without restriction of generality, we can suppose that $\mathbb{X}$ is locally summable with respect to the tensor products. By localization we can suppose that $\mathbb{X}$ is summable with respect to the tensor products and bounded. Let $s \in [0, T]$ and consider the following identity

$$\mathbb{X}_{s+\epsilon}^{\otimes^2} - \mathbb{X}_s^{\otimes^2} = \mathbb{X}_s \otimes (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) + (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes \mathbb{X}_s + (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2 . \tag{A.1}$$

Dividing (A.1) by ϵ and integrating from 0 to t in the Bochner sense we obtain

$$I_0(t, \epsilon) = I_1(t, \epsilon) + I_2(t, \epsilon) + \int_0^t \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2}{\epsilon} ds$$

where

$$I_0(t, \epsilon) = \int_0^t \frac{\mathbb{X}_{s+\epsilon}^{\otimes^2} - \mathbb{X}_s^{\otimes^2}}{\epsilon} ds\,, \quad I_1(t, \epsilon) = \int_0^t \frac{\mathbb{X}_s \otimes (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s)}{\epsilon} ds\,, \quad I_2(t, \epsilon) = \int_0^t \frac{(\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes \mathbb{X}_s}{\epsilon} ds\,.$$

Let $t \in [0, T]$. Obviously we get $\lim_{\epsilon \to 0} I_0(t, \epsilon) = \mathbb{X}_t^{\otimes^2} - \mathbb{X}_0^{\otimes^2}$.
By an elementary Fubini argument we can show that

$$I_1(t, \epsilon) = \int_0^t \left(\frac{1}{\epsilon} \int_{u-\epsilon}^u \mathbb{X}_s ds \right) \otimes d\mathbb{X}_u \,.$$

Since $\frac{1}{\epsilon} \int_{u-\epsilon}^u \mathbb{X}_s ds \longrightarrow \mathbb{X}_u$ for every $u \in [0, T]$ and $\omega \in \Omega$ and $\mathbb{X}$ being bounded, Theorem 1 in section 12. A of [8] allows to show that $I_1(t, \epsilon) \longrightarrow \int_0^t \mathbb{X}_s \otimes d\mathbb{X}_s$ in probability. Similarly one shows that $I_2(t, \epsilon) \longrightarrow \int_0^t d\mathbb{X}_s \otimes \mathbb{X}_s$. In conclusion $\mathbb{X}$ admits a tensor quadratic variation which equals

$$\mathbb{X}_t^{\otimes^2} - \int_0^t \mathbb{X}_s \otimes d\mathbb{X}_s - \int_0^t d\mathbb{X}_s \otimes \mathbb{X}_s \,.$$

□

Sketch of the proof of Proposition 1.7. Let H be the Hilbert space values of $\mathbb{X}$. Let $\mathbb{V}$ (resp. $\mathbb{Y}$) be an H-valued bounded variation (resp. continuous) process. Without restriction of generality we can suppose that $\mathbb{X}$ is an $(\mathcal{F}_t)$-local martingale. After localization one can suppose that $\mathbb{X}$ is an $(\mathcal{F}_t)$-square integrable martingale. Proceeding similarly as for the proof of Proposition 1.6, using Remark 14.b) of Chapter 6.23 of [8], it is possible to show that

$$\frac{1}{\epsilon} \int_0^t \|\mathbb{X}_{s+\epsilon} - \mathbb{X}_s\|_H^2 ds \xrightarrow[\epsilon \longrightarrow 0]{} \|\mathbb{X}_t\|_H^2 - 2 \int_0^t \langle \mathbb{X}_s, d\mathbb{X}_s \rangle_H \,.$$

The analogous of the bilinear forms considered in Proposition 1.6 proof will be the H inner product. □

Before writing the proof of Proposition 3.19 we need a technical lemma. In the sequel the indices χ and χ^* in the duality, will often be omitted.

Lemma A.1. Let $t \in [0,T]$. There is a subsequence of (n_k) still denoted by the same symbol and a null subset N of Ω such that

$$\widetilde{F}^{n_k}(\omega,t)(\phi) \longrightarrow_{k\to\infty} \widetilde{F}(\omega,t)(\phi) \quad \text{for every } \phi \in \chi \text{ and } \omega \notin N.$$

Proof of Lemma A.1 . Let $\mathcal{S}$ be a dense countable subset of χ. By a diagonalization principle for extracting subsequences, there is a subsequence (n_k), a null subset N of Ω such that for all $\omega \notin \Omega$,

$$\widetilde{F}_\infty(\omega,t)(\phi) := \lim_{k\to+\infty} \widetilde{F}^{n_k}(\omega,t)(\phi) \quad \text{exists for any } \phi \in \mathcal{S},\, \omega \notin N \text{ and } \forall\, t \in [0,T]. \tag{A.2}$$

By construction, for every $t \in [0,T]$, $\phi \in \mathcal{S}$

$$\widetilde{F}(\cdot,t)(\phi) = F(\phi)(\cdot,t) = \widetilde{F}_\infty(\cdot,t)(\phi) \qquad a.s.$$

Let $t \in [0,T]$ be fixed. Since $\phi \in \mathcal{S}$ countable, a slight modification of the null set N, yields that for every $\omega \notin N$,

$$\widetilde{F}(\omega,t)(\phi) = \widetilde{F}_\infty(\omega,t)(\phi) \qquad \forall\, \phi \in \mathcal{S}\ .$$

At this point (A.2) becomes

$$\widetilde{F}(\omega,t)(\phi) = \lim_{k\to+\infty} \widetilde{F}^{n_k}(\omega,t)(\phi) \quad \text{for every } \omega \notin N,\, \phi \in \mathcal{S}. \tag{A.3}$$

It remains to show that (A.3) still holds for $\phi \in \chi$. Therefore we fix $\phi \in \chi$, $\omega \notin N$. Let $\epsilon > 0$ and $\phi_\epsilon \in \mathcal{S}$ such that $\|\phi - \phi_\epsilon\|_\chi \le \epsilon$. We can write

$$\begin{aligned} \left|\widetilde{F}(\omega,t)(\phi) - \widetilde{F}^{n_k}(\omega,t)(\phi)\right| &\le \left|\widetilde{F}(\omega,t)(\phi-\phi_\epsilon)\right| + \left|\widetilde{F}(\omega,t)(\phi_\epsilon) - \widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon)\right| + \left|\widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon-\phi)\right| \le \\ &\le \left\|\widetilde{F}(\omega,t)\right\|_{\chi^*} \|\phi-\phi_\epsilon\|_\chi + \sup_k \left\|\widetilde{F}^{n_k}(\omega,t)\right\|_{\chi^*} \|\phi-\phi_\epsilon\|_\chi + \\ &\quad + \left|\widetilde{F}(\omega,t)(\phi_\epsilon) - \widetilde{F}^{n_k}(\omega,t)(\phi_\epsilon)\right|\ . \end{aligned}$$

Taking the $\limsup_{k\to+\infty}$ in previous expression and using (A.3) yields

$$\limsup_{k\to+\infty} \left|\widetilde{F}(\omega,t)(\phi) - \widetilde{F}^{n_k}(\omega,t)(\phi)\right| \le \left\|\widetilde{F}(\omega,t)\right\|_{\chi^*} \epsilon + \sup_k \left\|\widetilde{F}^{n_k}(\omega,\cdot)\right\|_{Var[0,T]} \epsilon\ .$$

Since $\epsilon > 0$ is arbitrary, the result follows. □

Proof of Proposition 3.19 . Let $t \in [0,T]$ be fixed. We denote

$$I(n)(\omega) := \int_0^t \langle H(\omega,s), d\widetilde{F}^n(\omega,s)\rangle - \int_0^t \langle H(\omega,s), d\widetilde{F}(\omega,s)\rangle\ .$$

Let $\delta > 0$ and a subdivision of $[0,t]$ given by $0 = t_0 < t_1 < \cdots < t_m = t$ whose mesh is smaller than δ. Let (n_k) be a sequence diverging to infinity. We need to exhibit a subsequence (n_{k_j}) such that

$$I(n_{k_j})(\omega) \longrightarrow 0 \qquad \text{a.s.} \tag{A.4}$$

Lemma A.1 implies the existence of a null set N, a subsequence (n_{k_j}) such that

$$\left|\widetilde{F}^{n_{k_j}}(\omega,t_l)(\phi)-\widetilde{F}(\omega,t_l)(\phi)\right| \xrightarrow[j\longrightarrow+\infty]{} 0 \qquad \forall\,\phi\in\chi \quad \text{and for every} \quad l\in\{0,\dots,m\}\,. \tag{A.5}$$

Let $\omega\notin N$. We have

$$\begin{aligned}
\left|I(n_{k_j})(\omega)\right| &= \left|\sum_{i=1}^m\left(\int_{t_{i-1}}^{t_i}\langle H(\omega,s),d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle-\langle H(\omega,s),d\widetilde{F}(\omega,s)\rangle\right)\right|\le\\
&\le\sum_{i=1}^m\left|\int_{t_{i-1}}^{t_i}\langle H(\omega,s)-H(\omega,t_{i-1})+H(\omega,t_{i-1}),d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle+\right.\\
&\qquad\qquad\left.-\int_{t_{i-1}}^{t_i}\langle H(\omega,s)-H(\omega,t_{i-1})+H(\omega,t_{i-1}),d\widetilde{F}(\omega,s)\rangle\right|\le\\
&\le I_1(n_{k_j})(\omega)+I_2(n_{k_j})(\omega)+I_3(n_{k_j})(\omega)\ ,
\end{aligned}$$

where

$$\begin{aligned}
I_1(n_{k_j})(\omega)&=\sum_{i=1}^m\left|\int_{t_{i-1}}^{t_i}\langle H(\omega,s)-H(\omega,t_{i-1}),d\widetilde{F}^{n_{k_j}}(\omega,s)\rangle\right|\le\varpi_{H(\omega,\cdot)}(\delta)\,\sup_j\|\widetilde{F}^{n_{k_j}}(\omega)\|_{Var[0,T]}\\
I_2(n_{k_j})(\omega)&=\sum_{i=1}^m\left|\int_{t_{i-1}}^{t_i}\langle H(\omega,s)-H(\omega,t_{i-1}),d\widetilde{F}(\omega,s)\rangle\right|\le\varpi_{H(\omega,\cdot)}(\delta)\,\|\widetilde{F}(\omega)\|_{Var[0,T]}\\
I_3(n_{k_j})(\omega)&=\sum_{i=1}^m\left|\int_{t_{i-1}}^{t_i}\langle H(\omega,t_{i-1}),d(\widetilde{F}^{n_{k_j}}(\omega,s)-\widetilde{F}(\omega,s))\rangle\right|=\\
&=\sum_{i=1}^m\left|\langle H(\omega,t_{i-1}),\widetilde{F}^{n_{k_j}}(\omega,t_i)-\widetilde{F}(\omega,t_i)-\widetilde{F}^{n_{k_j}}(\omega,t_{i-1})+\widetilde{F}(\omega,t_{i-1})\rangle\right|\le\\
&\le\sum_{i=1}^m|F^{n_{k_j}}(H(\omega,t_{i-1}))(\omega,t_i)-F(H(\omega,t_{i-1}))(\omega,t_i)|+\\
&\qquad\qquad\sum_{i=1}^m|F^{n_{k_j}}(H(\omega,t_{i-1}))(\omega,t_{i-1})-F(H(\omega,t_{i-1}))(\omega,t_{i-1})|\ .
\end{aligned}$$

The notation $\varpi_{H(\omega,\cdot)}$ indicates the modulus of continuity for H and it is a random variable; in fact it depends on ω in the sense that

$$\varpi_{H(\omega,\cdot)}(\delta)=\sup_{|s-t|\le\delta}\|H(\omega,s)-H(\omega,t)\|_\chi\ .$$

By (A.5) applied to $\phi=H(\omega,t_{i-1})$ we obtain

$$\limsup_{j\to\infty}|I(n_{k_j})(\omega)|\le\left(\sup_j\|\widetilde{F}^{n_{k_j}}(\omega)\|_{Var[0,T]}+\|\widetilde{F}(\omega)\|_{Var[0,T]}\right)\varpi_{H(\omega,\cdot)}(\delta)\,.$$

Since $\delta>0$ is arbitrary and H is uniformly continuous on $[0,t]$ so that $\varpi_{H(\omega,\cdot)}(\delta)\to0$ a.s. for $\delta\to0$, then $\limsup_{j\to\infty}|I(n_{k_j})(\cdot)|=0$ a.s..
This concludes (A.4) and the proof of Proposition 3.19. □

Proof of Theorem 3.22. Supposing **iv')**, Lemma 3.1 from [27] implies that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \mathcal{S}$, since for every $\phi \in \mathcal{S}$, $F(\phi)$ is an increasing process, so **iv)** is established. We only show the result considering **iv)**.

a) We recall that $\mathscr{C}([0,T])$ is an F-space. Let $\phi \in \chi$. Clearly $(F^n(\phi)(\cdot,t))_t$ and $\left(\tilde{F}^n(\cdot,t)(\phi)\right)_t$ are indistinguishable processes and so $\left(\tilde{F}^n(\phi)(\cdot,t)\right)_t$ is a continuous process. So it follows

$$\begin{aligned}\|F^n(\phi)\|_\infty &= \sup_{t\in[0,T]} |F^n(\phi)(t)| = \sup_{t\in[0,T]} |\tilde{F}^n(\cdot,t)(\phi)| \leq \\ &\leq \sup_{t\in[0,T]} \left\|\tilde{F}^n(\cdot,t)\right\|_{\chi^*} \|\phi\|_\chi \leq \sup_n \|\tilde{F}^n\|_{Var([0,T])}\|\phi\|_\chi < +\infty\end{aligned}$$

a.s. by the hypothesis. By Remark 3.21.2. and 3. it follows that the set $\{F^n(\phi)\}$ is a bounded subset of the F-space $\mathscr{C}([0,T])$ for every fixed $\phi \in \chi$.
We can apply the Banach-Steinhaus Theorem II.1.18, pag. 55 in [9] and point **iv)**, which imply the existence of $F : \chi \longrightarrow \mathscr{C}([0,T])$ linear and continuous such that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \chi$. So **a)** is established in both situations **1)** and **2)**.

b) It remains to show the rest in situation **1)**, i.e. when χ is separable.

b.1) We first prove the existence of a suitable version $\tilde{F}$ of F such that $\tilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ is weakly star continuous ω a.s.
Since χ is separable, we consider a dense countable subset $\mathcal{D} \subset \chi$. Point **a)** implies that for a fixed $\phi \in \mathcal{D}$ there is a subsequence (n_k) such that $F^{n_k}(\phi)(\omega,\cdot) \xrightarrow{C([0,T])} F(\phi)(\omega,\cdot)$ a.s. Since $\mathcal{D}$ is countable there is a null set N and a further subsequence still denoted by (n_k) such that

$$\tilde{F}^{n_k}(\omega,\cdot)(\phi) \xrightarrow{C([0,T])} F(\phi)(\omega,\cdot) \qquad \forall\, \phi \in \mathcal{D},\ \forall \omega \notin N\ . \tag{A.6}$$

For $\omega \notin N$, we set $\tilde{F}(\omega,t)(\phi) = F(\phi)(\omega,t)\ \forall\ \phi \in \mathcal{S},\ t \in [0,T]$. By a slight abuse of notation the sequence $\tilde{F}^{n_k}$ can be seen as applications

$$\tilde{F}^{n_k}(\omega,\cdot) : \chi \longrightarrow C([0,T])$$

which are linear continuous maps verifying the following.

- $\tilde{F}^{n_k}(\omega,\cdot)(\phi) \longrightarrow \tilde{F}(\omega,\cdot)(\phi)$ in $C([0,T])$ for all $\phi \in \mathcal{D}$, because of (A.6).
- For every $\phi \in \chi$, we have

$$\begin{aligned}\sup_k \sup_{t\leq T} |\tilde{F}^{n_k}(\omega,t)(\phi)| &\leq \sup_k \sup_{t\leq T} \sup_{\|\phi\|_\chi \leq 1} |\tilde{F}^{n_k}(\omega,t)(\phi)|\ \|\phi\|_\chi \leq \sup_k \sup_{t\leq T} \|\tilde{F}^{n_k}(\omega,t)\|\ \|\phi\|_\chi \\ &\leq \sup_k \|\tilde{F}^{n_k}(\omega,\cdot)\|_{Var([0,T])}\|\phi\|_\chi < +\infty.\end{aligned}$$

Banach-Steinhaus theorem implies the existence of a linear random continuous map

$$\tilde{F}(\omega,\cdot) : \chi \longrightarrow C([0,T])$$

extending previous map $\tilde{F}(\omega,\cdot)$ from $\mathcal{D}$ to χ with values on $C([0,T])$. Moreover

$$\tilde{F}^{n_k}(\omega,\cdot)(\phi) \xrightarrow{C([0,T])} \tilde{F}(\omega,\cdot)(\phi) \qquad \forall \phi \in \chi,\ \forall \omega \notin N$$

and for every $\omega \notin N$ the application

$$\tilde{F}(\omega, \cdot) : [0,T] \longrightarrow \chi^* \qquad t \mapsto \tilde{F}(\omega, t)$$

is weakly star continuous. $\tilde{F}$ is measurable from $\Omega \times [0,T]$ to χ^* being limit of measurable processes.

b.2) We prove now that the χ^*-valued process $\tilde{F}$ has bounded variation.
Let $\omega \notin N$ fixed again. Let $(t_i)_{i=0}^M$ be a subdivision of $[0,T]$ and let $\phi \in \chi$. Since the functions

$$F^{t_i,t_{i+1}} : \phi \longrightarrow \left(\tilde{F}(t_{i+1}) - \tilde{F}(t_i)\right)(\phi) \qquad F^{n_k,t_i,t_{i+1}} : \phi \longrightarrow \left(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\right)(\phi)$$

belong to χ^*, Banach-Steinhaus theorem says

$$\sup_{\|\phi\| \le 1} \left| \left(\tilde{F}(t_{i+1}) - \tilde{F}(t_i)\right)(\phi) \right| = \|F^{t_i,t_{i+1}}\|_{\chi^*} \le \lim\inf_{k\to\infty} \|F^{n_k,t_i,t_{i+1}}\|_{\chi^*} =$$

$$= \lim\inf_{k\to\infty} \sup_{\|\phi\|\le 1} \left| \left(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\right)(\phi) \right| .$$

Taking the sum over $i = 0, \ldots, (M-1)$ we get

$$\sum_{i=0}^{M-1} \sup_{\|\phi\|\le 1} \left| \left(\tilde{F}(t_{i+1}) - \tilde{F}(t_i)\right)(\phi) \right| \le \sum_{i=0}^{M-1} \lim\inf_{k\to\infty} \sup_{\|\phi\|\le 1} \left| \left(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\right)(\phi) \right| \le$$

$$\le \sup_k \sum_{i=0}^{M-1} \sup_{\|\phi\|\le 1} \left| \left(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\right)(\phi) \right| \le \sup_k \|\tilde{F}^{n_k}\|_{Var([0,T])} ,$$

where the second inequality is justified by the relation $\lim\inf a_i^n + \lim\inf b_i^n \le \sup(a_i^n + b_i^n)$.
Taking the sup over all subdivision $(t_i)_{i=0}^M$ we obtain

$$\|\tilde{F}\|_{Var([0,T])} \le \sup_k \|\tilde{F}^{n_k}\|_{Var([0,T])} < +\infty .$$

This shows finally the fact that $\tilde{F}(\omega, \cdot) : [0,T] \longrightarrow \chi^*$ has bounded variation.

□

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