

# The potential to improve the choice: list conflict-free coloring for geometric hypergraphs

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## Abstract

Given a geometric hypergraph (or a range-space)  $H = (V, \mathcal{E})$ , a coloring of its vertices is said to be conflict-free if for every hyperedge  $S \in \mathcal{E}$  there is at least one vertex in  $S$  whose color is distinct from the colors of all other vertices in  $S$ . The study of this notion is motivated by frequency assignment problems in wireless networks. We study the list-coloring (or choice) version of this notion. In this version, each vertex is associated with a set of (admissible) colors and it is allowed to be colored only with colors from its set. List coloring arises naturally in the context of wireless networks.

Our main result is a list coloring algorithm based on a new potential method. The algorithm produces a stronger unique-maximum coloring, in which colors are positive integers and the maximum color in every hyperedge occurs uniquely. As a corollary, we provide asymptotically sharp bounds on the size of the lists required to assure the existence of such unique-maximum colorings for many geometric hypergraphs (e.g., discs or pseudo-discs in the plane or points with respect to discs). Moreover, we provide an algorithm, such that, given a family of lists with the appropriate sizes, computes such a coloring from these lists.

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# 1 Introduction and preliminaries

Before introducing our results, let us start with several definitions and notations that will be used throughout the paper.

**Definition 1.1.** Let  $H = (V, \mathcal{E})$  be a hypergraph and let  $C$  be a coloring  $C: V \rightarrow \mathbb{N}$ :

- We say that  $C$  is a *proper coloring* if for every hyperedge  $S \in \mathcal{E}$  with  $|S| \geq 2$  there exist two vertices  $u, v \in S$  such that  $C(u) \neq C(v)$ . That is, every hyperedge with at least two vertices is non-monochromatic.
- We say that  $C$  is a *conflict-free coloring* (cf-coloring in short) if for every hyperedge  $S \in \mathcal{E}$  there exists a color  $i \in \mathbb{N}$  such that  $|S \cap C^{-1}(i)| = 1$ . That is, every hyperedge  $S \in \mathcal{E}$  contains some vertex whose color is unique in  $S$ .
- We say that  $C$  is a *unique-maximum coloring* (um-coloring in short) if for every hyperedge  $S \in \mathcal{E}$ ,  $|S \cap C^{-1}(\max_{v \in S} C(v))| = 1$ . That is, in every hyperedge  $S \in \mathcal{E}$  the maximum color in  $S$  is unique in  $S$ .

We denote by  $\chi(H)$ ,  $\chi_{\text{cf}}(H)$ ,  $\chi_{\text{um}}(H)$  the minimum integer  $k$  for which  $H$  admits a proper, a conflict-free, a unique-maximum coloring, respectively, with a total of  $k$  colors. Obviously, every um-coloring of  $H$  is a cf-coloring of  $H$  which is also a proper coloring of  $H$ , but the converse is not necessarily true. Thus, we have:  $\chi(H) \leq \chi_{\text{cf}}(H) \leq \chi_{\text{um}}(H)$ .

**Conflict-free coloring.** The study of cf-coloring was initiated in [17] and [27] and was further studied in many settings (see, e.g., [1, 2, 5, 6, 10, 11, 12, 18, 22, 25, 26]). The study was initially motivated by its application to frequency assignment for cellular networks. A cellular network consists of two kinds of nodes: *base stations* and *mobile clients*. Base stations have fixed positions and provide the backbone of the network; they can be modeled, say, as discs in the plane that represent the area covered by each base station's antenna. Every base station emits at a fixed frequency. If a client wants to establish a link with a base station, it has to tune itself to the base station's frequency. Clients, however, can be in the range of many different base stations. To avoid interference, the system must assign frequencies to base stations in the following way: For any point  $p$  in the plane (representing a possible location of a client), there must be at least one base station which covers  $p$  and with a frequency that is not used by any other base station covering  $p$ . Since frequencies are limited and costly, a scheme that reuses frequencies, where possible, is desirable. Let us formulate this in the language of hypergraph coloring. Let  $D$  be the set of discs representing the antennas. We thus seek the minimum number of colors  $k$  such that one can assign each disc with one of the  $k$  colors so that in every point  $p$  in the union of the discs in  $D$ , there is at least one disc  $d \in D$  that covers  $p$  and whose color is distinct from all the colors of other discs containing  $p$ . This is equivalent to finding the cf-chromatic number of a certain hypergraph  $H = H(D)$  whose vertex set is  $D$  and whose hyperedges are defined by the Venn diagram of  $D$ . Below, we give a formal definition for  $H(D)$ .

**Geometric hypergraphs.** Let  $P$  be a set of  $n$  points in the plane and let  $\mathcal{R}$  be a family of regions in the plane (such as all discs, all axis-parallel rectangles, etc.). We denote by  $H = H_{\mathcal{R}}(P)$  the hypergraph on the set  $P$  whose hyperedges are all subsets  $P'$  that can be cut off from  $P$  by a region in  $\mathcal{R}$ . That is, all subsets  $P'$  such that there exists some region  $r \in \mathcal{R}$  with  $r \cap P = P'$ . We refer to such a hypergraph as the hypergraph *induced by  $P$  with respect to  $\mathcal{R}$* .

For a finite family  $\mathcal{R}$  of planar regions, we denote by  $H(\mathcal{R})$  the hypergraph whose vertex set is  $\mathcal{R}$  and whose hyperedge set is the family  $\{\mathcal{R}_p \mid p \in \mathbb{R}^2\}$  where  $\mathcal{R}_p \subseteq \mathcal{R}$  is the subset of all regions in  $\mathcal{R}$  that contain  $p$ . We refer to such a hypergraph as the hypergraph *induced by*  $\mathcal{R}$ .

Consider, for example, the (infinite) family  $D$  of all discs in the plane. In [17], it was proved that for any finite set  $P$  of  $n$  points, we have  $\chi_{\text{cf}}(H_D(P)) = O(\log n)$ . Similar questions can be asked for other families of geometric hypergraphs where one is interested in bounds on any of the chromatic numbers defined earlier.

**Unique-maximum coloring.** Most cf-coloring algorithms in the literature produce unique-maximum colorings (which are stronger than conflict-free colorings). The main reason for this approach is that unique-maximum colorings have more structure than conflict-free colorings, and thus it seems easier to argue about them in proofs. One interesting question is how can a non-unique-maximum conflict-free coloring improve on a unique-maximum coloring, with respect to the number of colors used and this line of research has been pursued in [8, 9].

**List coloring.** Until now, research on cf-coloring was carried out under the assumption that we can use any color from some global set of colors. The goal was to minimize the total number of colors used. In real life, it makes sense to assume that each antenna in the wireless network is further restricted to use a subset of the available spectrum. This restriction might be local (depending, say, on the physical location of the antenna). Hence, different antennas may have different subsets of (admissible) frequencies available for them. Thus, it makes sense to study the list version of conflict-free coloring. That is, assume further that each antenna  $d \in D$  is associated with a subset  $L_d$  of frequencies. We want to assign to each antenna  $d$  a frequency that is taken from its allowed set  $L_d$ . The following problem thus arises: *What is the minimum number  $f = f(n)$  such that given any set  $D$  of  $n$  antennas (represented as discs) and any family of subsets of positive integers  $\mathcal{L} = \{L_d\}_{d \in D}$  associated with the antennas in  $D$ , the following holds: If each subset  $L_d$  is of cardinality  $f$ , then one can cf-color the hypergraph  $H = H(D)$  from  $\mathcal{L}$ .* In what follows, we give a formal definition of the coloring model.

**Definition 1.2.** Let  $H = (V, \mathcal{E})$  be a hypergraph and let  $\mathcal{L} = \{L_v\}_{v \in V}$  be a family of  $|V|$  subsets of positive integers. We say that  $H$  admits a cf-coloring from  $\mathcal{L}$  if there exists a cf-coloring  $C: V \rightarrow \mathbb{N}$  such that  $C(v) \in L_v$  for every  $v \in V$ . Analogous definitions apply for the notions of a hypergraph  $H$  admitting a proper or a um-coloring from  $\mathcal{L}$ .

**Definition 1.3.** We say that a hypergraph  $H = (V, \mathcal{E})$  is  $k$ -cf-choosable if for every family  $\mathcal{L} = \{L_v\}_{v \in V}$  such that  $|L_v| \geq k \forall v \in V$ ,  $H$  admits a cf-coloring from  $\mathcal{L}$ . Analogous definitions apply for the notions of a hypergraph  $H$  being  $k$ -choosable or  $k$ -um-choosable.

In this paper we are interested in the minimum number  $k$  for which a given geometric hypergraph is  $k$ -cf-choosable (respectively,  $k$ -choosable,  $k$ -um-choosable). We refer to this number as the *cf-choice number* (respectively, *choice number* and *um-choice number*) of  $H$  and denote it by  $ch_{\text{cf}}(H)$  (respectively,  $ch(H)$  and  $ch_{\text{um}}(H)$ ). Obviously, if the cf-choice number of  $H$  is  $k$  then  $\chi_{\text{cf}}(H) \leq k$ , as one can cf-color  $H$  from  $\mathcal{L} = \{L_v\}_{v \in V}$  where for every  $v$  we have  $L_v = \{1, \dots, k\}$  (the same can be said for proper and um colorings). Thus,

$$ch(H) \geq \chi(H), \quad ch_{\text{cf}}(H) \geq \chi_{\text{cf}}(H), \quad ch_{\text{um}}(H) \geq \chi_{\text{um}}(H). \quad (1)$$

It is also easy to see that all of those parameters are upper-bounded by the number of vertices of the underlying hypergraph.

The study of list coloring for the special case of graphs, i.e., 2-uniform hypergraphs, was initiated in [16, 30]. List proper coloring of hypergraphs has been studied more recently, as well; see, e.g., [21]. We refer the reader to the survey of Alon [4] for more on list coloring of graphs.

**Our results.** In this paper we study the choice number, the cf-choice number, and the um-choice number of hypergraphs. We focus mainly on geometric hypergraphs.

Our main result is an asymptotically tight bound of  $O(\log n)$  on the um-choice number of  $H(\mathcal{R})$  when  $\mathcal{R}$  is a family of  $n$  planar Jordan regions with linear union-complexity. In order to obtain the above result, in Section 2, we introduce a potential method for list um-coloring hypergraphs that has also other applications. In Section 3, we apply the potential method in list um-coloring several geometric hypergraphs of interest. In Section 4, we obtain an asymptotically tight upper bound on the cf-choice number of hypergraphs consisting of the vertices of a planar graph together with all subsets of vertices that form a simple path in the graph (see [9] for applications of this class of hypergraphs); it is not possible to prove a similar upper bound on the um-choice number and indeed we show that the um-choice number of a hypergraph induced by paths of a planar graph can be substantially higher. In Section 5, using the list coloring approach, we prove tight upper bounds on the um-choice number in terms of (a) the number of hyperedges in the hypergraph or (b) the maximum degree of a vertex. These results extend results of [7, 25]. Moreover, the list coloring approach allows us to provide a more concise proof. In Section 6, we provide a general bound on the cf-choice number of any hypergraph in terms of its cf-chromatic number. We show that for any hypergraph  $H$  (not necessarily of a geometric nature) with  $n$  vertices we have:  $ch_{cf}(H) \leq \chi_{cf}(H) \cdot \ln n + 1$ . The proof of this fact uses a probabilistic argument, which is an extension of a probabilistic argument first given in [16]. There, it was proved that the choice-number of every bipartite graph with  $n$  vertices is  $O(\log n)$ . Our argument can be generalized to a large natural class of colorings (i.e., not just conflict-free), however, we note that such a bound is not possible for  $ch_{um}$ . Finally, in Section A of the appendix (due to space considerations), we study the (proper) choice number of several geometric hypergraphs and show that many of the known bounds for the proper coloring of the underlying hypergraphs hold in the context of list-coloring as well.

## 2 A potential method for list um-coloring

Let us start with a simple example of a hypergraph which can be viewed as induced by points on the line with respect to all intervals. Let  $[n] = \{1, \dots, n\}$ . For  $s \leq t$ ,  $s, t \in [n]$ , we define the (discrete) interval  $[s, t] = \{i \mid s \leq i \leq t\}$ . The *discrete interval hypergraph*  $H_n$  has vertex set  $[n]$  and hyperedge set  $\{[s, t] \mid s \leq t, s, t \in [n]\}$ . It is not difficult to prove that  $\chi_{cf}(H_n) = \lfloor \log_2 n \rfloor + 1$  (see, e.g., [17, 27]). Therefore, from inequality (1), we have the lower bound  $ch_{cf}(H_n) \geq \lfloor \log_2 n \rfloor + 1$ . As a warmup, we prove that the above lower bound is tight:

**Proposition 2.1.** *For every  $n \geq 1$ ,  $ch_{cf}(H_n) \leq \lfloor \log_2 n \rfloor + 1$ .*

*Proof.* Assume, without loss of generality, that  $n = 2^{k+1} - 1$ . We will show that  $H_n$  is  $k + 1$  cf-choosable. The proof is by induction on  $k$ . Let  $\mathcal{L} = \{L_i\}_{i \in [n]}$ , such that  $|L_i| = k + 1$ , for every  $i$ . Consider the median vertex  $p = 2^k$ . Choose a color  $x \in L_p$  and assign it to  $p$ . Remove  $x$  from all other lists (for lists containing  $x$ ), i.e., consider  $\mathcal{L}' = \{L'_i\}_{i \in [n] \setminus p}$  where  $L'_i = L_i \setminus \{x\}$ . Note that all lists in  $\mathcal{L}'$  have size at least  $k$ . The induction hypothesis is that we can cf-color any set of points of size  $2^k - 1$  from lists of size  $k$ . Indeed, the number of vertices smaller (respectively, larger) than  $p$  is exactly  $2^k - 1$ . Thus, we cf-color vertices smaller than  $p$  and independently vertices larger than

$p$ , both using colors from the lists of  $\mathcal{L}'$ . Intervals that contain the median vertex  $p$  also have the conflict-free property, because color  $x$  is used only in  $p$ . This completes the induction step and hence the proof of the proposition.  $\square$

We now turn to the more difficult problem of bounding the um-choice number. Even for the discrete interval hypergraph  $H_n$ , a divide and conquer approach, along the lines of the proof of Proposition 2.1 is doomed to fail. In such an approach, some vertex close to the median must be found, a color must be assigned to it from its list, and this color must be deleted from all other lists. However, vertices close to the median might have only ‘low’ colors in their lists. Thus, while we are guaranteed that a vertex close to the median is uniquely colored for intervals containing it, such a unique color is not necessarily the maximal color for such intervals.

Instead, we use a new approach. Our approach provides a general framework for um-coloring hypergraphs from lists. Moreover, when applied to many geometric hypergraphs, it provides asymptotically tight bounds for the um-choice number. First, we need the definitions of an independent set and of an induced sub-hypergraph.

**Definition 2.2.** Given a hypergraph  $H = (V, \mathcal{E})$ , a subset  $U \subseteq V$  is called an *independent set* in  $H$  if it does not contain any hyperedge of cardinality at least 2, i.e., for every  $S \in \mathcal{E}$  with  $|S| \geq 2$ , we have  $S \not\subseteq U$ . Note that each color class of a proper coloring of  $H$  is an independent set in  $H$ .

**Definition 2.3.** Let  $H = (V, \mathcal{E})$  be a hypergraph. For a subset  $V' \subseteq V$ , we refer to  $H[V'] := (V', \{S \cap V' \mid S \in \mathcal{E}\})$  as the *sub-hypergraph* of  $H$  induced by  $V'$ .

Below, we give an informal description of the approach, which is then summarized in Algorithm 1.

We start by sorting the colors in the union of all lists in increasing order. Let  $c$  denote the minimum color. Let  $V_c \subseteq V$  denote the subset of vertices containing  $c$  in their lists. Note that  $V_c$  might contain very few vertices, in fact, it might be that  $|V_c| = 1$ . We simultaneously color a suitable subset  $U \subseteq V_c$  of vertices in  $V_c$  with  $c$ . We make sure that  $U$  is independent in the hypergraph  $H[V_c]$ . The exact way in which we choose  $U$  is crucial to the performance of the algorithm and is discussed below. Next, for the uncolored vertices in  $V_c \setminus U$ , we remove the color  $c$  from their lists. This is repeated for every color in the union  $\bigcup_{v \in V} L_v$  in increasing order of the colors. The algorithm stops when all vertices are colored. Notice that such an algorithm might run into a problem, when all colors in the list of some vertex are removed before this vertex is colored. Later, we show that if we choose the subset  $U \subseteq V_c$  in a clever way and the lists are sufficiently large, then we avoid such a problem.

As mentioned, Algorithm 1 might cause some lists to run out of colors before coloring all vertices. However, if this does not happen, we prove that it produces a um-coloring.

**Lemma 2.4.** *Provided that the lists associated with the vertices do not run out of colors during the execution of Algorithm 1, then the algorithm produces a um-coloring from  $\mathcal{L}$ .*

*Proof.* Consider any hyperedge  $S \in \mathcal{E}$ . Consider the last iteration  $t$  of the while loop for which some vertex of  $S$  was colored. Let  $c$  denote the color chosen in that iteration. Note that  $c$  is a maximal color in  $S$ . We need to prove that it is also unique in  $S$ . Let  $V_c$  denote the subset of uncolored vertices (until iteration  $t$ ) containing  $c$  in their lists and let  $U \subseteq V_c$  denote the independent set in  $H[V_c]$  chosen to be colored with  $c$ . Note that  $S \cap V_c$  is a hyperedge in  $H[V_c]$ . We need to show that  $|S \cap V_c| = 1$ . Indeed, assume to the contrary that  $|S \cap V_c| \geq 2$ . Then, since  $S \cap V_c$  is a hyperedge in  $H[V_c]$  and  $U$  is independent in  $H[V_c]$ , we must have  $S \cap V_c \not\subseteq U$ . Therefore, there must be a vertex

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**Algorithm 1**  $\text{UMColorGeneric}(H, \mathcal{L})$ : Unique-max color hypergraph  $H = (V, \mathcal{E})$  from family  $\mathcal{L}$

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while  $V \neq \emptyset$  do
   $c \leftarrow \min \bigcup_{v \in V} L_v$  (*  $c$  is the minimum color in the union of the lists *)
   $V_c \leftarrow \{v \in V \mid c \in L_v\}$  (*  $V_c$  is the subset of remaining vertices containing  $c$  in their lists *)
   $U \leftarrow$  a “good” independent subset of the induced hypergraph  $H[V_c]$ 
  for  $x \in U$  do
     $f(x) \leftarrow c$  (* color it with color  $c$  *)
  end for
  for  $v \in V_c \setminus U$  do (* for every uncolored vertex, remove  $c$  from its list *)
     $L_v \leftarrow L_v \setminus \{c\}$ 
  end for
   $V \leftarrow V \setminus U$  (* remove the colored vertices *)
end while
return  $f$ 

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$v \in (S \cap V_c) \setminus U$ . This means that such a vertex  $v \in S \cap V_c$  is not colored in iteration  $t$ . Hence, it is colored in a later iteration, a contradiction.  $\square$

The key ingredient, which will determine the necessary size of the lists of  $\mathcal{L}$ , is the particular choice of the independent set in the above algorithm. We assume that the hypergraph  $H = (V, \mathcal{E})$  is hereditarily  $k$ -colorable for some fixed positive integer  $k$ . That is, for every subset  $V' \subseteq V$ , the sub-hypergraph  $H[V']$  induced by  $V'$  admits a proper  $k$ -coloring. This is the case in many geometric hypergraphs. For example, hypergraphs induced by planar discs, or pseudo-discs, or, more generally, hypergraphs induced by regions having linear union complexity have such a hereditary colorability property for some small constant  $k$  (see Section 3 for details).

A natural choice of a good independent set would of course be the largest one. Unfortunately, such a naive approach does not work. Instead, we consider a potential function on subsets of uncolored vertices at the end of every iteration and in the next iteration, we choose the independent set with the highest potential. For an uncolored vertex  $v \in V$ , let  $d_t(v)$  denote the number of colors removed from the initial list  $L_v$  after the first  $t$  iterations of the algorithm. By convention, for every vertex  $v$ ,  $d_0(v) = 0$ . Obviously, the value of  $d_t(v)$  depends on the particular run of the algorithm. Let  $\lambda > 1$  be a fixed constant to be determined later. For a subset of uncolored vertices  $X \subseteq V$  at the end of iteration  $t$ , let  $P_t(X) := \sum_{v \in X} \lambda^{d_t(v)}$ . We define the potential at the end of iteration  $t$  to be  $P_t := P_t(V^t)$ , where  $V^t$  denotes the subset of all uncolored vertices at the end of iteration  $t$ . Notice that the value of the potential in the beginning of the algorithm is  $P_0 = \sum_{v \in V} \lambda^0 = n$ , where  $n$  is the number of vertices of the hypergraph. Our goal is to show that, with the right choice of  $\lambda$  and the independent set in each iteration, we can make sure that for every vertex  $v \in V$  the inequality  $d_t(v) \leq \log_\lambda n$  holds for every iteration  $t$ . Thus, if  $|L_v| > \log_\lambda n$ , the algorithm will never run out of colors.

In order to achieve this, we will show that, with the right choice of the subset of vertices colored in each iteration, the potential function  $P_t$  is non-increasing in  $t$ . This will imply that for every uncolored vertex  $u \in V^t$  and any iteration  $t$  we have:

$$\lambda^{d_t(u)} \leq P_t \leq P_0 = n,$$

as required.

Assume that the potential function is non-increasing up to iteration  $t$ . Let  $P_t$  be the value of the potential function at the end of iteration  $t$  and let  $c$  be the color associated with iteration

$t + 1$ . Recall that  $V^t$  denotes the set of uncolored vertices that are input to iteration  $t + 1$ , and  $V_c \subseteq V^t$  denotes the subset of uncolored vertices that contain the color  $c$  in their lists. Put  $P' = P_t(V^t \setminus V_c)$  and  $P'' = P_t(V_c)$ . Note that  $P_t = P' + P''$ . Let us describe the way in which we find the independent set of vertices to be colored at iteration  $t + 1$ . First, we find an auxiliary proper coloring of the hypergraph  $H[V_c]$  with  $k$  colors. Consider the color class  $U$  which has the largest potential  $P_t(U)$ . Since the vertices in  $V_c$  are partitioned into at most  $k$  independent subsets  $U_1, \dots, U_k$  and  $P'' = \sum_{i=1}^k P_t(U_i)$ , then by the pigeon-hole principle there is an index  $j$  for which  $P_t(U_j) \geq P''/k$ . We choose  $U = U_j$  as the independent set to be colored at iteration  $t + 1$ . Notice that, in this case, the value  $d_{t+1}(v) = d_t(v) + 1$  for vertices  $v \in V_c \setminus U$ , and all vertices in  $U$  are colored. For vertices in  $V^t \setminus V_c$ , there is no change in the size of their lists. Thus, the value  $P_{t+1}$  of the potential function at the end of iteration  $t + 1$  is  $P_{t+1} \leq P' + \lambda(1 - \frac{1}{k})P''$ . Choosing  $\lambda = \frac{k}{k-1}$  we have that  $P_{t+1} \leq P' + P'' = P_t$ , as required. We summarize in Algorithm 2.

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**Algorithm 2**  $\text{UMColor}(H, \mathcal{L})$ : Unique-max color the hypergraph  $H = (V, \mathcal{E})$  from  $\mathcal{L}$

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**Require:**  $H$ : a hereditarily  $k$ -colorable hypergraph

$$\lambda := \frac{k}{k-1}$$

**for**  $v \in V$  **do**

$d(v) \leftarrow 0$  (\* the number of colors removed from  $L_v$  is set to 0 \*)

**end for**

**while**  $V \neq \emptyset$  **do**

$c \leftarrow \min \bigcup_{v \in V} L_v$  (\*  $c$  is the minimum color in the union of the lists \*)

$V_c \leftarrow \{v \in V \mid c \in L_v\}$

compute a proper coloring of  $H[V_c]$  with at most  $k$  colors and color classes  $U_1, \dots, U_k$

$U \leftarrow$  a color class among  $U_1, \dots, U_k$  with  $\max_{i \in \{1, \dots, k\}} \sum_{v \in U_i} \lambda^{d(x)}$

**for**  $x \in U$  **do**

$f(x) \leftarrow c$

**end for**

**for**  $v \in V_c \setminus U$  **do**

$L_v \leftarrow L_v \setminus \{c\}$  (\* remove the color  $c$  from all lists of uncolored vertices in  $V_c$  \*)

$d(v) \leftarrow d(v) + 1$  (\* update the number of removed colors from the list  $L_v$  \*)

**end for**

$V \leftarrow V \setminus U$  (\* remove the colored vertices \*)

**end while**

**return**  $f$

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We thus have:

**Theorem 2.5.** *Let  $H = (V, \mathcal{E})$  be a hypergraph, which is hereditarily  $k$ -colorable. Let  $\mathcal{L} = \{L_v\}_{v \in V}$ . If for every vertex  $v \in V$ ,  $|L_v| \geq \log_{\frac{k}{k-1}} n + 1 \approx (k-1) \ln n + 1$ , then Algorithm 2 um-colors  $H$  from  $\mathcal{L}$ .*

Theorem 2.5 has an algorithmic version:

**Corollary 2.6.** *Let  $H = (V, \mathcal{E})$  be a hypergraph on  $n$  vertices, which is hereditarily  $k$ -colorable and let  $\mathcal{L} = \{L_v\}_{v \in V}$  so that  $|L_v| > (k-1) \ln n$  for every  $v \in V$ . Assume that we have an efficient algorithm for  $k$ -coloring every induced subhypergraph of  $H$ . Then we also have an efficient algorithm for um-coloring  $H$  from  $\mathcal{L}$ .*

### 3 Geometric hypergraphs

As a corollary of Theorem 2.5 we obtain asymptotically optimal bounds on the um-choice number (hence, also on the cf-choice number) of many geometric hypergraphs.

**Corollary 3.1.** *Let  $C$  be some absolute constant. Let  $\mathcal{R}$  be a (possibly infinite) family of simple planar Jordan regions such that, for any  $n$  and any subset  $\mathcal{R}' \subseteq \mathcal{R}$  of  $n$  regions, the union complexity of  $\mathcal{R}'$  is bounded by  $Cn$ . Let  $H = H(\mathcal{R}')$  be a hypergraph induced by a subset  $\mathcal{R}' \subseteq \mathcal{R}$  of  $n$  such regions. Then*

$$ch_{\text{um}}(H) = O(\log n).$$

This follows from the fact that such a hypergraph has chromatic number  $O(1)$  [28] combined with Theorem 2.5.

**Corollary 3.2.** *Let  $\mathcal{D}$  denote the (infinite) family of all planar discs.*

(i) *Let  $P$  be a set of  $n$  points in the plane and let  $H = H_{\mathcal{D}}(P)$  be the hypergraph induced by  $P$  (with respect to  $\mathcal{D}$ ). Then*

$$ch_{\text{um}}(H) \leq \log_{4/3} n + 1 \approx 2.41 \log_2 n + 1.$$

(ii) *Let  $\mathcal{D}' \subseteq \mathcal{D}$  be a set of  $n$  discs. Then*

$$ch_{\text{um}}(H(\mathcal{D}')) \leq \log_{4/3} n + 1.$$

This follows from combining the fact that such hypergraphs are hereditary 4-colorable [17, 28] together with Theorem 2.5.

### 4 List cf-coloring of planar graphs with respect to paths

Given a simple graph  $G = (V, E)$ , consider the hypergraph

$$H_G^{\text{path}} = (V, \{S \mid S \text{ is the vertex set of a simple path in } G\}).$$

A cf (respectively, um) coloring of  $H_G^{\text{path}}$  is called a cf (respectively, um) coloring of  $G$  with respect to paths. Unique-maximum coloring of a graph  $G$  with respect to paths is known in the literature as *vertex ranking* or *ordered coloring*. See e.g., [14, 19].

**Theorem 4.1.** *Let  $G$  be a planar graph with  $n$  vertices. Then  $ch_{\text{cf}}(H_G^{\text{path}}) = O(\sqrt{n})$ .*

*Proof.* The proof is constructive. Given a planar graph  $G$  on  $n$  vertices together with a family  $\mathcal{L} = \{L_v\}_{v \in V}$  of sets of size  $c\sqrt{n}$  where  $c$  is some absolute constant to be revealed later, we produce a cf-coloring  $C$  of  $G$  with respect to paths with colors from  $\mathcal{L}$ .

The algorithm is recursive. By the Lipton-Tarjan separator theorem [23] and in particular by the version of the separator theorem from [15], there exists a partition of the vertex set  $V = R \cup B \cup S$  such that  $\max(|R|, |B|) \leq 2n/3$  and  $|S| \leq \sqrt{6n}$  and such that there is no edge connecting a vertex in  $R$  with a vertex in  $B$ . Moreover, this partition can be computed efficiently.

We color all vertices in  $S$  with distinct colors. This can be done greedily as follows: Arbitrarily order the vertices in  $S$  and for each vertex  $v$  in this order choose a color from  $L_v$  to assign to  $v$  which is distinct from all colors assigned to previous vertices in  $S$ . This is possible if  $|L_v| = c\sqrt{n} \geq \sqrt{6n} \geq |S|$ . Next, for each vertex  $u \in R \cup B$  modify the lists  $\{L_u\}_{u \in R \cup B}$  by erasing all colors used



for  $S$ , namely put  $\mathcal{L}' = \{L_u \setminus \{C(v) \mid v \in S\}\}_{u \in R \cup B}$ . We recursively color  $G[B]$  and  $G[R]$  from  $\mathcal{L}'$ . Note that the colors assigned to vertices in  $R \cup B$  are distinct from all colors used for  $S$ . Note also that if this coloring is indeed a valid cf-coloring of  $G$  from  $\mathcal{L}$  then the function  $f(n)$  defined to be the maximum cf-choice number for a planar graph on  $n$  vertices satisfies the following recursive inequality:

$$f(n) \leq \sqrt{6}\sqrt{n} + f(2n/3) \leq \sum_{i=0}^{\infty} \sqrt{6 \cdot \left(\frac{2}{3}\right)^i} n = \frac{\sqrt{6}\sqrt{n}}{1 - \sqrt{2/3}} \approx 13.3485\sqrt{n}$$

Thus, we have  $f(n) \leq c\sqrt{n}$ , for  $c \approx 13.3485$ , as claimed.  $\square$

*Remark 4.2.* The upper bound  $O(\sqrt{n})$  is asymptotically tight, since for the  $\sqrt{n} \times \sqrt{n}$  grid graph  $G_{\sqrt{n}}$ , it was proved in [9] that  $\chi_{\text{cf}}(H_{G_{\sqrt{n}}}^{\text{path}}) = \Omega(\sqrt{n})$  and thus, from inequality (1), also  $ch_{\text{cf}}(H_{G_{\sqrt{n}}}^{\text{path}}) = \Omega(\sqrt{n})$ .

It is easily seen that an analog of Theorem 4.1 for  $ch_{\text{um}}$  does not hold. For example, consider the *star graph* on  $n > 2$  vertices  $K_{1,n-1}$ . It is easy to check that  $\chi_{\text{cf}}(H_{K_{1,n-1}}^{\text{path}}) = ch_{\text{cf}}(H_{K_{1,n-1}}^{\text{path}}) = \chi_{\text{um}}(H_{K_{1,n-1}}^{\text{path}}) = 2$ . However, consider a family  $\mathcal{L}$  of lists as follows: Associate the  $n-1$  leaves of the star with the same list of colors and associate the other vertex with a list of colors which are all lower than the colors appearing in the lists of the leaves. In a unique-maximum coloring with respect to paths from  $\mathcal{L}$ , no two leaves can get the same color, because then we do not have the unique maximum property for the path that connects these two leaves and hence,  $ch_{\text{um}}(H_{K_{1,n-1}}^{\text{path}}) \geq n-1$ .

## 5 List um-coloring hypergraphs with few edges

In this section we extend upper bounds on  $\chi_{\text{cf}}$  from [7, 25], making them hold also for  $ch_{\text{um}}$ . In other words, we extend the results in two ways making them hold for choice instead of chromatic number and for unique-maximum colorings instead of conflict-free colorings. Moreover, we provide a more concise proof. In order to state the results, we need the following definition.

**Definition 5.1.** For every hypergraph  $H$ , define  $s(H)$  to be the minimum positive integer  $s$  such that  $|E(H)| \leq s(s-1)/2$ .

**Fact 5.2.** For two hypergraphs  $H$  and  $H'$ , if  $|\mathcal{E}(H)| \geq |\mathcal{E}(H')|$  and  $s(H) = s(H') > 1$ , then

$$|\mathcal{E}(H)| - |\mathcal{E}(H')| < s(H) - 1.$$

*Proof.* We have  $(s(H) - 1)(s(H) - 2)/2 < |\mathcal{E}(H')| \leq |\mathcal{E}(H)| \leq s(H)(s(H) - 1)/2$ , which implies  $|\mathcal{E}(H)| - |\mathcal{E}(H')| < s(H) - 1$ .  $\square$

**Theorem 5.3.** Let  $H = (V, \mathcal{E})$  be a hypergraph and let  $\mathcal{L} = \{L_v\}_{v \in V}$  be a family of lists. If for every  $v \in V$   $|L_v| \geq \min(\deg_H(v) + 1, s(H))$ , then  $H$  admits a unique-maximum coloring from  $\mathcal{L}$ .

*Proof.* Notice that if  $s(H) = 1$ , the hypergraph has no hyperedge and thus if  $|L_v| \geq 1$  for every  $v \in V$ , then  $H$  admits a unique-maximum coloring from  $\mathcal{L}$ . The proof is by induction on  $|V|$ . If  $H$  has one vertex  $v$ , then  $\deg_H(v) = 0$  and  $s(H) = 1$ . Hence, if  $|L_v| \geq 1$ , then  $H$  admits a unique-maximum coloring from  $\mathcal{L}$ .

If  $|V| > 1$  and  $s(H) > 1$ , consider the maximum color occurring in the union of all lists, that is,  $c = \max \bigcup_{v \in V} L_v$ . Among these vertices which have  $c$  in their list, choose the vertex  $v$  with

maximum degree in the hypergraph. Consider the subset of hyperedges  $\mathcal{E}_v \subseteq \mathcal{E}$  that contain  $v$ . Put  $H' = (V', \mathcal{E}')$ , where  $V' = V \setminus \{v\}$ ,  $\mathcal{E}' = \mathcal{E} \setminus \mathcal{E}_v$ , and define  $\mathcal{L}' = \{L'_u\}_{u \in V'}$  such that

$$L'_u = \begin{cases} L_u \setminus \{c\} & \text{if } u \in \bigcup_{S \in \mathcal{E}_v} S, \\ L_u & \text{if } u \notin \bigcup_{S \in \mathcal{E}_v} S. \end{cases}$$

In order to apply the inductive hypothesis on  $H'$ , we prove that for every  $u \in V'$ ,

$$|L'_u| \geq \min(\deg_{H'}(u) + 1, s(H')), \quad (2)$$

If  $|L_u| = |L'_u|$ , that is, when  $u \notin \bigcup_{S \in \mathcal{E}_v} S$  or  $c \notin L_u$ , then condition (2) holds. Also, if  $c \in L_u$ ,  $u \in \bigcup_{S \in \mathcal{E}_v} S$ , and  $s(H') < s(H)$ , then condition (2) holds, since  $\deg_{H'}(u) < \deg_H(u)$ . If  $c \in L_u$ ,  $u \in \bigcup_{S \in \mathcal{E}_v} S$ , and  $s(H') = s(H)$ , then Fact 5.2 implies

$$s(H) > |\mathcal{E}| - |\mathcal{E}'| + 1 = \deg_H(v) + 1 \geq \deg_H(u) + 1.$$

This follows from the fact that  $\deg_H(v) = |\mathcal{E}| - |\mathcal{E}'|$  and  $\deg_H(v) \geq \deg_H(u)$ . Since  $s(H') = s(H)$  and  $\deg_H(u) \geq \deg_{H'}(u) + 1$ , we also have

$$s(H') > \deg_{H'}(u) + 1.$$

As a result,

$$\begin{aligned} |L'_u| &= |L_u| - 1 \geq \min(\deg_H(u) + 1, s(H)) - 1 = \deg_H(u) + 1 - 1 \\ &= \deg_{H'}(u) + 1 = \min(\deg_{H'}(u) + 1, s(H')). \end{aligned}$$

Finally, by the inductive hypothesis,  $H'$  admits a um-coloring from  $\mathcal{L}'$ . Extend this coloring by coloring  $v$  with  $c$  to get a um-coloring of  $H$ .  $\square$

**Corollary 5.4.** *For every hypergraph  $H$ ,  $ch_{\text{um}}(H) \leq \Delta(H) + 1$ .*

**Corollary 5.5.** *For every hypergraph  $H$ ,  $ch_{\text{um}}(H) \leq s(H)$ .*

## 6 A connection between choosability and colorability in general hypergraphs

**Definition 6.1.** We call  $C'$  a *refinement* of a coloring  $C$  if  $C(x) \neq C(y)$  implies  $C'(x) \neq C'(y)$ . A class  $\mathcal{C}$  of colorings is said to have *the refinement property* if every refinement of a coloring in the class is also in the class.

The class of conflict-free colorings and the class of proper colorings are examples of classes which have the refinement property. On the other hand, the class of unique-maximum colorings does not have this property.

For a class  $\mathcal{C}$  of colorings we can define as usual the notions of chromatic number  $\chi_{\mathcal{C}}$  and choice number  $ch_{\mathcal{C}}$ . Then, we can prove the following theorem for classes with the refinement property.

**Theorem 6.2.** *For every class of colorings  $\mathcal{C}$  that has the refinement property and every hypergraph  $H$  with  $n$  vertices,  $ch_{\mathcal{C}}(H) \leq \chi_{\mathcal{C}}(H) \cdot \ln n + 1$ .*

*Proof.* If  $k = \chi_{\mathcal{C}}(H)$ , there is a  $\mathcal{C}$ -coloring  $C$  of  $H$  with colors  $\{1, \dots, k\}$ , which induces a partition of  $V$  into  $k$  classes:  $V_1 \cup V_2 \cup \dots \cup V_k$ . Consider a family  $\mathcal{L} = \{L_v\}_{v \in V}$ , such that for every  $v$ ,  $|L_v| = k^* > k \cdot \ln n$ . We wish to find a family  $\mathcal{L}' = \{L'_v\}_{v \in V}$  with the following properties:

1. For every  $v \in V$ ,  $L'_v \subseteq L_v$ .
2. For every  $v \in V$ ,  $L'_v \neq \emptyset$ .
3. For every  $i \neq j$ , if  $v \in V_i$  and  $u \in V_j$ , then  $L'_v \cap L'_u = \emptyset$ .

Obviously, if such a family  $\mathcal{L}'$  exists, then there exists a  $\mathcal{C}$ -coloring from  $\mathcal{L}'$ : For each  $v \in V$ , pick a color  $x \in L'_v$  and assign it to  $v$ .

We create the family  $\mathcal{L}'$  randomly as follows: For each element in  $\cup \mathcal{L}$ , assign it uniformly at random to one of the  $k$  classes of the partition  $V_1 \cup \dots \cup V_k$ . For every vertex  $v \in V$ , say with  $v \in V_i$ , we create  $L'_v$ , by keeping only elements of  $L_v$  that were assigned through the above random process to  $v$ 's class,  $V_i$ .

The family  $\mathcal{L}'$  obviously has properties 1 and 3. We will prove that with positive probability it also has property 2.

For a fixed  $v$ , the probability that  $L'_v = \emptyset$  is at most

$$\left(1 - \frac{1}{k}\right)^{k^*} \leq e^{-k^*/k} < e^{-\ln n} = \frac{1}{n}$$

and therefore, using the union bound, the probability that for at least one vertex  $v$ ,  $L'_v = \emptyset$ , is at most

$$n \left(1 - \frac{1}{k}\right)^{k^*} < 1.$$

Thus, there is at least one family  $\mathcal{L}'$  where property 2 also holds, as claimed.  $\square$

**Corollary 6.3.** *For every hypergraph  $H$ ,  $ch_{\text{cf}}(H) \leq \chi_{\text{cf}}(H) \cdot \ln n + 1$ .*

**Corollary 6.4.** *For every hypergraph  $H$ ,  $ch(H) \leq \chi(H) \cdot \ln n + 1$ .*

The argument in the proof of Theorem 6.2 is a generalization of an argument first given in [16], proving that any bipartite graph with  $n$  vertices is  $O(\log n)$ -choosable (see also [3]).

We can not have an analog of Theorem 6.2 for unique maximum colorings. Again, as in the end of Section 4, the counterexample is the hypergraph with respect to paths of the star graph,  $H_{K_1, n-1}^{\text{path}}$ , for which  $\chi_{\text{um}}(H_{K_1, n-1}^{\text{path}}) = 2$ , whereas  $ch_{\text{um}}(H_{K_1, n-1}^{\text{path}}) \geq n - 1$ .

## 7 Open problems

We consider the following as an interesting problem left open here:

- Let  $H$  be a hypergraph induced by  $n$  axis-parallel rectangles in the plane. Is it true that  $ch(H) = O(\log n)$ ? It is known that  $\chi(H) = \Theta(\log n)$  [24, 28].

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## A Choice number of geometric hypergraphs

In this section, we provide near-optimal upper bounds on the choice number of several geometric hypergraphs. We need the following definitions:

**Definition A.1.** Let  $\mathcal{R}$  be a family of  $n$  simple Jordan regions in the plane. The *union complexity* of  $\mathcal{R}$  is the number of vertices (i.e., intersection of boundaries of pairs of regions in  $\mathcal{R}$ ) that lie on the boundary  $\partial \bigcup_{r \in \mathcal{R}} r$ .

**Definition A.2.** Let  $H = (V, \mathcal{E})$  be a hypergraph. Let  $G = (V, E)$  be the graph whose edges are all hyperedges of  $\mathcal{E}$  with cardinality two. We refer to  $G$  as the *Delaunay graph* of  $H$ .

**Theorem A.3.** (i) Let  $H$  be a hypergraph induced by a finite set of points in the plane with respect to discs. Then  $ch(H) \leq 5$ .

(ii) Let  $D$  be a finite family of discs in the plane. Then  $ch(H(D)) \leq 5$ .

(iii) Let  $\mathcal{R}$  be a set of  $n$  regions and let  $\mathcal{U} : \mathbb{N} \rightarrow \mathbb{N}$  be a function such that  $\mathcal{U}(m)$  is the maximum complexity of any  $k$  regions in  $\mathcal{R}$  over all  $k \leq m$ , for  $1 \leq m \leq n$ . We assume that  $\frac{\mathcal{U}(m)}{m}$  is a non-decreasing function. Then,  $ch(H(\mathcal{R})) = O\left(\frac{\mathcal{U}(n)}{n}\right)$ .

*Proof.* (i) Consider the Delaunay graph  $G = G(P)$  on  $P$ , where two points  $p$  and  $q$  form an edge in  $G$  if and only if there exists a disc  $d$  such that  $d \cap P = \{p, q\}$ . That is, there exists a disc  $d$  that cuts off  $p$  and  $q$  from  $P$ . The proof of (i) follows easily from the following known facts:

1. Every disc containing at least two points of  $P$  must also contain a Delaunay edge  $\{p, q\} \in E(G)$ . (see, e.g., [17]).
2.  $G$  is planar (see, e.g., [13]).
3. Every planar-graph is 5-choosable [29].

(ii) The proof of the second part follows from a reduction to three dimensions from [28] and Thomassen's result [29].

(iii) For the third part of the theorem, we need the following lemma from [28]:

**Lemma A.4.** [28] Let  $\mathcal{R}$  be a set of  $n$  regions and let  $\mathcal{U} : \mathbb{N} \rightarrow \mathbb{N}$  be a function such that  $\mathcal{U}(m)$  is the maximum complexity of any  $k$  regions in  $\mathcal{R}$  over all  $k \leq m$ , for  $1 \leq m \leq n$ . Then, the Delaunay graph  $G$  of the hypergraph  $H = H(\mathcal{R})$  has a vertex with degree at most  $c \frac{\mathcal{U}(n)}{n}$  where  $c$  is some absolute constant.

The proof is similar to the proof of [28] of the fact that  $\chi(H(\mathcal{R})) = O\left(\frac{\mathcal{U}(n)}{n}\right)$ . We prove that  $ch(H(\mathcal{R})) \leq c \cdot \frac{\mathcal{U}(n)}{n} + 1$ . Let  $\mathcal{L} = \{L_r\}_{r \in \mathcal{R}}$  be the sets associated with the regions of  $\mathcal{R}$ . The proof is by induction on  $n$ . Let  $r \in \mathcal{R}$  be a region with at most  $c \cdot \frac{\mathcal{U}(n)}{n}$  neighbors in  $G$ . By the induction hypothesis, the hypergraph  $H(\mathcal{R} \setminus \{r\})$  is  $c \cdot \frac{\mathcal{U}(n-1)}{n-1} + 1 \leq c \cdot \frac{\mathcal{U}(n)}{n} + 1$ -choosable (by our monotonicity assumption on  $\frac{\mathcal{U}(n)}{n}$ ). We need to choose a color (out of the  $c \cdot \frac{\mathcal{U}(n)}{n} + 1$  colors that are available for us in the set  $L_r$ ) for  $r$  such that the coloring of  $\mathcal{R}$  is valid. Obviously, points that are not covered by  $r$  are not affected by the coloring of  $r$ . Note also that any point  $p \in r$  that is contained in at least two regions of  $\mathcal{R} \setminus \{r\}$  is not affected by the color of  $r$  since, by induction, the set of regions in  $\mathcal{R} \setminus \{r\}$  containing such points is non-monochromatic. We thus only need to color  $r$  with a color that is different from the colors of all regions  $r' \in \mathcal{R} \setminus r$ , for which there is a point  $p$  that is contained only in  $r \cap r'$ . However, by our choice of  $r$ , there are at most  $c \cdot \frac{\mathcal{U}(n)}{n}$  such regions.

Thus, we can assign to  $r$  a color among the  $c \cdot \frac{\mathcal{U}(n)}{n} + 1$  colors available to us in  $L_r$  and keep the coloring of  $\mathcal{R}$  proper. This completes the inductive step.  $\square$

**Corollary A.5.** *Let  $\mathcal{P}$  be a family of  $n$  pseudo-discs (i.e., a family of simple closed Jordan regions, such that the boundaries of any two of them intersect at most twice). Then  $ch(H(\mathcal{P})) = O(1)$ .*

The corollary follows immediately from the fact that such a family  $\mathcal{P}$  has linear union complexity [20], combined with Theorem A.3.