

Space crossing numbers

Boris Bukh*

Alfredo Hubard†

Abstract

We define a variant of the crossing number for an embedding of a graph G into \mathbb{R}^3 , and prove a lower bound on it which almost implies the classical crossing lemma. We also give sharp bounds on the rectilinear space crossing numbers of pseudo-random graphs.

Introduction

All the graphs in this paper are simple, i.e. they contain no loops or multiple edges. The *crossing number* of a graph $G = (V, E)$ is the minimum number of crossings between edges of G among all the ways to draw G in the plane. It is denoted $\text{cr}(G)$. The edges in a drawing of G need not be line segments, they are allowed to be arbitrary continuous curves. If one restricts to the straight-line drawings, then one obtains the *rectilinear crossing number* $\text{lin-cr}(G)$. It is clear that $\text{cr}(G) \leq \text{lin-cr}(G)$, and there are examples where $\text{cr}(G) = 4$, but $\text{lin-cr}(G)$ is unbounded [BD93]. The principal result about crossing numbers is the crossing lemma of Ajtai–Chvátal–Newborn–Szemerédi and Leighton [ACNS82, Lei84] which states that

$$\text{cr}(G) \geq c \frac{|E|^3}{|V|^2} \quad \text{whenever } |E| \geq C|V|. \quad (1)$$

The inequality is sharp apart from the values of c and C (see [PT97] for the best known estimate on c). The most famous applications of the crossing lemma are short and elegant proofs by Székely [Szé97] of Szemerédi–Trotter theorem on point–line incidences and of Spencer–Szemerédi–Trotter theorem on the unit distances. Another remarkable application is the bound on the number of halving lines by Dey [Dey98]. In this paper we propose an extension of the crossing number to \mathbb{R}^3 , in such a way that the corresponding “space crossing lemma” (Theorem 2 below) implies (1) (up to a logarithmic factor).

A *spatial drawing* of a graph G is representation of vertices of G by points in \mathbb{R}^3 , and edges of G by continuous curves. A *space crossing* consists of a quadruple of vertex-disjoint edges (e_1, \dots, e_4) and a line l that meets these four edges. The *space crossing number* of G , denoted $\text{cr}_4(G)$ is the least number of crossings in any spatial drawing of G . As in the planar case, the

*B.Bukh@dpms.cam.ac.uk, University of Cambridge and Churchill College, United Kingdom.

†hubard@cims.nyu.edu. Courant Institute of Mathematical Sciences, New York University, United States.

spatial rectilinear crossing number $\text{lin-cr}_4(G)$ is obtained by restricting to straight-line spatial drawings.

For a graph G pick a drawing of G in the plane with the fewest crossings. By perturbing the drawing slightly, we may assume that there are no points where three vertex-disjoint edges meet. The drawing can be lifted to a drawing G on a large sphere without changing any of the crossings. Since no line meets the sphere in more than two points, every space crossing in the resulting spatial drawing comes from a pair of crossings in the planar drawing. Thus,

$$\text{cr}_4(G) \leq \binom{\text{cr}(G)}{2}. \quad (2)$$

Let us note that the space crossing number is not the usual crossing number in disguise, for the inequality in the reverse direction does not hold:

Proposition 1 (Proof is on p. 3). *For every natural number n there is a graph G with $\text{cr}_4(G) = 0$ and $\text{cr}(G) \geq n$.*

The principal result that justifies the introduction of the space crossing number is the following generalization of the crossing lemma.

Theorem 2 (Proof is on p. 3). *Let $G = (V, E)$ be an arbitrary graph, then*

$$\text{cr}_4(G) \geq \frac{|E|^6}{4^{179}|V|^4 \log_2^2 |V|},$$

whenever $|E| \geq 4^{41}|V|$.

Since (1) is sharp, in the light of the argument that led to (2) there are graphs on the sphere for which the bound in Theorem 2 is tight up to the logarithmic factor. In the drawings of these graphs, the edges are of course not straight. It turns out that there are also straight-line spatial drawings for which Theorem 2 is tight.

Theorem 3 (Proof is on p. 7). *For all positive integers m and n satisfying $m \leq \binom{n}{2}$ there is a graph G with n vertices and m edges, and rectilinear space crossing number at most $6720m^6/n^4$.*

The construction in the proof of Theorem 3 uses the idea of stair-convexity introduced in [BMN]. We shall briefly review the necessary background before the proof of Theorem 3.

Our final result is the lower bound on the space crossing number of (possibly sparse) pseudo-random graphs.

Theorem 4 (Proof is on p. 6). *There is an absolute constant $\varepsilon > 0$ such that the following holds. Let $G = (V, E)$ be a graph such that whenever V_1, V_2 are any two subsets of V of size $\varepsilon|V|$, the number of edges between V_1 and V_2 is at least N . Then $\text{lin-cr}_4(G) \geq N^4$.*

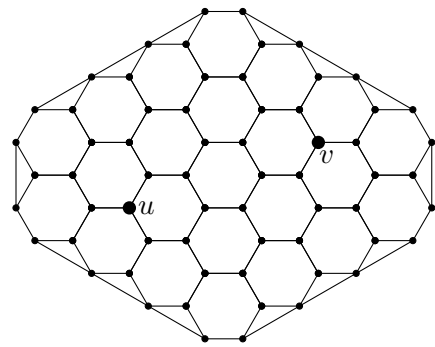
The condition of the theorem holds for several models of random graphs, as well as for (n, d, λ) -graphs (see for example [KS06, Theorem 2.11]).

Separation between crossing numbers and space crossing numbers

To construct graphs with $\text{cr}_4(G) = 0$ and unbounded $\text{cr}(G)$, we shall use the lower bound on crossing numbers due to Riskin. Recall that a 3-connected planar graph has a unique planar drawing [Die05, Theorem 4.3.1].

Lemma 5 (Theorem 4 in [Ris96]). *Suppose e is an edge in a graph G such that $H = G \setminus e$ is a 3-regular 3-connected planar graph. Then there is a drawing of G in the plane with $\text{cr}(G)$ crossings that is obtained from the unique planar drawing of H by adding the edge e .*

Proof of Proposition 1. Let H be the truncated n -by- n hexagonal grid drawn as in the picture on the right. The graph H is clearly 3-connected 3-regular planar graph. Pick two vertices $u, v \in H$ that are separated from one another by at least $n/4$ faces (the outer region is also a face). Then by the preceding lemma the graph $G = H \cup \{uv\}$ has crossing number at least $n/4$. On the other hand, there is a spatial drawing of G without any spatial crossings: Let H be drawn on the surface of the sphere without crossings, and represent the edge uv by a straight-line segment. Since every line meets the sphere in at most two vertex-disjoint edges, there are indeed no space crossings. \square



Lower bounds on the space crossing number

The naive strategy to prove Theorem 2 is to show that a graph without space crossing can have only $O(|V|)$ edges, derive from this a lower bound on the space crossing number of the form $c_1|E| - c_2|V|$, and then use random sampling to “boost” this to a stronger bound on cr_4 . Whereas, it is true that a space-crossing-free graph has only $O(|V|)$ edges (it follows from [Živ99, Corollary 3.5] that a graph with a $K_{6,6}$ -minor has a space crossing), this approach yields only $\text{cr}_4(G) \geq c|E|^7/|V|^6$. The reason is that to get $\text{cr}_4 \geq c|E|^6/|V|^4$ one needs to boost a stronger inequality $\text{cr}_4(G) \geq c_1|E|^2 - c_2|V|^2$. To obtain such an inequality we shall break the graph G into many small pieces, so that for each pair of pieces there is a space crossing that involves two edges from each piece. For that we need several known results, which we now state.

Recall that a subdivision of a graph G is a graph obtained from G by subdividing each edge of G into paths [Die05, p. 20].

Lemma 6 ([KP88]). *Let $\varepsilon > 0$ be arbitrary. Then every graph $G = (V, E)$ with $4^{t^2}|V|^{1+\varepsilon}$ edges contains a subdivision of K_t on at most $7t^2 \log t/\varepsilon$ vertices.*

Corollary 7. *Let $C \geq 3$. Suppose $G = (V, E)$ is a graph with at least $|E| \geq C4^{t^2}|V|$ edges. Then G contains at least $|E|/(16t^2 \log t \log_{C/2}|V|)$ edge-disjoint subdivisions of K_t .*

Proof. Define a nested sequence of graphs $G = G_0 \supset G_1 \supset G_2 \supset \dots \supset G_s$ on the vertex set V as follows. As long as $E(G_i) \geq (C/2)4^{t^2}|V|$ it follows by Lemma 6 with $\varepsilon = 1/\log_{C/2}|V|$ that G_i contains a subgraph H_i , which is a subdivision of K_t with $|V(H_i)| \leq 7t^2 \log t \log_{C/2}|V|$. Let G_{i+1} be the result of removing the edges of H_i from G_i . The sequence terminates once the number of edges in the graph falls below $(C/2)4^{t^2}|V|$. As

$$|E(G_i)| - |E(G_{i+1})| = |E(H_i)| \leq |E(K_t)| + |V(H_i)| \leq 8t^2 \log t \log_{C/2}|V|,$$

the number of terms in the sequence is at least $(|E|/2)/(8t^2 \log t \log_{C/2}|V|)$. Since the graphs $\{H_i\}$ are edge-disjoint subgraphs of G , the corollary follows. \square

The next is a version of [BS93, Theorem 3].

Lemma 8. *The vertex set of every graph $G = (V, E)$ can be partitioned into two classes $V = V_1 \cup V_2$ so that the number of edges in each of the induced subgraphs $G_i = G|_{V_i}$ is at least $|E|/4 - \sqrt{|V||E|}$.*

Proof. For each vertex v place v into V_1 or V_2 with equal probability independently of the other vertices. Let $X_i = |E(G_i)|$. Then $\mathbb{E}[X_i] = \frac{1}{4}|E|$ and

$$\begin{aligned} \mathbb{E}[X_i^2] &= \sum_{e_1, e_2 \in E} \Pr[e_1 \in E_i \wedge e_2 \in E_i] \\ &= \frac{1}{4}|E| + \frac{1}{8}|\{(e_1, e_2) \in E^2 : |e_1 \cap e_2| = 1\}| + \frac{1}{16}|\{(e_1, e_2) \in E^2 : e_1 \cap e_2 = \emptyset\}| \\ &\leq |E|^2/16 + \frac{1}{4} \sum_{v \in V} \deg(v)^2 \leq |E|^2/16 + \frac{1}{4}|V||E|. \end{aligned}$$

Hence $\mathbb{E}[(X_i - |E|/4)^2] \leq \frac{1}{4}|V||E|$, implying $\Pr[|X_i - |E|/4| > \sqrt{|V||E|}] < 1/4$. Therefore, the conclusion of the lemma holds for the random partition with probability at least $1/2$. \square

To find lines through four edges, we use two results from knot theory. We first recall the standard definitions. Two continuous injective maps $f_1, f_2: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ whose images are disjoint define a (two-component) *link* in \mathbb{R}^3 . The sets $C_1 = f_1(\mathbb{S}^1)$ and $C_2 = f_2(\mathbb{S}^1)$ are a pair of continuous closed curves (knots) in \mathbb{R}^3 . The linking number $\text{lk}(C_1, C_2)$ of the two curves is the degree¹ of the Gauss map

$$\begin{aligned} g: \mathbb{S}^1 \times \mathbb{S}^1 &\rightarrow \mathbb{S}^2 \\ g: (x, y) &\mapsto \frac{f_1(x) - f_2(y)}{\|f_1(x) - f_2(y)\|}. \end{aligned} \tag{3}$$

The linking number is an invariant of the knots, and if the functions f_1, f_2 are sufficiently nice, then it can also be defined by counting the number of signed crossings between C_1 and C_2 in a projection to a generic plane.

¹Implicit in the definition of the degree is the group of the coefficients for the homology. We use \mathbb{Z} coefficients throughout the paper.

Lemma 9 (Theorem 1 in [CG83] and independently in [Sac83]). *In every spatial drawing of K_6 there is a pair of vertex-disjoint triangles whose linking number is odd.*

Lemma 10. *If $C_1, C_2, C_3, C_4 \subset \mathbb{R}^3$ are four disjoint continuous closed curves, and $\text{lk}(C_1, C_2)$ and $\text{lk}(C_3, C_4)$ are non-zero, then there is at least one line that intersects all the four curves.*

This lemma is similar to Corollary 1 of Theorem 2 in [Vir09]. That corollary asserts that if the four curves are in addition smooth, and satisfy an appropriate general position requirement, then the number of lines through all four of them is at least $|\text{lk}(C_1, C_2) \text{lk}(C_3, C_4)|$. It is possible to derive Lemma 10 from the result in [Vir09], by a limiting argument. For completeness we include a short proof of Lemma 10, which uses a different idea.

Proof of Lemma 10. Let $T\mathbb{S}^2$ be the tangent bundle to \mathbb{S}^2 . An element $(p, v) \in T\mathbb{S}^2$ consists of a point $p \in \mathbb{S}^2$ and a tangent vector v to p . We shall think of $(p, v) \in T\mathbb{S}^2$ as a directed line in \mathbb{R}^3 in direction p which intersects the hyperplane $\{x : \langle x, p \rangle = 0\}$ in the point v .

For each $i = 1, \dots, 4$, let $f_i: \mathbb{S}^1 \rightarrow \mathbb{R}^3$ be a continuous injective map such that $f_i(\mathbb{S}^1) = C_i$. Consider the pair f_1, f_2 , and for $x, y \in \mathbb{S}^1$ let $h_{12}(x, y) \in T\mathbb{S}^2$ be the directed line that goes from $f_2(y)$ to $f_1(x)$. The result of composition of $h_{12}: \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow T\mathbb{S}^2$ with the projection map $\pi: T\mathbb{S}^2 \rightarrow \mathbb{S}^2$ is the Gauss map $g_{12} = \pi \circ h_{12}$ as defined in (3). By the assumption the degree of g_{12} is non-zero. Since \mathbb{S}^2 is a deformation retract of $T\mathbb{S}^2$, the projection map π induces isomorphism between the homology groups of $T\mathbb{S}^2$ and \mathbb{S}^2 , and hence the degree of h_{12} is non-zero.

Let $\mathcal{T} = \mathcal{T}(T\mathbb{S}^2)$ be the Thom space of $T\mathbb{S}^2$. It is a bundle over \mathbb{S}^2 obtained from $T\mathbb{S}^2$ replacing each fiber by its one-point compactification, and identifying all the new points into a single point (see page 367 of [Bre93] for the motivation and properties). Let $\sigma: T\mathbb{S}^2 \rightarrow \mathcal{T}$ be the inclusion map. Let $A_{12} \stackrel{\text{def}}{=} (\sigma \circ h_{12})(\mathbb{S}^1 \times \mathbb{S}^1)$ be the image of h_{12} in \mathcal{T} .

In the same way as we used f_1 and f_2 to define h_{12} and A_{12} , we define h_{34} and A_{34} using f_3 and f_4 . We shall exhibit two homology classes $\alpha_{12} \in H_2(A_{12}, \mathbb{Z})$ and $\alpha_{34} \in H_2(A_{34}, \mathbb{Z})$ whose intersection product in \mathcal{T} is non-zero. It will then follow by Theorem VI.11.10 from [Bre93] that $A_{12} \cap A_{34} \neq \emptyset$.

Since π induces an isomorphism between $H_2(T\mathbb{S}^2, \mathbb{Z})$ and $H_2(\mathbb{S}^2, \mathbb{Z})$, the definition of the linking number implies that the pushforward of the homology class $[\mathbb{S}^1 \times \mathbb{S}^1] \in H_2(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z})$ by h_{12} is the homology class $\text{lk}(C_1, C_2)[\mathbb{S}^2] \in H_2(T\mathbb{S}^2, \mathbb{Z})$. Let $D: H^k(M, \mathbb{Z}) \rightarrow H_{\dim M - k}(M, \mathbb{Z})$ be a Poincaré duality on orientable manifold M . The homology class $\sigma_*([\mathbb{S}^2])$ is the $D^{-1}(\tau)$, where $\tau \in H^2(\mathcal{T})$ is the Thom class of \mathcal{T} . The homology class $\alpha_{12} \stackrel{\text{def}}{=} (\sigma \circ h_{12})_*([\mathbb{S}^1 \times \mathbb{S}^1])$ is supported on A_{12} , and similarly defined class α_{34} is supported on A_{34} . The intersection product of α_{12} and α_{34} is then $\text{lk}(C_1, C_2) \text{lk}(C_3, C_4) D^{-1}(\tau^2)$. By the calculation on page 382 of [Bre93] $(\tau^2) \cap [T\mathbb{S}^2] = i_*(\chi \cap [\mathbb{S}^2])$, where χ is the Euler class of the bundle $T\mathbb{S}^2 \rightarrow \mathbb{S}^2$ and $i: \mathbb{S}^2 \rightarrow T\mathbb{S}^2$ is the zero section. Thus τ^2 is non-zero, and hence the intersection product of α_{12} and α_{34} is non-zero as well, as claimed. \square

The following lemma is analogous to the inequality $\text{cr}(G) \geq |E| - 3|V| + 6$ that is used in the proof of the usual crossing lemma.

Lemma 11. *Let $G = (V, E)$ be a graph with at least $|E| \geq 4^{39}|V|$ edges. Then $\text{cr}_4(G) \geq |E|^2/2^{28} \log_2^2|V|$.*

Proof. With foresight set $J = \lceil |E|/2^{14} \log_2|V| \rceil$. By Lemma 8 the graph splits into two vertex-disjoint graphs G_1, G_2 that have at least $|E|/4 - \sqrt{|V||E|} \geq |E|(1/4 - 1/4^{19}) \geq |E|/8$ edges each. By Corollary 7 each of G_i contains a family of $|E|/(8 \cdot 16 \cdot 6^2 \log 6 \log_2|V|) \geq J$ edge-disjoint subdivisions of K_6 . Thus, by Lemma 9 we obtain a family of J pairs of cycles $(C_{i,j}, C'_{i,j})_{j=1}^J$ in G_i , such that $C_{i,j}$ and $C'_{i,j}$ are vertex-disjoint, all the cycles are edge-disjoint, and $\text{lk}(C_{i,j}, C'_{i,j}) \geq 1$. By lemma 10 for every $1 \leq j_1, j_2 \leq J$ there is a line that intersects $C_{1,j_1}, C'_{1,j_1}, C_{2,j_2}, C'_{2,j_2}$. Furthermore, the four cycles are vertex disjoint. As all the cycles are edge-disjoint, the J^2 space crossings obtained in this manner are distinct. \square

Corollary 12. *If G is any graph, and $B \geq |V|$ then $\text{cr}_4(G) \geq \frac{|E|^2 - 4^{80}|V|^2}{2^{28} \log_2^2 B}$.*

Proof of Theorem 2. Given a graph $G = (V, E)$ with $|E| \geq 4^{41}|V|$ edges, let $p = 4^{41}|V|/|E|$. Let $V' \subset V$ be obtained by choosing each element of V independently with probability p . Let the $G' = (V', E')$ be the induced subgraph G on V' . By the preceding corollary with $B = |V|$ we have

$$\text{cr}_4(G') \geq \frac{|E'|^2 - 4^{80}|V'|^2}{2^{26} \log_2^2|V|}.$$

We shall estimate the expectation of both sides. On one hand, $\mathbb{E}[\text{cr}_4(G')] \leq p^8 \text{cr}_4(G)$ since a space crossing in a fixed drawing survives with probability p^8 . On the other hand, $\mathbb{E}[|E'|^2] \geq p^4|E|^2$ as every pair of edges survives with probability at least p^4 (the probability is higher if the two edges overlap). Furthermore, $\mathbb{E}[|V'|^2] = p^2|V|^2 + (p - p^2)|V| \leq 4p^2|V|^2$ by the choice of p . Hence,

$$p^8 \text{cr}_4(G) \geq \frac{p^4|E|^2 - 4^{81}p^2|V|^2}{2^{28} \log_2^2|V|},$$

and

$$\text{cr}_4(G) \geq \frac{4^{81}|V|^2}{2^{28}(4^{41}|V|/|E|)^6 \log_2^2|V|} = \frac{|E|^6}{4^{179}|V|^4 \log_2^2|V|} \quad \square$$

Remark: By using Lemma 11 instead of its corollary, and invoking large deviation inequalities, the above can be improved to $\text{cr}_4(G) \geq c \frac{|E|^6}{|V|^4 \log_2^2(|V|^2/|E|)}$. As the logarithmic factors are almost certainly superfluous, we chose the more transparent argument instead.

Rectilinear space crossing numbers of pseudo-random graphs

To prove Theorem 4 we shall need the same-type lemma for semi-algebraic relations. It is inspired by the same-type lemma of Bárány and Valtr [BV98, Theorem 2], and by the Szemerédi-type

result from [FGL⁺10]. In the final version of [FGL⁺10] (to appear in J. Reine Angew. Math), a more general result is proved independently. Our proof technique is borrowed from the previous results, with only minor pretense at novelty.

For a real number x its sign $\text{sgn } x$ is $-1, 0, +1$ according to whether x is negative, zero, or positive, respectively. A semi-algebraic relation on k -tuples of vectors x_1, \dots, x_k is an arbitrary logical formula (in the language of ordered fields) of the form

$$Q_1 t_1 \in \mathbb{R} Q_2 t_2 \in \mathbb{R} \cdots Q_l t_l \in \mathbb{R} (I_1 \wedge \cdots \wedge I_m)$$

where each of Q_1, \dots, Q_l is either \exists or \forall and each of I_1, \dots, I_m is of the form $\text{sgn } f(x_1, \dots, x_k, t_1, \dots, t_l) = s \in \{-1, 0, +1\}$, where f is a polynomial.

Lemma 13 (Proof is in on p. 11). *If R is a semi-algebraic relation in k variables, then there is a constant $\varepsilon = \varepsilon(R) > 0$ such that the following holds. For every collection of k finite sets $\mathcal{F}_1, \dots, \mathcal{F}_k$, there are subsets $\mathcal{F}'_i \subset \mathcal{F}_i$ such that:*

1. \mathcal{F}'_i are large: $|\mathcal{F}'_i| \geq \varepsilon |\mathcal{F}_i|$,
2. R is constant on $\mathcal{F}'_1 \times \cdots \times \mathcal{F}'_k$: either for all $(x_1, \dots, x_k) \in \mathcal{F}'_1 \times \cdots \times \mathcal{F}'_k$ the relation $R(x_1, \dots, x_k)$ holds, or for all $(x_1, \dots, x_k) \in \mathcal{F}'_1 \times \cdots \times \mathcal{F}'_k$ the relation $R(x_1, \dots, x_k)$ does not hold.

Proof of Theorem 4. Let the graph G with a rectilinear spatial drawing be given. Let R be the relation on 8-tuples $x_1, \dots, x_8 \in \mathbb{R}^3$ given by “the straight-line segments $x_1x_2, x_3x_4, x_5x_6, x_7x_8$ form a space crossing”. The relation is semi-algebraic. Indeed, it is given by

$$\begin{aligned} R(x_1, \dots, x_8) = & \exists t_1, t_2, t_3, t_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}, \exists y, v \in \mathbb{R}^3 \\ & (0 < t_1, t_2, t_3, t_4 < 1) \wedge \\ & (t_1x_1 + (1 - t_1)x_2 = y + \lambda_1v) \wedge (t_2x_3 + (1 - t_2)x_4 = y + \lambda_2v) \wedge \\ & (t_3x_5 + (1 - t_3)x_6 = y + \lambda_3v) \wedge (t_4x_7 + (1 - t_4)x_8 = y + \lambda_4v). \end{aligned}$$

By the preceding lemma applied $8! \binom{12}{8}$ times there are 12 subsets V_1, \dots, V_{12} of $V(G)$ such that $|V_i| \geq \varepsilon |V|$ for $i = 1, \dots, 12$ and R is constant on all the product sets of the form $V_{\sigma(1)} \times \cdots \times V_{\sigma(8)}$ for any injective map $\sigma: [8] \rightarrow [12]$. Pick any twelve points $x_1 \in V_1, \dots, x_{12} \in V_{12}$. Since graph K_{12} contains $K_{6,6}$, which has a positive space crossing by [Živ99, Corollary 3.5], there is a map $\sigma: [8] \rightarrow [12]$ such that $R(x_{\sigma(1)}, \dots, x_{\sigma(8)})$ holds. Since R is constant on $V_{\sigma(1)} \times \cdots \times V_{\sigma(8)}$, we obtain at least as many space crossings as the number of quadruples of edges of the form e_1, e_2, e_3, e_4 , where e_i is between V_{2i-1} and V_{2i} . \square

Straight-line spatial drawing with very few space crossings

Review of stair-convexity To prove Theorem 3 we employ stair-convexity, which is a method to make constructions in \mathbb{R}^d in such a way that convex sets, which are geometric objects, are replaced by their combinatorial cousins, stair-convex sets.

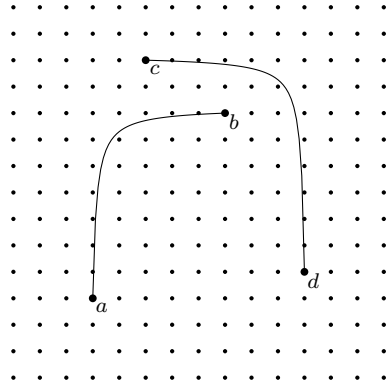


Figure 1: Image under π of line segments $[a, b]$ and $[c, d]$

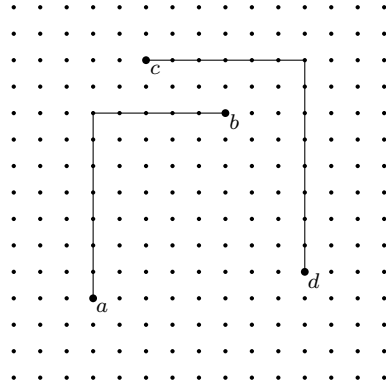


Figure 2: Image under π of stair-paths $\sigma(a, b)$ and $\sigma(c, d)$

The basis for the connection between convexity and stair-convexity is the *stretched grid* $G_s = G_s(n)$ which is the Cartesian product $X_1 \times X_2 \times \cdots \times X_d$, where X_1, X_2, \dots, X_d are “fast-growing” sequences, with each X_i growing much faster than X_{i-1} . Let $X_i = \{x_{i1}, \dots, x_{in}\}$. The actual choice of X_1, X_2, \dots, X_d is not important, as long as they grow quickly enough.

More precisely, for each coordinate $i = 1, \dots, d$ there is a relation \prec_i , such that the condition on the growth of X_i is that $1 = x_{i1} \prec_i x_{i2} \prec_i \cdots \prec_i x_{in}$. The relation \prec_i is not a linear relation, but it is transitive, and is compatible with the usual linear ordering on \mathbb{R} in the sense that $A \prec_i B$ implies $A < B$.

Since the coordinates in G_s grow very fast, to visualize and to work with the grid it is convenient to rescale G_s . Let $BB(G_s) = [1, x_{1m}] \times \cdots \times [1, x_{dm}]$ be the “bounding box” of G_s . Let the *uniform grid* be

$$G_u = G_u(n) \stackrel{\text{def}}{=} \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-1}{n-1}\right\}^d,$$

and pick a bijection $\pi: BB(G_s) \rightarrow [0, 1]^d$ that maps G_s onto G_u and preserves ordering in each coordinate.

The figure 1 above shows the image under π of two straight line segments connecting the grid points for $d = 2$. As the uniform grid becomes finer, the straight line segments become closer to a piecewise linear curve, the stair-path. A *stair-path* joining points $a = (a_1, a_2, \dots, a_d)$ and $b = (b_1, b_2, \dots, b_d)$ consists of at most d closed line segments, each parallel to a different coordinate axis. The definition goes by induction on d . For $d = 1$, $\sigma(a, b)$ is simply the segment ab . For $d \geq 2$, after possibly interchanging a and b , assume $a_d \leq b_d$. We set $a' = (a_1, a_2, \dots, a_{d-1}, b_d)$ and let $\sigma(a, b)$ be the union of the segment aa' and of the stair-path $\sigma(a', b)$, which is defined recursively after “forgetting” the (common) last coordinate of a' and b . A set $S \subseteq \mathbb{R}^d$ is *stair-convex* if for every $a, b \in S$ we have $\sigma(a, b) \subseteq S$. Since the intersection of stair-convex sets is stair-convex, we can define *stairconvex hull* of a set $S \subset \mathbb{R}^d$ as the intersection of all stair-convex sets containing S .

Two points (a_1, a_2, \dots, a_d) and (b_1, b_2, \dots, b_d) in $BB(G_s)$ are *k-far apart in i'th coordinate*, if there are $k - 1$ real numbers r_1, \dots, r_{k-1} such that either $a_i \prec_i r_1 \prec \dots \prec r_{k-1} \prec_i b_i$ or $b_i \prec_i r_1 \prec \dots \prec r_{k-1} \prec_i a_i$. Otherwise, we say that a and b are *k-close in i'th coordinate*. If a and b are *k-close in every coordinate*, then we say that a and b are *k-close*. If a and b lie on G_s , then they are *k-close* if $\pi(a)$ and $\pi(b)$, which are points of G_u , are separated by fewer than k points in each coordinate. In the picture above, the points a and b are 6-close, but not 5-close. For $a, b \in BB(G_s)$ put $\text{dist}(a, b)$ be the least integer k such that a and b are *k-close*. Note that dist satisfies the triangle inequality $\text{dist}(a, b) \leq \text{dist}(a, c) + \text{dist}(b, c)$.

Several results capture the intuition that the image of a convex set in $BB(G_s)$ looks like a stair-convex set. The following lemma of Nivasch is the form that we need.

Lemma 14 (Lemma 2.11 in [Niv09]). *Let a, b be two points in $BB(G_s)$, and let ab and $\sigma(a, b)$ be the line segment and the stair-path between a and b , respectively. Then every point in ab is 1-close to some point of $\sigma(a, b)$ and vice versa.*

Corollary 15. *Suppose a, b, a', b' are points in $BB(G_s)$. If the segments ab and $a'b'$ intersect, then there are points $c \in \sigma(a, b)$ and $c' \in \sigma(a', b')$ that are 2-close.*

Proof. Let c and c' be 1-close to $ab \cap a'b'$. Then $\text{dist}(c, c') \leq 2$ holds by the triangle inequality. \square

Proof of Theorem 3 We shall now describe a straight-line drawing with few space crossings. From now on we fix $d = 3$ and pick a particular choice of stretched grid $G_s = G_s(5n)$ with $(5n)^3$ points. We shall also work with the subgrid $G'_s \subset G_s$ that consists of the points of the form $(x_{1i_1}, x_{2i_2}, x_{3i_3})$ with $5 \mid i_1, i_2, i_3$. The subgrid G'_s has n^3 points. Let $p(i) = (x_{1i}, x_{2i}, x_{3i})$ be the i 'th point on the “diagonal” of G_s .

Let G be the graph on the vertex set $\{1, 2, 3, \dots, n\}$ with i and j forming an edge if $|i - j| \leq D$, where $D = 2m/n$. The *standard drawing of G* is one in which the vertex $i \in V(G)$ is represented by the point $p(5i)$, and all the edges are straight-line segments. Note that in this drawing all the vertices lie on the subgrid G'_s , and thus no pair of them is 5-close. Moreover, if the stair-paths $\sigma(a_1, b_1)$ and $\sigma(a_2, b_2)$ with $a_1, a_2, b_1, b_2 \in G'_s$ do not intersect, then no point of $\sigma(a_1, b_1)$ is 5-close to a point of $\sigma(a_2, b_2)$.

We say that four stair-paths form a *space stair-crossing* if there is another stair-path that meets all four stair-paths. The *standard stair-drawing of G* is one in which vertex $i \in V(G)$ is represented by the point $p(5i)$, and all edges are stair-paths.

The following two lemmas imply Theorem 3.

Lemma 16. *Let $s_1, t_1, \dots, s_4, t_4 \in V(G)$ be eight distinct vertices of G . Then the edges s_1t_1, \dots, s_4t_4 form a space crossing in the standard drawing of G only if they form a space stair-crossing in the standard stair-drawing of G .*

Lemma 17. *Let $s_1, t_1, \dots, s_4, t_4 \in [0, 1]$ be distinct vertices of G . Let $I_i = [s_i, t_i]$ be the interval spanned by s_i and t_i . Then the four vertex-disjoint edges s_1t_1, \dots, s_4t_4 form a stair-crossing in*

the standard stair-drawing of G only if for each $i = 1, \dots, 4$ there is at least one $j \neq i$ so that $I_i \cap I_j \neq \emptyset$

Proof that Lemmas 16 and 17 imply Theorem 3. The graph G has n vertices and more than $Dn/2 = m$ edges. The four vertex-disjoint edges $s_1t_1, \dots, s_4t_4 \in E(G)$ form a space crossing only if the union of the four intervals $[s_1, t_1], \dots, [s_4, t_4]$ has at most two connected components.

There are $\frac{1}{4!} \binom{2}{2} \binom{4}{2} \binom{6}{2} \binom{8}{2} = 105$ order types of four unlabeled endpoint-disjoint intervals (each order type corresponds to a perfect matching on 8 labeled points). Each order type that consists of $r \leq 2$ connected component gives rise to at most $n^r D^{8-r}$ space crossings in the standard drawing of G . Indeed, there are n^r ways to choose the leftmost points of the intervals, and once those are specified, it only suffices to specify the distances between the consecutive points in a connected component, and these distances are bounded by D . Thus, there are at most $105n^2 D^6 = 6720m^6/n^4$ space crossing in the standard drawing of G . \square

Proof of Lemma 16. Suppose s_1t_1, \dots, s_4t_4 are edges of G forming a space crossing, and let l be the line that meets these four edges. Let r_1 and r_2 be the intersection points of l with $BB(G'_s)$. Then by Corollary 15 the stair-path $\sigma(r_1, r_2)$ is 2-close to the stair-paths $\sigma(s_1, t_1), \dots, \sigma(s_4, t_4)$.

Let r'_1 and r'_2 be the points of G'_s such that $\text{dist}(r_1, r'_1) \leq 3$ and $\text{dist}(r_2, r'_2) \leq 3$. Since $\sigma(r_1, r_2)$ is 2-close to $\sigma(s_1, t_1)$, by the triangle inequality, $\sigma(r'_1, r'_2)$ is 5-close to $\sigma(s_1, t_1)$. Since r'_1, r'_2, s_1, t_1 belong to G'_s , that means that $\sigma(r'_1, r'_2)$ and $\sigma(s_1, t_1)$ intersect. Similarly, $\sigma(r'_1, r'_2)$ intersects $\sigma(s_i, t_i)$ for $i = 1, \dots, 4$, and the edges s_1t_1, \dots, s_4t_4 form a stair-crossing. \square

Proof of Lemma 17. Every stair-path is a subset of one of the three types of sets

1. $L_1(x_0, y_0, y_1, z_1) \stackrel{\text{def}}{=} \sigma((x_0, y_0, 0), (+\infty, y_1, z_1))$,
2. $L_2(x_0, y_0, y_1, z_1) \stackrel{\text{def}}{=} \sigma((x_0, y_0, 0), (-\infty, y_1, z_1))$,
3. $L_3(x_0, y_0, x_1, z_1) \stackrel{\text{def}}{=} \sigma((x_0, y_0, 0), (x_1, -\infty, z_1))$.

We call L_1, L_2, L_3 *stair-lines*. The numbers x_0, y_0, \dots are the *coordinates* of stair-lines.

Let L be a line that meets the four edges s_1t_1, \dots, s_4t_4 . We say that L shares the coordinate with the edge s_it_i if that coordinates is equal to either s_i or t_i . Note that since the edges are vertex-disjoint, no coordinate of L can be shared with two distinct edges. Since the edge s_it_i is represented by the stair-path $\sigma((s_i, s_i, s_i), (t_i, t_i, t_i))$, it meets L only if it shares a coordinate with L . It follows that each of the four coordinates of L is shared with a unique edge.

Suppose the edge s_it_i shares the coordinate c with L . Let p_i be an intersection point of L with s_it_i . The point p_i shares at least two coordinates with L , one of which is c . The other coordinate c' is between s_i and t_i . Thus if s_jt_j is the edge that shares the coordinate c' with L , then the intervals $[s_i, t_i]$ and $[s_j, t_j]$ intersect. \square

The proof of Lemma 13

To simplify the proof we will need to work in slightly greater generality than subsets of a fixed Euclidean space. First, we permit $\mathcal{F}_1, \mathcal{F}_2, \dots$ to be multisets (this will come handy in the proof of Theorem 18), and we permit different multisets to belong to different spaces. To keep track of these spaces, we introduce a bit of notation. For a k -tuple $\mathbf{d} = (d_1, \dots, d_k) \in \mathbb{N}^k$ we define $\mathbb{R}^{\mathbf{d}} \stackrel{\text{def}}{=} \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_k}$. For simplicity of notation we adopt the convention that \mathbb{R}^{d_i} denotes the i 'th component of $\mathbb{R}^{\mathbf{d}}$ even if there are several i that share the same value of d_i . The *number of terms* of a polynomial f on i 'th component, denoted $t_i(f)$, is the number of monomials of f when treated as a polynomial on \mathbb{R}^{d_i} (with other coordinates treated as fixed). For example, if $\mathbf{d} = (1, 1)$ and $f: \mathbb{R}^{\mathbf{d}} \rightarrow \mathbb{R}$ is defined by $f(x_1, x_2) = 4 + 2x_1x_2 + 3x_1x_2^2 + x_1x_2^3 + 7x_1^2x_2$, then $t_1(f) = 3$, and $t_2(f) = 4$, though f has five terms. The sign of a number $x \in \mathbb{R}$ is $+1$ if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.

By Tarski–Seidenberg theorem (see [BPR06, Theorem 2.77]) every semialgebraic relation is equivalent to a quantifier-free semi-algebraic relation. Thus the following result implies Lemma 13.

Theorem 18. *Let $f_1, \dots, f_J: \mathbb{R}^{\mathbf{d}} \rightarrow \mathbb{R}$ be a family of polynomials. Suppose $\mathcal{F}_i \subset \mathbb{R}^{d_i}$ for $i = 1, \dots, k$ are finite multisets of points. Then there are submultisets $\mathcal{F}'_i \subset \mathcal{F}_i$ such that*

1. \mathcal{F}'_i are large: $|\mathcal{F}'_i| \geq \varepsilon |\mathcal{F}_i|$, with $\varepsilon = \prod_{j=1}^J 3^{-3^{t_2(f_j) + \dots + t_k(f_j)}}$;
2. For each j the sign of $f_j(p_1, \dots, p_k)$ is same for all the choices of $p_i \in \mathcal{F}'_i$.

(The fact that the expression for ε does not depend on $t_1(f_j)$ is not a typo, but an artifact of the proof.)

We use a version of Yao–Yao lemma [YY85] due to Lehec [Leh09]. Recall that a *convex cone* of vectors v_1, \dots, v_r is the set of all their non-negative linear combinations, $\text{conv-cone}(v_1, \dots, v_r) \stackrel{\text{def}}{=} \sum a_i v_i$ with $a_i \geq 0$.

Lemma 19 (Theorem 3 and Proposition 4 in [Leh09]). *Let μ be a probability Borel measure on \mathbb{R}^d such that $\mu(H) = 0$ on every affine hyperplane H . Then there is a way choose the origin of the coordinates in \mathbb{R}^d and 2^d convex cones such that:*

1. The union of the cones is \mathbb{R}^d , and the cones are disjoint apart from the boundaries;
2. Each cone has measure $1/2^d$ with respect to μ ;
3. Every closed halfspace that contains the origin also contains one of the cones;
4. Each cone is a convex cone of only d vectors.

By the standard approximation argument it follows that if \mathcal{F} is any finite multiset of points in \mathbb{R}^d , then there is a partition of \mathbb{R}^d as in the lemma above such that each cone contains at least

$|\mathcal{F}|/2^d$ points of \mathcal{F} . Furthermore, if $\text{conv-cone}(\{v_1, \dots, v_d\})$ is one of the convex cones, then for large enough $t > 0$ we have $\text{conv-cone}(\{v_1, \dots, v_d\}) \cap P = \text{conv}(\{0, tv_1, \dots, tv_d\}) \cap P$. We thus obtain

Corollary 20. *Suppose \mathcal{F} is a finite multiset in \mathbb{R}^d . Then there is a point p and 2^d d -dimensional closed simplices $\Delta_1, \dots, \Delta_{2^d} \subset \mathbb{R}^d$ such that*

1. *The interiors of the simplices $\Delta_1, \dots, \Delta_{2^d}$ are disjoint;*
2. *For each $j = 1, \dots, 2^d$ the number of points in j 'th simplex is $|\mathcal{F} \cap \Delta_j| \geq |\mathcal{F}|/2^d$;*
3. *The point p is a vertex of each Δ_j for $j = 1, \dots, 2^d$;*
4. *Every closed halfspace that contains p also contains one of Δ_j 's.*

The following lemma is a minor variation on the standard linearization argument (see [AM92] for example).

Lemma 21. *Let R be a commutative ring. Let $f: \mathbb{R}^d \rightarrow R$ be a polynomial with t non-constant terms. Then there is a map $\pi: \mathbb{R}^d \rightarrow R^t$ and a linear polynomial $f': R^t \rightarrow R$ such that $f = f' \circ \pi$.*

Proof. Let $1 = g_0, g_1, \dots, g_t$ be the set of all monomials appearing in f . Let $f = \sum \alpha_i g_i$. Define $\pi(x) = (g_1(x), \dots, g_t(x))$, and $f'(z_0, z_1, \dots, z_t) = \sum \alpha_i z_i$. The identity $f = f' \circ \pi$ is clear. \square

Proof of Theorem 18. The proof is by induction on k . The base case $k = 1$ is trivial because for at least one third of all the points $x_1 \in \mathcal{F}_1$ the sign of $f_j(x_1)$ is the same. Suppose $k \geq 2$, and the theorem is known to hold for $k - 1$. It suffices to prove the result for a single polynomial, which we shall call f . Think of f as a polynomial with $t_k(f)$ terms on \mathbb{R}^{d_k} . By Lemma 21 there is map $\pi: \mathbb{R}^{d_k} \rightarrow \mathbb{R}^{t_k(f)}$ and a polynomial $f': \mathbb{R}^{d'} \rightarrow \mathbb{R}$, which is linear on $\mathbb{R}^{t_k(f)}$, such that

$$f(x_1, x_2, \dots, x_{k-1}, x_k) = f'(x_1, x_2, \dots, x_{k-1}, \pi(x_k)).$$

Apply Corollary 20 to the multiset $\pi(\mathcal{F}_k)$. Let $\Delta_1, \dots, \Delta_{2^{d'_k}} \subset \mathbb{R}^{t_k(f)}$ be the simplices whose existence the corollary guarantees. They have a total of at most $1 + t_k(f')2^{t_k(f)} \leq 3^{t_k(f)}$ vertices, which we denote by v_1, \dots, v_M where $M \leq 3^{t_k(f)}$. Each of the simplices contains at least $|\mathcal{F}_k|/2^{t_k(f)}$ points of $\pi(\mathcal{F}_k)$.

Since the polynomial f' is linear in x_k each choice of $x_i \in \mathbb{R}^{d_i}$ ($i = 1, \dots, k-1$) gives the hyperplane in $\mathbb{R}^{t_k(f)}$, namely the hyperplane $H(x_1, \dots, x_{k-1}) = \{x \in \mathbb{R}^{d'_k} : f'(x_1, \dots, x_{k-1}, x) = 0\}$. Define $H^+(x_1, \dots, x_{k-1})$ and $H^-(x_1, \dots, x_{k-1})$ as the two closed halfspaces that $H(x_1, \dots, x_{k-1})$ bounds in the obvious way. By Corollary 20 either H^- or H^+ contains some Δ_j .

For each point v_m define the polynomial g_m by $g_m(x_1, \dots, x_{k-1}) = f'(x_1, \dots, x_{k-1}, v_m)$. Note that the indices j for which Δ_j is contained in $H^+(x_1, \dots, x_{k-1})$ depends only on the signs of $g_m(x_1, \dots, x_{k-1})$, and similarly for the indices j for which $\Delta_j \subset H^-(x_1, \dots, x_{k-1})$.

²It is here that we use that \mathcal{F}_k is a multiset. If \mathcal{F}_k was defined to be a set, then $\pi(\mathcal{F}_k)$ might have had fewer elements than \mathcal{F}_k .

Since $t_i(g_m) \leq t_i(f)$ for each $i = 1, \dots, k-1$, by the induction hypothesis there are subsets $\mathcal{F}'_i \subset \mathcal{F}_i$ for $i = 1, \dots, k-1$ of size $|\mathcal{F}'_i| \geq 3^{-M \cdot 3^{t_2(f) + \dots + t_{k-1}(f)}} |\mathcal{F}_i|$ such that for all $m = 1, \dots, M$ the sign of $g_m(x_1, \dots, x_{k-1})$ does not depend on the choice of $x_i \in \mathcal{F}'_i$. Denote this sign by ϵ_m . Therefore there is a single j and a non-zero sign s such that Δ_j is contained in $H^s(x_1, \dots, x_{k-1})$ for each choice $x_i \in \mathcal{F}'_i$. Without loss of generality $s = +1$, which means that $\epsilon_m \geq 0$ for every vertex v_m of Δ_j . Let σ be the face of Δ spanned by the vertices v_m for which $\epsilon_m = 0$. Thus $f'(x_1, \dots, x_{k-1}, x_k) = 0$ ($x_i \in \mathcal{F}'_i$) if $x_k \in \sigma$ and $f'(x_1, \dots, x_{k-1}, x_k) > 0$ if $x_k \in \Delta_j \setminus \sigma$. One of the two alternatives holds for at least half of the points in $\pi(\mathcal{F}_k) \cap \Delta_j$, and since $2^{-t_k(f)-1} \geq 3^{-3^{t_k(f)}}$ the theorem follows. \square

Higher dimensions and other questions

Throughout the paper we spoke of “the” space crossing number, but it is but one of a family of similar quantities. Similarly to the way the crossing number measures the complexity of planar embeddings, these quantities measure the complexity of embeddings into higher-dimensional Euclidean spaces. We give some examples:

1. For an embedding of graph $G \rightarrow \mathbb{R}^3$ one can count not the quadruples, but the triples of edges crossed by a line. The methods in this paper easily adapt to show that the corresponding crossing number $\text{cr}_3(G)$ satisfies $\text{cr}_3(G) \geq c(|E|^4/|V|^2 \log^2|V|)$, and there are graphs G such that $\text{lin-cr}_3(G) \leq C(|E|^4/|V|^2)$.
2. For an embedding of a graph $G \rightarrow \mathbb{R}^4$ there are at least two kinds of objects to consider: the lines that pierce three edges of G , and 2-planes that pierce 6 edges of G . The simple dimension-counting shows that for generic embeddings, there are finitely many such lines and 2-planes.
3. More generally, one can count the number of $(d-2)$ -dimensional planes through $2(d-1)$ edges of $G \rightarrow \mathbb{R}^d$. The case $d = 2$ is the classical crossing number, whereas $d = 3$ is the space crossing number of the present paper. Theorem 3 from [Kar07] can be used to give lower bounds on these higher crossing numbers, but we lack the constructions for the upper bounds.
4. One can consider representations of a 3-uniform hypergraph in \mathbb{R}^4 by means of (topological) triangles, and count the number of triples of triangles that meet at a single point. However, it is an open problem even to show that in every 3-uniform hypergraph with more than Cn^2 edges there is a single pair of intersecting triangles!

There are several more question about the crossing numbers we defined:

1. A result about crossing number with many applications is the bisection width inequality (proved independently in [PSS96, Theorem 2.1], extending the proof in [Lei84] for the

bounded-degree graphs). The bisection width inequality states that $\text{cr}(G)^2 \geq c_1 b^2(G) - c_2 \sum_{v \in V(G)} \deg_G(v)^2$, where $b(G)$ is the bisection width of the graph G . Is there an analogous inequality for the space crossing number that is of comparable strength to Theorem 4?

2. Is the family of graphs with $\text{cr}_4(G) = 0$ a minor-closed family?
3. Is it true that $\text{cr}_4(G) = 0$ if and only if $\text{lin-cr}_4(G) = 0$?

We guess that the answers are 1) yes, 2) no, and 3) no.

Acknowledgments. We thank András Juhász for discussions about Lemma 10. We also grateful to him and Saul Schleimer for helping us to find [Vir09]. We thank János Pach and Boris Aronov for helpful conversations. The work was partly done with support of Discrete and Computational Program in Bernoulli Centre, EFPL, Lausanne. The second author acknowledges the support of Conacyt.

References

- [ACNS82] M. Ajtai, V. Chvátal, M. M. Newborn, and E. Szemerédi. Crossing-free subgraphs. In *Theory and practice of combinatorics*, volume 60 of *North-Holland Math. Stud.*, pages 9–12. North-Holland, Amsterdam, 1982.
- [AM92] Pankaj K. Agarwal and Jiří Matoušek. On range searching with semialgebraic sets. In *Mathematical foundations of computer science 1992 (Prague, 1992)*, volume 629 of *Lecture Notes in Comput. Sci.*, pages 1–13. Springer, Berlin, 1992.
- [BD93] Daniel Bienstock and Nathaniel Dean. Bounds for rectilinear crossing numbers. *J. Graph Theory*, 17(3):333–348, 1993.
- [BMN] Boris Bukh, Jiří Matoušek, and Gabriel Nivasch. Lower bounds for weak epsilon-nets and stair-convexity. *Israel J. Math.*, To appear. arXiv:0812.5039.
- [BPR06] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in real algebraic geometry*, volume 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, second edition, 2006. <http://perso.univ-rennes1.fr/marie-francoise.roy/bpr-ed2-posted2.html>.
- [Bre93] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993.
- [BS93] B. Bollobás and A. D. Scott. Judicious partitions of graphs. *Period. Math. Hungar.*, 26(2):125–137, 1993.

- [BV98] I. Bárány and P. Valtr. A positive fraction Erdős-Szekeres theorem. *Discrete Comput. Geom.*, 19(3, Special Issue):335–342, 1998. Dedicated to the memory of Paul Erdős.
- [CG83] J. H. Conway and C. McA. Gordon. Knots and links in spatial graphs. *J. Graph Theory*, 7(4):445–453, 1983.
- [Dey98] T. K. Dey. Improved bounds for planar k -sets and related problems. *Discrete Comput. Geom.*, 19(3, Special Issue):373–382, 1998. Dedicated to the memory of Paul Erdős.
- [Die05] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.
- [FGL⁺10] Jacob Fox, Mikhail Gromov, Vincent Lafforgue, Assaf Naor, and János Pach. Overlap properties of geometric expanders. arXiv:1005.1392, May 2010.
- [Kar07] Roman N. Karasev. Tverberg’s transversal conjecture and analogues of nonembeddability theorems for transversals. *Discrete Comput. Geom.*, 38(3):513–525, 2007.
- [KP88] A. Kostochka and L. Pyber. Small topological complete subgraphs of “dense” graphs. *Combinatorica*, 8(1):83–86, 1988.
- [KS06] M. Krivelevich and B. Sudakov. Pseudo-random graphs. In *More sets, graphs and numbers*, volume 15 of *Bolyai Soc. Math. Stud.*, pages 199–262. Springer, Berlin, 2006.
- [Leh09] Joseph Lehec. On the Yao-Yao partition theorem. *Arch. Math. (Basel)*, 92(4):366–376, 2009. arXiv:1011.2123.
- [Lei84] Frank Thomson Leighton. New lower bound techniques for VLSI. *Math. Systems Theory*, 17(1):47–70, 1984.
- [Niv09] Gabriel Nivasch. *Weak epsilon-nets, Davenport-Schinzel sequences, and related problems*. PhD thesis, Tel Aviv University, 2009. http://people.inf.ethz.ch/gnivasch/publications/papers/phd_thesis.pdf.
- [PSS96] J. Pach, F. Shahrokhi, and M. Szegedy. Applications of the crossing number. *Algorithmica*, 16(1):111–117, 1996. <http://www.cims.nyu.edu/~pach/publications/mario.pdf>.
- [PT97] János Pach and Géza Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997.
- [Ris96] A. Riskin. The crossing number of a cubic plane polyhedral map plus an edge. *Studia Sci. Math. Hungar.*, 31(4):405–413, 1996.

- [Sac83] Horst Sachs. On a spatial analogue of Kuratowski's theorem on planar graphs—an open problem. In *Graph theory (Łagów, 1981)*, volume 1018 of *Lecture Notes in Math.*, pages 230–241. Springer, Berlin, 1983.
- [Szé97] László A. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.*, 6(3):353–358, 1997.
- [Vir09] Julia Viro. Lines joining components of a link. *J. Knot Theory Ramifications*, 18(6):865–888, 2009. arXiv:math/0511527.
- [YY85] A C Yao and F F Yao. A general approach to d-dimensional geometric queries. In *Proceedings of the seventeenth annual ACM symposium on Theory of computing*, STOC '85, pages 163–168, New York, NY, USA, 1985. ACM.
- [Živ99] Rade T. Živaljević. The Tverberg-Vrećica problem and the combinatorial geometry on vector bundles. *Israel J. Math.*, 111:53–76, 1999.