

Multi-terminal Secrecy in a Linear Non-coherent Packetized Networks

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Abstract

We consider a group of $m + 1$ trusted nodes that aim to create a shared secret key \mathcal{K} over a network in the presence of a passive eavesdropper, Eve. We assume a linear non-coherent network coding broadcast channel (over a finite field \mathbb{F}_q) from one of the honest nodes (i.e., Alice) to the rest of them including Eve. All of the trusted nodes can also discuss over a cost-free public channel which is also overheard by Eve.

For this setup, we propose upper and lower bounds for the secret key generation capacity assuming that the field size q is very large. For the case of two trusted terminals ($m = 1$) our upper and lower bounds match and we have complete characterization for the secrecy capacity in the large field size regime.

I. INTRODUCTION

For communication over a network performing linear network coding, Cai and Yeung [1] introduced the problem of securing a multicast transmission against an eavesdropper. In particular, consider a network implementing linear network coding over a finite field \mathbb{F}_q . Let us assume that the min-cut value from the source to each receiver is c . From the main theorem of network coding [2], [3] we know that a source can send information at rate equal to the min-cut c to the destinations, in the absence of any malicious eavesdropper. Now, suppose there is a passive eavesdropper, Eve, who overhears ρ arbitrary edges in the network. The *secure network coding* problem is to design a coding scheme such that Eve does not obtain any information about the messages transmitted from the source to destinations. Cai and Yeung [1] showed that the secrecy capacity for this problem is $c - \rho$ and can be achieved if the field size q is sufficiently large. Later this problem formulation has been investigated in many other works. Feldman *et al.* [4] showed that by sacrificing a small amount of rate, one might find a secure scheme that requires much smaller field size. Rouayheb *et al.* [5] observed that this problem can be considered as a generalization of the Ozarow-Wyner wiretap channel of type II. Silva *et al.* [6] proposed a universal coding scheme that only employs encoding at the source.

In contrast to the previous work, in this paper we study the problem of secret key sharing among multiple terminals when nodes can send feedback over a public channel. We consider a source multicasting information over a network at rate equal to the min-cut c to the destinations. We also assume that the relay nodes in the network

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perform linear randomized network coding which is modeled by a non-coherent transmission scheme. Motivated by [7], [8], we model a non-coherent network coding scenario by a multiplicative matrix channel over a finite field \mathbb{F}_q with uniform and i.i.d. distribution over transfer matrices in every time-slot.

The problem of key agreement between a set of terminals with access to noisy broadcast channel and public discussion channel (visible to the eavesdropper) was studied in [9], where some achievable secrecy rates were established, assuming Eve does not have access to the noisy broadcast transmissions. This was generalized in [10], [11] by developing (non-computable) outer bounds for secrecy rates. However, to the best of our knowledge, ours is the first work to consider multi-terminal secret key agreement over networks employing randomized network coding, when a passive eavesdropper has access to the broadcast transmissions.

Our contributions in this paper are as follows. For the secret key sharing problem introduced above, we propose an asymptotic achievability scheme assuming that the field size q is large. This scheme is based on *subspace coding* and can be extended for arbitrary number of terminals. Using the result of [9], we derive an upper bound for this problem. For $m = 1$, the proposed lower bound matches the upper bound and the *secret key generation capacity* is characterized. However, for $m \geq 2$, depending on the channel parameters, the upper and lower bound might match or not.

The paper is organized as follows. In §II we introduce our notation and the problem formulation and present some preliminaries. In §III, we state a general upper bound for the key generation capacity and evaluate it for the non-coherent network coding broadcast channel. The main results of the paper are presented in §IV.

II. NOTATION AND SETUP

A. Notation

All vectors are column vectors unless otherwise stated. Fixed matrices are denoted by bold uppercase letters and normal uppercase letters denote the random matrices. We use $\langle \mathbf{A} \rangle$ to denote the row span of a matrix \mathbf{A} . For convenience, we also use $[i : j]$ to denote $\{i, i + 1, \dots, j\}$ where $i, j \in \mathbb{Z}$.

Let Π be an arbitrary vector space of finite dimension defined over a finite field \mathbb{F}_q . Suppose Π_1 and Π_2 are two subspaces of Π , i.e., $\Pi_1 \subseteq \Pi$ and $\Pi_2 \subseteq \Pi$. We use $\Pi_1 \cap \Pi_2$ to denote the common subspaces of both Π_1 and Π_2 and $\Pi_1 + \Pi_2$ as the smallest subspace that contains both Π_1 and Π_2 . Two subspaces Π_1 and Π_2 are called *orthogonal* if $\Pi_1 \cap \Pi_2 = \{\mathbf{0}\}$. Two subspaces Π_1 and Π_2 of Π are called *complementary* if they are orthogonal and $\Pi_1 + \Pi_2 = \Pi$.

Now, consider two subspaces Π_1 and Π_2 . We define the subtraction of Π_2 from Π_1 by $U = \Pi_1 \setminus_s \Pi_2$ where U is any subspace of Π_1 which is complementary with $\Pi_1 \cap \Pi_2$. Note that, given Π_1 and Π_2 , U is not uniquely defined.

For notational convenience, when \mathcal{J} is a set, by $\Pi_{\mathcal{J}}$ we mean $\Pi_{\mathcal{J}} \triangleq \bigcap_{i \in \mathcal{J}} \Pi_i$.

B. Preliminaries

Definition 1 (Grassmannian and Gaussian coefficient [12], [13]). *The Grassmannian $\text{Gr}(\ell, d)_q$ is the set of all d -dimensional subspaces of the ℓ -dimensional space over a finite field \mathbb{F}_q , namely,*

$$\text{Gr}(\ell, d)_q \triangleq \{\pi \subseteq \mathbb{F}_q^\ell : \dim(\pi) = d\}. \quad (1)$$

The cardinality of $\text{Gr}(\ell, d)_q$ is the Gaussian coefficient, namely,

$$\begin{bmatrix} \ell \\ d \end{bmatrix}_q \triangleq |\text{Gr}(\ell, d)_q| = \prod_{i=0}^{d-1} \frac{q^{\ell-i} - 1}{q^{d-i} - 1}. \quad (2)$$

Definition 2. *We define $\mathcal{S}(\ell, m)_q$ to be the set (sphere) of all subspaces of dimension at most m in the ℓ -dimensional space \mathbb{F}_q^ℓ , namely*

$$\mathcal{S}(\ell, m)_q \triangleq \bigcup_{d=0}^{\min[m, \ell]} \text{Gr}(\ell, d)_q = \{\pi \subseteq \mathbb{F}_q^\ell : \dim(\pi) \leq \min[m, \ell]\}. \quad (3)$$

Definition 3 (see [7]). *We denote by $\xi(n, d)$ the number of different $n \times \ell$ matrices with elements from a finite field \mathbb{F}_q , such that their rows span a specific subspace $\pi_d \subseteq \mathbb{F}_q^\ell$ of dimension d where $0 \leq d \leq \min[n, \ell]$. Note that $\xi(n, d)$ does not depend on ℓ and depends on π_d only through its dimension d [7, Lemma 2].*

For simplicity, in the rest of the paper, we will drop the subscript q in the previous definitions whenever it is obvious from the context.

Lemma 1. *Suppose that k subspaces Π_1, \dots, Π_k , with dimensions d_1, \dots, d_k , are chosen uniformly at random from \mathbb{F}_q^n . Then with high probability (probability of order $1 - O(q^{-1})$) we have*

$$\dim(\Pi_1 + \dots + \Pi_k) = \min[d_1 + \dots + d_k, n], \quad (4)$$

and

$$\dim(\Pi_1 \cap \dots \cap \Pi_k) = [d_1 + \dots + d_k - (k-1)n]^+. \quad (5)$$

Note that if one of the subspaces, for example Π_1 , be a fixed subspace then the above results still hold.

Proof: These results follow from [14, Corollary 1] by using induction on the number of subspaces. ■

C. Problem Statement

We consider a set of $m + 1 \geq 2$ honest nodes, T_0, \dots, T_m , (T stands for “terminal”) that aim to share a secret key \mathcal{K} among themselves while keeping it concealed from a passive adversary, Eve. Eve does not perform any transmissions, but is trying to eavesdrop on (overhear) the communications between the honest nodes. For convenience, sometimes we will refer to node T_0, T_1, T_2, \dots , as “Alice,” “Bob,” “Calvin,” and so on.

We assume that there exists a non-coherent network coding broadcast channel (which is going to be defined more precisely in the followig) from Alice to the other terminals (including Eve). Also we assume that the legitimate terminals can publicly discuss over a noiseless rate unlimited public channel.

Consider a non-coherent linear network coding communication scenario where at every time-slot t Alice injects a set of n_A vectors (packets) of length ℓ (over some finite field \mathbb{F}_q) into the network, denoted by the row vectors of the matrix $X_A[t] \in \mathbb{F}_q^{n_A \times \ell}$. Each terminal T_i receives n_i randomly chosen linear combinations of the transmitted vectors, namely for $r \in \{\mathsf{T}_1, \dots, \mathsf{T}_m, \mathsf{E}\}$, we have¹

$$X_r[t] = F_r[t]X_A[t], \quad (6)$$

where $F_r[t] \in \mathbb{F}_q^{n_r \times n_A}$ is chosen uniformly at random among all possible matrices and independently for each receiver and every time-slot. So for the channel transition probability we can write

$$P_{X_1 \dots X_m X_E | X_A}(x_1, \dots, x_m, x_E | x_A) = P_{X_E | X_A}(x_E | x_A) \prod_{i=1}^m P_{X_i | X_A}(x_i | x_A), \quad (7)$$

where for each $r \in \{\mathsf{T}_1, \dots, \mathsf{T}_m, \mathsf{E}\}$ we have [7, Sec IV-A]

$$P_{X_r | X_A}(x_r | x_A) \triangleq \begin{cases} q^{-n \dim(x_A)} & \text{if } \langle x_r \rangle \subseteq \langle x_A \rangle, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Note that in this setup we do not assume any CSI² at the transmitter or receivers.

III. UPPER BOUND

A. Secrecy Upper Bound for Independent Broadcast Channels

The secret key generation capacity among multiple terminals (without eavesdropper having access to the broadcast channel) is completely characterized in [9]. By using this result, it is possible to state an upper bound for the secrecy capacity of the key generation problem among multiple terminals where the eavesdropper has also access to the broadcast channel. This can be done by adding a dummy terminal to the first problem and giving all the eavesdropper's information to this dummy node and let it to participate in the key generation protocol. By doing so, the secret key generation rate does not decrease. Hence by combining [9, Theorem 4.1] and [9, Lemma 5.1], the following result can be stated.

Theorem 1. *The secret key generation capacity is upper bounded as follows*

$$C_s \leq \max_{P_{X_0}} \min_{\lambda \in \Lambda([0:m])} \left[H(X_{[0:m]} | X_E) - \sum_{B \subsetneq [0:m]} \lambda_B H(X_B | X_{B^c}, X_E) \right], \quad (9)$$

where $\Lambda([0:m])$ is the set of all collections $\lambda = \{\lambda_B : B \subsetneq [0:m], B \neq \emptyset\}$ of weights $0 \leq \lambda_B \leq 1$, satisfying

$$\sum_{B \subsetneq [0:m], i \in B} \lambda_B = 1, \quad \forall i \in [0:m]. \quad (10)$$

¹During the paper, we use T_i and i interchangeably when they are used as subscript. So instead of X_{T_i} we sometimes write X_i . At some points, we also use X_A, X_B, X_C , etc., to denote for X_0, X_1, X_2 , etc.

²Channel state information.

Note that in the above expression for the upper bound, it is possible to change the order of maximization and minimization [9, Theorem 4.1].

Now, for our problem where the channel from Alice to the other terminals are assumed to be independent, we can further simplify the upper bound given in Theorem 1, as stated in Corollary 1.

Corollary 1. *If the channels from Alice to the other terminals are independent, as described in (7), then the upper bound stated in Theorem 1, for the secret key generation capacity is simplified to*

$$C_s \leq \max_{P_{X_0}} \min_{j \in [1:m]} I(X_0; X_j | X_E) \quad (11)$$

$$\leq \min_{j \in [1:m]} \max_{P_{X_0}} I(X_0; X_j | X_E). \quad (12)$$

Proof: For the proof please refer to Appendix B. ■

Remark: Note that (11) is the best upper bound one might hope for an independent broadcast channel using the results of [9].

Remark: Using [15, Theorem 7] or [16, Theorem 2], we observe that the bound given in (12) is indeed tight for the two terminals problem where we have the Markov chains $X_B \leftrightarrow X_A \leftrightarrow X_E$ (when the channels are independent) or $X_A \leftrightarrow X_B \leftrightarrow X_E$ (when the channels are degraded).

B. Upper Bound for Non-coherent Channel

In the previous section, we have shown that the secret key generation rate for our problem can be upper bounded by (12). Now, we need to evaluate the above upper bound for the non-coherent network coding channel defined in §II-C.

Lemma 2. *For the joint distribution of the form*

$$P_{X_A X_i X_E}(x_A, x_i, x_E) = P_{X_A}(x_A) P_{X_i | X_A}(x_i | x_A) P_{X_E | X_A}(x_E | x_A) \quad (13)$$

the mutual information $I(X_A; X_i | X_E)$ is a concave function of $P_{X_A}(x_A)$ for fixed $P_{X_i | X_A}(x_i | x_A)$ and $P_{X_E | X_A}(x_E | x_A)$.

Proof: TBA. ■

Similar to [7, Definition 5], here we define an equivalent subspace broadcast channel from Alice to the rest of terminals as follows. We assume that Alice sends a subspace $\Pi_A \in \mathcal{S}(\ell, n_A)$ where $\Pi_A = \langle X_A \rangle$ and each of the legitimate terminals receives $\Pi_i \in \mathcal{S}(\ell, n_i)$ and Eve receives $\Pi_E \in \mathcal{S}(\ell, n_E)$ where $\Pi_i = \langle X_i \rangle$ and $\Pi_E = \langle X_E \rangle$, respectively. The channel transition probabilities are independent and for each receiver i is defined as follows

$$P_{\Pi_i | \Pi_A}(\pi_i | \pi_A) \triangleq \begin{cases} \xi(n_i, \dim(\pi_i)) q^{-n_i \dim(\pi_A)} & \text{if } \pi_i \sqsubseteq \pi_A, \\ 0 & \text{otherwise,} \end{cases} \quad (14)$$

where the function ξ is defined in §II-B.

Lemma 3. For every input distribution P_{X_A} there exists an input distribution P_{Π_A} such that $I(X_A; X_i|X_E) = I(\Pi_A; \Pi_i|\Pi_E)$ and vice-versa.

Proof: We can expand $I(X_A; X_i|X_E) = I(X_A; X_i X_E) - I(X_A; X_E)$. Using [7, Theorem 1], by defining $\Pi_A = \langle X_A \rangle$ and $\Pi_r = \langle X_r \rangle$ for $r \in \{\mathbb{T}_1, \dots, \mathbb{T}_m, \mathbb{T}_E\}$, we can write

$$\begin{aligned} I(X_A; X_i|X_E) &= I(X_A; X_i X_E) - I(X_A; X_E) \\ &= I(\Pi_A; \Pi_i + \Pi_E) - I(\Pi_A; \Pi_E) \\ &\stackrel{(a)}{\leq} I(\Pi_A; \Pi_i, \Pi_E) - I(\Pi_A; \Pi_E) \\ &= I(\Pi_A; \Pi_i|\Pi_E) \end{aligned} \tag{15}$$

where (a) is true because of the data processing inequality applied on the Markov chain $\Pi_i + \Pi_E \leftrightarrow (\Pi_i, \Pi_E) \leftrightarrow \Pi_A$. On the other hand, again by applying data processing inequality, we can write

$$\begin{aligned} I(X_A; X_i|X_E) &= I(X_A; X_i X_E) - I(X_A; X_E) \\ &\stackrel{(a)}{\geq} I(\Pi_A; \Pi_i, \Pi_E) - I(X_A; X_E) \\ &\stackrel{(b)}{=} I(\Pi_A; \Pi_i, \Pi_E) - I(\Pi_A; \Pi_E) \\ &= I(\Pi_A; \Pi_i|\Pi_E) \end{aligned} \tag{16}$$

where (a) is true because of the Markov chain $(\Pi_i, \Pi_E) \leftrightarrow (X_i, X_E) \leftrightarrow X_A \leftrightarrow \Pi_A$ and (b) is true because of [7, Theorem 1]. Hence we are done. ■

So by Lemma 3, in order to maximize $I(X_A; X_i|X_E)$ with respect to P_{X_A} it is sufficient to solve an equivalent problem, i.e., maximize $I(\Pi_A; \Pi_i|\Pi_E)$ with respect to P_{Π_A} ; which is seemingly a simpler optimization problem than the original one.

Lemma 4. The input distribution that maximizes $I(\Pi_A; \Pi_i|\Pi_E)$ is the one which is uniform over all subspaces having the same dimension.

Proof: By the concavity of $I(\Pi_A; \Pi_i|\Pi_E)$ with respect to P_{Π_A} , that is stated in Lemma 2, the proof follows by an argument very similar to [7, Lemma 8]. ■

Lemma 5. Asymptotically in the field size we have

$$\begin{aligned} \max_{P_{X_A}} I(X_A; X_i|X_E) &= \max_{P_{\Pi_A}} I(\Pi_A; \Pi_i|\Pi_E) \\ &= (\min[n_A, n_i + n_E] - n_E) (\ell - \min[n_A, n_i + n_E]) \log q. \end{aligned} \tag{17}$$

Proof: For the proof refer to Appendix B. ■

Thus, by using the upper bound given in (12) and Lemma 5 we have the following result for the upper bound on the secret key generation rate, as stated in Theorem 2.

Theorem 2. *The secret key generation rate in a non-coherent network coding scenario, which is defined in §II-C, is upper bounded by*

$$C_s \leq \min_{i \in [1:m]} [(\min[n_A, n_i + n_E] - n_E) (\ell - \min[n_A, n_i + n_E])] \log q. \quad (18)$$

Remark: Note that if $n_E = n_A$ then the secret key generation rate is zero because Eve is so powerful that she overhears all of the transmitted information.

IV. ASYMPTOTIC ACHIEVABILITY SCHEME FOR ARBITRARY NUMBER OF TERMINALS

Here in this section, we describe our achievability scheme for the secret key sharing problem among multiple terminals in a non-coherent network coding setup.

Without loss of generality, let us assume that³ $n_A < \ell$. Moreover, in this work we focus on the asymptotic regime where the field size is large. Suppose that Alice broadcasts a message $X_A[t]$ at time-slot t of the following form

$$X_A[t] = \begin{bmatrix} I_{n_A \times n_A} & M[t] \end{bmatrix}, \quad (19)$$

where $M[t] \in \mathbb{F}_q^{n_A \times (\ell - n_A)}$ is a uniformly at random distributed matrix. The rest of legitimate terminals and Eve receive a linear transformed version of $X_A[t]$ according to the channel introduced in (6).

For each terminal $r \in \{\mathsf{T}_0, \dots, \mathsf{T}_m, \mathsf{T}_E\}$, we define the subspace $\Pi_r \triangleq \langle X_r \rangle$. Then, for every $r \neq \mathsf{T}_0$ we have $\Pi_r \subseteq \Pi_A$. Because of (19), after broadcasting $X_A[t]$, the legitimate terminals learn the channel state and reveal the channel transfer matrices $F_r[t]$, $r \in [1 : m]$, publicly over the public channel. Thus Alice can also recover the subspaces Π_r for all of the legitimate terminals.

Now, for each non-empty subset $\mathcal{J} \subseteq [1 : m]$ of legitimate receivers, let us define the subspace $U_{\mathcal{J}}$ as follows

$$U_{\mathcal{J}} \triangleq \Pi_{\mathcal{J}} \setminus_s \left(\sum_{i \in \mathcal{J}^c} \Pi_{i\mathcal{J}} + \Pi_{E\mathcal{J}} \right), \quad (20)$$

where $\Pi_{\mathcal{J}} = \cap_{i \in \mathcal{J}} \Pi_i$, $\Pi_{i\mathcal{J}} = \Pi_i \cap \Pi_{\mathcal{J}}$, and $\Pi_{E\mathcal{J}} = \Pi_E \cap \Pi_{\mathcal{J}}$. By definition, $U_{\mathcal{J}}$ is the common subspace among the receivers in \mathcal{J} which is orthogonal to all of the subspaces of other terminals, i.e., it is orthogonal to Π_i , $i \in \mathcal{J}^c$, and Π_E (see also Fig. 1). Note that the subspaces $U_{\mathcal{J}}$'s are not uniquely defined. However, from the definition of the operator “ \setminus_s ”, it can be easily shown that the dimension of each $U_{\mathcal{J}}$ is uniquely determined and equal to

$$\dim(U_{\mathcal{J}}) = \dim(\Pi_{\mathcal{J}}) - \dim \left(\sum_{i \in \mathcal{J}^c} \Pi_{i\mathcal{J}} + \Pi_{E\mathcal{J}} \right). \quad (21)$$

If Alice had the subspace Π_E observed by Eve, she would be able to construct subspaces $U_{\mathcal{J}}$'s; but she does not have Π_E . However, because the subspaces Π_i 's and Π_E are chosen independently and uniformly at random from Π_A , and because the field size q is large, Alice, by applying Lemma 1, can find the dimension of each $U_{\mathcal{J}}$ w.h.p. Then it can be easily observed that (e.g., see [14, Lemma 3]) if Alice chooses a uniformly at random subspace of $\Pi_{\mathcal{J}}$ with dimension $\dim(U_{\mathcal{J}})$ then it satisfies (20) w.h.p., so it can be a possible candidate for $U_{\mathcal{J}}$.

³If $\ell \leq n_A$ then Alice can reduce the number of injected packets into the network from n_A to some smaller number n'_A where $n'_A < \ell$.

Now, consider $2^m - 1$ different non-empty subsets of $[1 : m]$. To each subset $\emptyset \neq \mathcal{J} \subseteq [1 : m]$, we assign a parameter $\theta_{\mathcal{J}} \geq 0$ such that the following set of inequalities hold,

$$\theta_{\mathcal{J}_1} + \cdots + \theta_{\mathcal{J}_k} \leq \dim(U_{\mathcal{J}_1} + \cdots + U_{\mathcal{J}_k} + \Pi_E) - \dim(\Pi_E), \quad (22)$$

for any $k \in [1 : 2^{(2^m-1)} - 1]$ and any different selection of subsets $\mathcal{J}_1, \dots, \mathcal{J}_k$. Note that the right hand side of the inequalities defined in (22) depend on the actual choice of subspaces $U_{\mathcal{J}}$'s. But, as described above, in the following we assume that $U_{\mathcal{J}}$'s are chosen uniformly at random from $\Pi_{\mathcal{J}}$.

If Alice knows the subspace Π_E , then we can state the following result.

Lemma 6. *There exists subspaces $U'_{\mathcal{J}} \subseteq U_{\mathcal{J}}$ such that $\dim(U'_{\mathcal{J}}) = \theta_{\mathcal{J}}$ for all $\emptyset \neq \mathcal{J} \subseteq [1 : m]$, and $U'_{\mathcal{J}}$'s and Π_E are orthogonal subspaces (i.e., $\dim(\Pi_E + \sum_i U'_{\mathcal{J}_i}) = \dim(\Pi_E) + \sum_i \theta_{\mathcal{J}_i}$) if and only if $\theta_{\mathcal{J}}$'s are non-negative integers and satisfy (22).*

Proof: The proof of this lemma is based on [17, Lemma 4] and can be found in Appendix B. ■

Fig.1 depicts pictorially the relation between subspaces introduced in the above discussions.

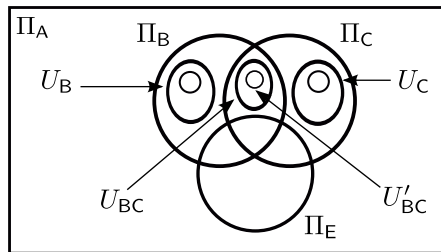


Fig. 1. The relations between subspaces Π 's, U 's, and U' 's for the case of $m = 2$.

Although in practice Alice only knows the dimension of Π_E (w.h.p.), but still she can find subspaces $U'_{\mathcal{J}} \subseteq U_{\mathcal{J}}$ such that the result of Lemma 6 holds w.h.p., as stated in Lemma 7.

Lemma 7. *Alice can find subspaces $U'_{\mathcal{J}} \subseteq U_{\mathcal{J}}$ such that $\dim(U'_{\mathcal{J}}) = \theta_{\mathcal{J}}$ for all $\emptyset \neq \mathcal{J} \subseteq [1 : m]$, and $U'_{\mathcal{J}}$'s are orthogonal subspaces and $U'_{\mathcal{J}}$'s and Π_E are orthogonal subspaces w.h.p., if and only if $\theta_{\mathcal{J}}$'s are non-negative integers and satisfy (22).*

Proof: For the proof refer to Appendix B. ■

Then, we have the following result.

Theorem 3. *The secret key sharing rate given by the solution of the following convex optimization problem can*

be achieved

$$\begin{aligned}
& \text{maximize} && \left[\min_{r \in [1:m]} \sum_{\mathcal{J} \ni r} \theta_{\mathcal{J}} \right] (\ell - n_A) \log q \\
& \text{subject to} && \theta_{\mathcal{J}} \geq 0, \quad \forall \mathcal{J} \subseteq [1:m], \mathcal{J} \neq \emptyset, \quad \text{and} \\
& && \theta_{\mathcal{J}_1} + \dots + \theta_{\mathcal{J}_k} \leq \\
& && \dim(U_{\mathcal{J}_1} + \dots + U_{\mathcal{J}_k} + \Pi_E) - \dim(\Pi_E) \\
& && \forall k, \forall \mathcal{J}_1, \dots, \mathcal{J}_k : \emptyset \neq \mathcal{J}_i \subseteq [1:m], \\
& && \mathcal{J}_i \neq \mathcal{J}_j \text{ if } i \neq j,
\end{aligned} \tag{23}$$

where for every \mathcal{J} , $U_{\mathcal{J}}$ is chosen uniformly at random from $\Pi_{\mathcal{J}}$ with the dimension calculated by (21) under the assumption that Π_1, \dots, Π_m , and Π_E are selected independently and uniformly at random from Π_A with dimensions n_1, \dots, n_m , and n_E respectively.

Proof: Let Alice use the broadcast channel N times by sending matrices $X_A[1], \dots, X_A[N]$ of the form (19). As mentioned before, in every time-slot t , each of the legitimate terminals sends publicly the channel transfer matrix it has received.

Then, let us define $\hat{\theta}_{\mathcal{J}} \triangleq \lfloor N\theta_{\mathcal{J}} \rfloor$ for all \mathcal{J} and consider the following set of inequalities

$$\begin{aligned}
& \hat{\theta}_{\mathcal{J}_1} + \dots + \hat{\theta}_{\mathcal{J}_k} + N \dim(\Pi_E) \leq \\
& \dim \left(\bigoplus_{t=1}^N U_{\mathcal{J}_1}[t] + \dots + \bigoplus_{t=1}^N U_{\mathcal{J}_k}[t] + \bigoplus_{t=1}^N \Pi_E[t] \right),
\end{aligned} \tag{24}$$

where “ \oplus ” is the direct sum operator. Each of $\hat{U}_{\mathcal{J}_i} \triangleq \bigoplus_{t=1}^N U_{\mathcal{J}_i}[t]$ is a subspace of an $N \times n_A$ dimensional space $\bigoplus_{t=1}^N \Pi_A[t]$. Similarly, we have $\hat{\Pi}_E \subseteq \bigoplus_{t=1}^N \Pi_A[t]$ where $\hat{\Pi}_E \triangleq \bigoplus_{t=1}^N \Pi_E[t]$. It can be easily seen that if the set of inequalities (22) are satisfied then the set of inequalities (24) are also satisfied.

Now, by using Lemma 7, Alice can find a set of orthogonal subspaces $\hat{U}'_{\mathcal{J}}$ with dimension $\hat{\theta}_{\mathcal{J}}$ (that are also orthogonal to $\hat{\Pi}_E$ w.h.p.). By applying Lemma 8, one would observe that if Alice uses a basis of $\hat{U}'_{\mathcal{J}}$ ($\hat{\theta}_{\mathcal{J}}$ linear independent vectors from $\hat{U}'_{\mathcal{J}}$) to share a secret key $\mathcal{K}_{\mathcal{J}}$ with all terminals in \mathcal{J} , then this key is secure from Eve and all other legitimate terminals in \mathcal{J}^c w.h.p. Using each key $\mathcal{K}_{\mathcal{J}}$, Alice can send a message of size $\hat{\theta}_{\mathcal{J}}(\ell - n_A) \log q$ secretly to the terminals in \mathcal{J} . In order to share the key $\mathcal{K}_{\mathcal{J}}$, Alice sends publicly a set of coefficients for each terminal in \mathcal{J} so that each of them can construct the subspace $\hat{U}_{\mathcal{J}}$ from their own received subspace. Note that even having these coefficients, Eve cannot recover any information regarding $\mathcal{K}_{\mathcal{J}}$ (for more discussion see [18]).

Up until now, the problem of sharing a key \mathcal{K} among legitimate terminals have been reduced to a multicast problem where Alice would like to transmit a message (i.e., the shared key \mathcal{K}) to a set of terminal where the r th one has a min-cut $\sum_{\mathcal{J} \ni r} \hat{\theta}_{\mathcal{J}}$. From the main theorem of network coding (e.g., see [2], [3], [19], [20]), we know that this problem can be solved by performing linear network coding where the achievable rate is as follows

$$R_s \leq \left[\frac{1}{N} \min_{r \in [1:m]} \sum_{\mathcal{J} \ni r} \hat{\theta}_{\mathcal{J}} \right] (\ell - n_A) \log q. \tag{25}$$

By increasing N , the achievable secrecy rate will be arbitrarily close to

$$R_s \leq \left[\min_{r \in [1:m]} \sum_{\mathcal{J} \ni r} \theta_{\mathcal{J}} \right] (\ell - n_A) \log q, \quad (26)$$

and we are done. \blacksquare

Lemma 8. Consider a set of n_A packets denoted by the rows of a matrix $X_A \in \mathbb{F}_q^{n_A \times \ell}$ of the form $X_A = [I \ M]$, where $M \sim \text{Uni}(\mathbb{F}_q^{n_A \times (\ell - n_A)})$. Assume that Eve has overheard n_E independent linear combinations of these packets, represented by the rows of a matrix $X_E \in \mathbb{F}_q^{n_E \times \ell}$. Then for every k packets y_1, \dots, y_k that are linear combinations of the rows of X_A , if the subspace $\Pi_Y = \langle y_1, \dots, y_k \rangle$ is orthogonal to $\langle X_E \rangle$ we have

$$I(y_1, \dots, y_k; X_E) = 0. \quad (27)$$

Proof: The proof is stated in Appendix B. \blacksquare

A. Special Case: Achievability Scheme for Two Terminals

For simplicity and without loss of generality we assume that $n_B \leq n_A$ and $n_E \leq n_A$. The key generation scheme starts by Alice broadcasting a message $X_A[t]$ at time t of the form of (19). Then, Theorem 3 states that the secrecy rate R_s is achievable if

$$R_s \leq [\dim(U_B + \Pi_E) - \dim(\Pi_E)] (\ell - n_A) \log q, \quad (28)$$

where $U_B = \Pi_B \setminus_s \Pi_E$ (for convenience we have replaced $U_{\{B\}}$ with U_B). Because $U_B \cap \Pi_E = \{\mathbf{0}\}$, we have

$$\begin{aligned} R_s &\leq [\dim(U_B)] (\ell - n_A) \log q \\ &= [\dim(\Pi_B) - \dim(\Pi_B \cap \Pi_E)] (\ell - n_A) \log q \\ &= [n_B - (n_B + n_E - n_A)^+] (\ell - n_A) \log q \\ &= [\min[n_A, n_B + n_E] - n_E] (\ell - n_A) \log q, \quad \text{bits/matrix channel use,} \end{aligned} \quad (29)$$

where this is the same as the upper bound given in Theorem 2. This is obvious when $n_A \leq n_B + n_E$. On the other hand, if $n_A > n_B + n_E$, we can reduce the number of injected packets by Alice in every time-slot from n_A to $n_B + n_E$ (there is no need to use more than $n_B + n_E$ degrees of freedom).

Remark: Note that in the above scheme, as long as $n_E < n_A$, the secrecy rate is non-zero.

Now, we compare the derived secrecy rate with the case where no feedback is allowed. First let us assume that $n_B \geq n_E$. Then, in the non-coherent network coding scenario introduced in §II-C, it can be easily verified that the channel from Alice to Eve is a *stochastically degraded* (for the definition refer to [21, p. 373]) version of the channel from Alice to Bob.

So by applying the result of [22] or [23, Theorem 3], for the secret key sharing capacity we can write

$$\begin{aligned} C_s &= \max_{P_{X_A}} [I(X_A; X_B) - I(X_A; X_E)] \\ &= \max_{P_{\Pi_A}} [I(\Pi_A; \Pi_B) - I(\Pi_A; \Pi_E)], \end{aligned} \quad (30)$$

where the sufficiency of optimization over subspaces follows from [7, Theorem 1]. Similar to the proof of Lemma 5 (because in the proof of Lemma 5, we also maximize an expression that contains subtraction of two mutual information similar to (30)), one can show that

$$C_s = [n_B - n_E](\ell - n_B) \log q, \quad (31)$$

which is positive only if $n_B > n_E$ (obviously for the case $n_B < n_E$ we have $C_s = 0$ as well, because even for a weaker eavesdropper, when $n_B = n_E$, we have $C_s = 0$). ■

The above comparison demonstrates the amount of improvement of the secret key generation rate we might gain by using feedback.

B. Special Case: Achievability Scheme for Three Terminals

As an another example, here we consider the three trusted terminals problem (i.e., $m = 2$). As before, we assume that $n_A < \ell$ and for the convenience we suppose that $n_B = n_C \leq n_A$ and $n_E \leq n_A$.

In order to characterize the achievable secrecy rate, we need to find the dimension of subspaces U_B , U_C , and U_{BC} and their sums (including Π_E as well). We assume that the field size q is large. We know that Π_B , Π_C , and Π_E are chosen uniformly at random from n_A -dimensional space Π_A . Subspaces Π_{BC} and Π_{BE} are also distributed independently and uniformly at random in Π_B . Similarly, the same is true for Π_{BC} and Π_{CE} in Π_C .

We have

$$\begin{cases} U_B = \Pi_B \setminus_s (\Pi_{BC} + \Pi_{BE}) \\ U_C = \Pi_C \setminus_s (\Pi_{BC} + \Pi_{CE}) \\ U_{BC} = \Pi_{BC} \setminus_s (\Pi_{BCE}), \end{cases} \quad (32)$$

so we can write

$$\begin{aligned} \dim(U_B) &= \dim(\Pi_B) - \dim(\Pi_{BC} + \Pi_{BE}) \\ &\stackrel{(a)}{=} \dim(\Pi_B) - \min [\dim(\Pi_{BC}) + \dim(\Pi_{BE}), \dim(\Pi_B)] \\ &\stackrel{(b)}{=} n_B - \min [\dim(\Pi_{BC}) + \dim(\Pi_{BE}), n_B] \\ &= [n_B - \dim(\Pi_{BC}) - \dim(\Pi_{BE})]^+ \\ &\stackrel{(c)}{=} [n_B - (2n_B - n_A)^+ - (n_B + n_E - n_A)^+]^+, \end{aligned} \quad (33)$$

where (a) follows from Lemma 1 because Π_{BC} and Π_{BE} are chosen independently and uniformly at random from Π_B , (b) is true because q is large, and (c) follows from Lemma 1. Note that because we have assumed $n_B = n_C$ it follows that $\dim(U_C) = \dim(U_B)$.

Similarly, for the dimension of U_{BC} we can write

$$\begin{aligned}
\dim(U_{BC}) &= \dim(\Pi_{BC}) - \dim(\Pi_{BCE}) \\
&= \dim(\Pi_{BC}) - [\dim(\Pi_{BC}) + n_E - n_A]^+ \\
&= \min[n_A - n_E, \dim(\Pi_{BC})] \\
&= \min[n_A - n_E, (2n_B - n_A)^+].
\end{aligned} \tag{34}$$

Proposition 1. *From the construction, the subspaces U_B , U_C , and U_{BC} are complementary. The same holds for U_B , U_{BC} , and Π_E and similarly for U_C , U_{BC} , and Π_E w.h.p.*

Now we may write the linear program stated in Theorem 3 as follows

$$\begin{aligned}
&\text{maximize} && \min[\theta_B + \theta_{BC}, \theta_C + \theta_{BC}] (\ell - n_A) \log q \\
&\text{subject to} && \theta_B \leq \dim(U_B + \Pi_E) - n_E \\
&&& \theta_C \leq \dim(U_C + \Pi_E) - n_E \\
&&& \theta_{BC} \leq \dim(U_{BC} + \Pi_E) - n_E \\
&&& \theta_B + \theta_C \leq \dim(U_B + U_C + \Pi_E) - n_E \\
&&& \theta_B + \theta_C + \theta_{BC} \leq \dim(U_B + U_C + U_{BC} + \Pi_E) - n_E.
\end{aligned} \tag{35}$$

Because of the symmetry in the problem ($n_B = n_C$), for the optimal solution we should have $\theta_B = \theta_C$. Knowing this and using Proposition 1, we may further simplify the above linear program as follows

$$\begin{aligned}
&\text{maximize} && [\theta_B + \theta_{BC}] (\ell - n_A) \log q \\
&\text{subject to} && \theta_B \leq \frac{1}{2} [\dim(U_B + U_C + \Pi_E) - n_E] \triangleq \alpha_1 \\
&&& \theta_{BC} \leq \dim(U_{BC}) \triangleq \alpha_2 \\
&&& 2\theta_B + \theta_{BC} \leq \dim(U_B + U_C + U_{BC} + \Pi_E) - n_E \triangleq \alpha_3.
\end{aligned} \tag{36}$$

From the definitions of α 's, we can easily observe that, $\alpha_3 \geq 2\alpha_1$, $\alpha_3 \geq \alpha_2$, and $\alpha_3 \leq 2\alpha_1 + \alpha_2$. Hence, $\theta_B + \theta_{BC}$ gets its maximum at the point $(\theta_B, \theta_{BC}) = (\frac{\alpha_3 - \alpha_2}{2}, \alpha_2)$. Thus, for the maximum achievable secrecy rate we have

$$R_s = \left\lceil \frac{\alpha_2 + \alpha_3}{2} \right\rceil (\ell - n_A) \log q. \tag{37}$$

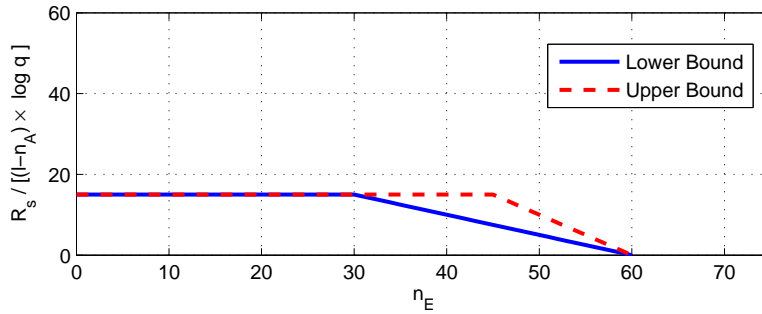
As mentioned before, we assume that subspaces $U_{\mathcal{J}}$'s are chosen uniformly at random from $\Pi_{\mathcal{J}}$. So Π_E and $U_{\mathcal{J}}$'s are independent and for α_3 we can write

$$\begin{aligned}
\alpha_3 &= \min[\dim(U_B) + \dim(U_C) + \dim(U_{BC}) + \dim(\Pi_E), n_A] - n_E \\
&= \min[\dim(U_B) + \dim(U_C) + \dim(U_{BC}), n_A - n_E] \\
&= \min[2 \dim(U_B) + \dim(U_{BC}), n_A - n_E].
\end{aligned} \tag{38}$$

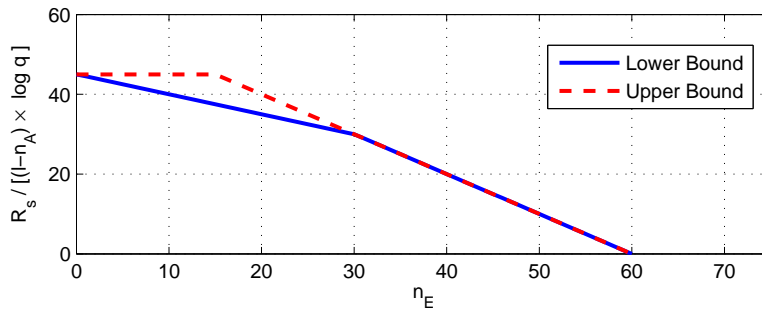
Finally, for the secrecy rate (achievable asymptotically when q goes to infinity) we have

$$R_s = \min \left[\dim(U_B) + \dim(U_{BC}), \frac{1}{2} (n_A + \dim(U_{BC}) - n_E) \right] (\ell - n_A) \log q. \tag{39}$$

Example 1. As an example, here we compare the achievable secret key sharing rate among three legitimate terminals (i.e., $m = 2$) as derived in (39) with the upper bound stated in Theorem 2. We consider two symmetric setup where for the first one we have $n_A = 60$, $n_B = n_C = 15$ (see Fig. 2(a)) and for the second one we have $n_A = 60$, $n_B = n_C = 45$ (see Fig. 2(b)). In each of these situations, we depict the upper and lower bounds on the secret key generation rate as a function of the number of packets (degrees of freedom) received by Eve, i.e., n_E .



(a) $m = 2$, $n_A = 60$, and $n_B = n_C = 15$.



(b) $m = 2$, $n_A = 60$, and $n_B = n_C = 45$.

Fig. 2. A comparison between the achievable secrecy rate of Theorem 3 and the upper bound given by Theorem 2 for two cases: (a) when $m = 2$, $n_A = 60$, and $n_B = n_C = 15$ and (b) when $m = 2$, $n_A = 60$, and $n_B = n_C = 45$.

APPENDIX A

SIMPLIFYING THE MUTUAL INFORMATION FOR A NON-COHERENT NETWORK CODING CHANNEL

Let us consider a non-coherent network coding channel described by the following matrix channel

$$Y[t] = F[t]X[t], \quad (40)$$

where $X[t] \in \mathbb{F}_q^{n_x \times \ell}$, $Y[t] \in \mathbb{F}_q^{n_y \times \ell}$, and $F[t] \in \mathbb{F}_q^{n_y \times n_x}$ is an uniformly at random chosen transfer matrix which is independently chosen for every time-slot t . For simplicity we assume that $\ell \geq \max[n_x, n_y]$.

As stated in [7, Theorem 1], in order to find the capacity of (40) we can instead focus on an equivalent subspace channel described by a transition probability as follows

$$P_{\Pi_Y|\Pi_X}(\pi_y|\pi_x) \triangleq \begin{cases} \xi(n_y, \dim(\pi_y))q^{-n_y \dim(\pi_x)} & \text{if } \pi_y \subseteq \pi_x, \\ 0 & \text{otherwise.} \end{cases} \quad (41)$$

In this work we focus on large q regime, so we can approximate the above transition probability as follows

$$P_{\Pi_Y|\Pi_X}(\pi_y|\pi_x) = \mathbb{1}_{\{\dim \pi_x \leq n_y\}} \mathbb{1}_{\{\pi_y = \pi_x\}} + \frac{\mathbb{1}_{\{\dim \pi_x > n_y\}} \mathbb{1}_{\{\dim \pi_y = n_y\}} \mathbb{1}_{\{\pi_y \subseteq \pi_x\}}}{\binom{\dim \pi_x}{n_y}}. \quad (42)$$

From here on we assume that the input distribution is uniform over all subspaces having the same dimension, namely

$$\mathbb{P}[\Pi_X = \pi_x] = \alpha_{d_x} \binom{\ell}{d_x}^{-1}, \quad (43)$$

where $d_x = \dim \pi_x$ and $\alpha_{d_x} = \mathbb{P}[\dim \Pi_X = d_x]$.

Then, for P_{Π_Y} we can write

$$\begin{aligned} P_{\Pi_Y}(\pi_y) &= \sum_{\pi_x} P_{\Pi_Y|\Pi_X}(\pi_y|\pi_x) P_{\Pi_X}(\pi_x) \\ &= \sum_{\substack{\pi_x: \\ \dim \pi_x \leq \min[n_x, n_y]}} \mathbb{1}_{\{\pi_y = \pi_x\}} P_{\Pi_X}(\pi_x) + \sum_{\substack{\pi_x: \\ n_y < \dim \pi_x \leq n_x}} \frac{\mathbb{1}_{\{\dim \pi_y = n_y\}} \mathbb{1}_{\{\pi_y \subseteq \pi_x\}}}{\binom{\dim \pi_x}{n_y}} P_{\Pi_X}(\pi_x) \\ &= P_{\Pi_X}(\pi_y) \mathbb{1}_{\{\dim \pi_y \leq \min[n_x, n_y]\}} + \sum_{d_x = n_y + 1}^{n_x} \sum_{\substack{\pi_x: \\ \pi_y \subseteq \pi_x, \\ \dim \pi_x = d_x}} \frac{\mathbb{1}_{\{\dim \pi_y = n_y\}}}{\binom{d_x}{n_y}} \cdot \frac{\alpha_{d_x}}{\binom{\ell}{d_x}} \\ &= P_{\Pi_X}(\pi_y) \mathbb{1}_{\{\dim \pi_y \leq \min[n_x, n_y]\}} + \sum_{d_x = n_y + 1}^{n_x} \binom{\ell - n_y}{d_x - n_y} \frac{\mathbb{1}_{\{\dim \pi_y = n_y\}}}{\binom{d_x}{n_y}} \cdot \frac{\alpha_{d_x}}{\binom{\ell}{d_x}}. \end{aligned} \quad (44)$$

Now, we use the following relation (see [7, Lemma 17], [13], and [24]) to further simplify the expression in front of the summation

$$\binom{\ell - n_y}{d_x - n_y} \binom{\ell}{n_y} = \binom{d_x}{n_y} \binom{\ell}{d_x}.$$

Then for P_{Π_Y} we have

$$\begin{aligned}
P_{\Pi_Y}(\pi_y) &= P_{\Pi_X}(\pi_y) \mathbb{1}_{\{\dim \pi_y \leq \min[n_x, n_y]\}} + \sum_{d_x=n_y+1}^{n_x} \mathbb{1}_{\{\dim \pi_y = n\}} \alpha_{d_x} \begin{bmatrix} \ell \\ n_y \end{bmatrix}^{-1} \\
&= P_{\Pi_X}(\pi_y) \mathbb{1}_{\{\dim \pi_y \leq \min[n_x, n_y]\}} + \mathbb{P}[\dim \Pi_X > n_y] \mathbb{1}_{\{\dim \pi_y = n_y\}} \begin{bmatrix} \ell \\ n_y \end{bmatrix}^{-1} \\
&= P_{\Pi_X}(\pi_y) \mathbb{1}_{\{\dim \pi_y \leq \min[n_x, n_y-1]\}} + \mathbb{P}[\dim \Pi_X \geq n_y] \mathbb{1}_{\{\dim \pi_y = n_y\}} \begin{bmatrix} \ell \\ n_y \end{bmatrix}^{-1}.
\end{aligned} \tag{45}$$

Hence, by definition, for the mutual information $I(\Pi_X; \Pi_Y)$ we can write

$$\begin{aligned}
I(\Pi_X; \Pi_Y) &= \sum_{d_x=0}^{n_x} \sum_{\substack{\pi_x: \\ \dim \pi_x = d_x}} \sum_{d_y=0}^{\min[n_y, d_x]} \sum_{\substack{\pi_y: \\ \dim \pi_y = d_y, \\ \pi_y \subseteq \pi_x}} P_{\Pi_X}(\pi_x) P_{\Pi_Y|\Pi_X}(\pi_y|\pi_x) \log \frac{P_{\Pi_Y|\Pi_X}(\pi_y|\pi_x)}{P_{\Pi_Y}(\pi_y)} \\
&= \sum_{d_x=0}^{\min[n_x, n_y]} \sum_{\substack{\pi_x: \\ \dim \pi_x = d_x}} \sum_{d_y=0}^{d_x} \sum_{\substack{\pi_y: \\ \dim \pi_y = d_y, \\ \pi_y \subseteq \pi_x}} \frac{\alpha_{d_x}}{\begin{bmatrix} \ell \\ d_x \end{bmatrix}} \mathbb{1}_{\{\pi_y = \pi_x\}} \log \frac{\mathbb{1}_{\{\pi_y = \pi_x\}}}{P_{\Pi_Y}(\pi_y)} \\
&\quad + \mathbb{1}_{\{n_y < n_x\}} \left\{ \sum_{d_x=n_y+1}^{n_x} \sum_{\substack{\pi_x: \\ \dim \pi_x = d_x}} \sum_{d_y=0}^{n_y} \sum_{\substack{\pi_y: \\ \dim \pi_y = d_y, \\ \pi_y \subseteq \pi_x}} \frac{\alpha_{d_x}}{\begin{bmatrix} \ell \\ d_x \end{bmatrix}} \frac{\mathbb{1}_{\{d_x > n_y\}} \mathbb{1}_{\{d_y = n_y\}}}{\begin{bmatrix} d_x \\ n_y \end{bmatrix}} \log \frac{\mathbb{1}_{\{d_x > n_y\}} \mathbb{1}_{\{d_y = n_y\}}}{P_{\Pi_Y}(\pi_y) \begin{bmatrix} d_x \\ n_y \end{bmatrix}} \right\} \\
&= - \sum_{d_x=0}^{\min[n_x, n_y]} \sum_{\substack{\pi_x: \\ \dim \pi_x = d_x}} \frac{\alpha_{d_x}}{\begin{bmatrix} \ell \\ d_x \end{bmatrix}} \log P_{\Pi_Y}(\pi_x) \\
&\quad - \mathbb{1}_{\{n_y < n_x\}} \left\{ \sum_{d_x=n_y+1}^{n_x} \sum_{\substack{\pi_x: \\ \dim \pi_x = d_x}} \frac{\alpha_{d_x}}{\begin{bmatrix} \ell \\ d_x \end{bmatrix}} \log \left[P_{\Pi_Y}(\pi_y) \begin{bmatrix} d_x \\ n_y \end{bmatrix} \right] \right\},
\end{aligned} \tag{46}$$

where in the last line π_y is an arbitrary subspace with $\dim \pi_y = n_y$. So finally we can write

$$I(\Pi_X; \Pi_Y) = - \sum_{d_x=0}^{\min[n_x, n_y]} \alpha_{d_x} \log \frac{\alpha_{d_x}}{\begin{bmatrix} \ell \\ d_x \end{bmatrix}} - \mathbb{1}_{\{n_y < n_x\}} \left\{ \sum_{d_x=n_y+1}^{n_x} \alpha_{d_x} \log \left[\frac{\begin{bmatrix} d_x \\ n_y \end{bmatrix}}{\begin{bmatrix} \ell \\ n_y \end{bmatrix}} \mathbb{P}[\dim \Pi_X \geq n_y] \right] \right\}. \tag{47}$$

APPENDIX B

PROOFS

Proof of Corollary 1: First, note that we can write

$$\begin{aligned}
H(X_{[0:m]}|X_E) &\stackrel{(a)}{=} \sum_{j=0}^m H(X_j|X_E, X_{[0:j-1]}) \\
&\stackrel{(b)}{=} H(X_0|X_E) + \sum_{j=1}^m H(X_j|X_0),
\end{aligned} \tag{48}$$

where (a) follows from the chain rule and (b) follows from the independence of the channels. Similarly, for every $B \subsetneq [0:m]$ we can expand $H(X_B|X_{B^c}, X_E)$ as follows

$$H(X_B|X_{B^c}, X_E) = H(X_0|X_{B^c}, X_E) + \sum_{j \in B} H(X_j|X_0). \tag{49}$$

Now, from Theorem 1, we know that for every $\lambda \in \Lambda([0 : m])$ there exists a distribution P_{X_0} such that C_s is upper bounded by

$$\begin{aligned}
C_s &\leq H(X_{[0:m]}|X_E) - \sum_{B \subsetneq [0:m]} \lambda_B H(X_B|X_{B^c}, X_E) \\
&= H(X_0|X_E) + \sum_{j=1}^m H(X_j|X_0) - \sum_{B \subsetneq [0:m]} \lambda_B \left[H(X_0|X_{B^c}, X_E) + \sum_{j \in B} H(X_j|X_0) \right] \\
&\stackrel{(a)}{=} H(X_0|X_E) - \sum_{B \subsetneq [0:m], 0 \in B} \lambda_B H(X_0|X_{B^c}, X_E) + \sum_{j=1}^m H(X_j|X_0) - \sum_{j=1}^m \sum_{B \subsetneq [0:m], j \in B} \lambda_B H(X_j|X_0) \\
&\stackrel{(b)}{=} H(X_0|X_E) - \sum_{B \subsetneq [0:m], 0 \in B} \lambda_B H(X_0|X_{B^c}, X_E) \\
&= \sum_{B \subsetneq [0:m], 0 \in B} \lambda_B I(X_0; X_{B^c}|X_E), \tag{50}
\end{aligned}$$

where in (a) we have changed the order of summation over j and B , and (b) follows from (10). In order to find the best upper bound we proceed as follows. For every λ and P_{X_0} we can write

$$\begin{aligned}
A &\triangleq \sum_{B \subsetneq [0:m], 0 \in B} \lambda_B I(X_0; X_{B^c}|X_E) \\
&\geq \sum_{B \subsetneq [0:m], 0 \in B} \lambda_B \min_{j \in B^c} I(X_0; X_j|X_E) \\
&\geq \sum_{B \subsetneq [0:m], 0 \in B} \lambda_B \min_{j \in [1:m]} I(X_0; X_j|X_E) \\
&= \min_{j \in [1:m]} I(X_0; X_j|X_E). \tag{51}
\end{aligned}$$

Let us define $i = \arg \min_{j \in [1:m]} I(X_0; X_j|X_E)$. Then, note that $\lambda_B = \lambda_{B^c} = 1$ where $B^c = \{i\}$ is a valid choice according to the condition of Theorem 1, i.e., they satisfy (10). Now, for this choice we have the chain of inequalities in (51) is satisfied with equalities.

Combining all of the above arguments, for the secrecy upper bound we can write

$$\begin{aligned}
C_s &\leq \max_{P_{X_0}} \min_{j \in [1:m]} I(X_0; X_j|X_E) \\
&\leq \min_{j \in [1:m]} \max_{P_{X_0}} I(X_0; X_j|X_E). \tag{52}
\end{aligned}$$

Proof of Lemma 5: By using Lemma 2 and Lemma 3, we conclude that in order to maximize $I(X_A; X_i|X_E)$ with respect to P_{X_A} , it is sufficient to maximize $I(\Pi_A; \Pi_i|\Pi_E)$ for an equivalent subspace channel introduced in (14). Also Lemma 4 indicates that considering input distributions that are uniform over all subspaces having the same dimension is sufficient.

Let us assume

$$\mathbb{P}[\Pi_A = \pi_A] = \alpha_d \begin{bmatrix} \ell \\ d \end{bmatrix}^{-1}, \tag{53}$$

where $d = \dim \pi_A$ and $\alpha_d = \mathbb{P}[\dim \Pi_A = d]$. Now, define

$$f \triangleq I(\Pi_A; \Pi_i | \Pi_E) = I(\Pi_A; \Pi_i \Pi_E) - I(\Pi_A; \Pi_E), \quad (54)$$

and the goal is to maximize f with respect to α_i s.

We consider two cases as follows.

First case: $n_i + n_E \leq n_A$

Then, by applying the results of Appendix A, specially (47), we can write

$$\begin{aligned} f = & - \sum_{d=n_E+1}^{n_i+n_E} \alpha_d \log \frac{\alpha_d}{\binom{\ell}{d}} - \sum_{d=n_i+n_E+1}^{n_A} \alpha_d \log \left[\frac{\binom{d}{n_i+n_E}}{\binom{\ell}{n_i+n_E}} \mathbb{P}[\dim \Pi_A \geq n_i + n_E] \right] \\ & + \sum_{d=n_E+1}^{n_A} \alpha_d \log \left[\frac{\binom{d}{n_E}}{\binom{\ell}{n_E}} \mathbb{P}[\dim \Pi_A \geq n_E] \right]. \end{aligned} \quad (55)$$

Now we have to maximize f with respect to the input distribution, α_i . We know that the mutual information is a concave function with respect to α_i 's. This allows us to use the Kuhn-Tucker theorem to solve the convex optimization problem. According to this theorem, the set of probabilities α_i^* , $0 \leq i \leq n_A$, maximize the mutual information if and only if there exists some constant λ such that

$$\begin{cases} \frac{\partial f}{\partial \alpha_k} \Big|_{\alpha^*} = \lambda & \forall k : \alpha_k^* > 0, \\ \frac{\partial f}{\partial \alpha_k} \Big|_{\alpha^*} \leq \lambda & \forall k : \alpha_k^* = 0, \end{cases} \quad (56)$$

where $0 \leq k \leq n_A$, $\sum_{i=0}^{n_A} \alpha_i^* = 1$, and α^* is the vector of the optimum input probabilities of choosing subspaces of certain dimension,

$$\alpha^* = \left[\alpha_0^* \quad \cdots \quad \alpha_{n_A}^* \right]^T. \quad (57)$$

Taking the derivative for $0 \leq k < n_E$ we have

$$\frac{\partial f}{\partial \alpha_k} = 0, \quad (58)$$

for $k = n_E$ we have

$$\frac{\partial f}{\partial \alpha_k} = \sum_{d=n_E+1}^{n_A} \alpha_d \left[\frac{\log e}{\mathbb{P}[\dim \Pi_A \geq n_E]} \right] = \frac{\mathbb{P}[\dim \Pi_A > n_E]}{\mathbb{P}[\dim \Pi_A \geq n_E]} \log e, \quad (59)$$

for $n_E < k < n_i + n_E$ we have

$$\frac{\partial f}{\partial \alpha_k} = -\log \frac{\alpha_k}{\binom{\ell}{k}} - \log e + \log \left[\frac{\binom{k}{n_E}}{\binom{\ell}{n_E}} \mathbb{P}[\dim \Pi_A \geq n_E] \right] + \frac{\mathbb{P}[\dim \Pi_A > n_E]}{\mathbb{P}[\dim \Pi_A \geq n_E]} \log e, \quad (60)$$

for $k = n_i + n_E$ we have

$$\begin{aligned} \frac{\partial f}{\partial \alpha_k} = & -\log \frac{\alpha_k}{\binom{\ell}{k}} - \log e + \log \left[\frac{\binom{k}{n_E}}{\binom{\ell}{n_E}} \mathbb{P}[\dim \Pi_A \geq n_E] \right] + \frac{\mathbb{P}[\dim \Pi_A > n_E]}{\mathbb{P}[\dim \Pi_A \geq n_E]} \log e \\ & - \frac{\mathbb{P}[\dim \Pi_A > n_i + n_E]}{\mathbb{P}[\dim \Pi_A \geq n_i + n_E]} \log e, \end{aligned} \quad (61)$$

and finally for $n_i + n_E < k \leq n_A$ we have

$$\begin{aligned} \frac{\partial f}{\partial \alpha_k} = & + \log \left[\frac{\binom{k}{n_E}}{\binom{\ell}{n_E}} \mathbb{P}[\dim \Pi_A \geq n_E] \right] + \frac{\mathbb{P}[\dim \Pi_A > n_E]}{\mathbb{P}[\dim \Pi_A \geq n_E]} \log e \\ & - \log \left[\frac{\binom{k}{n_i + n_E}}{\binom{\ell}{n_i + n_E}} \mathbb{P}[\dim \Pi_A \geq n_i + n_E] \right] - \frac{\mathbb{P}[\dim \Pi_A > n_i + n_E]}{\mathbb{P}[\dim \Pi_A \geq n_i + n_E]} \log e. \end{aligned} \quad (62)$$

We can easily check that for large q , the input distribution that has $\alpha_{n_i + n_E} = 1$ and $\alpha_i = 0$ for $i \neq n_i + n_E$ satisfies the Kuhn-Tucker conditions. For this distribution, we have

$$\left\{ \begin{array}{ll} 0 \leq k < n_E & : \quad \frac{\partial f}{\partial \alpha_k} = 0 < \lambda, \\ k = n_E & : \quad \frac{\partial f}{\partial \alpha_k} = \log e < \lambda, \\ n_E < k < n_i + n_E & : \quad \frac{\partial f}{\partial \alpha_k} = \log \frac{\binom{k}{n_E} \binom{\ell}{n_E}^{-1}}{\binom{\ell}{n_E}} < \lambda, \\ k = n_i + n_E & : \quad \frac{\partial f}{\partial \alpha_k} = \log \frac{\binom{\ell}{n_i + n_E} \binom{n_i + n_E}{n_E} \binom{\ell}{n_E}^{-1}}{\binom{\ell}{n_E}} = \lambda, \\ n_i + n_E < k \leq n_A & : \quad \frac{\partial f}{\partial \alpha_k} = \log e + \log \frac{\binom{k}{n_E} \binom{\ell}{n_E}^{-1}}{\binom{\ell}{n_E}} - \log \frac{\binom{k}{n_i + n_E} \binom{\ell}{n_i + n_E}^{-1}}{\binom{\ell}{n_i + n_E}} < \lambda. \end{array} \right. \quad (63)$$

So we have

$$\begin{aligned} \max_{P_{\Pi_A}} I(\Pi_A; \Pi_i | \Pi_E) &= \lambda \\ &= \log \frac{\binom{\ell}{n_i + n_E} \binom{n_i + n_E}{n_E} \binom{\ell}{n_E}^{-1}}{\binom{\ell}{n_E}} \\ &= \log \frac{\ell - n_E}{n_i} \\ &\approx n_i (\ell - n_i - n_E) \log q. \end{aligned} \quad (64)$$

Second case: $n_i + n_E > n_A$

For this case the function f becomes

$$f = - \sum_{d=n_E+1}^{n_A} \alpha_d \log \frac{\alpha_d}{\binom{\ell}{d}} + \sum_{d=n_E+1}^{n_A} \alpha_d \log \left[\frac{\binom{d}{n_E}}{\binom{\ell}{n_E}} \mathbb{P}[\dim \Pi_A \geq n_E] \right]. \quad (65)$$

Similar to the previous case, we can apply the Kuhn-Tucker theorem to find the optimal input distribution α_i^* 's.

Taking derivative for $0 \leq k < n_E$, we have

$$\frac{\partial f}{\partial \alpha_k} = 0, \quad (66)$$

for $k = n_E$ we have

$$\frac{\partial f}{\partial \alpha_k} = \sum_{d=n_E+1}^{n_A} \alpha_d \left[\frac{\log e}{\mathbb{P}[\dim \Pi_A \geq n_E]} \right] = \frac{\mathbb{P}[\dim \Pi_A > n_E]}{\mathbb{P}[\dim \Pi_A \geq n_E]} \log e, \quad (67)$$

and finally for $n_E < k \leq n_A$ we have

$$\frac{\partial f}{\partial \alpha_k} = - \log \frac{\alpha_k}{\binom{\ell}{k}} - \log e + \log \left[\frac{\binom{k}{n_E}}{\binom{\ell}{n_E}} \mathbb{P}[\dim \Pi_A \geq n_E] \right] + \frac{\mathbb{P}[\dim \Pi_A > n_E]}{\mathbb{P}[\dim \Pi_A \geq n_E]} \log_2 e. \quad (68)$$

We can easily check that for large q , the input distribution that has $\alpha_{n_A} = 1$ and $\alpha_i = 0$ for $i \neq n_A$ satisfies the Kuhn-Tucker conditions. For this distribution, we have

$$\begin{cases} 0 \leq k < n_E & : \frac{\partial f}{\partial \alpha_k} = 0 < \lambda, \\ k = n_E & : \frac{\partial f}{\partial \alpha_k} = \log e < \lambda, \\ n_E < k < n_A & : \frac{\partial f}{\partial \alpha_k} = \log \begin{bmatrix} k \\ n_E \end{bmatrix} \begin{bmatrix} \ell \\ n_E \end{bmatrix}^{-1} < \lambda, \\ k = n_A & : \frac{\partial f}{\partial \alpha_k} = \log \begin{bmatrix} \ell \\ n_A \end{bmatrix} \begin{bmatrix} n_A \\ n_E \end{bmatrix} \begin{bmatrix} \ell \\ n_E \end{bmatrix}^{-1} = \lambda. \end{cases} \quad (69)$$

So for the second case, we have

$$\begin{aligned} \max_{P_{\Pi_A}} I(\Pi_A; \Pi_i | \Pi_E) &= \lambda \\ &= \log \begin{bmatrix} \ell \\ n_A \end{bmatrix} \begin{bmatrix} n_A \\ n_E \end{bmatrix} \begin{bmatrix} \ell \\ n_E \end{bmatrix}^{-1} \\ &= \log \begin{bmatrix} \ell - n_E \\ n_A - n_E \end{bmatrix} \\ &\approx (n_A - n_E)(\ell - n_A) \log q. \end{aligned} \quad (70)$$

Combining the first and the second case we get the desired result, namely,

$$\max_{P_{\Pi_A}} I(\Pi_A; \Pi_i | \Pi_E) = (\min[n_A, n_i + n_E] - n_E)(\ell - \min[n_A, n_i + n_E]) \log q. \quad (71)$$

■

Proof of Lemma 6: Let us add $U_E \triangleq \Pi_E$ to $2^m - 1$ subspaces $U_{\mathcal{J}}$'s, where $\emptyset \neq \mathcal{J} \subseteq [1 : m]$. Then from the assumption of the lemma, for $k \in [1 : 2^m - 1]$ and any selection of subsets $\mathcal{J}_1, \dots, \mathcal{J}_k$, we have also

$$\begin{aligned} \theta_{\mathcal{J}_1} + \dots + \theta_{\mathcal{J}_k} &\leq \dim(U_{\mathcal{J}_1} + \dots + U_{\mathcal{J}_k} + \Pi_E) - \dim(\Pi_E) \\ &\leq \dim(U_{\mathcal{J}_1} + \dots + U_{\mathcal{J}_k}). \end{aligned} \quad (72)$$

Now by defining $\theta_E \triangleq \dim(\Pi_E)$, we can apply [17, Lemma 4] to the set of subspaces $U_{\mathcal{J}}$'s and U_E to show that there exist subspaces $U'_{\mathcal{J}} \subseteq U_{\mathcal{J}}$ such that $\dim(U'_{\mathcal{J}}) = \theta_{\mathcal{J}}$ for $\emptyset \neq \mathcal{J} \subseteq [1 : m]$, and $U'_E = \Pi_E$ where all of them are complementary. Note that in the above argument, we have $U'_E = \Pi_E$ because we set $\theta_E = \dim(\Pi_E)$ (which is an integer number). ■

Proof of Lemma 7: Let us assume that Alice has Π_E . Then she can create subspaces $U'_{\mathcal{J}}$'s such that by using Lemma 6, for $k \in [1 : 2^m - 1]$ and any selection of subsets $\mathcal{J}_1, \dots, \mathcal{J}_k$, we have

$$\dim(U'_{\mathcal{J}_1} + \dots + U'_{\mathcal{J}_k} + \Pi_E) = \theta_{\mathcal{J}_1} + \dots + \theta_{\mathcal{J}_k} + \dim(\Pi_E), \quad (73)$$

which means that

$$\theta_{\mathcal{J}_1} + \dots + \theta_{\mathcal{J}_k} + \dim(\Pi_E) \leq n_A. \quad (74)$$

Now, suppose that Alice does not have access to Π_E . From the proof of Lemma 6, we know that for any k and any subsets $\mathcal{J}_1, \dots, \mathcal{J}_k$ we have also

$$\theta_{\mathcal{J}_1} + \dots + \theta_{\mathcal{J}_k} \leq \dim(U_{\mathcal{J}_1} + \dots + U_{\mathcal{J}_k}), \quad (75)$$

so by using [17, Lemma 4] Alice can find subspaces $U'_{\mathcal{J}}$'s such that they are complementary and $\dim(U'_{\mathcal{J}}) = \theta_{\mathcal{J}}$ for every subset \mathcal{J} .

From Alice's point of view Π_E is chosen independently and uniformly at random from Π_A . So by (74) and applying Lemma 1, the subspace Π_E is complementary to all $U'_{\mathcal{J}}$'s w.h.p and we are done. ■

Proof of Lemma 8: Construct matrix Y that has as rows the packets y_1, \dots, y_k . Then note that we can write

$$\begin{bmatrix} Y \\ X_E \end{bmatrix} = \begin{bmatrix} A \\ F_E \end{bmatrix} X_A = \begin{bmatrix} A \\ F_E \end{bmatrix} \begin{bmatrix} I & M \end{bmatrix}, \quad (76)$$

where M is a uniformly random matrix and $A \in \mathbb{F}_q^{k \times n_A}$ is the coefficients of Y packets.

We now proceed by expanding $H(Y|X_E)$. We have

$$\begin{aligned} H(Y|X_E) &= H(Y, X_E) - H(X_E) \\ &= H(AM, F_E M) - H(F_E M) \\ &= [\text{rk}(B) - \text{rk}(F_E)] (\ell - n_A) \log q, \end{aligned} \quad (77)$$

where $B = \begin{bmatrix} A \\ F_E \end{bmatrix} \in \mathbb{F}_q^{n_A \times n_A}$. Now the only way that we have $H(Y|X_E) = H(Y)$ is that B becomes a full rank matrix.

Now, to prove the assertion of the lemma, we note that creating vectors y_i uniformly at random is equivalent to selecting the elements of matrix A independently and uniformly at random from the field \mathbb{F}_q . In this case we can write

$$\begin{aligned} \mathbb{P}[B \text{ is full-rank}] &= \frac{(q^{n_A} - q^{n_E}) \dots (q^{n_A} - q^{n_A-1})}{q^{n_A(n_A - n_E)}} \\ &= 1 - O(q^{-1}), \end{aligned} \quad (78)$$

which goes to 1 as q goes to infinity. ■

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