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ECPOL: equations and MATLAB tools for EC wave reflection and polarization calculations

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1 Introduction

This document is meant as supporting material for the ECPOL code. This code calculates the propagation of a polarized wave through a series of reflectors, of which some can be grooved mirrors altering the polarization of the wave. A wave is represented simply by a propagation vector, (complex-valued) electric field, and a frequency. A mirror is assumed infinitely large and only the normal vector defines its orientation. As such, effects such as finite mirror size, diffraction, losses, mode conversion, etc are not included. The code also includes some interface scripts to TCV equipment, including the X2 MOU, transmission lines, and launchers, and can be used to calculate the optimal MOU settings to obtain the required polarization at the plasma. Section 2 contains the main definitions and equations used in the code, and Section 3 gives a summary of the main functions. Most help is available through the matlab help interface directly.

2 Signs and definitions

All calculations are ultimately done in the *lab frame*, a fixed cartesian frame with unit vectors $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. All vectors are assumed to be in this frame, unless specified otherwise. All quantities are in SI units.

2.1 Electric field of a polarized wave

The plane propagating electromagnetic wave with wave vector $\mathbf{k} \in \mathbb{R}^3$ in $[1/\text{m}]$ has its electric field varying in space and time as $\mathbf{E}(\mathbf{r}, t) = \text{Re}\{\hat{\mathbf{E}} \exp(i(\mathbf{k} \cdot \mathbf{r} - \omega t))\}$. where $\hat{\mathbf{E}} \in \mathbb{C}^3$. Considering a fixed point in space this gives $\mathbf{E}(t) = \text{Re}\{\hat{\mathbf{E}} \exp(-i\omega t)\}$. This can be written explicitly as

$$\mathbf{E}(t) = \text{Re}\{\hat{\mathbf{E}}\} \cos(\omega t) + \text{Im}\{\hat{\mathbf{E}}\} \sin(\omega t)$$

This shows that the electric field vector will lie in a plane spanned by the two vectors $\{\text{Re}\{\hat{\mathbf{E}}\}, \text{Im}\{\hat{\mathbf{E}}\}\}$. These vectors are orthogonal to \mathbf{k} . Now assume for simplicity that $\mathbf{k} \parallel \mathbf{e}_z$ such that the \mathbf{E} vector lies in the xy plane. We can then write each of the complex x and y components in polar form as $\hat{\mathbf{E}}$ as $\hat{E}_x = |\hat{E}_x| e^{i\delta_x}$ and $\hat{E}_y = |\hat{E}_y| e^{i\delta_y}$. We now multiply both components by $e^{-i\delta_x}$. This has no global effect on the electric field since it represents an equal phase shift of the both components of the wave electric field vector. Defining $\delta = \delta_x - \delta_y$, $\hat{\mathbf{E}}$ is now given by $\hat{\mathbf{E}} = |\hat{E}_x| \mathbf{e}_x + |\hat{E}_y| e^{-i\delta} \mathbf{e}_y$. The temporal evolution of $\mathbf{E}(t)$ can then be written as

$$\mathbf{E}(t) = |\hat{E}_x| \cos(\omega t) \mathbf{e}_x + |\hat{E}_y| \cos(\omega t + \delta) \mathbf{e}_y$$

This equation describes an ellipse in the xy plane.

It can be seen that for $\delta < 0$, the E_y component *trails* the E_x component in time. The electric field thus moves from the x to the y direction, or in the right-hand direction with respect to \mathbf{k} . In the opposite case, for $\delta > 0$ the E_y component *leads* the E_x component, so the electric field moves in the left-hand sense with respect to \mathbf{k} .

It should be noted that this definition is the one used in [1] but is opposite to the one used in [2] which follows a different convention due to the historical evolution of optics. The notation chosen here is commonly used in plasma physics, where a wave propagating in the same direction as the static magnetic field is referred to as the R wave when it is right handed with respect to \mathbf{k} (and \mathbf{B}).

2.2 Polarization angles

One can also define polarization angles (α, β) defining the characteristics of the ellipse. The linear polarization angle α is defined as the inclination angle of the polarization ellipse with respect to the x axis. Following [2, Ch.1.4]¹ one can define the ratio of the magnitude of the field components $r = \frac{|E_y|}{|E_x|}$. The polarization angle $-\pi/2 \leq \alpha \leq \pi/2$ is defined as the orientation of the main axis of the ellipse with respect to \mathbf{e}_x . It can be shown that

$$\tan(2\alpha) = \frac{2r}{1-r^2} \cos(\delta)$$

Optionally, introducing the auxiliary angle $\chi = \arctan(r)$ one obtains the expression $\tan(2\alpha) = \tan(2\chi) \cos(\delta)$ which is also often encountered in the literature.

The elliptical polarization angle β is the arctangent of the ratio of minor to major axis of the ellipse. Its sign defines the polarization direction.

$$\sin(2\beta) = -\sin(2\chi) \sin(\delta) = -\frac{2r}{1+r^2} \sin(\delta)$$

since $r > 0$ clearly β and δ have opposite sign.

With these definitions we can summarize:

¹Note the sign difference: [2] uses the “traditional” notation rather than the “natural” notation for the polarization used here. The relation between α, β and the corresponding angles (called (ψ, χ) in [2]) is $\alpha \leftrightarrow \psi, \beta \leftrightarrow -\chi$

- *Right handed* polarization (R) is right handed with respect to \mathbf{k} , has $\delta < 0$ and $\beta > 0$. Looking from a position fixed in space, towards the direction of propagation of the wave (in \mathbf{k} direction), the \mathbf{E} vector rotates CW in time. Right handed perfectly circular polarization has $\delta = -\pi/2$, $\beta = 45^\circ$, $iE_x/E_y = 1$ (α can not be defined for circular polarization)
- *Left handed* polarization (L) is left handed with respect to \mathbf{k} , has $\delta > 0$ and $\beta < 0$. Looking from a position fixed in space, towards the direction of propagation of the wave (i.e. in the \mathbf{k} direction) the \mathbf{E} vector rotates CCW in time. Left handed perfectly circular polarization has $\delta = \pi/2$, $\beta = -45^\circ$, $iE_x/E_y = -1$ (α can not be defined for circular polarization)

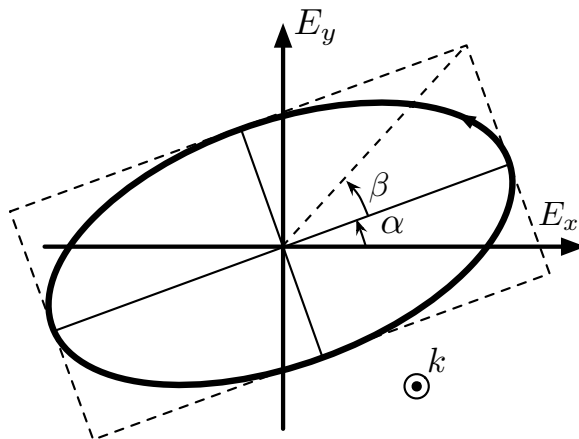


Figure 1: Trajectory swept out by electric field vector and definitions of polarizer angles. The direction of the electric field shown here corresponds to an R wave.

2.3 Reflection off a corrugated mirror

The polarization of the wave is altered upon reflection off a reflecting surface. A model will now be derived to calculate the effect of a general corrugated mirror with rectangular grooves on the electric field of the wave.

A (corrugated) mirror is defined by the following parameters.

- The normal vector \mathbf{n} points *into* the plane of the mirror and is orthogonal to its surface.
- The properties of the grooves are defined with the parameters h , c and d . Respectively the height, width and period of the grooves. If $h = 0$ the other parameters have no effect.

The effect of the grooved mirror depends on the above parameters, but also on the incidence angle of the incoming wave with respect to the groove direction. Given the incoming wave vector \mathbf{k}_{in} , the outgoing wave vector \mathbf{k}_{out} follows from geometrical optics as

$$\mathbf{k}_{out} = \mathbf{k}_{in} - 2(\mathbf{k}_{in} \cdot \mathbf{n})\mathbf{n}$$

The orientation of the grooves is defined with respect to the $\mathbf{n} \times \mathbf{k}_{in}$ vector:

- The rotation angle of the polarizer ϕ is defined as positive in the right-handed sense with respect to \mathbf{n} (which points into the mirror)
- $\phi = 0$ when the grooves are aligned with $\mathbf{n} \times \mathbf{k}$.

With knowledge of \mathbf{n} and $\mathbf{n} \times \mathbf{k}$, using these definitions, one can calculate the vector \mathbf{g} which is orthogonal to \mathbf{n} and aligned with the grooves.

$$\mathbf{g} = (\mathbf{n} \times \frac{\mathbf{k}}{|\mathbf{k}|}) \cos(\phi) + (\mathbf{n} \times (\mathbf{n} \times \frac{\mathbf{k}}{|\mathbf{k}|})) \sin(\phi)$$

a *local coordinate system* $\{x_m, y_m, z_m\}$ is defined for the mirror. z_m is aligned with the grooves, y_m is aligned with $-\mathbf{n}$ and x_m is chosen to make the system right handed

$$\mathbf{R} = (\mathbf{g} \times \mathbf{n} \quad -\mathbf{n} \quad \mathbf{g})^T$$

such that

$$\begin{pmatrix} x_m \\ y_m \\ z_m \end{pmatrix} = \mathbf{R} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}$$

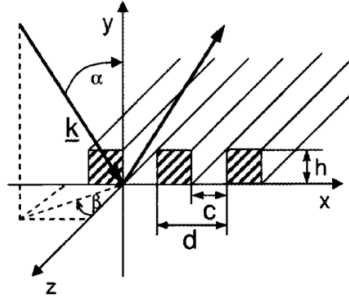


Figure 2: Local coordinates for polarizer element, from [3]

In this local coordinate system, the effect of the polarizer on the wave electric field is calculated by defining the matrix [3]

$$\bar{\mathbf{P}} = \begin{pmatrix} -s_o \frac{k_y^2}{k_x^2 + k_y^2} & -s_o \frac{k_x k_y}{k_x^2 + k_y^2} & -r_o \frac{k_x k_z}{k_x^2 + k_y^2} \\ s_o \frac{k_x k_y}{k_x^2 + k_y^2} & s_o \frac{k_x^2}{k_x^2 + k_y^2} & -r_o \frac{k_y k_z}{k_x^2 + k_y^2} \\ 0 & 0 & r_o \end{pmatrix}$$

where

$$r_o = -e^{-2ik_y h}$$

and

$$s_o = \frac{-\sqrt{k_x^2 + k_y^2} \cdot \frac{c}{d} \cdot \tan(\sqrt{k_x^2 + k_y^2} \cdot h) + ik_y}{\sqrt{k_x^2 + k_y^2} \cdot \frac{c}{d} \cdot \tan(\sqrt{k_x^2 + k_y^2} \cdot h) + ik_y} e^{-2ik_y h}$$

the relation between the incident and reflected electric field vectors is then given by

$$\hat{\mathbf{E}}_{r,m} = \bar{\mathbf{P}} \hat{\mathbf{E}}_{i,m}$$

. To verify this relation, note that for perpendicular incidence ($k_x = k_z = 0$) we obtain

$$\bar{\mathbf{P}} = \begin{pmatrix} -s_o & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_o \end{pmatrix}$$

Now examine some particular cases of polarizing mirrors

- For the case of a flat mirror $h = 0$ and we get $r_o = -1$, $s_o = 1$ which gives $\hat{\mathbf{E}}_r = -\hat{\mathbf{E}}_i$ as one would expect.
- For an ideal half-wave plate we know that $h = \lambda/4 = \frac{\pi}{2k}$ so $k_y h = \pi/2$. In this limit $r_o = 1$ and $s_o = 1$ which shows that only the x component of the E field is flipped.
- Finally for an ideal quarter-wave plate $k_y h = \pi/4$ in which case $r_o = i$ and $s_o = 1$. This results in a reflected field $E_{r,x} = -E_{i,x}$, $E_{r,z} = iE_{i,z}$, a phase shift of 90° .

Since the electric field vectors in this case are in the assumed in the local coordinate frame, the complete expression relating the incident and reflected electric field in the lab frame is given by

$$\hat{\mathbf{E}}_r = (\mathbf{R}^T \bar{\mathbf{P}} \mathbf{R}) \hat{\mathbf{E}}_i = \mathbf{P} \hat{\mathbf{E}}_i$$

$\mathbf{P} = \mathbf{R}^T \bar{\mathbf{P}} \mathbf{R}$ describes the total effect of the reflecting element on the wave electric field in the lab frame.

To simulate the effect of multiple elements, one needs to know for each element: \mathbf{n} , h , and if $h \neq 0$ also d and c . Using these parameters the procedure is as follows:

- Starting from the first element along the beam path, with knowledge of \mathbf{n} and \mathbf{k}_{in} one calculates \mathbf{k}_{out} and $\hat{\mathbf{E}}_r$
- set $\mathbf{k}_{in} = \mathbf{k}_{out}$, $\hat{\mathbf{E}}_i = \hat{\mathbf{E}}_r$ and continue to the next element.

2.4 Coupling to a Tokamak plasma

The coupling to the plasma LCFS can be calculated using equations from [1]. Defining a local cartesian coordinate system with k in the z direction and \mathbf{B} in the x - z plane.

$$\frac{iE_p}{E_y} = \frac{\Omega_e}{\omega} \left\{ \frac{\pm[\sin^4 \theta + 4 \cos^2 \theta / (\Omega_e / \omega)^2]^{1/2} - \sin^2 \theta}{2 \cos \theta} \right\}$$

here θ is the angle between the wave vector \mathbf{k} and the magnetic field at the last closed flux surface. E_p is the electric field in the plane containing \mathbf{k} and \mathbf{B} , E_y is the electric field perpendicular to \mathbf{k} and \mathbf{B} , Ω_e is the (positive) electron cyclotron frequency $|eB/m_e|$ at the local magnetic field strength and ω is the frequency of the wave. The upper sign is for X mode and the lower sign for O mode.

Distinguish the following special cases

- In the case where $\mathbf{k} \cdot \mathbf{B} = |\mathbf{k}||\mathbf{B}|$, $\theta = 0$ and one has for X mode $\frac{iE_p}{E_y} = 1$ so $E_y = iE_p$ which describes a right-handed wave with $\beta > 0$ according to the definitions above, gyrating in the same sense as the electrons.
- For $\mathbf{k} \cdot \mathbf{B} = |\mathbf{k}||\mathbf{B}|$ in O mode one has $\frac{iE_p}{E_y} = -1$ which is a left-handed wave with $\beta < 0$.
- Conversely, if $\mathbf{k} \cdot \mathbf{B} = -|\mathbf{k}||\mathbf{B}|$, $\theta = \pi$ yields $\frac{iE_p}{E_y} = -1$ for X mode and a left handed wave with $\beta < 0$
- Again, if $\mathbf{k} \cdot \mathbf{B} = -|\mathbf{k}||\mathbf{B}|$, $\theta = \pi$ yields $\frac{iE_p}{E_y} = 1$ for O mode and a right handed wave with $\beta > 0$
- Choosing $\mathbf{k} \perp \mathbf{B}$ one gets for X mode $\lim_{\theta \rightarrow \pi/2} \frac{iE_p}{E_y} = 0$ so the field is only in the y direction, perpendicular to \mathbf{B} and \mathbf{k} .
- Finally for O mode with $k \perp \mathbf{B}$, $\lim_{\theta \rightarrow \pi/2} \frac{iE_p}{E_y} = \infty$ giving a field in the E_p direction i.e. parallel to \mathbf{B}

The sign of the required β is flipped in any of these cases

- Change sign of B_ϕ
- Flip the toroidal injection angle (if the magnitude of the toroidal angle is large enough so that $\mathbf{k} \cdot \mathbf{B}$ flips sign)

When switching from X to O mode, α is rotated by 90° and β changes sign.

3 ECPOL

A suite of matlab routines has been written implementing the expressions above. They are hosted on the CRPP SVN <https://crppsvn.epfl.ch/repos/TCV/ECPOL/trunk/>

A quick summary of the main functions is given below. More detailed information is given in the help of each routine. Two classes are defined: “mirrors” and “waves”. Mirrors are defined in terms of their normal vector \mathbf{n} and groove properties, while waves are defined in terms of their wave vector \mathbf{k} and complex electric field vector \mathbf{E} (see above).

3.1 General functions

- `mirror` used to define a “mirror” object, including grooved polarizers with all its properties. Polarizer mirrors can be defined in three ways: as rectangular corrugations (described in this paper), as a phase delay expressed as a sum of cosines, or using a separate FORTRAN code by T-M. Tran for arbitrary groove shapes.
- `wave` used to define a “wave” object and its properties.
- `pos2mirrors` Define mirrors based on their positions rather than normal vectors
- `reflect` function that takes a mirror and a wave objects as input, calculates the reflection of the input wave by the mirror object and returns the wave object corresponding to the output wave.
- `propagate_wave` propagates a wave through an array of mirrors.
- `ox_coupling` Calculates the wave required at last closed flux surface for perfect coupling to X and O mode, given the local \mathbf{k} and \mathbf{B} vectors.
- `wave_coupling` Power coupling between waves of different polarization.
- `pol3d` Calculates the polarization parameters α, β for a given wave. One must specify the direction of the local \hat{y} explicitly.
- `p2ab` Same as `pol3d` but for 2D vectors. One must take great care that the input matches what is described in the help.
- `p2Ed` Calculates the absolute values of $|\hat{E}|$ and the phase delay δ for a given
- `draw_mirror` Plots a graphical representation of a mirror using matlab 3D plotting functions.
- `draw_wave` Plots a graphical representation of a wave and the ellipse described by its electric field using matlab 3D plotting functions.
- `alpha_beta_map` General tool to draw maps of α, β as a function of polarizer angles.
- `ECPOL_demo` Some illustrative demonstrations

3.2 TCV specific functions

- `ox_lau` Computes the wave that would be required at the launcher entrance for perfect X and O mode coupling using knowledge of the launcher angles and the plasma equilibrium to determine the LCFS intersection point.
- `sim_TCV_TL` Computes the wave at the launcher entrance from given MOU angles, simulates the TCV MOUs and transmission lines.
- `optimize_mou` Computes the optimal angles for the MOU for a given equilibrium
- `ox_fraction` Fraction coupling to O or X mode for a given shot
- `TCV_TL_setup` TCV transmission line setup
- `def_launcher` launcher geometry definitions
- `reqpol` Required polarization angles for a given shot and time. For backwards compatibility, superseded by `optimize_mou`
- `find_mou_angles` Find MOU angles for a give polarization at launcher entrance. For backwards compatibility, superseded by `optimize_mou`

4 Acknowledgements

Some of this work uses previous work and codes by T. Goodman and M. Henderson. In some cases, the routines may call the “grating” code developed and maintained by Trach-Minh Tran.

References

- [1] T.H. Stix. *Waves in plasmas*. Springer, New York, 1992.
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