# Wireless Network Simplification: the Gaussian $N$-Relay Diamond Network

Caner Nazaroglu
Middle East Technical University
Ankara, Turkey
e155529@metu.edu.tr

Ayfer Özgür, Christina Fragouli
EPFL, Lausanne
Switzerland
{ayfer.ozgur, christina.fragouli}@epfl.ch

***Abstract*—We consider the Gaussian $N$-relay diamond network, where a source wants to communicate to a destination node through a layer of $N$-relay nodes. We investigate the following question: What fraction of the capacity can we maintain by using only $k$ out of the $N$ available relays? We show that in every Gaussian $N$-relay diamond network, there exists a subset of $k$ relays which alone provide approximately $\frac{k}{k+1}$ of the total capacity. The result holds independent of the number of available relay nodes $N$, the channel configurations and the operating SNR. The result is tight in the sense that there exists channel configurations for $N$-relay diamond networks, where every subset of $k$ relays can provide at most $\frac{k}{k+1}$ of the total capacity. The approximation is within $3 \log N + 3k$ bits/s/Hz to the capacity.**

**This result also provides a new approximation to the capacity of the Gaussian $N$-relay diamond network which is up to a multiplicative gap of $\frac{1}{k+1}$ and additive gap of $3 \log N + 3k$. The current approximation results in the literature either aim to characterize the capacity within an additive gap by allowing no multiplicative gap or vice a versa. Our result suggests a new approximation approach where multiplicative and additive gaps are allowed simultaneously and are traded through an auxiliary parameter.[1]**

## I. Introduction

Consider a source that communicates to a destination with the help of relays in a wireless Gaussian network. The question we ask in this paper is, can we simplify the network by removing a (significant) number of the relays, while maintaining (a good part of) the capacity?

There are a number of important motivations, both practical and fundamental, to consider this question. Information theory traditionally aims to characterize the best communication rate we can achieve by optimally utilizing a set of available relay nodes [1], [3], [2], [4], [5]. However, complexity constraints in practice often limit the number of relay nodes we can employ in our relaying strategy. This necessitates to understand how closely we can achieve the capacity of the wireless network by using only a (small) subset of (perhaps a large number of) available relay nodes, for various network topologies. On the other hand, wireless networks are characterized by limited resources such as battery life, power and bandwidth. We can optimally utilize these resources if we know how much each relay node contributes to the end-to-end capacity. This introduces the notion of "capacity per relay use" as opposed to the traditional capacity notion, a better understanding of which can allow efficient resource utilization in wireless networks.

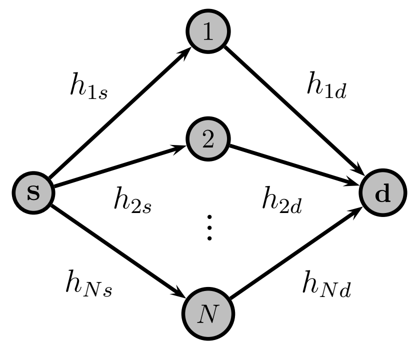


Fig. 1. The Gaussian $N$-relay diamond network. The source is connected to the relays through a broadcast channel, while the relays are connected to the destination through a multiple-access channel.

In this paper we consider a source that communicates to a destination over the Gaussian N-relay diamond network depicted in Fig. 1. This is a two-stage network, where the source node is connected to $N$ relays through a broadcast channel and the relays are connected to the destination through a multiple-access channel. The question we are asking is, what is the loss in the capacity if we simplify the network by removing all but $k$ of the $N$ relays (for example, if we remove all but one of the relays).

The performance loss can depend on the channel gains. Indeed, consider for example the diamond network ($N = 2$), and the example in Fig. 2. For the identical channel gains in Fig. 2(a) we can show that the communication rate achieved using only one of the relays is only 1 bit/s/Hz away from the cut-set upper bound on the capacity of the network; while for the anti-symmetrical channel gains as in Fig. 2(b) using only one of the relays achieves (within 1 bit/s/Hz) only *half* of the cutset upper bound on the capacity of the network.

To avoid channel-specific results, we can try to find what is the *largest* capacity loss, universally for all possible channel gains. For example, is it possible that, we can always find a single relay to use and still achieve *half* of the capacity of the diamond network within 1 bit/s/Hz (as was the case for the two examples in Fig. 2). We prove in this paper that this is indeed always the case. In fact, we show that even if we have an arbitrary number $N$ of relays, we can remove all but one

[1]This work was supported by the ERC Starting Grant Project NOWIRE ERC-2009-StG-240317.

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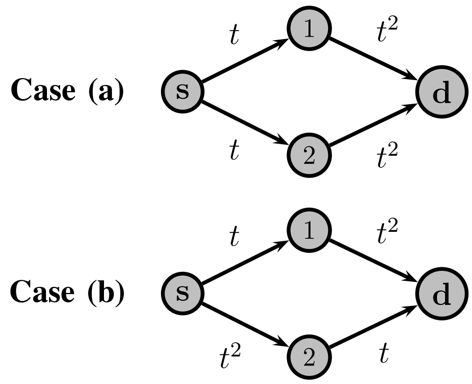


Fig. 2. Two instantiations of a diamond relay network.

of them and still achieve approximately half of the capacity.

Our main result is to show that in every Gaussian $N$-relay diamond network, there exist a $k$-relay sub-network whose capacity $C_k$ satisfies

$$C_k \geq \frac{k}{k+1}\overline{C} - G \tag{1}$$

where $\overline{C}$ is the cut-set upper bound on the capacity of the $N$-relay diamond network and $G = \max\left(3\log N - \log\frac{27}{4}, 2\log N\right)$ is a universal constant independent of the channel gains and the operating SNR. Intuitively, this holds because if all $k$-relay subnetworks have small capacity, the capacity of the whole network cannot be too large. As $k$ increases, the difference between the capacity of the best $k$-relay subnetwork and that of the whole network naturally decreases. The surprising and interesting outcome here is that the fraction of the capacity we can get with $k$ relays is independent of the number of available relay nodes $N$. Moreover, it increases quite quickly with $k$: in the high-capacity regime, we can get at least half-the capacity of every $N$-relay diamond network by simply routing information over the best relay, using 2 relays we achieve a fraction of $2/3$, etc.

We also show that the lower bound in (1) is tight in the multiplicative fraction, i.e., it is possible to find $N$-relay diamond networks where the capacity of *every* $k$-relay sub-diamond network is at most $C_k \leq \frac{k}{k+1}C + G'$, where $C$ is the capacity of the whole network and $G'$ is a constant linear in $k$ and independent of everything else. For the case $k = 1$ and $N = 2$, one such example is in Fig. 2 case (b).

## II. Related Work

Two lines of work have previously looked at a form of network simplification for wireless network. First, relay selection techniques in [10], [11], [12], design practical algorithms that allow to select the best single relay in an N-relay diamond network, and show that such algorithms provide cooperative diversity. Second, work in [8], [13], [14], [15] looks at selecting a subset of the best relays when restricted to utilize an amplify and forward strategy. Our work differs in that we do not restrict our attention to specific strategies (or number of relays) but instead provide universal capacity results for arbitrary strategies.

Our result can also be regarded as a new approximation to the capacity $C$ of the Gaussian $N$-Relay diamond network. We show that

$$\frac{k}{k+1}\overline{C} - 3k - \frac{k}{k+1}G \leq C \leq \overline{C} \quad \forall k,\ 1 \leq k \leq N-1, \tag{2}$$

where $\overline{C}$ denotes the cut-set upper bound. The earlier approximation results in [4], [7] yield

$$\overline{C} - 3N \leq C \leq \overline{C}. \tag{3}$$

for the $N$-Relay diamond network. Note that the lower bound we provide in (2) is tighter than (3) in the regime where $N$ is large. The auxiliary parameter $k$ in (2) allows us to optimize our lower bound as a function of $C$ and $N$ to make it tightest in different regimes. When $N$ is large, choosing a small $k$ reduces the additive gap from $O(N)$ in (3) to $O(\log N)$. When $C$ is large and $N$ is small increasing $k$ to $N$ we recover (3). Our result also suggests a new approximation approach to wireless networks where multiplicative and additive gaps are allowed simultaneously and are traded through an auxiliary parameter (in our case $k$). Earlier works in the literature either aim to characterize the capacity within an additive gap by allowing no multiplicative gap [4], [7], or vice-a-versa [8].

## III. Model

We consider the Gaussian $N$-relay diamond network depicted in Fig. 1 where the source node $s$ wants to communicate to the destination node $d$ with the help of $N$ relay nodes. Let $X_s[t]$ and $X_i[t]$ denote the signals transmitted by the source node $s$ and the relay node $i \in \{1, \ldots, N\}$ respectively at time instant $t \in \mathbb{N}$. Let $Y_d[t]$ and $Y_i[t]$ denote the signals received by the destination node $d$ and the relay node $i \in \{1, \ldots, N\}$ respectively at time instant $t$. The transmitted signal $X_i[t]$ by relay $i$ is a causal function of the corresponding received signal $Y_i[t]$. The received signals relate to the transmitted signals as

$$Y_i[t] = h_{is}X_s[t] + Z_i[t],$$
$$Y_d[t] = \sum_{i=1}^{N} h_{id}X_i[t] + Z[t],$$

where $h_{is}$ denotes the complex channel coefficient between the source node and the relay node $i$ and $h_{id}$ denotes the complex channel coefficient between the relay node $i$ and the destination node. $Z_i[t]$, $i = 1, \ldots, N$ and $Z[t]$ are independent and identically distributed white Gaussian random processes of power spectral density of $N_0/2$ Watts/Hz. All nodes are subject to an average power constraint $P$ and the narrow-band system is allocated a bandwidth of $W$. Note that the equal power constraint assumption is without loss of generality as the channel coefficients are arbitrary. We assume that the channel coefficients are known at all the nodes.

## IV. Main Result

The main result of this paper is summarized in the following two theorems.

*Theorem 1:* In every Gaussian $N$-relay diamond network, there exists a subset of $k$ relays, such that the capacity $C_k$

of the corresponding Gaussian $k$-relay diamond sub-network satisfies

$$C_k \geq \frac{k}{k+1}\overline{C} - 3k - \frac{k}{k+1}\max\left(3\log N - \log\frac{27}{4}, 2\log N\right), \quad (4)$$

where $\overline{C}$ denotes the cut-set upper bound on the capacity of the $N$-relay network. Moreover, there exist configurations of Gaussian $N$-relay diamond networks such that the capacity of *every* $k$-relay sub-network satisfies,

$$C_k \leq \frac{k}{k+1}C + 3k + \max\left(3\log k - \log\frac{27}{4}, 2\log k\right), \quad (5)$$

where $C$ is the capacity of the $N$-relay network.

*Remark 1:* For the case $k = 1$, we have the following tighter bound,

$$C_1 \geq \frac{1}{2}\overline{C} - \frac{1}{2}\max\left(3\log N - \log\frac{27}{4}, 2\log N\right).$$

This result is shown in two steps: We first show that in every Gaussian $N$-Relay diamond network, there exists a subset of $k$-relay nodes such that the cut-set upper bound on the capacity of the corresponding $k$-relay sub-network is larger than $\frac{k}{k+1}\overline{C} - G$. This step only involves the cut-set upper bound on the capacities of the corresponding networks. The second step uses the approach in [4], [7]. Namely, performing the quantize-map-and-forward strategy of [4], [7] with these $k$ relay nodes and keeping the remaining $N-k$ relays silent, we can achieve the cut-set upper bound of the $k$-relay network within $3k$ bits/s/Hz.

## V. Approximating the Cut-Set Upper Bound

In this section we derive upper and lower bounds on the cut-set upper bound, that essentially reduce calculating its value to a combinatorial problem.

Let $[N] \doteq \{1, 2, \cdots, N\}$ and for a subset $\Lambda \subseteq [N]$, $\overline{\Lambda} \doteq [N] \setminus \Lambda$. By the cut-set upper bound [6, Theorem 14.10.1], the capacity $C$ of the network is upper bounded by,

$$C \leq \overline{C} \doteq \max_{X_s, X_1, \ldots, X_N} \min_{\Lambda \subseteq [N]} I(X_s, X_\Lambda; Y_d, Y_{\overline{\Lambda}} \,|\, X_{\overline{\Lambda}}), \quad (6)$$

where the maximization is over the joint probability distribution of the random variables $X_s$ and $X_1, \ldots, X_N$ satisfying the power constraint $P$. For a set $S \subseteq [N]$, $X_S$ denotes the corresponding collection of random variables, i.e $X_S \doteq \{X_i\}_{i\in S}$.

### A. An Upper Bound for the Cut-Set Upper Bound

The cut-set upper bound in (6) can be upper bounded by exchanging the order of maximization and minimization in (6). For each cut $\Lambda$, the resulting maximization of the mutual information can be upper bounded by the capacities of the SIMO (single input multiple output) channel between $s$ and nodes in $\overline{\Lambda}$ and the MISO (multiple input single output) channel between nodes in $\Lambda$ and $d$. We have,

$$\begin{aligned}\overline{C} &\leq \min_{\Lambda\subseteq[N]} \sup_{X, X_\Lambda, X_{\overline{\Lambda}}} I(X, X_\Lambda; Y, Y_{\overline{\Lambda}} \,|\, X_{\overline{\Lambda}}) \\ &= \min_{\Lambda\subseteq[N]} \sup_{X} I(X; Y_{\overline{\Lambda}}) + \sup_{X_\Lambda} I(X_\Lambda; \sum_{i\in\Lambda} h_{id}X_i + Z), \\ &\leq \min_{\Lambda\subseteq[N]} C_{SIMO}(s; \overline{\Lambda}) + C_{MISO}(\Lambda; d).\end{aligned}$$

The capacities of the corresponding SIMO and MISO channels are well-known [9]. Plugging these expressions yields

$$\overline{C} \leq \min_{\Lambda\subseteq[N]} \log\Big(1 + \mathrm{SNR}\sum_{i\in\overline{\Lambda}} |h_{is}|^2\Big) + \log\Big(1 + \mathrm{SNR}\big(\sum_{i\in\Lambda} |h_{id}|\big)^2\Big), \quad (7)$$

where $\mathrm{SNR} \doteq \frac{P}{N_0 W}$. We will further develop a trivial upper bound on this expression by setting each summand in the above summations to the maximum of the variables that are summed. This gives us the upper bound,

$$\overline{C} \leq \min_{\Lambda\subseteq[N]} \left(\max_{i\in\Lambda} R_{id} + \max_{i\in\overline{\Lambda}} R_{is}\right) + G, \quad (8)$$

where $R_{id} = \log\left(1 + \mathrm{SNR}\,|h_{id}|^2\right)$ and $R_{is} = \log\left(1 + \mathrm{SNR}\,|h_{is}|^2\right)$ are the capacities of the corresponding point-to-point channels and

$$G \doteq \max\left(3\log N - \log\frac{27}{4}, 2\log N\right).$$

A detailed derivation of this upper bound can be found in [16].

### B. A Lower Bound on the Cut-Set Upper Bound

Consider a subset $\Gamma \subseteq [N]$ of the relay nodes such that $|\Gamma| = k$. Let $C_\Gamma$ be the capacity of the $k$-relay diamond sub-network where the source node $s$ wants to communicate to the destination node $d$ with the help of these $k$ relay nodes. The rest $N-k$ relay nodes are not used. The cut-set upper bound on the capacity of the $k$-relay network yields

$$C_\Gamma \leq \overline{C}_\Gamma \doteq \sup_{X, X_\Gamma} \min_{\Lambda\subseteq\Gamma} I(X, X_\Lambda; Y, Y_{\overline{\Lambda}} \,|\, X_{\overline{\Lambda}}), \quad (9)$$

where we slightly abuse notation by assuming that $\overline{\Lambda} = \Gamma \setminus \Lambda$ when $\Lambda \subseteq \Gamma$. The cut-set upper bound $\overline{C}_\Gamma$ above can be lower bounded by choosing $X, \{X_i\}_{i\in\Gamma}$ to be independent circularly-symmetric Gaussian random variables of variance $P$, in which case

$$\begin{aligned}&I(X, X_\Lambda; Y, Y_{\overline{\Lambda}} \,|\, X_{\overline{\Lambda}}) \\ &\quad = \log\Big(1 + \mathrm{SNR}\sum_{i\in\Lambda} |h_{id}|^2\Big) + \log\Big(1 + \mathrm{SNR}\sum_{i\in\overline{\Lambda}} |h_{is}|^2\Big).\end{aligned}$$

Retaining only the maximum terms in the summations, we obtain

$$\overline{C}_\Gamma \geq \min_{\Lambda\subseteq\Gamma} \left(\max_{i\in\Lambda} R_{id} + \max_{i\in\overline{\Lambda}} R_{is}\right). \quad (10)$$

Note that for $\Gamma = [N]$, this lower bound for $\overline{C}$ differs from the upper bound in (8) only by the gap term $G$. This implies that

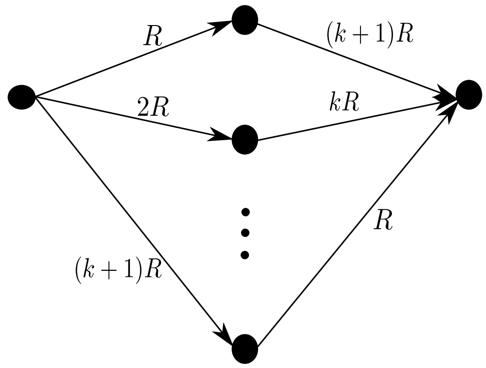


Fig. 3. A $(k+1)$-relay diamond network where every subset of $k$ relays achieve approximately $\frac{k}{k+1}$ of the capacity. The labels indicate the capacity of the corresponding links.

within a factor of $G$, the cut-set upper bound on the network capacity behaves like the lower bound in (10). This simpler form of the cut-set upper bound in terms of the point-to-point capacities of the individual channels is easier to work with and allows us to express our main problem in a combinatorial form in the next section.

Among all $\Gamma \subseteq [N]$ with $|\Gamma| = k$, consider the one that has largest cut-set upper bound $\overline{C}_\Gamma$. Let $\overline{C}_k$ denote the cut-set upper bound on the capacity of this best $k$-relay sub-network,

$$\overline{C}_k = \max_{\substack{\Gamma \subseteq [N] \\ |\Gamma| = k}} \overline{C}_\Gamma. \tag{11}$$

Combining (10) and (11), we have

$$\overline{C}_k \geq \max_{\substack{\Gamma \subseteq [N] \\ |\Gamma| = k}} \min_{\Lambda \subseteq \Gamma} \left( \max_{i \in \Lambda} R_{id} + \max_{i \in \overline{\Lambda}} R_{is} \right). \tag{12}$$

## VI. $k$ Relays Approximately Achieve $\frac{k}{k+1}$ Fraction of the Capacity

In this section, we prove the Theorem 1. The proof is based on the following two technical lemmas.

*Lemma 1:* Let $R_{id}$ and $R_{is}$ be arbitrary positive real numbers for $i = 1, 2, \cdots, N$. For $k \in [N]$, let

$$r_k \doteq \frac{\max\limits_{\substack{\Gamma \subseteq [N] \\ |\Gamma| = k}} \min\limits_{\Lambda \subseteq \Gamma} \left( \max\limits_{i \in \Lambda} R_{id} + \max\limits_{i \in \overline{\Lambda}} R_{is} \right)}{\min\limits_{\Lambda \subseteq [N]} \left( \max\limits_{i \in \Lambda} R_{id} + \max\limits_{i \in \overline{\Lambda}} R_{is} \right)}. \tag{13}$$

Then, $r_k \geq \frac{k}{k+1}$.

*Lemma 2:* Let $R_{is} = i\,R$ and $R_{id} = (k+2-i)\,R$ for every $i \in [k+1]$ where $R$ is an arbitrary positive number. Then,

$$r_k = \frac{k}{k+1}.$$

The configuration in Lemma 2 is depicted in Fig. 3.

*Proof of Theorem 1:* From (8) and (12), we have

$$\frac{\overline{C}_k}{\overline{C} - G} \geq r_k.$$

Combining this with the result of Lemma 1, we obtain

$$\overline{C}_k \geq \frac{k}{k+1}\overline{C} - \frac{k}{k+1}G. \tag{14}$$

This proves that in every $N$ relay diamond network, there exists a subset of $k$ relays, such that the cut-set upper bound on the capacity of the corresponding $k$ relay subnetwork is lower bounded by approximately a fraction $\frac{k}{k+1}$ of the cut-set upper bound on the capacity of the whole network. Let $C_k$ be the actual capacity of this $k$-relay sub-network, i.e. the maximizing term in (11). It is shown in [7] that $C_\Gamma \geq \overline{C}_\Gamma - 3k$, for any $k$-relay network via demonstrating that the quantize-map-and-forward strategy of [4] is able to achieve this rate. Therefore,

$$C_k \geq \overline{C}_k - 3k.$$

Together with (14) this yields the result (4) in Theorem 1.

Next, in order to prove the existence of a diamond N-relay network satisfying the inequality (5) for each of its k-relay subnetworks, we require an upper bound on $\overline{C}_k$ and a lower bound on $C$. The lower bound on $C$ follows by applying (10) for $\Gamma = [N]$ to obtain

$$\overline{C} \geq \min_{\Lambda \subseteq [N]} \left( \max_{i \in \Lambda} R_{id} + \max_{i \in \overline{\Lambda}} R_{is} \right), \tag{15}$$

and $C \geq \overline{C} - 3(k+1)$ by [7]. On the other hand, applying (8) for $\Gamma \subseteq [N]$, we obtain

$$\overline{C}_\Gamma \leq \min_{\Lambda \subseteq \Gamma} \left( \max_{i \in \Lambda} R_{id} + \max_{i \in \overline{\Lambda}} R_{is} \right) + G_k,$$

where $G_k \doteq \max\left(3\log k - \log \frac{27}{4}, 2\log k\right)$. Therefore,

$$\overline{C}_k \leq \max_{\substack{\Gamma \subseteq [N] \\ |\Gamma| = k}} \min_{\Lambda \subseteq \Gamma} \left( \max_{i \in \Lambda} R_{id} + \max_{i \in \overline{\Lambda}} R_{is} \right) + G_k. \tag{16}$$

Combining (15) and (16) with the result of Lemma 2, we obtain

$$\frac{\overline{C}_k - G_k}{C + 3(k+1)} \leq r_k = \frac{k}{k+1}.$$

This proves that there exist $k+1$ relay diamond networks such that each $k$-relay subnetwork satisfies the bound (5) in Theorem 1. To extend the proof for any $N > k$, simply consider augmenting the $k+1$ relay diamond network of Fig. 3 by adding relay nodes with zero capacities. $\square$

We will next prove Lemma 1 for the case where $k = 1$ and $k = 2$. The proof of Lemma 1 for $k > 2$ and the proof of Lemma 2 are provided in [16].

*Proof of Lemma 1:* We introduce the following notation. Let

$$\omega(\Gamma) \doteq \min_{\Lambda \subseteq \Gamma} \left( \max_{i \in \Lambda} R_{id} + \max_{i \in \overline{\Lambda}} R_{is} \right) \tag{17}$$

$$\bar{\omega} \doteq \min_{\Lambda \subseteq [N]} \left( \max_{i \in \Lambda} R_{id} + \max_{i \in \overline{\Lambda}} R_{is} \right), \tag{18}$$

and $\omega_k \doteq \max_{\substack{\Gamma\subseteq[N]\\|\Gamma|=k}} \omega(\Gamma)$. Note that $r_k$ in Lemma 1 is defined as $r_k = \frac{w_k}{\bar{\omega}}$.

The first thing we note is that $r_k \leq 1$. This follows from the fact that every subset of $\Gamma$ is necessarily contained in a subset of $[N]$. More precisely, let $\Gamma^* \subseteq [N]$ be such that $|\Gamma^*| = k$ and $\omega_k = \omega(\Gamma^*)$. Any $\Lambda \subseteq [N]$ can be expressed in the form $\Lambda = S_\Lambda \cup T_\Lambda$ such that $S_\Lambda \subseteq \Gamma^*$ and $\overline{\Lambda} = \overline{S_\Lambda} \cup R_\Lambda$ where $\overline{\Lambda} = [N] \setminus \Lambda$ and $\overline{S_\Lambda} = \Gamma^* \setminus S_\Lambda$. Therefore,

$$\begin{aligned}\bar{\omega} &= \min_{\Lambda\subseteq[N]} \left( \max_{i\in S_\Lambda\cup T_\Lambda} R_{id} + \max_{i\in\overline{S_\Lambda}\cup R_\Lambda} R_{is} \right) \\ &\geq \min_{\Lambda\subseteq[N]} \left( \max_{i\in S_\Lambda} R_{id} + \max_{i\in\overline{S_\Lambda}} R_{is} \right) \\ &= \omega(\Gamma^*).\end{aligned}$$

The same reasoning also implies that for $k_1 \geq k_2$ we have $r_{k_1} \geq r_{k_2}$, which is intuitively trivial; by allowing greater subsets we can not have smaller sums in the form $\max_{i\in\Lambda} R_{id} + \max_{i\in\overline{\Lambda}} R_{is}$.

- For $k = 1$, the lemma claims that $w_1 \geq \frac{1}{2}\bar{\omega}$. Since

$$w_1 = \max_{i\in[N]} \min\left(R_{id}, R_{is}\right),$$

this is equivalent to saying that $\exists\, y \in [N]$ s.t. $R_{id} \geq \frac{1}{2}\bar{\omega}$ and $R_{is} \geq \frac{1}{2}\bar{\omega}$. We will prove this by contradiction. Assume

$$\forall i \in [N],\ R_{id} < \frac{1}{2}\bar{\omega} \quad \text{or} \quad R_{is} < \frac{1}{2}\bar{\omega}.$$

Let $\Lambda_0 = \left\{i \in [N] : R_{id} < \frac{1}{2}\bar{\omega}\right\}$. Note by the assumption above we have $R_{is} < \frac{1}{2}\bar{\omega},\ \forall i \in \overline{\Lambda}_0$. Note that the cut-set upper bound in (18) can be further upper bounded by considering only the cut $\Lambda_0$ among all possible cuts $\Lambda \subseteq [N]$. We obtain

$$\bar{\omega} \leq \max_{i\in\Lambda_0} R_{id} + \max_{i\in\overline{\Lambda}_0} R_{is} < \bar{\omega}$$

since each of the two terms are strictly smaller than $\frac{1}{2}\bar{\omega}$. This contradiction proves the lemma for $k = 1$.

- For $k = 2$, the lemma claims that $w_2 \geq \frac{2}{3}\bar{\omega}$. We can prove this by establishing the following three properties for a network with $\bar{\omega}$.

*Property 1:* $\exists\, p \in [N]$ s.t. $R_{ps} \geq \frac{2}{3}\bar{\omega}$ and $R_{pd} \geq \frac{1}{3}\bar{\omega}$. We prove this by contradiction. Assume

$$\forall i \in [N],\ R_{is} < \frac{2}{3}\bar{\omega} \quad \text{or} \quad R_{id} < \frac{1}{3}\bar{\omega}.$$

Consider the cut $\Lambda_1 = \left\{i \in [N] : R_{id} < \frac{1}{3}\bar{\omega}\right\}$. Then $R_{is} < \frac{2}{3}\bar{\omega},\ \forall i \in \overline{\Lambda}_1$. Considering only the cut $\Lambda_1$ we obtain

$$\bar{\omega} \leq \max_{i\in\Lambda_0} R_{id} + \max_{i\in\overline{\Lambda}_0} R_{is} < \bar{\omega},$$

which is contradiction.

*Property 2:* $R_{pd} < \frac{2}{3}\bar{\omega}$. Otherwise the proof of the lemma is complete for $k = 2$, since in such a case we have $w_2 \geq w_1 \geq \frac{2}{3}\bar{\omega}$.

*Property 3:* $\exists\, m \in [N],\ m \neq p$ s.t. $R_{ms} \geq \frac{1}{3}\bar{\omega}$ and $R_{md} \geq \frac{2}{3}\bar{\omega}$. We can again prove this by contradiction. Assume the contrary and consider $\Lambda_2 = \left\{i \in [N] : R_{id} < \frac{2}{3}\bar{\omega}\right\}$. Note that $p \in \Lambda_2$ by Property 2 and $R_{is} < \frac{1}{3}\bar{\omega},\ \forall i \in \overline{\Lambda}_2$. The value of the cut $\Lambda_2$ is strictly smaller than $\bar{\omega}$, which is a contradiction.

Consider the 2-relay sub-network composed of $m$ and $p$. It can be easily verified that $\omega(\{m,p\}) \geq \frac{2}{3}\bar{\omega}$, completing the proof of the lemma for $k = 2$.

The proof of the lemma for the general case follows similar lines. The main idea is that if all $k$-relay subnetworks have value smaller than $\frac{k}{k+1}\bar{\omega}$, this allows us to construct a cut of the network which has value strictly smaller than $\bar{\omega}$. □

## VII. Conclusions

We showed that in an $N$-relay diamond network we can use $k$ of the $N$ relays and approximately maintain a $\frac{k}{k+1}$ fraction of the total capacity. In particular, we can use a single relay and approximately achieve half the capacity. Our proof was based on reducing the network simplification to a combinatorial problem.

## References


[1] T. M. Cover and A. El Gamal, *Capacity theorems for the relay channel*, IEEE Trans. on Information Theory, vol. 25, no. 5, pp. 572-584, 1979.

[2] G. Kramer, M. Gastpar, and P. Gupta, *Cooperative strategies and capacity theorems for relay networks,* IEEE Trans. on Information Theory, vol. 51, no. 9, pp. 3037-3063, Sept. 2005.

[3] G. Kramer, I. Maric, and R. Yates, *Cooperative Communications.* Foundations and Trends in Networking, 2006.

[4] S. Avestimehr, S. Diggavi and D. Tse, *Wireless network information flow: a deterministic approach*, e-print arXiv:0906.5394v2 - arxiv.org.

[5] S.H. Lim, Y.-H. Kim, A. El Gamal, S.-Y. Kim, "Noisy Network Coding", e-print arXiv:1002.3188v2 - arxiv.org.

[6] Cover, T.M, Thomas J. A., *Elements of Information Theory*, Wiley & Sons Inc., 1991.

[7] A. Özgür, S. Diggavi, *Approximately Achieving Gaussian Relay Network Capacity with Lattice Codes*, Proc. IEEE Int. Symposium on Information Theory, Austin, June 2010.

[8] U. Niesen, S. Diggavi, *The Approximate Capacity of the Gaussian N-Relay Diamond Network*, pre-print, 2010.

[9] D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*, Cambridge University Press, 2005.

[10] A. Bletsas, A. Khisti, D. Reed, and A. Lippman, *A simple cooperative diversity method based on network path selection,* IEEE Journal on Selected Areas in Communications, vol. 24, no. 3, pp. 659-672, March 2006.

[11] R. Tannious and A. Nosratinia, *Spectrally efficient relay selection with limited feedback,* IEEE Journal on Selected Areas in Communication, vol. 26, no. 8, Oct. 2008, pp. 1419-1428.

[12] A. Bletsas, H. Shin, and M. Z. Win, *Cooperative communications with outage-optimal opportunistic relaying,* IEEE Transactions on Wireless Communications, vol. 6, no. 9, pp. 3450-3460, Sep. 2007.

[13] J. Cai, S. Shen, J.W. Mark, and A.S. Alfa, *Semi-distributed user relaying algorithm for amplify-and-forward wireless relay networks,* IEEE Transactions on Wireless Communication, vol. 7, no. 4, pp. 1348-1357, April 2008.

[14] B. Wang, Z. Han, and K.J.R. Liu, *Distributed relay selection and power control for multiuser cooperative communication networks using buyer/seller game,* In Proc. IEEE INFOCOM, pp. 544-552, Anchorage, AL, May 2007.

[15] Y. Zhao, R.S. Adve, and T.J. Lim, *Improving amplify-and-forward relay networks: optimal power allocation versus selection,* in Proc. IEEE International Symposium on Information Theory, pp. 1234-1238, Seattle, WA, July 2006.

[16] C. Nazaroglu, A. Özgür, C. Fragouli, "Wireless Network Simplification: the Gaussian N-Relay Diamond Network", arXiv:1103.2046 - arxiv.org.