# THE DADE GROUP OF A FINITE GROUP

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Abstract. The aim of this paper is to construct an equivalent of the Dade group of a $p$-group for an arbitrary finite group $G$, whose elements are equivalences classes of endo-$p$-permutation modules. To achieve this goal we use the theory of relative projectivity with respect to a module and that of relative endotrivial modules.

## 1. Introduction

The construction of the Dade group $D(P)$ described by E. Dade in [Dad78a] is valid only in case the group $P$ is a $p$-group. This is linked to the facts that $kP$-permutation modules are indecomposable, whereas for an arbitrary group $G$, the $kG$-permutation modules are not indecomposable in general, and moreover that their direct summands need not be permutation modules. The classification of endo-permutation modules via the complete description of the structure of the Dade group $D(P)$ was completed in 2004 by S. Bouc with [Bou06]. It had started about 25 years earlier with the first papers and results by E. Dade in [Dad78a] and [Dad78b] in 1978, and the final classification was in fact achieved through the non-effortless combined work of several (co)-authors between 1998 and 2004, including J.L. Alperin, S. Bouc, J. Carlson, N. Mazza and J. Thévenaz. Yet, for an arbitrary finite group $G$, no satisfying equivalent group structure to the Dade group on a class of $kG$-modules has been defined so far.

One way to obtain a similar notion to that of the Dade Group for arbitrary groups is to consider endo-$p$-permutation modules as described by J.-M. Urfer in [Urf06]. He shows that if $P$ is a $p$-subgroup of a group $G$, this notion induces a group structure, denoted by $D_P(G)$, on a set of equivalence classes of indecomposable endo-$p$-permutation $kG$-modules with vertex $P$. (The equivalence relation being a generalisation of Dade's compatibility relation.) However, the main drawback of this approach resides in the fact that there is not a unique indecomposable representative, up to isomorphism, for the classes in $D_P(G)$. More precisely, $D_P(G)$ classifies the sources of the endo-$p$-permutation modules with vertex $P$, but not the modules themselves. Also note that if $P$ is a Sylow $p$-subgroup of $G$, then $D_P(G) \cong D(P, \mathcal{F}_P(G))$, where $D(P, \mathcal{F}_P(G))$ is the Dade group of the fusion system $\mathcal{F}_P(G)$ on $P$ defined in [LM09].

The aim of this piece of work is to show how the notion of relative endotrivial module, that we introduced in [Las11a], can generalise the Dade group in a more natural way. It is most interesting to note that crucial building pieces for the classification of endo-permutation modules are indeed the endotrivial modules, which are particular cases of endo-permutation modules. In some sense, we turn the problem upside down, and show how one can regard an endo-permutation module as an *endotrivial module*, of course not in the ordinary sense, but in the relative sense. This enables us to endow a well-chosen set of isomorphism classes of endo-$p$-permutation modules with a group structure, similar to that of the Dade group. We call this new group, the *generalised Dade group*

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*of the group* $G$, explicitly compute its structure and show how it is closely related to that of the $G$-stable points of the Dade group of a Sylow $p$-subgroup of $G$.

## 2. Preliminaries and definitions

Unless otherwise mentioned, throughout this text $k$ shall denote an algebraically closed field of prime characteristic $p$, $G$ a finite group whose order is divisible by $p$, all modules are finitely generated, $\mathsf{mod}(kG)$ denotes the category of finitely generated left $kG$-modules and $\mathsf{stmod}(kG)$ the corresponding stable category. We write $k$ for the one-dimensional trivial module. Moreover, $\otimes$ denotes the ordinary tensor product over $k$, $M^* = Hom_k(M, k)$ and $\Omega(M)$ the $k$-dual and the kernel of a projective cover of the $kG$-module $M$, respectively.

2.1. **Relative projectivity with respect to a module.** Projectivity relative to a $kG$-module was introduced by T. Okuyama [Oku91], then further developed and used in [Car96], [CP96], [CPW98], and also by the author in [Las11a]. This is a generalisation of the more classic projectivity relative to a subgroup widely used in the theory of vertices and sources. Moreover, it is also just a special case of the relative homological algebra defined for a projective class of epimorphisms or a pair of adjoint exact functors in [HS71, Chap. 10]. We recall here basic definitions and useful properties.

**Definition 2.1** ([Oku91])**.** Let $V$ be a $kG$-module. A finitely generated $kG$-module $M$ is termed *relatively $V$-projective*, or simply *$V$-projective*, if there exists a $kG$-module $N$ such that $M$ is isomorphic to a direct summand of $V \otimes_k N$.

**Proposition 2.2** (**Omnibus properties**, [Las11a], Prop. 2.0.2)**.** *Let $U, V$ be $kG$-modules.*

(a) *Any direct summand of a $V$-projective module is $V$-projective and if $U \in Proj(V)$, then $Proj(U) \subseteq Proj(V)$.*
(b) *If $p \nmid \dim_k(V)$ then $Proj(V) = \mathsf{mod}(kG)$. In particular $Proj(k) = \mathsf{mod}(kG)$.*
(c) $Proj(U \oplus V) = Proj(U) \oplus Proj(V)$.
(d) $Proj(U) \cap Proj(V) = Proj(U \otimes V) \supseteq Proj(U) \otimes Proj(V)$.
(e) $Proj(V) = Proj(V^*) = Proj(\Omega(V)) = Proj(\Omega^{-1}(V)) = Proj(V \oplus V) = Proj(V \otimes V)$.
(f) *If $P \in \mathsf{mod}(kG)$ is projective, then $Proj(P) = Proj(kG)$, which is equal to the whole class of projective modules in $\mathsf{mod}(kG)$. Moreover $Proj(kG) \subseteq Proj(V)$ for any $kG$-module $V$.*

**Remark 2.3.** The notion of relative projectivity with respect to a module encompasses the notion of projectivity relative to a subgroup, used in the theory of vertices and sources. More precisely, a $kG$-module $M$ is projective relatively to the subgroup $H$ of $G$ if and only if $M \in Proj(k\uparrow_H^G)$. Moreover, if $\mathcal{H}$ is a family of subgroups of $G$, then $M$ is projective relatively to the family $\mathcal{H}$ if and only if $M$ is projective relatively to the $kG$-module $V(\mathcal{H}) := \bigoplus_{H \in \mathcal{H}} k\uparrow_H^G$.

In the sequel, we will state results concerned with projectivity relative to subgroups and families of subgroups in terms of modules as described here. Translating projectivity relative to a subgroup in terms of modules we have the following well-known properties (see e.g. [CR90, §19]):
- if $H \leqslant G$, then $Proj(k\uparrow_H^G) = Proj(k\uparrow_{{}^gH}^G)$ for every $g \in G$;
- if $K \leqslant H \leqslant G$, then $Proj(k\uparrow_K^G) \subseteq Proj(k\uparrow_H^G)$.

Moreover, if $\mathcal{H} := \{H_1, \ldots, H_n\}$, $n \in \mathbb{N}$, is a family of subgroups of $G$, then, by the two preceding properties and the omnibus properties above, assuming that $H_i \nleqslant_G H_j \ \forall\, i \neq j, 1 \leqslant i, j \leqslant n$ does not alter $Proj(\mathcal{H})$.

In next subsection we describe how one can use projectivity with respect to a module to construct groups of *relative endotrivial modules*. This essentially relies on the following theorem by Benson and Carlson.

**Theorem 2.4** ([BC86], Thm. 2.1). *Let $k$ be an algebraically closed field of characteristic $p$ (possibly $p = 0$). Let $M, N$ be finite-dimensional indecomposable $kG$-modules, then*

$$k \mid M \otimes N \text{ if and only if } \begin{cases} (1)\ M \cong N^*\,; \\ (2)\ p \nmid \dim_k(N). \end{cases}$$

*Moreover, if $k$ is a direct summand of $N^* \otimes N$ then it has multiplicity one, i.e. $k \oplus k$ is not a summand.*

**Definition 2.5.** A $kG$-module $V \in \mathsf{mod}(kG)$ is called *absolutely $p$-divisible* if $p = \mathrm{char}(k)$ divides the $k$-dimension of every indecomposable direct summand of $V$.

**Proposition 2.6** ([Las11a], Prop. 2.2.2). *Let $V \in \mathsf{mod}(kG)$. Then, the following are equivalent:*

(a) *The trivial $kG$-module $k$ is not $V$-projective;*
(b) *$V$ is absolutely $p$-divisible;*
(c) *$Proj(V) \neq \mathsf{mod}(kG)$.*

2.2. **Relative endotrivial modules.** In [Las11a], we introduced and developed the notion of an *endotrivial module relative to a $kG$-module $V$*.

**Definition 2.7.** Let $V$ be an absolutely $p$-divisible $kG$-module. A module $M \in \mathsf{mod}(kG)$ is termed *endotrivial relative to the $kG$-module $V$* or simply *$V$-endotrivial* if

$$\mathrm{End}_k(M) \cong M^* \otimes M \cong k \oplus (V - proj)\,.$$

This definition is equivalent to requiring that $\mathrm{End}_k(M)$ is isomorphic to a trivial module in the relative stable category $\mathsf{stmod}_V(kG)$.

**Lemma 2.8** ([Las11a], Lem. 3.1.2, 3.2.1, 3.2.2, 3.2.3, 4.1.1). *Let $V \in \mathsf{mod}(kG)$ be an absolutely $p$-divisible module. Let $M, N \in \mathsf{mod}(kG)$ be $V$-endotrivial modules. Then:*

(a) $\dim_k(M)^2 \equiv 1 \mod p$.
(b) *The modules $M^*$, $M \otimes N$ and $\mathrm{Hom}_k(M, N)$ are $V$-endotrivial.*
(c) *If $M$ is indecomposable, then the vertices of $M$ are the Sylow $p$-subgroups of $G$. Moreover, if $(P, S)$ is a vertex-source pair for $M$, then $S$ is a $V\downarrow^G_P$-endotrivial module, and $S$ has multiplicity one as a direct summand of $M\downarrow^G_P$.*
(d) *There is a direct sum decomposition $M \cong M_0 \oplus (V - proj)$ where $M_0$ is the unique indecomposable summand of $M$ that is $V$-endotrivial.*
(e) *If $P$ is Sylow $p$-subgroup of $G$, then $L \in \mathsf{mod}(kG)$ is $V$-endotrivial if and only if $L\downarrow^G_P$ is $V\downarrow^G_P$-endotrivial.*

Now, if $V \in \mathsf{mod}(kG)$ is an absolutely $p$-divisible module, one can set an equivalence relation $\sim_V$ on the class of $V$-endotrivial $kG$-modules as follows: for $M$ and $N$ two $V$-endotrivial modules let

$$M \sim_V N \text{ if and only if } M_0 \cong N_0\,,$$

where $M_0$ and $N_0$ are the unique $V$-endotrivial indecomposable summands of $M$ and $N$, respectively, given by part (e) of Lemma 2.8. This amounts to requiring that $M$ and $N$ are isomorphic in $\mathsf{stmod}_V(kG)$. Then let $T_V(G)$ denote the resulting set of equivalence classes. In particular, any equivalence class in $T_V(G)$ consists of an indecomposable $V$-endotrivial module $M_0$ and all the modules of the form $M_0 \oplus (V - proj)$.

**Proposition 2.9** ([Las11a], Prop. 3.5.1). *The ordinary tensor product $\otimes_k$ induces an abelian group structure on the set $T_V(G)$ defined as follows:*

$$[M] + [N] := [M \otimes_k N]$$

*The zero element is $[k]$ and the opposite of a class $[M]$ is the class $[M^*]$.*

**Lemma 2.10** ([Las11a], Prop. 3.5.3)**.** *Let $V \in \mathsf{mod}(kG)$ be absolutely $p$-divisible. If $W \in Proj(V)$, then the group $T_W(G)$ can be identified with a subgroup of $T_U(G)$ via the injective group homomorphism $T_W(G) \longrightarrow T_V(G) : [M] \longmapsto [M]$. By abuse of notation, we write $T_V(G) \leqslant T_U(G)$ .*

Since $Proj(kG)$ is ordinary projectivity, the group of endotrivial modules is $T(G) = T_{kG}(G)$. Then, by the above and part (e) of Lemma 2.2, $T(G) \leqslant T_V(G)$ for every absolutely $p$-divisible $V \in \mathsf{mod}(kG)$.

One-dimensional representations: If $G$ is a finite group, denote by $X(G)$ the abelian group of all isomorphism classes of one-dimensional $kG$-modules endowed with the group law induced by $\otimes_k$, which can also be identified with the group $Hom(G, k^\times)$ of $k$-linear characters of $G$. This is a finite $p'$-group, isomorphic to the $p'$-part of the abelianisation $G/[G,G]$ of $G$.

Let $V \in \mathsf{mod}(kG)$ be an absolutely $p$-divisible module. Then any one-dimensional module $\chi$ is $V$-endotrivial, because $\chi^* \otimes \chi \cong k$. Therefore there is an embedding $X(G) \longrightarrow T_V(G) : \chi \longmapsto [\chi]$. Thus we can identify $X(G)$ with a subgroup of $T_V(G)$ and there is always a chain of subgroups:

$$X(G) \leqslant T(G) \leqslant T_V(G)$$

There are also several homomorphisms between groups of relative endotrivial modules induced by a change of group.

**Lemma 2.11** ([Las11a], Sect. 3.6)**.**
***1. Restriction.*** *Let $H$ be a subgroup of $G$ and let $V$ be an absolutely $p$-divisible $kG$-module, then restriction to $H$ induces a group homomorphism, called a* restriction map*:*

$$\begin{array}{rccc} \mathrm{Res}^G_H\colon & T_V(G) & \longrightarrow & T_{V\downarrow^G_H}(H) \\ & [M] & \longmapsto & [M\downarrow^G_H] \end{array}$$

*Moreover, if $H$ contains the normaliser $N_G(P)$ of a Sylow $p$-subgroup of $G$, then $\mathrm{Res}^G_{N_G(P)}$ is injective and sends the class of an indecomposable $kG$-module to the class of its $kH$-Green correspondents.*

***2. Inflation.*** *Let $N$ be a normal subgroup of a group $G$ such that $p \mid |G/N|$. If $V$ is an absolutely $p$-divisible $k[G/N]$-module, then inflation induces an injective group homomorphism:*

$$\begin{array}{rccc} \mathrm{Inf}^G_{G/N}\colon & T_V(G/N) & \hookrightarrow & T_{\mathrm{Inf}^G_{G/N}(V)}(G) \\ & [M] & \longmapsto & [\mathrm{Inf}^G_{G/N}(M)] \end{array}$$

***3. Isomorphism.*** *Let $\varphi : G_1 \longrightarrow G_2$ be a group isomorphism. If $M$ is a $kG_1$-module, then it can be seen as a $kG_2$-module via $\varphi^{-1}$ and is denoted $\mathrm{Iso}(\varphi)(M)$. Let $V$ be an absolutely $p$-divisible $kG_1$-module. Then there is a group isomorphism:*

$$\begin{array}{rccc} \mathrm{Iso}(\varphi)\colon & T_V(G_1) & \longrightarrow & T_{\mathrm{Iso}(\varphi)(V)}(G_2) \\ & [M] & \longmapsto & [\mathrm{Iso}(\varphi)(M)] \end{array}$$

**Lemma 2.12.** *Let $P$ be a Sylow $p$-subgroup of $G$ and let $H \leqslant G$ be a subgroup containing $N_G(P)$. Let $V \in \mathsf{mod}(kG)$ be an absolutely $p$-divisible module.*

(a) *The restriction map $\mathrm{Res}^G_H : T_V(G) \longrightarrow T_{V\downarrow^G_H}(H)$ is injective.*
(b) *If $Proj(V\downarrow^G_H) \supseteq Proj(V(\mathcal{Y}))$, where $\mathcal{Y} = \{{}^gP \cap H \mid g \in G \backslash H\}$, then the restriction map $\mathrm{Res}^G_H : T_V(G) \longrightarrow T_{V\downarrow^G_H}(H)$ is an isomorphism. Furthermore, the inverse map is induced by induction, so that*

$$T_V(G) = \{[M\uparrow^G_H] \mid [M] \in T_{V\downarrow^G_H}(H)\} \cong T_{V\downarrow^G_H}(H)\,.$$

*More accurately, on indecomposable $V\downarrow_H^G$-endotrivial modules, the inverse map is induced by the Green correspondence, that is, if $\Gamma(M)$ denotes the Green correspondent of an indecomposable $kH$-module $M$, then*

$$T_V(G) = \{[\Gamma(M)] \mid M \text{ is an indecomposable } V\downarrow_H^G \text{-endotrivial } kH\text{-module}\}.$$

**Lemma 2.13** ([Las11a], Lem. 4.4.1)**.** *Let $G$ be a finite group with a normal Sylow $p$-subgroup $P$ and let $V \in \mathsf{mod}(kG)$ be an absolutely $p$-divisible module. Then* $\ker(\mathrm{Res}_P^G) = X(G)$.

2.3. **Relative syzygy modules.** An important family of relative endotrivial modules is provided by the relative syzygies modules of the trivial module. We refer to [Car96, Sect. 8] for definitions of projective and injective resolutions with respect to a module $V \in \mathsf{mod}(kG)$.

**Definition 2.14.** Let $M \in \mathsf{mod}(kG)$, let $(P_*, \partial_*) \xrightarrow{\varepsilon} M$ and $M \xrightarrow{\imath} (I_*, \partial^*)$ be minimal $V$-projective and $V$-injective resolutions of $M$, respectively. Define for all $n \geqslant 1$: $\Omega_V^n(M) := \ker \partial_{n-1}$, $\Omega_V^{-n}(M) := \mathrm{Coker}(\partial^{n-1})$. Define $\Omega_V^0$ to be the $V$-projective free part of $M$. The module $\Omega_V^m(M)$, $m \in \mathbb{N}$ is called the $m$-th $V$-relative syzygy module of $M$.

**Notation.** We write $\Omega_V(M) := \Omega_V^1(M)$ and simply $\Omega^n(M) := \Omega_Q^n(M)$, $\Omega(M) := \Omega_Q^1(M)$ if the module $Q$ is projective. Moreover, if $\mathcal{H}$ is a family of subgroups of the group $G$, then we write $\Omega_{\mathcal{H}}(M)$ instead of $\Omega_{V(\mathcal{H})}(M)$. If $V \in \mathsf{mod}(kG)$ is absolutely $p$-divisible and $W \in Proj(V)$, we write $\Omega_W$ for the class of $\Omega_W(k)$ in $T_V(G)$ and we write $\Omega$ for the class of $\Omega(k)$ in $T_V(G)$.

**Lemma 2.15** ([Las11a], Lemmas 2.3.3, 2.3.4, 3.2.1)**.** *Let $M, V, W \in \mathsf{mod}(kG)$.*

(a) *If $Proj(V) = Proj(W)$, then $\Omega_V^n(M) \cong \Omega_W^n(M)$ for every $n \in \mathbb{Z}$.*
(b) *$\Omega_V \circ \Omega_W(M) \cong \Omega_{V\oplus W} \circ \Omega_{V\otimes W}(M)$ and if $\mathcal{H}, \mathcal{K}$ are families of subgroups of $G$, then this formula reads $\Omega_{\mathcal{H}} \circ \Omega_{\mathcal{K}}(M) \cong \Omega_{\mathcal{H}\cup\mathcal{K}} \circ \Omega_{{}^G\mathcal{H}\cap\mathcal{K}}(M)$ where ${}^G\mathcal{H}\cap\mathcal{K} = \{{}^gH \cap K \mid H \in \mathcal{H}, K \in \mathcal{K}\}$.*
(c) *If $M$ is a $V$-endotrivial $kG$-module and $W \in Proj(V)$, then the $kG$-modules $\Omega_W^n(M)$ are $V$-endotrivial modules for every $n \in \mathbb{Z}$.*
(d) *If $H \leqslant G$ and $V$ is absolutely $p$-divisible, then $Res_H^G(\Omega_V) = \Omega_{V\downarrow_H^G} \in T_{V\downarrow_H^G}(H)$.*

**Lemma 2.16** ([Oku91], [Las11b] Lem. 3.8.1)**.** *Let $n \geqslant 2$ be an integer and $V_1, \ldots, V_n \in \mathsf{mod}(kG)$ be pairwise non isomorphic absolutely $p$-divisble modules.*

(a) *In $T_{V_1\oplus V_2}(G)$, we have $\Omega_{V_1\oplus V_2} = \Omega_{V_1} + \Omega_{V_2} - \Omega_{V_1\otimes V_2}$ and $\Omega_{V_1\otimes V_2} = [\Omega_{V_1} \circ \Omega_{V_2}(k)]$.*
(b) *More generally, in $T_{V_1\oplus\ldots\oplus V_n}(G)$:*

$$\Omega_{V_1\oplus\ldots\oplus V_n} = \sum_{i=1}^n \Omega_{V_i} - \sum_{j=2}^n \Omega_{\oplus_{r=1}^{j-1} V_r \otimes V_j} = \sum_{s=1}^n (-1)^{s+1} \Big( \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} \Omega_{V_{i_1}\otimes\cdots\otimes V_{i_s}} \Big)$$

(c) *If $\mathcal{H} := \{H_1, \ldots, H_n\}$ is a family of subgroups of the group $G$ such that the $kG$-module $V(\mathcal{H})$ is absolutely $p$-divisible, then formula (b) reads*

$$\Omega_{\mathcal{H}} = \sum_{i=1}^n \Omega_{\{H_i\}} - \sum_{j=2}^n \Omega_{{}^G\{H_1,\ldots,H_{j-1}\}\cap\{H_j\}} \text{ in } T_{V(\mathcal{H})}(G).$$

2.4. **Endo-permutation modules and the Dade group.** If $P$ is a $p$-group, then a $kP$-module $M$ is called an *endo-permutation module* if its endomorphism algebra $\mathrm{End}_k(M)$ is a permutation $kP$-module. Furthermore, an endo-permutation module $M$ is called *capped* if it possesses an indecomposable summand with vertex $P$.

**Proposition 2.17** ([Dad78a]).

(a) *The class of capped endo-permutation modules is closed under taking direct summands, duals, tensor products (over $k$), Heller translates, restriction to a subgroup and tensor induction to an overgroup.*

(b) *An endo-permutation $kP$-module $M$ is capped if and only if the trivial module is a direct summand of $\mathrm{End}_k(M)$.*

(c) *If $M$ is capped, then any two indecomposable summands of $M$ with vertex $P$ are isomorphic. This unique summand, up to isomorphism, is called the cap of $M$ and is written $Cap(M)$.*

(d) *An equivalence relation $\sim$ on the class of endo-permutation module is defined by: $M \sim N$ if and only if $Cap(M) \cong Cap(N)$.*

(e) *Let $D(P)$ denote the resulting set of equivalence classes. Then $D(P)$ is an abelian group for the following law:*

$$[M] + [N] \cong [M \otimes N]$$

*The zero element is the class $[k]$ of the trivial $kP$-module, while the opposite of a class $[M]$ is the class of the dual module $[M^*]$. This group is called the Dade group of the group $P$.*

Note that in every equivalence class in $D(P)$, there is, up to isomorphism, a unique indecomposable module, namely the cap of any module in the class. Thus $D(P)$ is in bijection with the set of isomorphism classes of indecomposable endo-permutation $kP$-modules with vertex $P$ which becomes a group with the law $[M] + [N] := [Cap(M \otimes N)]$.

The classification of endo-permutation modules, through the description of the structure of the Dade group, started with [Dad78a], [Dad78b], and independently [Alp77]. It was completed in 2004 by S. Bouc in [Bou06]. Inbetween, crucial steps for this classification include the classification of the endotrivial modules of a $p$-group. All this was achieved through the work of [Pui90], [BT00], [CT00], [CT04], [CT05], [Bou04] and [BM04].

In [Las11a], we noted that a main reason of interest in relative endotrivial modules comes from the fact that they provide a way to define a group structure on collections of representations of an arbitrary finite group $G$ which gives a generalisation for the Dade group of a $p$-group. Indeed, endo-permutation modules can always be seen as relative endotrivial modules in the following sense:

**Theorem 2.18** ([Las11a], Thm 5.0.2). *Let $P$ be a $p$-group and let $V(\mathcal{F}_P) := \bigoplus_{Q \lneq P} k\uparrow_Q^P$. The Dade group $D(P)$ can be identified with a subgroup of $T_{V(\mathcal{F}_P)}(P)$ via the canonical injective homomorphism*

$$\begin{array}{ccc} D(P) & \longrightarrow & T_{V(\mathcal{F}_P)}(P) \\ [M] & \longmapsto & [Cap(M)] \end{array} .$$

## 3. Projectivity relative to the family of subgroups $\mathcal{F}_G$

Recall from the theory of vertices and sources that:
- If $H$ is a subgroup of $G$ and $Q$ is a Sylow $p$-subgroup of $H$, then $Proj(k\uparrow_H^G) = Proj(k\uparrow_Q^G)$.
- If $H \leqslant G$, then $Proj(k\uparrow_H^G) = \mathsf{mod}(kG)$ if and only if $H$ contains a Sylow $p$-subgroup of $G$.

Thus it follows from Remark 2.3 and Proposition 2.6 that a permutation module $k\uparrow_R^G$, for a subgroup $R \leqslant G$, is absolutely $p$-divisible if and only if $R$ has a Sylow $p$-subgroup $Q \lneq_G P$.

**Notation.** Given $G$ a finite group, fix a Sylow $p$-subgroup $P$ of $G$ and set $\mathcal{F}_G := \{Q \lneq P\}$. Then consider the associated module $V(\mathcal{F}_G) = \bigoplus_{Q \in \mathcal{F}_G} k\uparrow_Q^G$ and notice that by the above $Proj(V(\mathcal{F}_G))$ corresponds to projectivity relative to the family of all non maximal $p$-subgroups of $G$. We emphasise that $Proj(V(\mathcal{F}_G))$ does not depend on the choice of the Sylow $p$-subgroup $P$.

**Lemma 3.1.** *Let $H$ be a subgroup of $G$ that contains a Sylow-$p$ subgroup $P$ of $G$. Then:*

(a) $Proj(V(\mathcal{F}_G)\downarrow^G_H) = Proj(V(\mathcal{F}_H))$.
(b) $V(\mathcal{F}_H)$ *is absolutely* $p$*-divisible.*

*Proof.* The Mackey formula yields

$$V(\mathcal{F}_G)\downarrow^G_H = \bigoplus_{Q\in\mathcal{F}_G} k\uparrow^G_Q\downarrow^G_H \cong \bigoplus_{Q\in\mathcal{F}_G}\bigoplus_{x\in[H\backslash G/Q]} k\uparrow^H_{{}^xQ\cap H} = \Big(\bigoplus_{Q\lneq P} k\uparrow^H_Q\Big) \oplus X = V(\mathcal{F}_H) \oplus X$$

where $X$ is a direct sum of $kH$-modules of the form $k\uparrow^H_S$ with $S \lneq_G P$, so that $k\uparrow^H_S\in Proj(V(\mathcal{F}_H))$. Thus by Proposition 2.2 we obtain first that $Proj(X) \subseteq Proj(V(\mathcal{F}_H))$ and second that

$$Proj(V(\mathcal{F}_G)\downarrow^G_H) = Proj(V(\mathcal{F}_H)\oplus X) = Proj(V(\mathcal{F}_H))\,.$$

This proves (a). Now, by Green's indecomposability Criterion, the modules $k\uparrow^P_Q$ are indecomposable for every $Q \lneq P$, and moreover their dimension is divisible by $p$. In consequence, the module $V(\mathcal{F}_P) = \bigoplus_{Q\in\mathcal{F}_P} k\uparrow^P_Q$ is absolutely $p$-divisible and therefore so are the modules $V(\mathcal{F}_H)$ for every $P \leqslant H \leqslant G$.

Indeed, .This proves (b).

□

**Lemma 3.2.** *Let $N$ be a normal subgroup of the group $G$ such that $p\,|\,|G/N|$.Then*

$$Proj(\mathrm{Inf}^G_{G/N}(V(\mathcal{F}_{G/N}))) \subseteq Proj(V(\mathcal{F}_G))\,.$$

*Proof.* Let $P$ be a Sylow $p$-subgroup of $G$ and $PN/N$ the corresponding Sylow $p$-subgroup of $G/N$. By definition,

$$V(\mathcal{F}_{G/N}) = \bigoplus_{R\lneq PN/N} k\uparrow^{G/N}_R\ .$$

Moreover, if $R \lneq PN/N$, there exists a subgroup $Q$ such that $P\cap N \leqslant Q \lneq P$ and $R = QN/N$. Whence

$$\mathrm{Inf}^G_{G/N}(V(\mathcal{F}_{G/N})) = \bigoplus_{P\cap N\leqslant Q\lneq P} \mathrm{Inf}^G_{G/N}(k\uparrow^{G/N}_{QN/N}) = \bigoplus_{P\cap N\leqslant Q\lneq P} k\uparrow^G_{QN}\ .$$

Now, since $Q$ is a Sylow $p$-subgroup of $QN$, $Proj(k\uparrow^G_{QN}) = Proj(k\uparrow^G_Q)$ (see above). Whence

$$Proj(\mathrm{Inf}^G_{G/N}(V(\mathcal{F}_{G/N}))) = Proj(\bigoplus_{P\cap N\leqslant Q\lneq P} k\uparrow^G_{QN}) = \bigoplus_{P\cap N\leqslant Q\lneq P} Proj(k\uparrow^G_Q) \subseteq Proj(V(\mathcal{F}_G))\,.$$

where the last inclusion is obtain by Proposition 2.2, parts (a) and (c), and by definition of the family $\mathcal{F}_G$. □

## 4. $V(\mathcal{F}_G)$-endotrivial modules

Because the module $V(\mathcal{F}_G)$ is absolutely $p$-divisible, we obtain a well-defined group $T_{V(\mathcal{F}_G)}(G)$ of $V(\mathcal{F}_G)$-endotrivial modules. The following elementary properties of this group can easily be deduced from the general theory of relative endotrivial modules that is developed in [Las11a].

**Proposition 4.1.** *Let $P$ be a Sylow $p$-subgroup of $G$ and $H$ be a subgroup of $G$ such that $P \leqslant H \leqslant G$.*

(a) *There is a well-defined restriction map*

$$\begin{array}{rccc} \mathrm{Res}^G_H\colon & T_{V(\mathcal{F}_G)}(G) & \longrightarrow & T_{V(\mathcal{F}_H)}(H) \\ & [M] & \longmapsto & [M\downarrow^G_H]\,. \end{array}$$

(b) *If $H$ contains $N_G(P)$, then the restriction map $\mathrm{Res}^G_H : T_{V(\mathcal{F}_G)}(G) \xrightarrow{\cong} T_{V(\mathcal{F}_H)}(H)$ is an isomorphism, whose inverse is induced by the Green correspondence on the indecomposable $V(\mathcal{F}_H)$-endotrivial modules.*

(c) $\ker(\mathrm{Res}_P^{N_G(P)}) = X(N_G(P))$.

(d) *If* $\Gamma(X(N_G(P)))$ *denotes the subgroup of* $T_{V(\mathcal{F}_G)}(G)$ *made up of the classes of the* $kG$*-Green correspondents of the modules in* $X(N_G(P))$*, then* $\ker(\mathrm{Res}_P^G) = \Gamma(X(N_G(P)))$*, which is a finite group isomorphic to* $X(N_G(P))$*.*

*Proof.* (a) This follows from the definition of a restriction map (section 2.2) and part (a) of Lemma 3.1.

(b) By Lemma 3.1 $Proj(V(\mathcal{F}_G){\downarrow}_H^G) = Proj(V(\mathcal{F}_H))$. Therefore part (b) of Lemma 2.12 applies and yields the result. Indeed $\mathcal{Y} \subset \mathcal{F}_H$ (where $\mathcal{Y}$ is the family of subgroups in Lemma 2.12) and thus by the omnibus properties of relative projectivity, $Proj(V(\mathcal{F}_H)) \supseteq Proj(V(\mathcal{Y}))$.

(c) This is a straightforward application of Lemma 2.13.

(d) Since, by part (b), $\mathrm{Res}_{N_G(P)}^G$ is an isomorphism, (d) follows from (c).

□

**Example 4.2.** Thus far there are two obvious families of examples of $V(\mathcal{F}_G)$-endotrivial modules.

(a) The $kG$-Green correspondents of the one-dimensional representations of the normaliser $N_G(P)$, provided by part (d) of Lemma 4.1.

(b) The relative syzygies $\Omega_W^n(M)$ with $W \in Proj(V(\mathcal{F}_G))$, $n \in \mathbb{Z}$ and $M$ a $V(\mathcal{F}_G)$-endotrivial module as described in part (c) of Lemma 2.15. In particular if $\mathcal{H}$ is a family of subgroups of $G$ such that the associated module $V(\mathcal{H}) = \bigoplus_{H\in\mathcal{H}} k{\uparrow}_H^G$ (see Remark 2.3) is absolutely $p$-divisible, then $Proj(V(\mathcal{H})) \subseteq Proj(V(\mathcal{F}_G))$ and therefore the relative syzygy modules $\Omega_{\mathcal{H}}^n(k)$ of the trivial module are all $V(\mathcal{F}_G)$-endotrivial modules.

It is known from [Alp01] that the relative syzygies $\Omega_{\mathcal{H}}^n(k)$, for families of subgroups $\mathcal{H}$, are endo-permutation modules when $G$ is a $p$-group. In similar manner, [Urf06, Prop. 5.8] shows that they are endo-$p$-permutation modules when $G$ is arbitrary. We show in section 7 that the same is true for the modules in $\Gamma(X(N_G(P)))$. Therefore there are strong connections between $V(\mathcal{F}_G)$-endotrivial modules and endo-permutation modules as well as endo-$p$-permutation modules.

## 5. Endo-$p$-permutation modules and the Dade group of a finite group

An *endo-p-permutation* $kG$-module is a module $M \in \mathsf{mod}(kG)$ whose endomorphism algebra $\mathrm{End}_k(M)$ is a $p$-permutation[1] $kG$-module. I.e. if $\mathrm{End}_k(M) \cong \bigoplus_{i\in I} N_i$ where each $N_i$ is indecomposable, then for every $i \in I$, $N_i \,|\, k{\uparrow}_{Q_i}^G$ for some $p$-subgroup $Q_i$ of $G$. Equivalently, $M$ is endo-$p$-permutation if and only if $M{\downarrow}_Q^G$ is an endo-permutation $kQ$-module for every $p$-subgroup $Q$ of $G$. In addition, since $p$-permutation modules are preserved under conjugation and restriction, it is enough to check that $M{\downarrow}_P^G$ is an endo-permutation $kP$-module for $P$ a fixed Sylow $p$-subgroup of $G$. Other elementary properties of this class of modules are the following:

**Lemma 5.1.** *Let* $M \in \mathsf{mod}(kG)$ *be an indecomposable endo-p-permutation module with vertex* $P$*. Then:*

(a) $M{\downarrow}_P^G$ *is capped endo-permutation.*

(b) $p \nmid \dim_k M$*.*

(c) $k \,|\, \mathrm{End}_k(M)$ *with multiplicity 1.*

*Proof.* (a) It is easy to see that $M{\downarrow}_P^G$ is forced to have a summand with vertex $P$, thus it is capped endo-permutation. See [Urf06, Chapter 2] for details.

[1] In English, a $p$-permutation module is also often termed a *trivial source module*.

(b) Assume $M$ were an indecomposable $kG$-module with $k$-dimension divisible by $p$, that is absolutely $p$-divisible. Then, as a consequence of Theorem 2.4 (see [Las11a, Lem. 2.2.4] for a fully developed argument), so would be $M\downarrow^G_P$, which contradicts statement (a). Indeed, $M\downarrow^G_P$ being capped, it has got at least one direct summand with $k$-dimension not divisible by $p$, for according to the previous section, $Cap(M\downarrow^G_P)$ is an indecomposable endo-permutation module, hence $V(\mathcal{F}_P)$-endotrivial and thus $\dim_k Cap(M\downarrow^G_P) \equiv \pm 1 \mod (p)$.

(c) This is a consequence of (b) and Theorem 2.4.

□

It can be seen in [Urf07] that setting an equivalence relation on the whole class of endo-$p$-permutation modules with vertex $P$ given by a generalisation of Dade's compatibility relation (cf [Dad78a]) does not lead to a group structure induced by tensor product on the set of isomorphism classes of indecomposable endo-$p$-permutation modules with vertex $P$. The idea is then to find a subclass of this class which has more similarities with that of *capped endo-permutation* modules for a $p$-group, and secondly to obtain a group structure induced by tensor product which embeds naturally in $T_{V(\mathcal{F}_G)}(G)$, generalising the embedding $D(P) \leqslant T_{V(\mathcal{F}_P)}(P)$ of Theorem 2.18. In this respect we focus on endo-$p$-permutation modules which are, at the same time, $V(\mathcal{F}_G)$-endotrivial.

**Proposition 5.2.** *Let $M \in \mathsf{mod}(kG)$ be an endo-p-permutation module. The following conditions are equivalent:*

(a) *$M$ is $V(\mathcal{F}_G)$-endotrivial;*

(b) *$M\downarrow^G_P$ is $V(\mathcal{F}_P)$-endotrivial;*

(c) *$M$ has a unique indecomposable summand with vertex $P$, say $M_0$ and, in addition, if $S \in \mathsf{mod}(kP)$ is a source for $M_0$, then the multiplicity of $S$ as a direct summand of $M\downarrow^G_P$ is one;*

(d) *$\mathrm{End}_k(M) \cong k \oplus N$ where $N$ is a p-permutation kG-module, all of whose indecomposable summands have a vertex strictly contained in $P$.*

*Proof.* (a)⇔(b): By Lemma 3.1, $Proj(V(\mathcal{F}_G))\downarrow^G_P) = Proj(V(\mathcal{F}_P))$, therefore statements (a) and (b) are equivalent by part (e) of Lemma 2.8.

(a)⇒(c): Assuming (a), $M$ admits a decomposition $M \cong M_0 \oplus (V(\mathcal{F}_G) - proj)$ where $M_0$ is the unique indecomposable $V(\mathcal{F}_G)$-endotrivial summand of $M$. Then $\dim_k(M_0) \not\equiv 0 \pmod p$ and so $M_0$ is forced to have vertex $P$, whereas all the other summands of $M$ have vertices strictly smaller than $P$ by definition of $Proj(V(\mathcal{F}_G))$. Furthermore, by part (c) of Lemma 2.8, if $S \in \mathsf{mod}(kP)$ is a source for $M_0$, then $S$ has multiplicity one in $M_0\downarrow^G_P$. In consequence, since $M\downarrow^G_P$ is $V(\mathcal{F}_P)$-endotrivial we have

$$M\downarrow^G_P \cong M_0\downarrow^G_P \oplus (V(\mathcal{F}_P) - proj) \cong S \oplus (V(\mathcal{F}_P) - proj)$$

where the Krull-Schmidt Theorem forces $S$ to be isomorphic to the unique $V(\mathcal{F}_P)$-endotrivial summand of $M\downarrow^G_P$. Thus $S$ has multiplicity one in $M\downarrow^G_P$ as well.

(c)⇒(b): Write $M = M_0 \oplus L$ with $M_0$ indecomposoable with vertex $P$ and $L$ a module all of whose indecomposable summands have a vertex strictly smaller than $P$. Thus $L \in Proj(V(\mathcal{F}_G))$ and restricting $M$ to $P$ yields

$$M\downarrow^G_P \cong M_0\downarrow^G_P \oplus (V(\mathcal{F}_P) - proj)\,.$$

Now $M_0$ is endo-$p$-permutation as a direct summand of an endo-$p$-permutation module, therefore $M_0\downarrow^G_P$ is capped endo-permutation by Lemma 5.1. Moreover $S \mid M_0\downarrow^G_P$ and because $S$ has vertex $P$ too, we must have $S \cong Cap(M_0\downarrow^G_P)$, so that the fact that the multiplicity of $S$ is one forces all the remaining direct summands of $M_0\downarrow^G_P$ to have a vertex strictly smaller than $P$, that is to be $V(\mathcal{F}_P)$-ptojective. Hence $M\downarrow^G_P$ is $V(\mathcal{F}_P)$-endotrivial.

(a)⇔(d): Given that $M$ is endo-$p$-permutation, then $End_k(M)$ is a $p$-permutation module. Thus $M$ satisfies condition (d) if and only if it is $V(\mathcal{F}_G)$-endotrivial, by definition of the family $\mathcal{F}_G$. □

**Definition 5.3.** An endo-$p$-permutation $kG$-module $M$ is said to be *strongly capped* if it satisfies the equivalent conditions of Proposition 5.2. Moreover, the unique summand of $M$ with vertex $P$ given by condition (c) is called the *cap* of $M$ and denoted by $Cap(M)$.

The cap of a strongly capped endo-$p$-permutation module is its unique indecomposable direct summand which is itself strongly capped. Moreover, a strongly capped endo-$p$-permutation $kG$-module has a direct sum decomposition of the form $M \cong Cap(M) \oplus (V(\mathcal{F}_G) - proj)$ where the $V(\mathcal{F}_G)$-projective part is also an endo-$p$-permutation module, but not strongly capped.

**Lemma 5.4.** *The class of strongly capped endo-$p$-permutation $kG$-modules is closed under taking duals, tensor products and restrictions to a subgroup containing a Sylow $p$-subgroup.*

*Proof.* Taking duals and tensor products are stable operations for both the classes of endo-$p$-permutation modules and of $V(\mathcal{F}_G)$-endotrivial modules, therefore they are stable for strongly capped endo-$p$-permutation modules. Now if $H \leqslant G$ contains a Sylow $p$-subgroup of G, then the restriction to $H$ of an endo-$p$-permutation module is an endo-$p$-permutation module and the restriction to $H$ of a $V(\mathcal{F}_G)$-endotrivial module is a $V(\mathcal{F}_H)$-endotrivial module by Lemma 4.1. Thus the restriction to $H$ of a strongly capped endo-$p$-permutation module is strongly capped. □

Using a similar approach to that used by Dade for endo-permutation modules, one can define an equivalence relation $\sim$ on the class of all strongly capped endo-$p$-permutation modules by setting:

$$M \sim N \Leftrightarrow Cap(M) \cong Cap(N)$$

Write $[M]$ for the equivalence class of the module $M$ and let $D(G)$ denote the resulting set of equivalence classes.

Observe that this equivalence relation is the restriction to the class of strongly capped endo-$p$-permutation of the equivalence relation $\sim_{V(\mathcal{F}_G)}$ on $V(\mathcal{F}_G)$-endotrivial modules defined in Section 2.2. Thus the classes do not have the same meaning in $T_{V(\mathcal{F}_G)}(G)$ and in $D(G)$, and in general there are more representatives for a given class in $T_{V(\mathcal{F}_G)}(G)$ than in $D(G)$.

**Corollary-Definition 5.5.** The set $D(G)$ with the composition law

$$([M],[N]) \longmapsto [M]+[N] := [M \otimes N]\,,$$

is an abelian group called the generalised Dade group of $G$, or simply the Dade group of $G$. Moreover, $D(G)$ can be identified with a subgroup of $T_{V(\mathcal{F}_G)}(G)$ through the natural embedding

$$\begin{array}{cccc} \imath: & D(G) & \longrightarrow & T_{V(\mathcal{F}_G)}(G) \\ & [M] & \longmapsto & [M]\ . \end{array}$$

*Proof.* Lemma 5.4 and the uniqueness of the caps ensure that the assignment

$$([M],[N]) \longmapsto [M \otimes N]$$

is a well-defined composition law for $D(G)$. The zero element is the class $[k]$ of the trivial module, while the opposite of a class $[M]$ is the class $[M^*]$ of the dual module. The map $\imath$ is well-defined by the above observation on $\sim$ and $\sim_{V(\mathcal{F}_G)}$ and it is a homomorphism because the addition is induced by $\otimes_k$ on both sides. It is injective because $\ker(\imath) = \{[k]\}$. Indeed, if $\imath([M]) = [k]$, then $M \sim_{V(\mathcal{F}_G)} k$ which is equivalent to $M \sim k$ because both $M$ and $k$ are strongly capped endo-$p$-permutation modules. □

We identify $D(G)$ with its image $\imath(D(G))$ and view $D(G)$ as a subgroup of $T_{V(\mathcal{F}_G)}(G)$.

**Remark 5.6.** Notice that any ordinary endotrivial module is strongly capped, and in particular, so is any one-dimensional $kG$-module. Therefore, up to identifications, the groups $T(G)$ and $X(G)$ can also be viewed as subgroups of $D(G)$ and we have a series of subgroup inclusions:

$$X(G) \leqslant T(G) \leqslant D(G) \leqslant T_{V(\mathcal{F}_G)}(G)$$

The group $D^{\Omega}(G) = \langle \Omega_{\mathcal{H}}(k) \,|\, \mathcal{H} \subseteq \mathcal{F}_G \rangle$ is also a subgroup of $D(G)$ because of the next Lemma.

**Lemma 5.7.** *Let $\mathcal{H} \subseteq \mathcal{F}_G$. If $M$ is a strongly capped endo-$p$-permutation module, then $\Omega_{V(\mathcal{H})}(M)$ is a strongly capped endo-$p$-permutation $kG$-module.*

*Proof.* Since $M$ is assumed to be strongly capped, it is both endo-$p$-permutation and $V(\mathcal{F}_G)$-endotrivial. In consequence, on the one hand $\Omega_{V(\mathcal{H})}(M)$ is $V(\mathcal{F}_G)$-endotrivial by part (c) of Lemma 2.15, hence $V(\mathcal{H})$-endotrivial and on the second hand, it is shown in [Urf06, Proposition 5.8] that it is endo-$p$-permutation, hence strongly capped, as required. □

Finally, note that $D(G)$ can also be identified with set of isomorphism classes of indecomposable strongly capped endo-$p$-permutation $kG$-modules endowed with the group law $[M] + [N] := [Cap(M \otimes N)]$ (where the square brackets denote the isomorphism class of a module).

## 6. Group operations

The operations of restriction and inflation induce group homomorphisms between the generalised Dade groups, whereas, in contrast with ordinary Dade groups, tensor induction does not.

**Lemma 6.1.** *Let $P$ be a Sylow $p$-subgroup of $G$ and let $H$ be a subgroup of $G$ such that $P \leqslant H \leqslant G$. Then restriction induces a group homomorphism*

$$\begin{array}{rccc} \mathrm{Res}^G_H\colon & D(G) & \longrightarrow & D(H) \\ & [M] & \longmapsto & [M\!\downarrow^G_H] \end{array} .$$

*Furthermore, if $H$ contains the normaliser $N_G(P)$ of the Sylow $p$-subgroup $P$, then the map $Res^G_H$ is injective.*

*Proof.* As seen in 3.1, there is a restriction homomorphism for groups of relatively endotrivial modules

$$\begin{array}{rccc} \mathrm{Res}^G_H\colon & T_{V(\mathcal{F}_G)}(G) & \longrightarrow & T_{V(\mathcal{F}_H)}(H) \\ & [M] & \longmapsto & [M\!\downarrow^G_H] \end{array} ,$$

which is an isomorphism if $H$ contains $N_G(P)$. In consequence, it suffices to check that it maps $D(G)$ to a subgroup of $D(H)$. In fact, if $[M] \in D(G)$, then, $M\!\downarrow^G_H$ is strongly capped by Lemma 5.4 and so $[M\!\downarrow^G_H] \in D(H)$. Consequently, set $\mathrm{Res}^G_H : D(G) \longrightarrow D(H)$ to be the restriction (of maps) to $D(G)$ of the map $\mathrm{Res}^G_H : T_{V(\mathcal{F}_G)}(G) \longrightarrow T_{V(\mathcal{F}_H)}(H)$. It is injective if $H \geqslant N_G(P)$. □

The injectivity of the map $\mathrm{Res}^G_{N_G(P)} : D(G) \longrightarrow D(N_G(P))$ allows us to identify the Dade group $D(G)$ of a group $G$ with a subgroup of the Dade group $D(N_G(P))$.

**Lemma 6.2.** *Let $N$ be a normal subgroup of the group $G$ such that $G/N$ has order divisible by $p$. Then inflation induces a group homomorphism*

$$\begin{array}{rccc} \mathrm{Inf}^G_{G/N}\colon & D(G/N) & \longrightarrow & D(G) \\ & [M] & \longmapsto & [\mathrm{Inf}^G_{G/N}(M)] \,. \end{array}$$

*Proof.* Consider the composite map

$$T_{V(\mathcal{F}_{G/N})}(G/N) \xrightarrow{\mathrm{Inf}_{G/N}^{G}} T_{\mathrm{Inf}_{G/N}^{G}(V(\mathcal{F}_{G/N}))}(G) \xhookrightarrow{\imath} T_{V(\mathcal{F}_G)}(G)$$

where the first map is given by section 2.2 and the second map is given by Lemma 2.10 because

$$Proj(\mathrm{Inf}_{G/N}^{G}(V(\mathcal{F}_{G/N}))) \subseteq Proj(V(\mathcal{F}_G))$$

by Lemma 3.2. This composite maps $D(G/N)$ (viewed as a subgroup of $T_{V(\mathcal{F}_{G/N})}(G/N)$) to $D(G)$. Indeed, if $[M] \in D(G/N)$, then it only remains to check that $\mathrm{Inf}_{G/N}^{G}(M)$ is endo-$p$-permutation. But if we let $\varphi : P/P \cap N \xrightarrow{\cong} PN/N$ denote the canonical group morphism, then

$$\mathrm{Res}_{P}^{G} \circ \mathrm{Inf}_{G/N}^{G}(M) = \mathrm{Inf}_{P/P\cap N}^{P} \circ \mathrm{Iso}(\varphi^{-1}) \circ \mathrm{Res}_{PN/N}^{G/N}(M)$$

is endo-permutation because both isomorphism and inflation preserve endo-permutation modules. Hence $\mathrm{Inf}_{G/N}^{G}(M)$ is endo-$p$-permutation, as required. It follows that there is an inflation map $\mathrm{Inf}_{G/N}^{G} : D(G/N) \longrightarrow D(G)$ defined by restricting the map $\imath \circ \mathrm{Inf}_{G/N}^{G}$ to $D(G/N)$. □

Now, although the tensor induction of an endo-$p$-permutation module is an endo-$p$-permutation module (see [Urf07, Prop. 1.2]), the tensor induction of a strongly capped endo-$p$-permutation module is not necessarily strongly capped again.

**Counterexample 6.3.** Consider the 3-nilpotent group $G := C_7 \rtimes C_3$ with $k$ in characteristic 3. (If $C_7 :=< a >$ and $C_3 :=< u >$, then the action of $C_3$ on $C_7$ is given by $uau^{-1} = a^2$.) Then consider the module $\Omega(k) \in \mathsf{mod}(kC_3)$, which is endotrivial. However the tensor induced module

$$\Omega(k)\uparrow_{\otimes C_3}^{G}$$

is neither an endotrivial module nor a strongly capped endo-3-permutation module. In fact, there exists no absolutely 3-divisible $kG$-module $V$ such that the tensor induced module $\Omega(k)\uparrow_{\otimes C_3}^{G}$ is $V$-endotrivial. See [Las11b, Lem. 7.6.5] for detailed computations.

## 7. The structure of $D(G)$

Key tools to describe the structure of the group $D(G)$ are provided firstly by the following theorem proven by Dade but never published, and secondly by a characterisation of endo-$p$-permutation modules by Urfer.

**Theorem 7.1** ([Dad82], Theorem 7.1)**.** *Let $G$ be a finite group having a normal Sylow $p$-subgroup $P$. Let $M$ be an endo-permutation $kP$-module. Then $M$ extends to a $kG$-module if and only if $M$ is $G$-stable.*

**Theorem 7.2** ([Urf07], Thm 1.5)**.** *Let $G$ be a finite group. Let $M \in \mathsf{mod}(kG)$ be an indecomposable module with vertex $P$ and source $S \in \mathsf{mod}(kP)$. Then $M$ is an endo-$p$-permutation module if and only if $S$ is an endo-permutation module whose class $[S]$ in the Dade group $D(P)$ belongs to $D(P)^{G-st}$.*

Recall that $D(P)^{G-st}$, the subgroup of $G$-stable points of $D(P)$, consists of the classes $[M] \in D(P)$ such that $\mathrm{Res}_{^xP\cap P}^{P}([M]) = \mathrm{Res}_{^xP\cap P}^{^xP} \circ c_x([M])$, where $c_x$ denotes the conjugation by $x \in G$. In particular, $D(P)^{N_G(P)-st} = D(P)^{N_G(P)}$, the subgroup of fixed points of $D(P)$ under the action of the normaliser $N_G(P)$ by conjugation.

**Notation.** If $G$ is a finite group with a Sylow $p$-subgroup $P$, we write $X := X(N_G(P))$ for the group of one-dimensional representations of $N_G(P)$, identified with a subgroup of $D(N_G(P))$ by Remark 5.6, and we write $\Gamma(X) := \Gamma(X(N_G(P)))$ for the subgroup of $T_{V(\mathcal{F}_G)}(G)$ made up of the classes of the kG-Green correspondents of the modules in $X(N_G(P))$ defined in Lemma 4.1.

**Theorem 7.3.** *Let $G$ be a finite group with a non-trivial Sylow $p$-subgroup $P$. Then,*

(a) *restriction from $N_G(P)$ to $P$ yields an exact sequence*

$$0 \longrightarrow X \hookrightarrow D(N_G(P)) \xrightarrow{\operatorname{Res}^{N_G(P)}_P} D(P)^{N_G(P)} \longrightarrow 0\,;$$

(b) *restriction from $G$ to $P$ yields an exact sequence*

$$0 \longrightarrow \Gamma(X) \hookrightarrow D(G) \xrightarrow{\operatorname{Res}^G_P} D(P)^{G-st} \longrightarrow 0\,.$$

In the following proof, we denote by $R^G_H$ the map $\operatorname{Res}^G_H : T_{V(\mathcal{F}_G)}(G) \longrightarrow T_{V(\mathcal{F}_H)}(H)$ and keep the notation $\operatorname{Res}^G_H : D(G) \longrightarrow D(H)$ for the restriction maps at the level of the Dade groups.

*Proof.* First, it follows from Theorem 7.2, $\operatorname{Im}(\operatorname{Res}^G_P) \leqslant D(P)^{G-st}$. For if $M$ is an indecomposable strongly capped endo-$p$-permutation $kG$-module with source $S \in \mathsf{mod}(kP)$, then $\operatorname{Res}^G_P([M]) = [S]$. We claim that $\operatorname{Im}(\operatorname{Res}^G_P) = D(P)^{G-st}$. Let $[S] \in D(P)^{G-st}$ with $S$ indecomposable. Notice that $D(P)^{G-st} \subseteq D(P)^{N_G(P)}$, so that by Dade's Theorem $S \in \mathsf{mod}(kP)$ extends to a $kN_G(P)$-module $\widetilde{S}$. In other words, $\widetilde{S}\downarrow^{N_G(P)}_P \cong S$ and $S$ is a source for $\widetilde{S}$. By construction $\widetilde{S}$ is strongly capped endo-$p$-permutation because its source is endo-permutation and has multiplicity 1 in its restriction. Hence $[\widetilde{S}] \in D(N_G(P))$ and $\operatorname{Res}^{N_G(P)}_P([\widetilde{S}]) = [S]$. This proves the surjectivity of the map $Res^{N_G(P)}_P$ onto $D(P)^{N_G(P)}$.
Now if $\Gamma(\widetilde{S})$ is the $kG$-Green correspondent of $\widetilde{S}$, then it has source $S$ as well. Therefore $\Gamma(\widetilde{S})$ is endo-$p$-permutation by Theorem 7.2. It is moreover $V(\mathcal{F}_G)$-endotrivial by Lemma 4.1 because the restriction map $R^G_{N_G(P)}$ is an isomorphism whose inverse is induced by Green correspondence on indecomposable $kN_G(P)$-modules. Thus $[\Gamma(\widetilde{S})] \in D(G)$ and $\operatorname{Res}^G_P([\Gamma(\widetilde{S})]) = [S] \in D(P)^{G-st}$, as required.
Next we claim that the kernel of the restriction map $\operatorname{Res}^G_P : D(G) \longrightarrow D(P)$ is $\Gamma(X)$. It was established in Lemma 4.1 that $\ker(R^{N_G(P)}_P) = X$. Therefore

$$\ker(\operatorname{Res}^{N_G(P)}_P) = \ker(R^{N_G(P)}_P) \cap D(N_G(P)) = X \cap D(N_G(P)) = X$$

because $X \leqslant D(N_G(P))$ as noticed in Remark 5.6. Furthermore,

$$\begin{aligned}\ker(\operatorname{Res}^G_P) &= (\operatorname{Res}^G_{N_G(P)})^{-1}\left(\ker(\operatorname{Res}^{N_G(P)}_P)\right) = (\operatorname{Res}^G_{N_G(P)})^{-1}(X)\\ &= (R^G_{N_G(P)})^{-1}(X) \cap D(G) = \Gamma(X) \cap D(G)\end{aligned}$$

and it remains to show that $\Gamma(X) \leqslant D(G)$, i.e. that the indecomposable representatives of the classes in $\Gamma(X)$ are endo-$p$-permutation modules. Indeed, if $\chi \in X$, then its $kG$-Green correspondent $\Gamma(\chi)$ has the same source as $\chi$, that is the trivial module $k \in \mathsf{mod}(kP)$. Therefore $\Gamma(\chi) \mid k\uparrow^G_P$, or in other words, it is a $p$-permutation module and thus an endo-$p$-permutation module. Hence $\ker(\operatorname{Res}^G_P) = \Gamma(X)$. ☐

**Corollary 7.4.** *The generalised Dade group $D(G)$ of a finite group $G$ is finitely generated.*

*Proof.* The group $\Gamma(X) \cong X$ is finite. The group $D(P)^{G-st}$ is finitely generated as a subgroup of $D(P)$, which is finitely generated by [Pui90]. Thus the exact sequence

$$0 \longrightarrow \Gamma(X) \hookrightarrow D(G) \xrightarrow{\operatorname{Res}^G_P} D(P)^{G-st} \longrightarrow 0\,.$$

of Theorem 7.3 implies that $D(G)$ is finitely generated too. ☐

## 8. The generalised Dade group and control of $p$-fusion

The Dade group $D(G)$ may always be identified, via restriction, with a subgroup of the Dade group $D(N_G(P))$ of the normaliser of a Sylow $p$-subgroup $P$ of $G$. Then one may naturally ask when these groups are equal. The control of $p$-fusion in $G$ by a subgroup $H$ gives a partial answer to this question.

**Proposition 8.1.** *Let $H$ be a subgroup of $G$ such that $N_G(P) \leqslant H \leqslant G$. Then $D(G) = D(H)$ if and only if $D(P)^{G-st} = D(P)^{H-st}$.*

*Proof.* Since $H \leqslant G$, $D(P)^{G-st} \leqslant D(P)^{H-st}$. Thus, there is a commutative diagram with exact rows given by Theorem 7.3

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_G(X) & \longrightarrow & D(G) & \xrightarrow{\mathrm{Res}^G_P} & D(P)^{G-st} & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow{\mathrm{Res}^G_H} & & \downarrow{i} & & \\ 0 & \longrightarrow & \Gamma_H(X) & \longrightarrow & D(H) & \xrightarrow{\mathrm{Res}^H_P} & D(P)^{H-st} & \longrightarrow & 0 \end{array}$$

where $i$ denotes the inclusion of $D(P)^{G-st}$ in $D(P)^{H-st}$ as subgroup, and where $\Gamma_G(X) = \ker(\mathrm{Res}^G_P)$ and $\Gamma_H(X) = \ker(\mathrm{Res}^H_P)$. By part (d) of Proposition 4.1, $\Gamma_G(X) \cong X \cong \Gamma_H(X)$. Then, by the five-lemma the map $\mathrm{Res}^G_H$ is surjective if and only if the map $i$ is. Thus, up to identification, $D(G) = D(H)$ if and only if $D(P)^{G-st} = D(P)^{H-st}$. □

Links between control of $p$-fusion and the $G$-stable points of the Dade group of a $p$-group were already established in [Urf07]:

**Proposition 8.2** ([Urf07], Prop. 1.9)**.** *Let $P$ be a $p$-subgroup of $G$ and assume that $p$-fusion in $G$ is controlled by $H \leqslant G$. Then $D(P)^{G-st} = D(P)^{H-st}$.*

**Corollary 8.3.** *Assume that the $p$-fusion of $G$ is controlled by a subgroup $H \leqslant G$.*

(a) *If $G \geqslant H \geqslant N_G(P)$, then $D(G) = D(H)$.*
(b) *If $N_G(P) \geqslant H \geqslant P$, then $D(G) = D(N_G(P))$.*

*Proof.* (a) is a straightforward consequence of Propositions 8.1 and 8.2.
(b) If $N_G(P) \geqslant H \geqslant P$, and $H$ controls $p$-fusion then so does $N_G(P)$ and part (a) yields the result.

□

**Example 8.4.** For instance, if $G$ is a group with an abelian Sylow $p$-subgroup $P$, then the normaliser $N_G(P)$ controls $p$-fusion in $G$ by Burnside's Theorem. If $G$ is a $p$-nilpotent group, then $P$ controls $p$-fusion. If $p$ is odd and $G$ is a group with a metacyclic Sylow $p$-subgroup $P$, then $N_G(P)$ controls $p$-fusion in $G$ too (because such $p$-groups are resistant to fusion). Therefore, in all these cases it follows from the corollary that $D(G) = D(N_G(P))$.

**Example 8.5.** An example in which $D(G) \lneqq D(N_G(P))$ is provided by $G := GL_3(\mathbb{F}_3)$ and its extraspecial Sylow 3-subgroup $P$ of order 27 which consists of the upper unitriangular matrices. The subgroup of $P$ generated by the matrix

$$x := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is cyclic of order 3 and it is proven in [Urf07, Section 4] that the class in $D(P)$ of the relative syzygy module $\Omega_{k\uparrow^P_{<x>}}(k)$ is $N_G(P)$-stable but not $G$-stable. Thus $D(P)^{G-st} \lneqq D(P)^{N_G(P)}$ and it follows from Proposition 8.1 that $D(G) \lneqq D(N_G(P))$.

## 9. The $p$-nilpotent case

In this section, consider $G$ is a $p$-nilpotent group. In other words, $G$ is a semidirect product $G = N \rtimes P$, with $N = O_{p'}(G)$ and $P$ a Sylow $p$-subgroup of $G$. Thus $G = NP$, $N \cap P = \{1\}$, and we let $\varphi : P = P/N \cap P \longrightarrow NP/N = G/N$ be the canonical isomorphism. For the structure of the groups $T_V(G)$ of $V$-endotrivial modules for an arbitrary absolutely $p$-divisible $kG$-module $V$, we refer to [Las11b, Chap. 6].

**Theorem 9.1.** *Let $G = N \rtimes P$ be a $p$-nilpotent group. The restriction map $\mathrm{Res}^G_P : D(G) \longrightarrow D(P)$ is split surjective. In consequence there is a group isomorphism*

$$D(G) \cong X(N_G(P)) \oplus D(P)\,.$$

*Proof.* Since $G$ is $p$-nilpotent, the Sylow $p$-subgroup $P$ controls $p$-fusion in $G$, thus Proposition 8.2 yields $D(P)^{G-st} = D(P)^{P-st} = D(P)$. Now [Las11b, Thm. 6.2.2] states that the restriction map $\mathrm{Res}^G_P : T_{V(\mathcal{F}_G)}(G) \longrightarrow T_{V(\mathcal{F}_P)}(P)$ is split surjective and moreover that a section is provided by the map

$$T_{V(\mathcal{F}_P)}(P) \xrightarrow{\mathrm{Iso}(\varphi)} T_{V(\mathcal{F}_{G/N})}(G/N) \xrightarrow{\mathrm{Inf}^G_{G/N}} T_{V(\mathcal{F}_G)}(G)\,.$$

By Section 6, both these maps can be restricted to the Dade groups so that $\mathrm{Inf}^G_{G/N} \circ \mathrm{Iso}(\varphi) : D(P) \longrightarrow D(G)$ is a section for $\mathrm{Res}^G_P : D(G) \longrightarrow D(P)$. In consequence, in view of Theorem 7.3, $D(G)$ decomposes as a direct sum

$$D(G) \cong \Gamma(X) \oplus D(P)^{G-st} = \Gamma(X) \oplus D(P)\,.$$

Finally, $\Gamma(X) \cong X = X(N_G(P))$ by Lemma 4.1. The result follows. □

## 10. The cyclic case

Consider $G$ is a finite group with a non-trivial cyclic Sylow $p$-subgroup $P \cong C_{p^n}$, $n \geqslant 1$. In this case, the classification provided in [Las11a, Sect. 8] for the groups of relative endotrivial modules of $G$ allows us to determine the generalised Dade group $D(G)$ with ease.

**Proposition 10.1.** *Let $G$ be a finite group with a non-trivial cyclic Sylow $p$-subgroup $P \cong C_{p^n}$, $n \geqslant 1$, and for $0 \leqslant r \leqslant n-1$ let $Z_r$ denote the unique cyclic subgroup of order $p^r$ of $P$. Then*

$$D(G) = T_{V(\mathcal{F}_G)}(G) = T_{k\uparrow^G_{Z_{n-1}}}(G) \;=<\Gamma(X(N_G(P))), \{\Omega_{k\uparrow^G_{Z_s}} \,|\, 0 \leqslant s \leqslant n-1\}>\;.$$

*Proof.* Since $P$ is abelian, $N_G(P) =: N$ controls $p$-fusion by Burnside's Theorem. Therefore, by Corollary 8.3, $D(G) \cong D(N)$. Next we claim that $D(N) = T_{V(\mathcal{F}_N)}(N)$. By definition $V(\mathcal{F}_N) = \bigoplus_{s=0}^{n-1} k\uparrow^N_{Z_s}$ so that

$$Proj(V(\mathcal{F}_N)) = \bigoplus_{s=0}^{n-1} Proj(k\uparrow^N_{Z_s}) = Proj(k\uparrow^N_{Z_{n-1}})$$

because $Proj(k\uparrow^N_{Z_s}) \subseteq Proj(k\uparrow^N_{Z_{n-1}})$ for every $s \leqslant n-1$ as pointed out in remark 2.3. Therefore

$$T_{V(\mathcal{F}_N)}(N) = T_{k\uparrow^N_{Z_{n-1}}}(N)\,.$$

In addition, by [Las11a, Thm. 8.2.6], we have

$$T_{k\uparrow^N_{Z_{n-1}}}(N) \;=<X(N), \{\Omega_{k\uparrow^N_{Z_s}} \,|\, 0 \leqslant s \leqslant n-1\}>\;.$$

Now, $X(N) \leqslant D(N)$ by Remark 5.6 and the relative syzygy modules $\Omega_{k\uparrow_{Z_s}^N}(k)$ are endo-$p$-permutation modules by Lemma 5.7. Whence $D(N) = T_{V(\mathcal{F}_N)}(N)$. Finally, $T_{V(\mathcal{F}_G)}(G) \cong T_{V(\mathcal{F}_N)}(N)$ via restriction, by Lemma 4.1. Consequently, there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{V(\mathcal{F}_G)}(G) & \xrightarrow[\cong]{\mathrm{Res}^G_N} & T_{V(\mathcal{F}_N)}(N) & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \\
0 & \longrightarrow & D(G) & \xrightarrow[\cong]{\mathrm{Res}^G_N} & D(N) & \longrightarrow & 0
\end{array}$$

where the left-hand side vertical arrow is the inclusion as a subgroup of $D(G)$ in $T_{V(\mathcal{F}_G)}(G)$. Thus, the equality $D(G) = T_{V(\mathcal{F}_G)}(G)$ follows by the 5-Lemma. □

## 11. The Klein case

Consider $G$ is a finite group with a Sylow 2-subgroup $P \cong C_2 \times C_2$ and assume that the characteristic of the field $k$ is 2.

**Theorem 11.1** ([Las11a], Thm. 6.0.4)**.** *Let $G$ be a finite group with a normal Sylow 2-subgroup $P \cong C_2 \times C_2$. Let $V$ be any absolutely 2-divisible $kG$-module. Then there is a group isomorphism $\varphi : T_V(G) \longrightarrow T(G) : [M] \longmapsto [M_0]$ where $M \cong M_0 \oplus (V\text{–}proj)$ with $M_0$ the unique indecomposable and $V$-endotrivial summand of $M$.*

**Proposition 11.2.** *Let $G$ be a finite group with a Sylow 2-subgroup $P \cong C_2 \times C_2$.*

(a) *For any absolutely 2-divisible $kG$-module $V$, the group $T_V(G)$ identifies with a subgroup of $T_{V(\mathcal{F}_G)}(G) \cong T(N_G(P))$.*

(b) *Moreover $D(G) = T_{V(\mathcal{F}_G)}(G)$.*

*Proof.* Set $N := N_G(P)$.

(a) If $V \in \mathsf{mod}(kG)$ is absolutely 2-divisible, then, by Lemma 2.11, the restriction map $\mathrm{Res}^G_N : T_V(G) \longrightarrow T_{V\downarrow^G_N}(N)$ is injective and sends the class of an indecomposable $V$-endotrivial $kG$-module to the class of its $kN$-Green correspondent. By Lemma 4.1, the map $\mathrm{Res}^G_N : T_{V(\mathcal{F}_G)}(G) \longrightarrow T_{V(\mathcal{F}_N)}(N)$ is an isomorphism whose inverse is induced by Green correspondence on the indecomposable $V(\mathcal{F}_N)$-endotrivial modules. Furthermore, by Theorem 11.1, $T_{V(\mathcal{F}_N)}(N) \cong T(N) \cong T_{V\downarrow^G_N}(N)$. Therefore, the situation is as described in the following diagram:

$$\begin{array}{ccccc}
T_{V(\mathcal{F}_G)}(G) & \longleftarrow\!\!-\,-\,-\,-\,-\,-\,-\,-\!\!\hookleftarrow & & & T_V(G) \\
\cong \big\downarrow \mathrm{Res}^G_N & & & & \mathrm{Res}^G_N \big\downarrow \\
T_{V(\mathcal{F}_N)}(N) & \xleftrightarrow{\ \cong\ } & T(N) & \xleftrightarrow{\ \cong\ } & T_{V\downarrow^G_N}(N)
\end{array}$$

Thus, there is an injective group homomorphism $T_V(G) \longrightarrow T_{V(\mathcal{F}_G)}(G) : [L] \longmapsto [L]$, where $L$ denotes an indecomposable $V$-endotrivial module.

(b) The series of embeddings $T(N) \leqslant D(N) \leqslant T_{V(\mathcal{F}_N)}(N)$ and Theorem 11.1, which identifies $T(N)$ with $T_{V(\mathcal{F}_N)}(N)$, allow us to conclude that $D(N) = T_{V(\mathcal{F}_N)}(N)$. Then, to prove that $D(G) = T_{V(\mathcal{F}_G)}(G)$, use the same argument as in the proof of Proposition 10.1 in the cyclic case, because $P$ is also abelian and thus $N_G(P)$ controls $p$-fusion in $G$.

□

## 12. The group $D^{\Omega}(G)$

If $P$ is a $p$-group, with $p$ odd, then one of the main results of the classification of endo-permutation modules asserts that $D(P) = D^{\Omega}(P) = \langle\{\Omega_{k\uparrow_Q^P} \,|\, Q \lneq P\}\rangle$ (see [Bou06]). In this section we ask whether or not a similar result holds for the generalised Dade group.

Recall from 5.6 and 5.7 that the group $D^{\Omega}(G) = \langle \Omega_{\mathcal{H}}(k) \,|\, \mathcal{H} \subseteq \mathcal{F}_G \rangle$ is a subgroup of $D(G)$.

**Lemma 12.1.** *The group $D^{\Omega}(G)$ is generated by the relative syzygies $\Omega_{k\uparrow_Q^G}$, where $Q$ runs over the proper subgroups of $P$, that is $D^{\Omega}(G) = \langle\{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$.*

*Proof.* If $\mathcal{H} \subseteq \mathcal{F}_G$ is a family of subgroups, set $n_{\mathcal{H}} := \max\{|H| \,|\, H \in \mathcal{H}\}$. We claim that $\Omega_{\mathcal{H}} \in \langle\{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$ for every $\mathcal{H} \subseteq \mathcal{F}_G$ and the proof proceeds by induction on the natural number $n_{\mathcal{H}}$. First, if $n_{\mathcal{H}} = 1$, then $Proj(\mathcal{H})$ is projectivity relative to the trivial subgroup $\{1_G\}$, which is projectivity in the usual sense. Hence

$$\Omega_{\mathcal{H}} = \Omega_{k\uparrow_{\{1_G\}}^G} \in \langle\{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle .$$

Then, let $\mathcal{H} := \{H_1, \ldots, H_n\}$, $n \in \mathbb{N}$ be a subfamily of $\mathcal{F}_G$ such that $n_{\mathcal{H}} \geqslant 2$ and assume as induction hypothesis that $\Omega_{\mathcal{F}} \in \langle\{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$ for every subfamily $\mathcal{F} \subseteq \mathcal{F}_G$ such that $1 \leqslant n_{\mathcal{F}} < n_{\mathcal{H}}$. Furthermore, according to Remark 2.3, we may assume that $H_i \nleqslant_G H_j \;\; \forall\, i \neq j, 1 \leqslant i, j \leqslant n$. Then, according to Remark 2.16 we can write

$$\Omega_{\mathcal{H}} = \sum_{i=1}^{n} \Omega_{\{H_i\}} - \sum_{j=2}^{n} \Omega_{\,{}^G\{H_1,\ldots,H_{j-1}\} \cap \{H_j\}} \text{ in } T_{V(\mathcal{H})}(G)\,.$$

The sum $\sum_{i=1}^{n} \Omega_{\{H_i\}} \in \langle\{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$. Moreover, for every $2 \leqslant j \leqslant n$, the family of subgroups ${}^G\{H_1, \ldots, H_{j-1}\} \cap \{H_j\}$ is made up of the subgroups of the form ${}^gH_i \cap H_j$ with $g \in G$ and $1 \leqslant i \leqslant j-1$, which all satisfy ${}^gH_i \cap H_j \lneq H_j$ by the above assumption. In consequence, the sum $\sum_{j=2}^{n} \Omega_{\,{}^G\{H_1,\ldots,H_{j-1}\} \cap \{H_j\}}$ belongs to $\langle\{\Omega_{k\uparrow_Q^G} \,|\, Q \in \mathcal{F}_G\}\rangle$ by induction hypothesis, and the result follows. □

**Remark 12.2.** If $\mathcal{H}$ is a subfamily of $\mathcal{F}_G$, then it follows from the preceding proof that $\Omega_{\mathcal{H}} \in \langle\{\Omega_{k\uparrow_Q^G} \,|\, Q \leqslant H \text{ for some } H \in \mathcal{H}\}\rangle$.

**Question**: In case $G = P$ is a $p$-group and $p$ is odd, then $D(P) = D^{\Omega}(P)$ (see [Bou06]). Does a similar result hold in general for $D(G)$ when $G$ is an arbitrary finite group?

Because the one-dimensional representations are always in $D(G)$ this result obviously has to be adapted when $G$ is not a $p$-group. Nonetheless, we show that in the following cases, $D(G)$ is $D^{\Omega}(G)$ **modulo** the Green correspondents $\Gamma(X)$ of one-dimensional representations of $N_G(P)$ (with $P$ a Sylow $p$-subgroup of G):
(a) when $G$ has a cyclic Sylow $p$-subgroup;
(b) when $p$ is odd and $P$ is normal in $G$;
(c) when $N_G(P)$ controls $p$-fusion in $P$;
(d) it is also true for $G = GL_3(\mathbb{F}_p)$ with $p$ odd.
The question of determining if this result holds in general is left open.

**(a) The cyclic case.** In case the group $G$ has a cyclic Sylow $p$-subgroup $P$, then it was proven in Proposition 10.1 that

$$D(G) \cong T_{V(\mathcal{F}_G)}(G) = T_{k\uparrow_{Z_{n-1}}^G}(G) \; = <\Gamma(X(N_G(P))), \{\Omega_{k\uparrow_{Z_s}^G} \,|\, 0 \leqslant s \leqslant n-1\}> \; .$$

Hence $D(G)$ is indeed $D^{\Omega}(G)$ modulo $\Gamma(X)$.

**(b) The normal odd case:** In order to prove (b), we first recall that a set of generators for $D(P)^{G-st}$ is provided in [Urf06]:

**Proposition 12.3** ([Urf06], Cor. 3.7)**.** *Suppose that $p$ is an odd prime and $P$ is a Sylow $p$-subgroup of the group $G$. Then the abelian group $D(P)^{N_G(P)}$ is spanned by the elements*

$$f_Q := \sum_{g\in[N_G(P)/PN_G(P,Q)]} \Omega_{k\uparrow^P_{gQ}}$$

*where $N_G(P,Q) = \{g \in N_G(P) \mid {}^gQ = Q\}$ and $Q$ runs over $\mathcal{F}_G$.*

In what follows, we consider that $P \trianglelefteq G$, so that $N_G(P,Q) = N_G(Q)$ for every subgroup $Q \leqslant P$. We still need another technical result on projectivity relative to $p$-subgroups.

**Lemma 12.4.** *Let $G$ be a group with a normal Sylow $p$-subgroup $P$ and $R$ be a proper subgroup of $P$. Then*

$$Proj(k\uparrow^G_R\downarrow^G_P) = Proj(\bigoplus_{x\in[G/PN_G(R)]} k\uparrow^P_{{}^xR})\,.$$

*Proof.* The Mackey formula yields $Proj(k\uparrow^G_Q\downarrow^G_P) = Proj(\bigoplus_{x\in[G/P]} k\uparrow^P_{{}^xQ})$. Now, in order to obtain the equality of the statement, recall from Proposition 2.2 that if $V, W \in \mathsf{mod}(kG)$ and $Proj(V) = Proj(W)$ then $Proj(V \oplus W) = Proj(V)$. Therefore, in

$$Proj(\bigoplus_{x\in[G/P]} k\uparrow^P_{{}^xQ}) = \bigoplus_{x\in[G/P]} Proj(k\uparrow^P_{{}^xQ})$$

it is enough to keep only one copy of the summands generating the same relative projectivity. Thus, compute that for $x, y \in G$, $Proj(k\uparrow^P_{{}^xQ}) = Proj(k\uparrow^P_{{}^yQ})$ if and only if there exists $p \in P$ such that ${}^{px}Q = {}^yQ$ if and only if $y^{-1}x \in PN_G(Q)$ (since $P \lhd G$) if and only if $y \equiv x \mod PN_G(Q)$. Whence $Proj(k\uparrow^G_Q\downarrow^G_P) = Proj(\bigoplus_{x\in[G/PN_G(Q)]} k\uparrow^P_{{}^xQ})$. ☐

**Proposition 12.5.** *Let $p$ be an odd prime and $P$ be a normal Sylow $p$-subgroup of $G$. Then the restriction map $\mathrm{Res}^G_P : D^{\Omega}(G) \twoheadrightarrow D(P)^G$ is surjective.*
*More accurately, if $Q \lneq P$, then any generator $f_Q$ of $D(P)^G$ described in Proposition 12.3 can be expressed as*

$$f_Q = \sum_{g\in[G/PN_G(Q)]} \Omega_{k\uparrow^P_{{}^xQ}} = \mathrm{Res}^G_P(\Omega_{k\uparrow^G_Q}) + X$$

*where $X \in \langle\{f_R \in D(P)^G \mid R \lneq P \text{ and } |R| < |Q|\}\rangle$.*

*Proof.* The proof proceeds by induction on the order of the subgroup $Q$.
**Case** $|Q| = 1$: by part (d) of Lemma 2.15, $\mathrm{Res}^G_P(\Omega_{k\uparrow^G_{\{1\}}}) = \Omega_{k\uparrow^G_{\{1\}}} = f_{\{1\}}$. Hence $f_{\{1\}} \in \mathrm{Res}^G_P(D^{\Omega}(G))$.
**Induction step**: Let $Q \lneq P$ such that $|Q| > 1$ and assume as induction hypothesis that for every subgroup $S \lneq P$ such that $|S| < |Q|$, the generator $f_S = \sum_{x\in[G/PN_G(S)]} \Omega_{k\uparrow^G_{{}^xS}}$ of $D(P)^G$ belongs to $\mathrm{Res}^G_P(D^{\Omega}(G))$. Again by part (d) of Lemma 2.15, in $D(P)$ we have

$$\mathrm{Res}^G_P(\Omega_{k\uparrow^G_Q}) = \Omega_{k\uparrow^G_Q\downarrow^G_P} = \Omega_V$$

where $V := \bigoplus_{x\in[G/P]} k\uparrow^P_{{}^xQ}$, so that the second equality follows from the Mackey formula. Taking the vision of $P$-sets, $\Omega_V = \Omega_Y$ where $Y$ is the $P$-set defined by $Y := \bigsqcup_{x\in[G/P]} P/{}^xQ$. Then [Bou00, Lem. 5.2.3] yields the formula

$$\Omega_Y = \sum_{\substack{U,V\in[s_P]\\ U\leqslant_P V\\ Y^V\neq\varnothing}} \mu_P(U,V)\Omega_{P/U}$$

where $[s_P]$ is a set of representatives of conjugacy classes, under the action of $P$, of subgroups in $P$ and $\mu_P$ is the Möbius function of the poset $([s_P], \leqslant_P)$. Translating this in terms of $kP$-modules yields:

$$\begin{aligned}\Omega_V &= \sum_{\substack{U\in[s_P]\\ U\leqslant_G Q}} \Big( \sum_{\substack{V\in[s_P]\\ U\leqslant_P V\leqslant_G Q}} \mu_P(U,V)\Big)\Omega_{k\Uparrow_U^P} = \sum_{\substack{U\in[s_P]\\ U=_G Q}} \Omega_{k\Uparrow_U^P} + \sum_{\substack{U\in[s_P]\\ U\lneq_G Q}} \Big( \sum_{\substack{V\in[s_P]\\ U\leqslant_P V\leqslant_G Q}} \mu_P(U,V)\Big)\Omega_{k\Uparrow_U^P}\\ &= \sum_{x\in[G/PN_G(Q)]} \Omega_{k\Uparrow^P_{{}^xQ}} + \sum_{\substack{U\in[G\backslash[s_P]]\\ U<_G Q}} \Bigg(\Big( \sum_{\substack{V\in[s_P]\\ U\leqslant_P V\leqslant_G Q}} \mu_P(U,V)\Big) \sum_{x\in[G/PN_G(U)]} \Omega_{k\Uparrow^P_{{}^xU}}\Bigg)\\ &= f_Q + \sum_{\substack{U\in[G\backslash[s_P]]\\ U<_G Q}} \Big( \sum_{\substack{V\in[s_P]\\ U\leqslant_P V\leqslant_G Q}} \mu_P(U,V)\Big) f_U\end{aligned}$$

where $[G\backslash[s_P]]$ denotes a set of representatives of conjugacy classes of classes of subgroups in $[s_P]$ under the left action of $G$. Then set

$$X := - \sum_{\substack{U\in[G\backslash[s_P]]\\ U<_G Q}} \Big( \sum_{\substack{V\in[s_P]\\ U\leqslant_P V\leqslant_G Q}} \mu_P(U,V)\Big) f_U \in \langle\{f_R \in D(P)^G \mid R \lneq P, |R| < |Q|\}\rangle .$$

By induction hypothesis $X \in \mathrm{Res}^G_P(D^\Omega(G))$. It follows that $f_Q = \mathrm{Res}^G_P(\Omega_{k\Uparrow_Q^G}) + X \in \mathrm{Res}^G_P(D^\Omega(G))$, as required. □

**Theorem 12.6.** *Let $p$ be an odd prime and $G$ a finite group having a normal Sylow $p$-subgroup. Then*

$$D(G) = X(G) + D^\Omega(G) .$$

*Proof.* By Theorem 7.3 the restriction map $\mathrm{Res}^G_P : D(G) \longrightarrow D(P)^G$ induces an isomorphism $D(G)/X(G) \cong D(P)^G$. Moreover the previous proposition states that restriction of $\mathrm{Res}^G_P$ to $D^\Omega(G)$ is surjective onto $D(P)^G$. Hence the result. □

Notice that the sum $D(G) = X(G) + D^\Omega(G)$ of Theorem 12.6 need not be direct. A counterexample is provided by taking $G$ to be a group with a normal Sylow $p$-subgroup isomorphic to a cyclic $p$-group $C_{p^n}$ with $p, n \geqslant 3$. Indeed, Theorem 10.1 states that $D(G) = T_{k\Uparrow^G_{Z_{n-1}}}(G)$, and it can be seen from the description by generators and relations of this group given in [Las11a, Thm. 8.2.6] that there exists a class $[\nu] \in X(G)$, $[\nu] \neq [k]$, such that $2\Omega_{k\Uparrow^G_{Z_{n-1}}} = [\nu]$ and thus $[\nu]$ belongs to both $X(G)$ and $D^\Omega(G)$.

### (c) $D^\Omega$ and control of fusion.

**Lemma 12.7.** *Let $H$ be a subgroup of $G$ containing the Sylow $p$-subgroup $P$ of $G$ and assume that $H$ controls $p$-fusion in $G$. Then the restriction map $\mathrm{Res}^G_H : D^\Omega(G) \longrightarrow D^\Omega(H)$ is surjective. Moreover, if $H$ contains $N_G(P)$, then restriction induces an isomorphism $D^\Omega(G) \cong D^\Omega(H)$.*

*Proof.* The proof is similar to that of Proposition 12.5. We claim that for every subgroup $Q \lneq P$, $\mathrm{Res}^G_H(\Omega_{k\Uparrow_Q^G}) = \Omega_{k\Uparrow_Q^H} + X$ with $X \in \langle\{\Omega_{k\Uparrow_R^H} \in D^\Omega(H) \mid R \lneq P, |R| < |Q|\}\rangle$. We proceed by induction on the order of the subgroup $Q$.

**Case** $|Q| = 1$: $\Omega_{k\Uparrow_Q^G} = \Omega$ so that by Lemma 2.15, part (d), $\mathrm{Res}^G_H(\Omega) = \Omega = \Omega_{k\Uparrow^H_{\{1\}}} \in \mathrm{Res}^G_H(D^\Omega(G))$.

**Induction step**: Let $Q \lneq P$ be a subgroup such that $|Q| \geqslant 2$ and assume that $\mathrm{Res}^G_H(\Omega_{k\Uparrow_S^G})$ has the required form for every subgroup $S \lneq P$ such that $|S| < |Q|$. Compute, by part (d) of Lemma 2.15, that

$$\mathrm{Res}^G_H(\Omega_{k\Uparrow_Q^G}) = \Omega_{k\Uparrow_Q^G\downarrow_H^G} = \Omega_V ,$$

where by the Mackey Formula one can set $V := \bigoplus_{x\in[H\backslash G/Q]} k\uparrow^H_{{}^xQ\cap H}$. Then decompose

$$V = \bigoplus_{x\in[H\backslash G/Q]} k\uparrow^H_{{}^xQ\cap H} = \bigoplus_{\substack{x\in[H\backslash G/Q]\\ {}^xQ\leqslant H}} k\uparrow^H_{{}^xQ} \oplus \bigoplus_{\substack{x\in[H\backslash G/Q]\\ {}^xQ\nleqslant H}} k\uparrow^H_{{}^xQ\cap H} .$$

Write $V_1 := \bigoplus_{x\in[H\backslash G/Q],\, {}^xQ\leqslant H} k\uparrow^H_{{}^xQ}$ and $V_2 := \bigoplus_{x\in[H\backslash G/Q],\, {}^xQ\nleqslant H} k\uparrow^H_{{}^xQ\cap H}$. Then, by part (a) of Lemma 2.16, $\Omega_V = \Omega_{V_1} + \Omega_{V_2} - \Omega_{V_1\otimes V_2}$.

Now, firstly, since $H$ controls fusion, for every $x \in [H\backslash G/Q]$ such that ${}^xQ \leqslant H$, there exists $h \in H$, such that ${}^xQ = {}^hQ$. In consequence $Proj(V_1) = Proj(k\uparrow^H_Q)$ by Proposition 2.2 (c) and (e), and thus by part (a) of Lemma 2.15 we have $\Omega_{V_1} = \Omega_{k\uparrow^H_Q}$.

Secondly, $Proj(V_2)$ corresponds to projectivity relative to the family of subgroups $\mathcal{H} := \{{}^xQ \cap H \,|\, x \in [H\backslash G/Q],\, {}^xQ \nleqslant H\}$, all of whose elements have order strictly smaller than $|Q|$. Therefore Remark 12.2 states that

$$\Omega_{V_2} = \Omega_{\mathcal{H}} \in \langle\{\Omega_{k\uparrow^G_S} \,|\, S \lneqq P, |S| < |Q|\}\rangle .$$

Thirdly, by (b) of Lemma 2.15, $\Omega_{V_1\otimes V_2} = \Omega_{\mathcal{H}\cap\, {}^H\{Q\}}$. Since $\mathcal{H}$ consists of subgroups all of order strictly smaller than $|Q|$, so does the family $\mathcal{H} \cap {}^H\{Q\}$. Thus, the same argument as above yields

$$\Omega_{V_1\otimes V_2} = \Omega_{\mathcal{H}\cap\, {}^H\{Q\}} \in \langle\{\Omega_{k\uparrow^G_S} \,|\, S \lneqq P, |S| < |Q|\}\rangle .$$

Therefore set $X := -\Omega_{V_2} + \Omega_{V_1\otimes V_2}$ so that

$$\Omega_{k\uparrow^H_Q} = \mathrm{Res}^G_H(\Omega_{k\uparrow^G_Q}) + X$$

with $X \in \langle\{\Omega_{k\uparrow^H_R} \in D^\Omega(H) \,|\, R \lneqq P, |R| < |Q|\}\rangle$, as required. Then, by induction hypothesis, $X \in \mathrm{Res}^G_H(D^\Omega(G))$ and thus so does $\Omega_{k\uparrow^H_Q}$. In conclusion all the generators of $D^\Omega(H)$ are in $\mathrm{Res}^G_H(D^\Omega(G))$ and the surjectivity of $\mathrm{Res}^G_H : D^\Omega(G) \longrightarrow D^\Omega(H)$ follows.

Finally, if $N_G(P) \leqslant H \leqslant G$, the map $\mathrm{Res}^G_H : D(G) \longrightarrow D(H)$ is injective by Lemma 6.1. Hence the isomorphism follows. ☐

**Corollary 12.8.** *Let $p$ be an odd prime. If $P$ is a Sylow $p$-subgroup of $G$ and $N_G(P)$ controls $p$-fusion in $G$, then the Dade group decomposes as*

$$D(G) = D^\Omega(G) + \Gamma(X) .$$

*Proof.* Theorem 7.3 provides us with the exact sequence

$$0 \longrightarrow \Gamma(X) \hookrightarrow D(G) \xrightarrow{\mathrm{Res}^G_P} D(P)^{G-st} \longrightarrow 0 .$$

Thus it suffices to prove that the map $\mathrm{Res}^G_P : D^\Omega(G) \longrightarrow D(P)^{G-st}$ is surjective. Indeed, since $N_G(P)$ controls $p$-fusion, $D(P)^{N_G(P)} = D(P)^{G-st}$ by Proposition 8.2. Therefore, the map $\mathrm{Res}^G_P :$ $D^\Omega(G) \longrightarrow D(P)^{G-st}$ is equal to the composition

$$D^\Omega(G) \xrightarrow{\mathrm{Res}^G_{N_G(P)}} D^\Omega(N_G(P)) \xrightarrow{\mathrm{Res}^{N_G(P)}_P} D(P)^{N_G(P)} = D(P)^{G-st}$$

where $\mathrm{Res}^G_{N_G(P)}$ is surjective by the previous lemma and $\mathrm{Res}^{N_G(P)}_P$ is surjective by Proposition 12.5. Hence the result. ☐

**(d) The example of $GL_3(\mathbb{F}_p)$.** Let $G = GL_3(\mathbb{F}_p)$ with $p$ an odd prime. This group has an extraspecial Sylow $p$-subgroup $P$ of order $p^3$ consisting of the upper unitriangular matrices and generated by the three matrices

$$x := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ y := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } z := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since fusion in $P$ is the same under the action of $GL_3(\mathbb{F}_p)$ or under the action of $PSL_3(\mathbb{F}_p)$, $D(P)^{G-st} = D(P)^{PSL_3(\mathbb{F}_p)-st}$. This last group was computed in [LM09, Example 6.6], by the following general method:

$$D(P)^{G-st} = D(P)^{N_G(P)} \cap \bigcap_{\substack{E \leqslant P \\ E\ p\text{-essential}}} D(P)^{N_G(E)-st}.$$

(This is actually a consequence of Alperin's fusion Theorem.) In the current case $GL_3(\mathbb{F}_p)$ has exactly two $p$-essential subgroups, namely $E_1 :=< x, z >$ and $E_2 :=< y, z >$. Moreover

$$N_G(E_1) = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{$GL_2(\mathbb{F}_p)$}} & * \\ & & * \\ \hline 0 & 0 & \mathbb{F}_p^* \end{array}\right) =: H_p \text{ and } N_G(E_2) = \left(\begin{array}{c|cc} \mathbb{F}_p^* & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$GL_2(\mathbb{F}_p)$}} \\ 0 & & \end{array}\right) =: K_p\,.$$

which are the two maximal parabolic subgroups in $GL_3(\mathbb{F}_p)$ and both of which contain $N_G(P)$. Hence

$$D(P)^{G-st} = D(P)^{H_p-st} \cap D(P)^{K_p-st}\,.$$

**Proposition 12.9.** *Let $p$ be an odd prime. Let $G := GL_3(\mathbb{F}_p)$ and let $H_p$ and $K_p$ be as above. Then the three restriction maps $\mathrm{Res}^G_P : D^\Omega(G) \longrightarrow D(P)^{G-st}$, $\mathrm{Res}^{H_p}_P : D^\Omega(H_p) \longrightarrow D(P)^{H_p-st}$, and $\mathrm{Res}^{K_p}_P : D^\Omega(H_p) \longrightarrow D(P)^{K_p-st}$ are surjective.*

*Proof.* Detailed computations for the proof of this proposition can be found in [Las11b, Sect. 7.11]. The method is to find sets of generators for the groups $D(P)^{H_p-st}$ and $D(P)^{K_p-st}$ and thus for $D(P)^{G-st} = D(P)^{H_p-st} \cap D(P)^{K_p-st}$, and then show that all these generators are in the image of the corresponding restriction map.

□

Note that, in particular, the computations in [Las11b, Sect. 7.11] prove that the set of generators for $D(P)^{PSL_3(\mathbb{F}_p)-st}$ computed in [LM09, Example 6.6] misses one generator to be complete.

Finally, as above, the surjectivity of the three restriction maps $\mathrm{Res}^G_P$, $\mathrm{Res}^{H_p}_P$, and $\mathrm{Res}^{H_p}_P$ of Proposition 12.9 imply that

$$D(G) = D^\Omega(G) + \Gamma(X(N_G(P)))\,;$$
$$D(H_p) = D^\Omega(H_p) + \Gamma(X(N_{H_p}(P)))\,;$$
$$D(K_p) = D^\Omega(K_p) + \Gamma(X(N_{K_p}(P)))\,.$$

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